

Automatic calculation of plane loci using Groebner bases and integration into a Dynamic Geometry System

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Overview

JSXGraph - A short overview

Computing plane loci using Groebner bases

Implementing this algorithm in JSXGraph

Optimizations

Examples

```
<script src="http://www.jsxgraph.org/jsxgraph.js">  
var brd = JSXGraph.initBoard('box', {ax:  
var s = brd.createElement('slider', [[1,3], [5  
var a = brd.createElement('slider', [[1,2], [5  
var b = brd.createElement('slider', [[1,1], [5  
var f = function(x){ return Math.sin(x); }  
var plot = brd.createElement('functiongraph',  
var os = brd.createElement('riemannsum', {f:  
function(){ return s.Value(); }  
function(){return a.Value(); }  
function(){return b.Value(); }  
},  
{fillColor:'#ffff00',
```



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JSXGraph

```
<script src="/javascript">  
var brd = JSXGraph.initBoard('box', {ax:  
var s = brd.createElement('slider', [[1,3],[5  
var a = brd.createElement('slider', [[1,2],[5  
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{fillColor:'#ffff00'
```



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JSXGraph

What is JSXGraph?

- ▶ A library implemented in JavaScript
- ▶ Runs in recent versions of all major browsers
- ▶ No plugins required
- ▶ LGPL-Licensed

Main features

- ▶ Dynamic Geometry
- ▶ Interactive function plotting
- ▶ Turtle Graphics
- ▶ Charts

```
<script src="/js/javascript">  
var brd = JSXGraph.initBoard('box', {ax:  
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var a = brd.createElement('slider', [[1,2], [5  
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function() { return s.Value(); }  
function() { return a.Value(); }  
function() { return b.Value(); }  
},  
{fillColor: '#ffff00';
```



Supported Hardware

- ▶ PC (Windows, Linux, Mac)
- ▶ Mobile phones
- ▶ "Touchpads" like the Apple iPod and iPad
- ▶ Basically everything which runs at least one of the supported browsers

```
<script type="text/javascript">  
var g = JSXGraph({  
  board: JSXGraph.initBoard('box', {ax:  
    var a = brd.createElement('slider', [[1,3],[5,  
    var b = brd.createElement('slider', [[1,2],[5,  
    var f = function(x){ return Math.sin(x); }  
    var plot = brd.createElement('functiongraph',  
    var os = brd.createElement('riemannsum', {f:  
      function(){ return s.Value(); }  
      function(){return a.Value(); }  
      function(){return b.Value(); }  
    },  
    {fillColor:'#ffff00'  
  }  
});
```



Supported Browsers

- ▶ Firefox
- ▶ Chrome/Chromium
- ▶ Safari
- ▶ Internet Explorer
- ▶ Opera

```
<script src="/js/javascript">  
var brd = JSXGraph.initBoard('box', {ax:  
var s = brd.createElement('slider', [[1,3],[5  
var a = brd.createElement('slider', [[1,2],[5  
var b = brd.createElement('slider', [[1,1],[5  
var f = function(x){ return Math.sin(x); }  
var plot = brd.createElement('functiongraph',  
var os = brd.createElement('riemannsum', {f  
function(){ return s.Value(); }  
function(){return a.Value(); }  
function(){return b.Value(); }  
},  
{fillColor:'#ffff00'
```



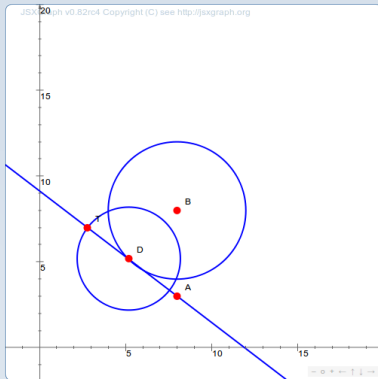
Example/Input

```
<link rel="stylesheet" type="text/css" href="css/jsxgraph.css" />
<script type="text/javascript" src="js/jsxgraphcore.js"></script>
[...]
```

```
<script type="text/javascript">
  /*  */
  board = JSXGraph.initBoard('jxgbox', {boundingbox: [-2, 20, 20, -2], axis:
    true, grid: false, keepaspectratio: true});
  p3 = board.create('point', [8, 3]);
  p4 = board.create('point', [8, 8]);
  c1 = board.create('circle', [p4, 4]);
  p6 = board.create('glider', [0, 0, c1], {name: 'D'});
  g = board.create('line', [p3, p6]);
  c2 = board.create('circle', [p6, 3]);
  p14_1 = board.create('intersection', [c2, g, 0], {name: 'T'});
  /*  */
</script>
```



Example/Output



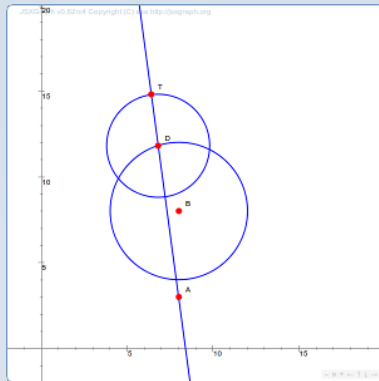
Computing plane loci using Groebner bases (in a nutshell)

```
<script src="/javascript">  
var brd = JSXGraph.initBoard('box', {ax:  
var s = brd.createElement('slider', [[1,3],[5  
var a = brd.createElement('slider', [[1,2],[5  
var b = brd.createElement('slider', [[1,1],[5  
var f = function(x){ return Math.sin(x); }  
var plot = brd.createElement('functiongraph',  
var os = brd.createElement('riemannsum', {f:  
function(){ return s.Value(); }  
function(){return a.Value(); }  
function(){return b.Value(); }  
},  
{fillColor:'#ffff00'
```



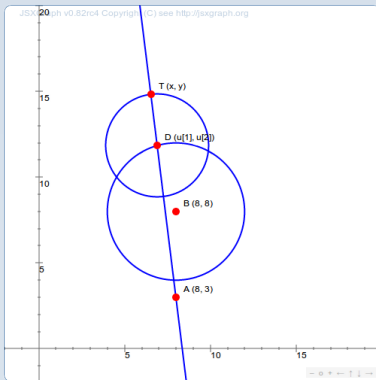
Computing plane loci using Groebner bases

- Given a set of free and dependent points,



Computing plane loci using Groebner bases

- ▶ we first choose a coordinate system,



Computing plane loci using Groebner bases

- ▶ translate geometric constraints into an algebraic form,

- ▶ $(u[1] - 8)^2 + (u[2] - 8)^2 - 16 = 0$

- ▶ $(x - u[1])^2 + (y - u[2])^2 - 9 = 0$

- ▶ $3x - 3u[1] + yu[1] - 8y + 8u[2] - xu[2] = 0$



Computing plane loci using Groebner bases

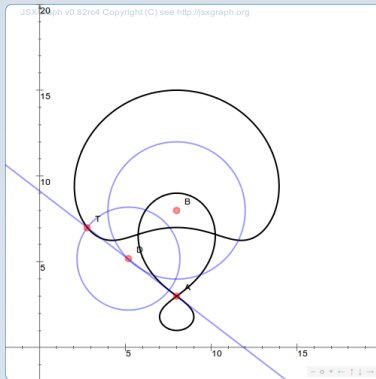
- ▶ calculate the Gröbner basis of the given ideal,

- ▶
$$x^6 + 3x^4y^2 + 3x^2y^4 + y^6 - 48x^5 - 38x^4y - 96x^3y^2 - 76x^2y^3 - 48xy^4 - 38y^5 + 1047x^4 + 1216x^3y + 1774x^2y^2 + 1216xy^3 + 727y^4 - 13024x^3 - 16596x^2y - 16096xy^2 - 8404y^3 + 97395x^2 + 109888xy + 63535y^2 - 415536x - 300806y + 790009 = 0$$



Computing plane loci using Groebner bases

- ▶ and finally plot the calculated implicit equation.



Implementing this algorithm in JSXGraph

```
<script type="text/javascript">  
var brd = JSXGraph.initBoard('box', {ax:  
var s = brd.createElement('slider', [[1,3],[5  
var a = brd.createElement('slider', [[1,2],[5  
var b = brd.createElement('slider', [[1,1],[5  
var f = function(x){ return Math.sin(x); }  
var plot = brd.createElement('functiongraph',  
var os = brd.createElement('riemannsum', {f:  
function(){ return s.Value(); }  
function(){return a.Value(); }  
function(){return b.Value(); }  
},  
{fillColor:'#ffff00'
```



Implementation

AJAX

- ▶ Transfer data (a)synchronously via HTTP with JavaScript

This enables us to

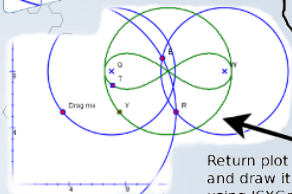
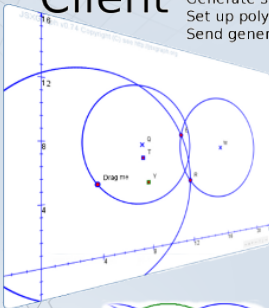
- ▶ use a computer algebra system on a (web) server for the expensive Gröbner basis calculations
- ▶ use a plotting tool/library for implicit plotting



Implementation

Client

Generate symbolic coordinates for free and dependent points.
Set up polynomials for the dependent ones describing their loci.
Send generated data to webserver for further calculations.



Return plot data
and draw it
using JSXGraph.

AJAX



Server

Several software packages are used server side:



python

is used to retrieve the data
and pass it on to

CoCoA handles the symbolic
algebra stuff and returns a
set of polynomials which are
plotted with the python library



matplotlib

Finally the locus curve is extracted as a list of
coordinates from the plots and is sent back to
JSXGraph where the data is used to plot the
locus directly in the geometric construction.



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Implementation

Example/Input

```
<link rel="stylesheet" type="text/css" href="css/jsxgraph.css" />
<script type="text/javascript" src="js/jsxgraphcore.js"></script>
[...]
```

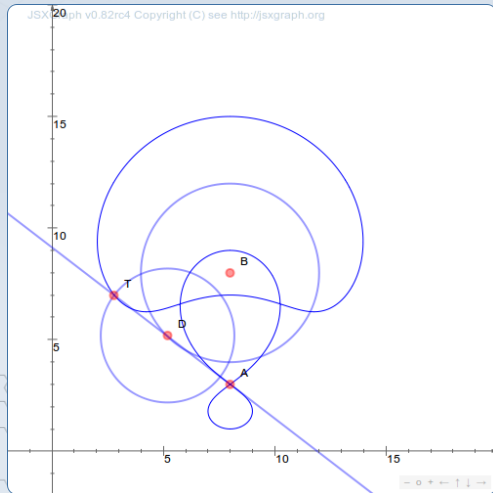
```
<div id="jxgbox" class="jxgbox" style="width:500px; height:500px;"></div>
```

```
<script type="text/javascript">
  /*  */
  board = JXG.JSXGraph.initBoard('jxgbox', {boundingbox: [-2, 20, 20, -2], axis:
    true, grid: false, keepaspectratio: true});
  p3 = board.create('point', [8, 3]);
  p4 = board.create('point', [8, 8]);
  c1 = board.create('circle', [p4, 4]);
  p6 = board.create('glider', [0, 0, c1], {name: 'D'});
  g = board.create('line', [p3, p6]);
  c2 = board.create('circle', [p6, 3]);
  p14_1 = board.create('intersection', [c2,g,0], {name: 'T'});

  locus = board.create('locus', [p14_1]);
  /* ]]&gt; */
&lt;/script&gt;</pre></div><div data-bbox="799 886 860 968" data-label="Image"><img alt="Logo of Universität Bayreuth, featuring a green square with a white stylized 'U' and 'B'."/></div><div data-bbox="865 902 994 957" data-label="Page-Footer"><p>UNIVERSITÄT<br/>BAYREUTH</p></div>
```

Implementation

Example/Output



Implementation

Ready-to-use elements

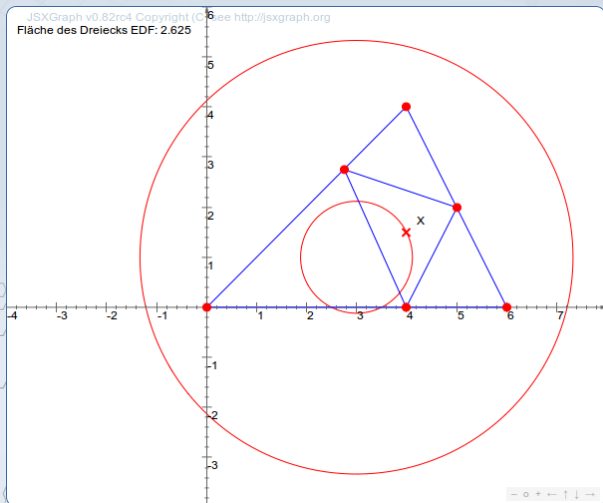
- ▶ Glider on circle and line
- ▶ Intersection points (circle/circle, circle/line, line/line)
- ▶ Midpoint
- ▶ Parallel line and point
- ▶ Perpendicular line and point
- ▶ Circumcircle and circumcenter

```
var brd = JSXGraph.initBoard('box', {ax.  
var s = brd.createElement('slider', [[1,3], [5  
var a = brd.createElement('slider', [[1,2], [5  
var b = brd.createElement('slider', [[1,1], [5  
var f = function(x) { return Math.sin(x); }  
var plot = brd.createElement('functiongraph',  
var cos = brd.createElement('riemannsum', {f  
function() { return s.Value(); }  
function() { return a.Value(); }  
function() { return b.Value(); }  
},  
{fillColor: '#ffff00'}
```



Implementation

Easy to extend



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Implementation

```
<link rel="stylesheet" type="text/css" href="css/jsxgraph.css" />
<script type="text/javascript" src="js/jsxgraphcore.js"></script>
[...]
```

```
<script type="text/javascript">
  /*  */
  board = JXG.JSXGraph.initBoard('jxgbox', {boundingbox:[-4, 6, 8, -4], axis:
    true, grid: false, keepaspectratio: true});
  A = board.createElement('point', [0, 0]);
  B = board.createElement('point', [6, 0]);
  C = board.createElement('point', [4, 4]);

  t1 = board.createElement('triangle', [A, B, C], {strokeWidth: '1px'});

  X = board.createElement('point', [4, 1.5], {name:"X"});

  L = board.createElement('perpendicularpoint', [X, t1.c]);
  M = board.createElement('perpendicularpoint', [X, t1.a]);
  N = board.createElement('perpendicularpoint', [X, t1.b]);

  t2 = board.createElement('triangle', [L, M, N], {strokeWidth: '1px'});

  [...]</pre></div><div data-bbox="799 885 860 968" data-label="Image"><img alt="Logo of Universität Bayreuth, featuring a stylized green and white design."/></div><div data-bbox="865 902 994 957" data-label="Page-Footer"><p>UNIVERSITÄT<br/>BAYREUTH</p></div>
```

Implementation

[...]

```
X.ancestors[L.id] = L;  
X.ancestors[M.id] = M;  
X.ancestors[N.id] = N;  
X.ancestors[A.id] = A;  
X.ancestors[B.id] = B;  
X.ancestors[C.id] = C;
```

```
X.generatePolynomial = function () {  
  var as16 = getTriangleArea(L, M, N),  
  as = '((( '+M.symbolic.x+')-('+N.symbolic.x+') )^2+(( '+M.symbolic.y+')-('+N.  
    symbolic.y+') )^2)',  
  bs = '((( '+L.symbolic.x+')-('+N.symbolic.x+') )^2+(( '+L.symbolic.y+')-('+N.  
    symbolic.y+') )^2)',  
  cs = '((( '+M.symbolic.x+')-('+L.symbolic.x+') )^2+(( '+M.symbolic.y+')-('+L.  
    symbolic.y+') )^2)',  
  
  return ['4*'+as+'*'+cs+'-('+as+'+'+cs+'-'+bs+')*('+as+'+'+cs+'-'+bs+')-('+  
    as16+'')'];  
};
```

```
locus = board.createElement('locus', [X], {strokeColor: 'red'});
```

```
/* ]]> */  
</script>
```



Implementation

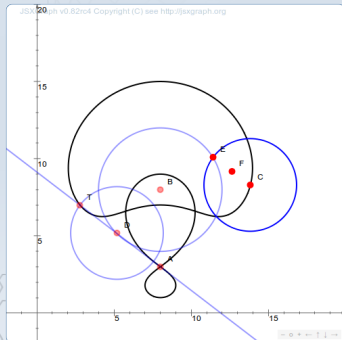
Re-using locus data: Discovered loci can be

- ▶ intersected with circles, lines, other curves, ...
- ▶ used as a base object for gliding points
- ▶ used for the discovery of other loci

```
var s = JXG.JSXGraph.initBoard('box', {ax:  
var a = brd.createElement('slider', [[1,3], [5  
var b = brd.createElement('slider', [[1,2], [5  
var f = function(x) { return Math.sin(x); }  
var plot = brd.createElement('functiongraph',  
var os = brd.createElement('riemannsum', {f:  
function() { return s.Value(); }  
function() { return a.Value(); }  
function() { return b.Value(); }  
},  
{fillColor: '#ffff00';
```



Implementation

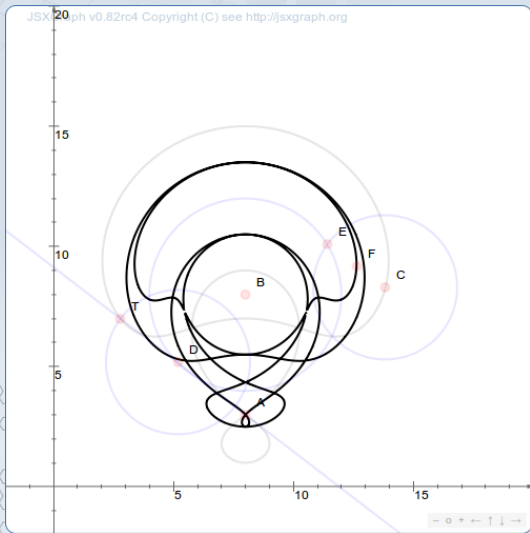


```
tg = board.create('glider', [loc]);  
  
tc = board.create('circle', [tg, 3]);  
ti = board.create('intersection', [tc, c1, 0]);  
tm = board.create('midpoint', [tg, ti]);
```

```
var brd = JSXGraph.initBoard('box', {ax:  
var s = brd.createElement('slider', [[1,3], [5  
var a = brd.createElement('slider', [[1,2], [5  
var b = brd.createElement('slider', [[1,1], [5,  
var f = function(x) { return Math.sin(x); }  
var plot = brd.createElement('functiongraph',  
var os = brd.createElement('riemannsum', {f:  
function() { return s.Value(); }  
function() { return a.Value(); }  
function() { return b.Value(); }  
},  
{fillColor: '#ffff00';
```



Implementation



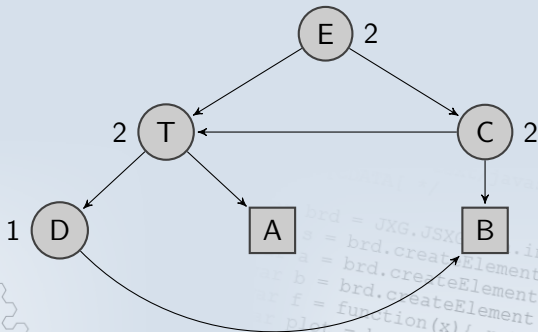
Optimization

```
.../javascript">  
var brd = JSXGraph.initBoard('box', {ax:  
var s = brd.createElement('slider', [[1,3],[5  
var a = brd.createElement('slider', [[1,2],[5  
var b = brd.createElement('slider', [[1,1],[5  
var f = function(x){ return Math.sin(x); }  
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var os = brd.createElement('riemannsum', {f:  
function(){ return s.Value(); }  
function(){return a.Value(); }  
function(){return b.Value(); }  
},  
{fillColor:'#ffff00'
```



Optimization

Idea: Divide and conquer



Optimization

Transformations

- ▶ Translate the construction moving one point to $(0,0)$
- ▶ Rotate the construction moving another point onto the x-axis
- ▶ After the Gröbner basis is calculated, the result is retransformed
- ▶ User can choose the two points or
- ▶ JSXGraph chooses two points (but sometimes not the best suited ones)



Examples

```
<script src="/javascript">  
var brd = JSXGraph.initBoard('box', {ax:  
var s = brd.createElement('slider', [[1,3],[5  
var a = brd.createElement('slider', [[1,2],[5  
var b = brd.createElement('slider', [[1,1],[5,  
var f = function(x){ return Math.sin(x); }  
var plot = brd.createElement('functiongraph',  
var os = brd.createElement('riemannsum', {f:  
function(){ return s.Value(); }  
function(){return a.Value(); }  
function(){return b.Value(); }  
},  
{fillColor:'#ffff00'
```



Last slide

Thank You

- ▶ <http://jsxgraph.org/>
- ▶ <http://jsxgraph.uni-bayreuth.de/wiki/>

```
<script type="text/javascript">  
var brd = JXG.JSXGraph.initBoard('box', {ax:  
var s = brd.createElement('slider', [[1,3],[5  
var a = brd.createElement('slider', [[1,2],[5  
var b = brd.createElement('slider', [[1,1],[5  
var f = function(x){ return Math.sin(x); }  
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function(){return a.Value(); }  
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{fillColor:'#ffff00',
```



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