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Exercise Sheet 03

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Total points: 12

1. Limiting degree distribution

(a) Consider a function **3P**

$$G_P(x; n, p) = \sum_{k=0}^{n} P_{n,p}(k)x^k \text{ with } P_{n,p}(k) = \binom{n}{k} \cdot p^k(1-p)^{n-k}$$

as well as a function

$$G_Q(x; \lambda) = \sum_{k=0}^{\infty} Q_\lambda(k)x^k \text{ with } Q_\lambda(k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

For $n \to \infty$ and $np \to \lambda$ show $G_P(x; n, p) \to G_Q(x; \lambda)$.

Hint: Use the binomial theorem to rewrite $G_P(x; n, p)$.

(b) In lecture L05 we have shown that the degrees of a random microstate generated by the $G(n, p)$ model follow a Binomial distribution. We further stated that for $n \to \infty$ and $np \to \lambda$ the degree distribution converges to the Poisson distribution with parameter λ. Investigate the definition of a so-called *probability generating function* and use it to explain how your proof from (a) can be used to prove the convergence to a Poisson degree distribution. **2P**

2. Diameter of random graphs

(a) Consider the $G(n, p)$ model of random graphs for a sufficiently large n and variable link probability p. Plot the dependency of the diameter of random realizations on the parameter p. Compare the observed dependency with the expected dependency based on the approximation of the diameter made in lecture L05. Explain possible discrepancies. **2P**

(b) Use the $G(n, p)$ model to generate a number of random microstates (undirected with self-loops) with a fixed expected degree $\langle k \rangle$. Plot the distribution of degrees of randomly chosen nodes in the generated microstates and compare it to the expected value $\langle k \rangle$. Repeat the same procedure for the degrees of *neighbors* of randomly chosen nodes. How do those distributions differ? Explain the phenomenon and how it might affect the approximation of the diameter made in lecture L05. **2P**

3. Watts-Strogatz Model

(a) Use the Watts-Strogatz model to generate microstates that exhibit the small-world property. Operationalize the definition of weak vs. strong ties from lecture L05, i.e. define an edge-level measure that captures to what extent an edge can be considered a weak tie. Use your measure to highlight weak ties in a plot of generated microstates. **1P**

(b) Show that the average clustering coefficient of a one-dimensional undirected ring lattice with parameter $s \geq 1$ is $\frac{3s-3}{4s-2}$. **1P**

(c) Use the Watts-Strogatz model to generate ring lattice networks (rewiring probability $p = 0$) with different lattice parameters $s \geq 1$. Plot the dependency of the average shortest path length of the ring lattice networks on the lattice parameter s. **1P**