دانشگاه صنعتی امیرکبیر
(پلی تکنیک تهران)

Electrical and Electronic Circuits

chapter 6. Basic RL and RC Circuits

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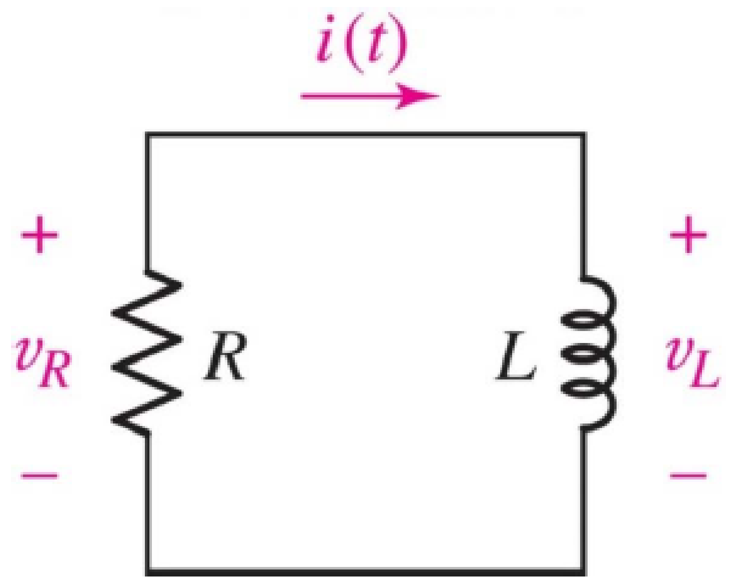
مهر ۱۴۰۳

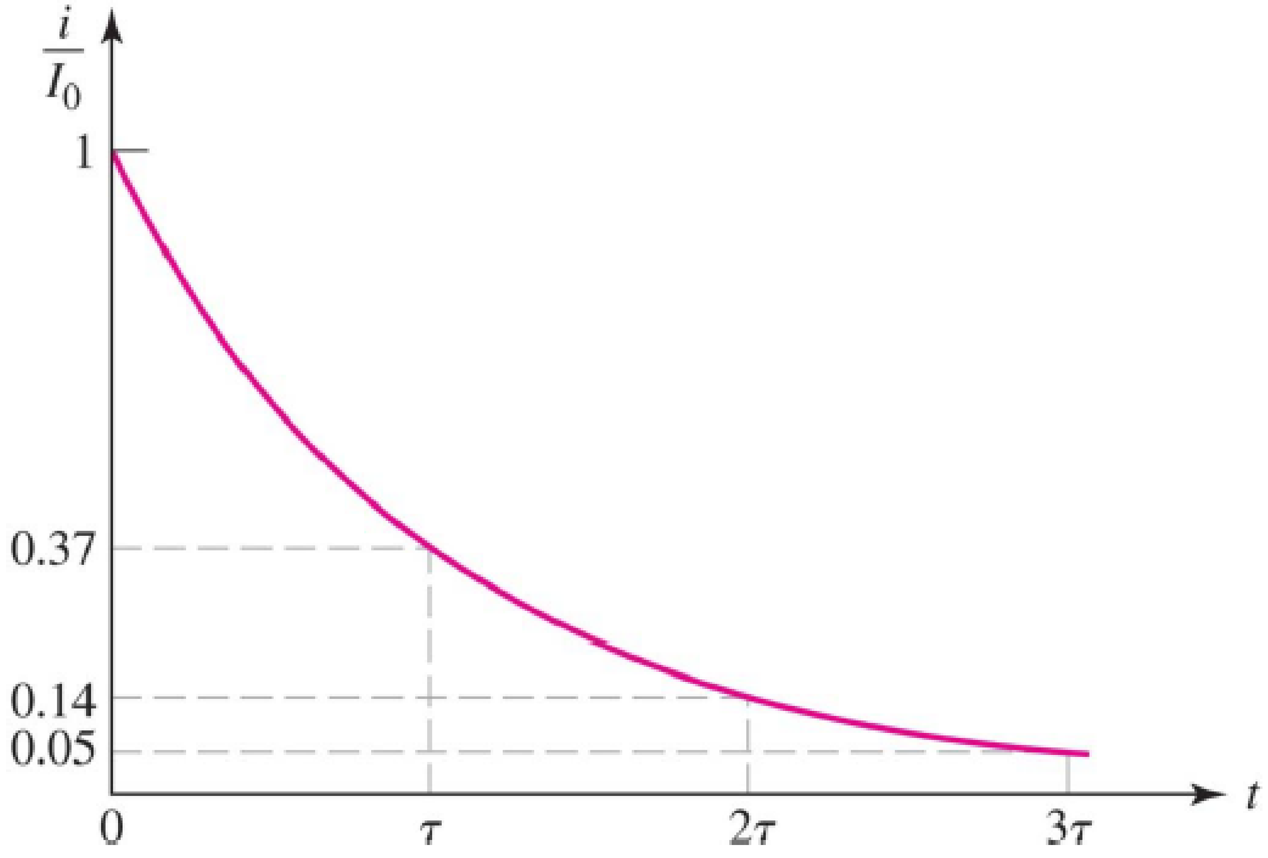
Objectives of the Lecture

- Determining the Time Response of First-Order Circuits
  - The Source-Free RL Circuit
  - The Source-Free RC Circuit
  - Driven RL Circuits
  - Driven RC Circuits

Objective

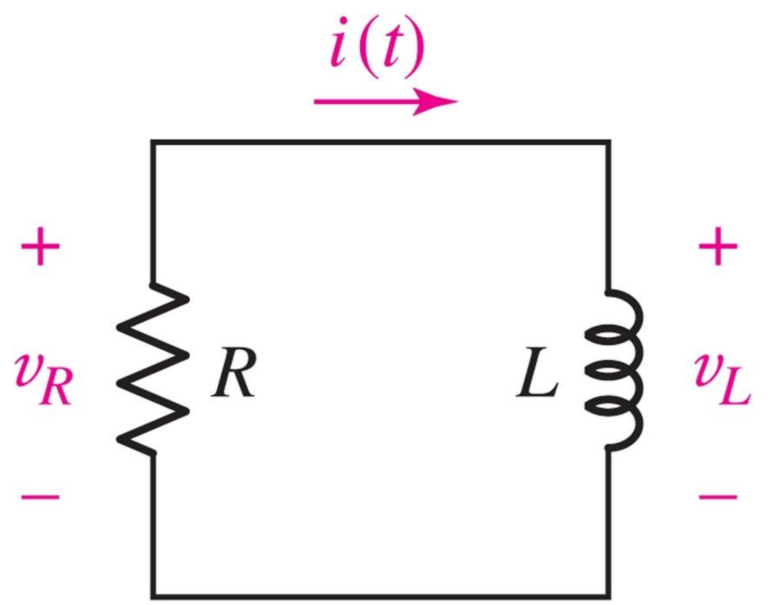
- ❖ Time Response Analysis of First-Order (RC or RL) Circuits
- ❖ Examining and analysing the charging or discharging behaviour of inductors and capacitors over time, and deriving a mathematical expression for this behaviour.





The Source-Free RL Circuit

- Applying KVL:



$$Ri + v_L = Ri + L\frac{di}{dt} = 0$$

$$\frac{di}{dt} + \frac{R}{L}i = 0$$

- We can solve for the *natural response* if we know the *initial condition* *i(0)=I$_0$:*

$$i(t) = I_0 e^{\frac{-Rt}{L}} \text{ for } t > 0$$

Natural response of the first order equation

➢ First solution:

$$\frac{di}{dt}+\frac{R}{L}i=0 \rightarrow \frac{di}{i}=-\frac{R}{L}\,dt$$

$$\int_{I_0}^{i(t)}\frac{di'}{i'}=\int_0^t -\frac{R}{L}\,dt'$$

$$\ln i-\ln I_0=-\frac{R}{L}(t-0)$$

$$\ln i=-\frac{Rt}{L}+k \longrightarrow i=Ae^{-\frac{Rt}{L}}$$

$$i(0)=I_0 \longrightarrow i(t)=I_0e^{\frac{-Rt}{L}}$$

Natural response of the first order equation

- The second solution: forming the characteristic equation
- Many of the differential equations encountered in circuit analysis have a solution which may be represented by the exponential function or by the sum of several exponential functions.

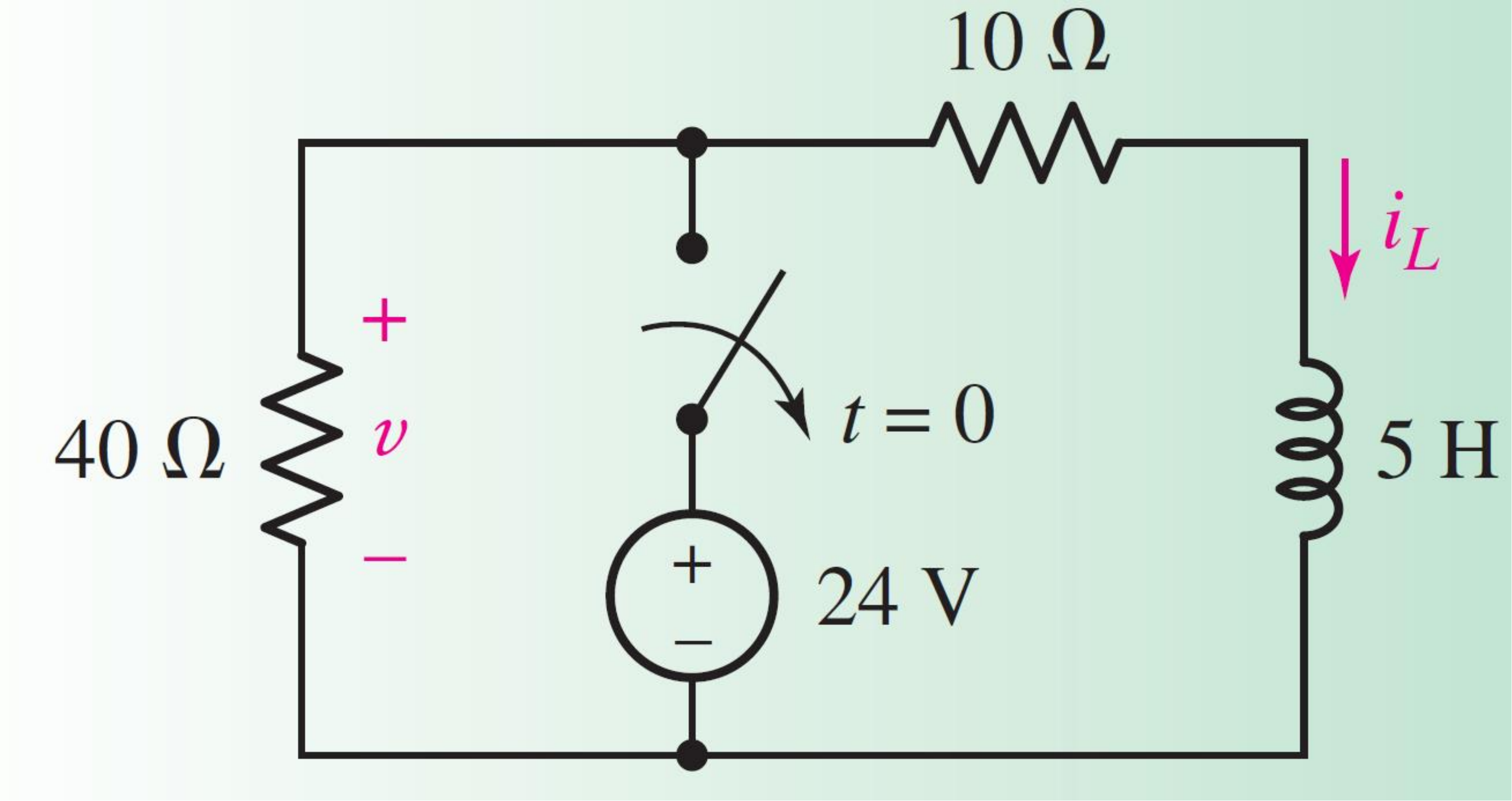
$$s + \frac{R}{L} = 0 \rightarrow s = -\frac{R}{L}$$

$$i = Ae^{st} \rightarrow i = Ae^{-\frac{Rt}{L}}$$

$$i(0) = I_0 \rightarrow i(t) = I_0 e^{\frac{-Rt}{L}}$$

Example: RL with a Switch

Show that the voltage v(t) will be -12.99 volts at t=200 ms.

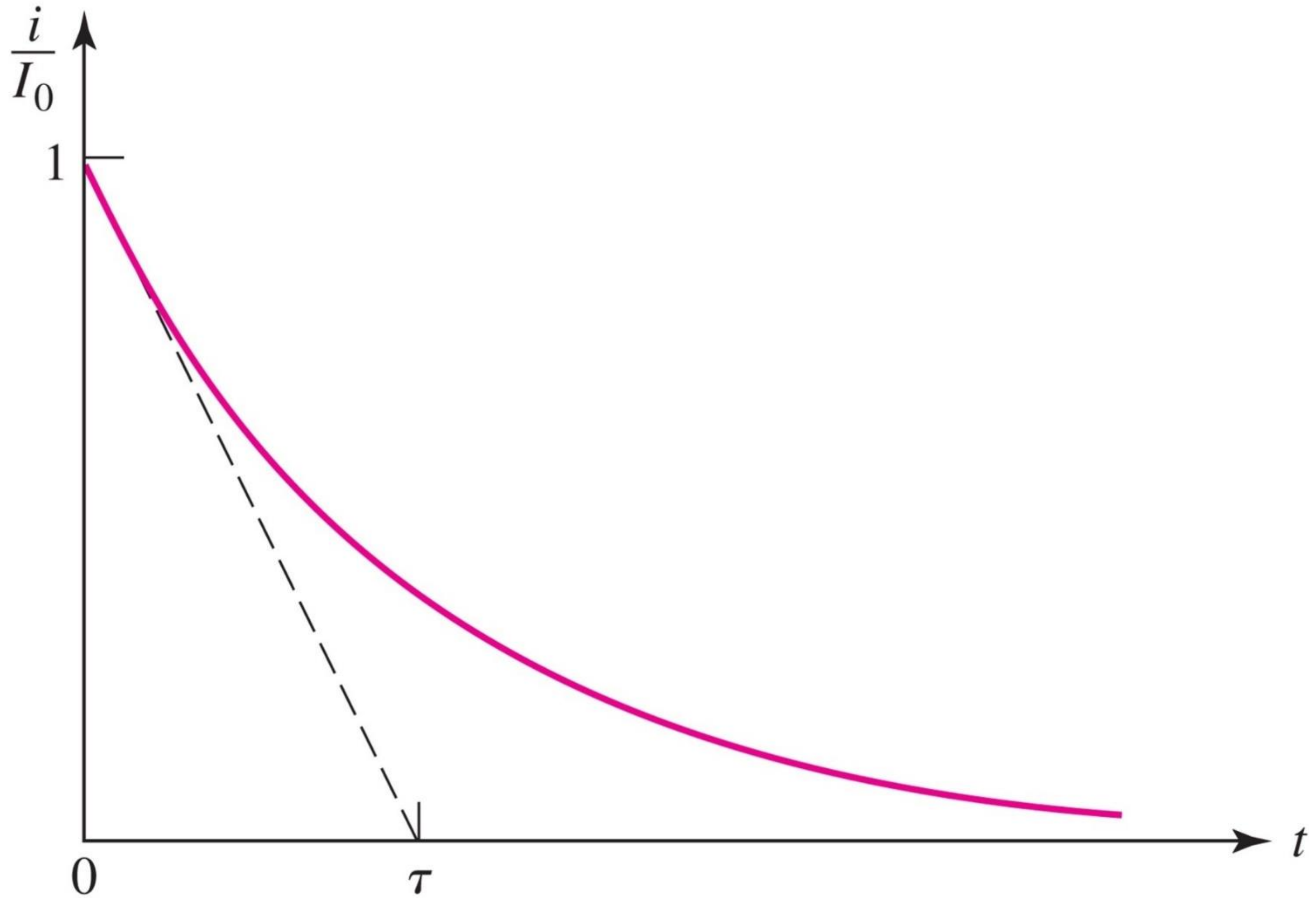


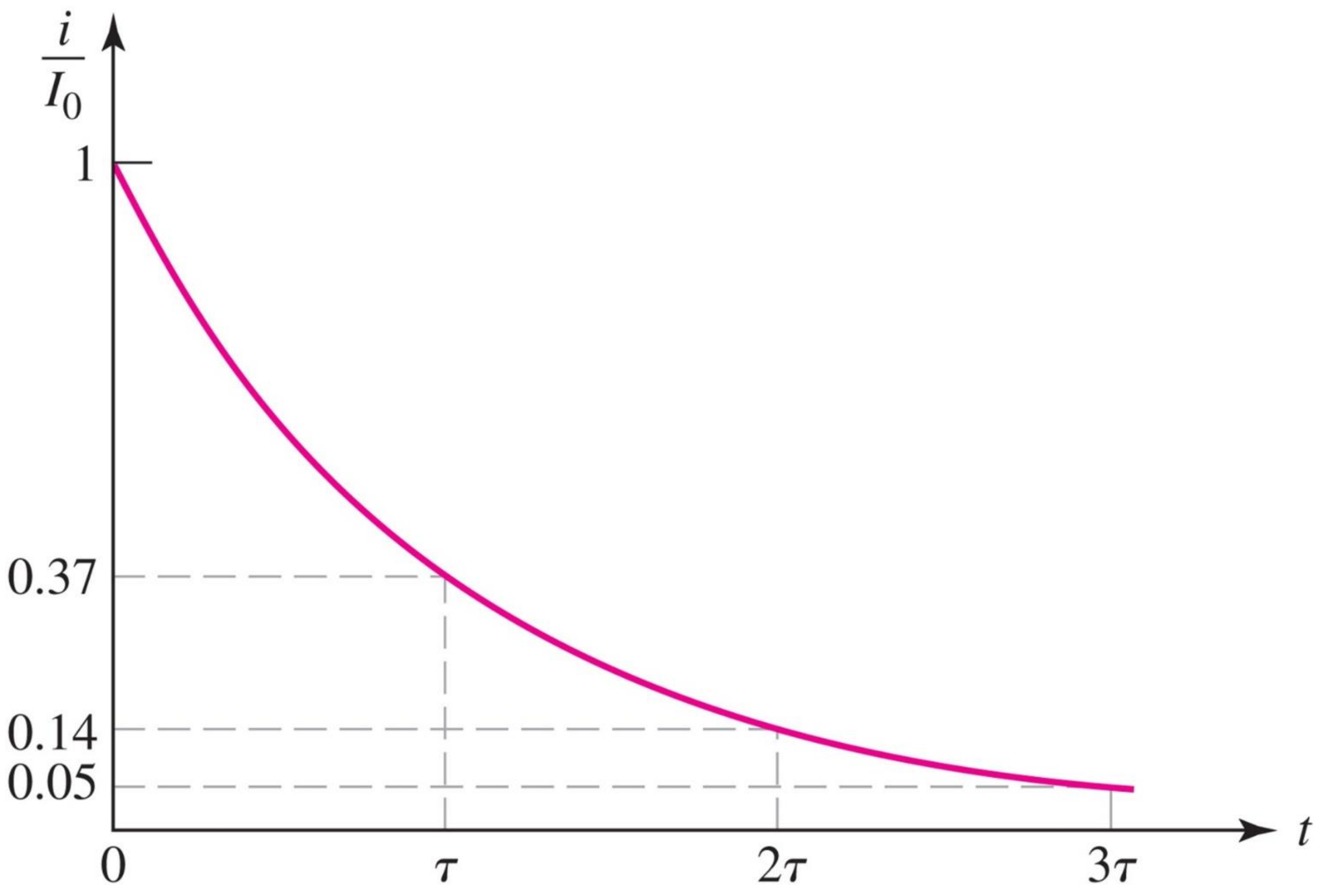
The Exponential Response

- ✓ The ratio L/R has the units of seconds, since the exponent $-Rt/L$ must be dimensionless.
- ✓ The time constant $\tau = L/R$ determines the rate of decay.
- ✓ The larger the value, the slower the function is damped.

$$i(t) = I_0 e^{-t/\tau}$$

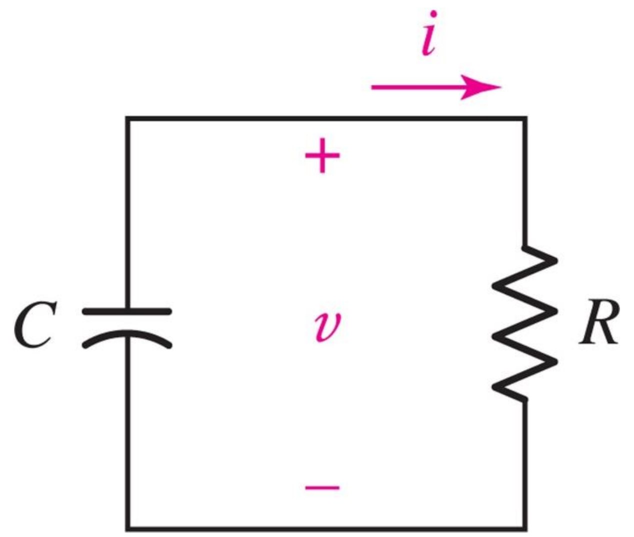
$$\frac{i(\tau)}{I_0} = e^{-1} = 0.3679 \qquad \text{or} \qquad i(\tau) = 0.3679 I_0$$





The Source-Free RC Circuit

Applying KCL:



$$C\frac{dv}{dt}+\frac{v}{R}=0$$

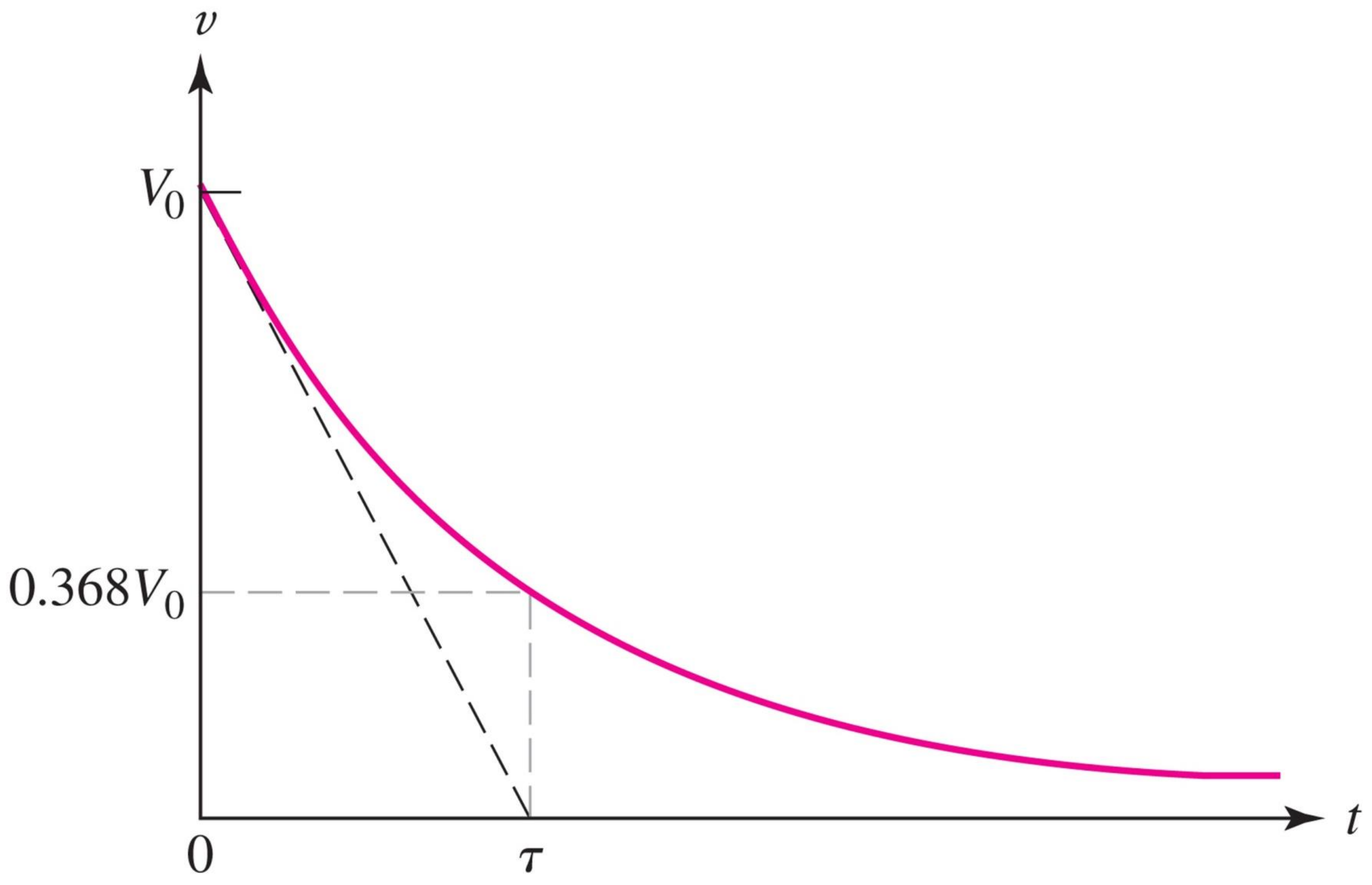
$$\frac{dv}{dt}+\frac{1}{RC}v=0$$

We can solve for the *natural response* if we know the *initial condition* $v(0)=V_0$

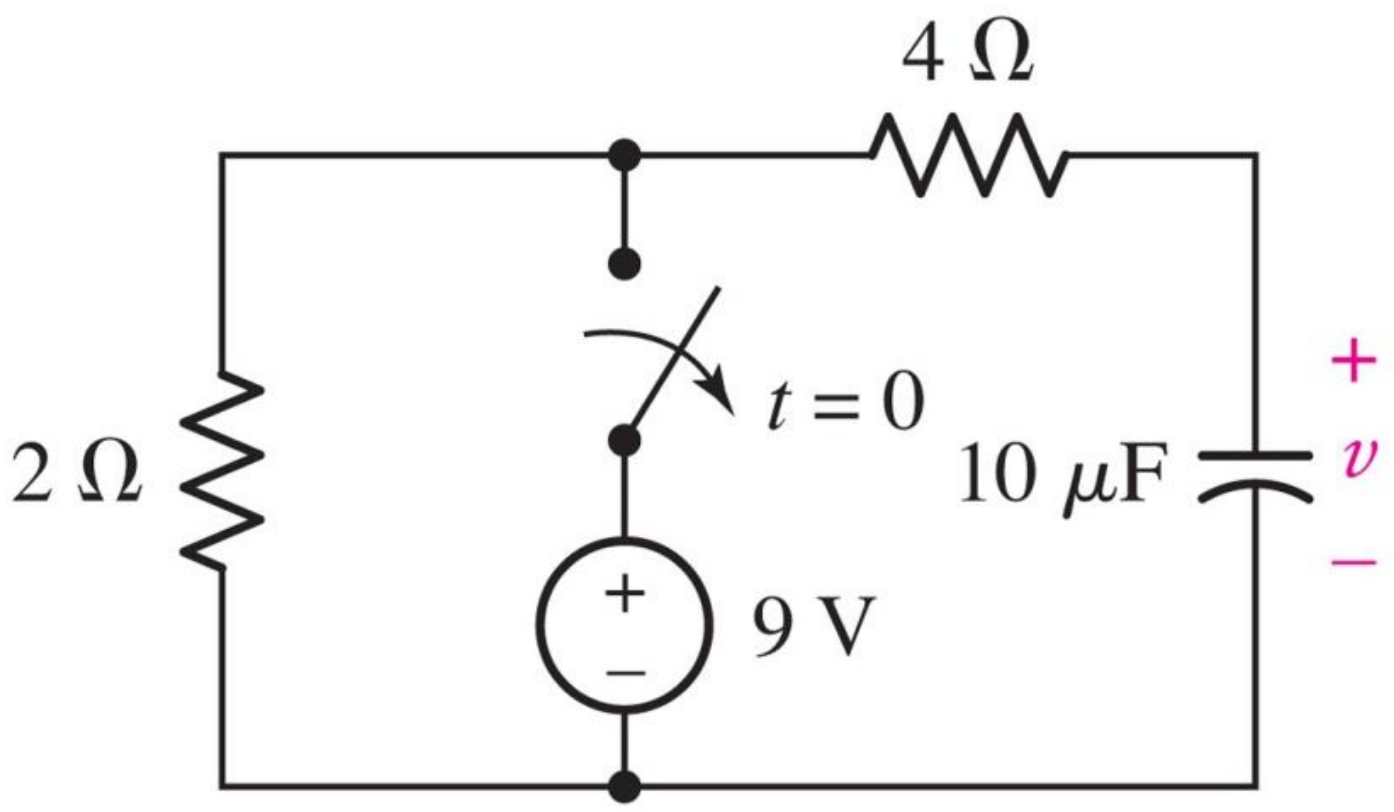
$$v(t)=v_0e^{\frac{-t}{RC}}\text{ for }\ t>0$$

RC Natural Response

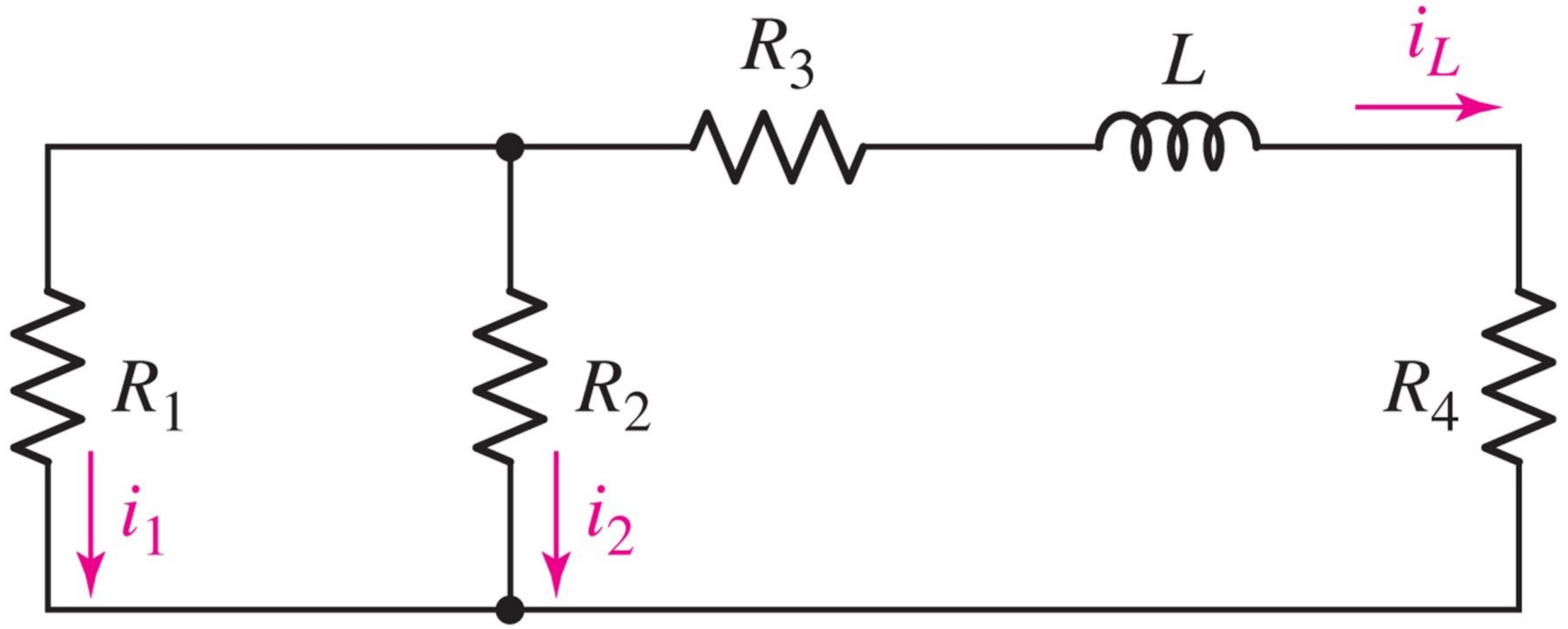
- ➢The time constant is τ=RC
- ➢The time at which the response has dropped to 37 percent of its initial value



Show that the voltage *v(t)* is 321 mV at t=200 µs.

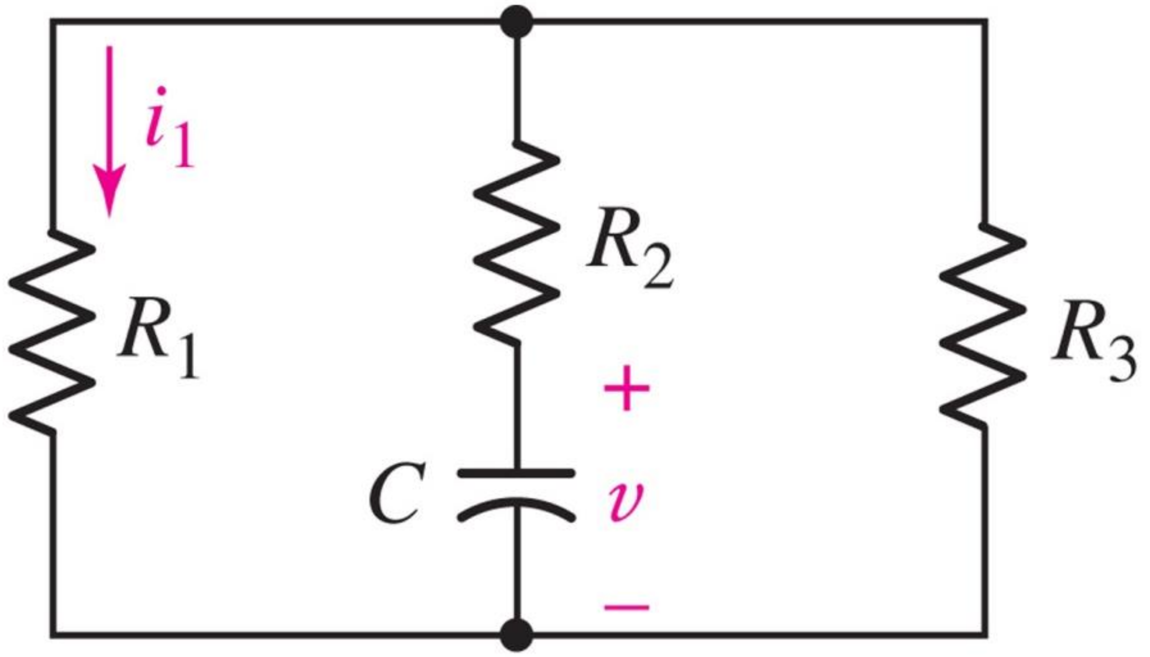


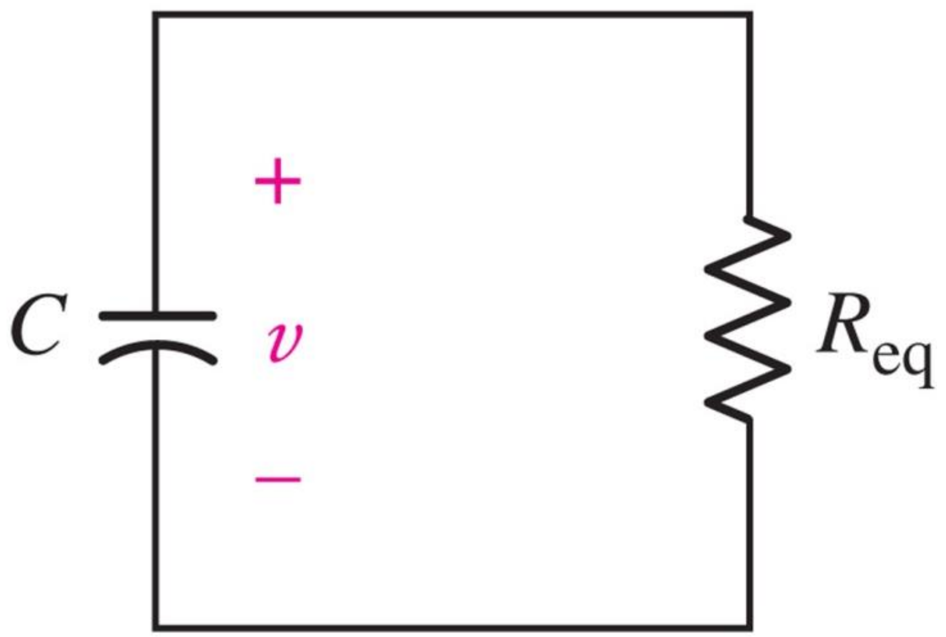
The time constant of a single-inductor circuit will be $\tau = L/R_{eq}$ where Req is the resistance seen by the inductor.



Example: $R_{eq} = R_3 + R_4 + \dfrac{R_1 R_2}{(R_1 + R_2)}$

The time constant of a single-capacitor circuit will be $\tau = R_{eq}C$ where Req is the resistance seen by the capacitor.





Example: $R_{eq} = R_3 + R_4 + \frac{R_1 R_2}{(R_1 + R_2)}$

1^{st} Order Response Observations

✓The voltage on a capacitor or the current through a inductor is the same *prior to* and *after* a switch at $t=0$.

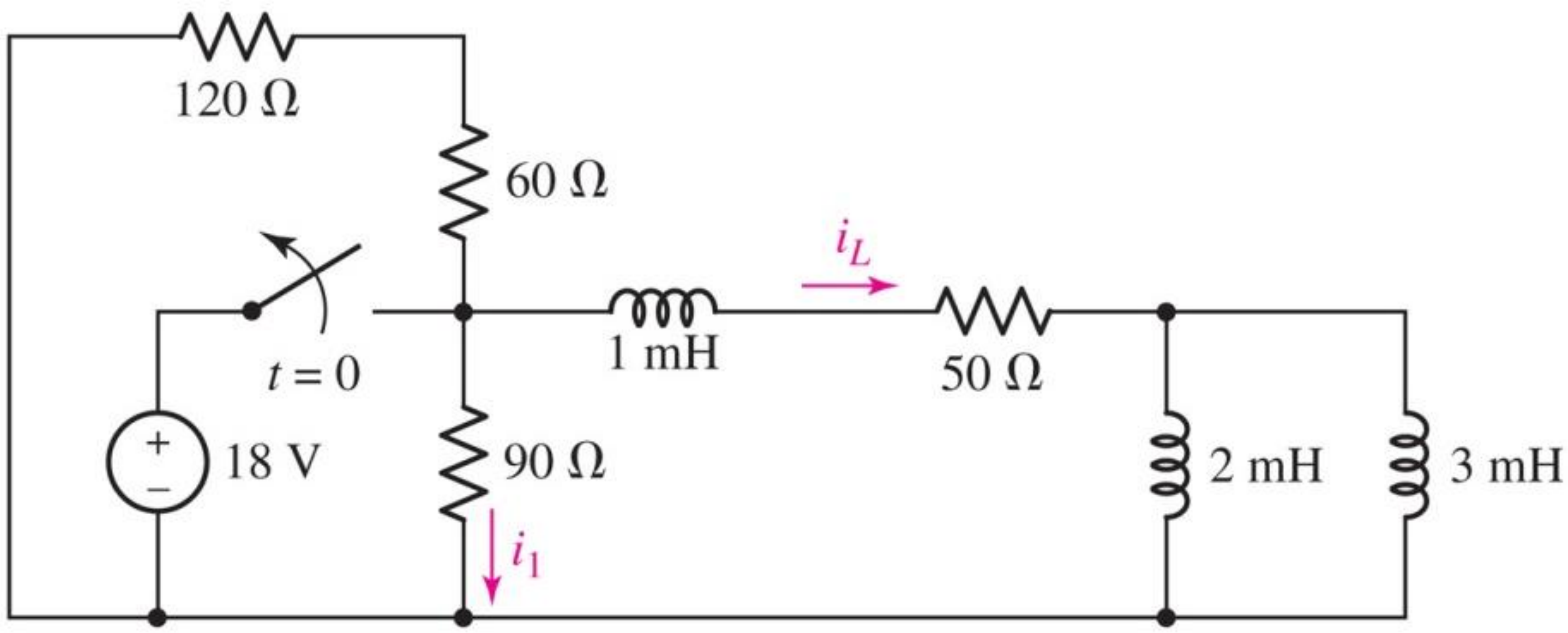
$$v_C(0^+) = v_C(0^-), \quad i_L(0^+) = i_L(0^-)$$

✓Resistor voltage (or current) prior to the switch $v(0^-)$ can be different from the voltage after the switch $v(0^+)$.

✓All voltages and currents in an RC or RL circuit follow the same natural response $e^{-t/\tau}$.

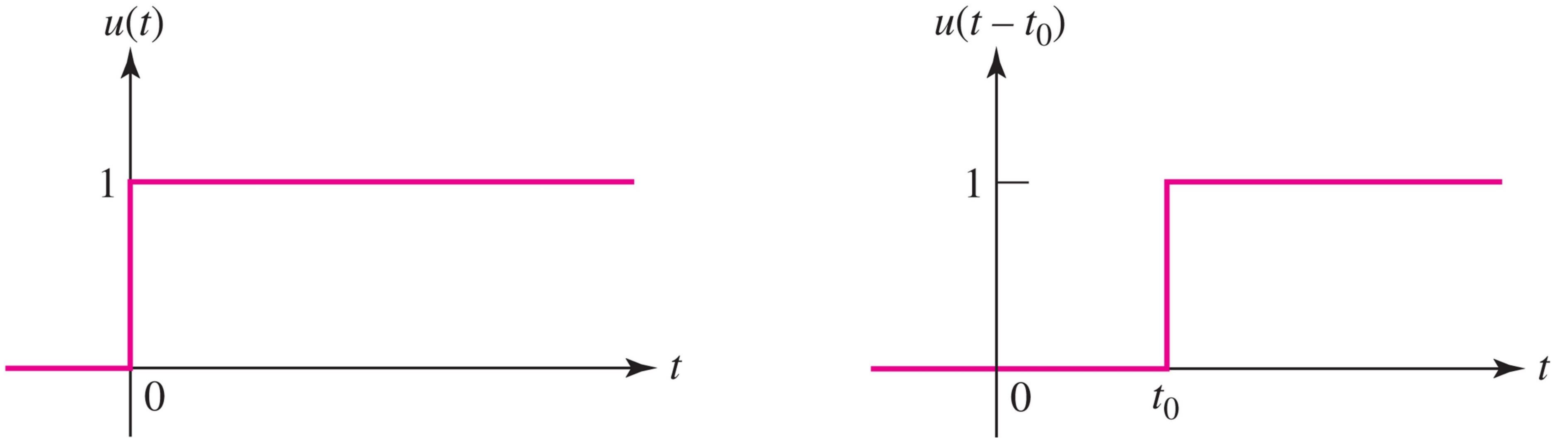
Example: L and R Current

Find $i_1(t)$ and $i_L(t)$ for $t>0$.



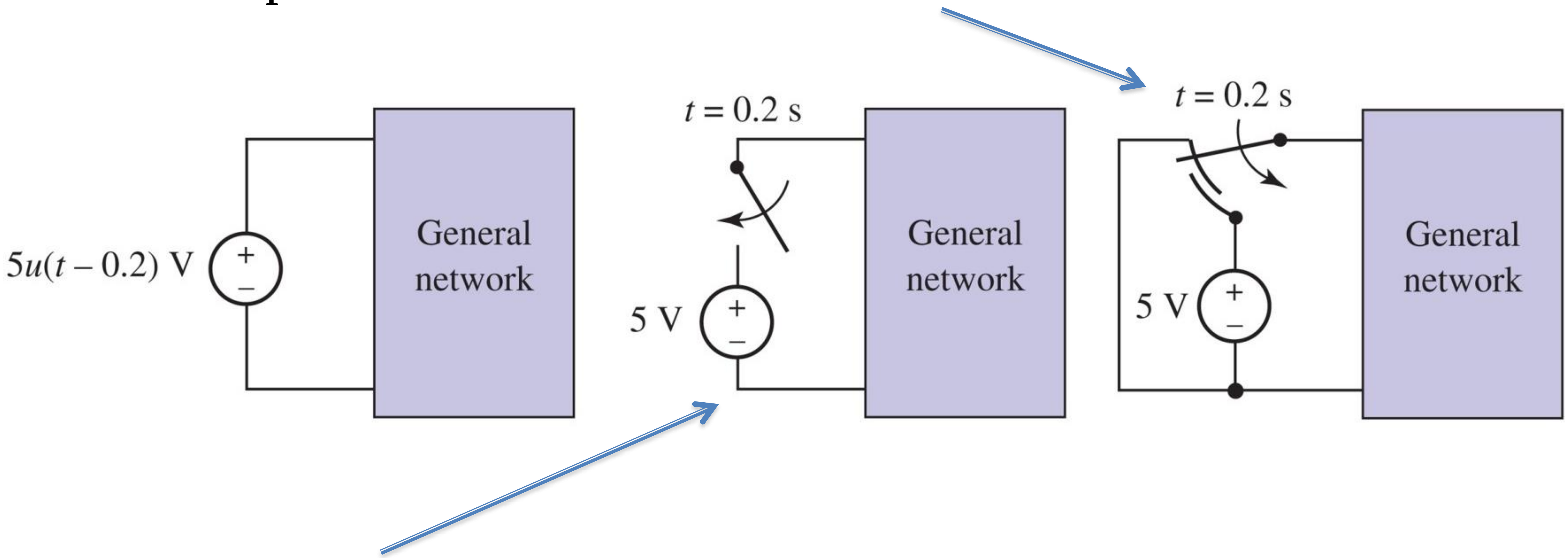
The Unit Step Function

The unit-step function *u(t)* is a convenient notation to represent change at t=0:



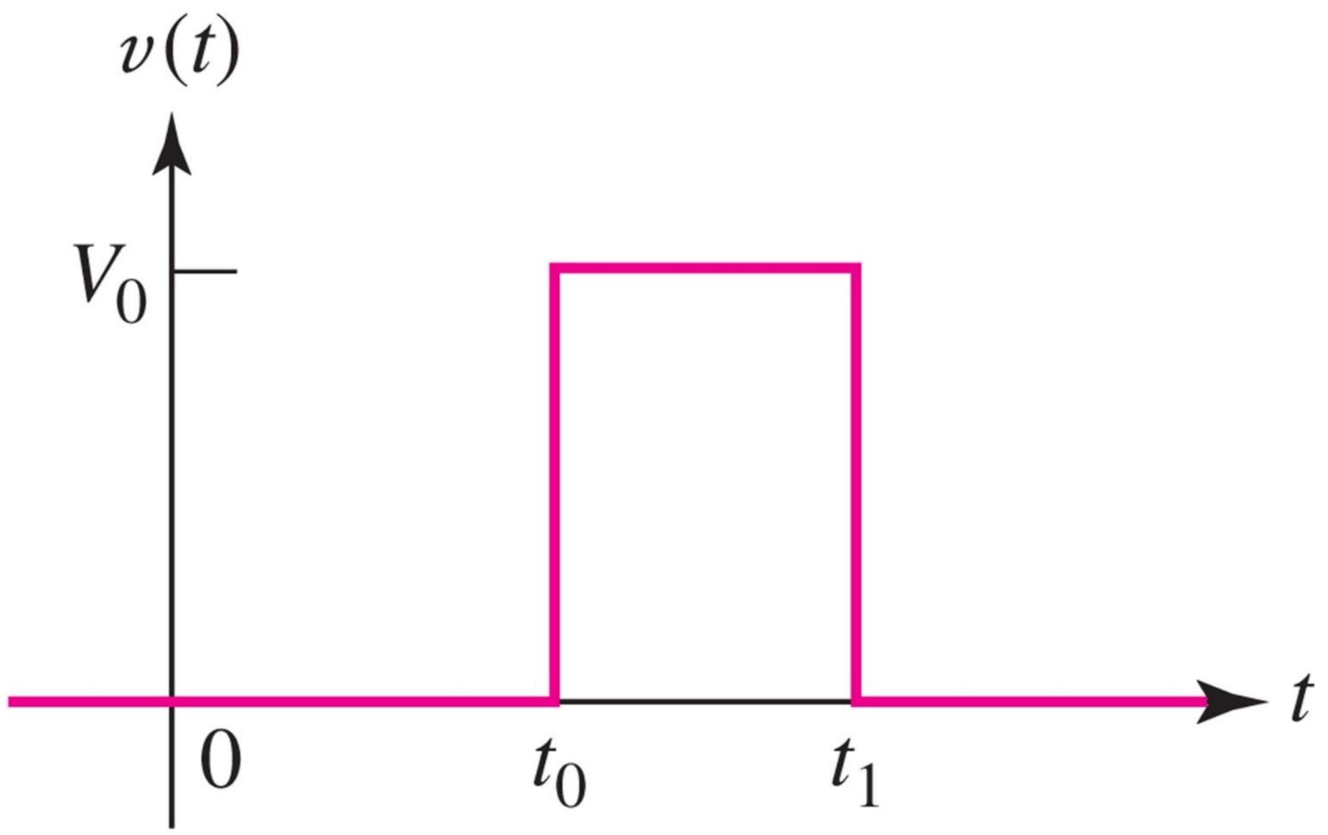
Switches and Steps

- The unit step models a double-throw switch.



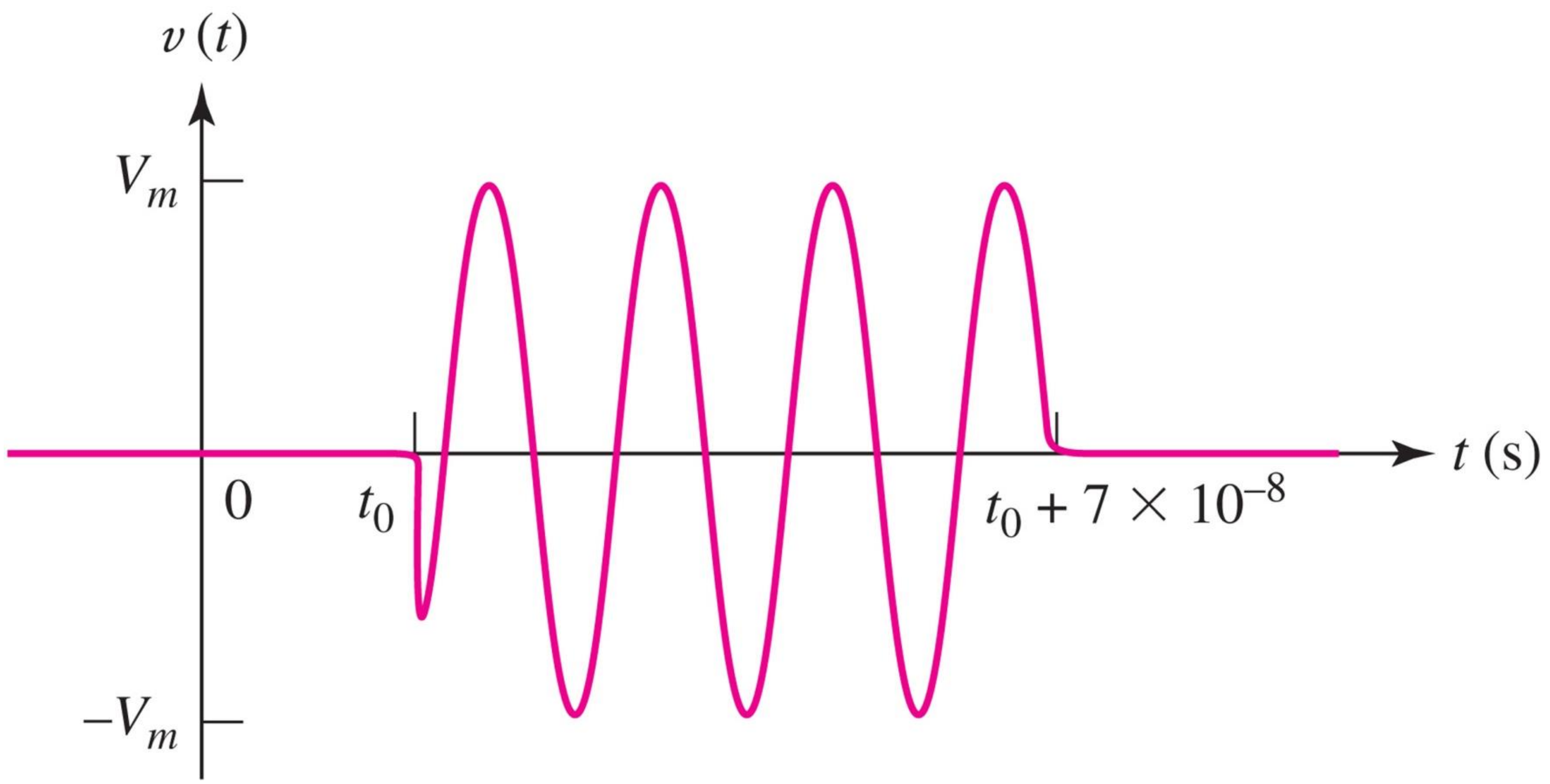
- A single-throw switch is open circuit for t<0, not short circuit.

Modeling Pulses using u(t)



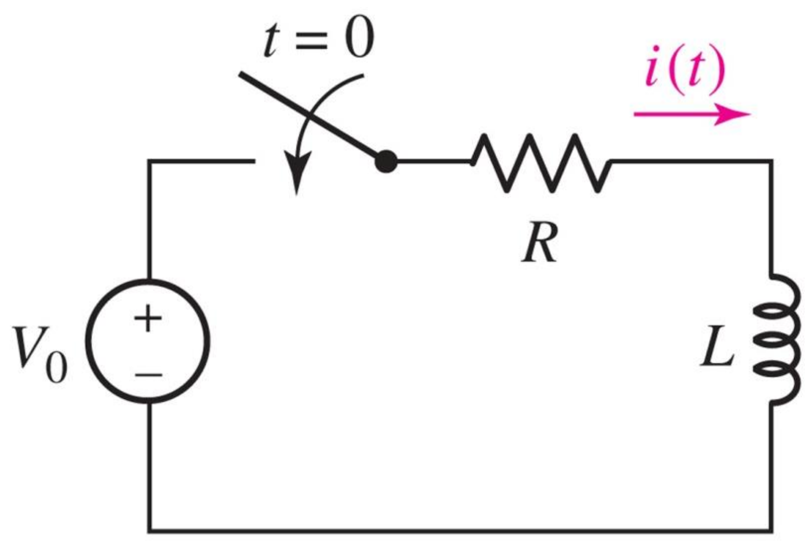
Rectangular pulse

Pulsed sinewave:

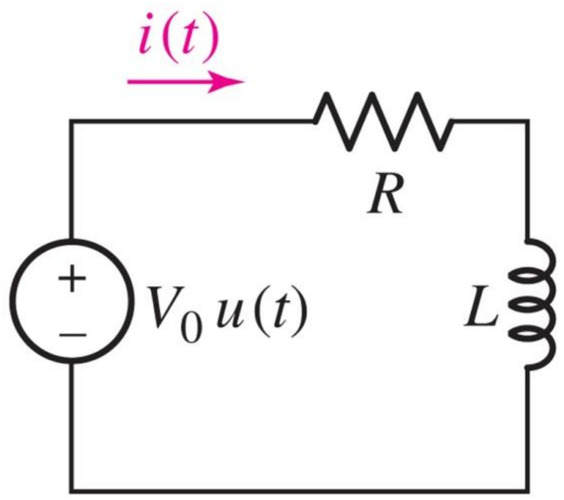


Driven RL Circuits

- The two circuits shown both have *i(t)=0* for *t<0* and are also the same for *t>0.*



(*a*)



(*b*)

- We now have to find both the *natural response* and the *forced response* due to the source V_0

$$Ri + L\frac{di}{dt} = V_0 u(t)$$

Driven RL Circuits

$$Ri + L\frac{di}{dt} = V_0 u(t) \qquad\qquad i(0^+) = 0$$

1. Natural response:

$$Ls + R = 0 \rightarrow s = -\frac{R}{L} \qquad\qquad i_n = Ae^{st} = Ae^{-\frac{Rt}{L}}$$

2. Forced response (of the same nature as the source)

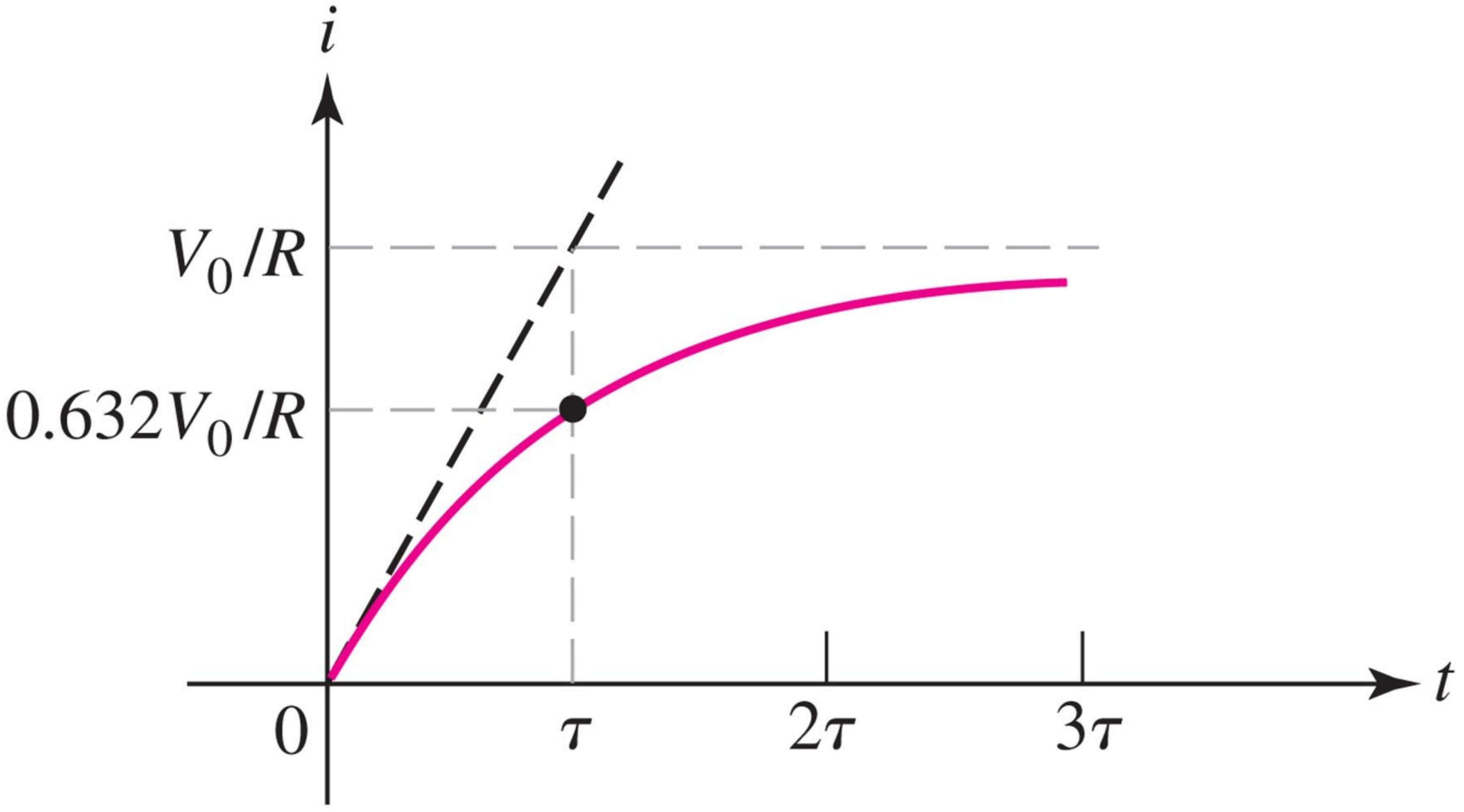
$$i_f = K \xrightarrow[Ri + L\frac{di}{dt} = V_0]{\text{صدق دادن در معادله}} i_f = \frac{V_0}{R}$$

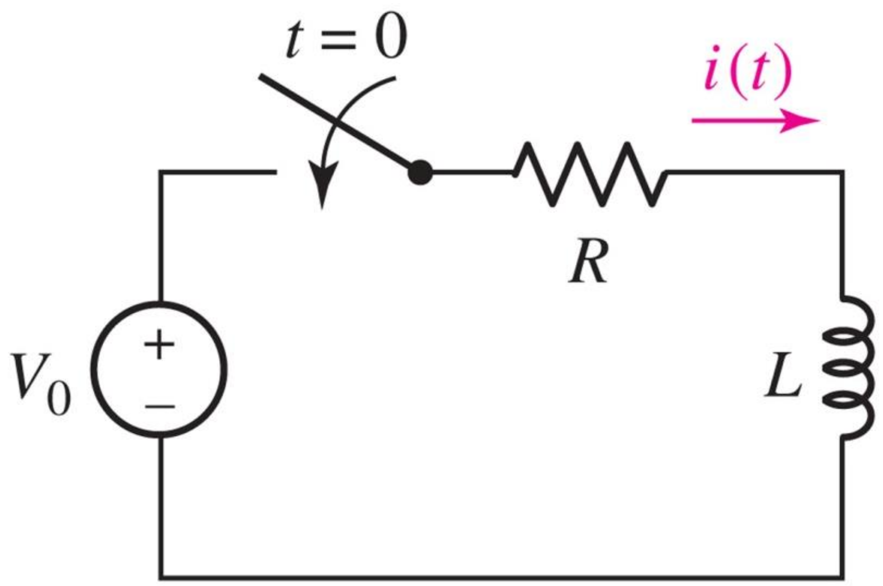
3. The complete response = natural response + forced response

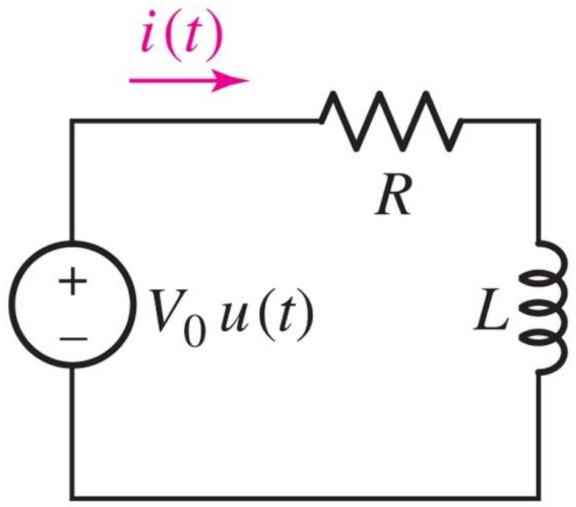
$$i(t) = Ae^{-\frac{Rt}{L}} + \frac{V_0}{R} \xrightarrow{\text{صدق دادن شرایط اولیه}} i(t) = \frac{V_0}{R}\left(1 - e^{-\frac{Rt}{L}}\right)u(t)$$

Driven RL Circuits

In this circuit, the inductor current is charged exponentially to the final value $\frac{V_0}{R}$.





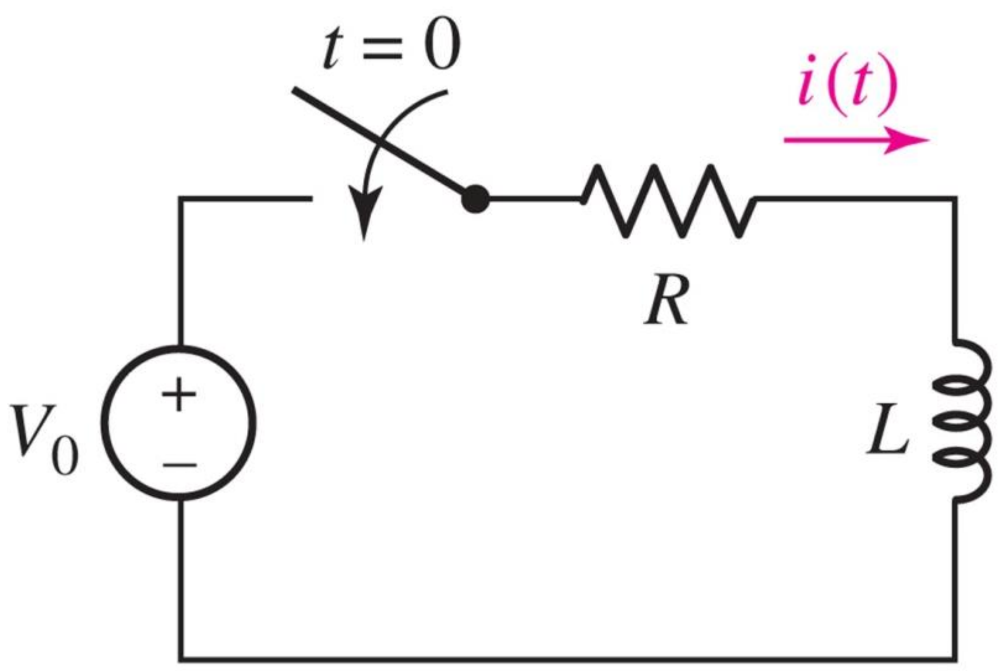


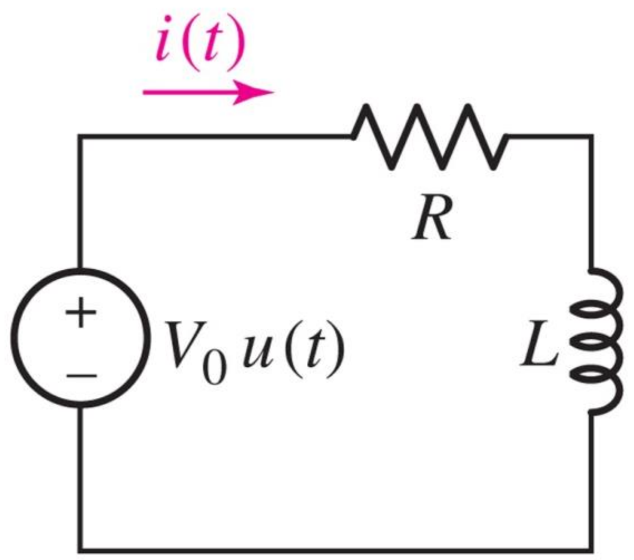
$$i(t) = \frac{V_0}{R}\left(1 - e^{-\frac{Rt}{L}}\right)u(t)$$

Driven RL Circuits

The total response is the combination of the transient/natural response and the forced response:

$$i(t) = \frac{V_0}{R}\left(1 - e^{-\frac{Rt}{L}}\right)u(t)$$





Driven RL Circuits and Initial condition

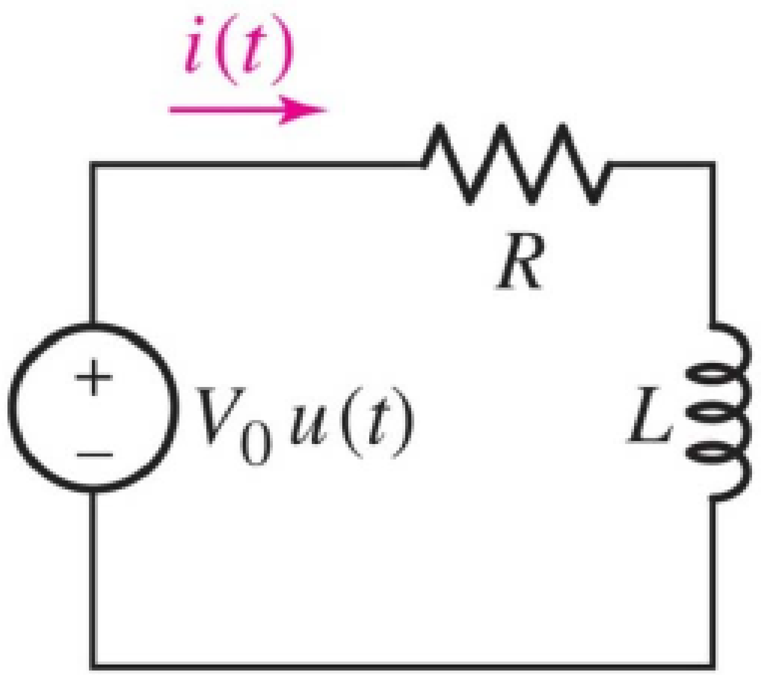
What if the circuit has both a source and an initial condition

- **Solve the differential equation with the given initial condition**:
  - Complete response = Natural response + Forced response
- **Use the principle of superposition**:
  - Complete response = Response of the circuit with the source (without the initial condition) + Response of the circuit without the source (with the initial condition)

Keep the sources but set the initial conditions to zero.

Remove the sources but keep the initial conditions.

Find i(t) if $i(0^-) = I_0$:



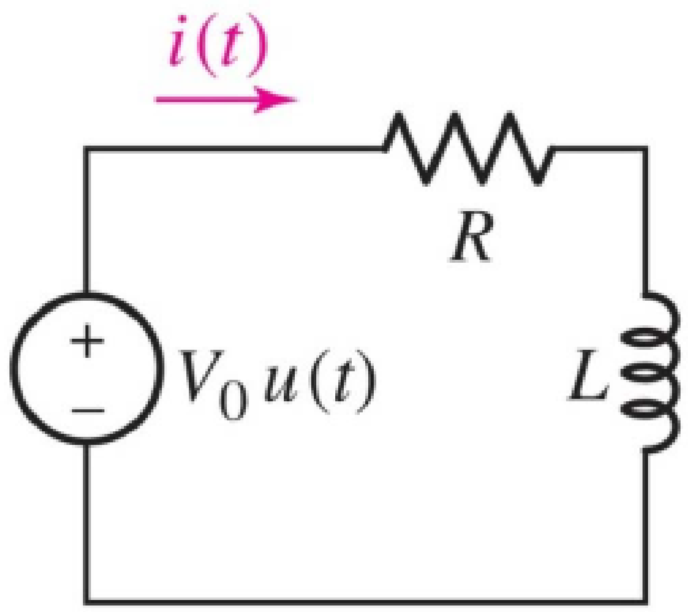
Superposition:

- Response without the source: $$i_{sf} = I_0 e^{-Rt/L}$$

- Response with the source: $$i_d = \frac{V_0}{R}(1 - e^{-Rt/L})$$

- Complete response: $$i(t) = I_0 e^{-Rt/L} + \frac{V_0}{R}(1 - e^{-Rt/L})$$

Find i(t) if $i(0^-) = I_0$:



$$i' + \frac{R}{L} i = \frac{V_0}{L}, \qquad i(0^+) = I_0$$

Solving the differential equation with the initial condition:

Natural response: $$i_n = K e^{-Rt/L}$$

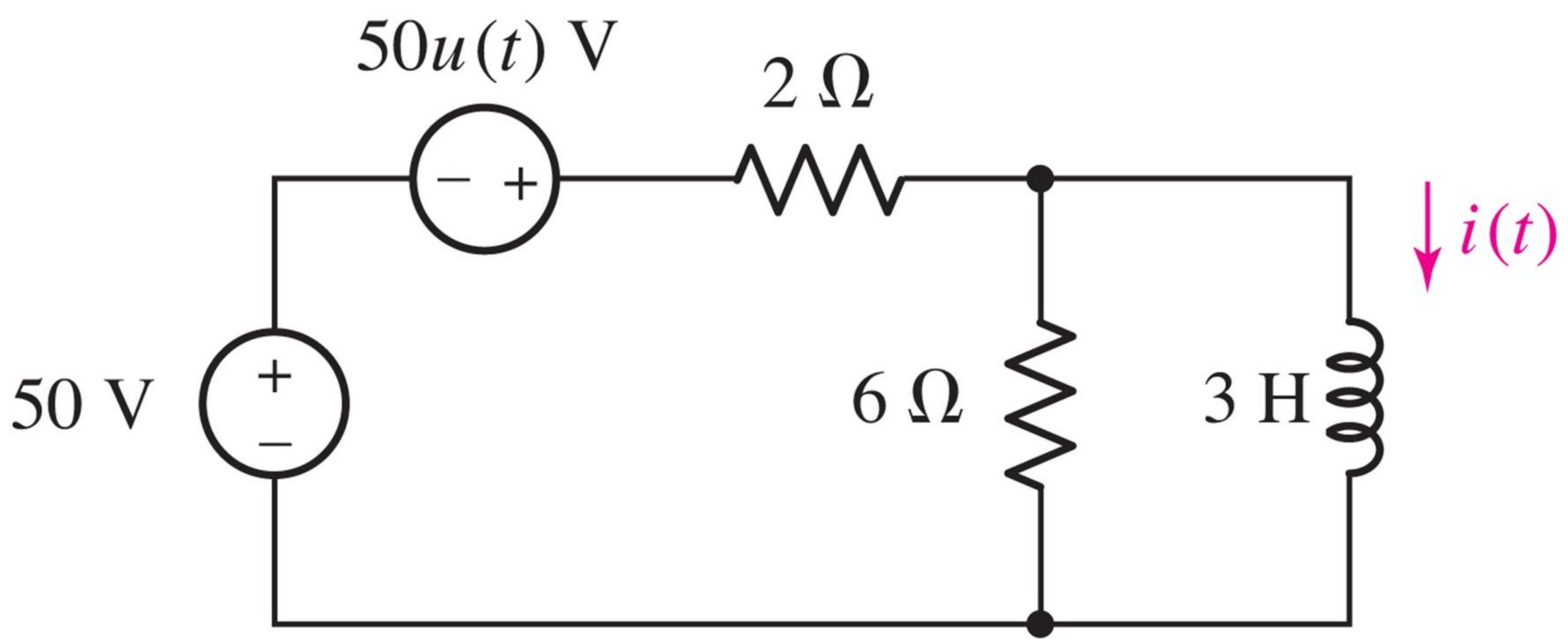
Forced response: $$i_f = \frac{V_0}{R}$$

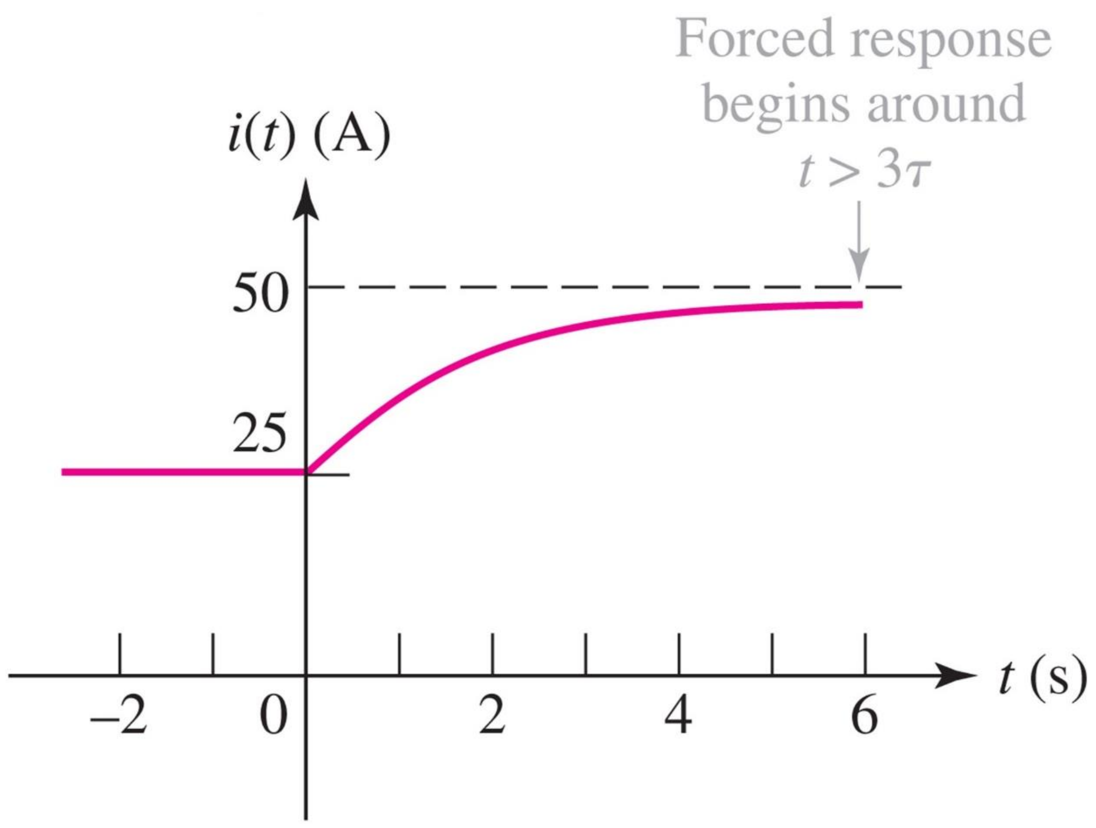
Complete response: $$i(t) = K e^{-Rt/L} + \frac{V_0}{R}$$

$$i(0) = I_0 \rightarrow K = I_0 - \frac{V_0}{R}, \qquad i(t) = \frac{V_0}{R} + (I_0 - \frac{V_0}{R}) e^{-Rt/L}$$

Example: RL Circuit with Step

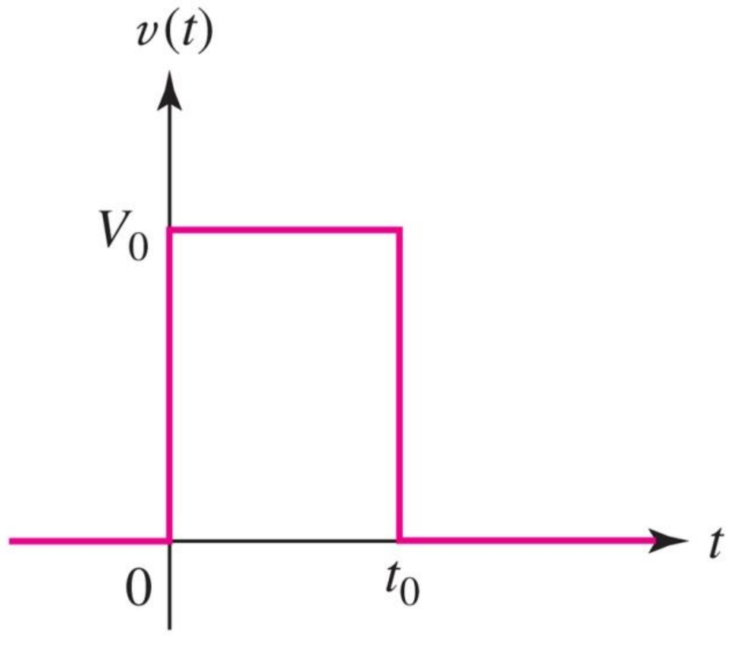
Show that $i(t)=25+25(1-e^{-t/2})u(t)$ A



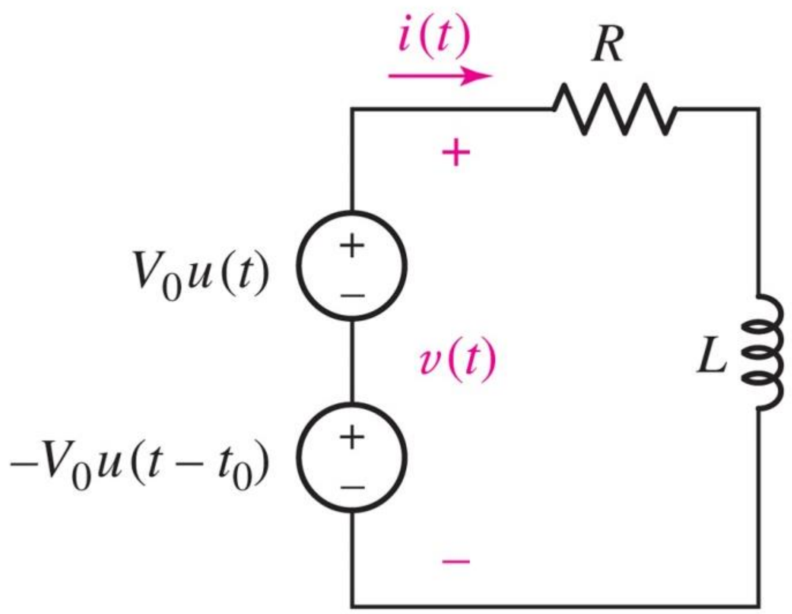


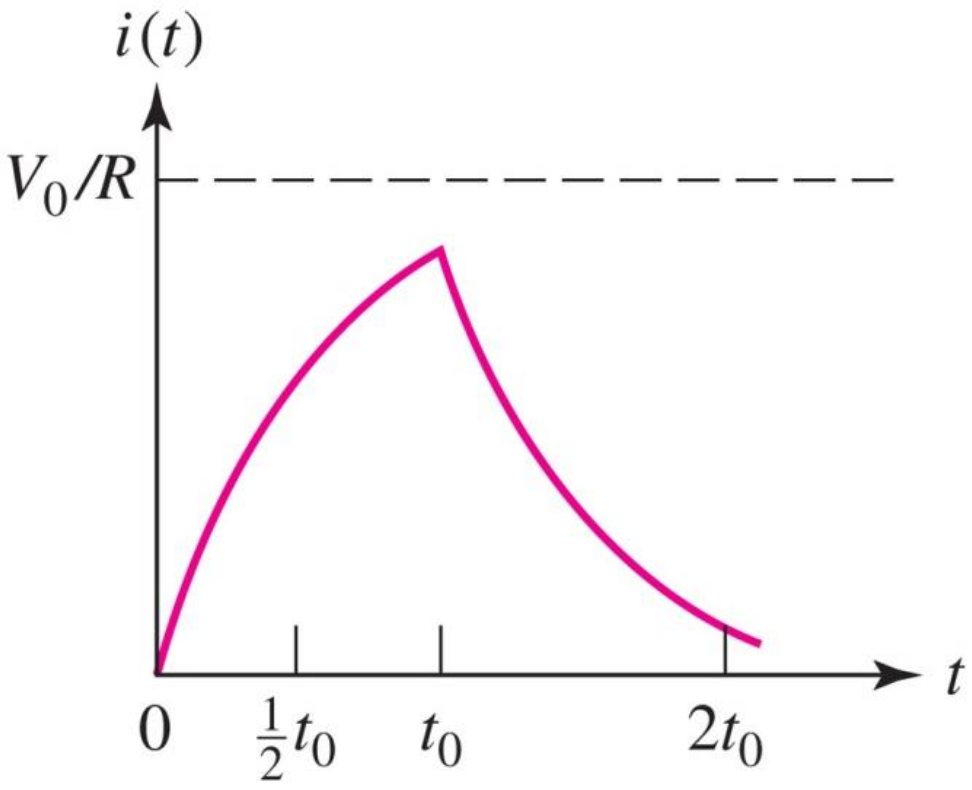
Example: Voltage Pulse

Assuming the given input voltage, find the current i(t).



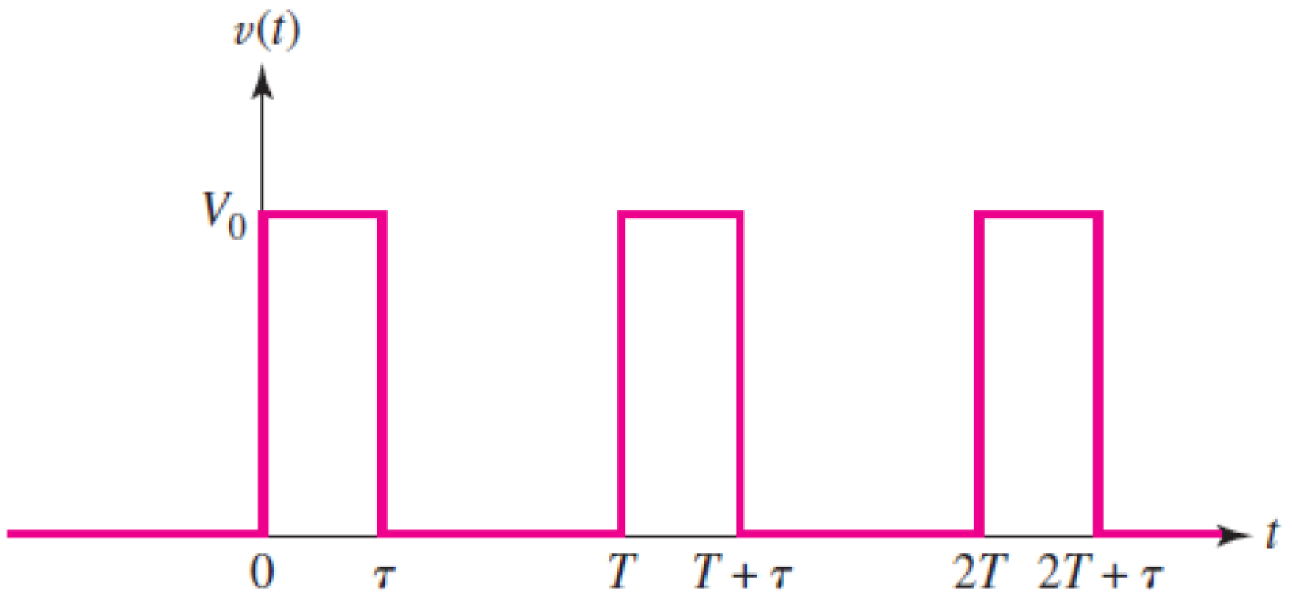
(a)



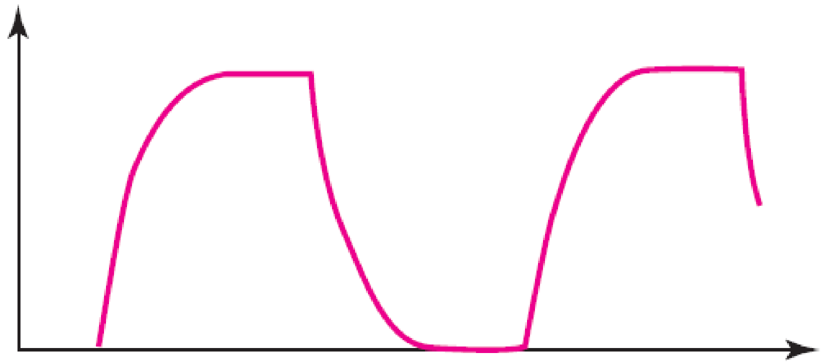


Response of RL or RC circuit to a pulse train input

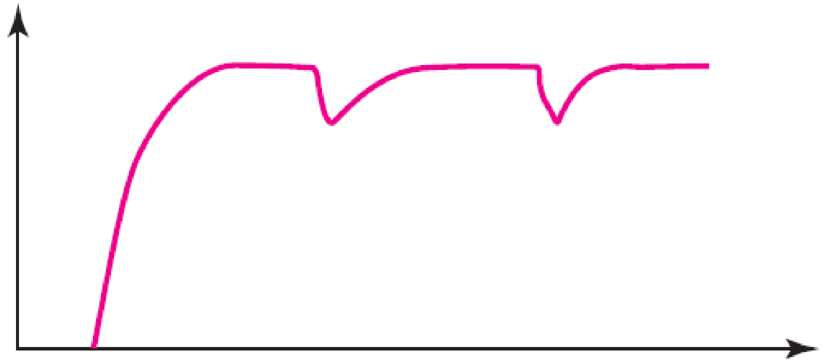
Depending on the values of pulse width (τ), pulse period (T), and the circuit time constant (RC), four different cases arise:



τ>>RC
T-τ>>RC



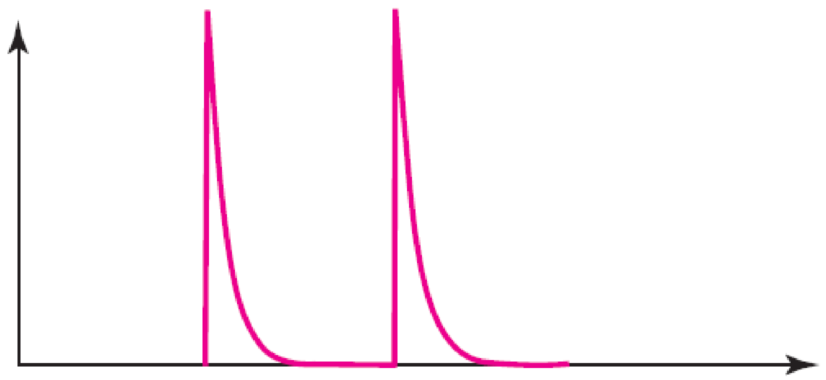
(a)



(b)

τ>>RC
T-τ<<RC

τ<<RC
T-τ>>RC

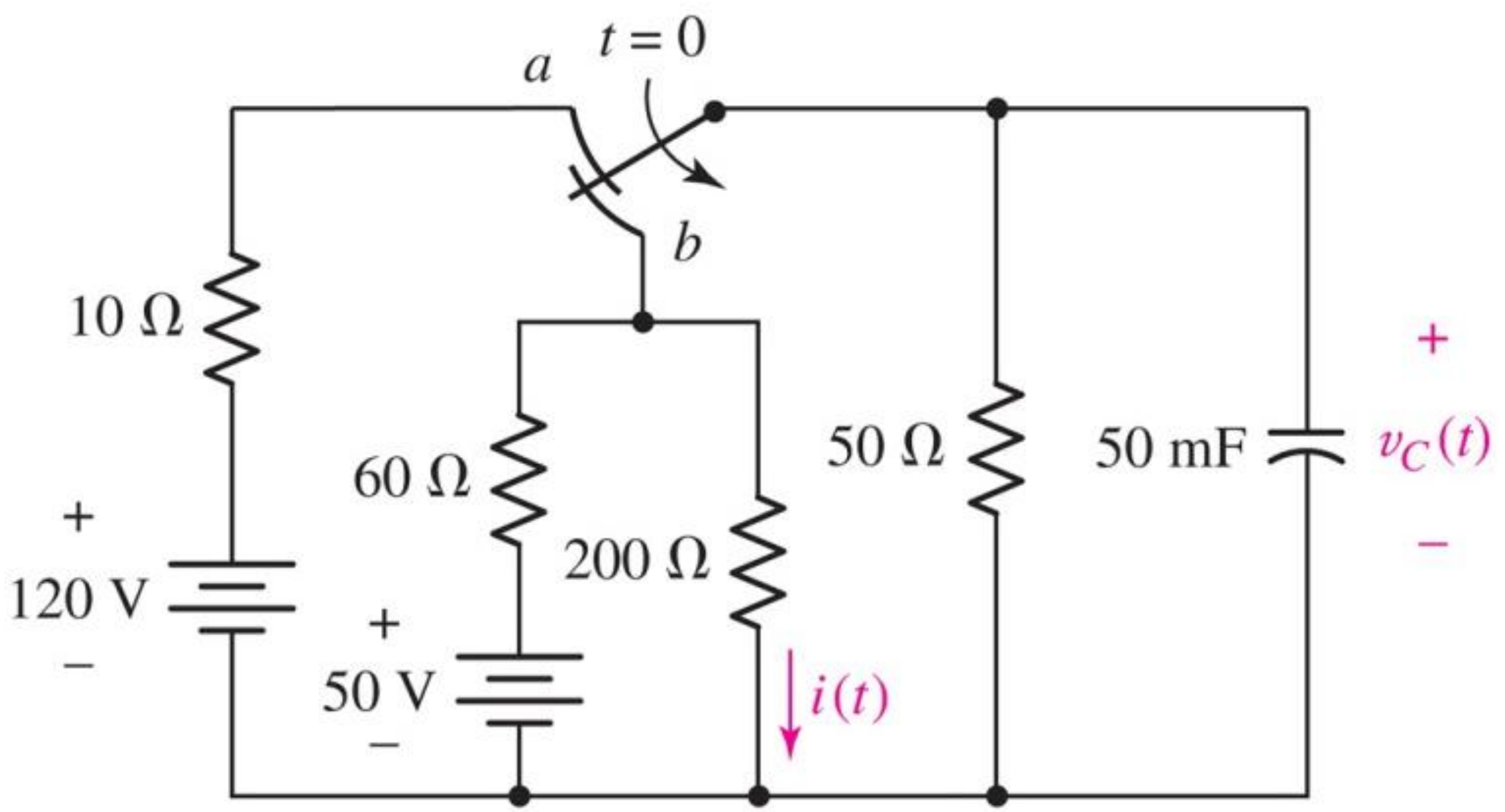


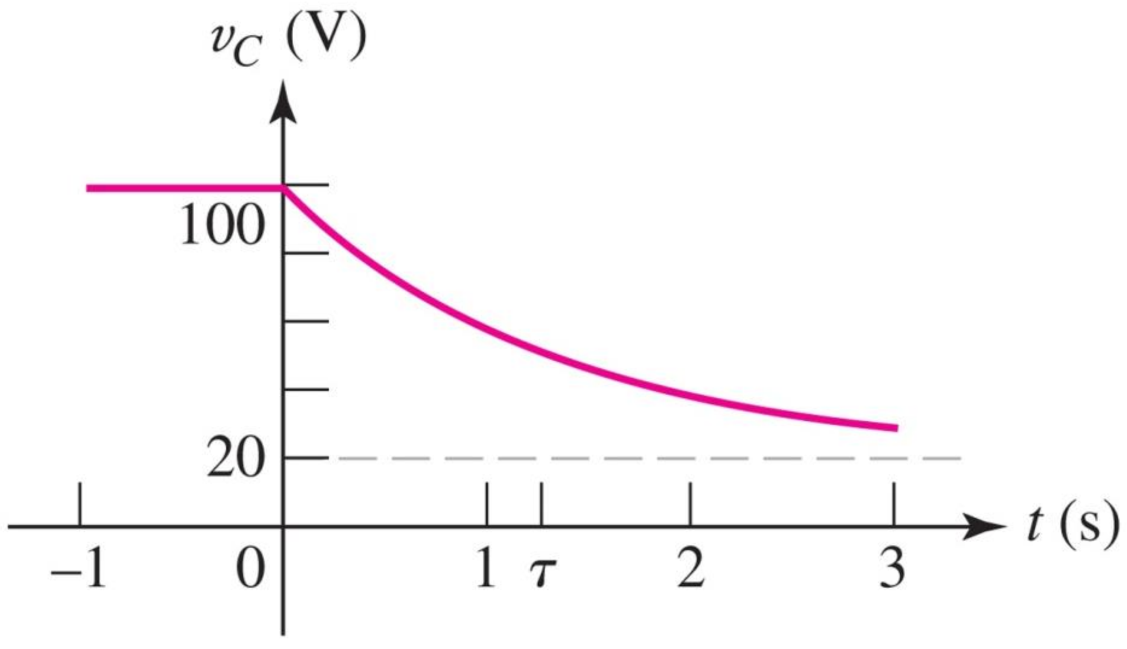
(c)

(d)

τ<<RC
T-τ<<RC

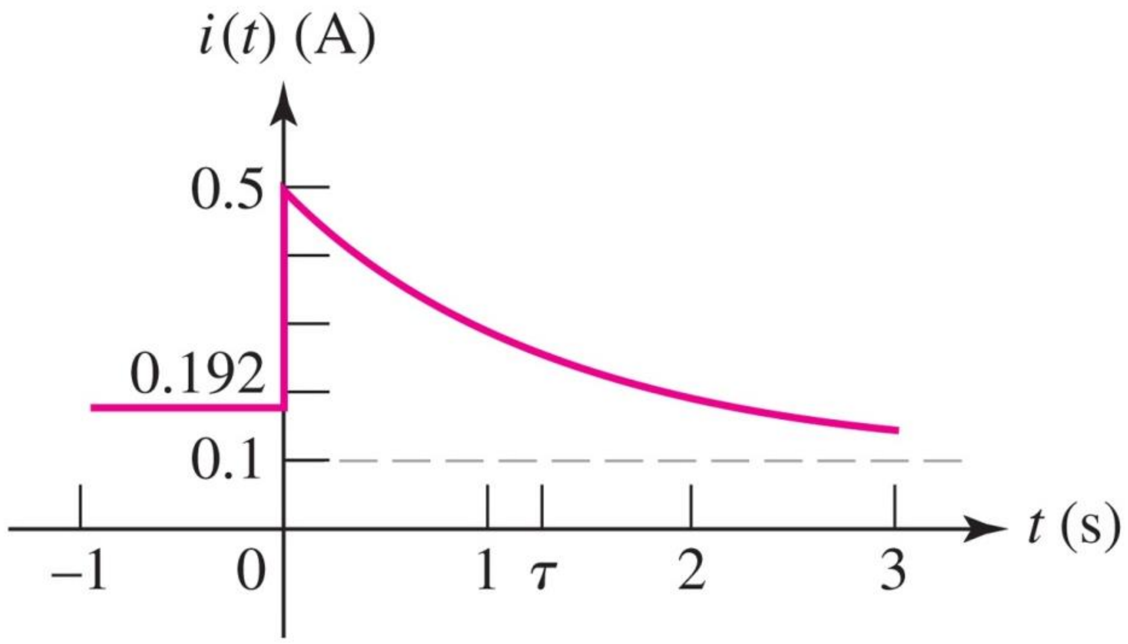
Driven RC Circuits (part 1 of 2)





(a)

$$v_C = 20 + 80e^{-t/1.2}\ \mathrm{V}$$

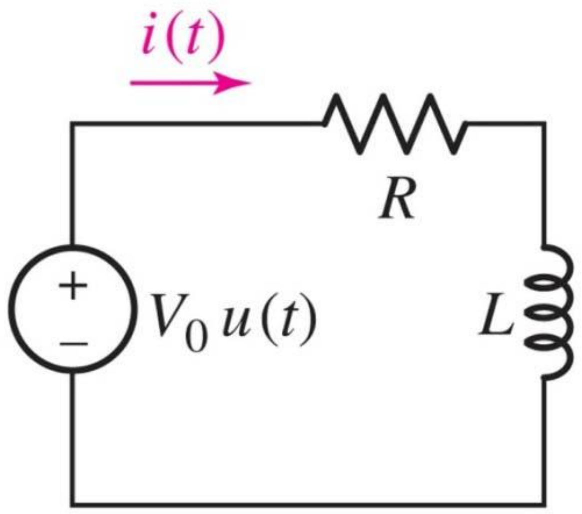


(b)

$$i = 0.1 + 0.4e^{-t/1.2}\ \mathrm{A}$$

✓ We have previously calculated the current response for an RL circuit as follows:

$$i(t) = \frac{V_0}{R} + (I_0 - \frac{V_0}{R})e^{-Rt/L}$$



✓ From the following relation, it can generally be shown that the step response of an RC or RL circuit is also obtainable:

$$x(t) = x(\infty) + [x(0^+) - x(\infty)]e^{-t/\tau}$$

- **What if there are no DC sources?**
- The natural response, which is independent of the source, will have the same form (Ke^{st}).
- The forced response has the same form as the source but with a different coefficient.

منبع	پاسخ اجباری
K	K'
$K_1 t + K_2$	$K't + k''$
$Ke^{bt}\ (b != s)$	$K'e^{bt}$
$Ke^{bt}\ (b == s)$	$K'te^{bt}$
$K\cos(wt + p)$	$K'\cos(wt + p')$

The roots of the characteristic equation of the circuit are s.

Complete response

Another method for solving the following first-order differential equation:

$$x'(t) + ax(t) = Q(t)$$

We multiply both sides of the equation by e^{at}:

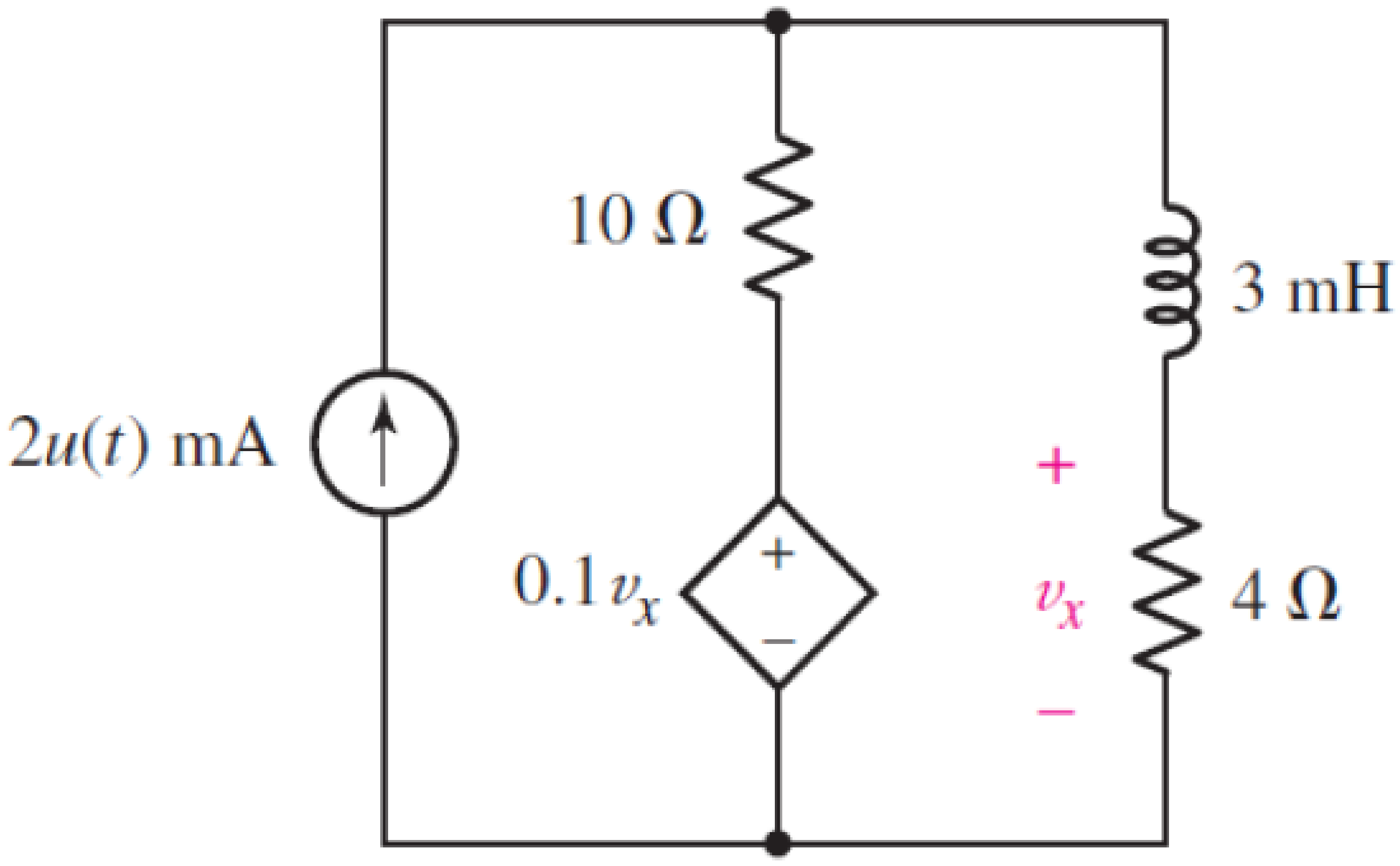
$$e^{at}x'(t) + ae^{at}\,x(t) = e^{at}Q(t)$$

$$\rightarrow \left(e^{at}x(t)\right)' = e^{at}Q(t)$$

$$\rightarrow x(t) = \underbrace{e^{-at}\int e^{at}Q(t)}_{\text{forced response}} + \underbrace{c\,e^{-at}}_{\text{natural response}}$$

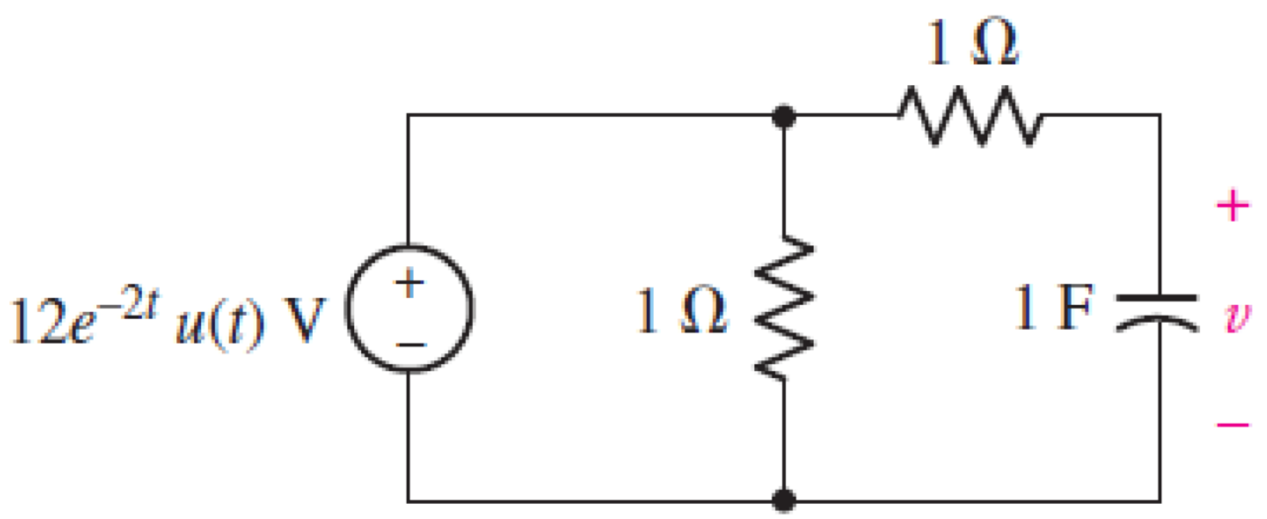
Practice 1

Find $v_x(t)$.



Practice 2

Find $v(t)$.



Soloution:

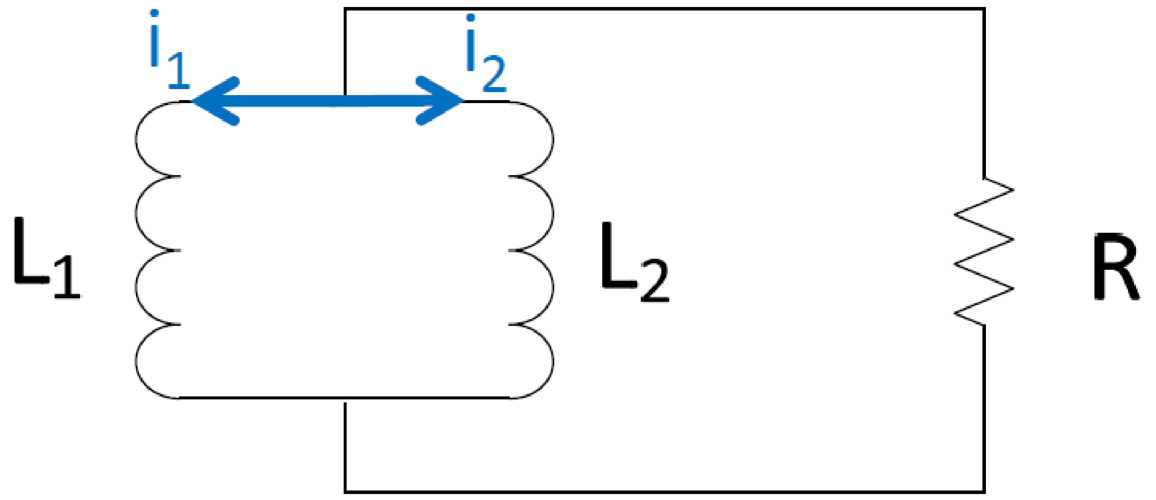
$$v(t) = 12(e^{-t} - e^{-2t})u(t)$$

What if the source value was equal to $12e^{-t}u(t)$?

$$v(t) = 12te^{-t}u(t)$$

Practice 3

Find $i_1(t)$ and $i_2(t)$.

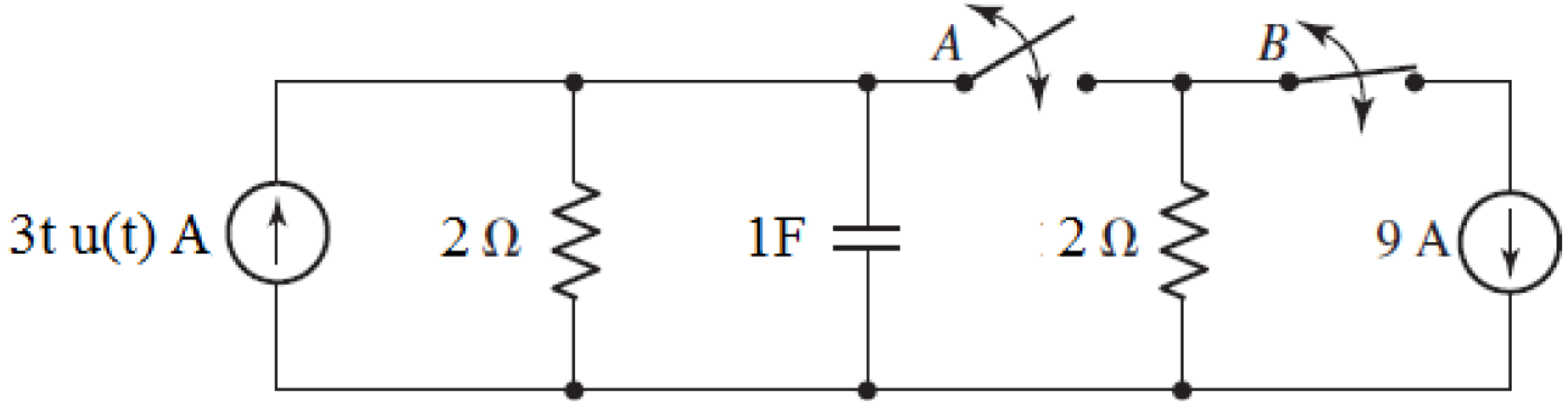


- $L_1 = 6H, i_1(0) = 2A$
- $L_2 = 3H, i_2(0) = 1A$
- $R = 6\Omega$

Hint: $V_{L_1} = V_{L_2} \rightarrow L_1 i_1' = L_2 i_2' \rightarrow L_1 i_1 = L_2 i_2 + K$

Practice 4

Find $v_c(t)$. Suppose keys A and B are initially closed and key A is opened in second 2 and key B is opened in second 1.

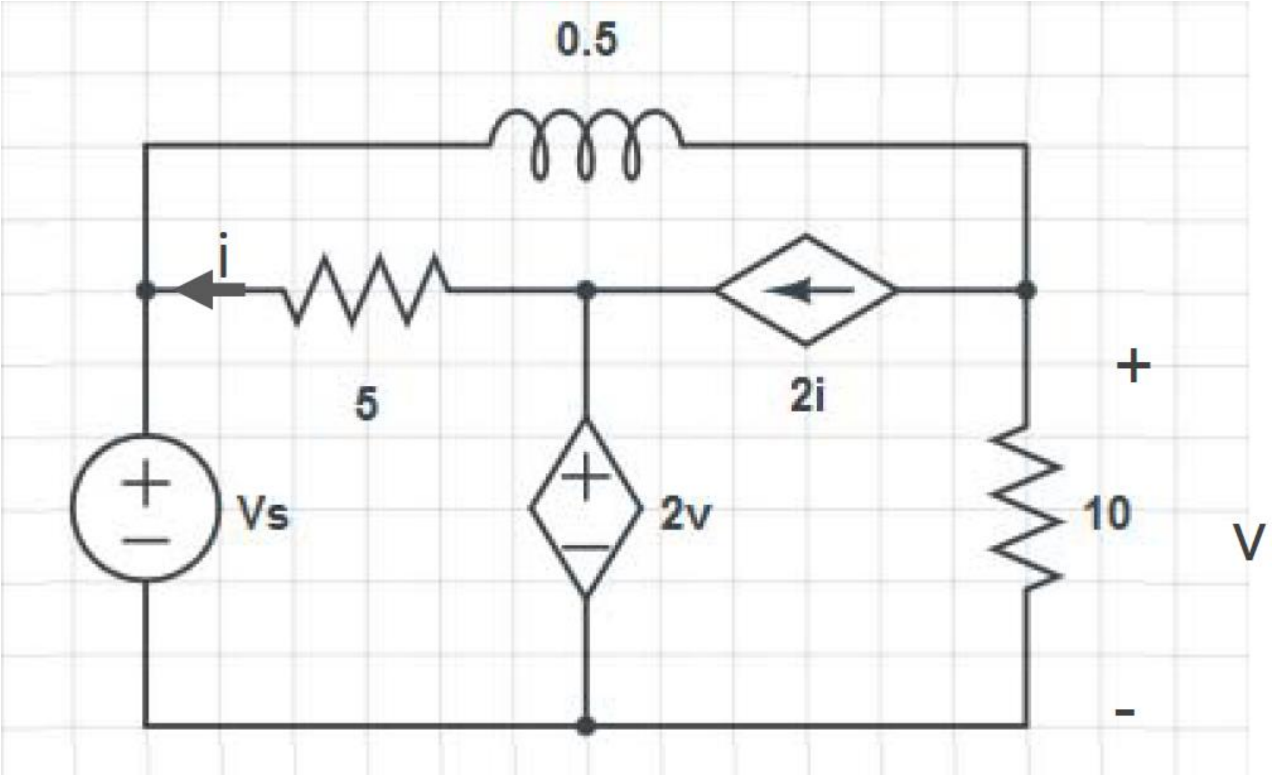


Practice 5

Find $v(t)$ if $v_s = u(t)$:

$$\left(1 - \frac{5}{9}e^{-\frac{20t}{9}}\right)u(t)$$

And if $v_s = \cos t \;\; u(t)$



Thanks