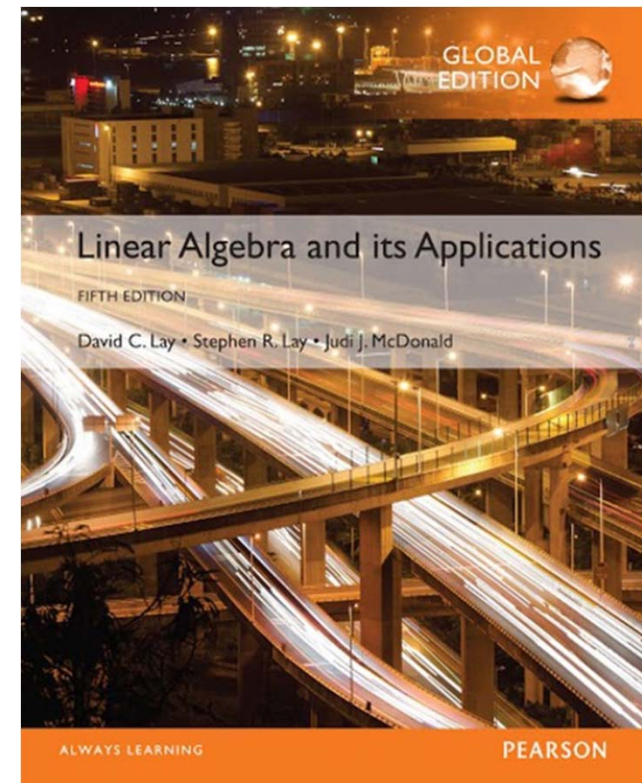


# 2

## Matrix Algebra

### 2.4

### PARTITIONED MATRICES



# PARTITIONED MATRICES

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- A key feature of our work with matrices has been the ability to regard matrix  $A$  as a list of column vectors rather than just a rectangular array of numbers.
- This point of view has been so useful that we wish to consider other **partitions** of  $A$ , indicated by horizontal and vertical dividing rules, as in Example 1 on the next slide.

# PARTITIONED MATRICES

- **Example 1** The matrix

$$A = \left[ \begin{array}{ccc|cc|c} 3 & 0 & -1 & 5 & 9 & -2 \\ -5 & 2 & 4 & 0 & -3 & 1 \\ \hline -8 & -6 & 3 & 1 & 7 & -4 \end{array} \right]$$

- Can also be written as the  $2 \times 3$  **partitioned** (or **block**) **matrix**

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

- Whose entries are the *blocks* (or *submatrices*)

$$A_{11} = \begin{bmatrix} 3 & 0 & -1 \\ -5 & 2 & 4 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 5 & 9 \\ 0 & -3 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} -8 & -6 & 3 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 1 & 7 \end{bmatrix}, \quad A_{23} = \begin{bmatrix} -4 \end{bmatrix}$$

## ADDITION AND SCALAR MULTIPLICATION

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- If matrices  $A$  and  $B$  are the same size and are partitioned in exactly the same way, then it is natural to make the same partition of the ordinary matrix sum  $A + B$ .
- In this case, each block of  $A + B$  is the (matrix) sum of the corresponding blocks of  $A$  and  $B$ .
- Multiplication of a partitioned matrix by a scalar is also computed block by block.

# MULTIPLICATION OF PARTITIONED MATRICES

- Partitioned matrices can be multiplied by the usual row—column rule as if the block entries were scalars, provided that for a product  $AB$ , the column partition of  $A$  matches the row partition of  $B$ .

- **Example 3** Let

$$A = \left[ \begin{array}{ccc|cc} 2 & -3 & 1 & 0 & -4 \\ 1 & 5 & -2 & 3 & -1 \\ 0 & -4 & -2 & 7 & -1 \end{array} \right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \left[ \begin{array}{cc} 6 & 4 \\ -2 & 1 \\ -3 & 7 \\ -1 & 3 \\ 5 & 2 \end{array} \right] = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

- The 5 columns of  $A$  are partitioned into a set of 3 columns and then a set of 2 columns. The 5 rows of  $B$  are partitioned in the same way—into a set of 3 rows and then a set of 2 rows.

# MULTIPLICATION OF PARTITIONED MATRICES

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- We say that the partitions of  $A$  and  $B$  are conformable for block multiplication. It can be shown that the ordinary product  $AB$  can be written as

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} A_{11}B_1 + A_{12}B_2 \\ A_{21}B_1 + A_{22}B_2 \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ -6 & 2 \\ \hline 2 & 1 \end{bmatrix}$$

- It is important for each smaller product in the expression for  $AB$  to be written with the submatrix from  $A$  on the left, since matrix multiplication is not commutative.

# MULTIPLICATION OF PARTITIONED MATRICES

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- For instance,

$$A_{11}B_1 = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 5 & -2 \end{bmatrix} \begin{bmatrix} 6 & 4 \\ -2 & 1 \\ -3 & 7 \end{bmatrix} = \begin{bmatrix} 15 & 12 \\ 2 & -5 \end{bmatrix}$$

$$A_{12}B_2 = \begin{bmatrix} 0 & -4 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} -20 & -8 \\ -8 & 7 \end{bmatrix}$$

- Hence the top block in  $AB$  is

$$A_{11}B_1 + A_{12}B_2 = \begin{bmatrix} 15 & 12 \\ 2 & -5 \end{bmatrix} + \begin{bmatrix} -20 & -8 \\ -8 & 7 \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ -6 & 2 \end{bmatrix}$$

# MULTIPLICATION OF PARTITIONED MATRICES

- **Theorem 10:** Column—Row Expansion of  $AB$

If  $A$  is  $m \times n$  and  $B$  is  $n \times p$ , then

$$AB = [\text{col}_1(A) \quad \text{col}_2(A) \quad \cdots \quad \text{col}_n(A)] \begin{bmatrix} \text{row}_1(B) \\ \text{row}_2(B) \\ \vdots \\ \text{row}_n(B) \end{bmatrix} \quad (1)$$
$$= \text{col}_1(A) \text{row}_1(B) + \cdots + \text{col}_n(A) \text{row}_n(B)$$



# INVERSES OF PARTITIONED MATRICES

- The next example illustrates calculations involving inverses and partitioned matrices.
- **Example 5** A matrix of the form

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

is said to be *block upper triangular*.

- Assume that  $A_{11}$  is  $p \times p$ ,  $A_{22}$  is  $q \times q$ , and  $A$  is invertible.
- Find a formula for  $A^{-1}$ .

# INVERSES OF PARTITIONED MATRICES

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- **Solution** Denote  $A^{-1}$  by  $B$  and partition  $B$  so that

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix} \quad (2)$$

- This matrix equation provides four equations that will lead to the unknown blocks  $B_{11}, \dots, B_{22}$ . Compute the product on the left side of equation (2), and equate each entry with the corresponding block in the identity matrix on the right.

# INVERSES OF PARTITIONED MATRICES

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- That is, set

$$A_{11}B_{11} + A_{12}B_{21} = I_p \quad (3)$$

$$A_{11}B_{12} + A_{12}B_{22} = 0 \quad (4)$$

$$A_{22}B_{21} = 0 \quad (5)$$

$$A_{22}B_{22} = I_q \quad (6)$$

- By itself, equation (6) does not show that  $A_{22}$  is invertible. However, since  $A_{22}$  is square, the Invertible Matrix Theorem and (6) together show that  $A_{22}$  is invertible and  $B_{22} = A_{22}^{-1}$ .

# INVERSES OF PARTITIONED MATRICES

- Next, left-multiply both sides of (5) by  $A_{22}^{-1}$  and obtain

$$B_{21} = A_{22}^{-1}0 = 0$$

- So that (3) simplifies to

$$A_{11}B_{11} + 0 = I_p$$

- Since  $A_{11}$  is square, this shows that  $A_{11}$  is invertible and  $B_{11} = A_{11}^{-1}$ . Finally, use these results with (4) to find that

$$A_{11}B_{12} = -A_{12}B_{22} = -A_{12}A_{22}^{-1} \quad \text{and} \quad B_{12} = -A_{11}^{-1}A_{12}A_{22}^{-1}$$

# INVERSES OF PARTITIONED MATRICES

■ Thus

$$A^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}$$

- A **block diagonal matrix** is a partitioned matrix with zero blocks off the main diagonal (of blocks). Such a matrix is invertible if and only if each block on the diagonal is invertible.