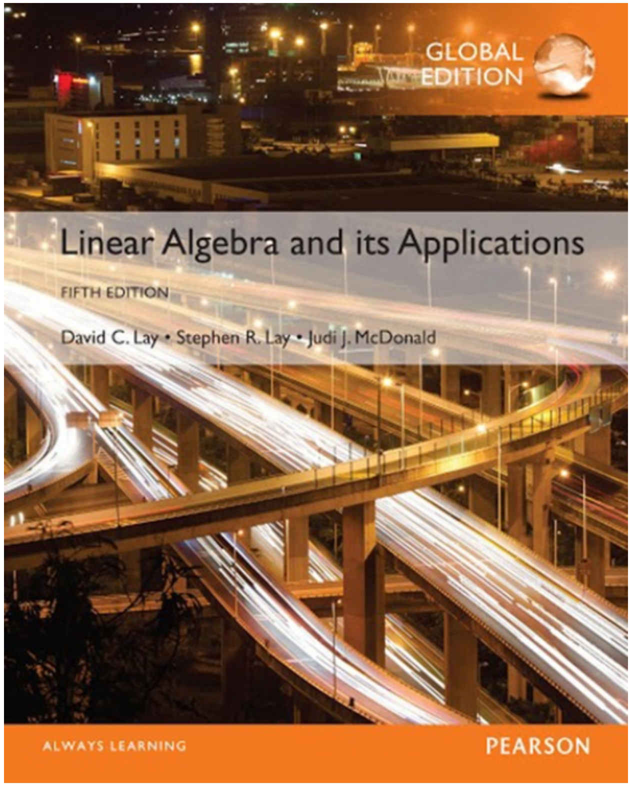
5 Eigenvalues and Eigenvectors

5.5

COMPLEX EIGENVALUES



COMPLEX EIGENVALUES

- The matrix eigenvalue-eigenvector theory already developed for $\mathbb{R}^n$ applies equally well to $\mathbb{C}^n$.

- So a complex scalar λ satisfied $\det(A-\lambda I) = 0$ if and only if there is a nonzero vector x in $\mathbb{C}^n$ such that $A\mathrm{x} = \lambda\mathrm{x}$.

- We call λ a **(complex) eigenvalue** and x a **(complex) eigenvector** corresponding to λ.

COMPLEX EIGENVALUES

- **Example 1** If $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, then the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ on $\mathbb{R}^2$ rotates the plane counterclockwise through a quarter-turn.
- The action of A is periodic, since after four quarter-turns, a vector is back where it started.
- Obviously, no nonzero vector is mapped into a multiple of itself, so A has no eigenvectors in $\mathbb{R}^2$ and hence no real eigenvalues.
- In fact, the characteristic equation of A is

$$\lambda^2 + 1 = 0$$

COMPLEX EIGENVALUES

- The only roots are complex: $\lambda = i$ and $\lambda = -i$. However, if we permit A to act on $\mathbb{C}^2$, then

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix} = i\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix} = -i\begin{bmatrix} 1 \\ i \end{bmatrix}$$

- Thus i and $-i$ are eigenvalues, with $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ i \end{bmatrix}$ as corresponding eigenvectors.

REAL AND IMAGINARY PARTS OF VECTORS

- The complex conjugate of a complex vector x in $\mathbb{C}^n$ is the vector $\bar{x}$ in $\mathbb{C}^n$ whose entries are the complex conjugates of the entries in x.

- The **real** and **imaginary parts** of a complex vector x are the vectors Re x and Im x in $\mathbb{R}^n$ formed from the real and imaginary parts of the entries of x.

REAL AND IMAGINARY PARTS OF VECTORS

- **Example 4** If $x = \begin{bmatrix} 3-i \\ i \\ 2+5i \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + i\begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$, then

$$\operatorname{Re} \mathrm{x} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix},\ \operatorname{Im} \mathrm{x} = \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}, \text{ and } \bar{x} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} - i\begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 3+i \\ -i \\ 2-5i \end{bmatrix}$$

EIGENVALUES AND EIGENVECTORS OF A REAL MATRIX THAT ACTS ON

- **Theorem 9**: Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ $(b \neq 0)$ and an associated eigenvector v in $\mathbb{C}^2$. Then

$$A = PCP^{-1}, \text{ where } P = [\text{Re v} \quad \text{Im v}] \text{ and } C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

PROOF

Let $\boldsymbol{v}$ be a vector in $\mathbb{C}^n$, and A be a real $n \times n$ matrix. First, we show that $Re(A\boldsymbol{v}) = A\ Re(\boldsymbol{v})$ and $Im(A\boldsymbol{v}) = A\ Im(\boldsymbol{v})$.

- We have: $\boldsymbol{v} = Re(\boldsymbol{v}) + Im(\boldsymbol{v})\ i$.
- So $A\boldsymbol{v} = A\ Re(\boldsymbol{v}) + A\ Im(\boldsymbol{v})\ i$.
- Since A is real, so are $A\ Re(\boldsymbol{v})$ and $A\ Im(\boldsymbol{v})$.
- Thus $A\ Re(\boldsymbol{v})$ is the real part of $A\boldsymbol{v}$ and $A\ Im(\boldsymbol{v})$ is the imaginary part of $A\boldsymbol{v}$.

PROOF

Let A be a real $n \times n$ matrix, and $\boldsymbol{v}$ be a complex eigenvector of it ($Re(\boldsymbol{v}), Im(\boldsymbol{v}) \neq \mathbf{o}$). Second, we show that $Re(\boldsymbol{v})$ and $Im(\boldsymbol{v})$ are linearly independent.

- Let $\overline{\boldsymbol{v}}$ be the complex conjugate of $\boldsymbol{v}$. It is easy to see that $\boldsymbol{v}$ and $\overline{\boldsymbol{v}}$ are linearly independent!
- So, the following has only trivial solution:

$$c_1(Re(\boldsymbol{v}) + Im(\boldsymbol{v})\, i) + c_2(Re(\overline{\boldsymbol{v}}) + Im(\overline{\boldsymbol{v}})\, i) = 0 \tag{1}$$

- Eq. 1 can be simplified as follows:

$$(c_1 + c_2)Re(\boldsymbol{v}) + i\,(c_1 - c_2)Im(\boldsymbol{v}) = 0 \tag{2}$$

  But the solution to Eq. 2 is equivalent to the solution of the following: $k_1\ Re(\boldsymbol{v}) + k_2\ Im(\boldsymbol{v}) = 0$.
- Therefore, $Re(\boldsymbol{v})$ and $Im(\boldsymbol{v})$ are linearly independent.

PROOF

- If $\lambda = a - b\,i$, then $A\boldsymbol{v} = \lambda \boldsymbol{v} = (a - b\,i)(Re(\boldsymbol{v}) + Im(\boldsymbol{v})\,i)$.
- This gives:
  $A\boldsymbol{v} = (a\,Re(\boldsymbol{v}) + b\,Im(\boldsymbol{v})) + (a\,Im(\boldsymbol{v}) - b\,Re(\boldsymbol{v}))\,i =$
  $Re(A\boldsymbol{v}) + Im(A\boldsymbol{v})\,i$.
- By the previous slide, we have:
  - $A\,Re(\boldsymbol{v}) = Re(A\boldsymbol{v}) = a\,Re(\boldsymbol{v}) + b\,Im(\boldsymbol{v})$
  - $A\,Im(\boldsymbol{v}) = Im(A\boldsymbol{v}) = -b\,Re(\boldsymbol{v}) + a\,Im(\boldsymbol{v})$
- Let $P = \begin{bmatrix} Re(\boldsymbol{v}) & Im(\boldsymbol{v}) \end{bmatrix}$. We have: $A\,Re(\boldsymbol{v}) = P\begin{pmatrix} a \\ b \end{pmatrix}$ and
  $A\,Im(\boldsymbol{v}) = P\begin{pmatrix} -b \\ a \end{pmatrix}$
- Therefore, $AP = A\begin{bmatrix} Re(\boldsymbol{v}) & Im(\boldsymbol{v}) \end{bmatrix} = \begin{bmatrix} P\begin{pmatrix} a \\ b \end{pmatrix} & P\begin{pmatrix} -b \\ a \end{pmatrix} \end{bmatrix} =$
  $P\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = P\,C$