



Discrete Mathematics

Session XV

# Recurrence Relations

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# Introduction

We introduced recurrence relations in the previous session.

It was also illustrated how one may derive a recurrence relation for a given (combinatorial) problem.

As discussed, there is no general method for solving recurrence relations.

In most cases, one could just try to guess a solution, through examining a number of terms of the given sequence. Then, they may verify the solution using mathematical induction.

Nevertheless, fortunately, there exists a systematic method for solving certain, useful classes of recurrence relations.

In this session, we elaborate on an approach to solving *linear homogeneous* recurrence relations (LHRRs) with *constant coefficients*.

We also give an overview of linear algebra so that one can understand the rationale behind the technique for solving LHRRs.



# Linear Recurrence Relations

Let  $k \in \mathbb{Z}^+$  and  $C_0, C_1, \dots, C_k \in \mathbb{C}$  be complex constants where  $C_0 \neq 0$  and  $C_k \neq 0$ . Let also  $f: \mathbb{Z}^{\geq 0} \rightarrow \mathbb{C}$  be a function. Then,

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \dots + C_k a_{n-k} = f(n), \quad n \geq k$$

is said to be a **linear recurrence relation** with **constant coefficients** of **order (degree)  $k$** .

**Linear** refers to the fact that  $a_n, a_{n-1}, \dots$ , and  $a_{n-k}$  appear in separate terms and to the first power.

**Constant Coefficients** refers to the fact that  $C_0, C_1, \dots, C_k$  are fixed (constant) complex numbers that do not depend on  $n$ .

**Order (Degree)  $k$**  refers to the fact that the relation relates any term  $a_n$  to its previous  $k$  terms.

If  $f(n) = 0$  for all  $n \in \mathbb{Z}^{\geq 0}$ , the relation is called **homogeneous**; otherwise, it is called **nonhomogeneous**.

A **solution** to a recurrence relation is a (complex) sequence whose elements satisfy the relation.

**Solving** a recurrence relation means to find explicit solutions for the recurrence relation. That is, to give the general term of the solution in terms of its index.



# Linear Recurrence Relations (Ctd.)

Consider the following recurrence relations.

$$a_n - 2a_{n-1} = 1 + i, \quad n \geq 1$$

linear
first-order

constant coefficients
nonhomogeneous

$$a_n - na_{n-1} = 0, \quad n \geq 1$$

linear
first-order

variable coefficients
homogeneous

$$a_n - 3.4a_{n-1} = 0, \quad n \geq 1$$

linear
first-order

constant coefficients
homogeneous

$$a_n - 2a_{n-1}^2 = 0, \quad n \geq 1$$

nonlinear
first-order

$$a_n a_{n-1} = 1, \quad n \geq 1$$

nonlinear
first-order

$$a_n - 2a_{n-1} + 6a_{n-2} = 0, \quad n \geq 2$$

linear
second-order

constant coefficients
homogeneous

$$a_n - ia_{n-2} + a_{n-5} = 0, \quad n \geq 5$$

linear
fifth-order

constant coefficients
homogeneous

$$a_n - 3a_{n-2} + a_{n-5} = 2^n + 3, \quad n \geq 5$$

linear
fifth-order

constant coefficients
nonhomogeneous



# Linear Homogeneous Recurrence Relations (LHRRs) with Constant Coefficients

Let us begin with an example.

**Example 1.** Solve the recurrence relation  $a_n - 5a_{n-1} + 6a_{n-2} = 0$  where  $n \geq 2$ .

**Solution.** A trivial solution to this recurrence relation is  $\{0\}_{n=0}^{\infty}$ . In fact,  $\{0\}_{n=0}^{\infty}$  is a solution to every linear homogeneous recurrence relation. Now, assume that  $\{r^n\}_{n=0}^{\infty}$  is a solution to the given recurrence relation where  $r \in \mathbb{C}$  and  $r \neq 0$ . Thus,

$$r^n - 5r^{n-1} + 6r^{n-2} = 0$$

must hold for all  $n \geq 2$ . It follows that

$$r^2 - 5r + 6 = 0.$$

**The Characteristic Equation**

The roots of the above equation are  $r_1 = 2$  and  $r_2 = 3$  (**characteristic roots**.) As a result,  $\{2^n\}_{n=0}^{\infty}$  and  $\{3^n\}_{n=0}^{\infty}$  are solutions to the relation. Because the recurrence relation is linear, all linear combinations of these two solutions, that is, all sequences

$$\{\alpha_1 2^n + \alpha_2 3^n\}_{n=0}^{\infty}$$

with  $\alpha_1, \alpha_2 \in \mathbb{C}$  are also solutions to the given recurrence relation.

Conversely, we can prove that for every solution  $\{b_n\}_{n=0}^{\infty}$  to the relation, there are constants  $\alpha_1$  and  $\alpha_2$  such that  $b_n = \alpha_1 2^n + \alpha_2 3^n$  holds for all  $n \geq 0$ . Assume that  $\{b_n\}_{n=0}^{\infty}$  is a solution to the relation. Let  $\alpha_1$  and  $\alpha_2$  be the **unique** constants that satisfy  $\alpha_1 + \alpha_2 = b_0$  and  $2\alpha_1 + 3\alpha_2 = b_1$ . By the strong mathematical induction, it can be proven that  $b_n = \alpha_1 2^n + \alpha_2 3^n$  holds for all  $n \geq 0$  (It is left to you.)



## LHRRs with Constant Coefficients (Ctd.)

**Example 2.** Solve the recurrence relation  $a_n - 6a_{n-1} + 9a_{n-2} = 0$  where  $n \geq 2$ .

**Solution.** Assume that  $\{r^n\}_{n=0}^{\infty}$  is a solution to the given recurrence relation where  $r \in \mathbb{C}$  and  $r \neq 0$ . Thus, the following must hold for all  $n \geq 2$ .

$$r^n - 6r^{n-1} + 9r^{n-2} = 0.$$

It follows that

$$r^2 - 6r + 9 = 0.$$

The characteristic roots are  $r_1 = r_2 = 3$ , that is, 3 is a **double** root of the characteristic equation. Because  $\{3^n\}_{n=0}^{\infty}$  is a nontrivial solution to the given recurrence relation, all sequences  $\{\alpha 3^n\}_{n=0}^{\infty}$  with  $\alpha \in \mathbb{C}$  are also solutions to the given relation.

However, unlike Example 1, it is not the case that every solution to the given recurrence relation is a linear combination of  $\{r_1^n\}_{n=0}^{\infty}$  and  $\{r_2^n\}_{n=0}^{\infty}$ , where  $r_1$  and  $r_2$  are the roots of the equation above, that is,  $\{\alpha 3^n\}_{n=0}^{\infty}$ . For example,  $\{n3^{n+1}\}_{n=0}^{\infty}$  is a solution to the given relation because

$$n3^{n+1} - 6(n-1)3^n + 9(n-2)3^{n-1} = 0$$

holds for all  $n \geq 2$ . As you see, it cannot be expressed as  $\{\alpha 3^n\}_{n=0}^{\infty}$  for any complex  $\alpha$ .

Nevertheless, by mathematical induction, one can prove that every solution to the given recurrence relation is a linear combination of the sequences  $\{3^n\}_{n=0}^{\infty}$  and  $\{n3^{n+1}\}_{n=0}^{\infty}$ . That is, the set of solutions to the given recurrence relation consists of the sequences  $\{\alpha_1 3^n + \alpha_2 n3^{n+1}\}_{n=0}^{\infty}$  where  $\alpha_1, \alpha_2 \in \mathbb{C}$ .



# Linear Algebra and LHRRs

Now, we quote some definitions and results from *linear algebra*.

A **vector space** over a **field**  $F$  is a set  $V$  (the elements of which are called **vectors**) with an **addition**  $\oplus$  and a **scalar multiplication**  $\odot$  satisfying the following properties for all  $u, v, w \in V$  and  $\alpha, \beta \in F$ :

- 1)  $v \oplus w \in V$ ,
- 2)  $v \oplus w = w \oplus v$ ,
- 3)  $(u \oplus v) \oplus w = u \oplus (v \oplus w)$ ,
- 4) there exists a vector  $\mathbf{0}$  in  $V$  such that  $v \oplus \mathbf{0} = v$ ,
- 5) for each vector  $v$  in  $V$ , there exists a vector  $-v$  in  $V$  such that  $v \oplus (-v) = \mathbf{0}$ ,
- 6)  $\alpha \odot v \in V$
- 7)  $\alpha \odot (v \oplus w) = (\alpha \odot v) \oplus (\alpha \odot w)$ ,
- 8)  $(\alpha + \beta) \odot v = (\alpha \odot v) \oplus (\beta \odot v)$ ,
- 9)  $(\alpha \cdot \beta) \odot v = \alpha \odot (\beta \odot v)$ ,
- 10) if 1 is the multiplicative identity of  $F$ , then  $1 \odot v = v$ .

The set  $\mathbb{R}^2$  of real pairs (the two-dimensional real coordinate space) with the following addition and scalar multiplication is a vector space over  $\mathbb{R}$  (Why?)

$$(x, y) \oplus (x', y') = (x + x', y + y').$$

$$\alpha \odot (x, y) = (\alpha x, \alpha y).$$



## Linear Algebra and LHRRs (Ctd.)

Consider the following second-order linear homogeneous recurrence relation.

$$a_n - 5a_{n-1} + 6a_{n-2} = 0 \quad (n \geq 2.)$$

The set of complex solutions to this relation is a vector space over  $\mathbb{C}$  with the operations  $\{a_n\}_{n=0}^{\infty} \oplus \{b_n\}_{n=0}^{\infty} = \{a_n + b_n\}_{n=0}^{\infty}$  and  $\alpha \odot \{a_n\}_{n=0}^{\infty} = \{\alpha \cdot a_n\}_{n=0}^{\infty}$ , where  $+$  and  $\cdot$  are the addition and multiplication operations on complex numbers, respectively (Why?)

Let  $V$  be a vector space over  $F$ . A **linear combination** of vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$  is a vector  $\mathbf{v} \in V$  that is equal to a sum of scalar multiples of  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ . That is, there exist  $\alpha_1, \alpha_2, \dots, \alpha_k \in F$  such that  $\mathbf{v} = (\alpha_1 \odot \mathbf{v}_1) \oplus (\alpha_2 \odot \mathbf{v}_2) \oplus \dots \oplus (\alpha_k \odot \mathbf{v}_k)$ .

Let  $V$  be a vector space over  $F$ . The vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$  are said to be **linearly independent** (or form a **linearly independent set**) if and only if the equation

$$(\alpha_1 \odot \mathbf{v}_1) \oplus (\alpha_2 \odot \mathbf{v}_2) \oplus \dots \oplus (\alpha_k \odot \mathbf{v}_k) = \mathbf{0}$$

has the unique solution  $\alpha_1 = \alpha_2 = \dots = \alpha_k = 0$ .

The sequences  $\{2^n\}_{n=0}^{\infty}$ ,  $\{3^n\}_{n=0}^{\infty}$ , and  $\{3^{n+1}\}_{n=0}^{\infty}$  are solutions to the recurrence relation  $a_n - 5a_{n-1} + 6a_{n-2} = 0$  where  $n \geq 2$ . The vectors  $\{2^n\}_{n=0}^{\infty}$  and  $\{3^n\}_{n=0}^{\infty}$  are linearly independent, whereas  $\{3^n\}_{n=0}^{\infty}$  and  $\{3^{n+1}\}_{n=0}^{\infty}$  are not because

$$(-3) \odot \{3^n\}_{n=0}^{\infty} \oplus 1 \odot \{3^{n+1}\}_{n=0}^{\infty} = \{0\}_{n=0}^{\infty}.$$





## Linear Algebra and LHRRs (Ctd.)

Let  $V$  be a vector space over  $F$ . A **subspace**  $V_0$  of  $V$  is said to be **spanned** by the set  $S = \{v_1, v_2, \dots, v_k\} \subseteq V$  iff it consists of all linear combinations of vectors in  $S$ . That is,

$$V_0 = \{(\alpha_1 \odot v_1) \oplus (\alpha_2 \odot v_2) \oplus \dots \oplus (\alpha_k \odot v_k) \mid \alpha_1, \alpha_2, \dots, \alpha_k \in F\}.$$

The set  $S$  in the above definition is also said to be a **spanning** set for the subspace  $V_0$  of  $V$ . Moreover, the subspace  $V_0$  is also denoted by  $\text{span}(S)$ .

A **basis** for a vector space  $V$  is a linearly independent spanning set for  $V$ . It is immediate that the number of vectors in all bases for a vector space is the same. The number of vectors in a basis for a vector space is called the **dimension** of that vector space. The dimension of a vector space  $V$  is denoted by  $\dim(V)$ .

Every linearly independent subset  $S$  of a vector space  $V$  with  $|S| = \dim(V)$  is a spanning set for  $V$ .

Let  $V$  be the complex vector space of complex sequences that solve the linear homogeneous recurrence relation

$$C_0 a_n + C_1 a_{n-1} + \dots + C_k a_{n-k} = 0$$

with constant coefficients where  $n \geq k$ ,  $C_i \in \mathbb{C}$ , and  $C_0, C_k \neq 0$ . Then,  $\dim(V) = k$ .

The set of all solutions to a linear homogeneous recurrence relation with constant coefficients of degree  $k$  equals the set of all linear combinations of any  $k$  linearly independent solutions to the relation.



## Solving LHRRs with Constant Coefficients

Let  $\{r^n\}_{n=0}^{\infty}$  be a nontrivial solution ( $r \neq 0$ ) to the LHRR

$$C_0 a_n + C_1 a_{n-1} + \cdots + C_k a_{n-k} = 0$$

With constant coefficients where  $n \geq k$ ,  $C_i \in \mathbb{C}$ , and  $C_0, C_k \neq 0$ .

Thus, we have

$$C_0 r^n + C_1 r^{n-1} + \cdots + C_k r^{n-k} = 0,$$

or

$$C_0 r^k + C_1 r^{k-1} + \cdots + C_k = 0,$$

which is called the **characteristic equation** for the given recurrence relation.

The characteristic equation for an LHRR of order  $k$  has exactly  $k$  (complex) roots, also called the **characteristic roots**. A characteristic root, however, may be **multiple**.

If the roots of the characteristic equation for an LHRR are **distinct**, say  $r_1, r_2, \dots, r_k$ , we obtain  $k$  linearly independent solutions to the LHRR. They are  $\{r_1^n\}_{n=0}^{\infty}, \{r_2^n\}_{n=0}^{\infty}, \dots, \{r_k^n\}_{n=0}^{\infty}$ . Any solution to the LHRR is, then, of the form

$$\{\alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n\}_{n=0}^{\infty},$$

where  $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{C}$ .

This is a corollary of the results from the linear algebra. You may also try to prove it using the strong mathematical induction.



## Solving LHRs with Constant Coefficients: Distinct Characteristic Roots

**Example 3.** Solve the recurrence relation

$$a_n - 2a_{n-1} - 5a_{n-2} + 6a_{n-3} = 0 \quad (n \geq 3,)$$

where  $a_0 = 0$ ,  $a_1 = -1$ , and  $a_2 = 11$ .

**Solution.** The characteristic equation is

$$r^3 - 2r^2 - 5r + 6 = 0.$$

Thus, the characteristic roots are  $r_1 = 1$ ,  $r_2 = -2$ , and  $r_3 = 3$ . Because the roots are distinct, the general term  $a_n$  of any solution  $\{a_n\}_{n=0}^{\infty}$  to the recurrence relation is of the form

$$\alpha_1(1)^n + \alpha_2(-2)^n + \alpha_3(3)^n,$$

where  $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ . According to the initial conditions, we have

$$\alpha_1 + \alpha_2 + \alpha_3 = 0,$$

$$\alpha_1 - 2\alpha_2 + 3\alpha_3 = -1,$$

$$\alpha_1 + 4\alpha_2 + 9\alpha_3 = 11.$$

It follows that  $\alpha_1 = -2$ ,  $\alpha_2 = 1$ , and  $\alpha_3 = 1$ . Hence, the unique solution to the given recurrence relation is

$$\{-2 + (-2)^n + 3^n\}_{n=0}^{\infty}.$$



**De Moivre's Theorem.** For any real number  $x$  and integer  $n$ , it holds that

$$(\cos x + i \sin x)^n = \cos(nx) + i \sin(nx) .$$

## Solving LHRs with Constant Coefficients Distinct Characteristic Roots (Ctd.)

**Example 4.** Solve the recurrence relation

$$a_n - 2a_{n-1} + 2a_{n-2} = 0 \quad (n \geq 2.)$$

where  $a_0 = 1$  and  $a_1 = 1$ .

**Solution.** The characteristic equation is

$$r^2 - 2r + 2 = 0,$$

whose roots are  $r_1 = 1 + i$ ,  $r_2 = 1 - i$ , where  $i$  is the imaginary unit. Equivalently,

$$r_1 = \sqrt{2} (\sqrt{2}/2 + \sqrt{2}/2 i) = \sqrt{2} (\cos \pi/4 + i \sin \pi/4), \text{ and}$$

$$r_2 = \sqrt{2} (\sqrt{2}/2 - \sqrt{2}/2 i) = \sqrt{2} (\cos (-\pi/4) + i \sin (-\pi/4)).$$

Because the roots are distinct, the general term  $a_n$  of any solution  $\{a_n\}_{n=0}^{\infty}$  to the recurrence relation is of the form

$$\alpha_1 r_1^n + \alpha_2 r_2^n = 2^{n/2} (\alpha_1 + \alpha_2) \cos(n\pi/4) + (2^{n/2} (\alpha_1 - \alpha_2) \sin(n\pi/4))i$$

where  $\alpha_1, \alpha_2 \in \mathbb{C}$ . With  $a_0 = 1$  and  $a_1 = 1$ ,  $\alpha_1$  and  $\alpha_2$  are determined as follows:

$$\alpha_1 + \alpha_2 = 1,$$

$$(\alpha_1 + \alpha_2) + (\alpha_1 - \alpha_2)i = 1.$$

It follows that  $\alpha_1 = \alpha_2 = 1/2$ . Hence, the unique solution to the given recurrence relation is

$$\{2^{n/2} \cos(n\pi/4)\}_{n=0}^{\infty}.$$



## Solving LHRRs with Constant Coefficients: Distinct Characteristic Roots (Ctd.)

**Example 5.** For  $b \in \mathbb{R}^+$ , find the determinant  $D_n$  of the following  $n \times n$  tridiagonal matrix

$$\begin{vmatrix} b & b & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ b & b & b & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & b & b & b & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & b & b & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & b & b & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & b & b & b & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b & b & b \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & b & b \end{vmatrix}$$

**solution.** It is immediate that  $D_1 = |b| = b$  and  $D_2 = \begin{vmatrix} b & b \\ b & b \end{vmatrix} = 0$ . For  $n \geq 3$ , expanding  $D_n$  by its first row, we have

$$D_n = bD_{n-1} - b \begin{vmatrix} 0 & b & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & b & b & b & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & b & b & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & b & b & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & b & b & b & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b & b & b \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & b & b \end{vmatrix} = bD_{n-1} - b(bD_{n-2})$$

$$D_n - bD_{n-1} + b^2D_{n-2} = 0$$



## Solving LHRRs with Constant Coefficients: Distinct Characteristic Roots (Ctd.)

Thus, we should solve the recurrence relation

$$D_n - bD_{n-1} + b^2D_{n-2} = 0 \quad (n \geq 3,)$$

with  $D_1 = b$  and  $D_2 = 0$ .

The characteristic equation for the given recurrence is  $r^2 - br + b^2 = 0$  whose roots are  $r_1 = b(1/2 + \sqrt{3}/2 i)$  and  $r_2 = b(1/2 - \sqrt{3}/2 i)$ . Thus, the general term of any solution to this relation is of the form

$$\begin{aligned} D_n &= \alpha_1 r_1^n + \alpha_2 r_2^n \\ &= \alpha_1 b^n (\cos(\pi/3) + i \sin(\pi/3))^n + \alpha_2 b^n (\cos(-\pi/3) + i \sin(-\pi/3))^n \\ &= \alpha_1 b^n (\cos(n\pi/3) + i \sin(n\pi/3)) + \alpha_2 b^n (\cos(n\pi/3) - i \sin(n\pi/3)) \\ &= (\alpha_1 + \alpha_2) b^n \cos(n\pi/3) + i(\alpha_1 - \alpha_2) b^n \sin(n\pi/3) \end{aligned}$$

With  $D_1 = b$  and  $D_2 = 0$ , we have

$$\begin{aligned} (1/2)(\alpha_1 + \alpha_2)b + i(\sqrt{3}/2)(\alpha_1 - \alpha_2)b &= b \\ (-1/2)(\alpha_1 + \alpha_2)b^2 + i(\sqrt{3}/2)(\alpha_1 - \alpha_2)b^2 &= 0 \end{aligned}$$

It follows that  $\alpha_1 + \alpha_2 = 1$  and  $\alpha_1 - \alpha_2 = -i(\sqrt{3}/3)$ . Thus,

$$D_n = b^n (\cos(n\pi/3) + (\sqrt{3}/3) \sin(n\pi/3)).$$



## Solving LHRRs with Constant Coefficients: Multiple Characteristic Roots

Consider the second-order LHRR

$$Aa_n + Ba_{n-1} + Ca_{n-2} = 0 \quad (n \geq 2)$$

with the **double** characteristic root  $r_0$ .

As  $r_0$  is double, we have  $Ar^2 + Br + C = A(r - r_0)^2 = Ar^2 - 2Ar_0r + Ar_0^2$ . It follows that  $B = -2Ar_0$  and  $C = Ar_0^2$ .

Now, we show that the sequence  $\{nr_0^n\}_{n=0}^\infty$  is a solution to the given relation.

$$\begin{aligned} Anr_0^n + B(n-1)r_0^{n-1} + C(n-2)r_0^{n-2} \\ &= n(Ar_0^n + Br_0^{n-1} + Cr_0^{n-2}) - Br_0^{n-1} - 2Cr_0^{n-2} \\ &= 0 - r_0^{n-2}(Br_0 + 2C) = r_0^{n-2}(2Ar_0^2 - 2Ar_0^2) = 0 \end{aligned}$$

Because  $\{r_0^n\}_{n=0}^\infty$  and  $\{nr_0^n\}_{n=0}^\infty$  are two linearly independent solutions to the relation, any solution  $\{a_n\}_{n=0}^\infty$  to the relation is a linear combination of  $\{r_0^n\}_{n=0}^\infty$  and  $\{nr_0^n\}_{n=0}^\infty$ . That is,

$$\{a_n\}_{n=0}^\infty = \{\alpha_1 r_0^n + \alpha_2 nr_0^n\}_{n=0}^\infty = \{(\alpha_1 + \alpha_2 n)r_0^n\}_{n=0}^\infty,$$

where  $\alpha_1, \alpha_2 \in \mathbb{C}$ .

For example, any solution to the recurrence relation  $a_n - 4a_{n-1} + 4a_{n-2} = 0$ , for all  $n \geq 2$ , is of the form  $\{(\alpha_1 + \alpha_2 n)2^n\}_{n=0}^\infty$ , where  $\alpha_1, \alpha_2 \in \mathbb{C}$ .



## Solving LHRRs with Constant Coefficients: Multiple Characteristic Roots (Ctd.)

Using the same technique, one can prove that  $\{nr_0^n\}_{n=0}^{\infty}$  and  $\{n^2r_0^n\}_{n=0}^{\infty}$  are solutions to the recurrence relation

$$Aa_n + Ba_{n-1} + Ca_{n-2} + Da_{n-3} = 0 \quad (n \geq 3)$$

if  $r_0$  is a **triple** characteristic root (It is left to you as an exercise.)

The above results can be generalized as follows:

Let  $C_0a_n + C_1a_{n-1} + \cdots + C_ka_{n-k} = 0 \quad (n \geq k)$  be an LHRR with constant coefficients of order  $k$ . The characteristic equation for this relation has exactly  $k$  complex roots. For  $1 \leq j \leq t$ , let  $r_j$  be a multiple characteristic root with multiplicity  $m_j$  (we have  $m_1 + m_2 + \cdots + m_t = k$ .) Then, the following are  $k$  linearly independent solutions to the relation.

$$\begin{aligned} &\{r_1^n\}_{n=0}^{\infty}, \{nr_1^n\}_{n=0}^{\infty}, \{n^2r_1^n\}_{n=0}^{\infty}, \dots, \{n^{m_1-1}r_1^n\}_{n=0}^{\infty}, \\ &\{r_2^n\}_{n=0}^{\infty}, \{nr_2^n\}_{n=0}^{\infty}, \{n^2r_2^n\}_{n=0}^{\infty}, \dots, \{n^{m_2-1}r_2^n\}_{n=0}^{\infty}, \\ &\quad \dots \\ &\{r_t^n\}_{n=0}^{\infty}, \{nr_t^n\}_{n=0}^{\infty}, \{n^2r_t^n\}_{n=0}^{\infty}, \dots, \{n^{m_t-1}r_t^n\}_{n=0}^{\infty}. \end{aligned}$$

It follows that every solution to the given recurrence relation is of the form

$$\left\{ \sum_{j=1}^t \sum_{s=0}^{m_j-1} \alpha_{js} n^s r_j^n \right\}_{n=0}^{\infty}.$$





## Solving LHRRs with Constant Coefficients: Multiple Characteristic Roots (Ctd.)

**Example 6.** Solve the recurrence relation

$$a_n - 11a_{n-1} + 49a_{n-2} - 113a_{n-3} + 142a_{n-4} - 92a_{n-5} + 24a_{n-6} = 0,$$

where  $n \geq 6$ ,  $a_0 = a_1 = a_2 = 0$ ,  $a_3 = 2$ ,  $a_4 = -1$ , and  $a_5 = 12$ .

**Solution.** The characteristic equation for the given recurrence relation is

$$r^6 - 11r^5 + 49r^4 - 113r^3 + 142r^2 - 92r + 24 = 0,$$

which can be factorized as

$$(r - 1)^2(r - 2)^3(r - 3) = 0.$$

As seen,  $r_1 = 1$  is a double,  $r_2 = 2$  is a triple, and  $r_3 = 3$  is a single characteristic root.

Thus, the general term of every solution  $\{a_n\}_{n=0}^{\infty}$  to the given recurrence relation is

$$a_n = \alpha_{11} + \alpha_{12}n + \alpha_{21}2^n + \alpha_{22}n2^n + \alpha_{23}n^22^n + \alpha_{31}3^n.$$

With  $a_0 = a_1 = a_2 = 0$ ,  $a_3 = 2$ ,  $a_4 = -1$ , and  $a_5 = 12$ , we have

$$\begin{aligned}\{a_n\}_{n=0}^{\infty} = & \{(-92.256) + (-53.103)n + (100.532)2^n \\ & + (-30.840)n2^n + (2.513)n^22^n + (0.316)3^n\}_{n=0}^{\infty}\end{aligned}$$





**Textbook: Ralph P. Grimaldi, Discrete and Combinatorial  
Mathematics**

**Please consult with Chapter 10 of your textbook.**