دانشگاه صنعتی امیرکبیر
(پلی تکنیک تهران)

Electrical and Electronic Circuits

chapter 7. RLC Circuits

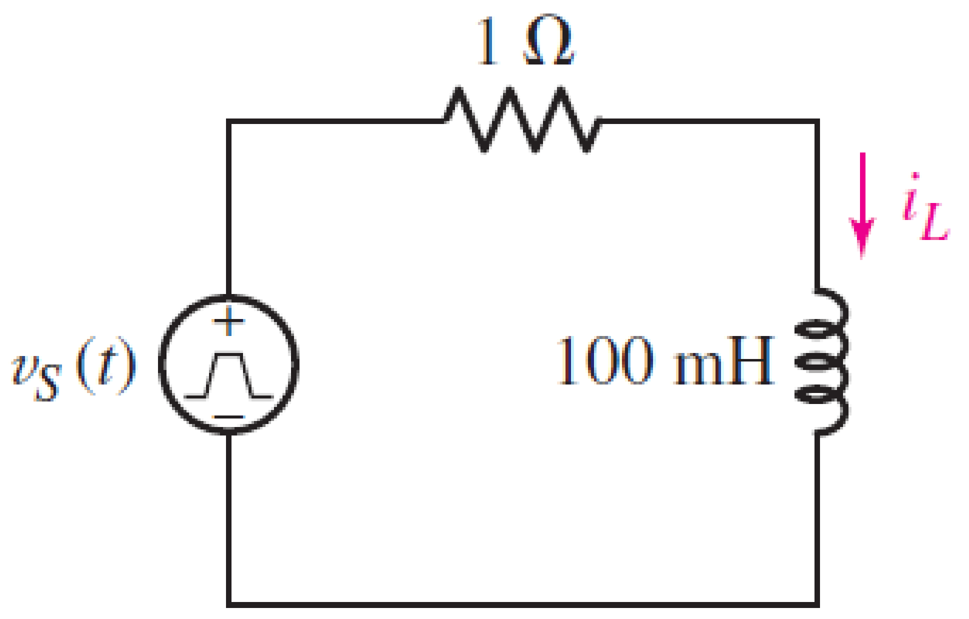
Afarghadan@aut.ac.ir

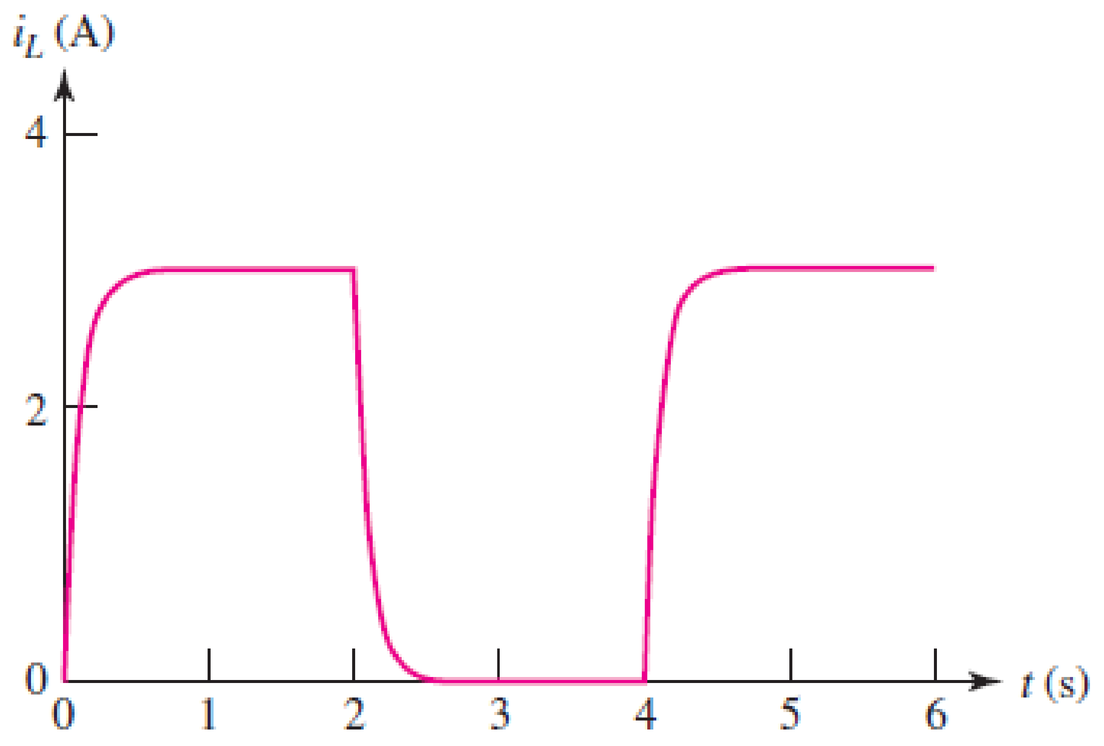
عظیم فرقدان

مهر ۱۴۰۳

Objectives of the Lecture

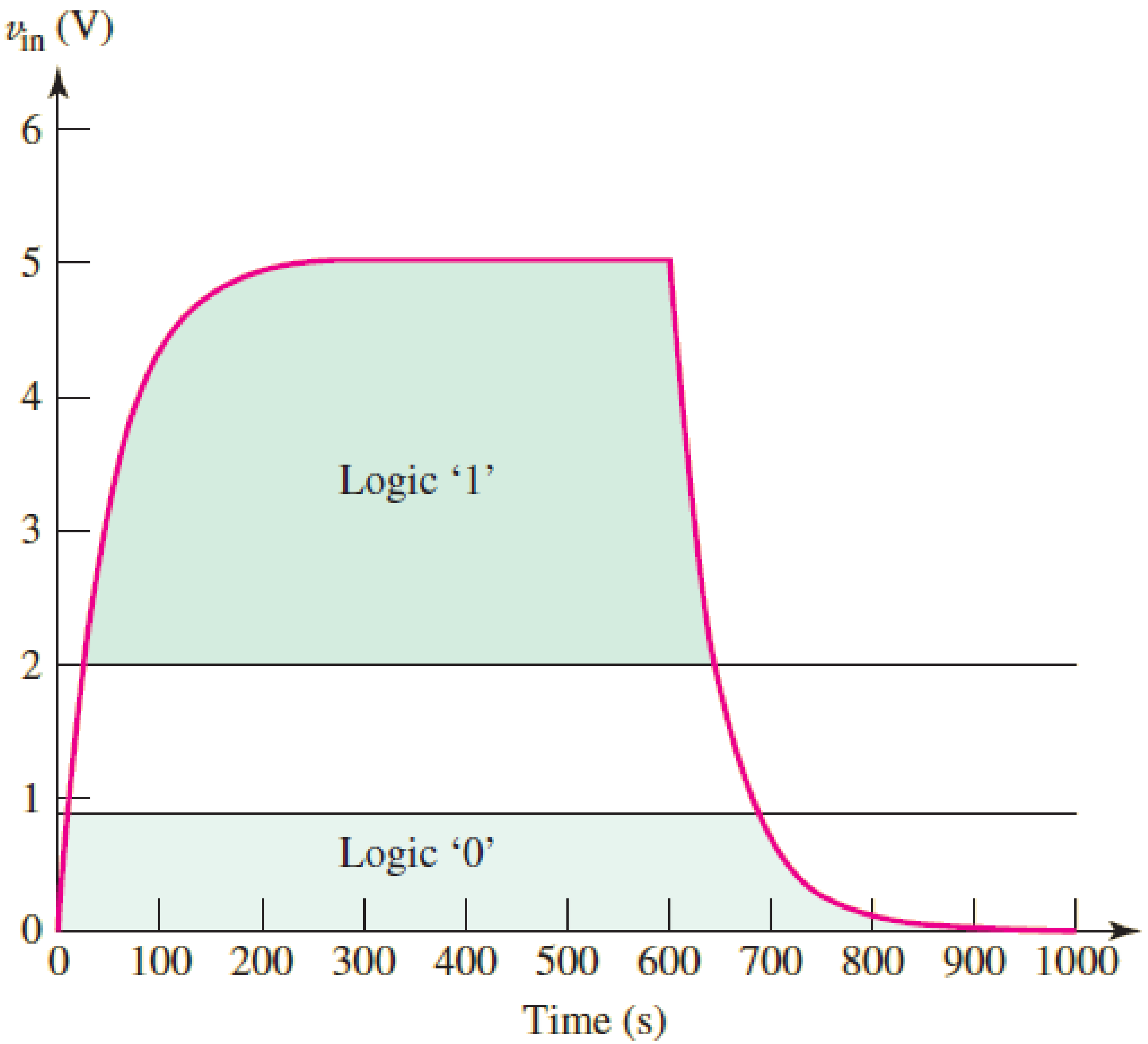
- Application of RL and RC circuits
- Second order circuits: RLC
  - Parallel RLC circuit without source
  - Series RLC circuit without source
  - Complete response in the presence of source and initial conditions
  - How to calculate initial conditions

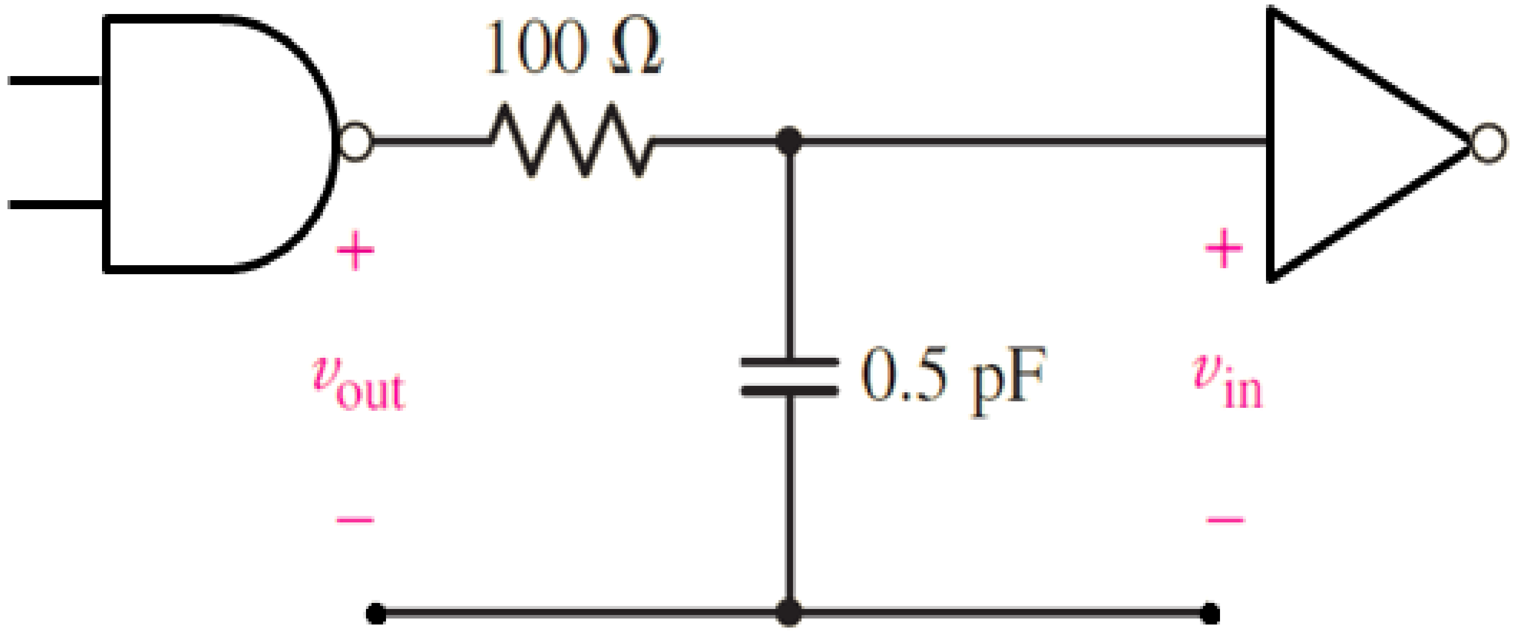




$$i_L(t)=\frac{V_0}{R}\left(1-e^{-Rt/L}\right)u(t)$$

Application in delay analysis of integrated circuits



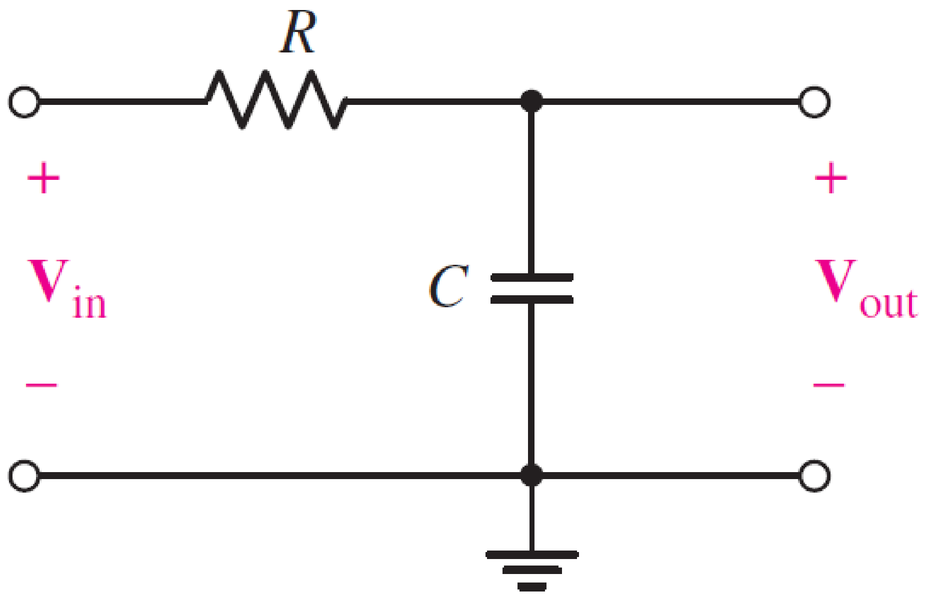


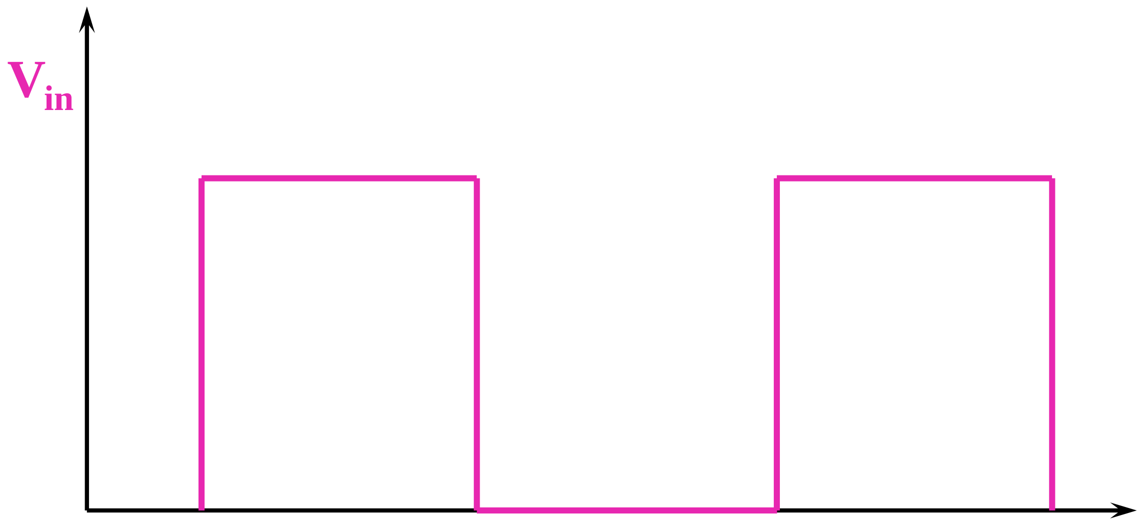
- The parasitic resistance and capacitor of the wire connecting two logic gates increases the delay.
- Maximum allowed frequency:

$$f_{max} = \frac{1}{2 \times 5\tau} = 2Ghz$$

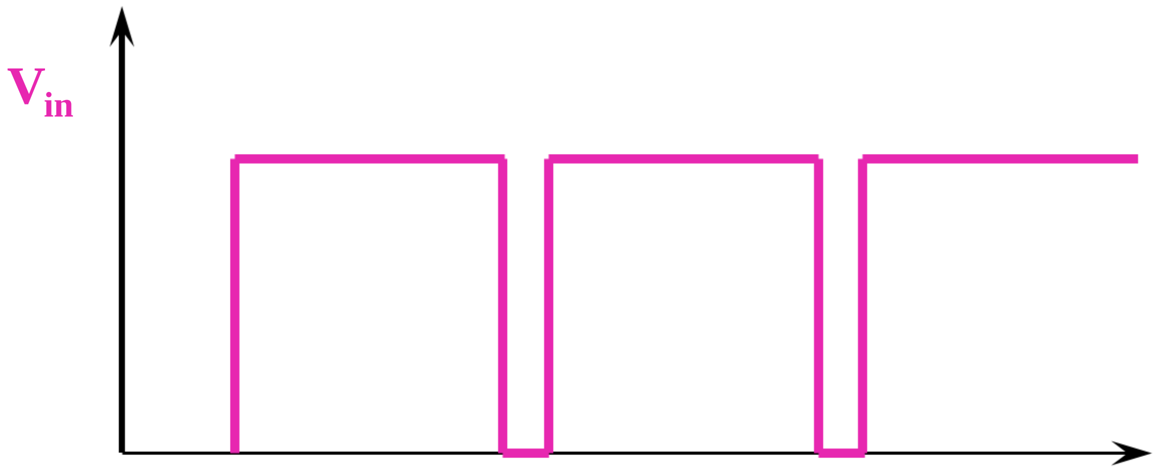
Application as a frequency filter

- A low-pass filter

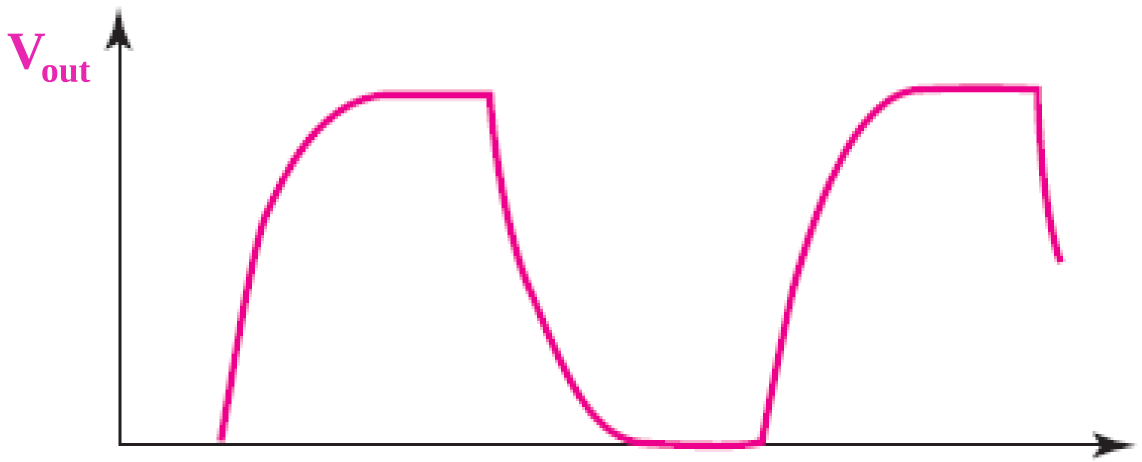


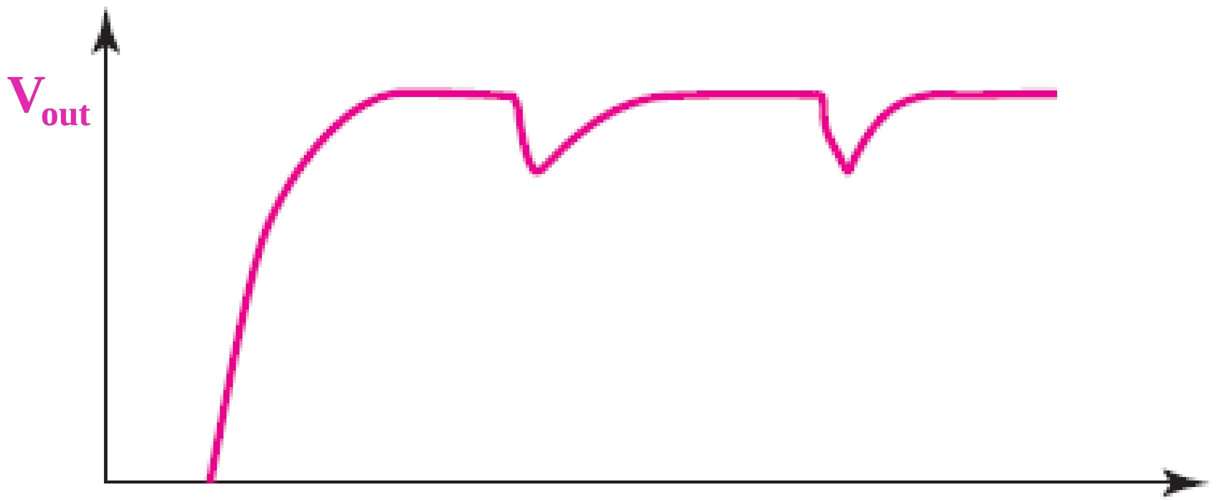


Low frequency

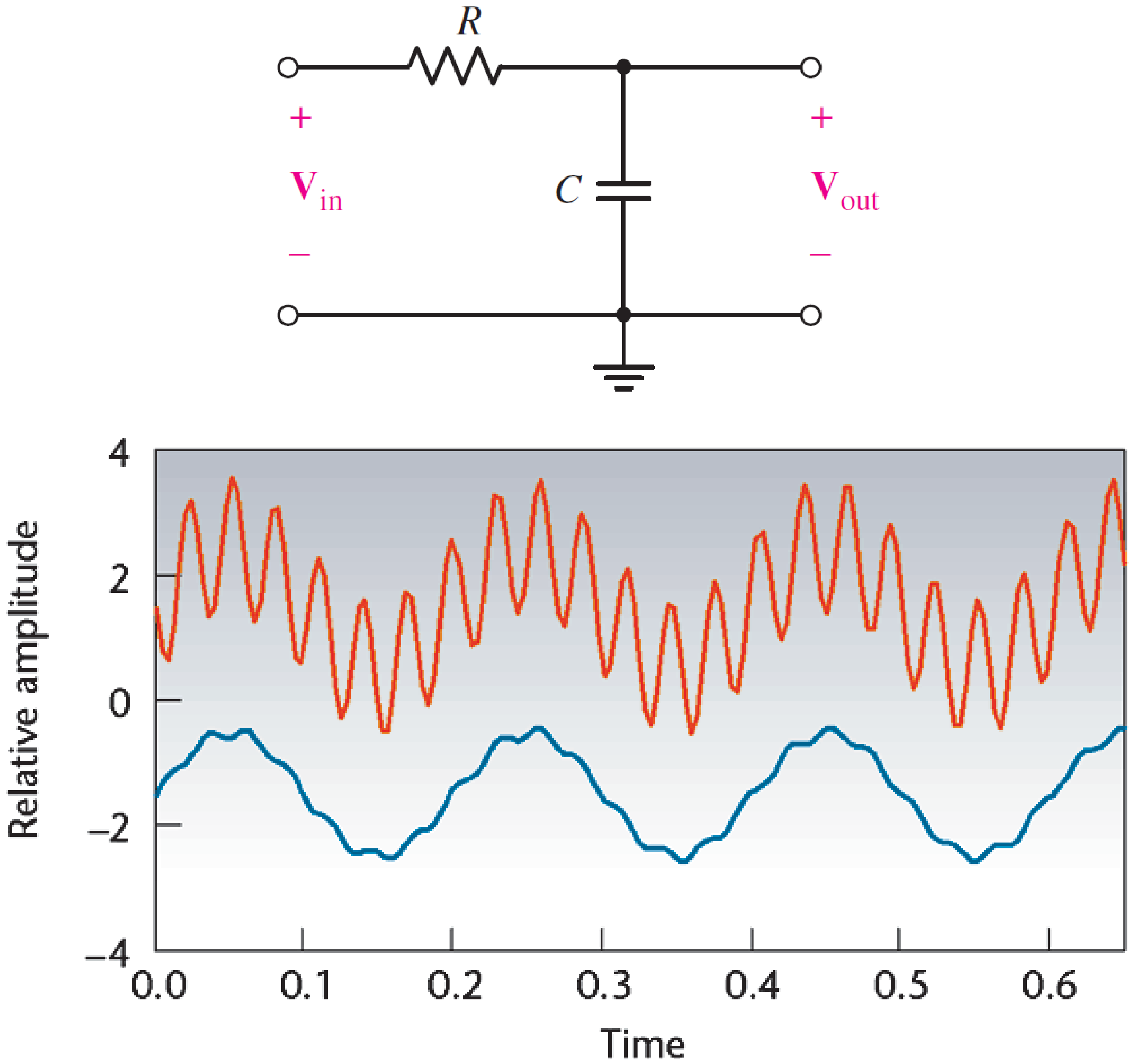


High frequency





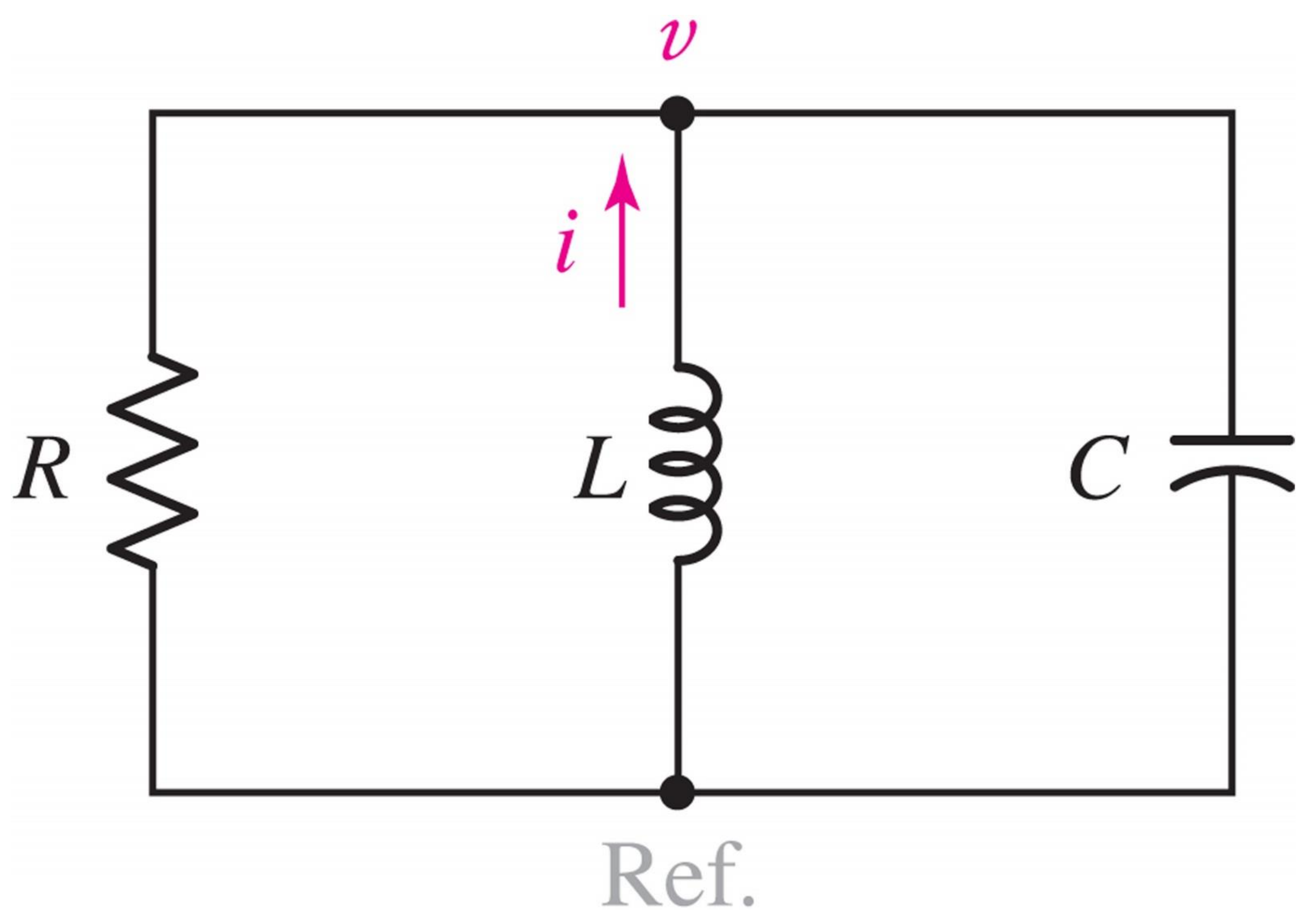
Application as a frequency filter: noise removal



The RLC circuit

- An RLC circuit has both an inductor and a capacitor.
- If it has only one inductor and one capacitor, the circuit will be second order.
- Of course, a second-order circuit can be made with two capacitors or two inductors.
- RLC circuits have many different applications:
- Oscillator: A circuit that produces an alternating pulse (to make a clock)
- Frequency filter: For example, to remove noise
- Analog radio receiver and...
- It is also possible to model the behaviour of car suspension system, elevator, airplane, temperature controller, etc. using an RLC circuit.

Applying KCL differentiate to show:



$$C\frac{d^2v}{dt^2}+\frac{1}{R}\frac{dv}{dt}+\frac{1}{L}v=0$$

Solving the Differential Equation

To solve, assume $v = Ae^{st}$.

The solution must then satisfy:

$$Cs^2 + \frac{1}{R}s + \frac{1}{L} = 0$$

which is called the *characteristic equation*.

If s_1 and s_2 are the solutions, then the natural response is

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Exploring the Solution

The solutions to the characteristic equation are:

$$s_1, s_2 = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

Define ω_0 *the resonant frequency*:

$$\omega_0 = 1/\sqrt{LC}$$

and α *the damping coefficient*:

$$\alpha = \frac{1}{2RC}$$

Exploring the Solution

With these definitions, the solutions can be expressed as:

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

The constants A_1 and A_2 are determined by the initial conditions.

Types of Responses

- If $\alpha>\omega_0$ the solutions are real, unequal and the response is termed ***overdamped***.
  - The response does not have an oscillatory state. Like dropping a pendulum in a container of grease, or dropping a very stiff spring.
- If $\alpha=\omega_0$ the solutions are real and equal and the response is termed ***critically damped***.
  - The circuit is on the verge of oscillating, but it is not yet oscillating.
- If $\alpha<\omega_0$ the solutions are complex conjugates and the response is termed ***underdamped.***
  - The response of the circuit is in the form of oscillatory damping. Like dropping a pendulum

Overdamped Parallel RLC $(\alpha>\omega_0)$

Both roots are real and distinct.

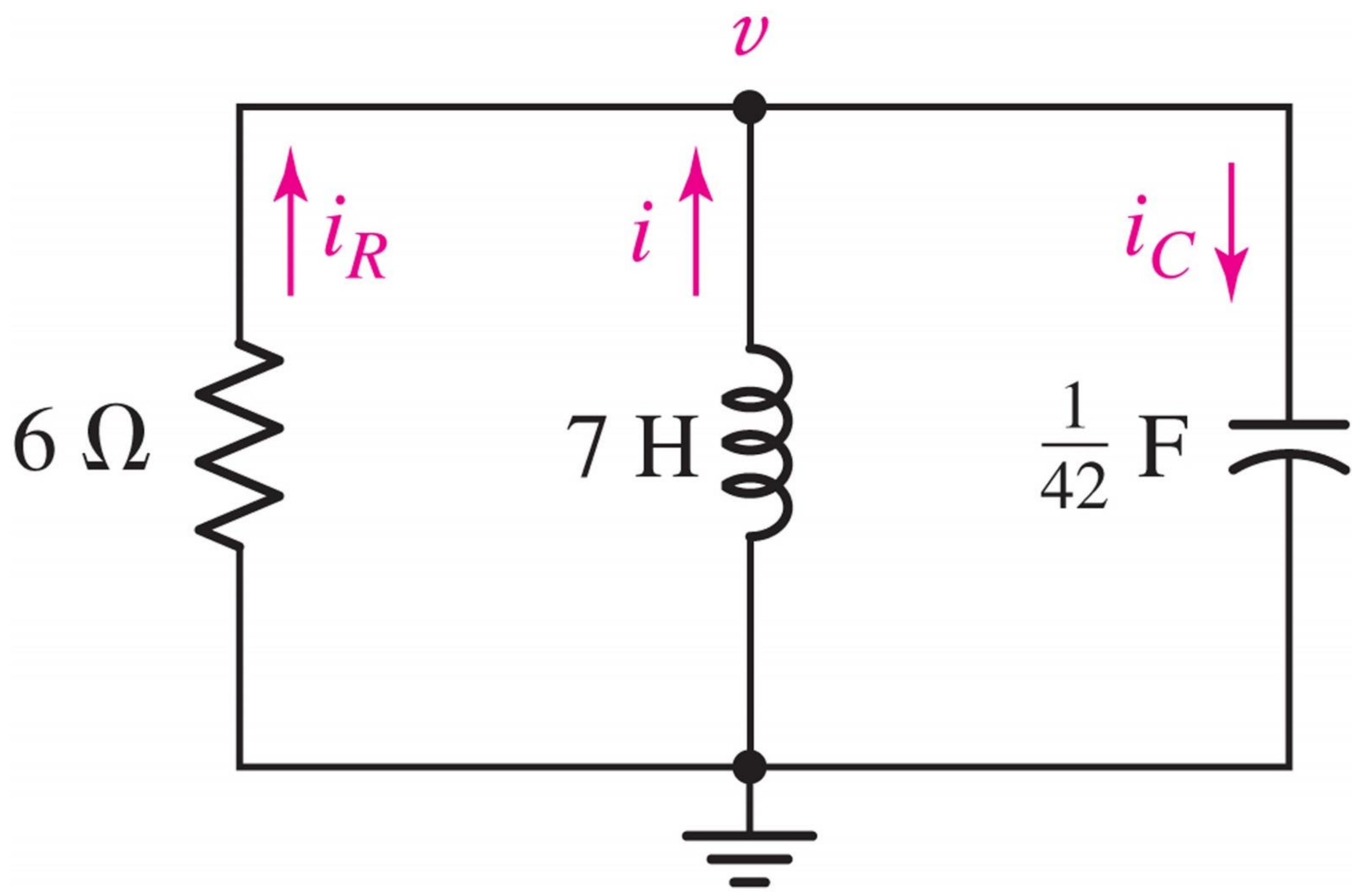
$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$
$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

The normal response form is as follows.

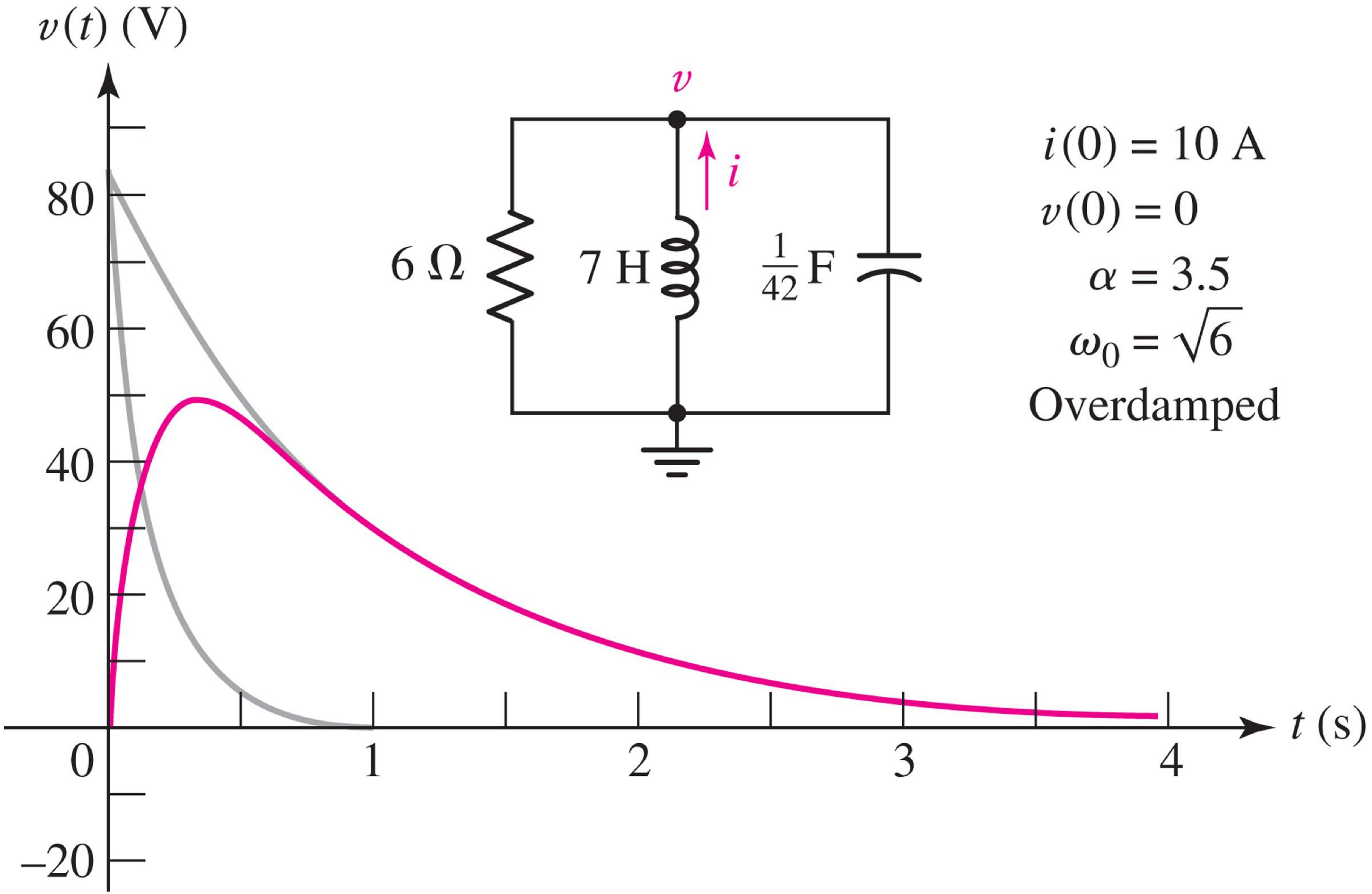
$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Overdamped Parallel RLC

Show that $v(t) = 84(e^{-t} - e^{-6t})$ *when* $i(0^+)=10$ A and $v(0^+)=0$ V.

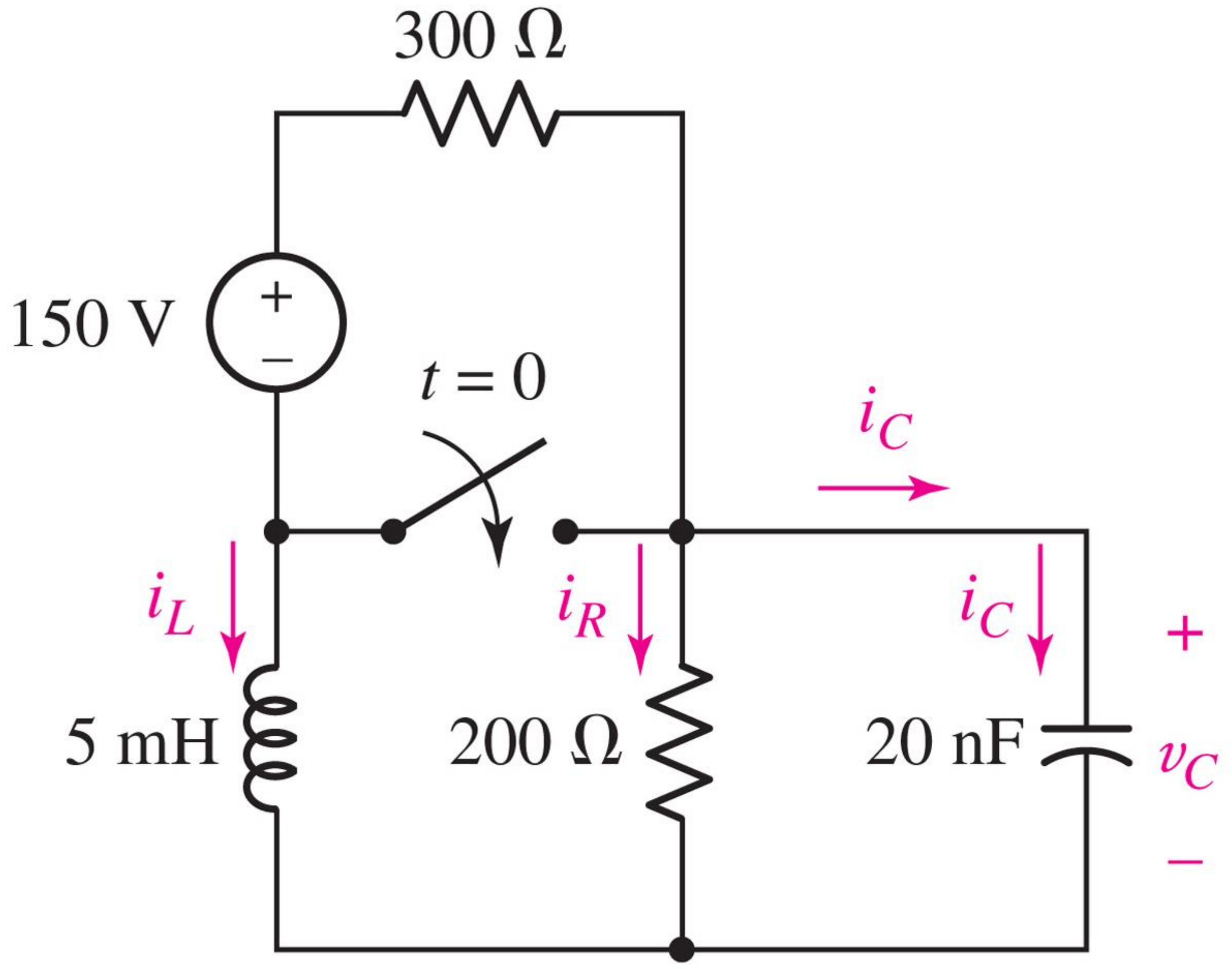


Graphing the Response



Example 2: overdamped RLC Circuit

Show that $v_C(t) = 80e^{-50{,}000t} - 20e^{-200{,}000t}$ V for t>0.



Critical Damping ($\alpha=\omega_0$)

A double real root:

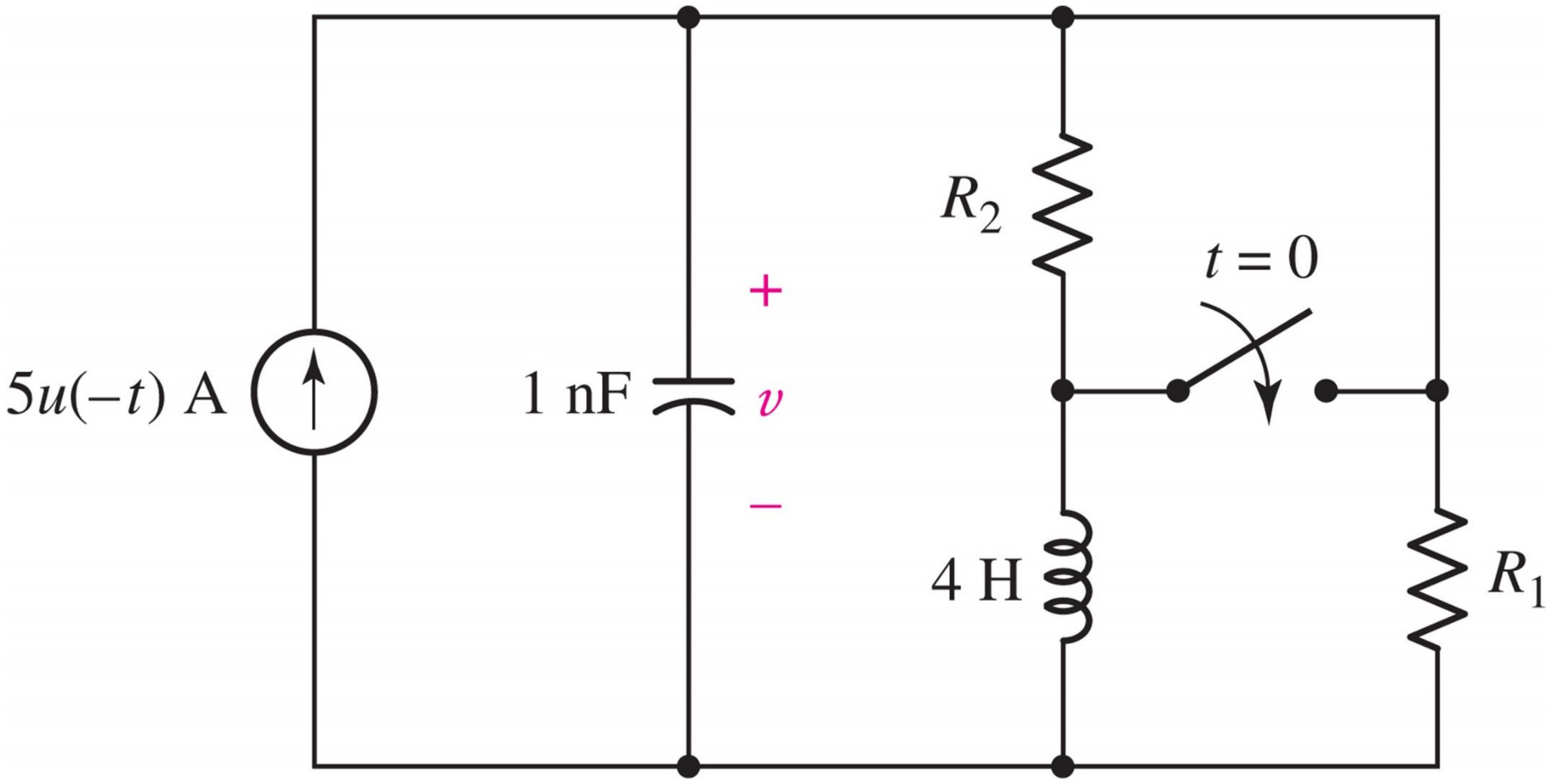
$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$
$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$s_1 = s_2 = -\alpha$$

The normal response form is as follows:

$$v(t) = e^{-\alpha t}\left(A_1 t + A_2\right)$$

Find R_1 such that the circuit is critically damped for $t>0$ and R_2 so that $v(0)=2\ V$.



Answer: $R_1 = 31.63\ k\Omega$, $R_2=0.4\Omega$

The Underdamped Response ($\alpha<\omega_0$)

If $\alpha<\omega_0$, define:

$$\omega_d = \sqrt{\omega_0^{\,2} - \alpha^2}$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

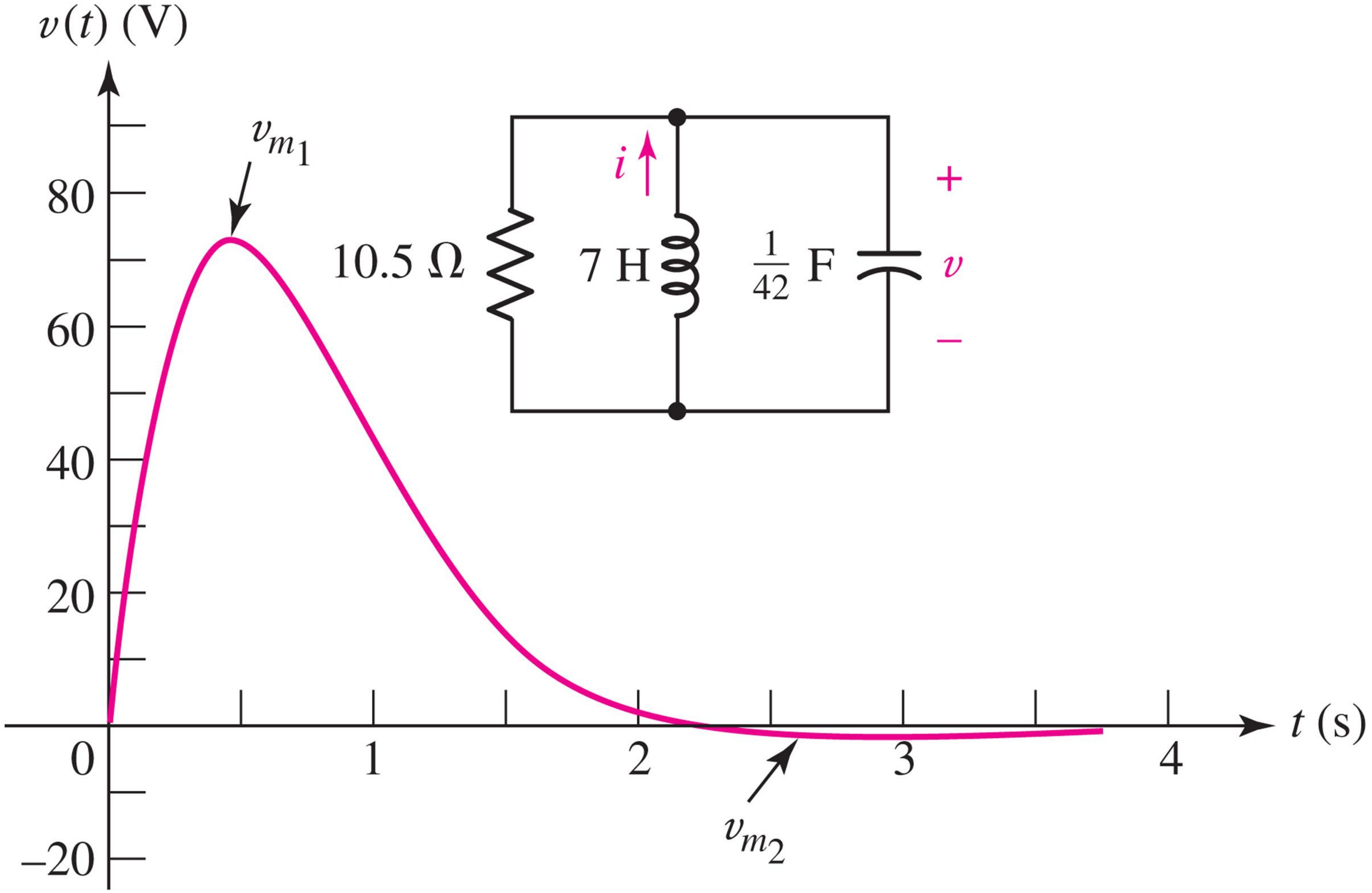
and the solution is

$$v(t) = e^{-\alpha t}\left(A_1 e^{j\omega_d t} + A_2 e^{-\omega_d t}\right)$$

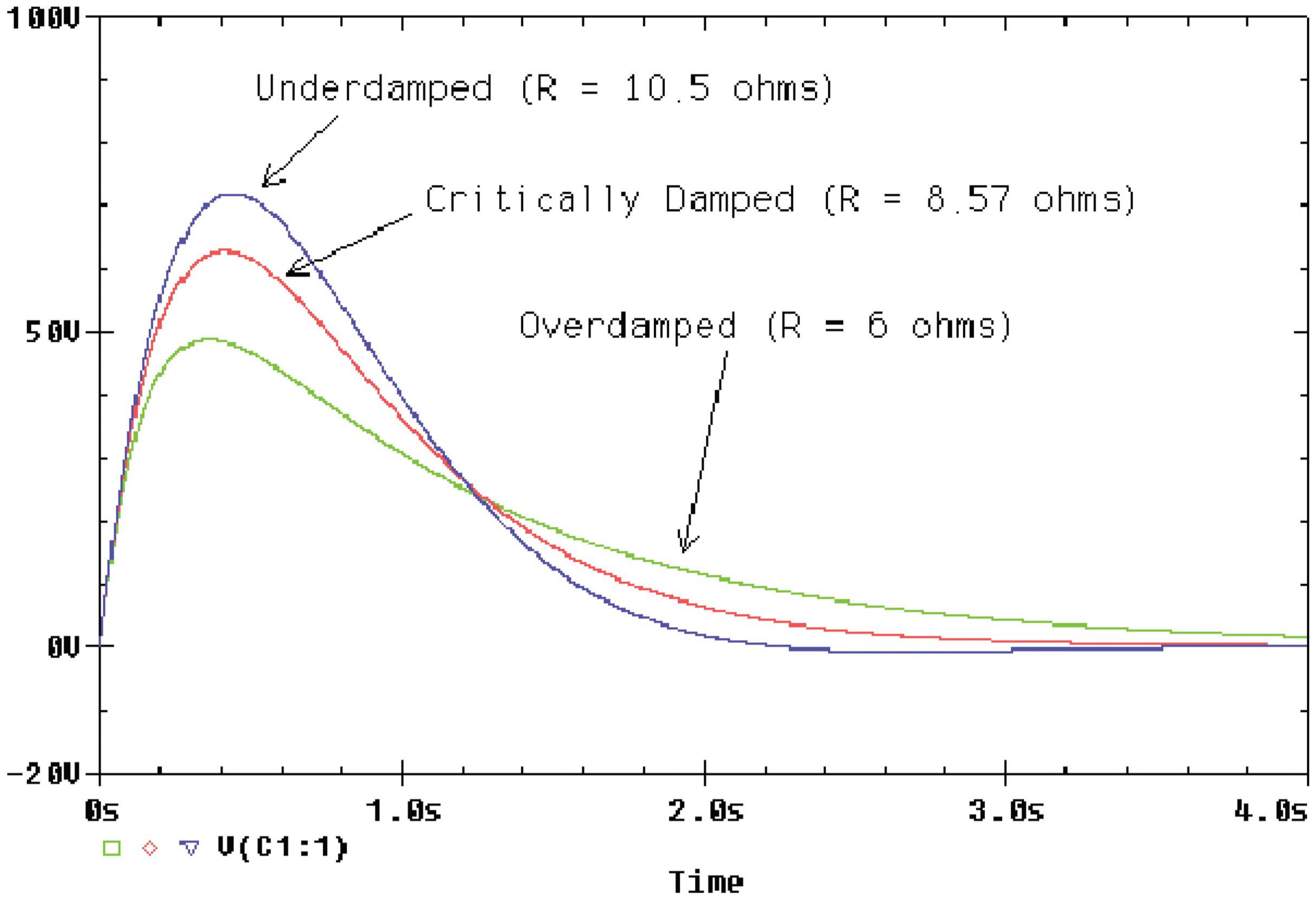
or equivalently

$$v(t) = e^{-\alpha t}\left(B_1 \cos(\omega_d t) + B_2 \sin(\omega_d t)\right)$$

Example: Underdamped Response



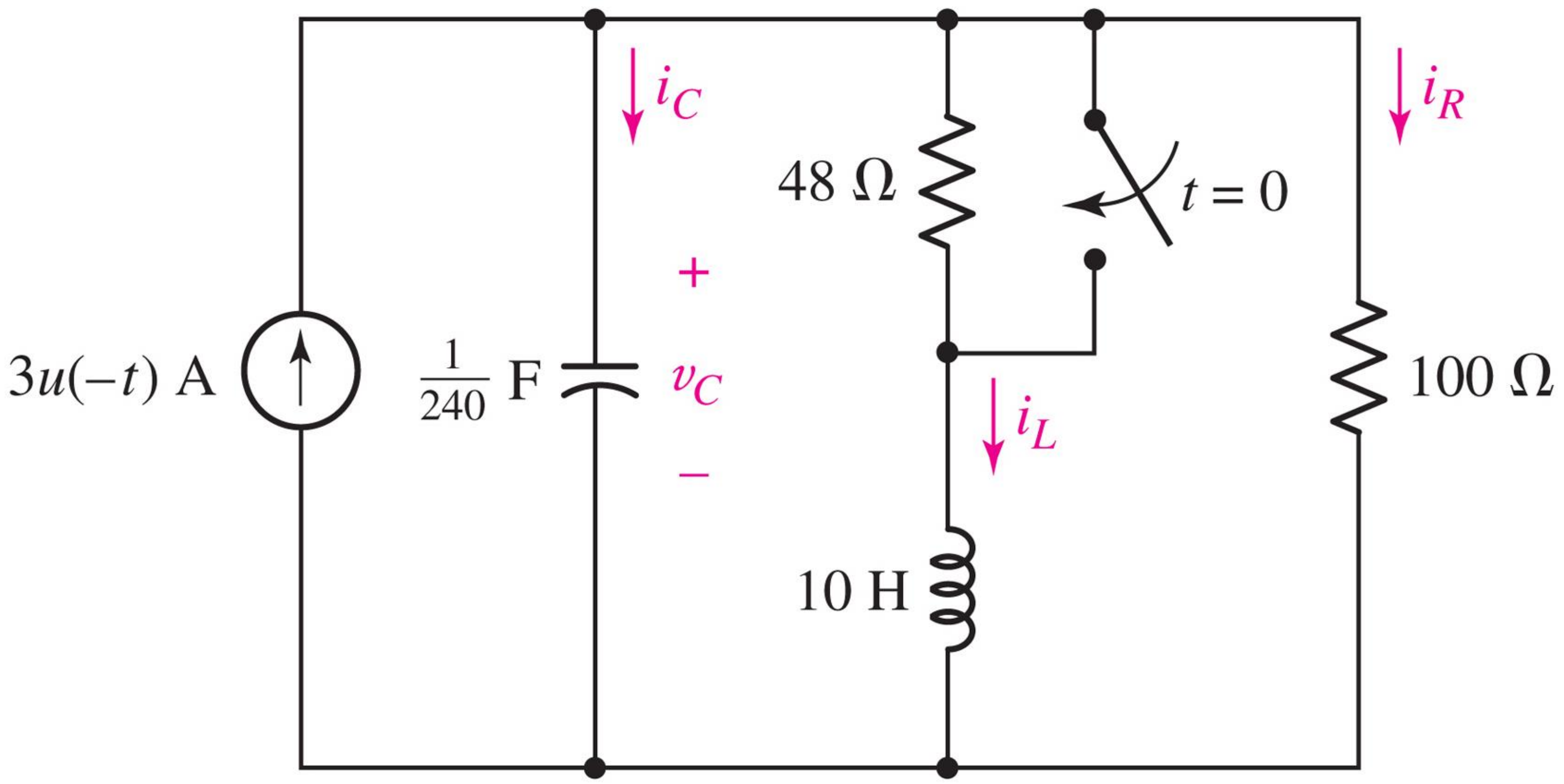
$$v(t) = 210\sqrt{2}e^{-2t}\sin\sqrt{2}t$$

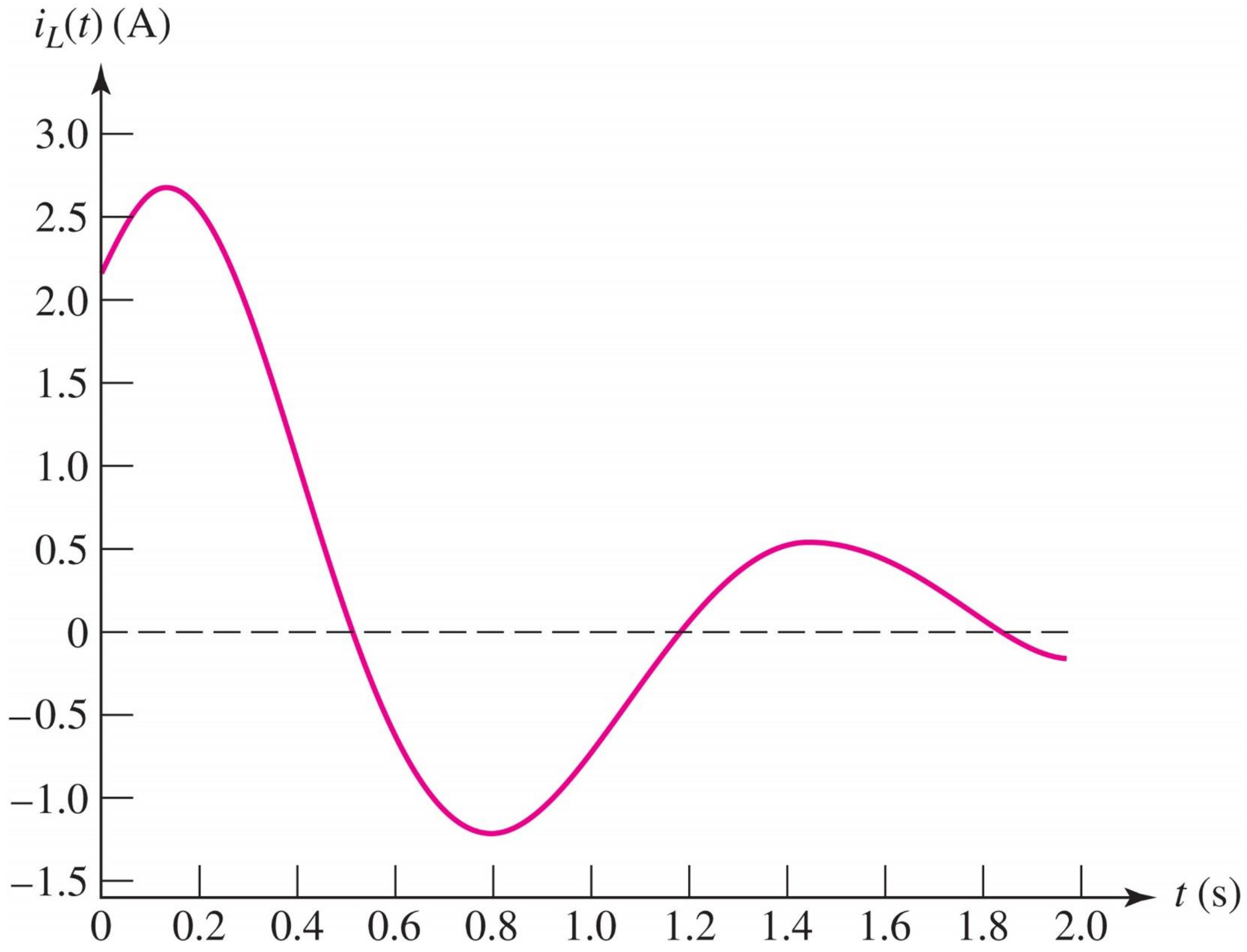


100V
50V
0V
-20V
Underdamped (R = 10.5 ohms)
Critically Damped (R = 8.57 ohms)
Overdamped (R = 6 ohms)
0s
1.0s
2.0s
3.0s
4.0s
V(C1:1)
Time

Show for *t>0*

$$i_L = e^{-1.2t}(2.027 \cos 4.75t + 2.561 \sin 4.75t)$$

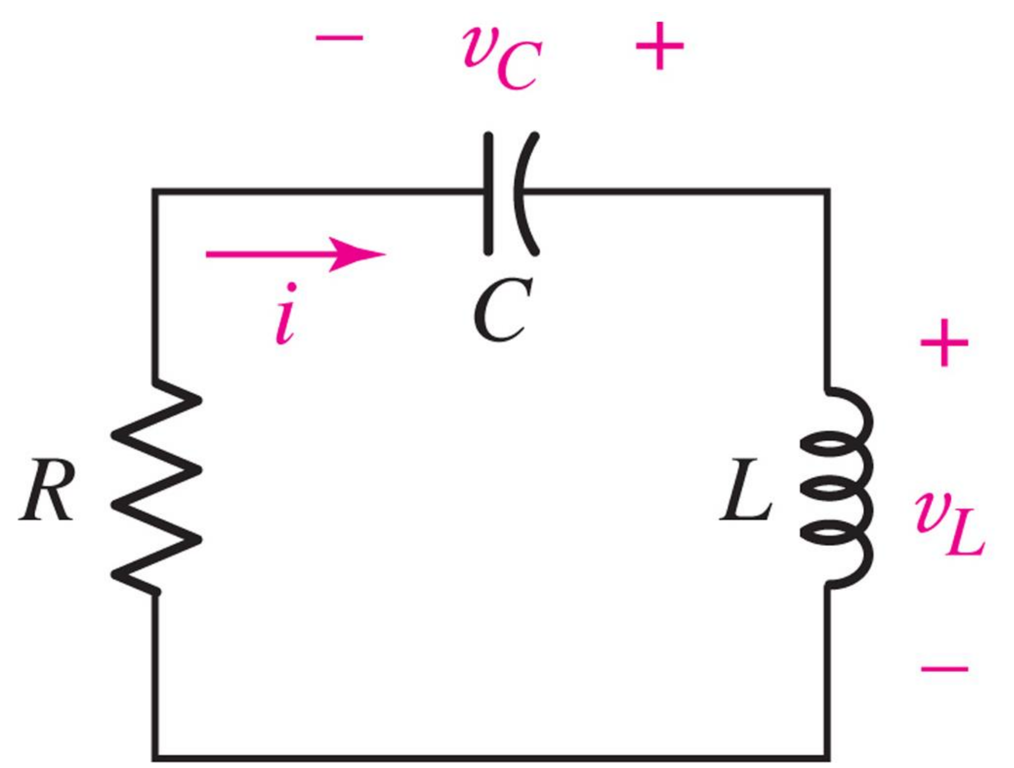




Source-Free Series RLC Circuit

For the series RLC circuit,

$$L\frac{d^2i}{dt^2}+R\frac{di}{dt}+\frac{1}{C}i=0$$



This circuit is the dual of the parallel RLC circuit.

Series RLC Differential Equation

The characteristic equation is:

$$Ls^2 + Rs + \frac{1}{C} = 0$$

Where:

$$s_1, s_2 = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

Series RLC Circuit Solution

Define $\omega_0 = 1/\sqrt{LC}$ and $\alpha = \dfrac{R}{2L}$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

Then if

$\alpha > \omega_0$ (overdamped):

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

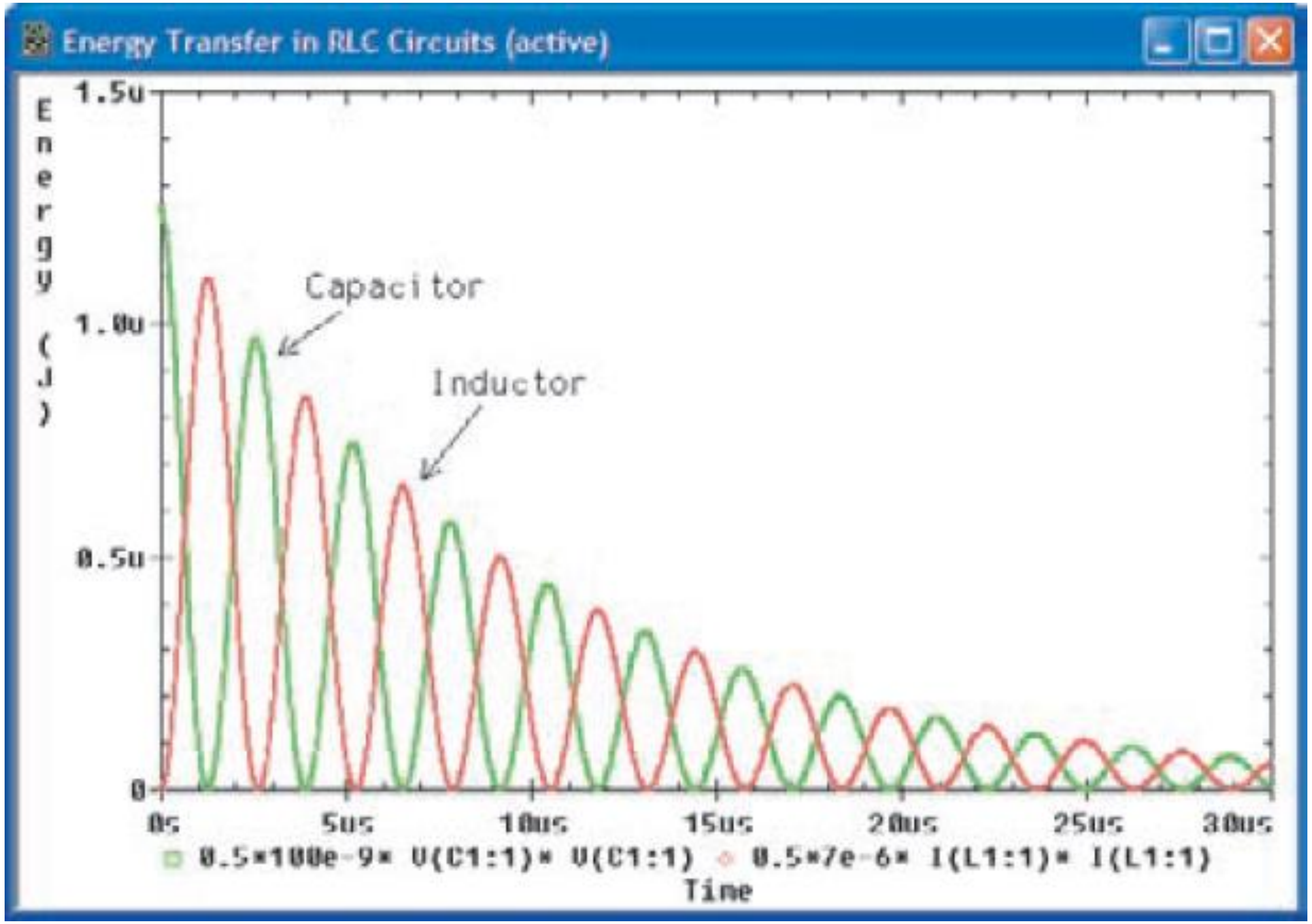
$\alpha = \omega_0$ (critically damped):

$$v(t) = e^{-\alpha t}\left(A_1 t + A_2\right)$$

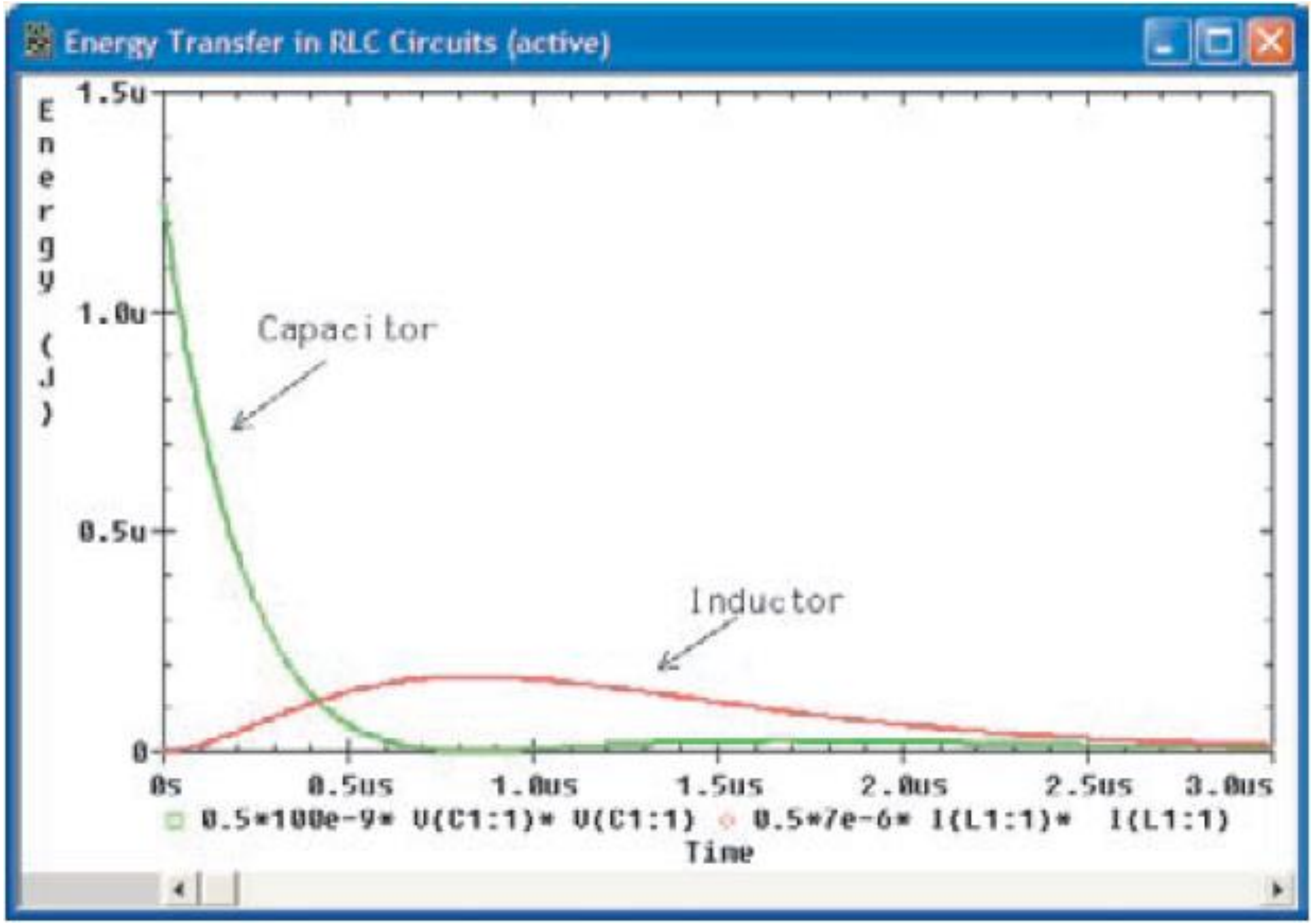
$\alpha < \omega_0$ (underdamped):

$$v(t) = e^{-\alpha t}\left(B_1 \cos(\omega_d t) + B_2 \sin(\omega_d t)\right)$$

Underdamped (R=100 Ω) $C = 100nF, L = 7\mu H$

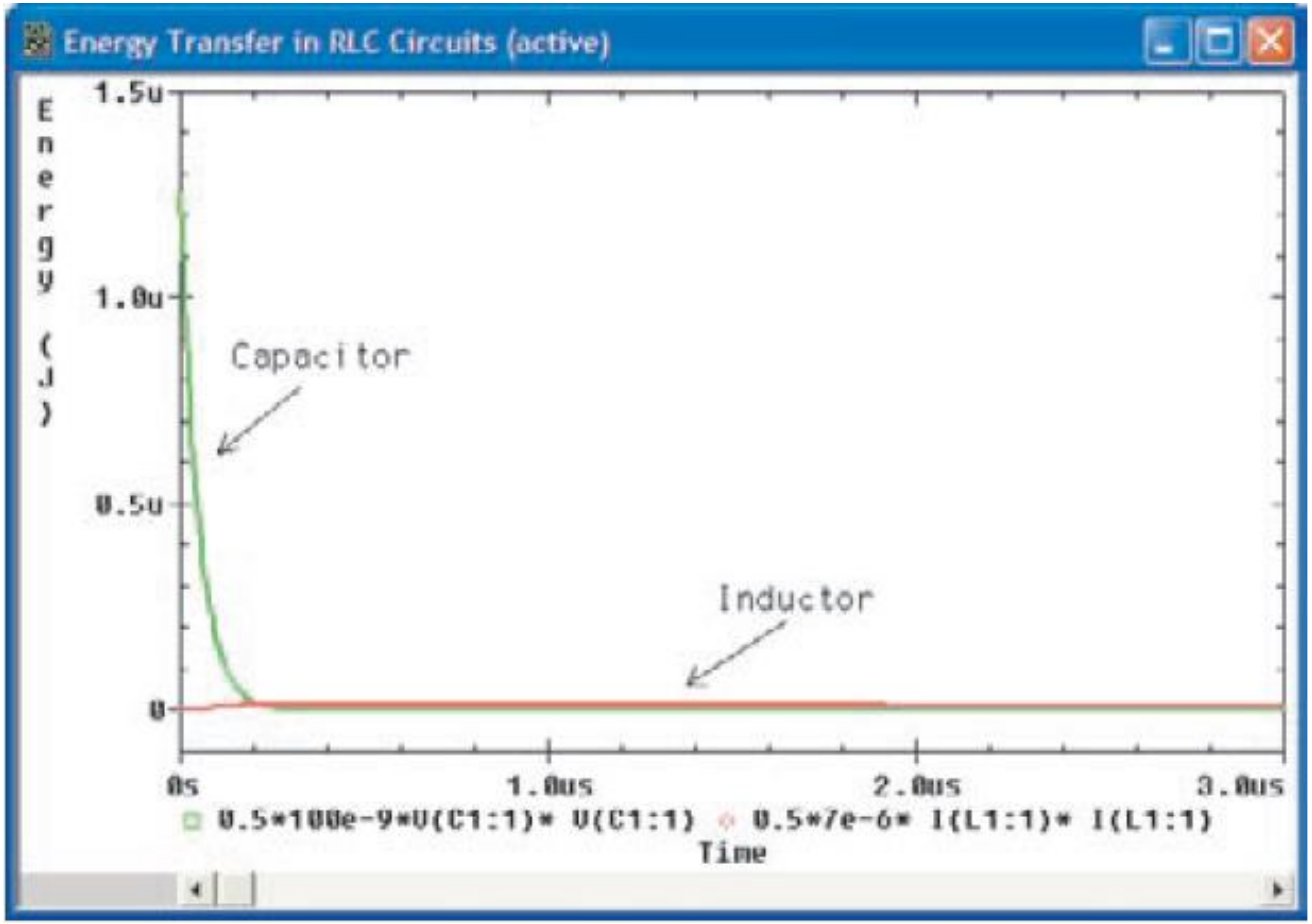


critically damped (R=100 Ω) $C = 100nF, L = 7\mu H$



Overdamped (R=100 Ω)

$$C = 100nF, L = 7\mu H$$



Summary of Source free RLC Circuits

نوع	وضعیت	شرط	α	ω_0	فرم پاسخ طبیعی
موازی	Overdamped	$\alpha > \omega_0$	$\frac{1}{2RC}$	$\frac{1}{\sqrt{LC}}$	$A_1e^{s_1t} + A_2e^{s_2t}$ $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$
سری			$\frac{R}{2L}$		
موازی	Critically damped	$\alpha = \omega_0$	$\frac{1}{2RC}$	$\frac{1}{\sqrt{LC}}$	$e^{-\alpha t}(A_1t + A_2)$
سری			$\frac{R}{2L}$		
موازی	Underdamped	$\alpha < \omega_0$	$\frac{1}{2RC}$	$\frac{1}{\sqrt{LC}}$	$e^{-\alpha t}(B_1cos\omega_dt + B_2sin\omega_dt)$ $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$
سری			$\frac{R}{2L}$		

The Complete response

The response of RLC circuits with DC sources and switches will consist of the natural response and the forced response:

$$v(t) = v_f(t)+v_n(t)$$

The complete response must satisfy both the initial conditions $v(0^+)$ and $\frac{dv}{dt}(0^+)$ or the forced response.

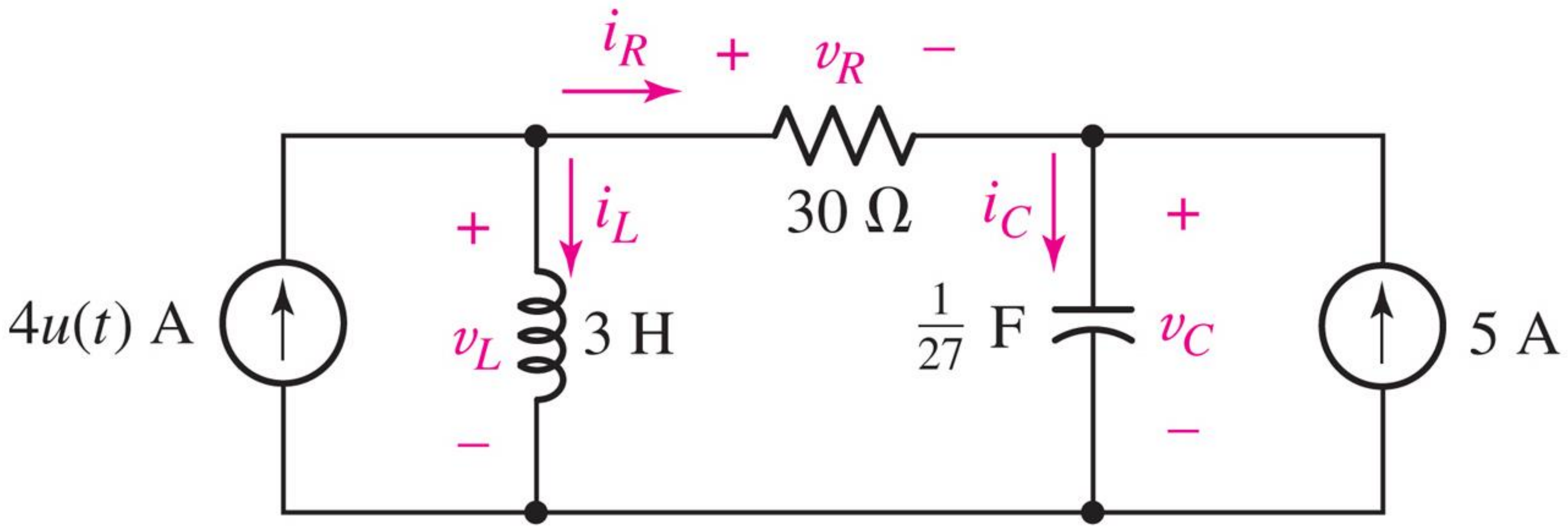
For example in the presence of DC sources:

A natural response : $v_n(t) = Ae^{s_1 t} + Be^{s_2 t}$

Full answer: $v(t) = K + Ae^{s_1 t} + Be^{s_2 t}$

Applying the basic conditions: $v(0^+) = V_f + A + B,\ \frac{dv}{dt}(0^+) = As_1 + Bs_2$

Find the labeled voltages and currents at $t=0^-$ and $t=0^+$.

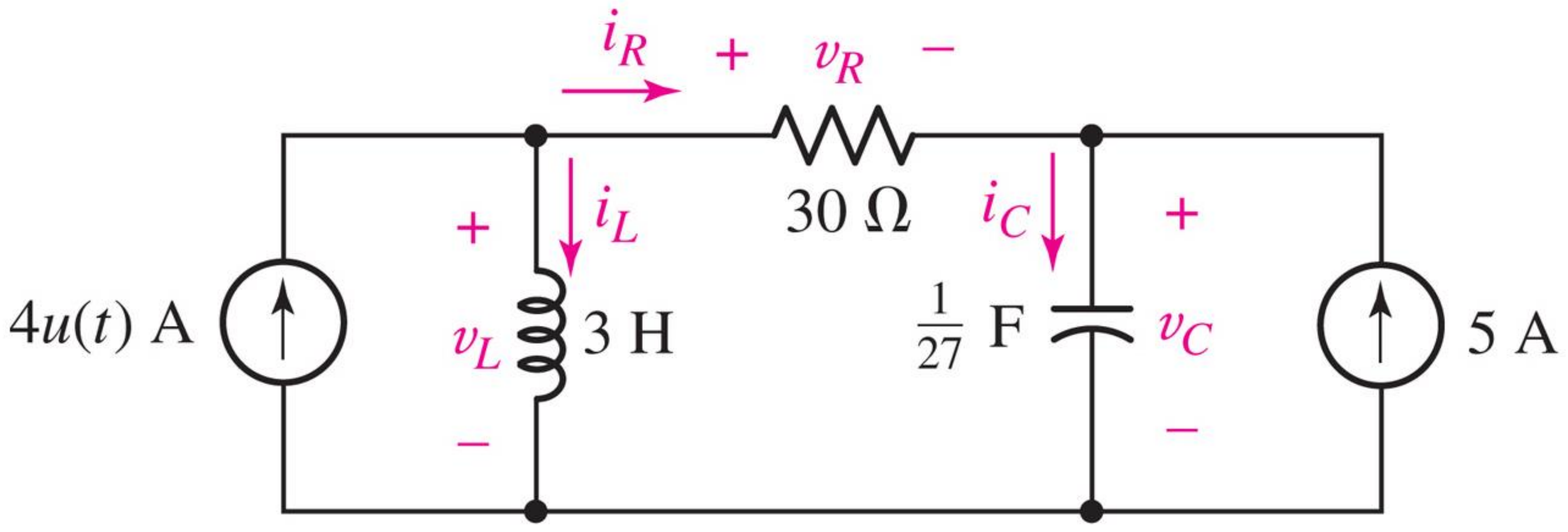


Answer:

$i_R(0^-) = -5$ A	$v_R(0^-) = -150$ V	$i_R(0^+) = -1$ A	$v_R(0^+) = -30$ V
$i_L(0^-) = 5$ A	$v_L(0^-) = 0$ V	$i_L(0^+) = 5$ A	$v_L(0^+) = 120$ V
$i_C(0^-) = 0$ A	$v_C(0^-) = 150$ V	$i_C(0^+) = 4$ A	$v_C(0^+) = 150$ V

Example: Initial Slopes

Find the first derivatives of the labeled voltages and currents at $t=0^+$.



Answer:

$di_R/dt(0^+) = -40$ A/s $\qquad dv_R/dt(0^+) = -1200$ V/s

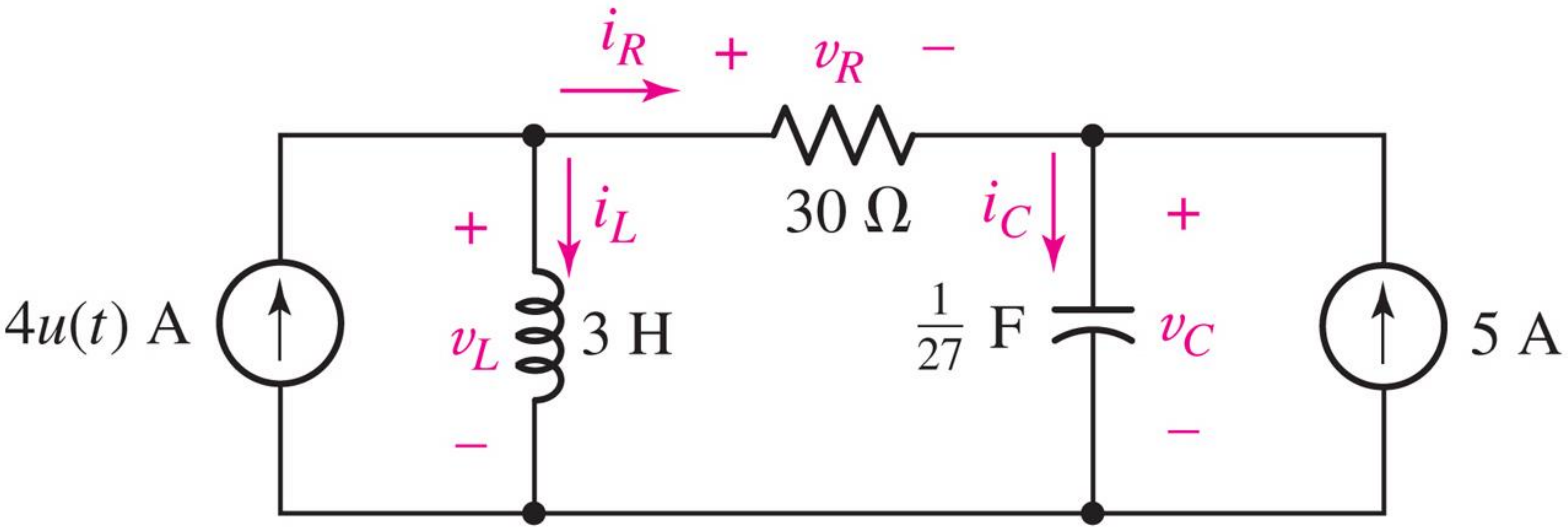
$di_L/dt(0^+) = 40$ A/s $\qquad dv_L/dt(0^+) = -1092$ V/s

$di_C/dt(0^+) = -40$ A/s $\qquad dv_C/dt(0^+) = 108$ V/s

Example: Complete Response

Show that for t>0

$$v_C(t) = 150 + 13.5(e^{-t} - e^{-9t}) \text{ volts}$$

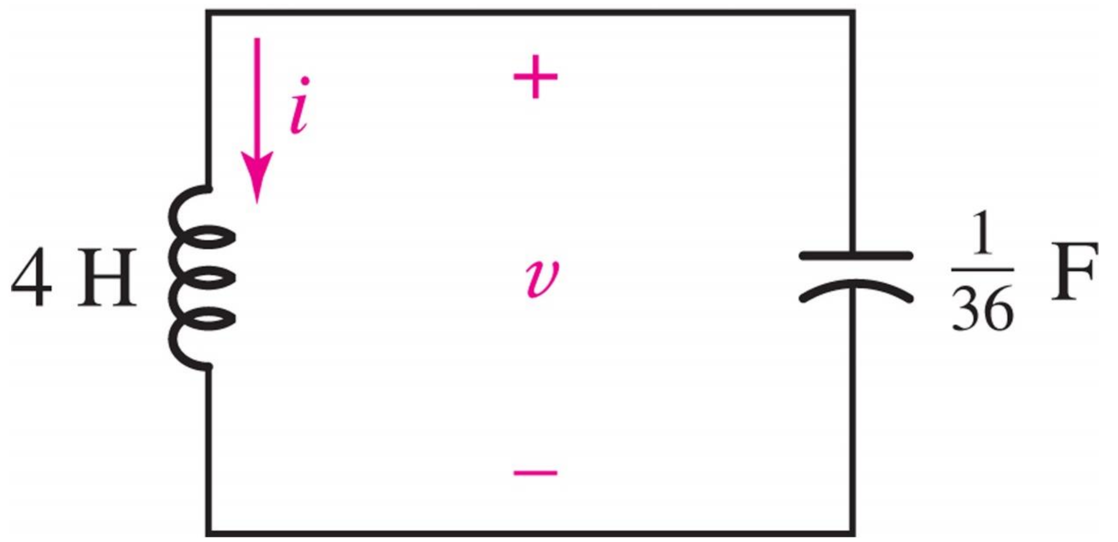


The Lossless LC Circuit

- The resistor in the RLC circuit serves to dissipate initial stored energy.
- When this resistor becomes 0 in the series RLC or infinite in the parallel RLC, the circuit will oscillate.

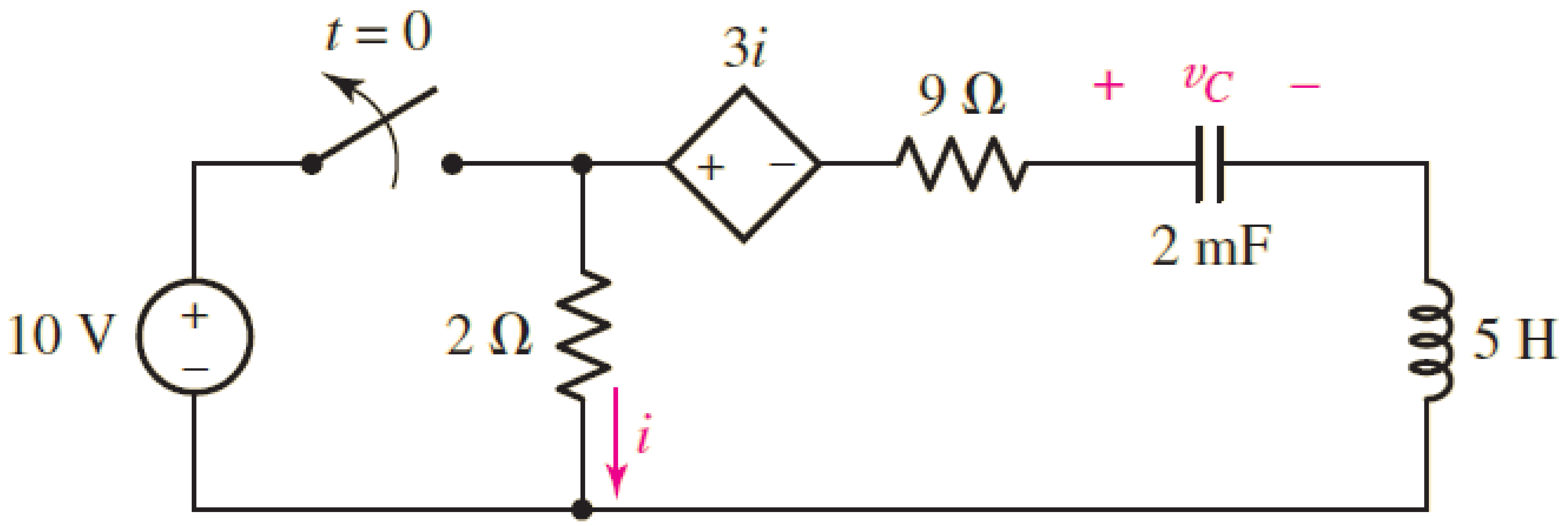
Example: for t>0, if i(0)=-1/6 A and v(0)=0 V

$$v(t) = 2 \sin 3t$$



Show that:

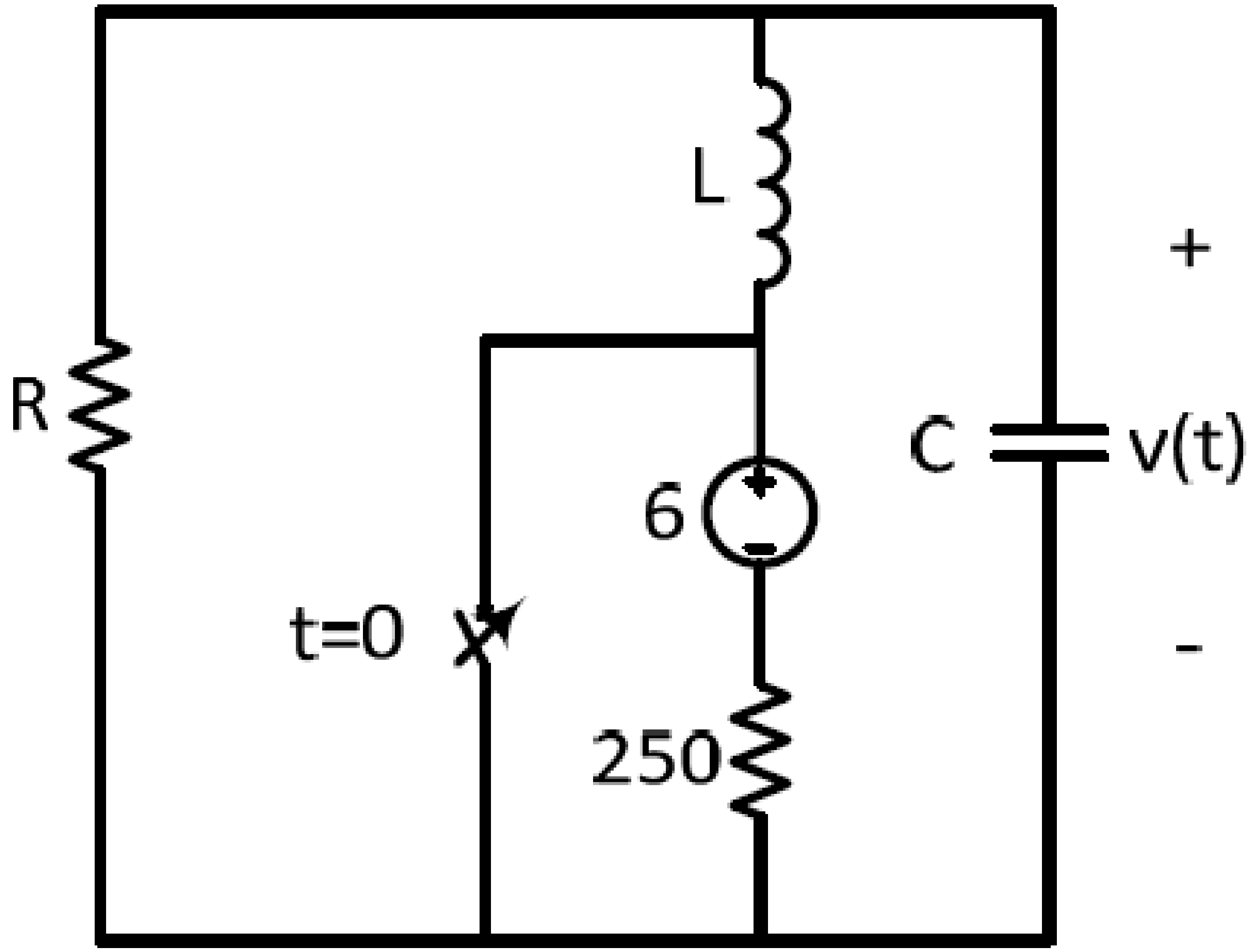
$$v_C(t) = -e^{-0.8t}(5\cos 9.97t + 0.4\sin 9.97t)V$$



Practice 2

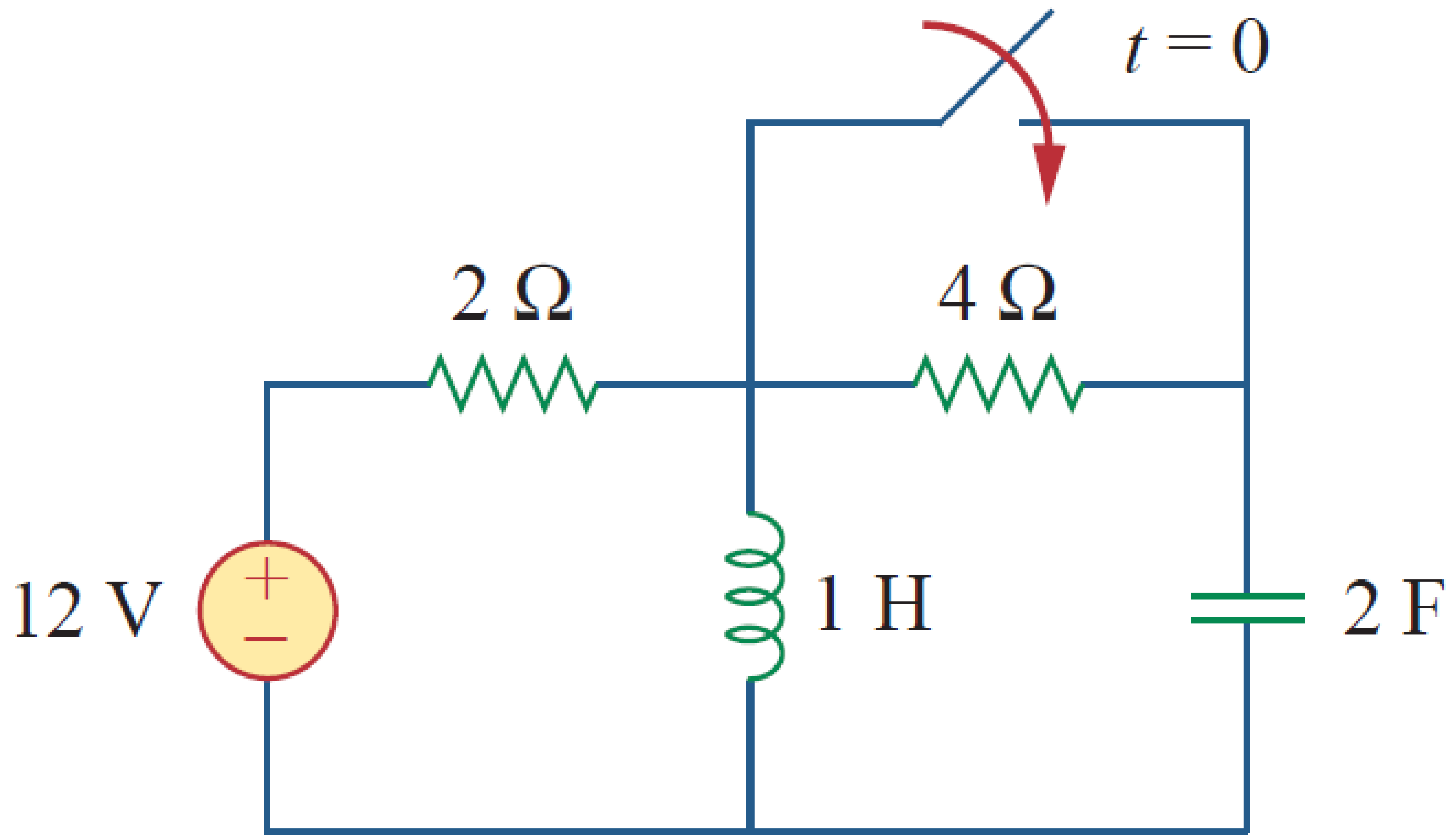
The key is closed at time t=0. Find R, L and C such that:

$$v(t) = 5e^{-400t}\cos(300t)$$



Practice 3

Find $vc(t)$.



Thanks