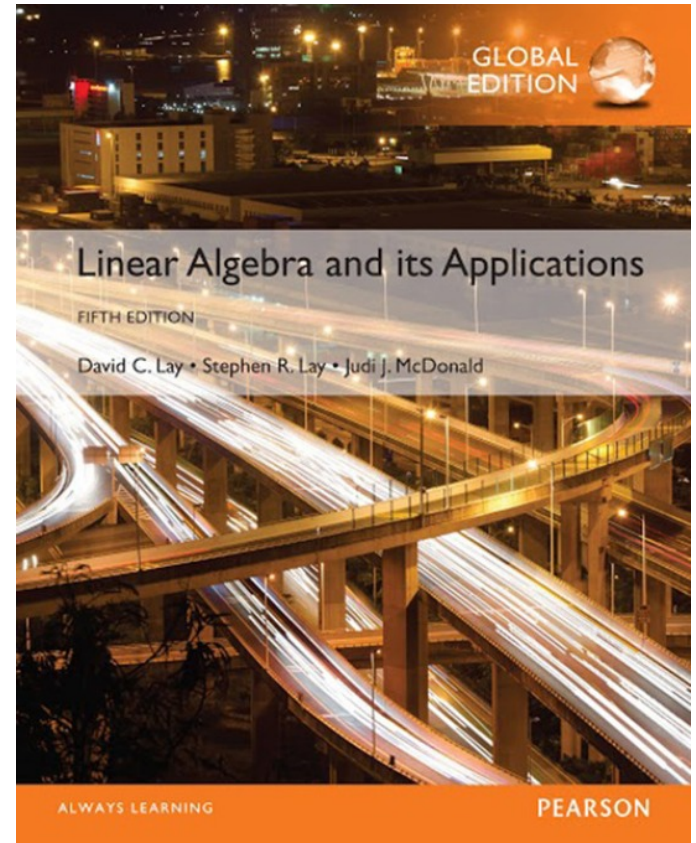


1

Linear Equations in Linear Algebra

1.9

THE MATRIX OF A LINEAR TRANSFORMATION



THE MATRIX OF A LINEAR TRANSFORMATION

- **Theorem 10:** Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(x) = Ax \text{ for all } x \text{ in } \mathbb{R}^n$$

- In fact, A is the $m \times n$ matrix whose j^{th} column is the vector $T(e_j)$, where e_j is the j^{th} column of the identity matrix in \mathbb{R}^n

$$A = [T(e_1) \dots T(e_n)] \quad (3)$$

THE MATRIX OF A LINEAR TRANSFORMATION

- **Proof:** Write $x = I_n x = [e_1 \dots e_n]x = x_1 e_1 + \dots + x_n e_n$, and use the linearity of T to compute

$$T(x) = T(x_1 e_1 + \dots + x_n e_n) = x_1 T(e_1) + \dots + x_n T(e_n)$$

$$= [T(e_1) \quad \dots \quad T(e_n)] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = Ax$$

THE MATRIX OF A LINEAR TRANSFORMATION

- The matrix A in (3) is called the **standard matrix for the linear transformation T** .
- We know now that every linear transformation from \mathbb{R}^n to \mathbb{R}^m can be viewed as a matrix transformation, and vice versa. The term *linear transformation* focuses on a property of a mapping, while *matrix transformation* describes how such a mapping is implemented, as the example on the next slide illustrates.

THE MATRIX OF A LINEAR TRANSFORMATION

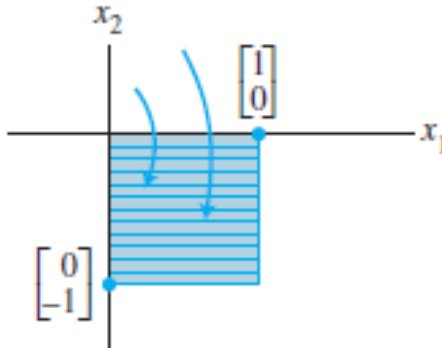
- **Example 2:** Find the standard matrix A for the dilation transformation $T(x) = 3x$, for x in \mathbb{R}^2 .
- **Solution:** Write

$$T(e_1) = e_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \text{ and } T(e_2) = e_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$
$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

GEOMETRIC LINEAR TRANSFORMATIONS OF \mathbb{R}^2

- Tables 1-4 illustrate other common geometric linear transformations of the plane.

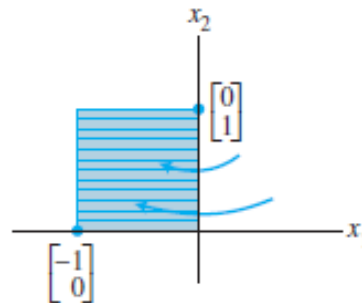
TABLE 1 Reflections

| Transformation | Image of the Unit Square | Standard Matrix |
|------------------------------------|---|---|
| Reflection through the x_1 -axis |  | $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ |

EXISTENCE AND UNIQUENESS QUESTIONS

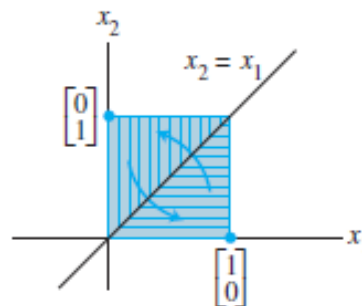
- Table 1 continued:

Reflection through
the x_2 -axis



$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Reflection through
the line $x_2 = x_1$

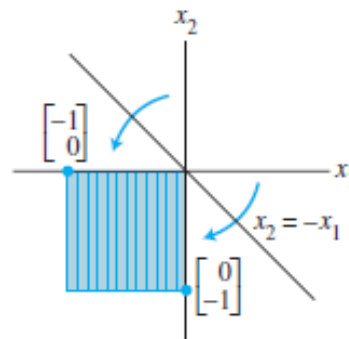


$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

EXISTENCE AND UNIQUENESS QUESTIONS

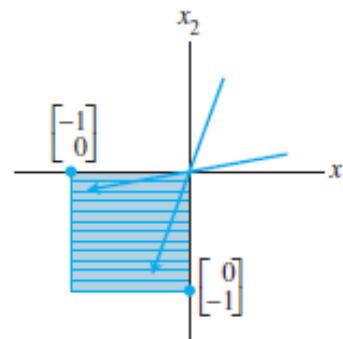
- Table 1 continued:

Reflection through
the line $x_2 = -x_1$



$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

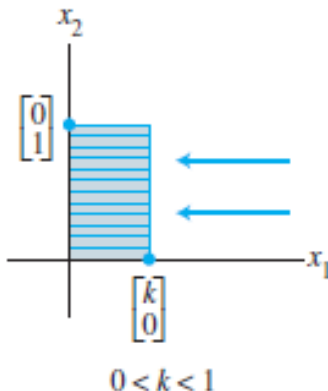
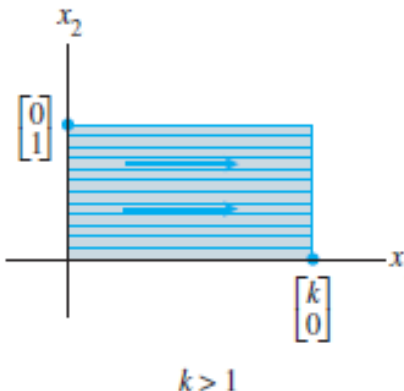
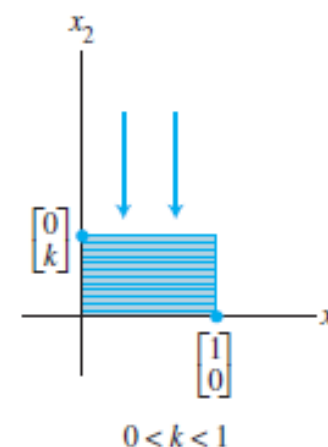
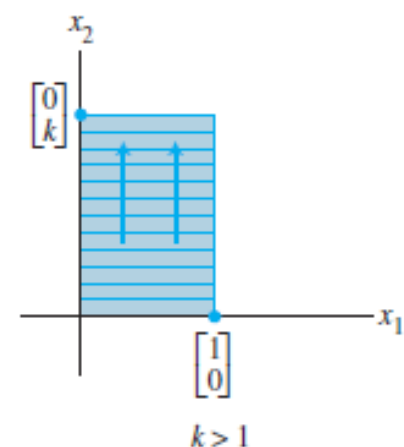
Reflection through
the origin



$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

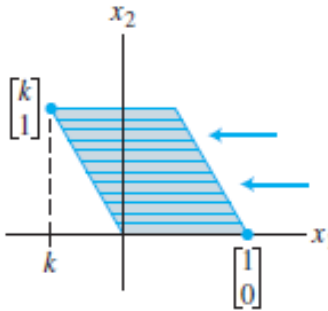
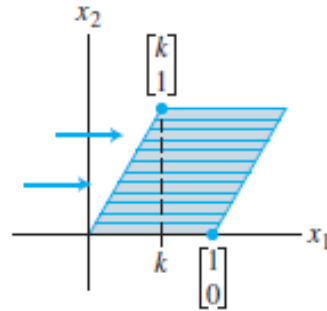
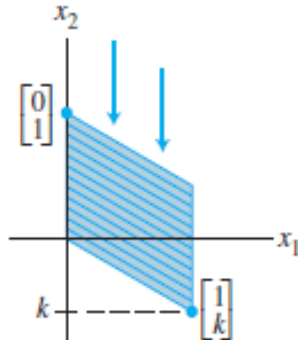
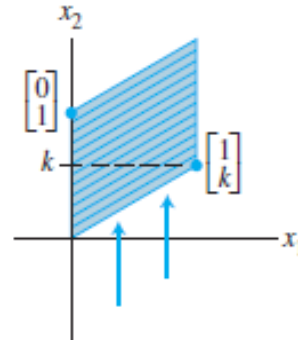
EXISTENCE AND UNIQUENESS QUESTIONS

TABLE 2 Contractions and Expansions

| Transformation | Image of the Unit Square | | Standard Matrix |
|--------------------------------------|--|--|--|
| Horizontal contraction and expansion |  <p>$0 < k < 1$</p> |  <p>$k > 1$</p> | $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ |
| | | | |
| Vertical contraction and expansion |  <p>$0 < k < 1$</p> |  <p>$k > 1$</p> | $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ |
| | | | |

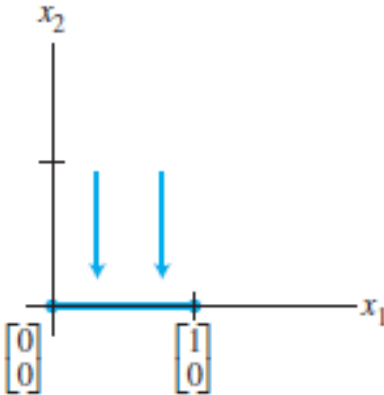
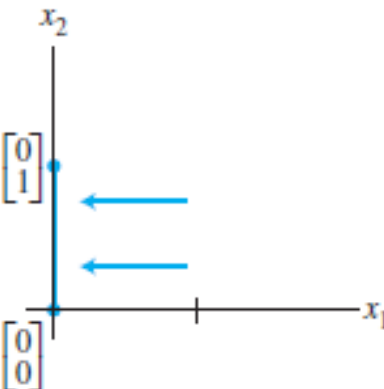
EXISTENCE AND UNIQUENESS QUESTIONS

TABLE 3 Shears

| Transformation | Image of the Unit Square | Standard Matrix |
|------------------|--|--|
| Horizontal shear |  <p style="text-align: center;">$k < 0$</p> | $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ |
| |  <p style="text-align: center;">$k > 0$</p> | |
| Vertical shear |  <p style="text-align: center;">$k < 0$</p> | $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ |
| |  <p style="text-align: center;">$k > 0$</p> | |

EXISTENCE AND UNIQUENESS QUESTIONS

TABLE 4 Projections

| Transformation | Image of the Unit Square | Standard Matrix |
|---------------------------------|---|--|
| Projection onto the x_1 -axis |  | $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ |
| Projection onto the x_2 -axis |  | $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ |

EXISTENCE AND UNIQUENESS QUESTIONS

- **Definition:** A mapping $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **onto** \mathbb{R}^m if each \mathbf{b} in \mathbb{R}^m is the image of *at least one* \mathbf{x} in \mathbb{R}^n .
- Equivalently, T is onto \mathbb{R}^m when the range of T is all of the codomain \mathbb{R}^m . That is, T maps \mathbb{R}^n onto \mathbb{R}^m if, for each \mathbf{b} in the codomain \mathbb{R}^m , there exists at least one solution of $T(\mathbf{x}) = \mathbf{b}$. “Does T map \mathbb{R}^n onto \mathbb{R}^m ?” is an existence question. The mapping T is not onto when there is some \mathbf{b} in \mathbb{R}^m for which the equation $T(\mathbf{x}) = \mathbf{b}$ has no solution. See the figure on the next slide.

EXISTENCE AND UNIQUENESS QUESTIONS

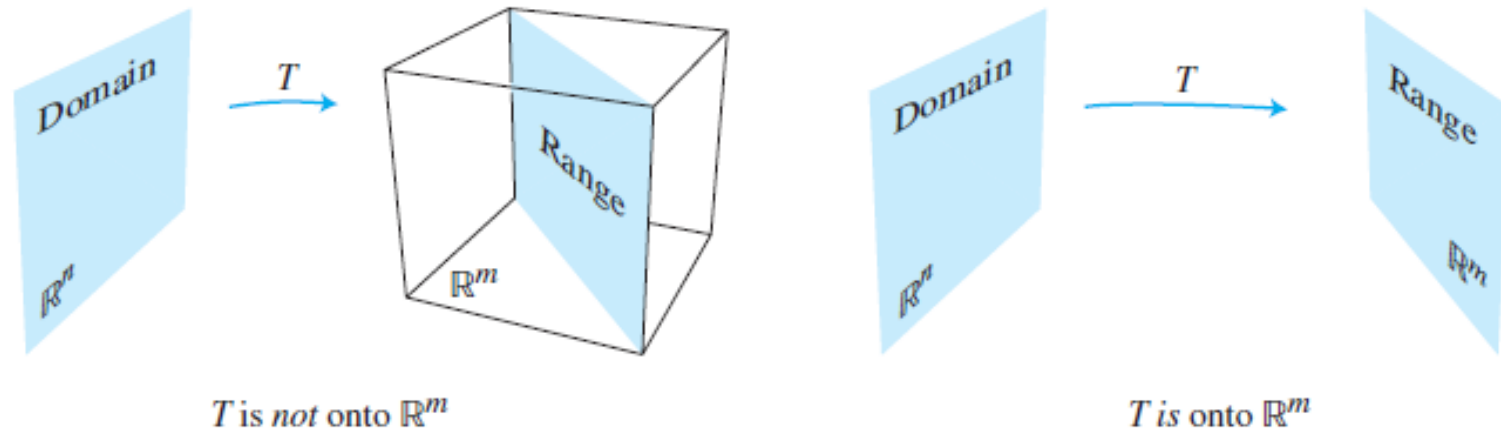


FIGURE 3 Is the range of T all of \mathbb{R}^m ?

- **Definition:** A mapping $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **one-to-one** if each \mathbf{b} in \mathbb{R}^m is the image of *at most one* \mathbf{x} in \mathbb{R}^n .

EXISTENCE AND UNIQUENESS QUESTIONS

- **Example 4:** Let T be the linear transformation whose standard matrix is

$$A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

- Does T map \mathbb{R}^4 onto \mathbb{R}^3 ? Is T a one-to-one mapping?

EXISTENCE AND UNIQUENESS QUESTIONS

- **Solution:** Since A happens to be in echelon form, we can see at once that A has a pivot position in each row. By Theorem 4 in Section 1.4, for each \mathbf{b} in \mathbb{R}^3 , the equation $A\mathbf{x}=\mathbf{b}$ is consistent. In other words, the linear transformation T maps \mathbb{R}^4 (its domain) onto \mathbb{R}^3 .
- However, since the equation $A\mathbf{x}=\mathbf{b}$ has a free variable (because there are four variables and only three basic variables), each \mathbf{b} is the image of more than one \mathbf{x} . This is, T is *not* one-to-one.

EXISTENCE AND UNIQUENESS QUESTIONS

- **Theorem 11:** Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(x)=0$ has only the trivial solution.
- **Proof:** Since T is linear, $T(0) = 0$. If T is one-to-one, then the equation $T(x)=0$ has at most one solution and hence only the trivial solution.
- If T is not one-to-one, then there is a b that is the image of at least two different vectors in \mathbb{R}^n --say, u and v . That is $T(u)=b$ and $T(v)=b$. But then, since T is linear,

$$T(u - v) = T(u) - T(v) = b - b = 0$$

EXISTENCE AND UNIQUENESS QUESTIONS

- The vector $\mathbf{u} - \mathbf{v}$ is not zero, since $u \neq v$. Hence the equation $T(\mathbf{x})=0$ has more than one solution. So, either the two conditions in the theorem are both true or they are both false.
- **Theorem 12:** Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T . Then:
 - a) T maps \mathbb{R}^n onto \mathbb{R}^m if and only if the columns of A span \mathbb{R}^m ;
 - b) T is one-to-one if and only if the columns of A are linearly independent.

EXISTENCE AND UNIQUENESS QUESTIONS

■ **Proof:**

- a) By Theorem 4 in Section 1.4, the columns of A span \mathbb{R}^m if and only if for each b in \mathbb{R}^m the equation $Ax=b$ is consistent—in other words, if and only if for every b , the equation $T(x)=b$ has at least one solution. This is true if and only if T maps \mathbb{R}^n onto \mathbb{R}^m .
- b) The equations $T(x)=0$ and $Ax=0$ are the same except for notation. So, by Theorem 11, T is one-to-one if and only if $Ax=0$ has only the trivial solution. This happens if and only if the columns of A are linearly independent.