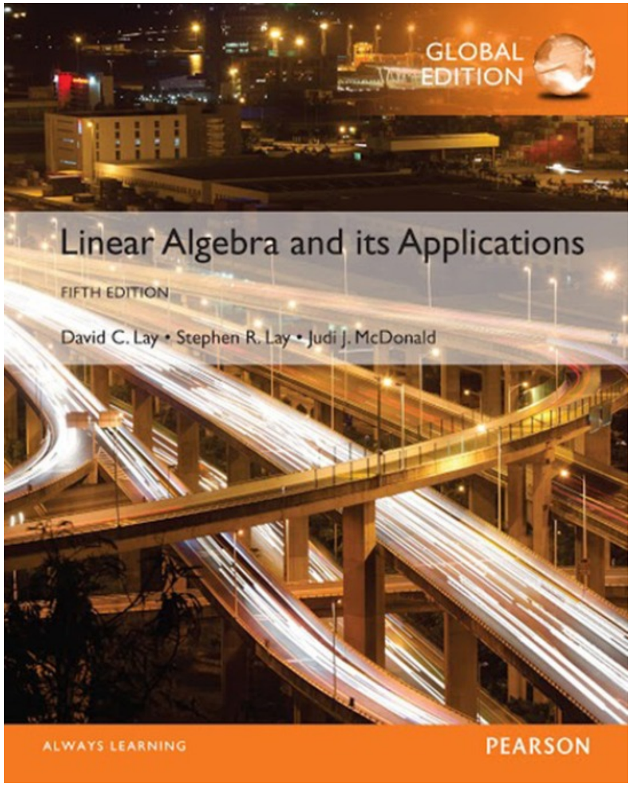
5 Eigenvalues and Eigenvectors

5.6

DISCRETE DYNAMICAL SYSTEMS



GRAPHICAL DESCRIPTION OF SOLUTIONS

- When A is 2×2, algebraic calculations can be supplemented by a geometric description of a system's evolution.
- We can view the equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$ as a description of what happens to an initial point $\mathbf{x}_0$ in $\mathbb{R}^2$ as it is transformed repeatedly by the mapping $\mathbf{x} \mapsto A\mathbf{x}$
- The graph of $\mathbf{x}_0, \mathbf{x}_1, \ldots$ is called a **trajectory** of the dynamical system.

GRAPHICAL DESCRIPTION OF SOLUTIONS

- **Example 2** Plot several trajectories of the dynamical system $x_{k+1} = Ax_k$, when

$$A = \begin{bmatrix} .80 & 0 \\ 0 & .64 \end{bmatrix}$$

- **Solution** The eigenvalues of A are .8 and .64, with eigenvectors $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. If $x_0 = c_1v_1 + c_2v_2$, then

$$x_k = c_1(.8)^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2(.64)^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Of course, x_k tends to 0 because $(.8)^k$ and $(.64)^k$ both approach 0 as $k \to \infty$. But the way x_k goes toward 0 is interesting. See Fig. 1 on the next slide.

GRAPHICAL DESCRIPTION OF SOLUTIONS

Figure 1 shows the first few terms of several trajectories that begin at points on the boundary of the box with corners at $(\pm 3, \pm 3)$. The points on each trajectory are connected by a thin curve, to make the trajectory easier to see.

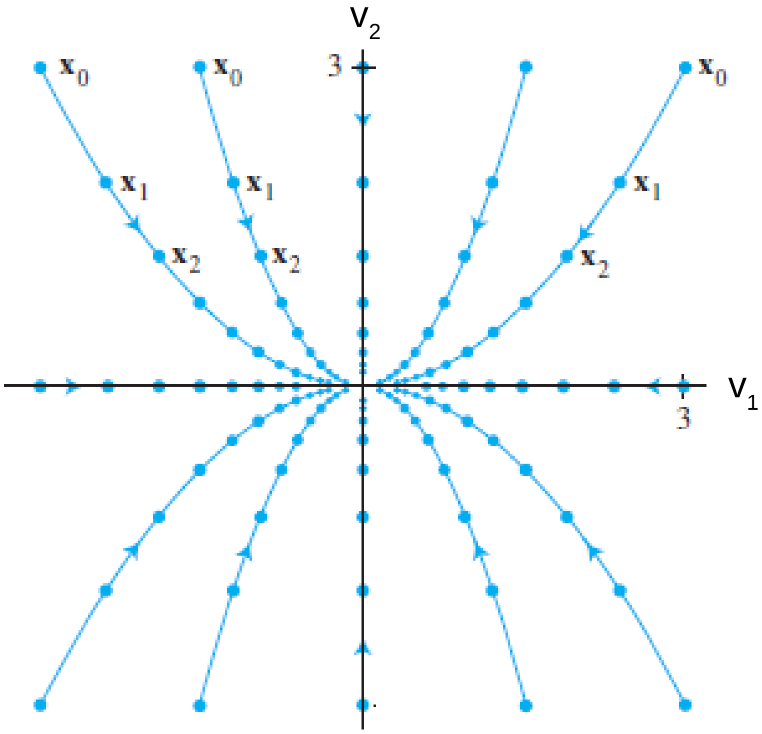


FIGURE 1 The origin as an attractor.

GRAPHICAL DESCRIPTION OF SOLUTIONS

- In example 2, the origin is called an **attractor** of the dynamical system because all trajectories tend toward 0.
- In the next example, both eigenvalues of A are larger than 1 in magnitude, and 0 is called a **repeller** of the dynamical system.

- **Example 3** Plot several typical solutions of the equation $x_{k+1} = Ax_k$, where

$$A = \begin{bmatrix} 1.44 & 0 \\ 0 & 1.2 \end{bmatrix}$$

GRAPHICAL DESCRIPTION OF SOLUTIONS

- **Solution** The eigenvalues of A are 1.44 and 1.2. If $x_0 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, then

$$x_k = c_1(1.44)^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2(1.2)^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Both terms grow in size, but the first term grows faster.
- So the direction of greatest repulsion is the line through 0 and the eigenvector for the eigenvalue of larger magnitude.
- Fig. 2 on the next slide shows several trajectories that begin at points quite close to 0.

GRAPHICAL DESCRIPTION OF SOLUTIONS

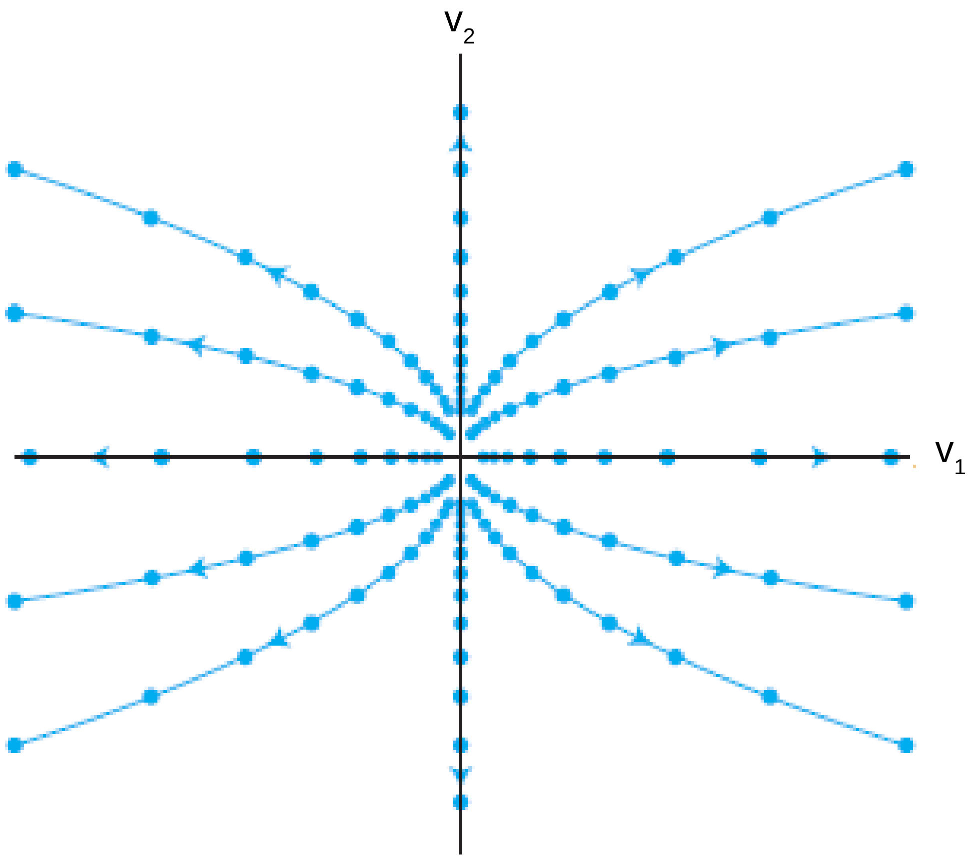


FIGURE 2 The origin as a repeller.