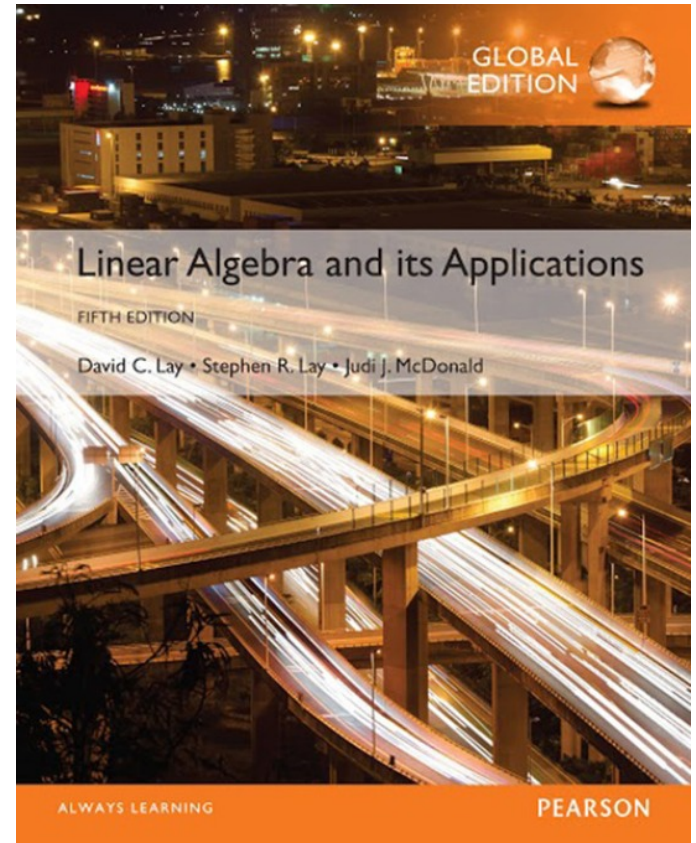


# 1

## Linear Equations in Linear Algebra

### 1.2

#### Row Reduction and Echelon Forms



# ROW REDUCTION AND ECHELON FORMS

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- In the definitions that follow, a *nonzero* row or column in a matrix means a row or column that contains at least one nonzero entry; a **leading entry** of a row refers to the leftmost nonzero entry (in a nonzero row).

# ECHELON FORM

- A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:
  1. All nonzero rows are above any rows of all zeros.
  2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
  3. All entries in a column below a leading entry are zeros.

# ECHELON FORM

- If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):
  4. The leading entry in each nonzero row is 1.
  5. Each leading 1 is the only nonzero entry in its column.
- An **echelon matrix** (respectively, **reduced echelon matrix**) is one that is in echelon form (respectively, reduced echelon form.)

# ECHELON FORM

- Any nonzero matrix may be **row reduced** (i.e., transformed by elementary row operations) into more than one matrix in echelon form, using different sequences of row operations. However, the reduced echelon form one obtains from a matrix is unique.

## THEOREM 1

### Uniqueness of the Reduced Echelon Form

Each matrix is row equivalent to one and only one reduced echelon matrix.

# PIVOT POSITION

- If a matrix  $A$  is row equivalent to an echelon matrix  $U$ , we call  $U$  **an echelon form** (or row echelon form) **of  $A$** ; if  $U$  is in reduced echelon form, we call  $U$  **the reduced echelon form of  $A$** .
- A **pivot position** in a matrix  $A$  is a location in  $A$  that corresponds to a leading 1 in the reduced echelon form of  $A$ . A **pivot column** is a column of  $A$  that contains a pivot position.

# PIVOT POSITION

- **Example 2:** Row reduce the matrix  $A$  below to echelon form, and locate the pivot columns of  $A$ .

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

- **Solution:** The top of the leftmost nonzero column is the first pivot position. A nonzero entry, or *pivot*, must be placed in this position.

# PIVOT POSITION

- Now, interchange rows 1 and 4.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Pivot

Pivot column

- Create zeros below the pivot, 1, by adding multiples of the first row to the rows below, and obtain the next matrix.



# PIVOT POSITION

- Choose the 2 in the second row as the next pivot.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Next pivot column

- Add  $-5/2$  times row 2 to row 3, and add  $3/2$  times row 2 to row 4.

# PIVOT POSITION

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

- There is no way to create a leading entry in column 3! However, if we interchange rows 3 and 4, we can produce a leading entry in column 4.

# PIVOT POSITION

The diagram shows a 4x5 matrix in echelon form. The first row has a leading 1 in the first column. The second row has a leading 2 in the second column. The third row has a leading -5 in the fourth column. The fourth row is all zeros. Blue arrows and labels identify the pivot and pivot columns. A blue arrow points from the label 'Pivot' to the element -5 in the third row, fourth column. A blue arrow points from the label 'Pivot columns' to the first, second, and fourth columns of the matrix.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot

Pivot columns

- The matrix is in echelon form and thus reveals that columns 1, 2, and 4 of  $A$  are pivot columns.

# PIVOT POSITION

Pivot positions

$$A = \begin{bmatrix} \boxed{0} & -3 & -6 & 4 & 9 \\ -1 & \boxed{-2} & -1 & 3 & 1 \\ -2 & -3 & 0 & \boxed{3} & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Pivot columns

The diagram shows a 4x5 matrix A. The first three rows have their first, second, and fourth elements boxed, respectively. Blue arrows point from the text 'Pivot positions' to these three boxed elements. Below the matrix, the first, second, and fourth columns are identified as pivot columns. Blue arrows point from the text 'Pivot columns' to these three columns.

- The pivots in the example are 1, 2 and  $-5$ .

# ROW REDUCTION ALGORITHM

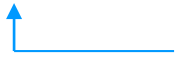
- **Example 3:** Apply elementary row operations to transform the following matrix first into echelon form and then into reduced echelon form:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

- **Solution:**
- **STEP 1:** Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

# ROW REDUCTION ALGORITHM

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

 Pivot column

- **STEP 2:** Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

# ROW REDUCTION ALGORITHM

- Interchange rows 1 and 3. (We could have interchanged rows 1 and 2 instead)

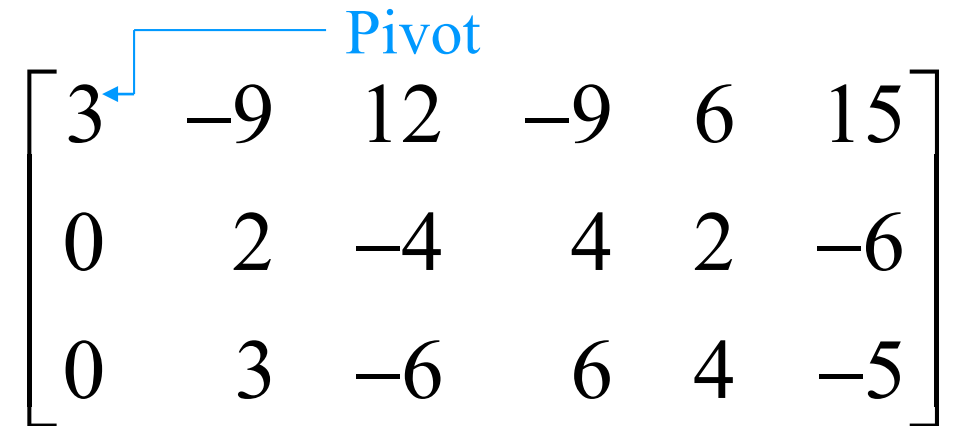
$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Pivot

- **STEP 3:** Use row replacement operations to create zeros in all positions below the pivot.

# ROW REDUCTION ALGORITHM

- We could have divided the top row by the pivot, 3, but with two 3s in column 1, it is just as easy to add  $-1$  times row 1 to row 2.


$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

- **STEP 4:** Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1–3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.



# ROW REDUCTION ALGORITHM

- With row 1 covered, step 1 shows that column 2 is the next pivot column; for step 2, select as a pivot the “top” entry in that column.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Pivot

New pivot column

- For step 3, we could insert an optional step of dividing the “top” row of the submatrix by the pivot, 2. Instead, we add  $-3/2$  times the “top” row to the row below.

# ROW REDUCTION ALGORITHM

- This produces the following matrix.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

- When we cover the row containing the second pivot position for step 4, we are left with a new submatrix that has only one row.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

# ROW REDUCTION ALGORITHM

- Steps 1–3 require no work for this submatrix, and we have reached an echelon form of the full matrix. We perform one more step to obtain the reduced echelon form.
- **STEP 5:** Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.
- The rightmost pivot is in row 3. Create zeros above it, adding suitable multiples of row 3 to rows 2 and 1.

# ROW REDUCTION ALGORITHM

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{array}{l} \leftarrow \text{Row 1} + (-6) \times \text{row 3} \\ \leftarrow \text{Row 2} + (-2) \times \text{row 3} \end{array}$$

- The next pivot is in row 2. Scale this row, dividing by the pivot.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \leftarrow \text{Row scaled by } \frac{1}{2}$$

# ROW REDUCTION ALGORITHM

- Create a zero in column 2 by adding 9 times row 2 to row 1.

$$\begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \leftarrow \text{Row 1} + (9) \times \text{row 2}$$

- Finally, scale row 1, dividing by the pivot, 3.

# ROW REDUCTION ALGORITHM

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \leftarrow \text{Row scaled by } \frac{1}{3}$$

- This is the reduced echelon form of the original matrix.
- The combination of steps 1–4 is called the **forward phase** of the row reduction algorithm. Step 5, which produces the unique reduced echelon form, is called the **backward phase**.

# SOLUTIONS OF LINEAR SYSTEMS

- The row reduction algorithm leads to an explicit description of the solution set of a linear system when the algorithm is applied to the augmented matrix of the system.
- Suppose, for example, that the augmented matrix of a linear system has been changed into the equivalent *reduced* echelon form.

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

# SOLUTIONS OF LINEAR SYSTEMS

- There are three variables because the augmented matrix has four columns. The associated system of equations

is

$$\begin{aligned}x_1 - 5x_3 &= 1 \\x_2 + x_3 &= 4 \\0 &= 0\end{aligned}\tag{4}$$

- The variables  $x_1$  and  $x_2$  corresponding to pivot columns in the matrix are called **basic variables**. The other variable,  $x_3$ , is called a **free variable**.



# SOLUTIONS OF LINEAR SYSTEMS

- Whenever a system is consistent, as in (4), the solution set can be described explicitly by solving the *reduced* system of equations for the basic variables in terms of the free variables.
- This operation is possible because the reduced echelon form places each basic variable in one and only one equation.
- In (4), solve the first and second equations for  $x_1$  and the second for  $x_2$ . (Ignore the third equation; it offers no restriction on the variables.)

# SOLUTIONS OF LINEAR SYSTEMS

$$x_1 = 1 + 5x_3$$

$$x_2 = 4 - x_3 \quad (5)$$

$x_3$  is free

- The statement “ $x_3$  is free” means that you are free to choose any value for  $x_3$ . Once that is done, the formulas in (5) determine the values for  $x_1$  and  $x_2$ . For instance, when  $x_3 = 0$ , the solution is  $(1, 4, 0)$ ; when  $x_3 = 1$ , the solution is  $(6, 3, 1)$ .
- *Each different choice of  $x_3$  determines a (different) solution of the system, and every solution of the system is determined by a choice of  $x_3$ .*

# PARAMETRIC DESCRIPTIONS OF SOLUTION SETS

- The descriptions in (5) and (7) are *parametric descriptions* of solutions sets in which the free variables act as parameters.
- *Solving a system* amounts to finding a parametric description of the solution set or determining that the solution set is empty.
- Whenever a system is consistent and has free variables, the solution set has many parametric descriptions.

# PARAMETRIC DESCRIPTIONS OF SOLUTION SETS

- For instance, in system (4), we may add 5 times equation 2 to equation 1 and obtain the following equivalent system

$$x_1 + 5x_2 = 21$$

$$x_2 + x_3 = 4$$

- We could treat  $x_2$  as a parameter and solve for  $x_1$  and  $x_3$  in terms of  $x_2$ , and we would have an accurate description of the solution set.
- Whenever a system is inconsistent, the solution set is empty, even when the system has free variables. In this case, the solution set has *no* parametric representation.

# EXISTENCE AND UNIQUENESS THEOREM

## THEOREM 2

### Existence and Uniqueness Theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column—i.e., if and only if an echelon form of the augmented matrix has *no* row of the form

$$[0 \ \dots \ 0 \ b] \text{ with } b \text{ nonzero.}$$

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

# ROW REDUCTION TO SOLVE A LINEAR SYSTEM

## Using Row Reduction to Solve a Linear System

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.

# ROW REDUCTION TO SOLVE A LINEAR SYSTEM

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5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.