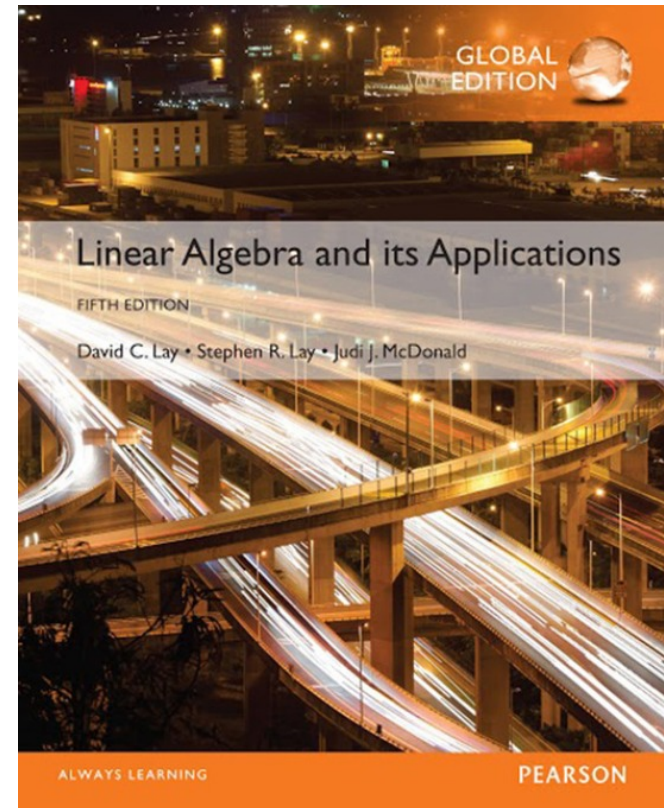


# 1

# Linear Equations in Linear Algebra

## 1.7

## LINEAR INDEPENDENCE



# LINEAR INDEPENDENCE

- **Definition:** An indexed set of vectors  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  in  $\mathbb{R}^n$  is said to be **linearly independent** if the vector equation

$$x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + \dots + x_p \mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is said to be **linearly dependent** if there exist weights  $c_1, \dots, c_p$ , not all zero, such that

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_p \mathbf{v}_p = \mathbf{0} \quad (2)$$

# LINEAR INDEPENDENCE

- Equation (2) is called a **linear dependence relation** among  $\mathbf{v}_1, \dots, \mathbf{v}_p$  when the weights are not all zero.
- An indexed set is linearly dependent if and only if it is not linearly independent.

- **Example 1:** Let  $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ ,  $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ , and  $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ .

# LINEAR INDEPENDENCE

---

- a. Determine if the set  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is linearly independent.
  - b. If possible, find a linear dependence relation among  $\mathbf{v}_1, \mathbf{v}_2$ , and  $\mathbf{v}_3$ .
- 
- **Solution:** We must determine if there is a nontrivial solution of the equation on the previous slide.

# LINEAR INDEPENDENCE

- Row operations on the associated augmented matrix show that

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

- $x_1$  and  $x_2$  are basic variables, and  $x_3$  is free.
- Each nonzero value of  $x_3$  determines a nontrivial solution of (1).
- Hence,  $\mathbf{v}_1$ ,  $\mathbf{v}_2$ ,  $\mathbf{v}_3$  are linearly dependent.

# LINEAR INDEPENDENCE

- b. To find a linear dependence relation among  $\mathbf{v}_1$ ,  $\mathbf{v}_2$ , and  $\mathbf{v}_3$ , completely row reduce the augmented matrix and write the new system:

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} x_1 - 2x_3 = 0 \\ x_2 + x_3 = 0 \\ 0 = 0 \end{array}$$

- Thus,  $x_1 = 2x_3$ ,  $x_2 = -x_3$ , and  $x_3$  is free.
- Choose any nonzero value for  $x_3$ —say,  $x_3 = 5$ .
- Then  $x_1 = 10$  and  $x_2 = -5$ .

# LINEAR INDEPENDENCE

---

- Substitute these values into equation (1) and obtain the equation below.

$$10\mathbf{v}_1 - 5\mathbf{v}_2 + 5\mathbf{v}_3 = 0$$

- This is one (out of infinitely many) possible linear dependence relations among  $\mathbf{v}_1$ ,  $\mathbf{v}_2$ , and  $\mathbf{v}_3$ .

# LINEAR INDEPENDENCE OF MATRIX COLUMNS

- Suppose that we begin with a matrix  $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$  instead of a set of vectors.
- The matrix equation  $A\mathbf{x} = \mathbf{0}$  can be written as
$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}.$$
- *Each linear dependence relation among the columns of  $A$  corresponds to a nontrivial solution of  $A\mathbf{x} = \mathbf{0}$*
- The columns of matrix  $A$  are linearly independent if and only if the equation  $A\mathbf{x} = \mathbf{0}$  has *only* the trivial solution.



# SETS OF ONE OR TWO VECTORS

---

- A set containing only one vector – say,  $\mathbf{v}$  – is linearly independent if and only if  $\mathbf{v}$  is not the zero vector.
- This is because the vector equation  $x_1 \mathbf{v} = \mathbf{0}$  has only the trivial solution when  $\mathbf{v} \neq \mathbf{0}$ .
- The zero vector is linearly dependent because  $x_1 \mathbf{0} = \mathbf{0}$  has many nontrivial solutions.

# SETS OF ONE OR TWO VECTORS

---

- A set of two vectors  $\{\mathbf{v}_1, \mathbf{v}_2\}$  is linearly dependent if at least one of the vectors is a multiple of the other.
- The set is linearly independent if and only if neither of the vectors is a multiple of the other.

# SETS OF TWO OR MORE VECTORS

## THEOREM 7

### **Characterization of Linearly Dependent Sets**

An indexed set  $S = \{v_1, \dots, v_p\}$  of two or more vectors is linearly dependent if and only if at least one of the vectors in  $S$  is a linear combination of the others. In fact, if  $S$  is linearly dependent and  $v_1 \neq 0$ , then some  $v_j$  (with  $j > 1$ ) is a linear combination of the preceding vectors,  $v_1, \dots, v_{j-1}$ .

# SETS OF TWO OR MORE VECTORS

- **Proof:** If some  $\mathbf{v}_j$  in  $S$  equals a linear combination of the other vectors, then  $\mathbf{v}_j$  can be subtracted from both sides of the equation, producing a linear dependence relation with a nonzero weight ( $-1$ ) on  $\mathbf{v}_j$ .
- [For instance, if  $\mathbf{v}_3 = c_2\mathbf{v}_2 + c_1\mathbf{v}_1$ , then  $0 = (-1)\mathbf{v}_3 + c_2\mathbf{v}_2 + c_1\mathbf{v}_1 + 0\mathbf{v}_4 + \dots + 0\mathbf{v}_p$ .]
- Thus  $S$  is linearly dependent.

First, assume some vector is a linear combination of its previous vectors



# SETS OF TWO OR MORE VECTORS

Second, assume that the vectors are linearly dependent



there exist weights  $c_1, \dots, c_p$ , not all zero, such that

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_p \mathbf{v}_p = \mathbf{0}.$$

- Let  $j$  be the largest subscript for which  $c_j \neq 0$ .
- If  $j = 1$ , then  $c_1 \mathbf{v}_1 = \mathbf{0}$ , which is impossible because  $\mathbf{v}_1 \neq \mathbf{0}$ .

# SETS OF TWO OR MORE VECTORS

- So  $j > 1$ , and

$$c_1 \mathbf{v}_1 + \dots + c_j \mathbf{v}_j + 0\mathbf{v}_{j+1} + 0\mathbf{v}_{j+2} + \dots + 0\mathbf{v}_p = \mathbf{0}$$

$$c_j \mathbf{v}_j = -c_1 \mathbf{v}_1 - \dots - c_{j-1} \mathbf{v}_{j-1}$$

$$\mathbf{v}_j = \left( -\frac{c_1}{c_j} \right) \mathbf{v}_1 + \dots + \left( -\frac{c_{j-1}}{c_j} \right) \mathbf{v}_{j-1}.$$

# SETS OF TWO OR MORE VECTORS

- Theorem 7 does *not* say that *every* vector in a linearly dependent set is a linear combination of the preceding vectors.
- A vector in a linearly dependent set may fail to be a linear combination of the other vectors.

- **Example 4:** Let  $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$ . Describe the

set spanned by  $\mathbf{u}$  and  $\mathbf{v}$ , and explain why a vector  $\mathbf{w}$  is in  $\text{Span } \{\mathbf{u}, \mathbf{v}\}$  if and only if  $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$  is linearly dependent.

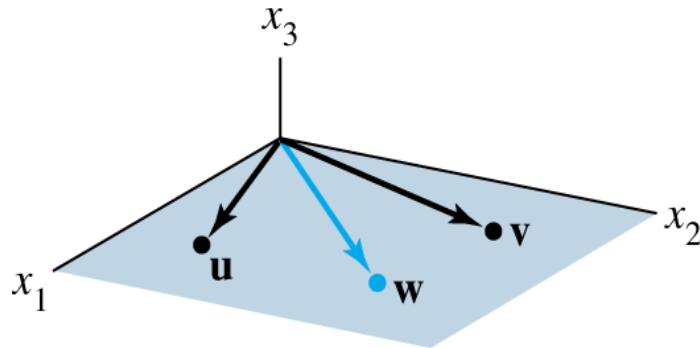
# SETS OF TWO OR MORE VECTORS

- **Solution:** The vectors  $\mathbf{u}$  and  $\mathbf{v}$  are linearly independent because neither vector is a multiple of the other, and so they span a plane in  $\mathbb{R}^3$ .
- Span  $\{\mathbf{u}, \mathbf{v}\}$  is the  $x_1x_2$ -plane (with  $x_3 = 0$ ).
- If  $\mathbf{w}$  is a linear combination of  $\mathbf{u}$  and  $\mathbf{v}$ , then  $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$  is linearly dependent, by Theorem 7.
- Conversely, suppose that  $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$  is linearly dependent.
- By theorem 7, some vector in  $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$  is a linear combination of the preceding vectors (since  $\mathbf{u} \neq \mathbf{0}$  ).
- That vector must be  $\mathbf{w}$ , since  $\mathbf{v}$  is not a multiple of  $\mathbf{u}$ .

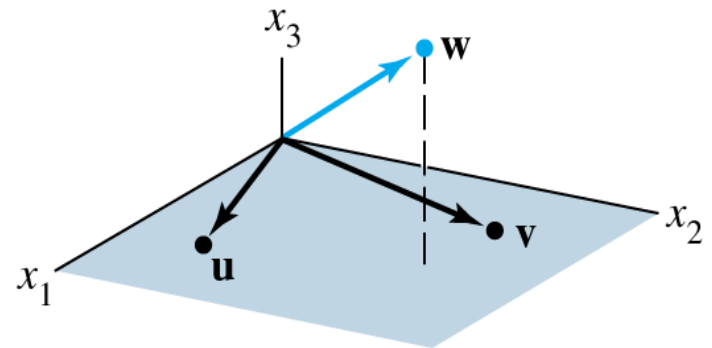


# SETS OF TWO OR MORE VECTORS

- So  $\mathbf{w}$  is in  $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ . Fig. 2 below



Linearly dependent,  
 $\mathbf{w}$  in  $\text{Span}\{\mathbf{u}, \mathbf{v}\}$



Linearly independent,  
 $\mathbf{w}$  not in  $\text{Span}\{\mathbf{u}, \mathbf{v}\}$

- Example 4 generalizes to any set  $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$  in  $\mathbb{R}^3$  with  $\mathbf{u}$  and  $\mathbf{v}$  linearly independent.
- The set  $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$  will be linearly dependent if and only if  $\mathbf{w}$  is in the plane spanned by  $\mathbf{u}$  and  $\mathbf{v}$ .

# SETS OF TWO OR MORE VECTORS

## THEOREM 8

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  in  $\mathbb{R}^n$  is linearly dependent if  $p > n$ .

- **Proof:** Let  $A = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_p \end{bmatrix}$ .
- Then  $A$  is  $n \times p$ , and the equation  $A\mathbf{x} = \mathbf{0}$  corresponds to a system of  $n$  equations in  $p$  unknowns.
- If  $p > n$ , there are more variables than equations, so there must be a free variable.

# SETS OF TWO OR MORE VECTORS

- Hence  $A\mathbf{x} = \mathbf{0}$  has a nontrivial solution, and the columns of  $A$  are linearly dependent.
- See the figure below for a matrix version of this theorem.

$$\begin{matrix} & p \\ n & \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \end{matrix}$$

If  $p > n$ , the columns are linearly dependent.

- Theorem 8 says nothing about the case in which the number of vectors in the set does *not* exceed the number of entries in each vector.

# SETS OF TWO OR MORE VECTORS

## THEOREM 9

If a set  $S = \{v_1, \dots, v_p\}$  in  $\mathbb{R}^n$  contains the zero vector, then the set is linearly dependent.

- **Proof:** By renumbering the vectors, we may suppose  $v_1 = 0$ .
- Then the equation  $1v_1 + 0v_2 + \dots + 0v_p = 0$  shows that  $S$  is linearly dependent.