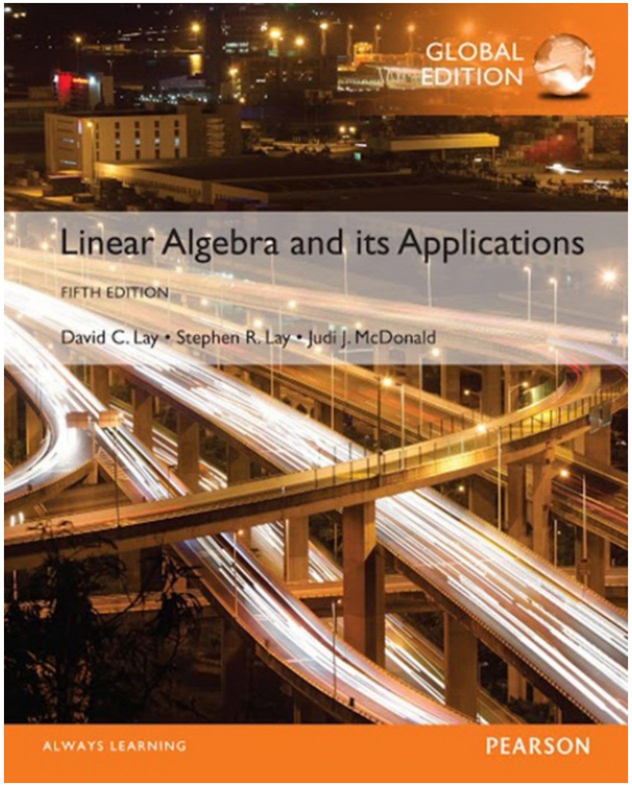
5 Eigenvalues and Eigenvectors

5.4

EIGENVECTORS AND LINEAR TRANSFORMATIONS



THE MATRIX OF A LINEAR TRANSFORMATION

- Given any x in V, the coordinate vector $[x]_\beta$ is in $\mathbb{R}^n$ and the coordinate vector of its image, $[T(x)]_C$, is in $\mathbb{R}^m$, as shown in Fig. 1 below.

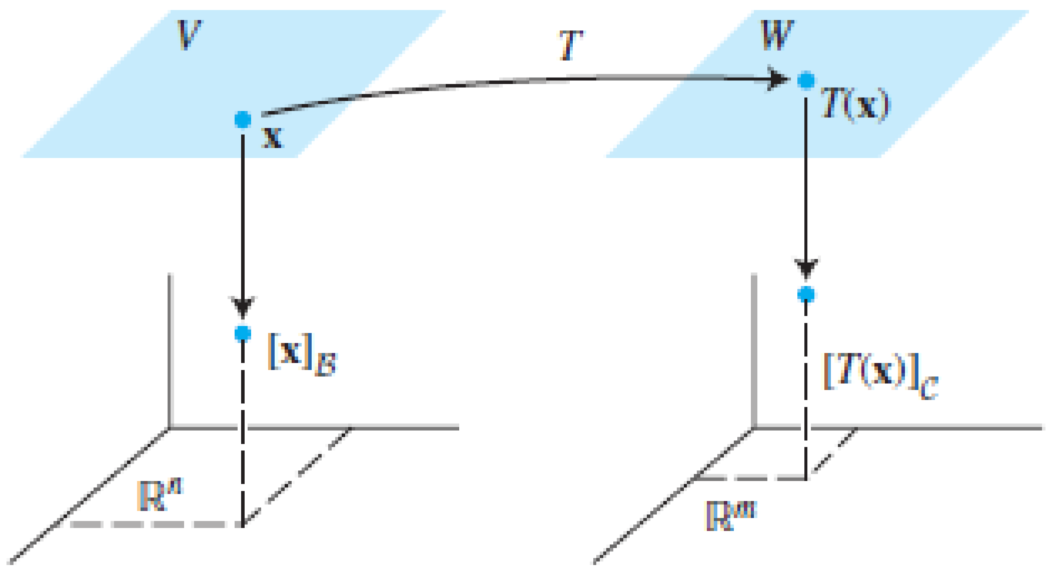


FIGURE 1 A linear transformation from V to W.

THE MATRIX OF A LINEAR TRANSFORMATION

- The connection between $[\mathrm{x}]_\beta$ and $[T(\mathrm{x})]_C$ is easy to find. Let $\{\mathrm{b}_1, \ldots, \mathrm{b}_2\}$ be the basis β for V. If $\mathrm{x} = r_1\mathrm{b}_1 + \ldots + r_n\mathrm{b}_n$, then,

$$[\mathrm{x}]_\beta = \begin{bmatrix} r_1 \\ \cdot \\ \cdot \\ \cdot \\ r_n \end{bmatrix}$$

- And

$$T(x) = T(r_1 b_1 + \cdots + r_n b_n) = r_1 T(b_1) + \cdots + r_n T(b_n) \quad (1)$$

- because T is linear.

THE MATRIX OF A LINEAR TRANSFORMATION

- Now, since the coordinate mapping from *W* to $\mathbb{R}^m$ is linear, equation (1) leads to

$$[T(x)]_C = r_1[T(b_1)]_C + \cdots + r_n[T(b_n)]_C \qquad (2)$$

- Since *C*-coordinate vectors are in $\mathbb{R}^m$, the vector equation (2) can be written as a matrix equation, namely,

$$[T(x)]_C = M[x]_\beta \qquad (3)$$

- where

$$M = [[T(b_1)]_C[T(b_2)]_C \ \ \ldots [T(b_n)]_C] \qquad (4)$$

THE MATRIX OF A LINEAR TRANSFORMATION

- The matrix M is a matrix representation of T, called the matrix for T relative to the bases β and C. See Fig. 2 below:

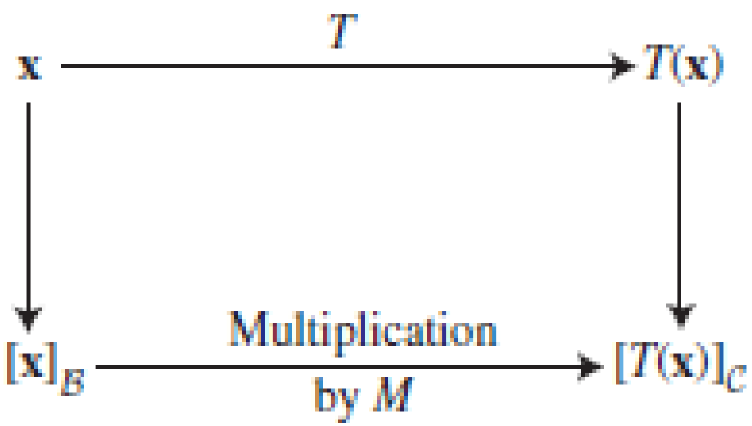


FIGURE 2

THE MATRIX OF A LINEAR TRANSFORMATION

- **Example 1** Suppose $\beta = \{b_1, b_2\}$ is a basis for V and $C = \{c_1, c_2, c_3\}$ is a basis for W. Let $T: V \rightarrow W$ be a linear transformation with the property that

$$T(b_1) = 3c_1 - 2c_2 + 5c_3 \quad \text{and} \quad T(b_2) = 4c_1 + 7c_2 - c_3$$

- Find the matrix M for T relative to β and C.

THE MATRIX OF A LINEAR TRANSFORMATION

- **Solution** The C-coordinate vectors of the images of b_1 and b_2 are

$$[T(b_1)]_C = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \text{ and } [T(b_2)]_C = \begin{bmatrix} 4 \\ 7 \\ -1 \end{bmatrix}$$

- Hence

$$M = \begin{bmatrix} 3 & 4 \\ -2 & 7 \\ 5 & -1 \end{bmatrix}$$

- If β and C are bases for the same space V and if T is the identity transformation T(x) = x for x in V, then matrix M in (4) is just a change-of-coordinates matrix.

LINEAR TRANSFORMATIONS FROM *V* INTO *V*

- In the common case where W is the same V and the basis C is the same as β, then the matrix M in (4) is called the **matrix for *T* relative to $\boldsymbol{\beta}$**, or simply the **$\boldsymbol{\beta}$ -matrix for *T***, and is denoted by $[T]_{\beta}$.
- See Fig. 3 below

$$
\begin{tikzcd}
\mathbf{x} \arrow[r, "T"] \arrow[d] & T(\mathbf{x}) \arrow[d] \\
{[\mathbf{x}]_{\mathcal{B}}} \arrow[r, "\text{Multiplication}", "\text{by } {[T]_{\mathcal{B}}}"'] & {[T(\mathbf{x})]_{\mathcal{B}}}
\end{tikzcd}
$$

FIGURE 3

LINEAR TRANSFORMATIONS ON $\mathbb{R}^n$

- **Theorem 8**: Suppose $A = PDP^{-1}$, where D is a diagonal $n \times n$ matrix. If β is the basis for $\mathbb{R}^n$ formed from the columns of P, then D is the β-matrix for the transformation $x \mapsto Ax$.

- **Proof** Denote the columns of P by $b_1, \ldots, b_n$, so that $\beta = \{b_1, \ldots, b_n\}$ and $P = [b_1 \ldots b_n]$. In this case, P is the change-of-coordinates matrix P_β discussed in Section 4.4, where

$$P[x]_\beta = x \text{ and } [x]_\beta = P^{-1}x$$

LINEAR TRANSFORMATIONS ON $\mathbb{R}^n$

- If $T(x) = Ax$ for x in $\mathbb{R}^n$, then

$$\begin{aligned}
[T]_\beta &= [[T(b_1)]_\beta \quad \dots \quad [T(b_n)]_\beta] && \text{Definition of } [T]_\beta \\
&= [[Ab_1]_\beta \quad \dots \quad [Ab_n]_\beta] && \text{Since } T(x) = Ax \\
&= [P^{-1}Ab_1 \quad \dots \quad P^{-1}Ab_n] && \text{Change of coordinates} \\
&= P^{-1}A[b_1 \quad \dots \quad b_n] && \text{Matrix multiplication} \\
&= P^{-1}AP
\end{aligned}$$

- Since $A = PDP^{-1}$, we have $[T]_\beta = P^{-1}AP = D$.

LINEAR TRANSFORMATIONS ON $\mathbb{R}^n$

- **Example 3** Define T: $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ by T(x)=Ax, where $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Find a basis β for $\mathbb{R}^2$ with the property that the β–matrix for T is a diagonal matrix.
- **Solution** From Example 2 in Section 5.3 we know that A= PDP^{-1}, where

$$P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \text{ and } D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

- The columns of P, call them b_1 and b_2, are eigenvectors of A. By Theorem 8, D is the β-matrix for T when $\beta = \{b_1, b_2\}$. The mappings $x \mapsto Ax$ and $u \mapsto Du$ describe the same linear transformation, relative to different bases.