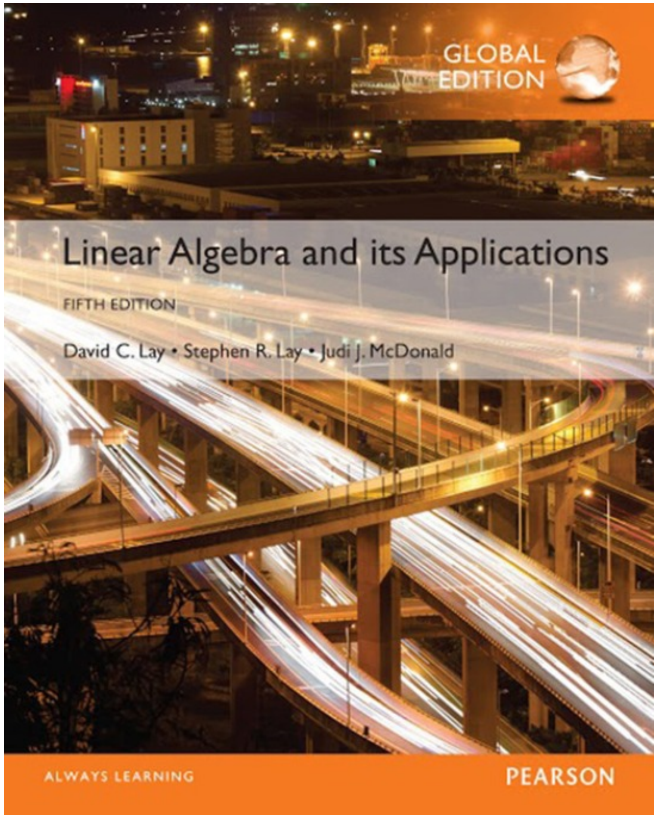
6 Orthogonality and Least Squares

6.1

INNER PRODUCT, LENGTH, AND ORTHOGONALITY



INNER PRODUCT

- If **u** and **v** are vectors in $\mathbb{R}^n$, then we regard **u** and **v** as $n \times 1$ matrices.
- The transpose $\mathbf{u}^T$ is a $1 \times n$ matrix, and the matrix product $\mathbf{u}^T\mathbf{v}$ is a 1×1 matrix, which we write as a single real number (a scalar) without brackets.
- The number $\mathbf{u}^T\mathbf{v}$ is called the **inner product** of **u** and **v**, and it is written as $u \cdot v$.
- This inner product is also referred to as a **dot product**.

INNER PRODUCT

- If u $= \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and v $= \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$,

  then the inner product of **u** and **v** is

$$\begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

INNER PRODUCT

- **Theorem 1:** Let **u**, **v**, and **w** be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

  a. $u \cdot v = v \cdot u$

  b. $(u+v) \cdot w = u \cdot w + v \cdot w$

  c. $(cu) \cdot v = c(u \cdot v) = u \cdot (cv)$

  d. $u \cdot u \geq 0$, and $u \cdot u = 0$if and only if $\mathbf{u} = 0$

- Properties (b) and (c) can be combined several times to produce the following useful

  $$(c_1u_1 + \ldots + c_pu_p) \cdot w = c_1(u_1 \cdot w) + \cdots + c_p(u_p \cdot w)$$

THE LENGTH OF A VECTOR

- If **v** is in $\mathbb{R}^n$, with entries $v_1, \ldots, v_n$, then the square root of $v \cdot v$ is defined because $v \cdot v$ is nonnegative.

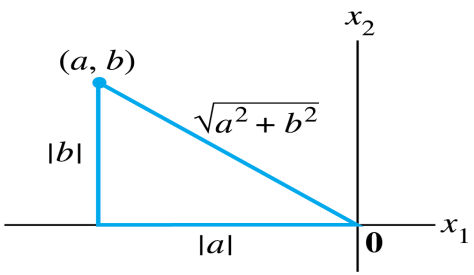
- **Definition:** The **length** (or **norm**) of **v** is the nonnegative scalar $\|\mathrm{v}\|$ defined by

$$\|v\| = \sqrt{v \cdot v} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}, \text{ and } \|v\|^2 = \sqrt{v \cdot v}$$

- Suppose **v** is in $\mathbb{R}^2$, say, $\mathrm{v} = \begin{bmatrix} a \\ b \end{bmatrix}$

THE LENGTH OF A VECTOR

- If we identify **v** with a geometric point in the plane, as usual, then $\|\mathrm{v}\|$ coincides with the standard notion of the length of the line segment from the origin to **v**.
- This follows from the Pythagorean Theorem applied to a triangle such as the one shown in the following figure.



Interpretation of $\|\mathbf{v}\|$ as length.

- For any scalar c, the length $c\mathbf{v}$ is $|c|$ times the length of v. That is,

$$\|c\mathrm{v}\| = |c|\|\mathrm{v}\|$$

THE LENGTH OF A VECTOR

- A vector whose length is 1 is called a **unit vector**.

- If we *divide* a nonzero vector **v** by its length—that is, multiply by $1/\|\mathbf{v}\|$—we obtain a unit vector **u** because the length of **u** is $(1/\|\mathbf{v}\|)\|\mathbf{v}\|$.

- The process of creating **u** from **v** is sometimes called **normalizing v**, and we say that **u** is *in the same direction* as **v**.

THE LENGTH OF A VECTOR

- **Example 2:** Let $\mathrm{v} = (1, -2, 2, 0)$. Find a unit vector **u** in the same direction as **v**.
- **Solution:** First, compute the length of **v**:

$$\|\mathrm{v}\|^2 = v \cdot v = (1)^2 + (-2)^2 + (2)^2 + (0)^2 = 9$$

$$\|\mathrm{v}\| = \sqrt{9} = 3$$

- Then, multiply **v** by $1/\|\mathrm{v}\|$ to obtain

$$\mathrm{u} = \frac{1}{\|\mathrm{v}\|}\mathrm{v} = \frac{1}{3}\mathrm{v} = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \\ 0 \end{bmatrix}$$

DISTANCE IN $\mathbb{R}^n$

- To check that $\|\mathbf{u}\| = 1$, it suffices to show that $\|\mathbf{u}\|^2 = 1$.

$$\|\mathbf{u}\|^2 = u \cdot u = \left(\frac{1}{3}\right)^2 + \left(-\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 + (0)^2$$

$$= \frac{1}{9} + \frac{4}{9} + \frac{4}{9} + 0 = 1$$

- **Definition:** For **u** and **v**in $\mathbb{R}^n$, the **distance between u and v**, written as dist (**u**, **v**), is the length of the vector $\mathbf{u} - \mathbf{v}$. That is,

$$\text{dist}\,(\mathbf{u},\mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

DISTANCE IN $\mathbb{R}^n$

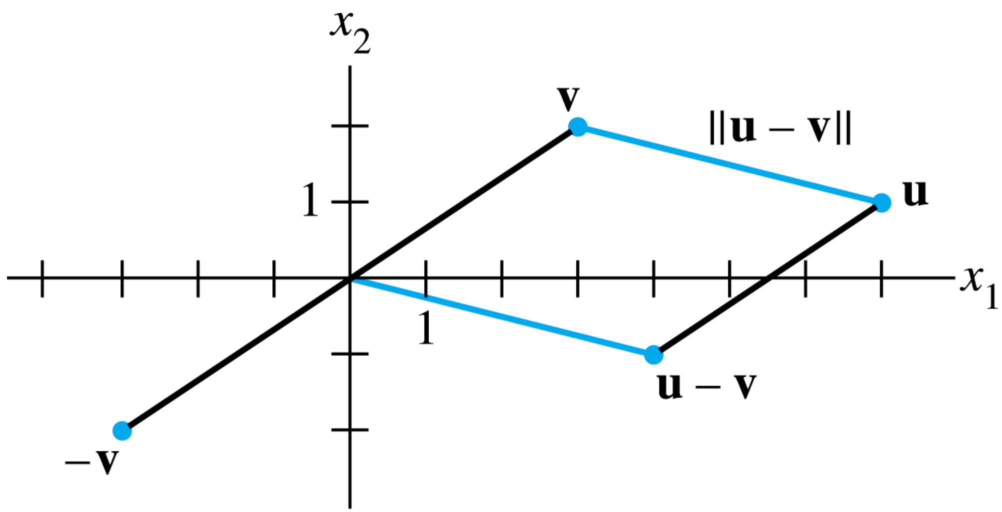
- **Example 4:** Compute the distance between the vectors $\mathbf{u} = (7,1)$ and $\mathbf{v} = (3,2)$.
- **Solution:** Calculate

$$\mathbf{u} - \mathbf{v} = \begin{bmatrix} 7 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$\|\mathbf{u} - \mathbf{v}\| = \sqrt{4^2 + (-1)^2} = \sqrt{17}$$

- The vectors **u**, **v**, and $\mathbf{u} - \mathbf{v}$ are shown in the figure on the next slide.
- When the vector $\mathbf{u} - \mathbf{v}$ is added to **v**, the result is **u**.

DISTANCE IN $\mathbb{R}^n$

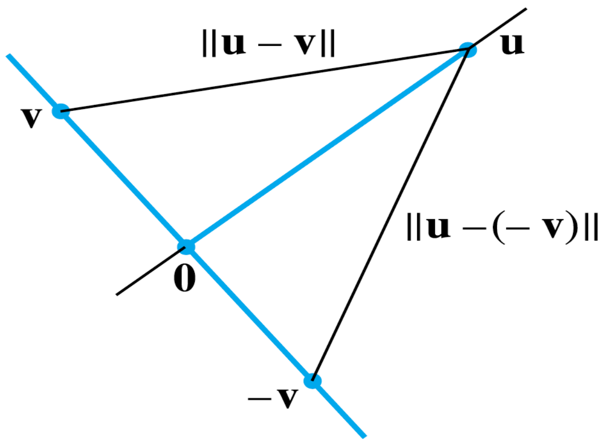


The distance between **u** and **v** is the length of $\mathbf{u} - \mathbf{v}$.

- Notice that the parallelogram in the above figure shows that the distance from **u** to **v** is the same as the distance from $\mathbf{u} - \mathbf{v}$ to **0**.

ORTHOGONAL VECTORS

- Consider $\mathbb{R}^2$ or $\mathbb{R}^3$ and two lines through the origin determined by vectors $\mathbf{u}$ and $\mathbf{v}$.
- See the figure below. The two lines shown in the figure are geometrically perpendicular if and only if the distance from $\mathbf{u}$ to $\mathbf{v}$ is the same as the distance from $\mathbf{u}$ to $-\mathrm{v}$.



- This is the same as requiring the squares of the distances to be the same.

ORTHOGONAL VECTORS

- Now

$$[dist(u,-v)]^2 = \|u-(-v)\|^2 = \|u+v)\|^2$$
$$= (u+v)\cdot(u+v)$$
$$= u\cdot(u+v) + v\cdot(u+v)$$ Theorem 1(b)
$$= u\cdot u + u\cdot v + \mathrm{v}\cdot u + v\cdot v$$ Theorem 1(a), (b)
$$= \|u\|^2 + \|v)\|^2 + 2\mathrm{u}\cdot v$$ Theorem 1(a)

- The same calculations with **v** and $-\mathrm{v}$ interchanged show that

$$[\mathrm{dist}(\mathrm{u},\mathrm{v})]^2 = \|\mathrm{u}\|^2 + \|-\mathrm{v}\|^2 + 2u\cdot(-v)$$
$$= \|\mathrm{u}\|^2 + \|\mathrm{v}\|^2 - 2u\cdot v$$

ORTHOGONAL VECTORS

- The two squared distances are equal if and only if $2u \cdot v = -2u \cdot v$, which happens if and only if $u \cdot v = 0$.
- This calculation shows that when vectors **u** and **v** are identified with geometric points, the corresponding lines through the points and the origin are perpendicular if and only if $u \cdot v = 0$.

- **Definition:** Two vectors **u** and **v** in $\mathbb{R}^n$ are **orthogonal** (to each other) if $u \cdot v = 0$.
- The zero vector is orthogonal to every vector in $\mathbb{R}^n$ because $0^T \mathrm{v} = 0$ for all **v**.

THE PYTHOGOREAN THEOREM

- **Theorem 2:** Two vectors **u** and **v** are orthogonal if and only if $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

Orthogonal Complements

- If a vector **z** is orthogonal to every vector in a subspace W of $\mathbb{R}^n$, then **z** is said to be **orthogonal to** W.
- The set of all vectors **z** that are orthogonal to W is called the **orthogonal complement** of W and is denoted by $W^{\perp}$ (and read as "W perpendicular" or simply "Wperp").

ORTHOGONAL COMPLEMENTS

1. A vector $\mathbf{x}$ is in $W^{\perp}$ if and only if $\mathbf{x}$ is orthogonal to every vector in a set that spans W.
2. $W^{\perp}$ is a subspace of $\mathbb{R}^n$.

- **Theorem 3:** Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T:

$$(\text{Row } A)^{\perp} = \text{Nul } A \quad \text{and} \quad (\text{Col } A)^{\perp} = \text{Nul } A^T$$

ORTHOGONAL COMPLEMENTS

- **Proof:** The row-column rule for computing $A\mathbf{x}$ shows that if $\mathbf{x}$ is in Nul$A$, then $\mathbf{x}$ is orthogonal to each row of A (with the rows treated as vectors in $\mathbb{R}^n$).

- Since the rows of A span the row space, $\mathbf{x}$ is orthogonal to Row A.

- Conversely, if $\mathbf{x}$ is orthogonal to Row A, then $\mathbf{x}$ is certainly orthogonal to each row of A, and hence $A\mathrm{x} = 0$.

- This proves the first statement of the theorem.

ORTHOGONAL COMPLEMENTS

- Since this statement is true for any matrix, it is true for A^T.
- That is, the orthogonal complement of the row space of A^T is the null space of A^T.
- This proves the second statement, because $\text{Row } A^T = \text{Col } A$.