TRIANGULATION AND TRILATERATION

The horizontal positions of points is a network developed to provide accurate control for topographic mapping, charting lakes, rivers and ocean coast lines, and for the surveys required for the design and construction of public and private works of large extent. The horizontal positions of the points can be obtained in a number of different ways and are:

1. Traversing
2. Triangulation
3. Trilateration
4. Intersection
5. Resection and
6. Satellite positioning

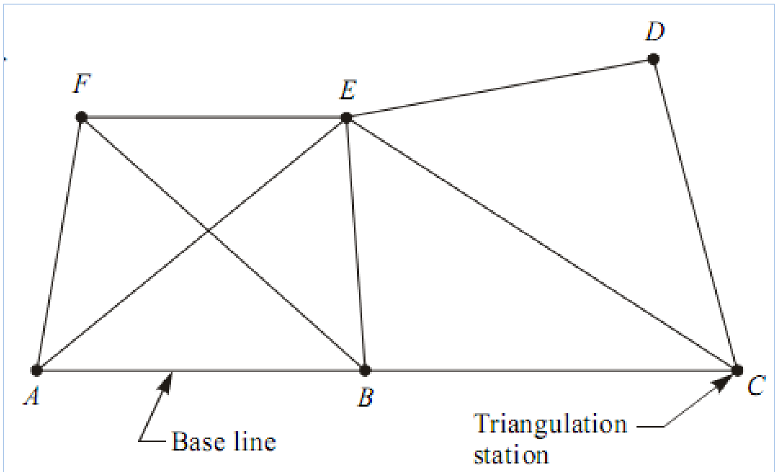

Fig. 1.1 Triangulation network

Triangulation, a method of surveying is based on the trigonometric proposition that if one side and two angles of a triangle are known, the remaining sides can be computed. Furthermore, if the direction of one side is known, the directions of the remaining sides can be determined. A triangulation system consists of a series of joined or overlapping triangles in which an occasional side is measured and remaining sides are calculated from angles measured at the vertices of the triangles. The vertices of the triangles are known as triangulation stations. The side of the triangle, whose length is predetermined, is called the base line. The lines of triangulation system form a network that ties together all the triangulation stations (Fig. 1.1).

A trilateration system also consists of a series of joined or overlapping triangles. However, for trilateration the lengths of all the sides of the triangle are measured and few directions or angles are measured to establish azimuth. Trilateration has become feasible with the development of electronic distance measuring (EDM) equipment which has made possible the measurement of all lengths with high order of accuracy under almost all field conditions.

A combined triangulation and trilateration system consists of a network of triangles in which all the angles and all the lengths are measured. Such a combined system represents the strongest network for creating horizontal control

Since a triangulation or trilateration system covers very large area, the curvature of the earth has to be taken into account. These surveys are, therefore, invariably geodetic. Triangulation surveys were first carried out by Snell, a Dutchman, in 1615.

Field procedures for the establishment of trilateration station are similar to the procedures used for triangulation, and therefore, henceforth in this chapter the term triangulation will only be used.

PRINCIPLE OF TRIANGULATION

Fig. 1.2 shows two interconnected triangles ABC and BCD. All the angles in both the triangles and the length L of the side AB have been measured. Also the azimuth θ of AB has been measured at the triangulation station A, whose coordinates (XA, YA), are known. The objective is to determine the coordinates of the triangulation stations B, C, and D by the method of triangulation. Let us first calculate the lengths of all the lines.

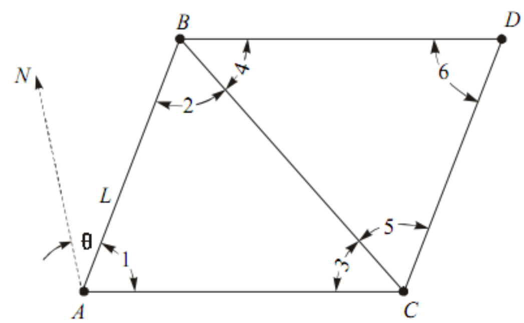

Fig. 1.2 Principle of triangulation

By sine rule in, ABC Δ we have

$$\frac{AB}{\sin 3} = \frac{BC}{\sin 1} = \frac{CA}{\sin 2}$$

We have $AB = L = l_{AB}$

or $BC = \dfrac{L \sin 1}{\sin 3} = l_{BC}$

and $CA = \dfrac{L \sin 2}{\sin 3} = l_{CA}$

Now the side BC being known in ΔBCD, by sine rule, we have

$$\frac{BC}{\sin 6} = \frac{CD}{\sin 4} = \frac{BD}{\sin 5}$$

We have $BC = \dfrac{L \sin 1}{\sin 3} = l_{BC}$

or $CD = \left(\dfrac{L \sin 1}{\sin 3}\right)\dfrac{\sin 4}{\sin 6} = l_{CD}$

and $BC = \left(\dfrac{L \sin 1}{\sin 3}\right)\dfrac{\sin 5}{\sin 6} = l_{BD}$

Let us now calculate the azimuths of all the lines.

Azimuth of $AB = \theta = \theta_{AB}$

Azimuth of $AC = \theta + \angle 1 = \theta_{AC}$

Azimuth of $BC = \theta + 180° - \angle 2 = \theta_{BC}$

Azimuth of $BD = \theta + 180° - (\angle 2 + \angle 4) = \theta_{BD}$

Azimuth of $CD = \theta - \angle 2 + \angle 5 = \theta_{CD}$

From the known length of the sides and azimuths, the consecutive and independent coordinates can be computed as:

Latitude of $AB = l_{AB} \cos \theta_{AB} = L_{AB}$

Departure of $AB = l_{AB} \sin \theta_{AB} = D_{AB}$

Latitude of $AC = l_{AC} \cos \theta_{AC} = L_{AC}$

Departure of $AC = l_{AC} \sin \theta_{AC} = D_{AC}$

Latitude of $BD = l_{BD} \cos \theta_{BD} = L_{BD}$

Departure of $BD = l_{BD} \sin \theta_{BD} = L_{BD}$

Latitude of $CD = l_{CD} \cos \theta_{CD} = L_{CD}$

Departure of $CD = l_{CD} \sin \theta_{CD} = D_{CD}$

The desired coordinates of the triangulation stations .

X-coordinate of B, $X_B = X_A + D_{AB}$

Y-coordinate of B, $Y_B = Y_B + L_{AB}$

X-coordinate of C, $X_C = X_A + D_{AC}$

Y-coordinate of C, $Y_C = Y_A + L_{AC}$

X-coordinate of D, $X_D = X_B + D_{BD}$

Y-coordinate of D, $Y_D = Y_B + L_{BD}$

It would be found that the length of side can be computed more than once following different routes, and therefore, to achieve a better accuracy, the mean of the computed lengths of a side is to be considered.

OBJECTIVES OF TRIANGULATION SURVEYS

The main objective of triangulation or trilateration surveys is to provide a number of stations whose relative and absolute positions, horizontal as well as vertical, are accurately established. More detailed location or engineering survey is then carried out from these stations.

The triangulation surveys are carried out

(i) To establish accurate control for plane and geodetic surveys of large areas, by terrestrial methods,
(ii) To establish accurate control for photogrammetric surveys of large areas,
(iii) To assist in the determination of the size and shape of the earth by making observations for latitude, longitude and gravity, and
(iv) To determine accurate locations of points in engineering works such as :

- (a) Fixing centre line and abutments of long bridges over large rivers.
- (b) Fixing centre line, terminal points, and shafts for long tunnels.
- (c) Transferring the control points across wide sea channels, large water bodies, etc.
- (d) Detection of crustal movements, etc.
- (e) Finding the direction of the movement of clouds.

CLASSIFICATION OF TRIANGULATION SYSTEM

Based on the extent and purpose of the survey, and consequently on the degree of accuracy desired, triangulation surveys are classified as first-order or primary, second-order or secondary, and third-order or tertiary.

Table 1.1 Triangulation system

S.No.	Characteristics	First-order triangulation	Second-order triangulation	Third-order triangulation
1.	Length of base lines	8 to 12 km	2 to 5 km	100 to 500 m
2.	Lengths of sides	16 to 150 km	10 to 25 km	2 to 10 km
3.	Average triangular error (after correction for spherical excess)	less than 1″	3″	12″
4.	Maximum station closure	not more than 3″	8″	15″
5.	Actual error of base	1 in 50,000	1 in 25,000	1 in 10,000
6.	Probable error of base	1 in 10,00,000	1 in 500,000	1 in 250,000
7.	Discrepancy between two measures (k is distance in kilometre)	$5\sqrt{k}$ mm	$10\sqrt{k}$ mm	$25\sqrt{k}$ mm
8.	Probable error of the computed distances	1 in 50,000 to 1 in 250,000	1 in 20,000 to 1 in 50,000	1 in 5,000 to 1 in 20,000
9.	Probable error in astronomical azimuth	0.5″	5″	10″

Table 1.1 presents the general specifications for the three types of triangulation systems.

First-order triangulation is used to determine the shape and size of the earth or to cover a vast area like a whole country with control points to which a second-order triangulation system can be connected. A second-order triangulation system consists of a network within a first-order triangulation. It is used to cover areas of the order of a region, small country, or province. A third-order triangulation is a framework fixed within and connected to a second-order triangulation system. It serves the purpose of furnishing the immediate control for detailed engineering and location surveys

TRIANGULATION FIGURES AND LAYOUTS:

The basic figures used in triangulation networks are the triangle, braced or geodetic quadrilateral, and the polygon with a central station (Fig 1.3)

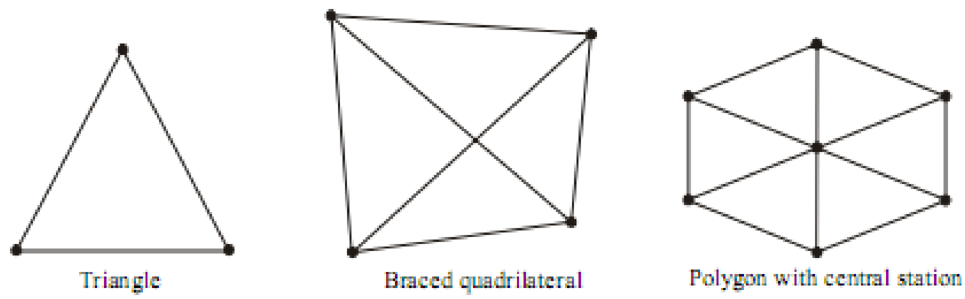

Fig. 1.3 Basic triangulation figures

The triangles in a triangulation system can be arranged in a number of ways: some of the commonly used arrangements, also called *layouts*, are as follows:

1. Single chain of triangles
2. Double chain of triangles
3. Braced quadrilaterals
4. Central triangles and polygons
5. A combination of above systems.

Single Chain of triangles:

When the control points are required to be established in a narrow strip of terrain such as a valley between ridges, a layout consisting of single chain of triangles is generally used as shown in Fig. 1.4. This system is rapid and economical due to its simplicity of sighting only four other stations, and does not involve observations of long diagonals. On the other hand, simple triangles of a triangulation system provide only one route through which distances can be computed, and hence, this system does not provide any check on the accuracy of observations. Check base lines and astronomical observations for azimuths have to be provided at frequent intervals to avoid excessive accumulation of errors in this layout

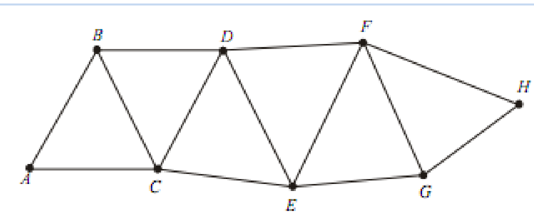

Fig. 1.4 Single of triangles

Double chain of triangles:

A layout of double chain of triangle is shown in fig 1.5. This arrangement is used for covering the larger width of belt. This system also has disadvantages of single chain of triangles system.

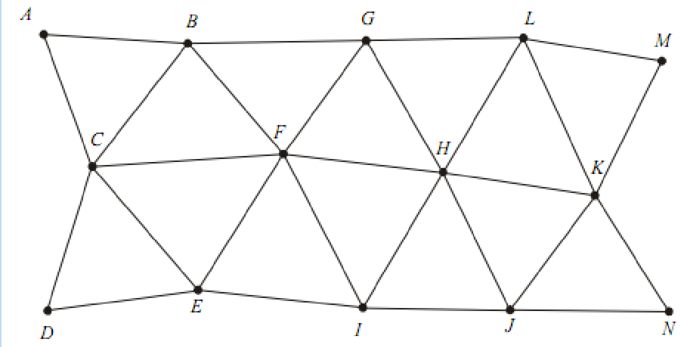

Fig. 1.5 Double chain of triangles

Braced Quadrilaterals:

A triangulation system consisting of figures containing four corner stations and observed diagonals shown in Fig. 1.6 is known as a layout of braced quadrilaterals. In fact, braced quadrilateral consists of overlapping triangles. This system is treated to be the strongest and the best arrangement of triangles, and it provides a means of computing the lengths of the sides using different combinations of sides and angles. Most of the triangulation systems use this arrangement.

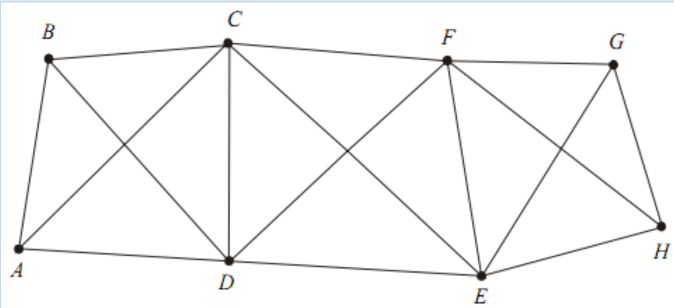

Fig. 1.6 Braced quadrilaterals

CENTERED TRIANGLES AND POLYGONS:

A triangulation system which consists of figures containing interior stations in triangle and polygon as shown in Fig. 1.7, is known as centered triangles and polygons. This layout in a triangulation system is generally used when vast area in all directions is required to be covered. The centered figures generally are quadrilaterals, pentagons, or hexagons with central stations. Though this system provides checks on the accuracy of the work, generally it is not as strong as the braced quadrilateral arrangement. Moreover, the progress of work is quite slow due to the fact that more settings of the instrument are required.

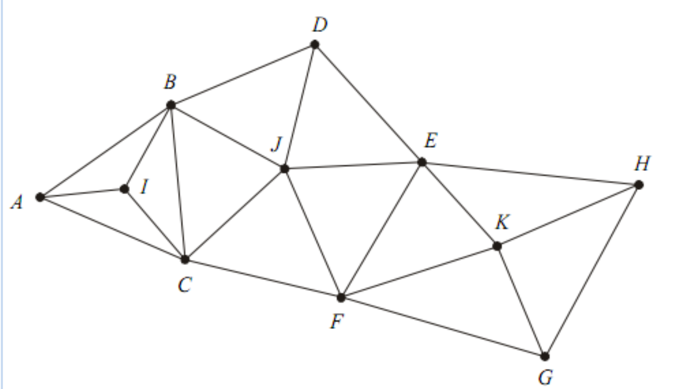

Fig. 1.7 Centered triangles and polygons

A COMBINATION OF ALL ABOVE SYSTEMS

Sometimes a combination of above systems may be used which may be according to the shape of the area and the accuracy requirements

LAYOUT OF PRIMARY TRIANGULATION FOR LARGE COUNTRIES

The following two types of frameworks of primary triangulation are provided for a large country to cover the entire area.

1. Grid iron system
2. Central system

Grid Iron System

In this system, the primary triangulation is laid in series of chains of triangles, which usually runs roughly along meridians (north-south) and along perpendiculars to the meridians (east-west), throughout the country (Fig. 1.8). The distance between two such chains may vary from 150 to 250 km. The area between the parallel and perpendicular series of primary triangulation, are filled by the secondary and tertiary triangulation systems. Grid iron system has been adopted in India and other countries like Austria, Spain, France, etc.

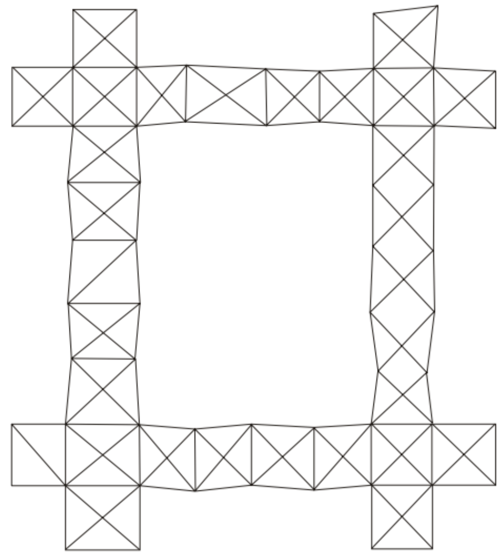

Fig. 1.8 Grid iron system of triangulation

Central system

In this system, the whole area is covered by a network of primary triangulation extending in all directions from the initial triangulation figure ABC, which is generally laid at the centre of the country (Fig. 1.9). This system is generally used for the survey of an area of moderate extent. It has been adopted in United Kingdom and various other countries.

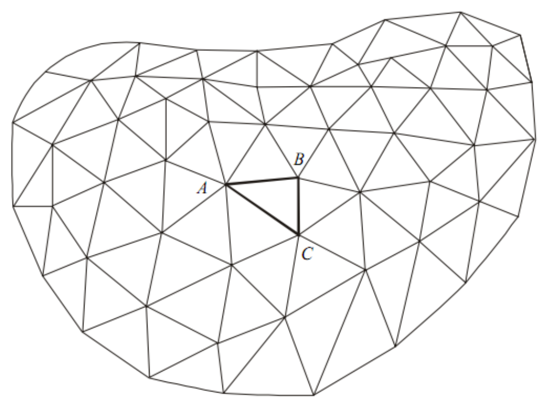

Fig. 1.9 Central system of triangulation

CRITERIA FOR SELECTION OF THE LAYOUT OF TRIANGLES

The under mentioned points should be considered while deciding and selecting a suitable layout of triangles.

1. Simple triangles should be preferably equilateral.
2. Braced quadrilaterals should be preferably approximate squares.
3. Centered polygons should be regular.
4. The arrangement should be such that the computations can be done through two or more independent routes.
5. The arrangement should be such that at least one route and preferably two routes form well-conditioned triangles.

6. No angle of the figure, opposite a known side should be small, whichever end of the series is used for computation.
7. Angles of simple triangles should not be less than 45°, and in the case of quadrilaterals, no angle should be less than 30°. In the case of centered polygons, no angle should be less than 40°.
8. The sides of the figures should be of comparable lengths. Very long lines and very short lines should be avoided.
9. The layout should be such that it requires least work to achieve maximum progress.
10. As far as possible, complex figures should not involve more than 12 conditions.

It may be noted that if a very small angle of a triangle does not fall opposite the known side it does not affect the accuracy of triangulation.

WELL-CONDITIONED TRIANGLES

The accuracy of a triangulation system is greatly affected by the arrangement of triangles in the layout and the magnitude of the angles in individual triangles. The triangles of such a shape, in which any error in angular measurement has a minimum effect upon the computed lengths, is known as well-conditioned triangle. In any triangle of a triangulation system, the length of one side is generally obtained from computation of the adjacent triangle. The error in the other two sides if any, will affect the sides of the triangles whose computation is based upon their values. Due to accumulated errors, entire triangulation system is thus affected thereafter. To ensure that two sides of any triangle are equally affected, these should, therefore, be equal in length. This condition suggests that all the triangles must, therefore, be isoceles. Let us consider an isosceles triangle ABC whose one side AB is of known length (Fig. 1.10). Let A, B, and C be the three angles of the triangle and a, b, and c are the three sides opposite to the angles, respectively.

As the triangle is isosceles, let the sides a and b be equal.

Applying sine rule to ΔABC, we have

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

or

$$a = c\frac{\sin A}{\sin C} \quad \text{.................1.2}$$

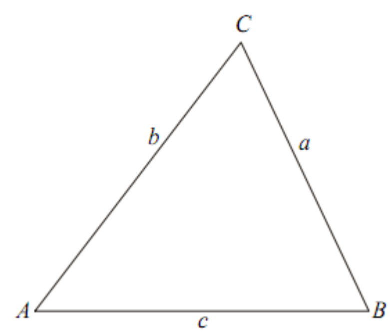

Fig. 1.10 Triangle in a triangulation system

If an error of δA in the angle A, and δC in angle C introduce the errors δa_1 and δa_2, respectively, in the side a, then differentiating above equation partially, we get

$$\delta a_1 = c\frac{\cos A\ \delta A}{\sin C} \quad \text{....................................1.3}$$

and

$$\delta a_2 = -c\frac{\sin A \cos C\ \delta C}{\sin^2 C} \quad \text{....................................1.4}$$

Dividing Eq. (1.3) by Eq. (1.2), we get

$$\frac{\delta a_1}{a} = \delta A \cot A$$

Dividing Eq. (1.4) by Eq. (1.2), we get

$$\frac{\delta a_2}{a} = -\delta C \cot C$$

If $\delta A = \delta C = \pm\alpha$, is the probable error in the angles, then the probable errors in the side a are

$$\frac{\delta a}{a} = \pm\alpha\sqrt{\cot^2 A + \cot^2 C}$$

But $\quad C = 180° - (A + B)$

or $\quad = 180° - 2A, \quad A$ being equal to B.

Therefore $$\frac{\delta a}{a} = \pm\alpha\sqrt{\cot^2 A + \cot^2 2A} \quad ...(1.7)$$

From Eq. (1.7), we find that, if $\frac{\delta a}{a}$ is to be minimum, $(\cot^2 A + \cot^2 2A)$ should be a minimum.

Differentiating $\cot^2 A + \cos^2 2A$ with respect to A, and equating to zero, we have

$$4\cos^4 A + 2\cos^2 A - 1 = 0 \quad ...(1.8)$$

Solving Eq. (1.8), for $\cos A$, we get

$$A = 56°14' (approximately)$$

Hence, the best shape of an isosceles triangle is that triangle whose base angels are $56^0 14'$ each. However trom practical considerations, an equilateral triangle may be treated as a well-conditional triangle. In actual practice, the triangles having an angle less than 30^0 or more than 120^0 should not be considered.

ROUTINE OF TRIANGULATION SURVEY

The routine of triangulation survey, broadly consists of

a) Field work and b) computations.

The field work of triangulation is divided into the following operations:

1. Reconnaissance
2. Erection of signals and towers

SIGNALS

Signals are centered vertically over the station mark, and the observations are made to these signals from other stations. The accuracy of triangulation is entirely dependent on the degree of accuracy of centering the signals. Therefore, it is very essential that the signals are truly vertical, and centered over the station mark. Greatest care of centering the transit over the station mark will be useless, unless some degree of care in centering the signal is impressed upon.

Classification of signals

The signals may be classified as under:

(i) Non-luminous, opaque or daylight signals

(ii) Luminous signals.

Non-luminous signals

Non-luminous signals are used during day time and for short distances. These are of various types, and the most commonly used are of following types.

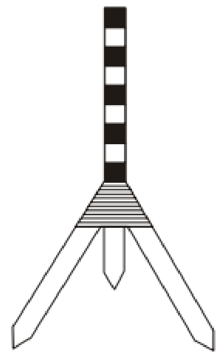

Fig. 1.24 Pole signal

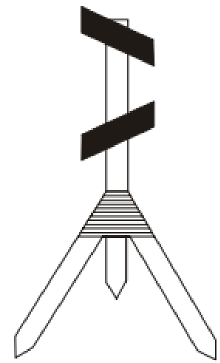

Fig. 1.25 Target signal

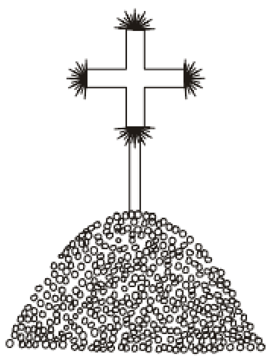

Fig. 1.26 Pole and brush signal

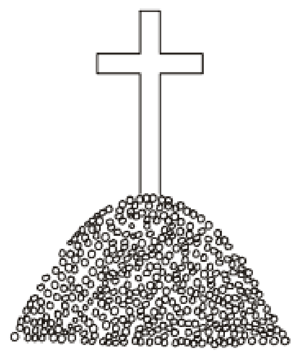

Fig. 1.27 Stone cairn

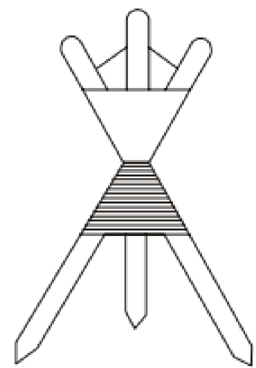

Fig. 1.28 Beacon

(a) Pole signal (Fig. 1.24): It consists of a round pole painted black and white in alternate strips, and is supported vertically over the station mark, generally on a tripod. Pole signals are suitable up to a distance of about 6 km.

(b) Target signal (Fig. 1.25): It consists of a pole carrying two squares or rectangular targets placed at right angles to each other. The targets are generally made of cloth stretched on wooden frames. Target signals are suitable up to a distance of 30 km

(c) Pole and brush signal (Fig. 1.26): It consists of a straight pole about 2.5 m long with a bunch of long grass tied symmetrically round the top making a cross. The signal is erected vertically over the station mark by heaping a pile of stones, up to 1.7 m round the pole. A rough coat of white wash is given to make it more conspicuous to be seen against black background. These signals are very useful, and must be erected over every station of observation during reconnaissance.

(d) Stone cairn (Fig. 1.27): A pile of stone heaped in a conical shape about 3 m high with a cross shape signal erected over the stone heap, is stone cairn. This white washed opaque signal is very useful if the background is dark

(e) Beacons (Fig. 1.28): It consists of red and white cloth tied round the three straight poles. The beacon can easily be centered over the station mark. It is very useful for making simultaneous observations

Luminous signals

Luminous signals may be classified into two types:

(i) Sun signals
(ii) Night signals.

(a) Sun signals (Fig. 1.29): Sun signals reflect the rays of the sun towards the station of observation, and are also known as heliotropes. Such signals can be used only in day time in clear weather

(b) Night signals: When the observations are required to be made at night, the night signals of following types may be used.

1. Various forms of oil lamps with parabolic reflectors for sights less than 80 km.
2. Acetylene lamp designed by Capt. McCaw for sights more than 80 km.
3. Magnesium lamp with parabolic reflectors for long sights.
4. Drummond's light consisting of a small ball of lime placed at the focus of the parabolic reflector, and raised to a very high temperature by impinging on it a stream of oxygen.
5. Electric lamps.

TOWERS

A tower is erected at the triangulation station when the station or the signal or both are to be elevated to make the observations possible form other stations in case of problem of intervisibility. The height of tower depends upon the character of the terrain and the length of the sight. The towers generally have two independent structures. The outer structure is for supporting the observer and the signal whereas the inner one is for supporting the instrument only. The two structures are made entirely independent of each other so that the movement of the observer does not disturb the instrument setting. The two towers may be made of masonary, timber or steel. For small heights, masonary towers are most suitable. Timber scaffolds are most commonly used, and have been constructed to heights over 50 m. Steel towers made of light sections are very portable, and can be easily erected and dismantled

3. Measurement of base line

The accuracy of an entire triangulation system depends on that attained in the measurement of the base line and, therefore, the measurement of base line forms the most important part of the triangulation operations. As base line forms the basis for computations of triangulation system it is laid down with great accuracy in its measurement and alignment. The length of the base line depends upon the grade of the triangulation. The length of the base is also determined by the desirability of securing strong figures in the base net. Ordinarily the longer base, the easier it will be found to secure strong figures.

4. Measurement of horizontal angles

The instruments used for triangulation surveys, require great degree of precision. Horizontal angles are generally measured with an optical or electronic theodolite in primary and secondary triangulation. For tertiary triangulation generally transit or Engineer's transit having least count of 20" is used. The horizontal angles of a triangulation system can be observed by the following methods:

(i) Repetition method

(ii) Reiteration method

5. Measurement of vertical angles

   Measurement of vertical angles is required to compute the elevation of the triangulation stations

6. Astronomical observations to determine the azimuth of the lines.

   To determine the azimuth of the initial side, intermediate sides, and the last side of the triangulation net, astronomical observations are made