JUMPING BEANS

Abstract
Problem: A simple toy called a "jumping bean" can be constructed by putting a metal ball inside of a pill capsule. Placed on an inclined surface at a certain inclination, the jumping bean will tumble down in a rather surprising way, seemingly standing up-right, flipping end to end, instead of rolling. Investigate its motion. Find the dimensions of the fastest and slowest beans for a given inclination.

Introduction

Why should the capsule topple instead of rolling?
When the capsule rolls, one side tilts due to the presence of the ball. This causes the the heavier end to move faster hence aligning the capsule towards a toppling motion. Once aligned, toppling is favored due to the constrained motion of the ball. And thus the capsule topples instead of rolling given that the ball transfers sufficient energy for topple. Otherwise, it falls sideways and rolls down.

Theoretical Model

The tumbling of the ball down an incline can be divided into two sub-systems. System 1 is when the ball rolls from one end of the static capsule to the other end with a velocity $\dot{\theta}_1$. As soon as the ball collides with the capsule, assuming perfectly elastic collision and taking into account the energy loss due to the presence of friction between the ball and the capsule, there is seen a transition from system 1 to 2. System 2: The ball capsule system now moves with a velocity $\dot{\theta}_2$ and the capsule topples with the centre of toppling coincident with the centre of rolling of the ball. As soon as the capsule falls down, assuming that the capsule-ball system collision with the surface of incline is significantly inelastic, the capsule rests static and system 1 is re-achieved.

Reference Formulas

Reye's hypothesis [1]:

$$G = N\left[\int_{s_1} \frac{dA}{v^r(x)}\right]^{-1} \int_{s_1}\left[\frac{dA}{v^r(x)}\int^{v^r(x)} \mu dv\right]$$

which is the Rayleigh's dissipation equation such that dissipation force $F = -\nabla_v G$, where v: velocity of object, v^r: velocity of object relative to surface, dA: area element on object, μ: coefficient of dissipation. In this jumping beans problem, μ has been assumed to be independent of speed

Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \frac{\partial G}{\partial \dot{q}_j} = \frac{\partial L}{\partial q_j}$$

Lagrangian:

$$L = T - U,\; E = T + U$$

where T = Kinetic Energy and U = Potential Energy

Modified Lagrangian:
L' and Energy E':

$$L' = L + \int G dt,\; E' = E - \int G dt$$

such that:

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}_j} = \frac{\partial L'}{\partial q_j}$$

System 1

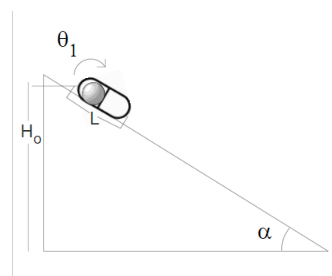

Figure 1. System 1

$$G_1 = \mu_1 mgR\dot{\theta}_1 \cos\alpha,\; T_1 = \frac{1}{2}I_{ball}\dot{\theta}_1^2 + \frac{1}{2}mR^2\dot{\theta}_1{}^2,\; I_{ball} = \frac{2}{5}mR^2$$

$$U_1 = mg\left[H_0 - R\sin\alpha\right] - mgR\theta_1 \sin\alpha + Mg\left[H_0 - \frac{L}{2}\sin\alpha\right]$$

Therefore equation of motion is:

$$\ddot{\theta}_1 = \frac{-mgR\left[\mu_1\cos\alpha - \sin\alpha\right]}{mR^2 + I_{ball}}$$

The total energy after subtracting the dissipated energy is:

$$E_1' = E_1 - \mu_1\, \mathrm{mg}R\theta_1 \cos\alpha,$$

where α: angle of inclination, H_0: reference height of capsule's upper edge, m, M: mass of ball, capsule, I_{ball}: moment of inertia of ball, L: total length of capsule, R: radius of ball, capsule (throughout assumed same), θ_1: position coordinate of ball.

System 2

$$G_2 = \mu_2(m+M)gR\dot{\theta}_2\cos\alpha,\; T_2 = \frac{1}{2}I_{sys}\dot{\theta}_2^2 + \frac{1}{2}(m+M)R^2\dot{\theta}_2^2$$

$$I_{sys} = I_{ball} + I_{cap},\; I_{cap} = I_{lss} + I_{cyl} + I_{rss}$$

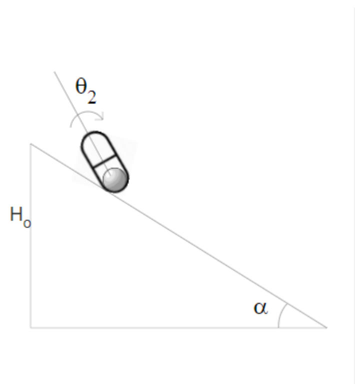

Figure 2. System 2

$$I_{cyl} = \frac{1}{4}M_{cyl}R^2 + \frac{1}{3}M_{cyl}(L-2R)^2,\ M_{cyl} = 2\pi R(L-2R)\sigma_{cap}$$

$$I_{rss} = \frac{1}{3}M_{rss}R^2,\ M_{rss} = 2\pi R^2\sigma_{cap}$$

$$I_{lss} = \frac{1}{12}M_{lss}R^2 + M_{lss}(\frac{R}{2} + (L-2R)^2,\ M_{lss} = 2\pi R^2\sigma_{cap}$$

$$U_2 = mg\left[H_0 - R\sin\alpha - (L-2R)\sin\alpha\right] - mgR\theta_2\sin\alpha$$

$$+Mg\left[H_0 - \frac{L}{2}\sin\alpha\right] + (m+M)gh - MgR\theta_2\sin\alpha$$

where I_{sys}: moment of inertia of system 2, I_{cap}: moment of inertia of the capsule, I_{lss}, I_{rss}: moment of inertia of the left, right semi sphere of the capsule, I_{cyl}: moment of inertia of the cylindrical body of capsule (all about the centre of toppling), θ_2: position coordinate of the rolling ball, toppling capsule, $h = (d-b)sin(\alpha+\theta_2)$: rising height of COM of capsule.

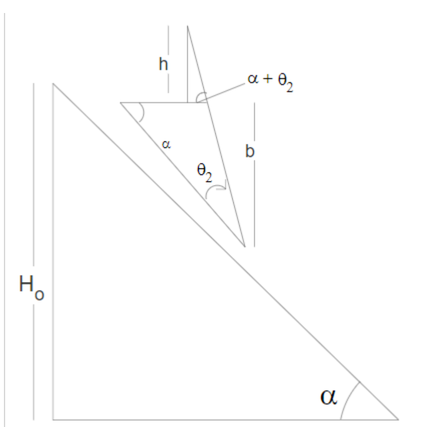

Figure 3. Vertical change in height of COM of system

Distance to the center of mass of the system:

$$d = \left[\frac{mR + \frac{ML}{2}}{m+M} - R\right] = \frac{M\left[\frac{1}{2}[L-2R]\right]}{m+M}$$

Through geometry: (law of sines)

$$b = \frac{d\sin\alpha}{\sin(\alpha+\theta_2)} \Rightarrow h = d\left(\sin(\alpha+\theta_2) - \sin\alpha\right)$$

$$\Rightarrow U_2 = (m+M)g\left[H_0 - (L-2R)\sin\alpha - R\sin\alpha\right]$$

$$-(m+M)gR\theta_2\sin\alpha + Mg\left[\frac{1}{2}[L-2R]\sin(\alpha+\theta_2)\right]$$

Therefore the equation of motion is:

$$\ddot{\theta}_2 = P + Q\cos(\alpha+\theta_2)$$

$$P = \frac{(m+M)gR\sin\alpha - \mu_2(m+M)gR\cos\alpha}{I_{sys} + (m+M)R^2}$$

$$Q = \frac{-Mg\left[\frac{1}{2}[L-2R])\right]}{I_{sys} + (m+M)R^2}$$

We assume totally elastic collision between the ball and the capsule in the transition from system 1 to 2, considering the energy lost due to friction between the ball and the capsule, $E_2 = T_2 + U_2$ must be set numerically equal to E_1' at $\theta_2 = 0$ which gives:

$$\dot{\theta}_2{}^2 = \frac{\left[E_1' - (m+M)g\left[H_1 - R\sin\alpha\right] - Mg\left[\frac{1}{2}(L-2R)\sin\alpha\right]\right.}{\left[\frac{1}{2}I_{sys} + \frac{1}{2}(m+M)R^2\right]}$$

Experimental setup

A wooden board and two metal stands with clamps were used to make an inclined surface. We used a mobile app "Smart Kit" to measure the angle of inclination and to ensure that the board is levelled. The app was calibrated with manual readings initially.

For the board to have a uniform coefficient of friction we used emery paper to polish the wooden board's surface.The starting point was marked on the board and it was held constant throughout all the experiments. The sideways trajectory of the capsule was filmed via a camera. The time taken was noted down by a tracking software.

Different sizes of jumping beans were released (perpendicular to the board) from the marked line at the top of the board with the help of a wooden barrier, so that all the specimens have the same initial conditions. No initial push was given to any of the capsules.This was repeated for different inclinations for which tumbling motion was seen.

The method of inclined plane was used to calculate the coefficients of friction between the ball and the capsule while it rolls, and the capsule-ball system and the wooden board while it topples.

Numerical Solutions for Time

Algorithm and Assumptions:

The obtained equations of motions $\ddot{\theta}_1$ and $\ddot{\theta}_2$ are solved numerically using Euler method to get the time taken to topple down a certain incline. [2] There are some assumptions we made which are under discussion ahead:

Assumption 1: The collision of the ball with the lower end of capsule is assumed to be totally elastic. This assumption restricts $\dot{\theta}_2$ to be such that at $\theta_2 = 0$, $E_2 = E_1'$.

Assumption 2: We take this assumption that the collision of the toppling capsule-ball system with the surface is significantly inelastic. This implies that a certain fraction of energy is left in the system and it decides what value $\dot{\theta}_1$ will take whenever a transition occurs from system 1 to 2. One may approach to solve for this fraction empirically but for that we did not have high speed cameras and thus an attempt to find that fraction which also seems to be dependent on the angle of inclination was not made (evident from the relative time differences between the experimental times and the simulated times (fig.4); as angle alpha increases the experimental curve converges to the theoretical curve).

Algorithm 1: Assuming degree of inelastic collision

```
Data: :
m, M, L, R, H_0, α, μ_1, μ_2
initialization of I_ball, I_sys
pos_ref = length of incline surface, dt = 0.000001
while True do
    if pos_ball − pos_ref >= L − 2R then
        initialize E'_1
        initialize θ̇_2
        while 0 <= θ_2 <= π do
            dθ_2 = θ_2
            θ_2 += θ̇_2 * dt
            dθ_2 = θ_2 − dθ_2
            θ̇_2 += θ̈_2 * dt
            t += dt
            pos_ball += R * dθ_2
            if pos_ball > length of board then
                break;
        θ_1 = 0
        if completely inelastic collision assumed then
            θ̇_1 = 0
        if significantly inelastic collision assumed then
            θ̇_1 is such that a significant fraction of energy gets
              dissipated and a certain fraction of energy left in
              the system decides what value θ̇_1 will take
        H = H − (L − 2R)sin(α) − πRsin(α)
        pos_ref = pos_ball
        if pos_ball > length of board then
            break
        ‘
    if pos_ball > length of board then
        break
    dθ_1 = θ_1
    θ_1 += θ̇_1 * dt
    dθ_1 = θ_1 − dθ_1
    θ̇_1 += θ̈_1 * dt
    t += dt
    pos_ball += R * dθ_2
Result: time taken to topple down incline: t
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Assumption 2.1: Using assumption 1 and the fact that $T_{1cap} = 0$; $T_{1ball} = T_{2cap} + T_{2ball}$. Assuming the energy of capsule to completely dissipate during transition from system 1 to 2, the initial kinetic energy of the ball after this transition can be roughly approximated to be equal to the final kinetic energy of the ball and hence $\dot{\theta}_{1i}(present\,system\,1) = \dot{\theta}_{1f}(previous\,system\,1)$. Applying this assumption, the simulated $t\ vs\ \alpha$ plot is an asymptotic curve to the experimental data.

Assumption 2.2: Up-till this point, we had not included the alpha dependency of energy dissipation. More energy seems to be dissipated at lower angles of inclination. For this we introduce an alpha dependent data fitting factor.

Data Fitting Factor: We catered to this alpha dependency by introducing an alpha dependent function in the dissipation equation. $f(\alpha) = 1/sin\alpha$ To introduce this factor in our simulation, since we noted that for a certain α, this data fitting factor is always accompanied by μ, we initialed $\mu(\alpha)$, a data fitting constant for a given α, as $\mu(\alpha) = \mu/sin\alpha$

Comparative Times

Fig.5 shows the averaged experimental data for several capsules. An exponential decay in the total times can be seen with an increase in inclination for all capsules. All the graph lines reach an asymptotic behavior, which can be explained by the fact that at large angles, all the capsules fall the same way under the influence of gravity (free fall), yielding approximately equal total times.

Fig.6 shows the simulated data for the same capsules using the same conditions as those present in the experiments. The theoretical prediction shows the same general behavior as the experimental one, yielding the same converging asymptotic behavior at large inclinations. This prediction can be modified with better assumptions used in the theoretical model.

Optimization

Now since the model is fitting with experimental data for all the experiments that were conducted. The equation of motion can be used to see what effect varying the parameters of the capsule system has on the total time taken. Hence dimensions for fastest and slowest beans can be calculated, with the assumption that the densities of the ball and capsule are kept fixed, and the ball has approximately the same diameter as the capsule.

Another way to find the dimensions for the minimum time numerically, is to write a program that takes an n-dimensional function and minimizes it, where n is the number of parameters we wish to test for our system. For example, length of the capsule, diameter of the ball, density of the ball/capsule or the given inclination. We take random points in the n dimensional hyperspace and treating each point as the starting point, minimize the function for each. We make sure that the initial values taken are within respectable ranges i.e they don't represent a totally unphysical situation (we checked for only two parameters, the length of the capsule and the diameter of the ball) and are generally within the confines of experiment. The program

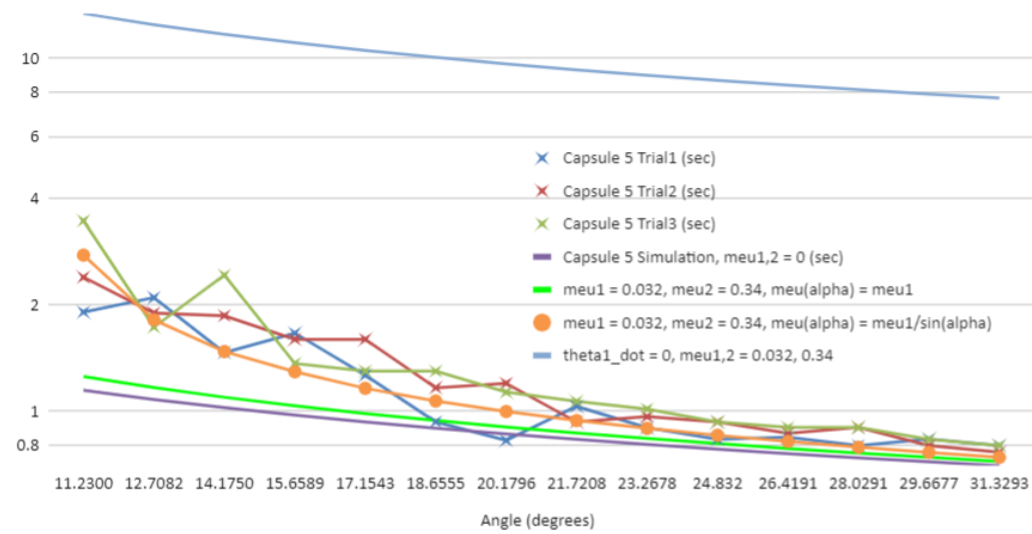

Figure 4. Experimental Times: x, Simulated Times: -, Best Fit Simulated Times: o

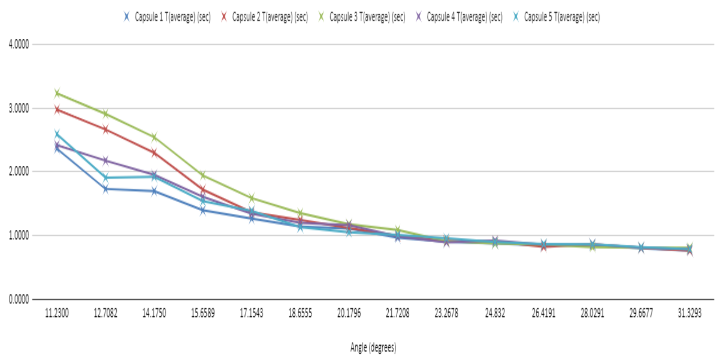

Figure 5. Experimental data for total time

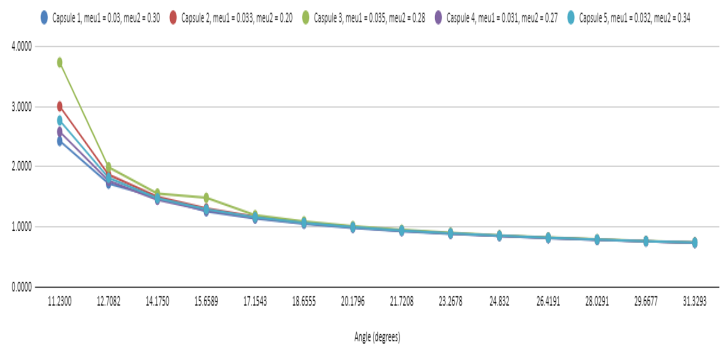

Figure 6. Simulated data for total time

then gives locals minimas for each starting point [given it finds one within a given number of iterations (say 50) and tolerance]. The successful trials are kept and amongst those we check for the least time taken. If we restrict ourselves to the ranges we tested experimentally, the results match the experiment. This can be checked against the experimental analysis as well. By plotting a 3D contour map or a 3D surface plot you can see where the time minimum lies, which we do in the experiment analysis section. The optimization code is attached [2].

Experimental Results

To have a variety of capsules for optimizing the problem at hand, we used aluminum foil to customize the dimensions, the aluminum foil proved a good choice to make our capsules even though it deformed in slightly different orientations, thus varying the dimensions of the capsules a little bit from our required dimensions. We used sticks of 3 different diameters and wrapped the aluminium foil twice around them to make the capsules. We did many experiments, varying length of the capsule, the diameter of the ball and the inclination of the board, in the following ranges:

- diameter from 4.8 mm to 9.5 mm.
- length from 17 mm to 32.5 mm.
- inclination from 5 degrees to 31 degrees.
- density of the ball and the capsule were approximately constant.

Length:
As the length of the capsule increases, the ball remains under the influence of the frictional force for a longer duration which takes away a larger portion of its energy. The moment of inertia of the capsule also increases and hence, it requires greater energy to topple. After a certain length the capsule wont flip (for a given inclination). As seen in the figure 7; the time on average increases with increasing length. The ball takes longer to travel the length of the capsule and, as stated above, the energy transferred to the system after collision is less, so it takes longer for the system to topple.

Diameter:
As the diameter of the ball increases, so does its mass, assuming constant density. So, for the same length on average, the time decreases (figure 8). More mass means more energy is attained by the ball in its motion spanning the length of the capsule. That energy is then transferred to the system, in the next step as the capsule topples. As that initial energy increases, the capsule will topple over more quickly hence decreasing the overall time. As in figure, here with constant length and density, the ball with the largest diameter takes the least time. This pattern holds for all inclinations.

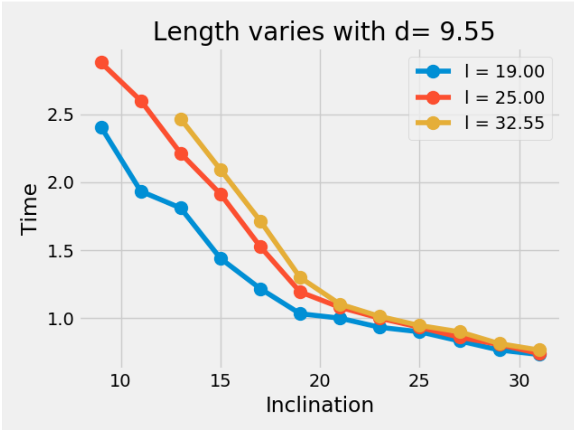

Figure 7. Constant diameter for variable length.

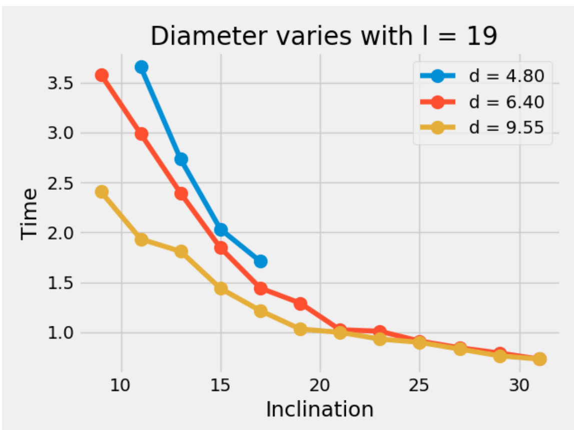

Figure 8. Constant length with variable diameter

Diameter and Length:

Now studied independently the two parameters show the behavior discussed above. What happens when both vary? What is that optimum length and diameter for which the system will complete its motion the fastest. A larger diameter doesn't mean the that time will increase regardless. The length of the capsule will put restrictions on how much the time will increase . That is to say, there comes a certain value when the length is too much that even a bigger diameter doesn't guarantee that the capsule will get there faster. From all the experiment we have conducted this comes out to be in somewhere between 17.5 ± 0.5 mm to 19.5 ± 0.5 mm of length and the diameter corresponding to that length was 9.55 mm.

Discussion

A few important parts ought to be revisited in the theoretical model. One is the assumption of the inclination varying friction and the other is assumption for the value of $\dot{\theta}_1$ as System 1 starts again after one flip.

The board inclination method allows for only one type of friction to be calculated; it does not cater to the surface variation of friction and of the contact forces.

Ideally, the energy transfer fractions should be calculated which would experimentally give the amount of energy transferred to System 1 after the capsule-ball system hits the surface. This could be done through the use of motion tracking software, and was tried for a number of capsules with the use of the Tracker software.However, the limitations of the camera frame speed and the lack of a better equipment questions the validity of such attempts, giving large uncertainty. Catering to this part of the problem might nullify the need for the two assumptions made in the theoretical model.

The simulation has been tested for the majority of the experiments conducted and yields closely fitting data, thus aptly predicting the solution to the problem question.

The consistency of aluminum could be improved.The surface structure of the aluminum capsule went under some changes over the course of the experiments due to repeated trials. However, the results still followed an explicit trend.

The aluminum capsules were prepared with the method used in the video linked with the problem.

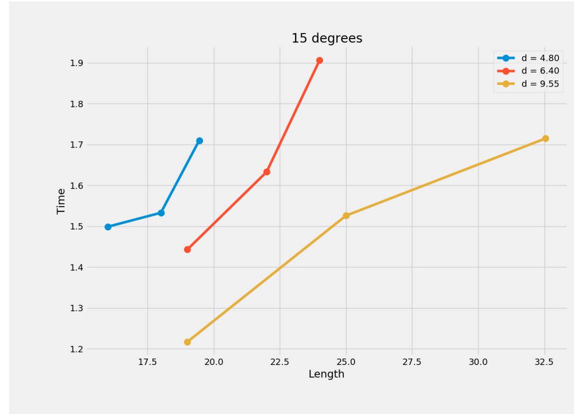

Figure 9. At 15 degrees, the variation between time and length for three different diameters. Here only one case is plotted but in general, this holds for every inclination.

Conclusion

Jumping beans were analyzed experimentally and a model was built around the observations and the equations were modified and simulated to give the close fitting results.The characteristic motion of the jumping bean was conferred to be due to the periodic energy transfers down the incline. We found, for our range of experiments, that the smaller the length and greater the diameter of the ball, the faster it travels down the incline.

References

[1] E. Minguzzi, *Rayleigh's dissipation function at work*. arXiv:1409.4041v2 [physics.class-ph] (2015)

[2] Data, some further plots, calculations plus the optimization and simulation codes.