Jumping Beans

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Problem statement

A simple toy called a "jumping bean" can be constructed by putting a metal ball inside of a pill capsule. Placed on an inclined surface at a certain inclination, the jumping bean will tumble down in a rather surprising way, seemingly standing up-right, flipping end to end, instead of rolling. Investigate its motion. Find the dimensions of the fastest and slowest beans for a given inclination.

- Why the capsule topples instead of rolling?
- Explain the dynamics of the tumbling motion.
- For a given inclination, what are the dimensions for the fastest and slowest beans as observed experimentally?

Theoretical Model

Reference Formulas

Reye's hypothesis: Rayleigh's dissipation equation[1]

$$G = N\left[\int_{s_1} \frac{dA}{v^r(x)}\right]^{-1} \int_{s_1} \left[\frac{dA}{v^r(x)} \int^{v^r(x)} \mu dv\right], F = -\nabla_v G$$

Modified Lagrangian: L' and Energy E':

$$L' = L + \int G dt, \;\; E' = E - \int G dt$$

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}_j} = \frac{\partial L'}{\partial q_j}$$

here G: dissipation function, v: velocity of object,
v^r: velocity of object relative to surface, dA: area element on object,
μ: coefficient of dissipation (μ has been assumed to be independent of speed)
q_j: generalized coordinate

[1] E. Minguzzi. "Rayleigh's dissipation function at work". In: *arXiv:1409.4041v2 [physics.class-ph])* (2015).

Why Topple instead of Roll:

- The capsule can roll downward, with the ball inside rolling downward as well.
- A slight shift of the ball from the center of the capsule will cause the tilted end of the capsule to come forward, and this would yield a topple, continuing it again and again.

The tumbling of the ball down an incline can be divided into two sub-systems.

- System 1
- System 2

$$G_1 = \mu_1 mgR\dot{\theta}_1 \cos\alpha$$

Equation of motion:

$$\ddot{\theta}_1 = \frac{-mgR\left[\mu_1 \cos\alpha - \sin\alpha\right]}{mR^2 + I_{ball}}$$

The total energy after subtracting the dissipated energy:

$$E_1' = E_1 - \mu_1 \,\mathrm{mg}\, R\theta_1 \cos\alpha,$$

m, M: mass of ball, capsule, I_{ball}: moment of inertia of ball,
L: total length of capsule,
R: radius of ball = of capsule

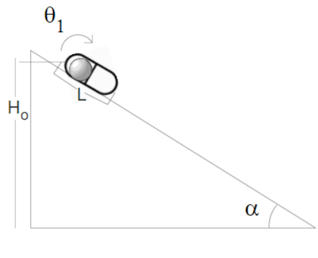


Figure: System 1

here,

$H_\circ$: height of fall of capsule

α: angle of inclination

θ_1: position coordinate of ball.

System 2

$$G_2 = \mu_2(m + M)gR\dot{\theta}_2 \cos\alpha$$

$$\ddot{\theta}_2 = P + Q\cos(\alpha + \theta_2)$$

$$Q = \frac{-Mg\left[\frac{1}{2}[L - 2R])\right]}{I_{sys} + (m + M)R^2}$$

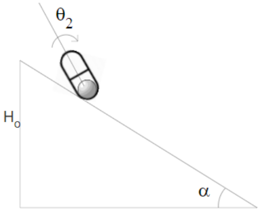


Figure: System 2

$$P = \frac{(m + M)gR\sin\alpha - \mu_2(m + M)gR\cos\alpha}{I_{sys} + (m + M)R^2}$$

here, θ_2: position coordinate of the rolling ball, toppling capsule

I_{sys}: moment of inertia of system 2

Numerical Solutions

The obtained equations of motions $\ddot{\theta}_1$ and $\ddot{\theta}_2$ are solved numerically using **Euler method** to get the time taken to topple down a certain incline.

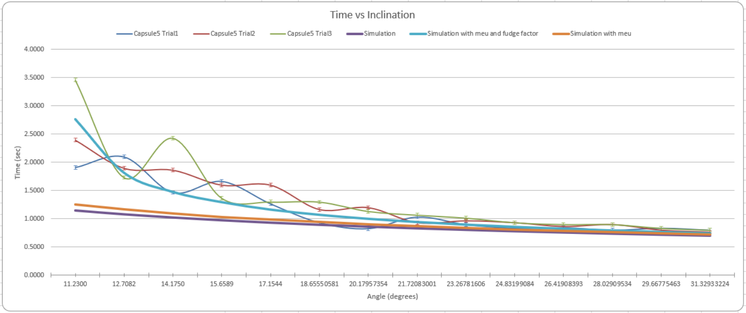
Initialization of $\dot{\theta}_2$: The collision of the ball with the lower end of capsule is assumed to be totally elastic. This assumption restricts $\dot{\theta}_2$ to be such that at $\theta_2 = 0$ (the moment of transition from system 1 to 2), $E_2 = E_1'$.

$$\dot{\theta}_2{}^2 = \frac{\left[E_1' - (m+M)g\left[H_1 - R\sin\alpha\right] - Mg\left[\frac{1}{2}(L-2R)\sin\alpha\right]\right.}{\left[\frac{1}{2}I_{sys} + \frac{1}{2}(m+M)R^2\right]}$$

Initialization of $\dot{\theta}_1$: In transition from system 2 to 1,
$\dot{\theta}_1$(of current system 1) = $\dot{\theta}_1$(of previous system 1)
Applying this assumption, the simulated t *vs* α plot is an asymptotic curve to the experimental data.

Numerical Solutions for Time

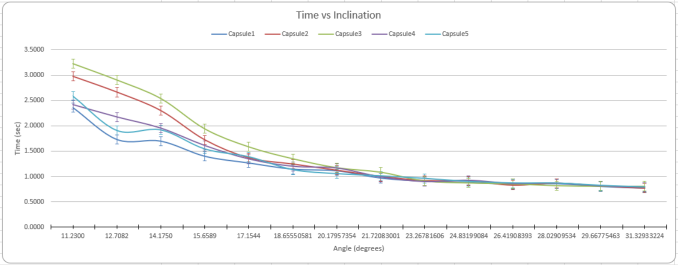
Fudge Factor



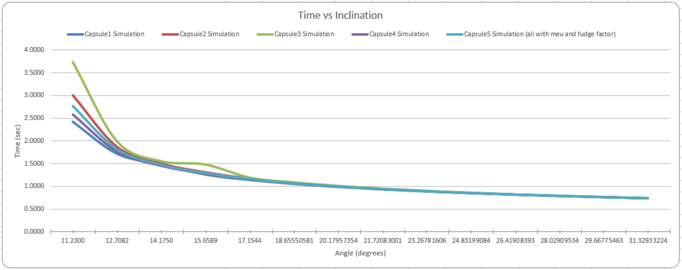
Fudge Factor:

- As the angle α increases the experimental curve converges to the theoretical curve.
- We introduce $f(\alpha) = 1/sin\alpha$, a data fitting factor, in the dissipation equation.

The simulation now fits with the experimental results.

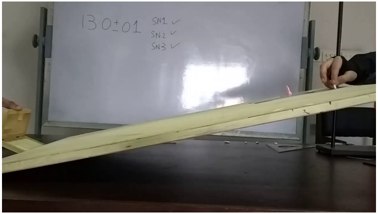


Time vs Inclination
Capsule1
Capsule2
Capsule3
Capsule4
Capsule5
Time (sec)
3.5000
3.0000
2.5000
2.0000
1.5000
1.0000
0.5000
0.0000
11.2300
12.7082
14.1750
15.6589
17.1544
18.65550581
20.17957354
21.72083001
23.26781606
24.83199084
26.41908393
28.02909534
29.66775463
31.32933224
Angle (degrees)



Time vs Inclination
Capsule1 Simulation
Capsule2 Simulation
Capsule3 Simulation
Capsule4 Simulation
Capsule5 Simulation (all with meu and fudge factor)
Time (sec)
4.0000
3.5000
3.0000
2.5000
2.0000
1.5000
1.0000
0.5000
0.0000
11.2300
12.7082
14.1750
15.6589
17.1544
18.65550581
20.17957354
21.72083001
23.26781606
24.83199084
26.41908393
28.02909534
29.66775463
31.32933224
Angle (degrees)

Standardized Experiments



Apparatus:

- Aluminium Foil
- Ball bearings of different radii
- Wooden Plank
- Two Stands

Parameters Varied:

- Diameter (4.8mm - 9.5mm)
- Length (17mm - 32.5mm)
- Inclination (5° - 31°)

Standardized Experiments

Diameter and Length

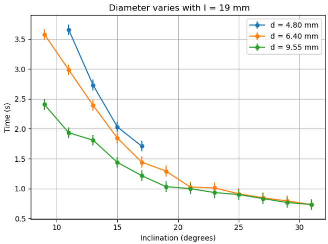


Figure: Constant length with variable diameter

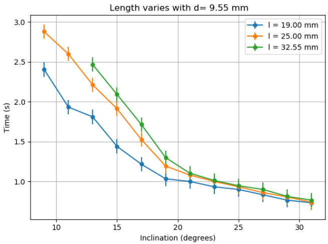
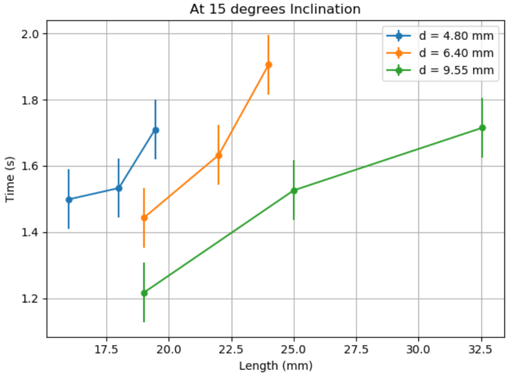


Figure: Constant diameter for variable length.



The variable parameters of the experiment are not mutually exclusive, and thus cannot be kept constant while changing a related quantity. e.g mass changes with length and diameter etc.

Standardized Experiments

Extreme times at different inclinations

	Time Taken to Topple Down Incline (sec)						colour key
Angle	9 deg	11 deg	13 deg	15 deg	17 deg	19 deg	slowest
SN1	3.954	2.921	2.333	1.888	1.499	1.355	fastest
SN2	4.054	3.188	2.462	2.01	1.533	1.299	
SN3	-	3.654	2.732	2.032	1.71	-	
M1	3.245	2.392	2.039	1.633	1.293	1	
M2	3.892	3.219	2.772	2.119	1.906	1.372	
L1	2.406	1.933	1.81	-	1.217	1.033	
L2	2.879	2.599	2.212	1.193	1.525	1.193	
L3	-	-	2.466	2.092	1.715	1.299	

Standardized Experiments

Dimensions of Different Sets

SN1		
Parameters	Values	Uncertainities
Length of Capsule / mm	16	0.05
Diameter of Capsule / mm	6	0.05
Diameter of Ball / mm	4.8	0.05
Mass of Cap + Ball / g	484	1
Mass of Ball/g	0.45	0.01
Mass of cap / mg	34	1

SN2		
Parameters	Values	Uncertainities
Length of Capsule/mm	18	0.05
Diameter of Capsule/mm	6.1	0.05
Diameter of Ball/mm	4.8	0.05
Mass of Cap + Ball/mg	486	1
Mass of Ball/g	0.45	0.01
Mass of cap / mg	36	1

SN3		
Parameters	Values	Uncertainities
Length of Capsule/mm	19.45	0.05
Diameter of Capsule/mm	6.1	0.05
Diameter of Ball/mm	4.8	0.05
Mass of Cap + Ball/mg	487	1
Mass of Ball/g	0.45	0.01
Mass of cap / mg	37	1

M1		
Parameters	Values	Uncertainities
Length of Capsule/mm	22	0.05
Diameter of Capsule/mm	8	0.05
Diameter of Ball/mm	6.4	0.05
Mass of Cap + Ball/mg	1138	1
Mass of Ball/g	1.05	0.01
Mass of cap / mg	88	1

M2		
Parameters	Values	Uncertainities
Length of Capsule/mm	24	0.05
Diameter of Capsule/mm	8.25	0.05
Diameter of Ball/mm	6.4	0.05
Mass of Cap + Ball/mg	1149	1
Mass of Ball/g	1.05	0.01
Mass of cap / mg	99	1

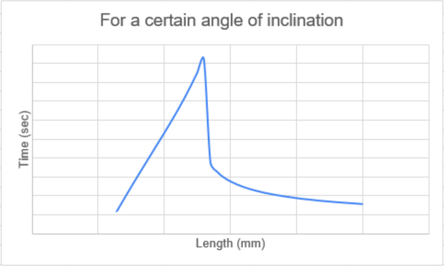
L1		
Parameters	Values	Uncertainities
Length of Capsule/mm	19	0.05
Diameter of Capsule/mm	10.7	0.05
Diameter of Ball/mm	9.55	0.05
Mass of Cap + Ball/mg	3783	1
Mass of Ball/g	3.53	0.01
Mass of cap / mg	253	1

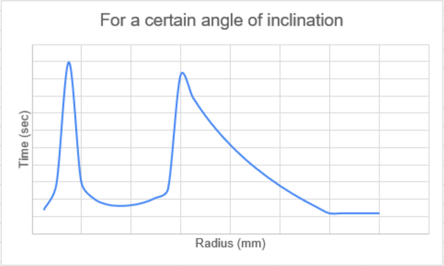
L2		
Parameters	Values	Uncertainities
Length of Capsule/mm	25	0.05
Diameter of Capsule/mm	10.7	0.05
Diameter of Ball/mm	9.55	0.05
Mass of Cap + Ball/mg	3827	1
Mass of Ball/g	3.53	0.01
Mass of cap / mg	297	1

L3		
Parameters	Values	Uncertainities
Length of Capsule/mm	32.55	0.05
Diameter of Capsule/mm	10.7	0.05
Diameter of Ball/mm	9.55	0.05
Mass of Cap + Ball/mg	3892	1
Mass of Ball/g	3.53	0.01
Mass of cap / mg	342	1

Numerical optimization

We can fix certain parameters in the simulation code and run it for a certain varying parameter, giving us a general trend for that parameter.





- We can follow the same approach as mentioned before, and optimize each variable parameter separately.
- We can combine the results to give us a complete optimization involving all parameters in N-dimensions.
- This way we can get the dimensions corresponding to extreme times.

Improvements

Improvements

- A better fudge factor can be thought of by analysing the fraction of energy transferred in every fall of the capsule using high speed cameras.
- The consistency of aluminum could be improved as the surface structure of the aluminum capsule deformed over the course of the experiments due to repeated trials.

THANK YOU

Appendix

Reye's hypothesis:

$$G = N\left[\int_{s_1} \frac{dA}{v^r(x)}\right]^{-1} \int_{s_1} \left[\frac{dA}{v^r(x)} \int^{v^r(x)} \mu dv\right]$$

which is the Rayleigh's dissipation equation such that dissipation force $F = -\nabla_v G$,

here

G: dissipation function

v: velocity of object,

v^r: velocity of object relative to surface,

dA: area element on object,

μ: coefficient of dissipation.

μ has been assumed to be independent of speed

$$G_1 = \mu_1 mgR\dot{\theta}_1 \cos\alpha, \; G_2 = \mu_2(m+M)gR\dot{\theta}_2 \cos\alpha$$

Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \frac{\partial G}{\partial \dot{q}_j} = \frac{\partial L}{\partial q_j}$$

Lagrangian:

$$L = T - U, \; E = T + U$$

here,
T = Kinetic Energy, U = Potential Energy, E = Total Energy
q_j = generalized coordinate, $\dot{q}_j = \frac{dq_j}{dt}$ = generalized velocity

Modified Lagrangian: L' and Energy E':

$$L' = L + \int G dt, \; E' = E - \int G dt$$

such that:

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}_j} = \frac{\partial L'}{\partial q_j}$$

Theoretical Model

The tumbling of the ball down an incline can be divided into two sub-systems.

- ***System* – 1** is when the ball rolls from one end of the static capsule to the other end with a velocity $\dot{\theta}_1$.
- As soon as the ball collides with the capsule, assuming perfectly elastic collision and taking into account the energy loss due to the presence of friction between the ball and the capsule, there is seen a transition from system 1 to 2.
- ***System* – 2**: The ball capsule system now moves with a velocity $\dot{\theta}_2$ and the capsule topples with the centre of toppling coincident with the centre of rolling of the ball.
- As soon as the capsule falls down, assuming that the capsule-ball system collision with the surface of incline is significantly inelastic, the capsule rests static and system 1 is re-achieved.

$$G_1 = \mu_1 mgR\dot{\theta}_1 \cos\alpha$$

$$T_1 = \frac{1}{2} I_{ball}\dot{\theta}_1^2 + \frac{1}{2} mR^2\dot{\theta}_1{}^2$$

$$I_{ball} = \frac{2}{5} mR^2$$

$$U_1 = mg\left[H_0 - R\sin\alpha\right]$$

$$-mgR\theta_1 \sin\alpha$$

$$+Mg\left[H_0 - \frac{L}{2}\sin\alpha\right]$$

m, M: mass of ball, capsule, I_{ball}: moment of inertia of ball,
L: total length of capsule,

R: radius of ball = of capsule

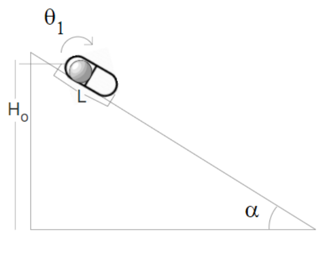


Figure: System 1

here,

$H_\circ$: height of fall of capsule

α: angle of inclination

θ_1: position coordinate of ball.

System 1

Using the Euler-Lagrange Equations, we get the equation of motion as follows:

$$\ddot{\theta}_1 = \frac{-mgR\left[\mu_1 \cos\alpha - \sin\alpha\right]}{mR^2 + I_{ball}}$$

The total energy after subtracting the dissipated energy (i.e. the modified energy) is:

$$E_1' = E_1 - \mu_1 \,\mathrm{mg}\, R\theta_1 \cos\alpha,$$

System 2

$$G_2 = \mu_2(m+M)gR\dot{\theta}_2 \cos\alpha$$

$$T_2 = \frac{1}{2} I_{sys}\dot{\theta}_2^2 + \frac{1}{2}(m+M)R^2\dot{\theta}_2^2$$

$$I_{sys} = I_{ball} + I_{cap}$$

$$I_{cap} = I_{lss} + I_{cyl} + I_{rss}$$

I_{sys}: moment of inertia of system 2, I_{cap}: moment of inertia of the capsule, I_{lss}, I_{rss}: moment of inertia of the left, right semi sphere of the capsule,

I_{cyl}: moment of inertia of the cylindrical body of capsule (all about the centre of toppling)

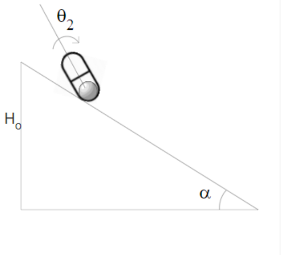


Figure: System 2

here,

θ_2: position coordinate of the rolling ball, toppling capsule

System 2

$$I_{cyl} = \frac{1}{4}M_{cyl}R^2 + \frac{1}{3}M_{cyl}(L-2R)^2,\ M_{cyl} = 2\pi R(L-2R)\sigma_{cap}$$

$$I_{rss} = \frac{1}{3}M_{rss}R^2,\ M_{rss} = 2\pi R^2\sigma_{cap}$$

$$I_{lss} = \frac{1}{12}M_{lss}R^2 + M_{lss}(\frac{R}{2} + (L-2R)^2,\ M_{lss} = 2\pi R^2\sigma_{cap}$$

$$U_2 = mg\left[H_0 - R\sin\alpha - (L-2R)\sin\alpha\right] - mgR\theta_2\sin\alpha$$

$$+Mg\left[H_0 - \frac{L}{2}\sin\alpha\right] + (m+M)gh - MgR\theta_2\sin\alpha$$

here,
M_{lss}, M_{rss}: mass of the left, right semi sphere of the capsule,
M_{cyl}: mass of the cylindrical body of capsule
σ_{cap}: surface mass density of capsule

$h = (d-b)sin(\alpha+\theta_2)$: rising height of COM of capsule

System 2

Distance to the center of mass of the system:

$$d = \left[\frac{mR + \frac{ML}{2}}{m + M} - R \right]$$

$$= \frac{M \left[\frac{1}{2} [L - 2R] \right]}{m + M}$$

Through geometry: (law of sines)

$$b = \frac{d \sin \alpha}{\sin(\alpha + \theta_2)}$$

$$\Rightarrow h = d \left(\sin(\alpha + \theta_2) - \sin \alpha \right)$$

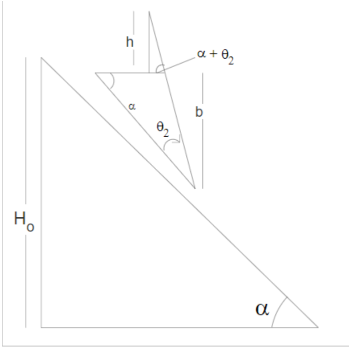


Figure: Vertical change in height of COM of system

$$\Rightarrow U_2 = (m+M)g\left[H_0 - (L-2R)\sin\alpha - R\sin\alpha\right]$$

$$-(m+M)gR\theta_2\sin\alpha + Mg\left[\frac{1}{2}[L-2R]\sin(\alpha+\theta_2)\right]$$

Therefore from the Euler-Lagrange Equations, the equation of motion is as follows:

$$\ddot{\theta}_2 = P + Q\cos(\alpha+\theta_2)$$

$$P = \frac{(m+M)gR\sin\alpha - \mu_2(m+M)gR\cos\alpha}{I_{sys} + (m+M)R^2}$$

$$Q = \frac{-Mg\left[\frac{1}{2}[L-2R])\right]}{I_{sys} + (m+M)R^2}$$

System 2

We assume totally elastic collision between the ball and the capsule in the transition from system 1 to 2, considering the energy lost due to friction between the ball and the capsule, $E_2 = T_2 + U_2$ must be set numerically equal to E_1' at $\theta_2 = 0$ which gives:

$$\dot{\theta}_2{}^2 = \frac{\left[E_1' - (m+M)g\left[H_1 - R\sin\alpha\right] - Mg\left[\frac{1}{2}(L-2R)\sin\alpha\right]\right]}{\left[\frac{1}{2}I_{sys} + \frac{1}{2}(m+M)R^2\right]}$$

The obtained equations of motions $\ddot{\theta}_1$ and $\ddot{\theta}_2$ are solved numerically using **Euler method** to get the time taken to topple down a certain incline.

The first assumption we made is as follows:

Initialization of $\dot{\theta}_2$: The collision of the ball with the lower end of capsule is assumed to be totally elastic. This assumption restricts $\dot{\theta}_2$ to be such that at $\theta_2 = 0$ (the moment of transition from system 1 to 2), $E_2 = E_1'$.

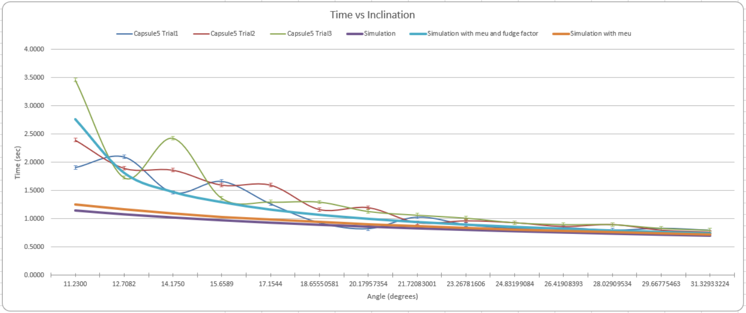
Initialization of $\dot{\theta}_1$: Note that

- $T_{1cap} = 0$ (at all times t)
- T_{1ball}(at moment of transition from system 1 to 2) $= T_{2cap} + T_{2ball}$ (T_{2cap} and T_{2ball} are time dependent)
- T_{2cap}(at moment of transition from system 2 to 1') $= 0$
- Thus, T_{2ball}(at moment of transition from system 2 to 1') $= T_{1'ball}$
- And hence, T_{1ball}(at moment of transition from system 1 to 2) $= T_{1'ball}$
- This implies $\dot{\theta}_{1'initial} = \dot{\theta}_{1final}$

Applying this assumption, the simulated t *vs* α plot is an asymptotic curve to the experimental data.

Numerical Solutions for Time

Time vs Angle Graph



152	mass of ball (mg)	454.1													
153	diameter of ball bearing (mm)	4.8													
154								CAPSULE 5							
155	mass of capsule (mg)	62.4													
156	length of capsule (mm)	12.6													
157															
158	**Angle (degrees)**	**11.2300**	**12.7082**	**14.1750**	**15.6589**	**17.1543**	**18.6555**	**20.1796**	**21.7208**	**23.2678**	**24.832**	**26.4191**	**28.0291**	**29.6677**	**31.3293**
159	**Capsule 5 Trial1 (sec)**	1.9080	2.0970	1.4650	1.6640	1.2650	0.9320	0.8260	1.0320	0.8980	0.8320	0.8430	0.7990	0.8330	0.7990
160	**Capsule 5 Trial2 (sec)**	2.3970	1.8970	1.8640	1.5980	1.5980	1.1650	1.1990	0.9320	0.9650	0.9320	0.8650	0.8990	0.7990	0.7650
161	**Capsule 5 Trial3 (sec)**	3.4620	1.7310	2.4300	1.3650	1.2990	1.2980	1.1320	1.0660	1.0140	0.9320	0.8990	0.8990	0.8320	0.7990
162	**Capsule 5 Simulation, meu1,2 = 0 (sec)**	1.1470	1.0790	1.0226	0.9739	0.9316	0.8945	0.8613	0.8315	0.8048	0.7805	0.7582	0.7378	0.7188	0.7013
163	**meu1 = 0.032, meu2 = 0.34, meu(alpha) = meu1**	1.2534	1.1656	1.0948	1.0352	0.9844	0.9405	0.9017	0.8673	0.8367	0.8091	0.784	0.7611	0.74	0.7206
164	**meu1 = 0.032, meu2 = 0.34, meu(alpha) = meu1/sin(alpha)**	2.7673	1.8147	1.4750	1.2930	1.1607	1.0684	0.9975	0.9406	0.8940	0.8544	0.8203	0.7903	0.7637	0.7399
165	**theta1_dot = 0, meu1,2 = 0.032, 0.34**	13.3947	12.4603	11.7063	11.0710	10.5284	10.0600	9.6461	9.2787	8.9522	8.6578	8.3899	8.1451	7.9197	7.7123
166	**Angle (radians)**	**0.1960**	**0.2218**	**0.2474**	**0.2733**	**0.2994**	**0.3256**	**0.3522**	**0.3791**	**0.4061**	**0.4334**	**0.4611**	**0.4892**	**0.5178**	**0.5468**

Fudge Factor: From the graph it is evident from the relative time differences between the experimental times and the simulated times that the fraction of energy left in the system is dependent on the angle of inclination; as the angle α increases the experimental curve converges to the theoretical curve,
Thus we introduced an α dependent data fitting factor in the dissipation equation: $f(\alpha) = 1/sin\alpha$.
The simulation now gives a best fit to our experimental results.

Data: :
$m, M, L, R, H_0, \alpha, \mu_1, \mu_2$
initialization of I_{ball}, I_{sys}
$pos_{ref} = length\,of\,incline\,surface,\ dt = 0.000001$
while $True$ **do**
  if $pos_{ball} - pos_{ref} >= L - 2R$ **then**
    initialize E_1'
    initialize $\dot{\theta}_2$
    while $0 <= \theta_2 <= \pi$ **do**
      $d\theta_2 = \theta_2$
      $\theta_2 + = \dot{\theta}_2 * dt$
      $d\theta_2 = \theta_2 - d\theta_2$
      $\dot{\theta}_2 + = \ddot{\theta}_2 * dt$
      $t + = dt$
      $pos_{ball} + = R * d\theta_2$
      if $pos_{ball} > length\,of\,board$ **then**
        break;
    $\theta_1 = 0$

$\theta_1 = 0$
if *completely inelastic collision assumed* **then**
  $\dot{\theta}_1 = 0$
if *significantly inelastic collision assumed* **then**
  $\dot{\theta}_1$ is such that a significant fraction of energy gets dissipated and a certain fraction of energy left in the system decides what value $\dot{\theta}_1$ will take
$H = H - (L - 2R)sin(\alpha) - \pi R sin(\alpha)$
$pos_{ref} = pos_{ball}$
if $pos_{ball} > length\,of\,board$ **then**
  break

if $pos_{ball} > length\,of\,board$ **then**
  break
$d\theta_1 = \theta_1$
$\theta_1 + = \dot{\theta}_1 * dt$
$d\theta_1 = \theta_1 - d\theta_1$
$\dot{\theta}_1 + = \ddot{\theta}_1 * dt$
$t + = dt$
$pos_{ball} + = R * d\theta_2$

Result: time taken to topple down incline: t