Homework 2

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1. $\frac{\partial u_i}{\partial z_i}$ · Let $f = z_i - \mu_B$ s.t. $\frac{\partial f}{\partial z_i} = 1$, $g = \sqrt{\sigma_B^2 + \epsilon}$ s.t. $\frac{\partial g}{\partial z_i} = \emptyset$

$$\rightarrow u_i = \frac{f}{g} \rightarrow \frac{\partial u_i}{\partial z_i} = \frac{\frac{\partial f}{\partial z_i} \cdot g - f \cdot \frac{\partial g}{\partial z_i}}{g^2} = \frac{1 \cdot \sqrt{\sigma_B^2 + \epsilon} - (z_i - \mu_B) \cdot \emptyset}{(\sigma_B^2 + \epsilon)}$$

$$= \frac{\sqrt{\sigma_B^2 + \epsilon}}{\sigma_B^2 + \epsilon} = \frac{1}{\sqrt{\sigma_B^2 + \epsilon}}$$

2. $\frac{\partial u_i}{\partial \mu_B}$ · Let $f = z_i - \mu_B$ s.t. $\frac{\partial f}{\partial \mu_B} = -1$, $g = \sqrt{\sigma_B^2 + \epsilon}$ s.t. $\frac{\partial g}{\partial \mu_B} = \emptyset$

$$\rightarrow \frac{\partial u_i}{\partial \mu_B} = \frac{\frac{\partial f}{\partial \mu_B} \cdot g - f \cdot \frac{\partial g}{\partial \mu_B}}{g^2} = \frac{-1 \cdot \sqrt{\sigma_B^2 + \epsilon} - (z_i - \mu_B) \cdot \emptyset}{(\sigma_B^2 + \epsilon)}$$

$$= \frac{-\sqrt{\sigma_B^2 + \epsilon}}{\sigma_B^2 + \epsilon} = -\frac{1}{\sqrt{\sigma_B^2 + \epsilon}}$$

3. $\frac{\partial \mu_B}{\partial z_i}$ · z_i appears once in the sum of μ_B $\left(\frac{1}{B} \sum_{j=1}^{B} z_j\right)$ and its coefficient is $1/B$.

· All other z_j $(j \neq i)$ are independent of z_i $\rightarrow$ derivatives are zero.

$$\rightarrow \frac{\partial \mu_B}{\partial z_i} = \frac{1}{B}$$

4. $\frac{\partial u_i}{\partial \sigma_B^2}$ · Let $f = z_i - \mu_B$ s.t. $\frac{\partial f}{\partial \sigma_B^2} = \emptyset$, $g = \sqrt{\sigma_B^2 + \epsilon}$ s.t. $\frac{\partial g}{\partial \sigma_B^2} = \frac{1}{2}(\sigma_B^2 + \epsilon)^{-1/2} \cdot 1 = \frac{1}{2\sqrt{\sigma_B^2 + \epsilon}}$

$$\rightarrow \frac{\partial u_i}{\partial \sigma_B^2} = \frac{\emptyset \cdot \sqrt{\sigma_B^2 + \epsilon} - (z_i - \mu_B) \cdot \frac{1}{2\sqrt{\sigma_B^2 + \epsilon}}}{(\sigma_B^2 + \epsilon)} = \frac{-(z_i - \mu_B) \cdot \frac{1}{2\sqrt{\sigma_B^2 + \epsilon}}}{\sigma_B^2 + \epsilon}$$

$$= \frac{-z_i - \mu_B}{2(\sigma_B^2 + \epsilon)^{3/2}}$$

5. $\frac{\partial \sigma_B^2}{\partial z_i} \longrightarrow \frac{\partial \sigma_B^2}{\partial z_i} = \frac{1}{B} \sum_{K=1}^{B} \frac{\partial}{\partial z_i} (z_K - \mu_B)^2 \longrightarrow$

- $\frac{\partial (z_K - \mu_B)^2}{\partial z_i} = 2(z_K - \mu_B) \cdot \frac{\partial (z_K - \mu_B)}{\partial z_i}$
- If $K = i$, $\frac{\partial (z_i - \mu_B)}{\partial z_i} = 1 - \frac{\partial \mu_B}{\partial z_i} = 1 - \frac{1}{B} = \frac{B-1}{B}$
- If $K \neq i$, $\frac{\partial (z_K - \mu_B)}{\partial z_i} = -\frac{\partial \mu_B}{\partial z_i} = -\frac{1}{B}$
- $\sum_{K \neq i} (z_K - \mu_B) = \sum_{K=1}^{B} (z_K - \mu_B) - (z_i - \mu_B) = \emptyset - (z_i - \mu_B) = -(z_i - \mu_B)$

$$\rightarrow \frac{\partial \sigma_B^2}{\partial z_i} = \frac{1}{B}\left[2(z_i - \mu_B) \cdot \frac{B-1}{B} + 2(-(z_i - \mu_B))\left(-\frac{1}{B}\right)\right]$$

$$= \frac{1}{B}\left[2(z_i - \mu_B) \cdot \frac{B-1}{B} + 2(z_i - \mu_B)\frac{1}{B}\right] = \frac{2(z_i - \mu_B)}{B}$$

Case 1: $j = i$

$$\rightarrow \frac{du_i}{dz_i} = \frac{1}{\sqrt{\sigma_B^2 + \epsilon}} + \frac{-1}{B\sqrt{\sigma_B^2 + \epsilon}} + \left(-\frac{z_i - \mu_B}{2(\sigma_B^2 + \epsilon)^{3/2}}\right) \cdot \frac{2(z_i - \mu_B)}{B}$$

$$= \frac{1}{\sqrt{\sigma_B^2 + \epsilon}} + \frac{1}{B\sqrt{\sigma_B^2 + \epsilon}} + \frac{-(z_i - \mu_B)^2}{B(\sigma_B^2 + \epsilon)^{3/2}} \quad (\text{if } j = i)$$

Case 2: $j \neq i$

$$\rightarrow \frac{du_j}{dz_i} = \frac{\partial u_j}{\partial z_i} + \frac{\partial u_j}{\partial \mu_B} \cdot \frac{\partial \mu_B}{\partial z_i} + \frac{\partial u_j}{\partial \sigma_B^2} \cdot \frac{\partial \sigma_B^2}{\partial z_i} \longrightarrow$$

- $\frac{\partial u_j}{\partial z_i} = \emptyset$ (since $j \neq i$)
- $\frac{\partial u_j}{\partial \mu_B} \cdot \frac{\partial \mu_B}{\partial z_i} = \left(-\frac{1}{\sqrt{\sigma_B^2 + \epsilon}}\right) \cdot \frac{1}{B} = -\frac{1}{B\sqrt{\sigma_B^2 + \epsilon}}$
- $\frac{\partial u_j}{\partial \sigma_B^2} \cdot \frac{\partial \sigma_B^2}{\partial z_i} = \left(-\frac{z_j - \mu_B}{2(\sigma_B^2 + \epsilon)^{3/2}}\right) \cdot \frac{2(z_i - \mu_B)}{B} = -\frac{(z_j - \mu_B)(z_i - \mu_B)}{B(\sigma_B^2 + \epsilon)^{3/2}}$

$$\rightarrow \frac{du_j}{dz_i} = \emptyset - \frac{1}{B\sqrt{\sigma_B^2 + \epsilon}} - \frac{(z_j - \mu_B)(z_i - \mu_B)}{B(\sigma_B^2 + \epsilon)^{3/2}} \quad (\text{for } j \neq i)$$