zonal flow/geodesic acoustic mode

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1 closure for fluid moments model

1.1 Hammett-Perkins closure

First consider the simplest case of linear one-dimensional electrostatic waves.Here's the Vlasov equations with cold ions,

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{eE}{m}\frac{\partial f}{\partial v} = 0 \tag{1}$$

Consider the linear response $f = f_0(v) + \tilde{f}(x,v,t)$ to a small driving electric field $E = -\partial\tilde{\phi}/\partial x$ and linearize the equation with $\tilde{f}(x,v,t) \sim \exp(ikx - i\omega t)$, the exact linear response is

$$\tilde{n} = \int \tilde{f}dv \equiv -n_0\frac{e\tilde{\phi}}{T_0}R(\zeta) = \frac{e\tilde{\phi}}{T_0}kv_t^2\int\frac{\partial f_0/\partial v}{kv-\omega} \tag{2}$$

where $R(\zeta)$ is the normalized reponse function and $\zeta = \omega/(|k|\sqrt{2}v_t)$ is a normalized frequency, $T_0 = mv_t^2 = m\int f_0v^2dv/\int f_0dv$. For the maxwellian disribution $f_0 = f_M$, response funtion is $R(\zeta) = 1 + \zeta Z(\zeta) = -Z'(\zeta)$, where $Z(\zeta) = \pi^{-1/2}\int dt\exp(-t^2)/(t-\zeta)$ is the usual plasma dispersion funtion.

Expand $Z'(\zeta)$. For $|\zeta| \gg 1$,

$$\begin{aligned} Z'(\zeta) &= -i\frac{k}{|k|}\sqrt{\pi}2\zeta e^{-\zeta^2} + \frac{1}{\zeta^2} + \frac{3}{2}\frac{1}{\zeta^4} + \frac{15}{4}\frac{1}{\zeta^6} + ... \\ &\simeq \frac{1}{\zeta^2} + \frac{3}{2}\frac{1}{\zeta^4} + \frac{15}{4}\frac{1}{\zeta^6} + ... \end{aligned} \tag{3}$$

for $|\zeta| \ll 1$,

$$\begin{aligned} Z'(\zeta) &= -i\frac{k}{|k|}\sqrt{\pi}2\zeta e^{-\zeta^2} - 2 + 4\zeta^2 - \frac{8}{3}\zeta^4 + ... \\ &\simeq -2 + (-i\frac{k}{|k|}\sqrt{\pi}2)\zeta + 4\zeta^2 + (i\frac{k}{|k|}\sqrt{\pi}2)\zeta^3 - \frac{8}{3}\zeta^4 + (-i\frac{k}{|k|}\sqrt{\pi})\zeta^5 + ... \end{aligned} \tag{4}$$

Considering the moment equation sets for the particle density $n = \int fdv$, the momentum density $mnu = m\int fvdv$, and the pressure $p = m\int f(v-u)^2dv$,

$$\begin{aligned} &\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(un) = 0 \\ &\frac{\partial}{\partial t}(mnu) + \frac{\partial}{\partial x}(umnu) = -\frac{\partial p}{\partial x} + enE - \frac{\partial S}{\partial x} \\ &\frac{\partial p}{\partial t} + \frac{\partial}{\partial x}(up) = -(\Gamma - 1)(p + S)\frac{\partial u}{\partial x} - \frac{\partial q}{\partial x} \end{aligned} \tag{5}$$

The heat flux moment is $q = m\int f(v-u)^3dv$. And then what we should do is to obtain the unkown variable by compare the linear dispersion relation with the result of Vlasov equation. Here the unknown variables include S, Γ, q. First we assume linear closure approximations by

$$\begin{aligned} \tilde{q_k} &= -n_0\chi_1\frac{\sqrt{2}v_t}{|k|}ik\tilde{T}_k \\ \tilde{S}_k &= -mn_0\mu_1\sqrt{2}(v_t/|k|)ik\tilde{u}_k \end{aligned} \tag{6}$$

where χ_1 and μ_1 are dimensionless coefficients, and $\tilde{T} = (\tilde{p} - T_0\tilde{n})/n_0$ is the pertubed temperature.

and then linearing the equation sets leads to the following three-moment fluid model for the response function,

$$R_3 = \frac{\chi_1 - i\zeta}{\chi_1 - i\Gamma\zeta - 2i\chi_1\mu_1\zeta - 2\chi_1\zeta^2 - 2\mu_1\zeta^2 + 2i\zeta^3} \tag{7}$$

Note that the three-moment fluid model yields a three-pole approximation for the response function(and therefore for the plasma dispersion Z). The asymptotic expansion of R_3 in the cold-plasma limit $|\zeta| \gg 1$ is $R_3 \simeq -1/2\zeta^2 + i\mu_1/2\zeta^3 + ...$According to the Eq.(3), this implies that R_3 is equivalent to an $f_0(v) \sim const/v^2$ and hence has an infinite-pressure moment. Therefore $\mu_1 = 0$. Carrying the asymptotic expansion to higher order then gives us $R_3 \sim -1/2\zeta^2 - \Gamma/4\zeta^4 + i(\Gamma-1)\chi_1/4\zeta^5 + ...$, which is equivalent to a more physival $f_0(v) \sim 1/v^4$. Setting $\Gamma = 3$ puts the proper amount of compressional $p\partial u/\partial z$ heating into the evolution of pressure to conserve total energy. Expanding R_3 for $|\zeta| \ll 1$ leads to $R_3 \simeq 1 + i2\zeta/\chi_1$. Requiring this to match the Maxwellian R for small ζ leads to the condition $\chi_1 = 2/\sqrt{\pi}$.Although χ_1 is chosen to fit the low-frequency limit, the closure is used in fluid equations which are automatically valid in the high-frequency limit, and the resulting R_3 does a fair job of approximating the Maxwellian R over the full frequency range, and is equivalent to an $f_0(v)$ which is fairly close to Maxwellian.

2 zonal flow

3 residula flow

3.1 history

- gyrofluid simulation [Beer,Waltz]: a total collisionless decay of poloidal rotation
- gyrokinetic analysis by [RH 1998]: linear collisionless kinetic mechanisms do not damp the zonal flows completely
- verification by various gyrokinetic codes
- considering shaping effect [Xiao 2006]
- modification of zonal flow closures in gyrofluid model[Mandell,Sugama]

3.2 description

Considering the polarization drift in plasma,

$$v_{pj} = \frac{m_j}{e_j B^2}\frac{d\boldsymbol{E}}{dt} \tag{8}$$

this gives rise to the polarization current,

$$j^{cl} = \sum_j e_j n_j \boldsymbol{v}_{pj} = \epsilon_0 \epsilon^{cl}\frac{d\boldsymbol{E}}{dt} \tag{9}$$

where $\epsilon^{cl} = m_i n_i/\epsilon_0 B^2 = (\omega_{pi}/\Omega_i)^2 = (k_{Di}\rho_i)^2 >> 1$. This polarization current originates from delayed ion gyro motion from time varying electric field and ϵ^{cl} is called classical polarization, which is a low frenquency dielectric constant perpendicular to the magnetic field.

And in tokamak, considering the toroidal effect, we should include the traped and passing particals. Some fraction of charged particals($f_t \sim \sqrt{\epsilon}, \epsilon = r/R$) are trapped by the magnetic mirror and have a radial excursion by

$$\Delta_t = \sqrt{\epsilon}\rho_{pi} = \frac{q\rho_i}{\sqrt{\epsilon}} \tag{10}$$

and as for passing partical the excursion is

$$\Delta_p = q\rho_i \tag{11}$$

So during this trapped partial orbit motion, we alse have the similar polarization effect if we have radial electric field E_r,

$$j^{nc} = \epsilon_0\epsilon^{nc}\frac{dE_r}{dt} \tag{12}$$

$$\epsilon^{nc} = \sqrt{\epsilon}k_{Di}^2\Delta_t^2 = \frac{q^2}{\sqrt{\epsilon}}\epsilon^{cl} \tag{13}$$

and Hinton-Rosenbluth give a expression for the polarization as,

$$\epsilon = \epsilon^{cl} + \epsilon^{nc} = \frac{\omega_{pi}^2}{\Omega_i^2}(1 + \frac{1.6q^2}{\sqrt{\epsilon}}) \tag{14}$$

The factor 1.6 comes from detialed kinetic calculation including passing partical contribution.

Then we consider the continuity equation of the polarization current,

$$\frac{\partial \rho_p}{\partial t} + \nabla \cdot \boldsymbol{j}_p = S \tag{15}$$

S is the external source density.Taking the flux surface average and Fourier expansion in space,

$$\frac{\partial}{\partial t} < \rho_p(\boldsymbol{k}) > + < i\boldsymbol{k}_\perp \cdot \boldsymbol{j}_p(\boldsymbol{k}) > = < S(\boldsymbol{k}) > \tag{16}$$

$$\epsilon_0\epsilon_p < k_\perp^2 > \Phi_k =< \rho_p > - \int < S(\boldsymbol{k}) > dt \tag{17}$$

Consider an initial source perturbation $< S(\boldsymbol{k}) >= \delta_k(0)\delta(t)$. when the time scale is in a few gyro motion and much shorter than the bounce time of trapped partical.we have,

$$\epsilon_0\epsilon^{cl}\Phi_k(t = +0) = -e_i\delta n_k(0) \tag{18}$$

when the time scale is longer than the bounce time of trapped particals, the electrostatic potential is further shielded by the addition of the neoclassical polarization, then we have,

$$\epsilon_0(\epsilon^{cl} + \epsilon^{nc})\Phi_k(t = +\infty) = -e_i\delta n_k(0) \tag{19}$$

Therefore, the ratio of the long term zonal flow potential to the initial zonal flow potential is given by,

$$\frac{\Phi_k(t = \infty)}{\Phi_k(t = 0)} = \frac{\epsilon^{cl}}{\epsilon^{cl} + \epsilon^{nc}} \tag{20}$$

Using the Hinton formula, we have,

$$\frac{\Phi_k(t = \infty)}{\Phi_k(t = 0)} = \frac{1}{1 + 1.6q^2/\sqrt{\epsilon}} \tag{21}$$

[reference]

1. Kikuchi ebook
2. Diamond PPCF 2005
3. Hinton PRL 1998
4. Idomura ...

4 Geodesic acoustic mode

4.1 history

- first prediction by [Winsor 1968]
- Fully forgotten between 1968 and 1996
- dispersion relation, frequency, radial structure and propagation
- close relation to alfven eigen mode
- experimental observations of GAM, H1-heliac,[Shats PRL 2002]; DIII- D, [McKee PoP 2003]

4.2 description

4.2.1 Winsor1968

The starting equations are the follows:

$$
\begin{aligned}
&\rho_0\frac{\partial \boldsymbol{v}}{\partial t} = \frac{1}{c}\boldsymbol{J}\times\boldsymbol{B} - \nabla p\\
&\frac{\partial \rho}{\partial t} + \nabla\cdot\rho_0\boldsymbol{v} = 0\\
&\nabla\phi = \frac{1}{c}\boldsymbol{b}\times\boldsymbol{B}\\
&\nabla\cdot\boldsymbol{J} = 0\\
&\rho_0^{-\gamma} - \gamma p_0\rho_0^{-\gamma-1}\frac{\partial\rho}{\partial t} + \boldsymbol{v}\cdot\nabla(p_0\rho_0^{-\gamma}) = 0
\end{aligned}
\tag{22}
$$

finally, we can deduce the dispersion relation:

$$
\omega^2\int|\rho|^2\mathcal{J}dS = \frac{\gamma p_0}{\rho_0}(|\int\rho_0\frac{\boldsymbol{B}\times\nabla\psi\cdot\nabla B^2}{B^4}\mathcal{J}dS|^2/\int\frac{|\nabla\psi|^2}{B^2}\mathcal{J}dS + \int\frac{|\boldsymbol{B}\cdot\nabla\rho|^2}{B^2}\mathcal{J}dS) \tag{23}
$$

The first term is due to motion in the $\boldsymbol{B}\times\nabla\psi$direction.It is associated with geodesic curvature, i.e., the surface component of the magnetic field line curvature. And the second **ordinary sound propagation propagating along the field lines**.

The physical mechanism of geodesic acoustic mode is explained as follows. An electric field pertubation E_ψ causes a flow $\boldsymbol{v}_\perp = \boldsymbol{E}\times\boldsymbol{B}/B^2$ and since the compressibility, the flow causes a density accumulation proportional to $-\nabla\cdot\boldsymbol{v}_\perp = \boldsymbol{E}\times\boldsymbol{B}\cdot\nabla B^2/B^4$.This n generates a current $\boldsymbol{J} = \boldsymbol{B}\times\nabla n/B^2$ which transports charge across the magnetic surface and acts to recerse E_ψ

In the limit of circular cross section with large aspect ratio($r \ll R$), the dispersion relation becomes,

$$
\omega^2 = \frac{2\gamma p_0}{\rho_0 R^2}[1 + 1/(2q^2)] = 2\frac{C_s^2}{R^2}(1 + 1/(2q^2)) \tag{24}
$$

with the defination of sound wave velocity in neutral gas $C_s = (\gamma p_0/\rho_0)^{1/2}$.

4.2.2 GAM in 5-field landau fluid model

[reference]

1. Diamond PPCF 2005
2. Winsor POF 1968