Energy transfer in ITG/KBM system

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1 model equation

$$\frac{dn}{dt} = a\frac{dn_{eq}}{dr}\nabla_\theta\phi - n_{eq}\nabla_\parallel v + \nabla_\parallel j + \omega_d(n_{eq}\phi - p_e) + D_n\nabla_\perp^2 n \tag{1}$$

$$\frac{d\nabla_\perp^2\phi}{dt} = -aT_{eq}(\frac{1}{n}\frac{dn_{eq}}{dr} + \frac{1}{T_{eq}}\frac{dT_{eq}}{dr})\nabla_\theta\nabla_\perp^2\phi + \frac{1}{n_{eq}}\nabla_\parallel j - \omega_d(T_i + \frac{T_{eq}}{n_{eq}}n + \frac{p_e}{n_{eq}}) + D_U\nabla_\perp^4\phi \tag{2}$$

$$\frac{dv}{dt} = -\nabla_\parallel T_i - 2\frac{T_{eq}}{n_{eq}}\nabla_\parallel n - \beta aT_{eq}(\frac{2}{n_{eq}}\frac{dn_{eq}}{dr} + \frac{1}{T_{eq}}\frac{dT_{eq}}{dr})\nabla_\theta A + D_v\nabla_\perp^2 v \tag{3}$$

$$\beta\frac{\partial A}{\partial t} = -\nabla_\parallel\phi + \frac{T_{eq}}{n_{eq}}\nabla_\parallel n + \beta aT_{eq}\frac{1}{n_{eq}}\frac{dn_{eq}}{dr}\nabla_\theta A + \sqrt{\frac{\pi}{2}\frac{m_e}{m_i}}|\nabla_\parallel|(v - \frac{j}{n_{eq}}) - \eta j \tag{4}$$

$$\frac{dT_i}{dt} = a\frac{\partial T_{eq}}{\partial r}\nabla_\theta\phi - (\Gamma-1)T_{eq}\nabla_\parallel v + T_{eq}\omega_d((\Gamma-1)\phi + (2\Gamma-1)T_i + (\Gamma-1)\frac{T_{eq}}{n_{eq}}n) - (\Gamma-1)\sqrt{\frac{8T_{eq}}{\pi}}|\nabla_\parallel|T_i + D_T\nabla_\perp^2 T_i \tag{5}$$

2 energy balance equation

energy in system

$$\begin{aligned}
E_k &= \frac{1}{2} < |\nabla_\perp\phi|^2 > \\
E_m &= \frac{1}{2} < \frac{\beta}{n_{eq}}|\nabla_\perp\psi|^2 > \\
E_n &= \frac{1}{2} < \tau\frac{T_{eq}}{n_{eq}^2}|n|^2 > \\
E_v &= \frac{1}{2} < \frac{\tau}{\tau+1}|v|^2 > \\
E_t &= \frac{1}{2} < \frac{\tau}{(\Gamma-1)(\tau+1)T_{eq}}|T|^2 >
\end{aligned} \tag{6}$$

where

$$< f >= \frac{\int f dV}{\int dV} = \frac{\int f \ rdrd\theta d\zeta}{\int \ rdrd\theta d\zeta} \tag{7}$$

total energy

$$E_{total} = E_k + E_m + E_n + E_v + E_t \tag{8}$$

time dependence of energy

$$\frac{d}{dt}E_i = \Gamma_F + \Gamma_Q + \Gamma_C + \Gamma_{LD} + \Gamma_{NL} + \Gamma_S \tag{9}$$

radial transport items

$$\begin{aligned}
\Gamma_{Q,n} &=< \tau a\frac{T_{eq}}{n_{eq}^2}n\nabla_\theta\phi > + < \beta\tau\frac{T_{eq}}{n_{eq}^2}\frac{\partial j_0}{\partial r}n\nabla_\theta A > \\
\Gamma_{Q,v} &= - < \frac{\tau}{\tau+1}\beta a T_{eq}(\frac{2}{n_{eq}}\frac{dn_{eq}}{dr} + \frac{1}{T_{eq}}\frac{dT_{eq}}{dr})v\nabla_\theta A > \\
\Gamma_{Q,t} &=< \frac{\tau}{(\Gamma-1)(\tau+1)T_{eq}}a\frac{\partial T_{eq}}{\partial r}T\nabla_\theta\phi > \\
\Gamma_{Q,k} &=< -aT_{eq}(\frac{1}{n}\frac{dn_{eq}}{dr} + \frac{1}{T_{eq}}\frac{dT_{eq}}{dr})\phi\nabla_\theta\nabla_\perp^2\phi > + < \frac{\beta}{n_{eq}}\frac{\partial j_0}{\partial r}\phi\nabla_\theta A > \\
\Gamma_{Q,m} &=< \frac{\beta}{n_{eq}}aT_{eq}\frac{1}{n_{eq}}\frac{dn_{eq}}{dr}j\nabla_\theta A >
\end{aligned} \tag{10}$$

parallel transport items

$$\Gamma_{F,k} =<> \tag{11}$$

curvature items

$$\Gamma_{C,k} =<> \tag{12}$$

landau damping items

$$\begin{aligned}
D_m &=< \frac{1}{n_{eq}}\sqrt{\frac{\pi\tau}{2}\frac{m_e}{m_i}}j\nabla_\parallel|(v - \frac{j}{n_{eq}}) > \\
D_t &= - < \frac{\tau}{(\tau+1)T_{eq}}\sqrt{\frac{8T_{eq}}{\pi}}T|\nabla_\parallel|T >
\end{aligned} \tag{13}$$

diffusion items

$$\begin{aligned}
\Gamma_{S,k} &= -D_U < |\nabla_\perp^2\phi|^2 > \\
\Gamma_{S,m} &= -\frac{1}{n_{eq}}D_m < |j|^2 > \\
\Gamma_{S,n} &= \tau\frac{T_{eq}}{n_{eq}^2}D_n < |\nabla_\perp n|^2 > \\
\Gamma_{S,v} &= -\frac{\tau}{\tau+1}D_v < |\nabla_\perp v|^2 > \\
\Gamma_{S,t} &= -\frac{\tau}{(\Gamma-1)(\tau+1)T_{eq}}D_t < |\nabla_\perp T|^2 >
\end{aligned} \tag{14}$$

time dependence of energy

$$\frac{\partial}{\partial t}E_{total} = \Sigma_i[\frac{1}{2}(\Gamma_{Q,i} + \Gamma_{C,i} + \Gamma_{LD,i}) + \Gamma_{S,i}] \tag{15}$$

3 energy drive of $E_{k,m=0,n=0}$ and $E_{p,m=1,n=0}$

for zonal flow energy:

$$\frac{\partial}{\partial t}E_k = \Gamma_{NLk,k} + \Gamma_{NLm,k} + \Gamma_{C,k} \tag{16}$$

for pressure:

$$p = p_i + p_e = n_0T + (1+\tau)T_0n \tag{17}$$

$$\frac{\partial}{\partial t}E_p = \Gamma_{Q,p} + \Gamma_{F,p} + \Gamma_{C,p} + \Gamma_{LD,p} + \Gamma_{NL,p} + \Gamma_{S,p} \tag{18}$$

$$
\begin{aligned}
&\Gamma_{Q,p} =< a[(1+\tau)T_{eq}\frac{\partial n_e q}{\partial r} + n_{eq}\frac{\partial T_{eq}}{\partial r}]p\nabla_\theta \phi > + < (1+\tau)\beta T_{eq}\frac{\partial j_0}{\partial r}p\nabla_\theta A > \\
&\Gamma_{F,p} = - < (\Gamma+\tau)n_{eq}T_{eq}p\nabla_\parallel v > + < (1+\tau)T_{eq}p\nabla_\parallel j > \\
&\Gamma_{LD,p} = - < (\Gamma-1)\sqrt{\frac{8T_{eq}}{\pi}}n_{eq}p|\nabla_\parallel|T > \\
&\Gamma_{C,p} =< p\omega_d((\Gamma+\tau)n_{eq}T_{eq}\phi) > + < p\omega_d((2\Gamma-1)n_{eq}T_{eq}T) > + < p\omega_d(((\Gamma-1)-\tau(\tau+1))T_{eq}^2 n) > \\
&\Gamma_{NL,p} = - < p((1+\tau)T_{eq}[\phi,n] + n_{eq}[\phi,T]) > + < p(\Gamma+\tau)n_{eq}T_{eq}\beta[A,v] > - < p(1+\tau)T_{eq}\beta[A,j] > \\
&\Gamma_{S,p} =< p[(1+\tau)T_{eq}D_n\nabla_\perp^2 n + n_{eq}D_T\nabla_\perp^2 T] >
\end{aligned} \tag{19}
$$

4 Appendix: equation for thr evolution of zonal flow

define flux average as

$$
< f >= \frac{\int\int \ f d\theta d\zeta}{\int\int \ d\theta d\zeta} \tag{20}
$$

and the flux-averaged derivation term of vorticity is written as:

$$
\begin{aligned}
\frac{\partial}{\partial t} < \nabla_\perp^2 \phi > &= \frac{\partial}{\partial t} < \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial \phi}{\partial r}) > \\
&= \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial t}(\frac{\partial < \phi >}{\partial r}) \\
&= \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial t}(v_E)
\end{aligned} \tag{21}
$$

the other flux-averaged term can be written as

$$
- < [\phi,\Omega] >= \frac{1}{r} < \frac{\partial \phi}{\partial \theta}\frac{\partial \Omega}{\partial r} - \frac{\partial \phi}{\partial r}\frac{\partial \Omega}{\partial \theta} > \tag{22}
$$

$$
< \frac{\partial \phi}{\partial \theta}\frac{\partial \Omega}{\partial r} >= \int \frac{\partial \phi}{\partial \theta}\frac{\partial \Omega}{\partial r}d\theta = \int \frac{\partial}{\partial r}(\frac{\partial \phi}{\partial \theta}\Omega)d\theta - \int \frac{\partial^2 \phi}{\partial r \partial \theta}\Omega d\theta \tag{23}
$$

$$
< \frac{\partial \phi}{\partial r}\frac{\partial \Omega}{\partial \theta} >= \int \frac{\partial \phi}{\partial r}\frac{\partial \Omega}{\partial \theta}d\theta = [\Omega\frac{\partial \phi}{\partial r}]_0^{2\pi} - \int \frac{\partial^2 \phi}{\partial r \partial \theta}\Omega d\theta = - \int \frac{\partial^2 \phi}{\partial r \partial \theta}\Omega d\theta \tag{24}
$$

then:

$$
\begin{aligned}
- < [\phi,\Omega] > &= \frac{1}{r}\frac{\partial}{\partial r} < \Omega\frac{\partial \phi}{\partial \theta} > \\
&= \frac{1}{r}\frac{\partial}{\partial r} < \phi\frac{\partial \Omega}{\partial \theta} >
\end{aligned} \tag{25}
$$

and we can convert it to the form like Reynolds stress as follows:

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial \theta}\nabla_\perp^2 \phi d\theta &= \int \frac{\partial \phi}{\partial \theta}\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial \phi}{\partial r}d\theta + \int \frac{\partial \phi}{\partial \theta}\frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}d\theta \\
&= \int \frac{\partial \phi}{\partial \theta}\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial \phi}{\partial r}d\theta + \frac{1}{2r^2}\int \frac{\partial}{\partial \theta}(\frac{\partial \phi}{\partial \theta})^2 d\theta \\
&= \int \frac{\partial \phi}{\partial \theta}\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial \phi}{\partial r}d\theta + \int \frac{1}{r}(r\frac{\partial \phi}{\partial r}\frac{\partial^2 \phi}{\partial r \partial \theta})d\theta \\
&= \int \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial \phi}{\partial r}\frac{\partial \phi}{\partial \theta})d\theta \\
&= \frac{1}{r}\frac{\partial}{\partial r} < r\frac{\partial \phi}{\partial r}\frac{\partial \phi}{\partial \theta} > \\
&= -\frac{1}{r}\frac{\partial}{\partial r}r^2 < v_\theta v_r >
\end{aligned} \tag{26}
$$

and we can deduce the maxwell stress in a similiar way:

$$
\begin{aligned}
-\frac{\beta}{n_{eq}} < [A, j] > &= \frac{\beta}{n_{eq}} < [A, \nabla_\perp^2 A] > \\
&= \frac{\beta}{n_{eq}} \frac{1}{r} \frac{\partial}{\partial r} < j \frac{\partial A}{\partial \theta} > \\
&= \frac{\beta}{n_{eq}} \frac{1}{r} \frac{\partial}{\partial r} < A \frac{\partial j}{\partial \theta} > \\
&= \frac{\beta}{n_{eq}} \frac{1}{r} \frac{\partial}{\partial r} r^2 < B_\theta B_r >
\end{aligned} \tag{27}
$$

as for curvature terms,

$$
\begin{aligned}
- < \omega_d \cdot f > &= -2\epsilon < [r\cos\theta, f] > \\
&= 2\epsilon < \frac{1}{r} (\frac{\partial (r\cos\theta)}{\partial \theta} \frac{\partial f}{\partial r} - \frac{\partial (r\cos\theta)}{\partial r} \frac{\partial f}{\partial \theta}) > \\
&= \frac{2\epsilon}{r} (- < r\sin\theta \frac{\partial f}{\partial r} > - < \cos\theta \frac{\partial f}{\partial \theta} >) \\
&= \frac{2\epsilon}{r} (- < r\sin\theta \frac{\partial f}{\partial r} > - < f\sin\theta >) \\
&= -2\epsilon < (\frac{\partial f}{\partial r} + \frac{f}{r}) \sin\theta > \\
&= -2\epsilon \frac{1}{r} \frac{\partial}{\partial r} r < f\sin\theta >
\end{aligned} \tag{28}
$$

Here,

$$
f = T_i + \frac{T_{eq}}{n_{eq}} n + \frac{p_e}{n_{eq}} = \frac{p}{n_{eq}} \tag{29}
$$

then,

$$
- < \omega \cdot \frac{p}{n_{eq}} >= -\frac{2\epsilon}{n_{eq}} \frac{1}{r} \frac{\partial}{\partial r} r < p\sin\theta > \tag{30}
$$

so the evolution of zonal flow in the limit of collisionless situation is as follows,

$$
\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial t} (v_E) = -\frac{1}{r} \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} r^2 < v_\theta v_r > + \frac{1}{r} \frac{\partial}{\partial r} \frac{\beta}{n_{eq}} \frac{1}{r} \frac{\partial}{\partial r} r^2 < B_\theta B_r > -\frac{2\epsilon}{n_{eq}} \frac{1}{r} \frac{\partial}{\partial r} r < p\sin\theta > \tag{31}
$$

eliminate the commom part, we get,

$$
\frac{\partial}{\partial t} < v_E >= -\frac{1}{r^2} \frac{\partial}{\partial r} r^2 < v_\theta v_r > + \frac{\beta}{n_{eq}} \frac{1}{r^2} \frac{\partial}{\partial r} r^2 < B_\theta B_r > -\frac{2\epsilon}{n_{eq}} < p\sin\theta > \tag{32}
$$

and in the code, we use the alternative expression,

$$
\frac{\partial}{\partial t} < v_E >= \frac{1}{r} < \phi \frac{\partial \Omega}{\partial \theta} > + \frac{\beta}{n_{eq}} \frac{1}{r} < A \frac{\partial A}{\partial \theta} > -\frac{2\epsilon}{n_{eq}} < p\sin\theta > \tag{33}
$$

when we use the Fourier expansion,

$$
\frac{\partial}{\partial t} < v_E >= \sum_{m1+m2=0} \phi_{m1} \Omega_{\theta,m2} + \sum_{m1+m2=0} A_{m1} j_{\theta,m2} - \frac{i\epsilon}{n_{eq}} (p_{1,0} - p_{-1,0}) \tag{34}
$$

$$
< p\sin\theta >= \frac{i}{2} (p_{1,0} - p_{-1,0}) \tag{35}
$$

5 Appendix: equation for thr evolution of $< p\sin\theta >$

we can deduce the evolution of p with the drivative equation of n and T,

$$\frac{\partial}{\partial t}p = R_Q + R_F + R_C + R_{NL} + R_{LD} + R_S \tag{36}$$

where,

$$\begin{aligned}
R_Q &= a[(1+\tau)T_{eq}\frac{\partial n_{eq}}{\partial r} + n_{eq}\frac{\partial T_{eq}}{\partial r}]\nabla_\theta\phi + (1+\tau)\beta T_{eq}\frac{\partial j_0}{\partial r}\nabla_\theta A \\
R_F &= -(\Gamma+\tau)n_{eq}T_{eq}\nabla_\parallel v + (1+\tau)T_{eq}\nabla_\parallel j \\
R_C &= \omega_d((\Gamma+\tau)n_{eq}T_{eq}\phi) + \omega_d((2\Gamma-1)n_{eq}T_{eq}T) + \omega_d(((\Gamma-1)-\tau(\tau+1))T_{eq}^2 n) \\
R_{NL} &= -(1+\tau)T_{eq}[\phi,n] - n_{eq}[\phi,T] + (\Gamma+\tau)n_{eq}T_{eq}\beta[A,v] - (1+\tau)T_{eq}\beta[A,j] \\
R_{LD} &= -(\Gamma-1)\sqrt{\frac{8T_{eq}}{\pi}}n_{eq}|\nabla_\parallel|T \\
R_S &= (1+\tau)T_{eq}D_n\nabla_\perp^2 n + n_{eq}D_T\nabla_\perp^2 T
\end{aligned} \tag{37}$$

with,

$$\begin{aligned}
< \nabla_\theta f\sin\theta > &= \int \frac{1}{r}\frac{\partial f}{\partial\theta}\sin\theta d\theta \\
&= -\frac{1}{r} < f\cos\theta >
\end{aligned} \tag{38}$$

$$\begin{aligned}
< \nabla_\parallel f\sin\theta > &= \int \frac{a}{R}(f_\theta/q + f_\zeta)\sin\theta d\theta d\zeta \\
&\approx -\frac{\epsilon}{q} < f\cos\theta >
\end{aligned} \tag{39}$$

$$\begin{aligned}
< (\omega_d\cdot f)\sin\theta > &= 2\epsilon < [r\cos\theta, f]\sin\theta > \\
&= \frac{2\epsilon}{r}(< \frac{\partial f}{\partial\theta}\cos\theta\sin\theta > - < r\frac{\partial f}{\partial r}sin^2\theta >) \\
&= \epsilon(< \frac{\partial f}{\partial r} > - < (\frac{2f}{r} + \frac{\partial f}{\partial r})\cos 2\theta >)
\end{aligned} \tag{40}$$

we can convert the terms form to,

$$\begin{aligned}
< R_Q\sin\theta > &= a[(1+\tau)T_{eq}\frac{\partial n_{eq}}{\partial r} + n_{eq}\frac{\partial T_{eq}}{\partial r}] < \phi\cos\theta > +(1+\tau)\beta T_{eq}\frac{\partial j_0}{\partial r} < A\cos\theta > \\
< R_F\sin\theta > &= (\Gamma+\tau)n_{eq}T_{eq}\frac{\epsilon}{q} < v\cos\theta > -(1+\tau)T_{eq}\frac{\epsilon}{q} < j\cos\theta > \\
< R_C\sin\theta > &= \epsilon(\Gamma+\tau)p_{eq} < v_\theta > +\epsilon(2\Gamma-1)p_{eq} < \frac{\partial T}{\partial r} > +\epsilon((\Gamma-1)-\tau(\tau+1))T_{eq}^2 < \frac{\partial n}{\partial r} > \\
< R_{LD}\sin\theta > &= -(\Gamma-1)\sqrt{\frac{8T_{eq}}{\pi}}n_{eq}\frac{\epsilon}{q} < T\sin\theta > \\
< R_{NL}\sin\theta > &= - < ((1+\tau)T_{eq}[\phi,n] + n_{eq}[\phi,T])\sin\theta > \\
&\quad + < (\Gamma+\tau)n_{eq}T_{eq}\beta[A,v]\sin\theta > - < (1+\tau)T_{eq}\beta[A,j]\sin\theta >
\end{aligned} \tag{41}$$