Energy transfer in ITG/KBM system

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1 Appendix: equation for thr evolution of zonal flow

define flux average as

$$< f >= \frac{\int\int \ f d\theta d\zeta}{\int\int \ d\theta d\zeta} \tag{1}$$

and the flux-averaged derivation term of vorticity is written as:

$$\begin{aligned} \frac{\partial}{\partial t} < \nabla_\perp^2 \phi > &= \frac{\partial}{\partial t} < \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial \phi}{\partial r}) > \\ &= \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial t}(\frac{\partial < \phi >}{\partial r}) \\ &= \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial t}(v_E) \end{aligned} \tag{2}$$

the other flux-averaged term can be written as

$$- < [\phi, \Omega] >= \frac{1}{r} < \frac{\partial \phi}{\partial \theta}\frac{\partial \Omega}{\partial r} - \frac{\partial \phi}{\partial r}\frac{\partial \Omega}{\partial \theta} > \tag{3}$$

$$< \frac{\partial \phi}{\partial \theta}\frac{\partial \Omega}{\partial r} >= \int \frac{\partial \phi}{\partial \theta}\frac{\partial \Omega}{\partial r} d\theta = \int \frac{\partial}{\partial r}(\frac{\partial \phi}{\partial \theta}\Omega) d\theta - \int \frac{\partial^2 \phi}{\partial r \partial \theta}\Omega d\theta \tag{4}$$

$$< \frac{\partial \phi}{\partial r}\frac{\partial \Omega}{\partial \theta} >= \int \frac{\partial \phi}{\partial r}\frac{\partial \Omega}{\partial \theta} d\theta = [\Omega \frac{\partial \phi}{\partial r}]_0^{2\pi} - \int \frac{\partial^2 \phi}{\partial r \partial \theta}\Omega d\theta = -\int \frac{\partial^2 \phi}{\partial r \partial \theta}\Omega d\theta \tag{5}$$

then:

$$\begin{aligned} - < [\phi, \Omega] > &= \frac{1}{r}\frac{\partial}{\partial r} < \Omega \frac{\partial \phi}{\partial \theta} > \\ &= \frac{1}{r}\frac{\partial}{\partial r} < \phi \frac{\partial \Omega}{\partial \theta} > \end{aligned} \tag{6}$$

and we can convert it to the form like Reynolds stress as follows:

$$\begin{aligned} \int \frac{\partial \phi}{\partial \theta}\nabla_\perp^2 \phi d\theta &= \int \frac{\partial \phi}{\partial \theta}\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial \phi}{\partial r} d\theta + \int \frac{\partial \phi}{\partial \theta}\frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} d\theta \\ &= \int \frac{\partial \phi}{\partial \theta}\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial \phi}{\partial r} d\theta + \frac{1}{2r^2}\int \frac{\partial}{\partial \theta}(\frac{\partial \phi}{\partial \theta})^2 d\theta \\ &= \int \frac{\partial \phi}{\partial \theta}\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial \phi}{\partial r} d\theta + \int \frac{1}{r}(r\frac{\partial \phi}{\partial r}\frac{\partial^2 \phi}{\partial r \partial \theta}) d\theta \\ &= \int \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial \phi}{\partial r}\frac{\partial \phi}{\partial \theta}) d\theta \\ &= \frac{1}{r}\frac{\partial}{\partial r} < r\frac{\partial \phi}{\partial r}\frac{\partial \phi}{\partial \theta} > \\ &= -\frac{1}{r}\frac{\partial}{\partial r}r^2 < v_\theta v_r > \end{aligned} \tag{7}$$

and we can deduce the maxwell stress in a similiar way:

$$
\begin{aligned}
-\frac{\beta}{n_{eq}} < [A, j] > &= \frac{\beta}{n_{eq}} < [A, \nabla_\perp^2 A] > \\
&= \frac{\beta}{n_{eq}} \frac{1}{r} \frac{\partial}{\partial r} < j \frac{\partial A}{\partial \theta} > \\
&= \frac{\beta}{n_{eq}} \frac{1}{r} \frac{\partial}{\partial r} < A \frac{\partial j}{\partial \theta} > \\
&= \frac{\beta}{n_{eq}} \frac{1}{r} \frac{\partial}{\partial r} r^2 < B_\theta B_r >
\end{aligned} \tag{8}
$$

as for curvature terms,

$$
\begin{aligned}
- < \omega_d \cdot f > &= -2\epsilon < [r\cos\theta, f] > \\
&= 2\epsilon < \frac{1}{r} \left(\frac{\partial (r\cos\theta)}{\partial \theta} \frac{\partial f}{\partial r} - \frac{\partial (r\cos\theta)}{\partial r} \frac{\partial f}{\partial \theta} \right) > \\
&= \frac{2\epsilon}{r} (- < r\sin\theta \frac{\partial f}{\partial r} > - < \cos\theta \frac{\partial f}{\partial \theta} >) \\
&= \frac{2\epsilon}{r} (- < r\sin\theta \frac{\partial f}{\partial r} > - < f\sin\theta >) \\
&= -2\epsilon < (\frac{\partial f}{\partial r} + \frac{f}{r}) \sin\theta > \\
&= -2\epsilon \frac{1}{r} \frac{\partial}{\partial r} r < f\sin\theta >
\end{aligned} \tag{9}
$$

Here,

$$
f = T_i + \frac{T_{eq}}{n_{eq}} n + \frac{p_e}{n_{eq}} = \frac{p}{n_{eq}} \tag{10}
$$

then,

$$
- < \omega \cdot \frac{p}{n_{eq}} >= -\frac{2\epsilon}{n_{eq}} \frac{1}{r} \frac{\partial}{\partial r} r < p\sin\theta > \tag{11}
$$

so the evolution of zonal flow in the limit of collisionless situation is as follows,

$$
\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial t} (v_E) = -\frac{1}{r} \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} r^2 < v_\theta v_r > + \frac{1}{r} \frac{\partial}{\partial r} \frac{\beta}{n_{eq}} \frac{1}{r} \frac{\partial}{\partial r} r^2 < B_\theta B_r > - \frac{2\epsilon}{n_{eq}} \frac{1}{r} \frac{\partial}{\partial r} r < p\sin\theta > \tag{12}
$$

eliminate the commom part, we get,

$$
\frac{\partial}{\partial t} < v_E >= -\frac{1}{r^2} \frac{\partial}{\partial r} r^2 < v_\theta v_r > + \frac{\beta}{n_{eq}} \frac{1}{r^2} \frac{\partial}{\partial r} r^2 < B_\theta B_r > - \frac{2\epsilon}{n_{eq}} < p\sin\theta > \tag{13}
$$

and in the code, we use the alternative expression,

$$
\frac{\partial}{\partial t} < v_E >= \frac{1}{r} < \phi \frac{\partial \Omega}{\partial \theta} > + \frac{\beta}{n_{eq}} \frac{1}{r} < A \frac{\partial A}{\partial \theta} > - \frac{2\epsilon}{n_{eq}} < p\sin\theta > \tag{14}
$$

when we use the Fourier expansion,

$$
\frac{\partial}{\partial t} < v_E >= \sum_{m1+m2=0} \phi_{m1} \Omega_{\theta,m2} + \sum_{m1+m2=0} A_{m1} j_{\theta,m2} - \frac{i\epsilon}{n_{eq}} (p_{1,0} - p_{-1,0}) \tag{15}
$$

$$
< p\sin\theta >= \frac{i}{2} (p_{1,0} - p_{-1,0}) \tag{16}
$$

2 Appendix: equation for thr evolution of $< p \sin\theta >$

we can deduce the evolution of p with the drivative equation of n and T,

$$\frac{\partial}{\partial t} p = R_Q + R_F + R_C + R_{NL} + R_{LD} + R_S \tag{17}$$

where,

$$\begin{aligned}
R_Q &= a[(1+\tau)T_{eq}\frac{\partial n_{eq}}{\partial r} + n_{eq}\frac{\partial T_{eq}}{\partial r}]\nabla_\theta \phi + (1+\tau)\beta T_{eq}\frac{\partial j_0}{\partial r}\nabla_\theta A \\
R_F &= -(\Gamma+\tau)n_{eq}T_{eq}\nabla_\parallel v + (1+\tau)T_{eq}\nabla_\parallel j \\
R_C &= \omega_d((\Gamma+\tau)n_{eq}T_{eq}\phi) + \omega_d((2\Gamma-1)n_{eq}T_{eq}T) + \omega_d(((\Gamma-1)-\tau(\tau+1))T_{eq}^2 n) \\
R_{NL} &= -(1+\tau)T_{eq}[\phi, n] - n_{eq}[\phi, T] + (\Gamma+\tau)n_{eq}T_{eq}\beta[A, v] - (1+\tau)T_{eq}\beta[A, j] \\
R_{LD} &= -(\Gamma-1)\sqrt{\frac{8T_{eq}}{\pi}} n_{eq}|\nabla_\parallel| T \\
R_S &= (1+\tau)T_{eq}D_n\nabla_\perp^2 n + n_{eq}D_T\nabla_\perp^2 T
\end{aligned} \tag{18}$$

with,

$$\begin{aligned}
< \nabla_\theta f \sin\theta > &= \int \frac{1}{r}\frac{\partial f}{\partial \theta}\sin\theta d\theta \\
&= -\frac{1}{r} < f\cos\theta >
\end{aligned} \tag{19}$$

$$\begin{aligned}
< \nabla_\parallel f \sin\theta > &= \int \frac{a}{R}(f_\theta/q + f_\zeta)\sin\theta d\theta d\zeta \\
&\approx -\frac{\epsilon}{q} < f\cos\theta >
\end{aligned} \tag{20}$$

$$\begin{aligned}
< (\omega_d \cdot f)\sin\theta > &= 2\epsilon < [r\cos\theta, f]\sin\theta > \\
&= \frac{2\epsilon}{r}(< \frac{\partial f}{\partial \theta}\cos\theta\sin\theta > - < r\frac{\partial f}{\partial r} sin^2\theta >) \\
&= \epsilon(< \frac{\partial f}{\partial r} > - < (\frac{2f}{r} + \frac{\partial f}{\partial r})\cos 2\theta >)
\end{aligned} \tag{21}$$

we can convert the terms form to,

$$\begin{aligned}
< R_Q\sin\theta > &= a[(1+\tau)T_{eq}\frac{\partial n_{eq}}{\partial r} + n_{eq}\frac{\partial T_{eq}}{\partial r}] < \phi\cos\theta > +(1+\tau)\beta T_{eq}\frac{\partial j_0}{\partial r} < A\cos\theta > \\
< R_F\sin\theta > &= (\Gamma+\tau)n_{eq}T_{eq}\frac{\epsilon}{q} < v\cos\theta > -(1+\tau)T_{eq}\frac{\epsilon}{q} < j\cos\theta > \\
< R_C\sin\theta > &= \epsilon(\Gamma+\tau)p_{eq} < v_\theta > +\epsilon(2\Gamma-1)p_{eq} < \frac{\partial T}{\partial r} > +\epsilon((\Gamma-1)-\tau(\tau+1))T_{eq}^2 < \frac{\partial n}{\partial r} > \\
< R_{LD}\sin\theta > &= -(\Gamma-1)\sqrt{\frac{8T_{eq}}{\pi}} n_{eq}\frac{\epsilon}{q} < T\sin\theta >
\end{aligned} \tag{22}$$