July 16, 2023, Set F2

The joint PMF of two discrete random variables X and Y is given in the following table:

$Y \backslash X$	0	1	2	$f_Y(y)$
0	$\frac{1}{6}$	a	b	$\frac{1}{3}$
1	c	d	$\frac{1}{9}$	$\frac{2}{3}$
$f_X(x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	1

Marginal

Marginal

Joint PMF of X and Y

Which of the following options is correct?

Options :

6406531926954. ✗ $2a = b = 6c = 4d$

6406531926955. ✗ $2a = b = 4c = 6d$

6406531926956. ✓ $6a = 12b = 2c = 3d$

6406531926957. ✗ $9a = 18b = 3c = 2d$

$$\frac{1}{6} + c = \frac{1}{2}$$

$$\Rightarrow c = \frac{3-1}{6} = \frac{1}{3}$$

$$c + d + \frac{1}{9} = \frac{2}{3}$$

$$\Rightarrow d = \frac{2}{3} - \left(\frac{1}{9} + c\right)$$

$$\Rightarrow d = \frac{2}{3} - \frac{1}{9} - \frac{1}{3}$$

$$\Rightarrow d = \frac{3-1}{9} = \frac{2}{9}$$

$$b + \frac{1}{9} = \frac{1}{6}$$

$$\Rightarrow b = \frac{3-2}{18}$$

$$\Rightarrow b = \frac{1}{18}$$

$$a + d = \frac{1}{3}$$

$$\Rightarrow a = \frac{1}{3} - \frac{2}{9}$$

$$\Rightarrow a = \frac{1}{9}$$

Let $X \sim \text{Binomial}(n, p)$. If the expected value and variance of X are 2 and $\frac{3}{2}$, respectively, find the value of $P(X = 2)$.

Options :

6406531926967. ✓ $\frac{7 \times 3^6}{4^7}$

6406531926968. ✗ $\frac{7 \times 3^6}{4^8}$

6406531926969. ✗ ${}^8C_2 \left(\frac{3^2}{4^8}\right)$

6406531926970. ✗ ${}^4C_2 \left(\frac{3^2}{4^4}\right)$

From formula table,

			μ	σ^2
Binomial(n,p)	${}^nC_k p^k (1-p)^{n-k}$, $k = 0, 1, \ldots, n$	$\begin{cases} 0 & x < 0 \\ \sum_{i=0}^{k} {}^nC_i p^i (1-p)^{n-i} & k \le x < k+1 \\ & k = 0, 1, \ldots, n \\ 1 & x \ge n \end{cases}$	np	$np(1-p)$

Given,

$\mu = 2$

$\sigma^2 = \frac{3}{2}$

① $\Rightarrow np = 2$

$np(1-p) = \frac{3}{2}$

From ① $\Rightarrow 2(1-p) = \frac{3}{2}$

$\Rightarrow 1 - p = \frac{3}{4}$

$p = \frac{1}{4}$

If $p = 1/4$

$\Rightarrow np = 2$ [putting $p = 1/4$ in ①]

$\Rightarrow n\left(\frac{1}{4}\right) = 2$

$\Rightarrow n = 8$

$\therefore P(X=2) = {}^8C_2 \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^6$

$\Rightarrow \frac{8!}{2!\,6!} \left(\frac{3^6}{4^8}\right)$

$\Rightarrow \frac{8 \times 7 \times 6!}{2 \times 6!} \left(\frac{3^6}{4^8}\right)$

$\Rightarrow \frac{\cancel{28}^{7} \times 3^6}{\cancel{4} \times 4^7}$

$\Rightarrow \underline{\underline{\frac{7 \times 3^6}{4^7}}}$

Suppose two fair dice are rolled. Let a random variable X denote the number obtained on the first die and let a random variable Y denote the number obtained on the second die. Define a new random variable $U = X + Y - 1$.

Find the range of U.

Options :

6406531926958. ✖ $T_U = \{0, 1, 2, \ldots, 12\}$

6406531926959. ✖ $T_U = \{1, 2, \ldots, 12\}$

6406531926960. ✖ $T_U = \{0, 1, 2, \ldots, 11\}$

6406531926961. ✔ $T_U = \{1, 2, \ldots, 11\}$

Minimum value X can take = 1

Minimum value Y can take = 1

$\Rightarrow$ Minimum value of $U = X + Y - 1$

$\Rightarrow 1 + 1 - 1$

$\Rightarrow \underline{\underline{1}}$

Maximum value X can take = 6

Maximum value Y can take = 6

$\Rightarrow$ Maximum value of $U = X + Y - 1$

$\Rightarrow 6 + 6 - 1$

$\Rightarrow \underline{\underline{11}}$

$\therefore$ Range of $U = T_U = \{1, 2, \ldots \ldots 11\}$

Find the value of $P(X = 4,\ U = 8)$.

Options :

6406531926962. ✖ $\frac{1}{6}$

6406531926963. ✖ $\frac{2}{3}$

6406531926964. ✔

If $X = 4$ and $U = 8$

$\Rightarrow U = X + Y - 1$

$\Rightarrow 8 = 4 + Y - 1$

$\Rightarrow 5 = Y$

$\Rightarrow P(X=4, U=8) = P(X=4, Y=5)$

$\frac{1}{36}$

6406531926965. ✖ $\frac{1}{3}$

$\Rightarrow P(X=4, Y=5) = P(X=4) \times P(Y=5)$

$\Rightarrow \frac{1}{6} \times \frac{1}{6}$

$\Rightarrow \underline{\underline{\frac{1}{36}}}$

The probability mass function of a random variable X is given as

x	-3	6	9
$P(X=x)$	$1/6$	$1/2$	$1/3$

Define $Y = (2X+1)^2$. Find the expected value of Y.

Response Type : Numeric

Evaluation Required For SA : Yes

Show Word Count : Yes

Answers Type : Equal

Text Areas : PlainText

Possible Answers :

209

Given,

$Y = (2X+1)^2$

$\Rightarrow$ If $X = -3$

$Y = ((2\times -3)+1)^2 = 25$

$\Rightarrow$ If $X = 6$

$Y = ((2\times 6)+1)^2 = 169$

$\Rightarrow$ If $X = 9$

$Y = ((2\times 9)+1)^2 = 361$

PMF of Y

y	25	169	361
$P(Y=y)$	$1/6$	$1/2$	$1/3$

$\Rightarrow E[Y] = \frac{25}{6} + \frac{169}{2} + \frac{361}{3}$

$\Rightarrow \frac{25}{6} + \frac{169\times 3}{6} + \frac{361\times 2}{6}$

$\Rightarrow \frac{25+507+722}{6} = \frac{1254}{6} = \underline{\underline{209}}$

Let X be a continuous random variable with the following PDF:

$$f_X(x) = \begin{cases} \frac{k}{(1+x)^2}, & 0 \le x \le 4, \\ 0, & \text{otherwise.} \end{cases}$$

Find the value of k. Enter the answer correct to two decimal places.

Hint: $\int \frac{1}{(a+bx)^2} dx = \frac{-1}{b(a+bx)}$

Response Type : Numeric

Evaluation Required For SA : Yes

Show Word Count : Yes

Answers Type : Range

Text Areas : PlainText

Possible Answers :

1.23 to 1.27

Given,

$f_x(x) = \frac{k}{(1+x)^2} \qquad 0 \le x \le 4$

We know PDF should integrate/sum upto 1

$\Rightarrow \int_0^4 \frac{k}{(1+x)^2} = 1$

$\Rightarrow k\left[\int_0^4 \frac{1}{(1+x)^2}\right] = 1$

$\Rightarrow k\left[\frac{-1}{(1+x)}\right]_0^4 = 1$

$\Rightarrow k\left[\frac{-1}{1+4} - \frac{-1}{1+0}\right] = 1$

$\Rightarrow k\left[1 - \frac{1}{5}\right] = 1$

$\Rightarrow k\left(\frac{4}{5}\right) = 1$

$\Rightarrow k = \frac{5}{4} = 1.25$

The joint PMF of two discrete random variables X and Y is

$$f_{XY}(x,y) = \begin{cases} \frac{1}{32}(x^2+y), & x \in \{0,1,2,3\}, y \in \{0,1\}, \\ 0, & \text{otherwise.} \end{cases}$$

Identify the correct joint PMF table of X and Y:

Options :

$Y \backslash X$	0	1	2	3
0	0	$\frac{1}{32}$	$\frac{4}{32}$	$\frac{9}{32}$
1	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{5}{32}$	$\frac{10}{32}$

6406531926972. ✔

$Y \backslash X$	0	1	2	3
0	0	$\frac{1}{32}$	$\frac{4}{32}$	$\frac{10}{32}$
1	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{4}{32}$	$\frac{10}{32}$

6406531926973. ✖

$Y \backslash X$	0	1	2	3
0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{5}{32}$	$\frac{10}{32}$
1	0	$\frac{1}{32}$	$\frac{4}{32}$	$\frac{9}{32}$

6406531926974. ✖

$X \backslash Y$	0	1	2	3
0	0	$\frac{1}{32}$	$\frac{4}{32}$	$\frac{9}{32}$
1	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{5}{32}$	$\frac{10}{32}$

6406531926975. ✖

Only way to do this is to check for respective values of x and y

$$P(X=2, Y=1) = \frac{1}{32}(2^2+1)$$

$$\Rightarrow \frac{5}{32}$$

$$P(X=3, Y=0) = \frac{1}{32}(3^2+0)$$

$$\Rightarrow \frac{9}{32}$$

Find $P\left(\frac{1}{2} < X < \frac{5}{2} \mid X > 1\right)$. Enter the answer correct to 2 decimal places.

Response Type : Numeric

Evaluation Required For SA : Yes

Show Word Count : Yes

Answers Type : Range

Text Areas : PlainText

Possible Answers :

0.30 to 0.34

$$P\left(\frac{1}{2} < X < \frac{5}{2} \,\middle|\, X > 1\right) = \frac{P\left(\frac{1}{2} < X < \frac{5}{2} \cap X > 1\right)}{P(X>1)}$$

Given X is greater than 1,

$\Rightarrow$ Actual Range of $X = 1 < X < \frac{5}{2}$

$$\Rightarrow P\left(\frac{1}{2} < X < \frac{5}{2} \cap X > 1\right) = P\left(1 < X < \frac{5}{2}\right)$$

$$\Rightarrow P(1 < X < 2.5)$$

As X is a discrete random variable, it does not take the value of 2.5

The closest possible value less than 2.5, that X takes is 2.

$$P(1 < X < 2.5) = P(1 < X \leq 2)$$

$$\therefore P\left(\frac{1}{2} < X < \frac{5}{2} \,\middle|\, X > 1\right) = \frac{P(1 < X \leq 2)}{P(X > 1)}$$

[Only value which is greater than 1 and less than or equal to 2 is 2.]

$$\Rightarrow \frac{P(X = 2)}{P(X=2) + P(X=3)}$$

$$\Rightarrow \frac{9/32}{9/32 + 19/32}$$

$$\Rightarrow \frac{9}{28} \simeq \underline{\underline{0.3214}}$$

Ten students from classes 9 and 10 have been nominated to form the school committee.

The table below provides the number of boys and girls selected from each class:

	class 9	class 10
Boys	1	5
Girls	3	1

The committee will consist of four students, with two students selected from each class uniformly at random.

Let a random variable G represent the number of girls selected for the committee. Find the range of G.

Options :

6406531926977. ✖ $T_G = \{0, 1, 2, 3\}$

6406531926978. ✔ $T_G = \{1, 2, 3\}$

6406531926979. ✖ $T_G = \{1, 2, 3, 4\}$

6406531926980. ✖ $T_G = \{0, 1, 2, 3, 4\}$

Committee consists of two students chosen from each class uniformly at random.

In class 9,

Boys = 1

Girls = 3

$\Rightarrow$ Minimum number of girls that can be chosen = 1

As there is only 1 boy and 2 students must be chosen to form a committee, 1 girl must be chosen along with 1 boy.

As the committee consists of 2 students from each class, even if 0 girls are chosen from class 10, 1 girl atleast will always be chosen from class 9.

[Class 9: G B   class 10: B B]

$\therefore$ Minimum number of girls in the committee = 1

For maximum number of girls,

Similarly, 1 boy will always be chosen from class 10, as there is only 1 girl in class 10 and committee consists of 2 students.

Even if 2 girls are chosen from class 9, 1 boy will be atleast chosen from class 10.

$\therefore$ Maximum number of girls in the committee = 3 [class 9: G G, class 10: G B]

$\therefore$ Range of girls in committee $= T_G = \{1, 2, 3\}$

Find the expected number of girls selected for the committee. Enter the answer correct to two decimal places.

Response Type : Numeric

Evaluation Required For SA : Yes

Show Word Count : Yes

Answers Type : Range

Text Areas : PlainText

Possible Answers :

1.81 to 1.85

In a bookstore, there are two book types: Type 1 and Type 2. Let X and Y be independent random variables representing the number of Type 1 and Type 2 books sold in a week, respectively. Suppose X and Y follow the Poisson distribution with averages of 2 and 3, respectively. Define a new random variable $Z = X + Y$.

If $Z = 5$, then which of the following options are true?

Options :

6406531926982. ✖ $(Y|Z) \sim \text{Binomial}(5, \frac{2}{5})$.

6406531926983. ✔ $(X|Z) \sim \text{Binomial}(5, \frac{2}{5})$.

6406531926984. ✖ $(X|Z) \sim \text{Binomial}(5, \frac{3}{5})$.

6406531926985. ✔ $(Y|Z) \sim \text{Binomial}(5, \frac{3}{5})$.

Two Independent Poisson

Sum

$Z = X + Y$

Sum of 2 poisson is poisson random variable

$$f_Z(Z) = \frac{e^{-(\lambda_1+\lambda_2)} \times (\lambda_1+\lambda_2)^Z}{Z!}$$

Conditional distribution of X|Z

$$P(X = k|Z = n) = \frac{n!}{k!(n-k)!} \times (\frac{\lambda_1}{\lambda_1+\lambda_2})^k \times (\frac{\lambda_2}{\lambda_1+\lambda_2})^{n-k}$$

which is also equals to

condition distribution is binomial

$$P(X = k|Z = n) = Binomial(n, \frac{\lambda_1}{\lambda_1+\lambda_2})$$

given that $X|Z \sim Binomial(n, \frac{\lambda_1}{\lambda_1+\lambda_2})$

Here, λ_1 = mean/average of $X = 2$

λ_2 = mean/average of $Y = 3$

Also, $Z = 5$

$\therefore$ According to formula,

$$P(X|Z) = \text{Binomial}\left(n, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$$

$$\Rightarrow \text{Binomial}\left(5, \frac{2}{2+3}\right) \qquad [n = Z = 5]$$

Similarly,

$$P(Y|Z) = \text{Binomial}\left(n, \frac{\lambda_2}{\lambda_1+\lambda_2}\right)$$

$$\Rightarrow \text{Binomial}\left(5, \frac{\lambda_2}{\lambda_1+\lambda_2}\right) \qquad [n = Z = 5]$$

Question Label : Short Answer Question

Find the value of $P(X = 1|Z = 5)$.

Enter the answer correct to two decimal places.

Response Type : Numeric

Evaluation Required For SA : Yes

Show Word Count : Yes

Answers Type : Range

Text Areas : PlainText

Possible Answers :

0.24 to 0.28

$$P(X=1|Z=5) = \text{Binomial}\left(5, \frac{2}{5}\right)$$

$$\Rightarrow {}^5C_1 \times \left(\frac{2}{5}\right)^1 \times \left(\frac{3}{5}\right)^4$$

$$\Rightarrow \cancel{5} \times \frac{2}{\cancel{5}} \times \left(\frac{3}{5}\right)^4$$

$$\Rightarrow 2 \times 0.1296 \simeq \underline{\underline{0.2592}}$$

Suppose $X \sim \text{Binomial}\left(n, \frac{1}{2}\right)$.

Find the value of n for which $\frac{1}{30}P(X = 3) = P(X = 2)$.

Options :

6406531926987. ✖ 90

6406531926988. ✔ 92

6406531926989. ✖ 30

6406531926990. ✖ 32

A.T.Q.

$$\frac{1}{30} P(X=3) = P(X=2)$$

$$\Rightarrow \frac{1}{30} \times {}^nC_3 \times \left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2}\right)^{n-3} = {}^nC_2 \times \left(\frac{1}{2}\right)^2 \times \left(\frac{1}{2}\right)^{n-2}$$

$$\Rightarrow \frac{1}{30} \times {}^nC_3 \times \left(\frac{1}{2}\right)^{3+n-3} = {}^nC_2 \times \left(\frac{1}{2}\right)^{2+n-2}$$

$$\Rightarrow \frac{1}{30} \times {}^nC_3 \times \cancel{\left(\frac{1}{2}\right)^n} = {}^nC_2 \times \cancel{\left(\frac{1}{2}\right)^n}$$

$$\Rightarrow \frac{1}{30} \times \frac{\cancel{n!}}{3!(n-3)!} = \frac{\cancel{n!}}{2!(n-2)!}$$

$$\Rightarrow \frac{1}{30} \times 2!(n-2)! = 3!(n-3)!$$

$$\Rightarrow \frac{1}{30} \times 2(n-2)\cancel{(n-3)!} = 3!\cancel{(n-3)!}$$

[$(n-2)!$ can be written as $(n-2)(n-2-1)!$]

$$\Rightarrow (n-2) = \frac{\overset{3}{\cancel{6}} \times 30}{\cancel{2}}$$

$$\Rightarrow n-2 = 90$$

$$\Rightarrow n = 92$$

Using the Chebyshev's inequality, find a lower bound for $P(-2\sigma \leq X-\mu \leq 2\sigma)$, where μ and σ^2 are mean and variance of X. Enter the answer correct to 2 decimal places

Response Type : Numeric

Evaluation Required For SA : Yes

Show Word Count : Yes

Answers Type : Equal

Text Areas : PlainText

Possible Answers :

0.75

From formula table,

2. **Chebyshev's inequality:** Let X be a discrete random variable with a finite mean μ and a finite variance σ^2. Then,

$$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$$

$$\Rightarrow P(|X-\mu| < k\sigma) \geqslant 1 - \frac{1}{k^2}$$

$$\Rightarrow P\left(|X-\mu| < 2\sigma\right) \geqslant 1 - \frac{1}{2^2}$$

$$\Rightarrow P\left(|X-\mu| < 2\sigma\right) \geqslant \frac{3}{4}$$

Note by expanding the modulus in the above equation we get the same thing given in the question.

$$P(-2\sigma \leqslant X-\mu \leqslant 2\sigma) = P(|X-\sigma| < 2\sigma)$$

Sruthi throws a dart onto a circular board. Let a random variable X denote the distance from the center to the point where the dart hits the board. Suppose the PDF of X is

$$f_X(x) = \begin{cases} kx(1-x^2), & 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find the value of k.

Response Type : Numeric

Evaluation Required For SA : Yes

Show Word Count : Yes

Answers Type : Equal

We know,

integral of a PDF over full range must always be 1.

$$\Rightarrow \int_0^1 kx(1-x^2) = 1$$

$$\Rightarrow k\int_0^1 x - x^3 = 1$$

$$\Rightarrow k\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = 1$$

$$\Rightarrow k\left[\frac{1}{2} - \frac{1}{4} - \left(\frac{0}{2} - \frac{0}{4}\right)\right] = 1$$

Text Areas : PlainText

Possible Answers :

4

$$\Rightarrow k\left[\frac{2}{4}-\frac{1}{4}\right]=1$$

$$\Rightarrow k=4$$

Find the value of $P\left(\frac{1}{4}\leq X\leq\frac{3}{4}\right)$. Enter the answer correct to two decimal places.

Response Type : Numeric

Evaluation Required For SA : Yes

Show Word Count : Yes

Answers Type : Range

Text Areas : PlainText

Possible Answers :

0.66 to 0.72

$$P\left(\frac{1}{4}\leq X\leq\frac{3}{4}\right)$$

$\Rightarrow$ X lies in the range of $\left[\frac{1}{4},\frac{3}{4}\right]$

$\Rightarrow$ Lower limit $=\frac{1}{4}$

Upper limit $=\frac{3}{4}$

$$\Rightarrow \int_{1/4}^{3/4} 4(x)(1-x^2)$$

$$\Rightarrow 4\int_{1/4}^{3/4} x-x^3$$

$$\Rightarrow 4\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_{1/4}^{3/4}$$

$$\Rightarrow 4\left[\left(\frac{(3/4)^2}{2}-\frac{(3/4)^4}{4}\right)-\left(\frac{(1/4)^2}{2}-\frac{(1/4)^4}{4}\right)\right]$$

$$\Rightarrow 4\left[\left(\frac{9}{16\times2}-\frac{81}{256\times4}\right)-\left(\frac{1}{16\times2}-\frac{1}{256\times4}\right)\right]$$

$$\Rightarrow 4\left[\frac{9-1}{32}+\frac{-81+1}{1024}\right]$$

$$\Rightarrow 4\left[\frac{\cancel{8}^{1}}{\cancel{32}_{4}}-\frac{80}{1024}\right]$$

$$\Rightarrow 4(0.1718)\simeq 0.6872$$