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1.

$f: V \to V'$ aplicación lineal. $\bar{f}: V/\ker(f) \to \operatorname{Im}(f)$

$$\bar{f}(v+\ker(f)) = f(v) \quad v \in V$$

Veamos que $\bar{f}$ está bien definida. Para ello, $\bar{f}(v+\ker(f))$ no depende de v sino de su clase.

Supongamos: $x, y \in V$

$x+\ker(f) = y+\ker(f) \Rightarrow x-y \in \ker(f) \Rightarrow f(x-y)=0 \Rightarrow f(x)-f(y)=0$

y así, $f(x) = f(y)$

Ahora veamos que $\bar{f}$ es lineal. $\forall \alpha, \beta \in K \ \forall (x+\ker(f)), (y+\ker(f)) \in V/\ker(f)$

$\bar{f}[\alpha(x+\ker(f)) + \beta\cdot(y+\ker(f))] = \bar{f}[(\alpha x+\beta y)+\ker(f)] = f(\alpha x+\beta y) =$

$= \alpha\cdot f(x) + \beta\cdot f(y) = \alpha\cdot \bar{f}[x+\ker(f)] + \beta\cdot \bar{f}[y+\ker(f)]$

Ahora comprobemos que $\bar{f}$ es monomorfismo.

$\ker(\bar{f}) = \{v+\ker(f) \in V/\ker(f) : \bar{f}(v+\ker(f)) = f(v) = 0\}$

$\ker(\bar{f}) = \ker(f) = \{0+\ker(f)\} \to$ vector nulo.

El núcleo de $\bar{f}$ se reduce al vector nulo de $V/\ker(f)$ así $\bar{f}$ es inyectiva.

Ahora veamos que $\bar{f}$ es epimorfismo.

Si $v' \in \operatorname{Im}(f) \Rightarrow \exists v \in V : f(v) = v'$, $\Rightarrow v' = \bar{f}(v+\ker(f))$. Así $\bar{f}$ es sobreyectiva.

Siendo $\bar{f}$ inyectiva y sobreyectiva, es biyectiva y isomorfismo.

$M_2(\mathbb{R})$ $\quad \varphi\begin{pmatrix} x & y \\ z & t \end{pmatrix} = x+t \quad \psi\begin{pmatrix} x & y \\ z & t \end{pmatrix} = x+y+z+t$

• Ampliar $\{\varphi, \psi\}$ a una base de espacio dual

Sea $B_u = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ la base usual de $M_2(\mathbb{R})$

$B_u^* = \{\phi^1, \phi^2, \phi^3, \phi^4\}$ la base usual del espacio dual $\phi^i(e_j) = \delta_{ij}$

$\begin{pmatrix} x & y \\ z & t \end{pmatrix} = \alpha \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \beta \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \gamma \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \sigma \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \sigma \end{pmatrix} \Rightarrow \begin{matrix} \alpha = x & \gamma = z \\ \beta = y & \sigma = t \end{matrix}$

$\phi^1\begin{pmatrix} x & y \\ z & t \end{pmatrix} = x \cdot \phi^1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + y \cdot \phi^1\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + z\phi^1\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + t \cdot \phi^1\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\phi^1 = x$ Análogamente, $\phi^2 = y \quad \phi^3 = z \quad \phi^4 = t$

* Disponemos las matrices como vectores para facilitar el cálculo

$B_u^* = \{x, y, z, t\} = \{\phi^1, \phi^2, \phi^3, \phi^4\}$

$\varphi = a_1 \cdot \phi^1 + a_2 \phi^2 + a_3 \phi^3 + a_4 \phi^4 \quad \varphi\begin{pmatrix} x & y \\ z & t \end{pmatrix} = x+t$

$a_1 + a_4 = x+t \qquad \varphi = (\phi^1, \phi^2, \phi^3, \phi^4)_{B_u^*} \Rightarrow (1,0,0,1)$

$\psi = b_1\phi^1 + b_2\phi^2 + b_3\phi^3 + b_4\phi^4 \quad \psi\begin{pmatrix} x & y \\ z & t \end{pmatrix} = x+y+z+t$

$\psi = (\phi^1, \phi^2, \phi^3, \phi^4)_{B_u^*} \Rightarrow \psi = (1,1,1,1)_{B_u^*}$

Ahora ampliamos

$B^* = \{(1,0,0,1), (1,1,1,1), (1,0,0,0), (0,1,0,0)\}$ * Tomamos coordenadas de la base usual de $\mathbb{R}^4$ para comprobar si son L.i

$\begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{vmatrix} = -1 \neq 0$ Son L.i y forman base.

$B^* = \{x+t, x+y+z+t, x, y\} = \{\varphi, \psi, \phi^1, \phi^2\}$

• Hallar la matriz de cambio de base entre la base usual de la usual de $M_2(\mathbb{R})$ y la base obtenida en el apartado anterior.

$B_U^* = \{\phi^1, \phi^2, \phi^3, \phi^4\}$ $\quad B^* = \{\varphi, \psi, \phi^1, \phi^2\}$

$B_U^* = \{x, y, z, t\}$ $\quad B^* = \{x+t, x+y+z+t, x, y\}$

Ahora trabajamos en coordenadas de la base usual de $M_2(\mathbb{R})$

$B^* = \{(1,0,0,1), (1,1,1,1), (1,0,0,0), (0,1,0,0)\}$

$B_U^* = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

$$M(Id_{(M_2(\mathbb{R}))^*}, B_U^* \text{ a } B^*) = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

$(1,0,0,0) = \alpha \cdot (1,0,0,1) + \beta \cdot (1,1,1,1) + \gamma (1,0,0,0) + \lambda (0,1,0,0)$

$$\left.\begin{aligned} \alpha+\beta+\gamma &= 1 \\ \beta+\lambda &= 0 \\ \beta &= 0 \\ \alpha+\beta &= 0 \end{aligned}\right\} \begin{aligned} \alpha &= 0 \\ \beta &= 0 \\ \gamma &= 1 \\ \lambda &= 0 \end{aligned} \qquad (1,0,0,0)_{B_U^*} = (0,0,1,0)_{B^*}$$

$(0,1,0,0) = \alpha \cdot (1,0,0,1) + \beta (1,1,1,1) + \gamma (1,0,0,0) + \lambda (0,1,0,0)$

$$\left.\begin{aligned} \alpha+\beta+\gamma &= 0 \\ \beta+\lambda &= 1 \\ \beta &= 0 \\ \alpha+\beta &= 0 \end{aligned}\right\} \begin{aligned} \alpha &= 0 \\ \beta &= 0 \\ \gamma &= 0 \\ \lambda &= 1 \end{aligned} \qquad (0,1,0,0)_{B_U^*} = (0,0,0,1)_{B^*}$$

$(0,0,1,0) = \alpha \cdot (1,0,0,1) + \beta \cdot (1,1,1,1) + \gamma \cdot (1,0,0,0) + \lambda (0,1,0,0)$

$$\left.\begin{aligned} \alpha+\beta+\gamma &= 0 \\ \beta+\lambda &= 0 \\ \beta &= 1 \\ \alpha+\beta &= 0 \end{aligned}\right\} \begin{aligned} \alpha &= -1 \\ \beta &= 1 \\ \gamma &= 0 \\ \lambda &= -1 \end{aligned} \qquad (0,0,1,0)_{B_U^*} = (-1,1,0,-1)_{B^*}$$

$(0,0,0,1) = \alpha \cdot (1,0,0,1) + \beta \cdot (1,1,1,1) + \gamma \cdot (1,0,0,0) + \lambda (0,1,0,0)$

$$\left.\begin{aligned} \alpha+\beta+\gamma &= 0 \\ \beta+\lambda &= 0 \\ \beta &= 0 \\ \alpha+\beta &= 1 \end{aligned}\right\} \begin{aligned} \alpha &= 1 \\ \beta &= 0 \\ \gamma &= -1 \\ \lambda &= 0 \end{aligned} \qquad (0,0,0,1)_{B_U^*} = (1,0,-1,0)_{B^*}$$

$$M(Id_{(M_2(\mathbb{R}))^*}, B^* \text{ a } B_U^*) = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$