n) $\int \frac{x+1}{x^4-1}\,dx$

Hallamos las raíces del denominador:

$x = \sqrt[4]{1} \left\langle \begin{matrix} 1 \\ -1 \end{matrix} \right\}$ Raíces dobles

Simplificamos el cociente:

$$\int \frac{x+1}{x^4-1}\,dx = \int \frac{x+1}{(x^2-1)(x^2+1)}\,dx = \int \frac{\cancel{x+1}}{(\cancel{x+1})(x-1)(x^2+1)}\,dx = \int \frac{1}{(x-1)(x^2+1)}\,dx$$

Raíces del denominador:

- $x = 1 \longrightarrow$ Raíz real
- x^2+1 no tiene raíces reales

Aplicamos el método de coeficientes indeterminados:

$$\frac{1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$\Downarrow$$

$$\int \frac{1}{(x-1)(x^2+1)}\,dx = \int \left[\frac{A}{x-1} + \frac{Bx+C}{x^2+1} \right] dx \underset{\substack{\uparrow \\ \text{por linealidad}}}{=} A \int \frac{1}{x-1}\,dx + \int \frac{Bx+C}{x^2+1}\,dx$$

$$= A \ln|x-1| + B \int \frac{x}{x^2+1}\,dx + C \int \frac{1}{x^2+1}\,dx =$$

$$= A \ln|x-1| + B \cdot \frac{1}{2} \ln|x^2+1| + C \cdot \operatorname{arctg}(x) + C$$

Calculamos A, B y C:

$$\frac{1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} \Longrightarrow 1 = A(x^2+1) + (Bx+C)(x-1)$$

- Si $x = 1$: $1 = 2A \longrightarrow A = \frac{1}{2}$
- Si $x = 0$: $1 = A - C = \frac{1}{2} - C \Longrightarrow C = \frac{-1}{2}$
- Si $x = 2$: $1 = 5A + 2B + C = \frac{5}{2} + 2B - \frac{1}{2} \Longrightarrow B = \frac{-1}{2}$

Por tanto, la integral nos queda:

$$\int \frac{1}{(x-1)(x^2+1)}\,dx = \frac{1}{2} \ln|x-1| - \frac{1}{4} \ln|x^2+1| - \frac{1}{2} \operatorname{arctg}(x) + C$$

ñ) $\int \frac{x-2}{x(x+1)(x-1)}\,dx$

Al igual que antes, aplicamos el método de coeficientes indeterminados:

$$\frac{x-2}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$$

$$\Downarrow$$

$$\int \frac{x-2}{x(x+1)(x-1)}\,dx = A\int \frac{1}{x}\,dx + B\int \frac{1}{x+1}\,dx + C\int \frac{1}{x-1}\,dx =$$

$$= A\ln|x| + B\ln|x+1| + C\ln|x-1| + C$$

Calculamos A, B y C:

$$\frac{x-2}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1} \Longrightarrow x-2 = A(x+1)(x-1) + Bx(x-1) + Cx(x+1)$$

- Si $x=0$: $-2 = -A + B\cdot 0 + C\cdot 0 \Longrightarrow A = 2$
- Si $x=1$: $-1 = A\cdot 0 + B\cdot 0 + C\cdot 2 \Longrightarrow C = \frac{-1}{2}$
- Si $x=-1$: $-3 = A\cdot 0 + B\cdot 2 + C\cdot 0 \Longrightarrow B = \frac{-3}{2}$

En consecuencia:

$$\int \frac{x-2}{x(x+1)(x-1)}\,dx = 2\ln|x| - \frac{3}{2}\ln|x+1| - \frac{1}{2}\ln|x-1| + C$$