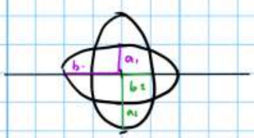
Ejercicio 7.15: Calcular el área comprendida entre las elipses $\frac{x^2}{1}+\frac{y^2}{4}=1$ y $\frac{x^2}{4}+\frac{y^2}{1}=1$.

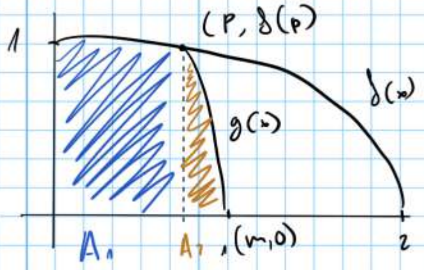
Sabemos que ambas están centradas en el origen y que tienen la a y la b intercambiadas, es decir,



$a_1 = b_2 = 1$

$a_2 = b_1 = \sqrt{4} = 2$

Por esto, puedo aprovechar la simetría y calcular el área del primer cuadrante.



$g(x) = \sqrt{4(1-x^2)}$

$f(x) = \sqrt{1-\frac{x^2}{4}}$

$p \in \mathbb{R} \mid f(p) = g(p)$

$m \in \mathbb{R} \mid g(m) = 0$

$A = A_1 + A_2$

$A = \int_0^p f(x)\,dx + \int_p^m g(x)\,dx$

hayo los puntos necesarios

p:

$\sqrt{4(1-x^2)} = \sqrt{1-\frac{x^2}{4}} \iff 4-4x^2 = 1-\frac{x^2}{4} \iff \frac{15x^2}{4} = +3$

$x = \sqrt{\frac{4}{5}} = \frac{2\sqrt{5}}{5} \Rightarrow \boxed{p = \left(\frac{2\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right)}$

m:

$\sqrt{4-4x^2} = 0 \iff 4-4x^2 = 0 \quad x = 1$

$\boxed{m = (1,0)}$

$$A = \int_0^{\frac{2\sqrt{5}}{5}} \sqrt{1-\frac{x^2}{4}}\,dx + \int_{\frac{2\sqrt{5}}{5}}^{1} \sqrt{4-4x^2}\,dx$$

$$\int \sqrt{1-\frac{x^2}{4}}\,dx = \left[\begin{array}{l} x = 2\,\text{sen}(t) \\ dx = 2\cos(t)dt \end{array}\right] = \int \sqrt{1-\text{sen}^2(t)}\; 2\cos(t)\,dt = \int 2\cos^2(t)\,dt$$

$\cos^2 + \text{sen}^2 = 1$

$\cos^2 - \text{sen}^2 = \cos(2x)$

$2\cos^2 = 1 + \cos(2x)$

$$\int 1+\cos(2t)\,dt = t + \frac{\text{sen}(2t)}{2} = \text{arc sen}\left(\frac{x}{2}\right) + \frac{x}{2}\sqrt{1-\frac{x^2}{4}}$$

$t = \text{arcsen}\left(\frac{x}{2}\right)$

$\text{sen}(2t) = 2\,\text{sen}(t)\cos(t)$

$$\int 2\sqrt{1-x^2}\,dx = 2\int \sqrt{1-x^2}\,dx = \left[\begin{array}{l} x = sen(t) \\ dx = \cos(t)\,dt \end{array}\right] = 2\int \sqrt{1-sen^2}\,\cos(t)\,dt = \int 2\cos(t)^2\,dt$$

$$= \int 1+\cos(2t)\,dt = t + \frac{sen(2t)}{2} = \underline{arc\,sen(x) + x\sqrt{1-x^2}}$$

$t = arcsen(x)$

$sen(2t) = 2\,sen\,cos$

$2x\sqrt{1-x^2}$

$$A = \int_0^{\frac{2\sqrt{5}}{5}} \sqrt{1-\frac{x^2}{4}}\,dx + \int_{\frac{2\sqrt{5}}{5}}^{1} \sqrt{4-4x^2}\,dx = arc\,sen\left(\frac{\sqrt{5}}{5}\right) + \frac{\sqrt{5}}{5}\sqrt{1-\frac{1}{5}} + \left(\frac{\pi}{2}\right) - arsen\left(\frac{2\sqrt{5}}{5}\right) - \frac{2\sqrt{5}}{5}\sqrt{1-\frac{4}{5}}$$

$\approx \underline{0,927}$

luego el área total es

$$\boxed{4A \approx 3,7091}$$