6.1. Calcular las siguientes integrales:

5) $\int \sec(x)\,dx$

$\sec(x)$ es equivalente a $\sec(x) \cdot \underbrace{\frac{\sec(x)+tg(x)}{\sec(x)+tg(x)}}_{1}$

y por lo tanto resolver la anterior integral es equivalente a resolver la siguiente: $\int \frac{\sec^2(x)+\sec(x)tg(x)}{\sec(x)+tg(x)}\,dx$

Apliquemos el siguiente cambio de variable:

$$t = \sec(x)+tg(x)$$

$$1 \cdot dt = \left(\frac{sen(x)}{\cos^2(x)} + \frac{1}{\cos^2(x)}\right) dx$$

$$1 \cdot dt = \left(\frac{sen(x)}{\cos(x)} \cdot \frac{1}{\cos(x)} + \frac{1}{\cos(x)} \cdot \frac{1}{\cos(x)}\right) dx$$

$$1 \cdot dt = (tg(x) \cdot \sec(x) + \sec^2(x))\,dx$$

$$\boxed{dx = \frac{1}{tg(x) \cdot \sec(x)+\sec^2(x)}\,dt}$$

Y sustituimos:

$$\int \frac{\sec^2(x)+\sec(x)tg(x)}{\sec(x)+tg(x)}\,dx = \int \frac{\sec^2(x)+\sec(x)tg(x)}{t(\sec^2(x)+\sec(x)tg(x))}\,dt =$$

$$= \int \frac{1}{t}\,dt = \ln|t| + C = \ln|\sec(x)+tg(x)| + C$$

Deshacemos el cambio

t) $\int \frac{1}{\sqrt{x^2+x+1}}\,dx$

Apliquemos el siguiente cambio de variable:

$$t-x=\sqrt{x^2+x+1}$$
$$t^2+x^2-2tx=x^2+x+1$$
$$t^2-1=2tx+x$$
$$\boxed{x=\frac{t^2-1}{2t+1}}$$

$$1\cdot dx=\frac{2t(2t+1)-2(t^2-1)}{(2t+1)^2}\,dt \iff dx=\frac{2t^2+2t+2}{(2t+1)^2}\,dt$$

Y sustituimos:

$$\int\frac{dx}{\sqrt{x^2+x+1}}=\int\frac{\frac{2t^2+2t+2}{(2t+1)^2}}{t-x}\,dt=\int\frac{\frac{2t^2+2t+2}{(2t+1)^2}}{t-\frac{t^2-1}{2t+1}}\,dt=$$

$$=\int\frac{\frac{2t^2+2t+2}{(2t+1)^2}}{\frac{t^2+t+1}{2t+1}}\,dt=\int\frac{2(t^2+t+1)(2t+1)}{(t^2+t+1)(2t+1)^2}\,dt=$$

$$=\int\frac{2}{2t+1}\,dt=\left(\begin{array}{l}\text{Nuevo cambio de variable:}\\ 2t+1=r \iff 2\,dt=dr \iff dt=\frac{dr}{2}\end{array}\right)$$

Deshacemos el cambio

$$=\int\frac{2}{r}\cdot\frac{dr}{2}=\int\frac{1}{r}\,dr=\ln|r|+C=\ln|1+2t|+C=$$

$$=\ln\left|1+2x+2\sqrt{x^2+x+1}\right|+C$$

Deshacemos el cambio