Ejercicios propuestos 26/03/2021

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7 de abril de 2021

Ejercicio 1. Demuestra que:

a) $\overline{x} = \sum_{j=1}^{p} f_{.j}\overline{x}_j$

$$\sum_{j=1}^{p} f_{.j}\overline{x}_j = \sum_{j=1}^{p} f_{.j} \sum_{i=1}^{k} f_i^j x_i = \sum_{j=1}^{p}\sum_{i=1}^{k} f_{.j} f_i^j x_i = \sum_{j=1}^{p}\sum_{i=1}^{k} \frac{n_{.j}}{n}\frac{n_{ij}}{n_{.j}} x_i = \sum_{j=1}^{p}\sum_{i=1}^{k} \frac{n_{ij}}{n} x_i =$$

$$\frac{1}{n}\sum_{j=1}^{p} n_{ij} \sum_{i=1}^{k} x_i = \frac{1}{n}\sum_{i=1}^{k} n_{i.} x_i = \sum_{i=1}^{k} f_{i.} x_i = \overline{x}$$

y queda demostrada la igualdad inicial.

b) $\overline{y} = \sum_{i=1}^{k} f_{i.}\overline{y}_i$

$$\sum_{i=1}^{k} f_{i.}\overline{y}_i = \sum_{j=1}^{p} f_j^i y_j \sum_{i=1}^{k} f_{i.} = \sum_{j=1}^{p}\sum_{i=1}^{k} f_{i.} f_j^i y_j = \sum_{j=1}^{p}\sum_{i=1}^{k} \frac{n_{i.}}{n}\frac{n_{ij}}{n_{i.}} y_j = \sum_{j=1}^{p}\sum_{i=1}^{k} \frac{n_{ij}}{n} y_j =$$

$$\frac{1}{n}\sum_{j=1}^{p} y_j \sum_{i=1}^{k} n_{ij} = \frac{1}{n}\sum_{j=1}^{p} n_{.j} y_j = \sum_{j=1}^{p} f_{.j} y_j = \overline{y}$$

Ejercicio 2. Demuestra que:

a) $\sigma_x^2 = \sum_{i=1}^{k} f_{i.}\overline{y}_i$

$$\sigma_x^2 = \sum_{i=1}^{k} f_{i.}(x_i - \overline{x})^2 = \sum_{i=1}^{k} \frac{n_{i.}}{n}(x_i - \overline{x})^2 = \sum_{j=1}^{p}\sum_{i=1}^{k} \frac{n_{ij}}{n}(x_i - \overline{x})^2 = \sum_{j=1}^{p}\sum_{i=1}^{k} \frac{n_{.j}}{n}\frac{n_{ij}}{n_{.j}}(x_i - \overline{x})^2 =$$

$$\sum_{j=1}^{p}\sum_{i=1}^{k} f_{.j} f_i^j (x_i - \overline{x})^2 = \sum_{j=1}^{p} f_{.j}\left[\sum_{i=1}^{k} f_i^j (x_i - \overline{x})^2\right] = \sum_{j=1}^{p} f_{.j}\left[\sum_{i=1}^{k} f_i^j (x_i - \overline{x}_j + \overline{x}_j - \overline{x})^2\right] =$$

$$\sum_{j=1}^{p} f_{.j}\left[\sum_{i=1}^{k} f_i^j (x_i - \overline{x}_j)^2 + \sum_{i=1}^{k} f_i^j (\overline{x}_j - \overline{x})^2 + 2\sum_{i=1}^{k} f_i^j (x_i - \overline{x}_j)(\overline{x}_j - \overline{x})\right] \overset{*}{=}$$

$$\sum_{j=1}^{p} f_{.j}\left[\sum_{i=1}^{k} f_i^j (x_i - \overline{x}_j)^2 + \sum_{i=1}^{k} f_i^j (\overline{x}_j - \overline{x})^2\right] = \sum_{j=1}^{p} f_{.j}\left[\sigma_{x,j}^2 + (\overline{x}_j - \overline{x})^2 \sum_{i=1}^{k} f_i^j\right] =$$

$$\sum_{j=1}^{p} f_{.j}\left[\sigma_{x,j}^2 + (\overline{x}_j - \overline{x})^2\right] = \underbrace{\sum_{j=1}^{p} f_{.j}\sigma_{x,j}^2}_{**} + \underbrace{\sum_{j=1}^{p} f_{.j}(\overline{x}_j - \overline{x})^2}_{***}$$

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$$2\sum_{i=1}^{k} f_i^j (x_i - \overline{x}_j)(\overline{x}_j - \overline{x}) = 2(\overline{x}_j - \overline{x})\sum_{i=1}^{k} f_i^j (x_i - \overline{x}_j) = 2(\overline{x}_j - \overline{x})\left[\sum_{i=1}^{k} f_i^j x_i - \sum_{i=1}^{k} f_i^j \overline{x}_j\right] =$$

$$2(\overline{x}_j - \overline{x})\left[\overline{x}_j - \overline{x}_j \sum_{i=1}^{k} f_i^j\right] = 2(\overline{x}_j - \overline{x})\left[\overline{x}_j - \overline{x}_j \frac{1}{n_{.j}}\sum_{i=1}^{k} n_{ij}\right] = 2(\overline{x}_j - \overline{x})\left[\overline{x}_j - \overline{x}_j \frac{n_{.j}}{n_{.j}}\right] = 0$$

** Media ponderada varianza condicionadas.

*** Varianzas ponderadas medias condicionadas.

b) $\sigma_y^2 = \sum_{i=1}^{k} f_{i.}\sigma_{y,i}^2 + \sum_{i=1}^{k} f_{i.}(\overline{y}_i - \overline{y})^2$

$$\sigma_y^2 = \sum_{j=1}^{p} f_{.j}(y_j - \overline{y})^2 = \sum_{j=1}^{p} \frac{n_{.j}}{n}(y_j - \overline{y})^2 = \sum_{j=1}^{p}\sum_{i=1}^{k} \frac{n_{ij}}{n}(y_j - \overline{y})^2 = \sum_{j=1}^{p}\sum_{i=1}^{k} \frac{n_{i.}}{n}\frac{n_{ij}}{n_{i.}}(y_j - \overline{y})^2 =$$

$$\sum_{j=1}^{p}\sum_{i=1}^{k} f_{i.} f_j^i (y_j - \overline{y})^2 = \sum_{i=1}^{k} f_{i.}\left[\sum_{j=1}^{p} f_j^i (y_j - \overline{y})^2\right] = \sum_{i=1}^{k} f_{i.}\left[\sum_{j=1}^{p} f_j^i (y_j - \overline{y}_i + \overline{y}_i - \overline{y})^2\right] =$$

$$\sum_{i=1}^{k} f_{i.}\left[\sum_{j=1}^{p} f_j^i (y_j - \overline{y}_i)^2 + \sum_{j=1}^{p} f_j^i (\overline{y}_i - \overline{y})^2 + 2\sum_{j=1}^{p} f_j^i (y_j - \overline{y}_i)(\overline{y}_i - \overline{y})\right] \stackrel{*}{=}$$

$$\sum_{i=1}^{k} f_{i.}\left[\sum_{j=1}^{p} f_j^i (y_j - \overline{y}_i)^2 + \sum_{j=1}^{p} f_j^i (\overline{y}_i - \overline{y})^2\right] = \sum_{i=1}^{k} f_{i.}\left[\sigma_{y,i}^2 + (\overline{y}_i - \overline{y})^2 \sum_{j=1}^{p} f_j^i\right] =$$

$$\sum_{i=1}^{k} f_{i.}\left[\sigma_{y,i}^2 + (\overline{y}_i - \overline{y})^2\right] = \underbrace{\sum_{i=1}^{k} f_{i.}\sigma_{y,i}^2}_{**} + \underbrace{\sum_{i=1}^{k} f_{i.}(\overline{y}_i - \overline{y})^2}_{***}$$

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$$2\sum_{j=1}^{p} f_j^i (y_j - \overline{y}_i)(\overline{y}_i - \overline{y}) = 2(\overline{y}_j - \overline{y})\sum_{j=1}^{p} f_j^i (y_j - \overline{y}_i) = 2(\overline{y}_i - \overline{y})\left[\sum_{j=1}^{p} f_j^i y_j - \sum_{j=1}^{p} f_j^i \overline{y}_i\right] =$$

$$2(\overline{y}_i - \overline{y})\left[\overline{y}_i - \overline{y}_i \sum_{j=1}^{p} f_j^i\right] = 2(\overline{y}_i - \overline{y})\left[\overline{y}_i - \overline{y}_i \frac{1}{n_{i.}}\sum_{j=1}^{p} n_{ij}\right] = 2(\overline{y}_i - \overline{y})\left[\overline{y}_i - \overline{y}_i \frac{n_i}{n_{i.}}\right] = 0$$

** Media ponderada varianza condicionadas.

*** Varianzas ponderadas medias condicionadas.

Ejercicio 3. Demuestra que $\mu_{11} = m_{11} - m_{10}m_{01}$ donde:

$$\mu_{11} = \sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}(x_i - \overline{x})(y_j - \overline{y}) \qquad m_{11} = \sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij} x_i y_j$$

$$m_{10} = \sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij} x_i \qquad m_{01} = \sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij} y_j$$

Comenzemos:

$$\mu_{11} = \sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}(x_i - \overline{x})(y_j - \overline{y}) = \sum_{i=1}^{k}\sum_{j=1}^{p}(f_{ij}x_iy_j + f_{ij}\overline{xy} - f_{ij}y_j\overline{x} - f_{ij}x_i\overline{y}) =$$

$$\sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}x_iy_j + \sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}\overline{x}\;\overline{y} - \sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}y_j\overline{x} - \sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}x_i\overline{y} =$$

$$m_{11} + \overline{x}\;\overline{y} - \overline{x}\sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}y_j - \overline{y}\sum_{i=1}^{k} x_i \sum_{j=1}^{p} f_{ij} = m_{11} + \overline{x}\;\overline{y} - \overline{x}\sum_{j=1}^{p} y_j \sum_{i=1}^{k} f_{ij} - \overline{y}\sum_{i=1}^{k} x_i \sum_{j=1}^{p} f_{ij} =$$

$$m_{11} + \overline{x}\;\overline{y} - \overline{x}\sum_{j=1}^{p} f_{.j}y_j - \overline{y}\sum_{i=1}^{k} f_{i.}x_i = m_{11} + \overline{x}\;\overline{y} - \overline{x}\;\overline{y} - \overline{x}\;\overline{y} = m_{11} - \overline{x}\;\overline{y} =$$

$$m_{11} - \left[\sum_{i=1}^{k} x_i \sum_{j=1}^{p} f_{ij}\right]\left[\sum_{j=1}^{p} y_j \sum_{i=1}^{k} f_{ij}\right] = m_{11} - \left[\sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}x_i\right]\left[\sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}y_j\right] =$$

$$m_{11} - \left[\sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}x_i\right]\left[\sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}y_j\right] = m_{11} - m_{10}m_{01}$$

Ejercicio 4. Si X e Y son independentes, demostrar que:

a) $m_{rs} = m_{r0}m_{0s}$, $\mu_{rs} = \mu_{r0}\mu_{0s}$

Cabe recordar un resultado obtenido en clase cuando X e Y son independientes: $f_{ij} = f_{i.}f_{.j}$

Comencemos:

$$m_{rs} = \sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}x_i^r y_j^s = \sum_{i=1}^{k}\sum_{j=1}^{p} f_{i.}f_{.j}x_i^r y_j^s = \left[\sum_{i=1}^{k} f_{i.}x_i^r\right]\left[\sum_{j=1}^{p} f_{.j}y_j^s\right] =$$

$$\left[\sum_{i=1}^{k} x_i^r \sum_{j=1}^{p} f_{ij}\right]\left[\sum_{j=1}^{p} y_j^s \sum_{i=1}^{k} f_{ij}\right] = \left[\sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}x_i^r\right]\left[\sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}y_j^s\right] =$$

$$= m_{r0}m_{0s}$$

Por otro lado:

$$\mu_{rs} = \sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}(x_i - \overline{x})^r(y_j - \overline{y})^s = \sum_{i=1}^{k}\sum_{j=1}^{p} f_{i.}f_{.j}(x_i - \overline{x})^r(y_j - \overline{y})^s =$$

$$\left[\sum_{i=1}^{k} f_{i.}(x_i - \overline{x})^r\right]\left[\sum_{j=1}^{p} f_{.j}(y_j - \overline{y})^s\right] = \left[\sum_{i=1}^{k}(x_i - \overline{x})^r \sum_{j=1}^{p} f_{ij}\right]\left[\sum_{j=1}^{p}(y_j - \overline{y})^s \sum_{i=1}^{k} f_{ij}\right] =$$

$$\left[\sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}(x_i - \overline{x})^r\right]\left[\sum_{i=1}^{k}\sum_{j=1}^{p} f_{ij}(y_j - \overline{y})^s\right] = \mu_{r0}\mu_{0s}$$

b) $\mu_{11} = 0$

Esta demostración es inmmediata usando lo ya demostrado anteriormente.

En el ejercicio hemos demostrado que $\mu_{11} = m_{11} - m_{10}m_{01}$ y por otro lado hemos visto que en el apartado anterior que $m_{rs} = m_{r0}m_{0s}$, esto es, $m_{rs} - m_{r0}m_{0s} = 0$. Así:

$$\mu_{11} = m_{11} - m_{10}m_{01} = 0$$

y queda demostrado.