Ejercicio 6.5 Calcular las siguientes integrales

q) $\int_a^b \frac{dx}{(x+a)(x+b)}$ $(0<a<b)$ Calculamos la factorización parcial

$$\int_a^b \frac{A}{(x+a)} + \frac{B}{(x+b)}\, dx \Rightarrow$$

$$Ax + Ab + Bx + Ba = 1 = \begin{cases} A+B=0 \Rightarrow A=-B \Rightarrow A = \frac{-1}{a-b} \\ Ab+Ba = 1 \Rightarrow -Bb+Ba=1 \Rightarrow B(-b+a)=1 \Rightarrow B=\frac{1}{a-b}\end{cases}$$

$$\Rightarrow \int_a^b -\frac{1/a-b}{(x+a)} + \frac{1/a-b}{(x+b)}\, dx = \frac{-1}{a-b}\int_a^b \frac{1}{x+a}dx + \frac{1}{a-b}\int_a^b \frac{1}{x+b}\, dx =$$

$$= \frac{-1}{a-b} \ln|x+a|\Big]_a^b + \frac{1}{a-b}\ln|x+b|\Big]_a^b = -\frac{1}{a-b}\left(\ln|b+a| - \ln(2a)\right) + \frac{1}{a-b}\left(\ln|2b| - \ln|a+b|\right) + C =$$

$$= \frac{1}{a-b}\left(-\ln|b+a| + \ln(2a) + \ln|2b| - \ln|a+b|\right) + C = \boxed{\frac{1}{a-b}\cdot \ln\left|\frac{2a\,2b}{(a+b)^2}\right| + C}$$

r) $\int_0^1 \frac{1}{x^3+1}\, dx$ Calculamos la factorización parcial

$$\begin{array}{r|rrrr} & 1 & 0 & 0 & 1 \\ -1 & & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & \boxed{0} \end{array}$$

$$\int_0^1 \frac{A}{(x+1)} + \frac{Bx+C}{(x^2-x+1)}\, dx \Rightarrow$$

$$Ax^2 - Ax + A + Bx^2 + Bx + Cx + C = 1 \begin{cases} A+C=1 \\ -A+B+C=0 \\ A+B\quad =0\end{cases} \Rightarrow \left(\begin{array}{ccc|c} 1&0&1&1\\-1&1&1&0\\1&1&0&0\end{array}\right) \sim \left(\begin{array}{ccc|c}1&0&1&1\\0&1&2&1\\0&1&-1&-1\end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c}1&0&1&1\\0&1&2&1\\0&0&-3&-2\end{array}\right) \Rightarrow \begin{array}{l} A=1/3 \\ B=-1/3 \\ C=2/3 \end{array}$$

$$\Rightarrow \int_0^1 \frac{1/3}{(x+1)} + \frac{(-1/3)x+(2/3)}{(x^2-x+1)}\,dx = \frac{1}{3}\ln|x+1|\Big]_0^1 + \frac{1}{3}\int_0^1 \frac{-x+2}{x^2-x+1}\,dx =$$

$$= \frac{1}{3}\ln(2) - \frac{1}{3}\cdot\frac{1}{2}\int_0^1 \frac{2x-1-3}{x^2-x+1}\,dx = \frac{1}{3}\ln(2) - \frac{1}{6}\overbrace{\left(\ln(1) - \ln|-1|\right)}^{0} - \frac{1}{6}\int_0^1 \frac{-3}{x^2-x+1}\,dx =$$

$$\frac{1}{3}\ln(2) + \frac{1}{2}\int_0^1 \frac{1}{(x-\frac12)^2+\frac34}\,dx = \frac13\ln(2) + \frac12\frac43\int_0^1 \frac{dx}{1+\left(\frac{2}{\sqrt3}(x-\frac12)\right)^2} = \frac13 \ln(2) + \frac23\, \text{arctg}\left(\frac{2x}{\sqrt3} - \frac{1}{\sqrt3}\right)\Big]_0^1 + C$$

$$= \frac13 \ln 2 + \frac23\, \text{arctg}\,\frac{1}{\sqrt3} - \frac23\,\text{arctg}\left(-\frac{1}{\sqrt3}\right) = \frac13\ln(2) + \frac{1}{\sqrt3}\left(\frac{\pi}{6}+\frac{\pi}{6}\right) = \boxed{\frac13\ln(2) + \frac{\sqrt3}{9}\pi + C}$$