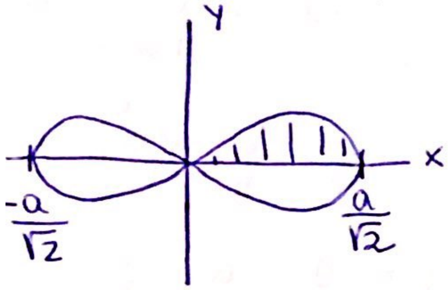
Cálculo II : Clara Ortega Sevilla

Relación 7: ejercicio 24.

* Calcular la longitud de arco de la curva dada por la ecuación $8a^2y^2 = x^2(a^2-2x^2)$



$$\frac{\ell}{4} = \int_0^{a\sqrt{2}} \sqrt{1+[f'(x)]^2}\, dx$$

$$f(x) = \sqrt{\frac{x^2}{8a^2}(a^2-2x^2)}$$

$$f'(x) = \frac{\frac{2}{8a^2}x(a^2-2x^2) + \frac{x^2}{8a^2}(-4x)}{2\sqrt{\frac{x^2}{8a^2}(a^2-2x^2)}} = \frac{\frac{x}{8a^2}(a^2-2x^2) - \frac{2x^3}{8a^2}}{\sqrt{\frac{x^2}{8a^2}(a^2-2x^2)}} =$$

$$= \frac{\frac{x}{8} - \frac{4x^3}{8a^2}}{\sqrt{\frac{x^2}{8a^2}(a^2-2x^2)}} = \frac{\frac{xa^2-4x^3}{8a^2}}{\frac{x}{\sqrt{8}a}\cdot\sqrt{a^2-2x^2}} = \frac{\sqrt{8}\cancel{a}(xa^2-4x^3)}{8\cdot a^{\cancel{2}}x\sqrt{a^2-2x^2}} = \frac{\sqrt{2}\cancel{x}\cdot(a^2-4x^2)}{4a\cancel{x}\sqrt{a^2-2x^2}}$$

$$[f'(x)]^2 = \frac{(\sqrt{2}(a^2-4x^2))^2}{(4a\sqrt{a^2-2x^2})^2} = \frac{2(a^2-4x^2)^2}{16a^2(a^2-2x^2)} = \frac{(a^2-4x^2)^2}{8a^2(a^2-2x^2)}$$

$$1+[f'(x)]^2 = \frac{8a^2(a^2-2x^2)+(a^2-4x^2)^2}{8a^2(a^2-2x^2)} = \frac{8a^4-16a^2x^2+a^4-8a^2x^2+16x^4}{8a^2(a^2-2x^2)} =$$

$$= \frac{9a^4-24a^2x^2+16x^4}{8a^2(a^2-2x^2)} = \frac{(3a^2-4x^2)^2}{8a^2(a^2-2x^2)}$$

$$\frac{\ell}{4} = \int_0^{a\sqrt{2}} \sqrt{\frac{(3a^2-4x^2)^2}{8a^2(a^2-2x^2)}} = \int_0^{a\sqrt{2}} \frac{(3a^2-4x^2)}{\sqrt{8}a\sqrt{a^2-2x^2}} \rightarrow \begin{array}{l} x = \frac{a}{\sqrt{2}}\operatorname{sen} t\, dt \\ dx = \frac{a}{\sqrt{2}}\cos t\, dt \end{array}$$

$$\frac{\ell}{4} = \int_0^{\pi/2} \frac{3a^2 - 4\cdot\left(\frac{a}{\sqrt{2}}\operatorname{sen} t\right)^2}{\sqrt{8}a\sqrt{a^2-2\cdot\left(\frac{a}{\sqrt{2}}\operatorname{sen} t\right)^2}}\, dt\, \frac{a}{\sqrt{2}}\cos t = \int_0^{\pi/2} \frac{3a^2-2a^2\operatorname{sen}^2 t\, dt}{\sqrt{8}a\sqrt{a^2-a^2\operatorname{sen}^2 t}}\, \frac{a}{\sqrt{2}}\cos t =$$

$$= \int_0^{\pi/2} \frac{3a^2-2a^2\operatorname{sen}^2 t}{\sqrt{8}a\sqrt{a^2(\cos^2 t)}}\, dt\, \frac{a}{\sqrt{2}}\cos t = \int_0^{\pi/2} \frac{3a^2-2a^2\operatorname{sen}^2 t}{\sqrt{8}a\cdot a\cancel{\cos t}}\, dt\, \frac{a}{\sqrt{2}}\cancel{\cos t} =$$

$$\int_0^{\pi/2} \frac{\cancel{a^2}(3-2\operatorname{sen}^2 t)}{\sqrt{8}\,\cancel{a^2}}\,dt\,\frac{a}{\sqrt{2}} = \int_0^{\pi/2} \frac{3a}{4} - \frac{2a}{4}\operatorname{sen}^2 t = \int_0^{\pi/2} \frac{3a}{4} - \int_0^{\pi/2} \frac{2a}{4}\operatorname{sen}^2 t =$$

$$= \frac{3a}{4}\cdot[x]_0^{\pi/2} - \frac{2a}{4}\int_0^{\pi/2} \frac{1-\cos(2t)}{2} = \frac{3a}{4}[x]_0^{\pi/2} - \frac{\cancel{2}a}{4}\cdot\frac{1}{\cancel{2}}[x]_0^{\pi/2} - \frac{\cancel{2}a}{4}\frac{1}{\cancel{2}}\int_0^{\pi/2} 2\cos 2t =$$

$$= \frac{3a}{4}[x]_0^{\pi/2} - \frac{a}{4}[x]_0^{\pi/2} - \frac{a}{4}[\operatorname{sen}(2t)]_0^{\pi/2} = \frac{3a}{4}\cdot\frac{\pi}{2} - \frac{a}{4}\frac{\pi}{2} - \cancel{\frac{a}{4}\operatorname{sen}(\pi/2\cdot 2)}^{\,0} -$$

$$\cancel{\frac{a}{4}\operatorname{sen} 0}_{\,0} = \frac{3a\pi}{8} - \frac{a\pi}{8} = a\left(\frac{3\pi}{8} - \frac{\pi}{8}\right) = \frac{a\pi}{4}$$

$$\underline{\text{Longitud}} = \cancel{4}\cdot\frac{a\pi}{\cancel{4}} = \boxed{a\pi}$$