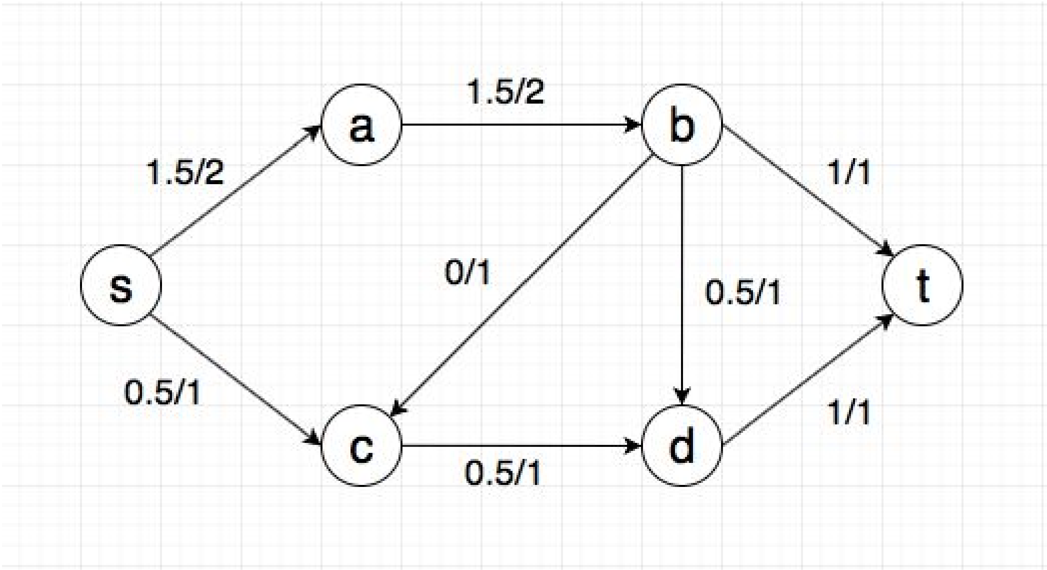
Q1-a

Below is an example of a maximum flow with integer capacities having non-integer edge value.



Proof for f is a maximum flow in G:

First, prove that **f is a flow**.

(1) Capacity:

it is clear that the value of every edge does not go beyond every edge's capacities. Therefore, f satisfies the capacity constriction.

(2) Conservation:

values in a = 1.5 and values out of a = 1.5 ⇒ values in a = values out of a

values in b = 1.5 and values out of b = 1.5 ⇒ values in b = values out of b

values in c = 0.5 and values out of c = 0.5 ⇒ values in c = values out of c

values in d = 1.0 and values out of d = 1.0 ⇒ values in d = values out of d

Therefore, f satisfies the conservation constriction.

Hence **f is a flow**.

Second, prove that **f is a maximum flow**.

Let S* be the set of nodes reachable from s in the residual graph of G.

S* = {s, a, b, c, d}

T* = V - S* ⇒ T* = {t}

⇒ (S*, T*) is a cut

V(f) = 1.5 + 0.5 = 2

c(S*, T*) = 1 + 1 = 2

⇒ V(f) = c(S*, T*)

By Lemma 4 in Lecture, if V(f) = c(S*, T*), then f is a maximum flow.

Hence, f is a maximum flow as wanted.

Q1-b
Proof by contradiction.

Suppose all edges in the flow network are not positive integers.
Let f be a maximum flow.
Let S be the set of nodes reachable from s in G_f(the residual graph of G).
T = V - S ⇒ (S, T) is a **minimum cut**, by the ALGO shown in the lecture.

By Lemma 2 in the lecture, V(f) = $\sum_{e \in out(S) \cap in(T)} f(e) - \sum_{e \in out(T) \cap in(S)} f(e)$

By the definition of capacity of cut, c(S, T) = $\sum_{u \in S \cap v \in T} c(u, v) = \sum_{e \in out(S) \cap in(T)} c(e)$

V(f) = $\sum_{e \in out(S) \cap in(T)} f(e) - \sum_{e \in out(T) \cap in(S)} f(e)$

$\leq \sum_{e \in out(S) \cap in(T)} f(e)$ [Since every edge is positive]

$< \sum_{e \in out(S) \cap in(T)} c(e)$ [By the assumption all edges are not integers and the capacity constraint]

= c(S, T) [By definition of capacity of cut]

However by the Max Flow Min Cut Theorem, V(f) = c(S, T), which **contradicts to** the result from the above step that V(f) < c(S, T).

Therefore, Suppose there is an edge in the flow network are positive integers, as wanted.

Q1-c

Proof by induction.

Basis: $k = 0$

If the value of every edge is 0, then $V(f) = 0$.

Hence, there is a flow f in F that has value 0, as wanted.

Induction Hypothesis:

There is a flow f in F that has value $(k - 1)$, where $0 \leq (k - 1) < k \leq m$ and m is the value of maximum flow.

Induction Step: Let $k > 0$.

$(k - 1) < k \leq m \Rightarrow (k - 1) < m$

$\Rightarrow$ there is a path **p** from s to t in the residual graph G_f

Constructing the flow f* based on original graph F as follows:

- Increase flow through edges in the path p traversed in forward direction by 1
- Decrease flow through edges in the path p traversed backward by 1
- Remain the same for the rest of the edges

Then try to prove that f* is a flow with value k.

Constructing the residual graph of f* denotes as G_{f*}.

First, prove that f* is a flow.

Since the construction only changes the edge values on the path, and by I.H., the edges not on the path would obey the capacity and conservation constraint, we can only check the changed edges on the path p in the residual graph G_{f*}.

1. Capacity:

   There are two cases for the changed edges:

   a. If (u, v) is a forward edge, then

      $f^*(u, v) = f(u, v) + 1 \leq c(u, v)$ [since there exists a path in the residual graph]

   b. If (u,v) is a backward edge, then

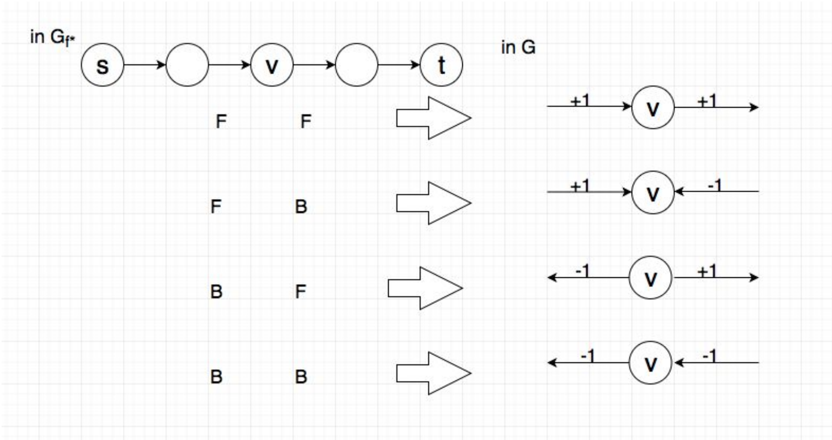
      $f^*(u, v) = f(u, v) - 1 \geq 0$

      $\leq c(u, v)$ [since there exists a path in the residual graph]

   Therefore, f satisfies the capacity constriction.

2. Conservation:
   Since G_{f*} is derived from flow f* respectively, we can prove f* holds conservation constraint by showing G_{f*} holds the constraint.

   There are four cases for the change of $\sum_{e \in in(v)} f(e)$ and $\sum_{e \in out(v)} f(e)$, where node v is on the path p, shows as follows:



   All four cases would satisfy $\sum_{e \in in(v)} f(e) = \sum_{e \in out(v)} f(e)$ by I.H. and the construction of the new flow for all nodes on the path p.
   Therefore, f satisfies the conservation constriction.

Hence **f* is a flow**.

Second prove that **f* has value k.**
Let the node out of s in the path p denote as d.
By the above construction, since G has no edge into s, edge (s ➝ d) is a forward edge having increased by 1.

Therefore, $V(f^*) = \sum_{e \in out(s)} f^*(e)$ [By the definition of value of flow]

$= \sum_{e \in out(s) - d} f^*(e) + f^*(d)$ [Extract d out]

$= \sum_{e \in out(s) - d} f(e) + f(d) + 1$ [By the construction of f*]

$= \sum_{e \in out(s)} f(e) + 1$ [Combine d with the rest of nodes out of s]

$= (k - 1) + 1$ [By I.H.]

$= k$

Hence, **f* is a flow with value k** as wanted.

Q2-a
True
Proof by contradiction:
Given: f is a max flow and (S, T) is a min cut.
Suppose ∃ an edge that crosses (S, T) is not saturated.

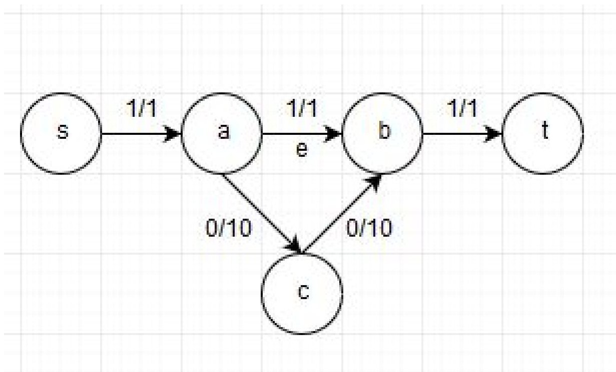
$$V(\mathrm{f}) = \sum_{e \in out(S) \cap in(T)} f(e) \; - \sum_{e \in out(T) \cap in(S)} f(e)$$

$$\leq \sum_{e \in out(S) \cap in(T)} f(e) \quad \text{[Since every edge is positive]}$$

$$< \sum_{e \in out(S) \cap in(T)} c(e) \quad \text{[By the assumption ∃ an edge that crosses (S, T) is not saturated]}$$

$= c(\mathrm{S, T})$ [By definition of capacity of cut]
However, by max flow min cut theorem, if f is a max flow, then $V(\mathrm{f}) = c(\mathrm{S, T})$.
Therefore by contradiction, all edges that crosses(S, T) is saturated.

Q2-b
False



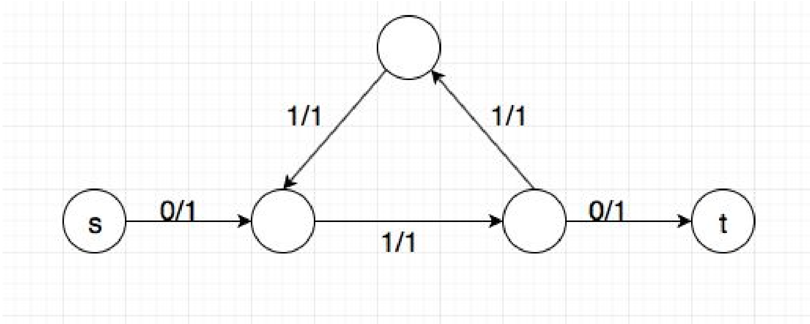
From the above example f in F we can see, e = (a, b) is saturated and V(f) = c(S*, T*) = 1 ($S^* = \{s\}$ or $S^* = \{s, a, b, c\}$ and $T^* = V - S^*$). Therefore we say **f is max flow**, by corollary 4.
In order to let e crosses some minimum cut (S, T), a **has to** ∈ S and b **has to** ∈ T.
There are two cases:

(1) S = {s, a, c} , T = {b, t}:
c(S, T) = c(a,b) + c(c,b) = 11 > 1
⇒ (S, T) is not a minimum cut

(2) S = {s, a} , T = {b, c, t}:
c(S, T) = c(a, b) + c(a, c) = 11 > 1
⇒ (S, T) is not a minimum cut

Therefore, we show that ∀ **cut** that e crossed is **not** a minimum.
Which is equivalent to that e crosses no minimum cut.

Q2-c
False



WTS: above f in F is a flow:

Capacity: $\forall$ edge e, $0 \le f(e) \le c(e)$. holds

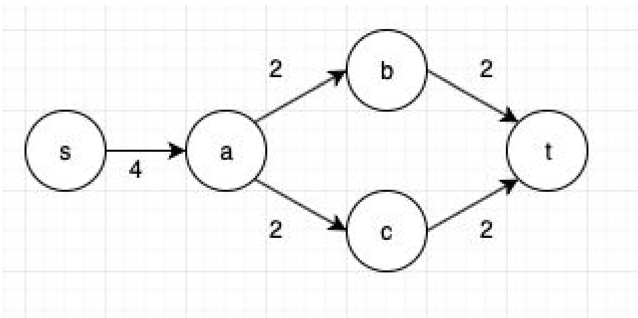
Conservation: $\forall$ node $n \in V - \{s, t\}$, $\sum_{e \in out(n)} f(e) = \sum_{e \in in(n)} f(e)$. holds

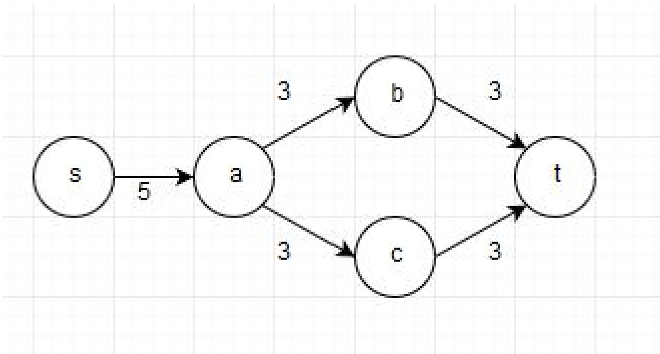
Therefore f is a flow.

In addition, $V(f) = \sum_{e \in out(s)} f(e) = 0$ is obvious.

But there are three edges e that $f(e) = 1 \neq 0$ is also obvious.

Q2-d
False





The above picture is F and the below picture is F^+. And the labels on edges are **capacities**.

S = {s,a}, T = {b, c, t} is a minimum cut of F and c(S, T) = 4 is obvious.
But S = {s, a}, T = {b, c, t} is not a minimum cut of F^+, because c(S, T) = 6 and we have a better cut $S^* = \{s\}$, $T^* = \{a, b, c, t\}$ s.t. $c(S^*, T^*) = 5 < c(S, T)$.

Q3-a

Prove that the value of a maximum flow F^+ is V(f) or V(f) + 1.

Given: f is an **integral** flow in flow network F.

⇒ we know Capacity: $\forall$ e $\in$ E, 0 <= f(e) <= c(e)

Conservation: $\forall$ v$\in$ V-{s,t}, $\sum_{e \in out(n)} f(e) = \sum_{e \in in(n)} f(e)$

⇒ f is also a flow in F^+

Since $\forall$ e $\in$ E, 0 <= f(e) <= c(e) <= c(e) + 1 <= c^+(e) and still preserve conservation.

Define the value of maximum flow of F^+ is V(f^+).

Constructing the **residual graph of F^+** with flow f (the original flow) denotes as $G_f|F^+$.

Case 1: if there is no path p in the residual graph $G_f|F^+$:

⇒ there is no way we can improve f

⇒ V(f^+) = V(f)

Case 2: if there is a path p in the residual graph $G_f|F^+$:

⇒ b = min residual capacity of edges on p = 1

Given: F is a flow network with integer capacity and f is an integral flow.

⇒ F^+ is a flow network with integer capacity

⇒ residual capacities of G_f are all integers

⇒ b >= 1 (the minimum integer for an edge exist)

We also want to show b <= 1:

Assume $\exists$ a integer b > 1.

An integer b' = min residual capacity of edges on a path of the residual graph of $G_f|F$ (the original f and orignal F) >= b - 1 > 0

⇒ there is a path on $G_f|F$

⇒ f is not a maximum flow of F

By contradiction, b <= 1

Therefore b = 1

Let f^+ = augment(f, p) with b = 1.

By lemma 1, f^+ is a better flow and V(f^+) = V(f) + b = V(f) + 1

Therefore the value of a maximum flow F^+ is V(f) or V(f) + 1.

Prove that the value of a maximum flow F^- is V(f) or V(f) - 1.

Similar to above, all capacities or residual capacities are **integer**, and min integer is 1.

⇒ if c(S, T) < c(S', T') ⟺ c(S, T) <= c(S', T') - 1

Define the value of maximum flow of F^- is V(f^-).

Define the capacity of cut (S, T) of F^- is c^-(S, T).

Case1: if $\exists$ a minimum cut (S^*, T^*) of F and e $\in$ out(S^*) ∩ in(T^*):

Let cut (S^*, T^*) be the **min cut crossed by e** and (S,T) be any cut.

Claim: the same (S^*, T^*) is also a minimum cut of F^-

c^-(S^*, T^*) = c(S^*, T^*) - 1 [since e $\in$ out(S^*) ∩ in(T^*)]

$<= c(S, T) - 1$ [since (S^*,T^*) is min cut of F]
$<= (c^-(S, T) + 1) - 1$ [since e may **not** $\in$ out(S) $\cap$ in(T)]
$= c^-(S, T)$

$\Rightarrow c^-(S^*, T^*) <= c^-(S, T)$ for any cut (S, T)
Therefore, (S^*, T^*) is a minimum cut of F^-
By max flow min cut theorem:
$V(f^-) = c^-(S^*, T^*) = c(S^*, T^*) - 1 = V(f) - 1$

Case2: if $\forall$ minimum cut (S^*, T^*) of F s.t e $\notin$ out(S^*) $\cap$ in(T^*):
Let (S^*, T^*) be any min cut and (S, T) be any **non-min** cut.
Claim: the same (S^*, T^*) is also a minimum cut of F^-
$c^-(S^*, T^*) = c(S^*, T^*)$ [since e $\notin$ out(S^*) $\cap$ in(T^*)]
$< c(S, T)$ [since (S^*, T^*) is a cut and (S, T) is not]
$<= c(S, T) - 1$
$<= (c^-(S, T) + 1) - 1$ [since e may **not** $\in$ out(S) $\cap$ in(T)]
$<= c^-(S, T)$

$\Rightarrow c^-(S^*, T^*) < c^-(S, T)$ for any non-min cut (S, T)
Therefore, (S^*, T^*) is a minimum cut of F^-
By max flow min cut theorem:
$V(f^-) = c^-(S^*, T^*) = c(S^*, T^*) = V(f)$

Therefore the value of a maximum flow F^+ is V(f) or V(f) + 1.

Q3-b
find_f^+(F, f, e):
1. c(e) = c(e) + 1
2. construct corresponding residual graph G_f after the increment of c(e)
3. if there is a path in the residual graph then
4 p = simple s $\rightarrow$ t path of G_f
5. f = augment(f, p)
6. return f

Proof of Correctness:
Let f* be the flow returned by find_f^+.
Construct the residual graph of f* denoted as G_{f^*}.

Claim 1: There is no s $\rightarrow$ t path in the residual graph G_{f^*}.
Proof: Recall f is the integral maximum flow in F.
After increasing the capacity of edge (u, v) by 1, it would add 1 to edge (u, v) in the residual graph.

Case 1: if the corresponding change produces a path from s → t in the residual graph, it would execute Line 4 - 5 of the algorithm.
Denote the path from s → t in the residual graph G_{f*} as p.
Since f is an integral maximum flow in F, there is no s → t path in the residual graph G_f of f.
By increasing 1 to edge (u, v) in the residual graph, it creates a path from s → t.
⇒ there is no value on (u, v) in G_f [Otherwise it would form a path ⇒ contradiction]
⇒ value of edge (u, v) in the residual graph G_{f*} of f* is 1
Therefore, the minimum residual capacity of edges on p would be 1.
Then after augmenting the path p by 1, it would run out of the value on edge (u, v) in the residual graph G_{f*}.
⇒ There is no path from s → t in the residual graph after augmenting the flow in Line 5.
Case 2: if there is no path in the residual graph aftering increasing the capacity of edge e, it would return the flow directly, as wanted.
Hence, **there is no s → t path in the residual graph G_{f*}.**

Let S* be the set of nodes reachable from s in G_{f*}.
By Claim 1, there is no s → t path in the residual graph G_{f*}
⇒ there exists T* such that T* = V - S* and t ∈ T*.
⇒ (S*, T*) is a cut.

Claim 2: ∀(u, v) ∈ out(S*) ∩ in(T*), f*(u,v) = c(u, v)
Proof by contradiction.
Suppose f*(u, v) < c(u, v).
G_{f*} has forward edge (u, v)
v is reachable from s in G_{f*}
v ∈ S* ⇒ contradicts to v ∈ T* since S* and T* have no intersection.

Claim 3: ∀(u, v) ∈ out(T*) ∩ in(S*), f*(u,v) = 0
Proof by contradiction.
Suppose f*(u, v) > 0.
G_{f*} has backward edge (v, u)
u is reachable from s in G_{f*}.
u ∈ S* ⇒ contradicts to u ∈ T* since S* and T* have no intersection.

$$V(f^*) = \sum_{e \in out(S^*) \cap in(T^*)} f^*(e) - \sum_{e \in out(T^*) \cap in(S^*)} f^*(e)$$

$$= \sum_{e \in out(S^*) \cap in(T^*)} c(e) - 0 \qquad \text{[By Claim 2 and 3]}$$

$$= \sum_{e \in out(S^*) \cap in(T^*)} c(e)$$

$$= C(S, T)$$

By Lemma 4 in the lecture, **f* is the maximum flow**.

Analysis for Running Time:
Let m = |E| and n = |V|.
Time for construct residual graph in line 2: O(m + n)
Time for look for s → t path in G_f in line 3: O(m + n)
Time for update f in line 5: O(n)
Total time complexity: O(m + n)
If assume all nodes reachable from s, total time complexity would be O(m).