Q1-a

Variables: x_e, where x_e is the flow in the edge, $\forall\ e \in E$

Objective Function: max $\sum_{e \in in(t)} x_e$

Constraint:

1. capacity: $\begin{cases} x_e \geq 0 & \forall\ e \in E \\ x_e \leq c(e) & \forall\ e \in E \end{cases}$

2. conservation : $\forall\ v \subset V - \{s, t\}$

$$l_u \sum_{e \in in(v)} x_e - \sum_{e \in out(v)} x_e = 0$$

Q1-b

Variables: x_e, where x_e is the flow in the edge, $\forall\ e \in E$

Objective Function: min $\sum_{e \in E} x_e\, p(e)$

Constraint:

1. capacity: $\begin{cases} x_e \geq 0 & \forall\ e \in E \\ x_e \leq c(e) & \forall\ e \in E \end{cases}$

2. conservation : $\forall\ v \subset V - \{s, t\}$

$$\sum_{e \in in(v)} x_e - \sum_{e \in out(v)} x_e = 0$$

3. $\sum_{e \in in(t)} x_e = d$

Q1-c

Variables: $x_i(e)$, where $x_i(e)$ is the number of product i go through e, $\forall\ e \in E$, $\forall\ i \in \{1,...,k\}$

Objective Function: max $\sum_{i=1}^{k} \sum_{e \in out(si)} x_i(e)$

Constraint:

1. capacity: $\begin{cases} x_i(e) \geq 0 & \forall\ e \in E,\ \forall\ i \in \{1,...,k\} \\ \sum_{i=1}^{k} x_i(e) \leq c(e) & \forall\ e \in E \end{cases}$

2. conservation : $\forall\ i \in \{1,...,k\}$, $\forall\ v \subset V - \{s_i, t_i\}$

$$\sum_{e \in in(v)} x_i(e) - \sum_{e \in out(v)} x_i(e) = 0$$

Q2-a

Variables: x(u, v)

where x(u, v) = 1, $\forall$ u, v $\in$ V and u $\neq$ v **iff** there is a G = (V, E) that realizes the given set of degree pairs and the edge (u, v) $\in$ E.

Object Function: Nothing.

Any solution will be fine, as long as it satisfies the below constraints. (i.e. there is only feasible or infeasible solutions and no such thing as optimal solution)

Constraints:

1. x(u, v) $\in$ {0, 1}, $\forall$ u, v $\in$ V
2. in(u) - $\sum_{v \in V-\{u\}}$ x(v, u) = 0, $\forall$ u $\in$ V
3. out(u) - $\sum_{v \in V-\{u\}}$ x(u, v) = 0, $\forall$ u $\in$ V

Construct the graph:

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We are given V
Let X be the set of all variables defined above.
E = {}
for x(u, v) in X:
    if x(u,v) == 1:
        E = E U {(u,v)}
return G = (V, E)
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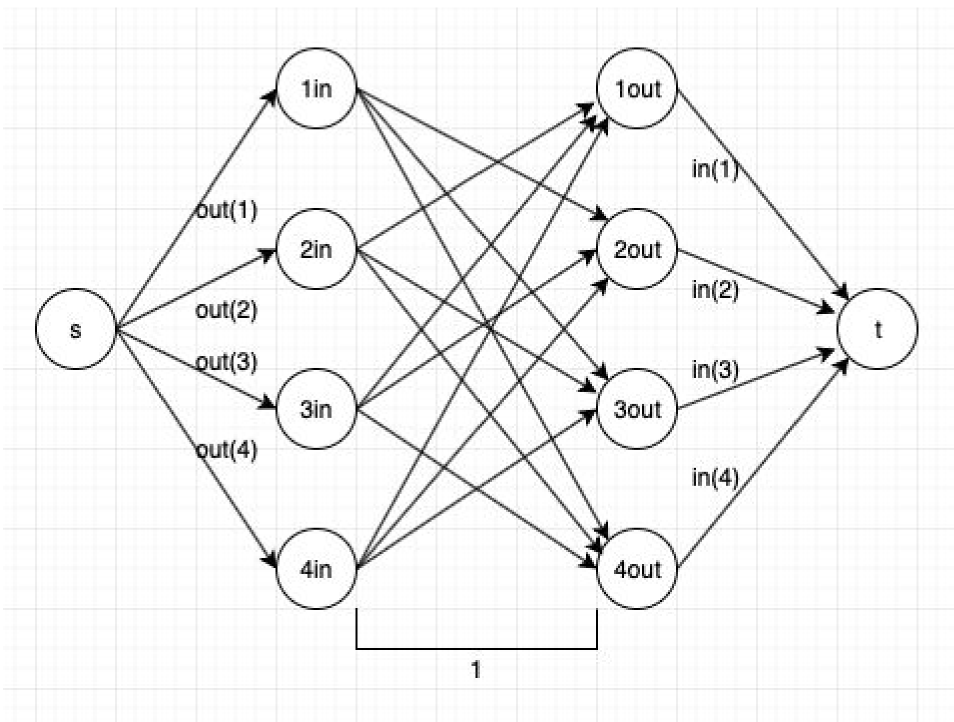
Q2-b



Construction Instruction:

1. $V_F = \{v_{in}, v_{out} \mid \forall\ v \in V\} \cup \{s, t\}$
2. $E_F = \{(s, v_{in}) \mid \forall\ v \in V\} \cup \{(u_{in}, v_{out}) \mid \forall\ u, v \in V\ \&\ u \neq v\} \cup \{(v_{out}, t) \mid \forall\ v \in V\}$
3. $c(e) = \begin{cases} out(v) & \text{if } e = (s, v_{in}) \\ in(v) & \text{if } e = (v_{out}, t) \\ 1 & \text{O/W} \end{cases}$

Let $F = (G_F=(V_F, E_F), s, t, c)$

Algorithm:

Realize(V, (in(u), out(u))):

construct flow net work F from V. $\Leftarrow O(n+m)$

f = MAXFLOW(F) (integral) $\Leftarrow O(nm)$

if $V(f) \neq \sum_{u \in V} out(u)$:

return non-realizable

else:

$E = \{(u, v) \in E_F \mid u \neq s \text{ and } v \neq t \text{ and } f(u,v) = 1\}$ $\Leftarrow O(m)$

return G = (V, E)

Running Time:

Note: from the construction we can see that $\mathbf{m = n^2}$.

$O(n+m) + O(nm) + O(m) = O(n+n^2) + O(n^3) + O(n^2) \in O(n^3)$

Correctness:

realizable set of degree (a)$\Rightarrow$ Integral max flow of

pairs V, (in(u), out(u)) $\Leftarrow$(b) value $\sum_{u \in V} out(u)$ in F

(a) Given a realizable set of degree pairs V, (in(u), out(u))

There is a G = (V, E) that realizes the given set of degree pairs.

And construct $F = (G_F = (V_F, E_F), s, t, c)$ from G

f is the max flow in F by "MAXFLOW(F)"

WTS, V(f) is indeed $\sum_{u \in V} out(u)$:

$f(u_{in}, v_{out}) \begin{cases} = 1 & \text{if there is a edge } (u, v) \in E \text{ of } G \\ = 0 & \text{o/w} \end{cases}$

$\forall\ u \in V$, total flow out of u = $\sum_{v \in V - \{u\}} f(u_{in}, v_{out})$

= out(u) (i.e out degree of u) [G realizes the given set of degree pairs]

= total flow in u [by conservation of a flow]

= $f(s, u_{in})$ (*)

$V(f) = \sum_{u \in V} f(s, u_{in})$ [by definition of value of flow]

$= \sum_{u \in V} out(u)$ [by *]

(b) Given an integral max flow of value $\sum_{u \in V} out(u)$ in F

Let $E = \{(u, v) \in E_F \mid u \neq s \text{ and } v \neq t \text{ and } f(u,v) = 1\}$
Let G = (V, E)
WST: $\forall\ u \in V, |\{(u, v) \mid (u, v) \in E\}| = out(u)$
and $\forall\ u \in V, |\{(v, u) \mid (v, u) \in E\}| = in(u)$

Fact: $\forall\ u \in V, f(s, u_{in}) = out(u)$

we know $\sum_{u \in V} f(s, u_{in}) = \sum_{u \in V} out(u)$ [given]

we know $\sum_{u \in V} c(s, u_{in}) = \sum_{u \in V} out(u)$ [since $\forall\ u \in V, c(s, u_{in}) = \sum_{u \in V} out(u)$]

we can say $\forall\ u \in V, f(s, u_{in}) = c(s, u_{in}) = out(u)$

Proof by contradiction:

$\forall\ u \in V, f(s, u_{in}) <= c(s, u_{in})$(**) [by capacity]

If $\exists\ u \in V$ s.t $f(s, u_{in}) \neq c(s, u_{in})$,

$\Rightarrow f(s, u_{in}) < c(s, u_{in})$ (***)

$\Rightarrow \sum_{u \in V} f(s, u_{in}) < \sum_{u \in V} c(s, u_{in}) = \sum_{u \in V} out(u)$ [by ** & ***]

Which contradict to $\sum_{u \in V} f(s, u_{in}) = \sum_{u \in V} out(u)$

Recall def of f: $\forall\ u \in V$,

$$f(u_{in}, v_{out}) \begin{cases} = 1 & \text{if there is a edge } (u, v) \in E \text{ of } G \\ = 0 & \text{o/w} \end{cases}$$

$|\{(u, v) \mid (u, v) \in E\}|$

$= \sum_{v \in V - \{u\}} f(u_{in}, v_{out})$ [by def of f and $f \in \{0,1\}$]

$= f(s, u_{in})$ [by conservation]

$= c(s, u_{in})$ [by fact]

$= out(u)$ [by construction of the F]

Similar for proof $|\{(v, u) \mid (v, u) \in E\}| = in(u)$

$\Rightarrow G = (V, E)$ realize the set of degree pairs.

Therefore the set of degree pairs is realizable.