18.335 Take-Home Midterm Exam: Spring 2022

Posted 1pm Wednesday April 6, due **1pm Friday April 8.**

Problem 0: Honor code

Copy and sign the following in your solutions:

I have not used any resources to complete this exam other than my own 18.335 notes, the textbook, running my own Julia code, and posted 18.335 course materials.

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your signature

Problem 1: (10+10+8+5 points)

Consider the two real vectors $u = (u_1, u_2, \dots, u_n)^T$ and $v = (v_1, v_2, \dots, v_n)^T$ and recall that the 1-norm of a vector x is defined as $\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|$.

(a) Show that computing the dot product $f(u, v) = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$ in floating point arithmetic with left to right summation is backward stable, and in fact that $\hat{f}(u, v) = f(u + \delta u, v) = f(u, v + \delta v)$ for some δu and δv satisfying $\|\delta u\|_1 \leq \|u\|_1 \mathcal{O}(\varepsilon_{\text{mach}})$ and $\|\delta v\|_1 \leq \|v\|_1 \mathcal{O}(\varepsilon_{\text{mach}})$.

(b) Calculate the relative condition number of $f(u, v)$ with respect to its first input u in the 1-norm.

(c) Using your results in parts (a) and (b), derive a bound on the forward error in $f(u, v)$. You should write the first order term explicitly (don't use "big-oh" notation).

(d) Explain why computing the outer product $F(u, v) = uv^T$ is not backward stable.

Problem 2: (9 + 9 + 9 + 6 points)

Consider a tall skinny matrix X with m rows and $n \ll m$ linearly independent columns. Let $X = QR$ be the reduced QR factorization of X and let $A = LL^T$ be the Cholesky factorization of the $n \times n$ symmetric positive definite matrix $A = X^T X$.

(a) Prove that the triangular factors in the two decompositions are equal, i.e., that $R = L^T$.

(b) Using (a), describe an algorithm that computes both Q and R from the Cholesky factorization of A.

(c) How many flops does each step from your algorithm in (b) require? Include the cost of forming A and computing the Cholesky decomposition. You should write the leading order terms explicitly, i.e. $\sim 2n^2$, rather than $\mathcal{O}(n^2)$.

(d) Explain why the orthogonal factor $\hat{Q}$ computed by the algorithm in (b) in floating point may not be close to an orthogonal matrix when X is ill-conditioned.

Remark: Although Cholesky-based QR appears unstable, it has excellent communication costs. Remarkably, the instability explored in (d) can be overcome by, essentially, iterating the algorithm three times. The result is a state-of-the-art communication-minimizing algorithm!

Problem 3: (10 + 18 + 6 points)

In analogy to the QR algorithm for the standard eigenvalue problem $Av = \lambda v$, the QZ algorithm is used to compute generalized eigenvalues of the generalized eigenvalue problem $Av = \lambda Bv$, where A and B are $n \times n$ matrices. Like QR, it proceeds in two stages: use orthogonal transformations (e.g., Householder transformations) applied from the left and right to (i) reduce A and B to upper Hessenberg form and upper triangular form, and (ii) perform QZ iterations that drive the subdiagonal entries of the upper Hessenberg matrix A to zero. In this problem, you will work through stage (i) of the QZ algorithm.

(a) Apply a sequence of Householder transformations on the left of A and B that triangularize B.

(b) Apply a sequence of Householder transformations on the left and right of A and B that make A upper Hessenberg and keep B upper triangular. (Hint: use transformations from the left to introduce zeros in A and from the right to keep B upper triangular.)

(c) Is stage (i) backward stable? Give a brief explanation (one or two sentences).

Your answer should include diagrams that show the pattern of zeros and nonzeros after each Householder transformation is applied, as well as Julia code to execute stage (i). You may use Julia code from the notes folder on the course page to compute and apply Householder reflectors.