A mathematical description of pelotons
18.355 Research Paper

Javier Chico Vázquez
Massachussets Insitute of Technology

December 2021

Contents

1 Abstract

In professional cycling events drafting is of the utmost importance, as wind resistance accounts for 90% of the drag a cyclist experiences [1]. For this reason, the main group of cyclists normally completes the courses of main races as a single, large group known as the peloton, as in Figure 1b. In the present paper several mathematical aspects of pelotons will be explored, with a strong focus on practical applications in competitive cycling.

Firstly, the optimal shape of a peloton in order to minimize the total drag it experiences will be discussed. Although results from this section are not readily applicable to standard cycling races - where the objective is to minimize your own individual drag to maximize your chance of winning - this can be applied to team time trial (TTT) stages, where the whole group of riders shares the common objective of travelling from point A to B in the shortest time. For a no crosswind case with strong assumptions on the properties of the drag decay the optimality of the straight line is proven, and the optimal solution for a small crosswind is hypothesized.

Then, the question of what is the optimal individual position in the peloton will be considered. Although from a drag perspective this is obviously at the end of the peloton, there are other non-trivial factors such as the probability of being involved in a crash, of reacting to attacks from other riders...

The last section is a brief theoretical foray into a possible application of ideas from porous flow to pelotons and the flow around them.

2 Optimizing the peloton shape

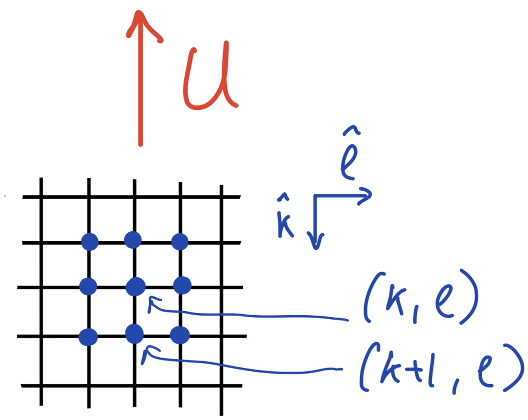
For this section we assume there are no winners in cycling and the object is for a group of N riders to get from point A to point B using the smallest possible energy, i.e. experience the smallest possible drag. Before proceeding, we must give a clear mathematical definition of what we mean by peloton and by smallest possible drag.

Assume we have an M by M grid of "spots" where the riders could be, and assume we have N riders, with $N < M^2$ and for the time being $N < M$, which can be justified by saying that in real life the road is much longer than the peloton. Furthermore assume all cyclists and their gear are identical in shape and size. I will use the following notation and conventions to describe the arrangement of cyclists:

Definition Let $A_{kl} = 1$ if there is a cyclist in the position (k, l), and zero if the spot is empty, where $\hat{k}$ is the axis parallel to the direction of travel. Call $\mathbf{A}$ the **peloton matrix**/tensor, as it encodes the geometry of the pack of riders. Mathematically, when we refer to a peloton we mean a peloton tensor $\mathbf{A}$. For a graphical representation, please refer to Figure 1a.

Definition In a similar way, let D_{kl} denote the (non-dimensional) drag experienced by a cyclist in position (k, l). By non-dimensional it is meant that for an isolated cyclist moving at the same speed they will have $D = 1$, and hence the drag of individual cyclists in the peloton are measured with respect to an individual rider. Note that $A_{kl} = 0$ implies trivially that $D_{kl} = 0$. Our aim is to minimize the total drag of the peloton, which will be

$$D_{\text{total}} = \sum_{k,l} D_{kl}$$



(a) Grid construction of the peloton

(b) Cyclists ride in a tich pack known as a peloton

Figure 1: Grid description of the peloton. Credit: Ciclismo a Fondo

2.1 No crosswind

As there is no crosswind we know the optimal trajectory (the one that minimizes the time it takes to travel, and hence the energy used) will be a straight line, so we can treat this as a one dimensional problem in terms of the trajectory. Now we will model D_{kl}, as $D(\eta_{kl})$, where η_{kl} is a variable that counts the number of riders ahead of position (k, l). We will use the following convention to express the direction of travel of the pack of cyclists: they travel in $-\hat{k}$ direction, in other words the rider with the lowest value of k (which we will commonly set to zero) is the first rider. This means that we can express η_{kl} as

$$\eta_{kl} = \sum_{k'=0}^{k-1} A_{k'l}$$

In particular, we assume $D(\eta_{kl})$ is **only a function of the number cyclists in front of cyclist** (k, l). Although if someone is riding next to you they will have some effect on the flow, the main pressure drop will come from the obstacle directly in the direction of travel of the flow. We can get a relatively arbitrary $D(\eta)$ by considering what properties it must have that make sense physically.

- Clearly $D(\eta) > 0$ as drag will always act against motion.
- We are modelling the drag force non dimensionally in the sense that we take as the reference drag the drag experience by an isolated cyclist. Hence the first rider will have $D(\eta = 0)$=1, as they have no aerodynamic shield, as if they were riding alone.[1]
- From wind tunnel experiments and CFD simulations [1], $D(\eta)$ is a decreasing function of η, as the more riders in front of you the less drag you experience.

Hence, if we assume the above properties real analysis tells us that $D(\eta) \to b$ as $\eta \to \infty$, and $0 < b < 1$. Wind tunnel results indicate that b is between 0.05 and 0.1 [1]. Furthermore, this means we can split D in the following way:

$$D(\eta) = \bar{D}(\eta) + b$$

[1] In real life, this is not the case, as the air pushed forwards from the trailing cyclists will reduce the drag experienced by the front rider, sometimes significantly.[1, 2]

where $\bar{D}(\eta)$ is a decreasing function and

$$\lim_{\eta\to\infty} \bar{D}(\eta) = 0$$

From experiments $\bar{D}(\eta) \approx \exp(-\lambda\eta)$ [1]. Physically this makes sense, as we can assume that the pressure drop across a single cyclist is $\sim \rho v^2$, say some fraction $0 < \epsilon < 1$ of $\sim \rho v^2$. The cyclist right after them will induce another pressure change, but instead of the original $\sim \rho v^2$, to $\sim \epsilon\rho v^2$, and hence the pressure drop across two riders will be $\sim \epsilon^2 \rho v^2$, and here lies the genesis of the experimental/geometric decay.

Our final assumption is that peloton is **dense**: there are no gaps in the peloton. This makes sense as the whole point of riding as a peloton is minimizing your *individual* drag and to do so you want to be as close as is safely possible to the previous rider. If a cyclist is not strong enough to bridge the gap and catch up with the previous rider they have been dropped from the group and the peloton has split, which is precisely what we assume that can not happen. Mathematically, this means that if $\exists k_1, k_2, l$ such that $A_{k_1 l} = A_{k_2 l} = 1$ then for all $k_1 \leq k \leq k_2$ we must have $A_{kl} = 1$. With all our assumptions clearly stated and defined we can begin tackling the problem.

2.1.1 Drag for a single line

As we have $N < M$, we can have a single line configuration of the peloton. In this case, without loss of generality say that the peloton occupies positions $(k, 0)$ for $1 \leq k \leq N$ (ie a vertical line at column 0 of the grid in Figure 1a). The total drag for this configuration is (A double sum is not needed as $D_{kl} = 0$ for all $l \neq 0$)

$$D^1_{\text{total}} = \sum_{k'=1}^{N} D(\eta(k')) = \sum_{k'=1}^{N} (\bar{D}(k'-1) + b) = Nb + \sum_{k'=1}^{N} \bar{D}(k'-1)$$

If we restrict ourselves to the exponential assumption ($\bar{D}(\eta) = \gamma e^{-k\eta}$), we can explicitly calculate

$$D^1_{\text{total}} = Nb + \gamma \sum_{k'=1}^{N} e^{-\lambda\eta} = Nb + \frac{\gamma}{1+e^{-\lambda}}(1 - e^{-\lambda N})$$

To measure the relative performance of an arrangement we define the quality factor, the ratio between the drag for a particular arrangement and the drag if all the cyclists would be riding in isolation, $D^I_{\text{total}} = N$

$$Q_f(\text{arrangement}) = \frac{D_{\text{total}}(\text{arrangement})}{D^I_{\text{total}}}$$

for the exponential model

$$Q_f = b + \frac{1}{N}\frac{\gamma}{1+e^{-\lambda}}(1 - e^{-\lambda N})$$

as a remark, $Q_f \to b$ as $N, M \to \infty$.

2.1.2 Optimality of the single line

Intuitively, under the above assumptions, it makes sense that the single line is the optimal arrangement, mainly because any perturbation from this arrangement will invoke the creation of a second line where the first rider will experience maximum wind resistance. As an example for N even, consider an arrangement consisting of two parallel lines of cyclists. The total drag in this case is (the drag on each of the lines is the same)

$$D^2_{\text{total}} = 2 = 2\sum_{k'=1}^{N/2} D(\eta(k')) = 2\sum_{k'=1}^{N/2}(\bar{D}(k'-1)+b) = Nb + 2\sum_{k'=1}^{N/2}\bar{D}(k'-1)$$

Claim The drag for a single file is smaller than for two lines: $D^1_{\text{total}} < D^2_{\text{total}}$.

Proof. We can see that

$$D^1_{\text{total}} < D^2_{\text{total}} \iff \sum_{k''=1}^{N}\bar{D}(k''-1) < 2\sum_{k'=1}^{N/2}\bar{D}(k'-1) \iff \sum_{k''=N/2+1}^{N}\bar{D}(k''-1) < \sum_{k'=1}^{N/2}\bar{D}(k'-1)$$

Now as $\bar{D}$ is a decreasing function, for $x < y$, $\bar{D}(x) > \bar{D}(y)$. In this case, we can bound the LHS by

$$\sum_{k''=N/2+1}^{N}\bar{D}(k''-1) < \sum_{k''=N/2+1}^{N}\bar{D}(N/2) = N/2\bar{D}(N/2)$$

We know $k'-1 < N/2$ for all k', and thus $\bar{D}(k'-1) > \bar{D}(N/2)$, so when we sum over k'

$$\sum_{k'=1}^{N/2}\bar{D}(k'-1) > N/2\bar{D}(N/2) > \sum_{k''=N/2+1}^{N}\bar{D}(k''-1)$$

and show we have shown $D^1_{\text{total}} < D^2_{\text{total}}$. $\square$

We can actually see that any single perturbation to the single line configuration yields higher total drag

Lemma Consider a single line configuration, for which $\exists! l^*$ such that $A_{kl^*} = 1$ for $1 \leq k \leq N$ and $A_{kl} = 0$ for $l \neq l^*$. Without loss of generality we can take $l^* = 0$ as before. Now consider a unique perturbation to this configuration, ie a unique $A_{k^*l} = 1$ for some $l \neq 0$. (Picture this as a rider that changes to another column, on their own). Let D^P_{total} denote the total drag for the perturbed configuration. Then $D^1_{\text{total}} < D^P_{\text{total}}$.

Proof. We begin by noting that as soon as the configuration is perturbed, say some unique rider moves from position $(k^*, 0) \to (k^*, l)$ all the cyclist originally behind them move up to minimize their own drag (and not violate our assumption that the peloton is dense). Then the total drag for the perturbed state will be the drag for a single line of length $N-1$ and a single isolated rider

$$D^P_{\text{total}} = \sum_{k'=1}^{N-1}\bar{D}(k'-1) + (N-1)b + b + \bar{D}(0)$$

as $\bar{D}$ is a decreasing function, $\bar{D}(0) > \bar{D}(N-1)$. Hence

$$D^P_{\text{total}} = \sum_{k'=1}^{N-1}\bar{D}(k'-1) + Nb + \bar{D}(0) > \sum_{k'=1}^{N-1}\bar{D}(k'-1) + Nb + \bar{D}(N-1) = Nb + \sum_{k'=1}^{N}\bar{D}(k'-1) = D^1_{\text{total}}$$

So we have shown that $D^1_{\text{total}} < D^P_{\text{total}}$.$\square$

Can we generalize this result and conclude that the single line configuration has optimal drag? We actually can. The intuitive idea behind the proof is the following: we can imagine an arbitrary dense configuration of the peloton as a perturbation of the single line, but instead of only perturbing a single rider, we perturb an arbitrary number, say T cyclists, where $N-T$ is the length of the largest single line in the configuration.

Theorem A single line configuration minimizes the total drag of a dense peloton.

Proof. We will proof the Theorem by induction on T, the number of perturbed riders. Conveniently, the base case $T=1$ is just the above lemma, so that's done. Now assume the that any configuration where T riders have been perturbed from the longest single line (which has length $N-T$, and without loss of generality is $l=0$), with total drag say D^T_{total} has $D^T_{\text{total}} > D^1_{\text{total}}$. Now consider that another rider leaves the longest single line, and goes from $(k^*, 0) \to (k^*, l^*)$. For convenience, we will define the length of each line of riders, L_l, which can be expressed as

$$L_l = \sum_{k=1}^{M} A_{kl} = \max_k(\eta_{kl}) + 1$$

Now consider the following cases:

- Most common: The rider moves into a line that was shorter than the longest line. This means $L_{l^*} < L_0 = N-T$. Now the total drag of the riders in that line is:
  - Before the perturbation (we can assume wlog that each line starts at $k=0$, as the drag on a line of cyclists is invariant under a translation of the whole line):

$$D^T_l = \sum_{k=0}^{L_{l*}} D(\eta_{kl})$$

  - After the perturbation

$$D^{T+1}_l = \sum_{k=0}^{L_{l*}+1} D(\eta_{kl})$$

  As for the total drag on the riders in the longest line, this is
  - Before the perturbation

$$D^T_0 = \sum_{k=0}^{N-T} D(\eta_{k0})$$

  - After the perturbation

$$D^{T+1}_0 = \sum_{k=0}^{N-T-1} D(\eta_{k0})$$

  Now as we know we are working on dense lines that start from 0, $\eta_{kl} = k-1$. Furthermore for this case we have $L_{l^*}+1 \leq N-T$ This implies

$$D(L_{l^*}+1) \geq D(N-T) \Longrightarrow D^{T+1}_0 = \sum_{k=0}^{N-T-1} D(\eta_{k0}) + \sum_{k=0}^{L_{l^*}+1} D(\eta_{kl}) \geq \sum_{k=0}^{N-T} D(\eta_{k0}) + \sum_{k=0}^{L_{l^*}} D(\eta_{kl})$$

  This means $D^{T+1}_0 + D^{T+1}_l \geq D^T_0 + D^T_l$. As all the other lines are invariant, this means that the total drag is also larger for the $T+1$ perturbed configuration, but we also had the induction hypothesis, which gives us $D_{\text{total}}^{T+1} \geq D_{\text{total}}^{T} > D_{\text{total}}^{1}$, precisely what we want to show.

- We still have the case $L_{l^*} = N - T$, meaning there's more than two lines with maximum length. But if the riders moves to a line of the same length as the longest one it will now be longer, in particular of length $N - T + 1$, and if we recenter the problem on that line the system will correspond to one with $T - 1$ perturbed riders, instead of $T + 1$, and hence by the induction hypothesis it has more drag than the single line
- As the case $L_{l^*} > N - T$ makes no sense because the longest line has length $N - T$, we have exhausted all possible cases and hence by the principle of mathematical induction the statement has been proved. □

2.1.3 Application: Team Time trials

The clearest application from the result in the preceding section are Team Time Trial (TTT) Events, where cyclists from the same team must compete together to achieve reach the end of the course in the shortest time possible. In Figures 2a and 2b we can confirm the no crosswind solutions as cyclists form a straight line, broken only for seconds at a time to relegate the first rider.

(a) Team Astana

(b) Team Deuckenick-Quick step

Figure 2: Riders forming a straight line during a team time trial (TTT). Credit: Ciclismo a Fondo

2.2 Arbitrary background wind

Suppose now that the pack of riders moves with speed U in the same direction as before. However, there is a background wind with components (w_k, w_l). Hence the apparent wind for riders will be $(U + w_k, w_l)$. Without loss of generality we can assume $w_k = 0$, as we can simply absorb the strength of a pure tail/head wind into U, and we can simplify the problem to an *apparent wind* (U, w_l), as displayed in Figure 3a. It will be useful to define the relative strength of the crosswind, r, as

$$r = \frac{w_l}{U} = \tan\theta$$

Were θ is the angle the free-stream makes with the direction of travel. Furthermore, we can assume the crosswind always comes from the same side of the peloton as displayed in Figure 3a. If it comes from the opposite side we can always reverse the solution of the problem, for it is symmetric with respect to the $\hat{k}$ axis. The drag force on an isolated cyclist will act on the direction of (U, w_l), and have strength $\sim \frac{1}{2}\rho A(U + w_l)^2$.

We can decompose this force into our coordinates $\hat{k}, \hat{l}$ by

$$D_A \sim \frac{1}{2}\rho A(U^2 + w_l^2)\cos\theta \qquad D_S \sim \frac{1}{2}\rho A(U^2 + w_l^2)\sin\theta$$

Where we have D_A, the drag from ahead and D_S, the drag from the side. We have $\cos^2\theta = (1+r^2)^{-1}$, $\sin^2\theta = \frac{r^2}{1+r^2}$. Hence,

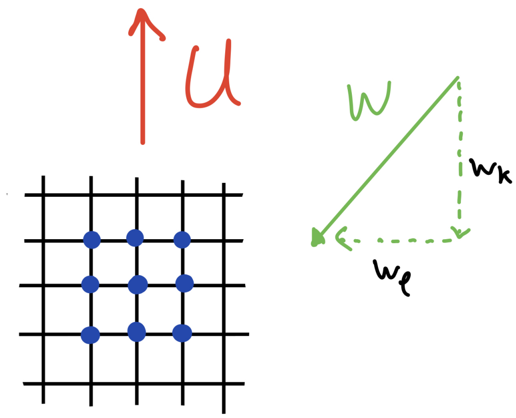
$$D_A \sim \frac{1}{2}\rho A U^2\sqrt{1+r^2} \sim \sqrt{1+r^2} \quad {}_S \sim \frac{1}{2}\rho A r\sqrt{1+r^2} \sim r\sqrt{1+r^2}$$

Where the we keep last relationship because we are just interested in the relative strength of each drag component. In most cases, the cyclists are moving quickly so we expect $r < 1$. However, in instances with extraordinarily strong crosswinds r might be close to one. Any higher value of the crosswind would imply detrimental atmospheric conditions, and hence any cycling event would be called off, as is commonplace when weather becomes exceedingly dangerous (Stage 19 and 20 of the 2019 Tour de France for instance). Hence we can restrict our analysis to $r < 1$. We will model the drag for the pack of riders in a similar way as for the no crosswind section. Instead of a single drag decay function, we will have two of them. Firstly, the drag from ahead, D_A, which is still only a function of the number of riders ahead of (k, l), which as in the previous section we denote as η_{kl}. Secondly, the drag from the side, D_S, which will be only a function to the number of cyclists to the right of (k, l), which we denote ξ_{kl}. As bikes are not streamlined in the direction normal to travel, we expect D_S to decay more rapidly than D_A, as the flow is more likely to separate in the case of D_S and hence we expect a larger pressure drop. Hence a reasonable model for this functions are exponential decays:

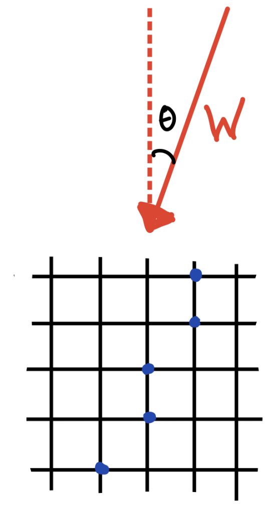
$$D_A(\eta) = b + \gamma e^{-\lambda\eta}$$

$$D_S(\xi) = c + \beta e^{-\sigma\xi}$$

And we expect $\sigma > \lambda$ for D_S to decay more rapidly than D_A. Once we have gotten all of this out of the way we can start discussing the optimal peloton arrangement.



(a) Crosswind Orientation Convention



(b) Approximating the wind vector with the grid we get close to the optimal arrangement

Figure 3: Optimal solutions for crosswind

2.2.1 Non optimality of the Single line

Theorem For all N and for all D_A, D_S there exists a a value of r such that the the single line configuration does not minimize the total drag of the peloton.

Proof. We will show that for a certain crosswind value the two line configuration offers smaller drag than the single line. We can start by computing the drag for a single line. The tangential or "ahead" component will be

$$D_A^1 = \sqrt{1+r^2}\sum_{k=1}^{N} D_A(k-1)$$

As for the side drag, all cyclist have 0 cyclists to their right (they are in a straight line) so $\xi_{kl} = 0$ for all the cyclists. As $D_S(0) = 1$, we have that

$$D_S^1 = r\sqrt{1+r^2}N$$

And the square magnitude total drag can be combined by computing the square modulus of the drag vector,

$$(D_{\text{total}}^1)^2 = (\sqrt{1+r^2}\sum_{k=1}^{N} D_A(k-1))^2 + (r\sqrt{1+r^2}N)^2$$

As for two lines, D_A^2 is computed much the same way as for the single line, except we only sum until $N/2$ and we multiply times 2 to account for the two lines. Hence

$$D_A^2 = 2\sqrt{1+r^2}\sum_{k=1}^{N/2} D_A(k-1)$$

As for the side drag, riders in the right most line have 0 cyclists to their right, and cyclists in the leftmost lane have 1. Hence the sideways drag will be

$$D_S^2 = r\sqrt{1+r^2}\frac{N}{2}(D_S(0) + D_S(1))$$

And the square magnitude of the total drag is

$$(D^2_{\text{total}})^2 = (1+r^2)(2\sum_{k=1}^{N/2} D_A(k-1))^2 + r^2(1+r^2)(\frac{N}{2}(D_S(0)+D_S(1)))^2$$

We are interested in seeing if $(D^2_{\text{total}})^2 < (D^1_{\text{total}})^2$ for some combination of the parameters and $r < 1$. If we restrict ourselves to D_A, D_S with the exponential model, the inequality reads

$$\frac{N^2r^2}{4}\left(1+\beta e^{-\sigma}+c\right)^2 + \left(2\gamma\frac{(1-e^{\lambda N/2})}{1-e^{-\lambda}} + bN\right)^2 < N^2r^2 + \left(\gamma\frac{(1-e^{\lambda N})}{1-e^{-\lambda}} + bN\right)^2$$

We can see this is true for a very reasonable set of parameters: $r = 0.5$, $N = 100$, $\lambda = 0.3$, $b = c = 0.1$, $\sigma = 0.6$, $\gamma = \beta = 0.9$. Indeed we can solve this inequality analytically. For this, let's re-express the above inequality as

$$\frac{N^2r^2}{4}(1+\beta e^{-\sigma}+c)^2 + A < N^2r^2 + B$$

With A, B, being equal to what was in their place in the previous expression. Then we have

$$r^2N^2(\frac{1}{4}(1+\beta e^{-\sigma}+c)^2 - 1) < B - A$$

From our no crosswind discussion we have that $A > B$, but we also have $\frac{1}{4}(1+\beta e^{-\sigma}+c)^2 < 1)$, meaning we can solve for r_c as

$$r_c = \frac{1}{N}\sqrt{\frac{B-A}{\frac{1}{4}(1+\beta e^{-\sigma}+c)^2 - 1)}}$$

With the actual expressions for A and B this turns out to be:

$$r_c = \frac{2e^{(-\frac{3N\lambda}{2}+\frac{\lambda}{2}+\sigma)}}{N\left(e^{\lambda}-1\right)}\Bigg[-\frac{\gamma}{\left(\beta+ce^{\sigma}-e^{\sigma}\right)\left(\beta+ce^{\sigma}+3e^{\sigma}\right)}\Big(4Nbe^{\frac{5N\lambda}{2}} - 2Nbe^{3N\lambda} - 2Nbe^{2N\lambda}$$

$$+2Nbe^{2N\lambda+\lambda} - 4Nbe^{\frac{5N\lambda}{2}+\lambda} + 2Nbe^{3N\lambda+\lambda} - \gamma e^{N\lambda+\lambda} + 6\gamma e^{2N\lambda+\lambda} - 8\gamma e^{\frac{5N\lambda}{2}+\lambda} + 3\gamma e^{3N\lambda+\lambda}\Big)^{1/2}$$

For our parameters this turns out to be $r_c \approx 0.17$ Hence we have shown that the single line is not always optimal.

2.2.2 Hypothesizing the solution: Approximating the correct line on our grid

With our current assumptions we expect the optimal solution to be a straight line in the direction of the total wind. With the present grid configuration, we expect this optimal solution to be represented by the closest possible fit to such a straight line in the direction of the free-stream wind. Hence for $r = 1$ we would expect a straight line at 45 degrees with respect to the direction of travel ($\hat{k}$), and for general r, at an angle of $\theta = \arctan r$ with respect to $\hat{k}$. An approximate example of this is given in Figure 3b

2.2.3 Applications: crosswinds and *echelons*

The main application of the crosswind solution are echelons in cycling. Echelons occurs when there is a strong crosswind, and drafting cyclists position themselves not directly behind the previous cyclist, but to a slight angle from the preceding rider as well, in the direction opposite to the crosswind, see Figure 4a, as this is the place that offers them the most protection from wind resistance. However, as the road width is finite, there will be a point where it is not possible to position yourself at an angle from the previous rider, and hence you will need to experience more drag if you want to keep up with the pack. See Figure 4b, where the peloton has stretched as a result of the echelon, and hence there is a larger time difference between the leading and trailing cyclists. At this point it is very likely that the unlucky cyclists at this final positions will get dropped. This makes for very interesting racing strategies and dynamics.

(a) Lead riders ride into the wind

(b) The peloton stretches as a result of the crosswinds

Figure 4: When crosswinds are strong, teams coordinate themselves to force adversaries in their wake to slow down when the optimal drag position is outside of the road. Credit: Ciclismo a Fondo

2.3 An alternative Characterization: the Protection Function

We can define how protected against the free-stream/wind a particular rider is by defining the protection P_{kl} as

$$P_{kl} \sim \frac{1}{D_{kl}}$$

We can construct a "second order" model for the protection P_{kl} that takes into account also the cyclists in front of the rider on the right/left as well, call that variable ϕ_{kl}, and a drag decay function in direction given by $D_D(\phi)$, by defining

$$P_{kl} = D_A(\eta_{kl})\cos\theta + D_D(\phi_{kl})\sin 2\theta + D_S(\xi_{kl})\sin\theta$$

So we see the trigonometric terms 'activate' the drag decay function which is closest to the direction of the wind resistance vector. This characterization would allow us to have a second order accurate solution to the problem, however due to time constraints it is left for future research.

3 Optimizing your position within the peloton

In this section I aim to discuss what is the optimal position within a peloton when acting as an individual solely interested in winning a certain stage or race.

This problem is relevant to standard cycling events, where as a necessary requirement for a stage win a cyclist must minimize their drag as much as possible, which, using results from the first section of this paper would imply that they must race at the back of the peloton. However, there are non trivial factors which would make a cyclist be closer to the front, mainly the probability of being involved in a crash and being able to respond to attacks. We will focus on the probability of crashing, as this is what makes or breaks a Grand Tour victory attempt.

We will deal with the 1D case with all riders in a single line. Again, we assume that drag decays as a function of the number of riders in front, η, with $D \sim \frac{1}{2}\rho A U^2 D(\eta)$, but we will keep the non dimensional drag function only, to make sure the drag is between 0 and 1, and we will take into account the free-stream velocity and density by incorporating them into the parameters of our drag decay function and our crash probability models.

Now we have to model the probability of crashing. We will do this as the probability of crashing *given* a crash has occurred, as a function of your position in the peloton, j. We call this event f_j, and we are interested in $F(j) = \mathbb{P}(f_j|\text{crash}) = Pr(f_j)$

3.1 Modelling crash probability

Our procedure for modelling the crash probability is the following. Firstly determine $\mathbb{P}(c_k)$ the probability that someone crashes at position k, provided that there has been a crash. We will provide several reasonable models for the distribution of c_k (the cyclist we are interested in is j, not necessarily the same as k). Next, we are interested in what happens after the crash, and in particular how it propagates through the peloton. We can model this with $\mathbb{P}(f_j|c_k)$, the probability that rider j (our rider) crashes/falls, given that k has crashed first. We will also provide several reasonable models for this probability. We can trivially see that for $j < k$, $\mathbb{P}(f_j|c_k) = 0$, as rider j is ahead of the crash. Similarly, another constraint on our model will always be the case $j = k$ for which $\mathbb{P}(f_j|c_j) = 1$, as we already know rider k has crashed. I will list the models by order of complexity. Once we have modelled all the above, we can compute the probability that rider j crashes by the law of total probability:

$$F(j) = \mathbb{P}(f_j) = \sum_{k=1}^{N} \mathbb{P}(f_j|c_k)\mathbb{P}(c_k)$$

Now we can construct a function to optimize. We want to minimize both the drag and the probability of crashing, so it is reasonable to combine them additively, into a function G to be optimized. However in many practical scenarios in cycling, both elements do not have the same importance, so we will give them weights. This is very clear in the case of a general classification contender, for whom it is far more important to avoid a crash than to minimize some drag so they might weight this as $80 - 20$ in favour of not crashing. On the other hand, a domestique[2] who is on their rest day might be more interested in saving energy for the upcoming stages than crashing and loosing time, as they are not a contender to win the race, and only play a support role, so they might assign a $20 - 80$ weight in favor of drag reduction. We believe including these weight to the model is of paramount importance if it is to have any practical relevance in competitive cycling. Let the weights be (w_C, w_D), for crash and drag respectively. Hence

$$G = w_C F + w_D D$$

To get an idea of the possible models for $\mathbb{P}(f_j|c_k)$ and $\mathbb{P}(c_k)$ we can look at several pictures of big peloton crashes (known as "pile ups") in Figure 5a and 5b, as well as using common sense.

[2] A rider who works for the leader of their team, and has no interest in winning the race

(a) Cyclists ahead of the first cyclist to crash are never affected (b) Riders behind the crash are very likely to be loose time

Figure 5: A big crash resulting in a "pile up". Credit: Ciclismo a Fondo

In Figure 5a we see that all riders ahead of the initial crash position are safe and not affected. Hence, we can say that $\mathbb{P}(f_j|c_k) = 0$ when $j < k$, i.e. when rider j is ahead of initial crash position k. In Figure 5b we see that riders behind k can be affected by the crash, but how many will be affected (i.e. how far back the crash propagates) will ultimately depend on the braking distance, which is itself a function of speed. In the upcoming sections we will try to model this crash propagation for different scenarios.

3.1.1 Model I: Uniform c_k, total crash propagation

We assume all riders are equally likely to crash first, so $c_k = 1/N$. Next we assume that all the cyclists positioned after a crash will crash too, i.e., the crash propagates completely. This might be reasonable for an extremely high speed case, where the braking distance is far too great to break in time. An example of this would be a crash in the descent of a mountain climb, where speeds regularly exceed 100 Km/h. Mathematically, this means

$$\mathbb{P}(f_j|c_k) = 0 \text{ if } j < k \quad \mathbb{P}(f_j|c_k) = 1 \text{ if } j \geq k$$

Hence $f_j|c_k \sim I(k \leq j)$, so we get

$$\mathbb{P}(f_j) = \sum_{k=1}^{N} \mathbb{P}(f_j|c_k)\mathbb{P}(c_k) = \frac{1}{N}\sum_{k=1}^{j} = \frac{j}{N}$$

So now we can optimize G, say for equal weights to simplify the algebra.

$$G = D\,(j-1) + \frac{j}{N}$$

We can optimize this simply by setting its derivative wrt j equal to zero.

$$\frac{\partial G}{\partial j} = \frac{d}{dj}D\,(j-1) + \frac{1}{N} = 0$$

For an explicit solution, let D^t be approximated by an exponential trendline as in previous sections. Hence,

$$\frac{-N\gamma\lambda\rho e^{\lambda(1-j)} + 1}{N} = 0$$

and we can solve for j, constrained to $1 \leq j \leq N$ (if we get a larger/smaller number then just set to 1 or N)

$$j^* = 1 + \frac{1}{\lambda} \log(\gamma \lambda N)$$

or when weighted

$$j^* = 1 + \frac{1}{\lambda} \log\left(\frac{\gamma \lambda N w_D}{w_C}\right)$$

3.1.2 Model II: Uniform c_k, deterministic partial crash propagation

In this case, given that k has crashed first only the next Γ will crash. In general Γ will depend on the braking distance, which is a function predominantly of the speed U, so we will likely have $\Gamma(U)$, although I suspect $\Gamma(U, j)$ too. Hence $f_j | c_k \sim \mathbf{I}(k \leq j \leq k + \Gamma)$, so we can compute

$$\mathbb{P}(f_j) = \frac{1}{N} \sum_{k=1}^{N} \mathbf{I}(k \leq j \leq k + \Gamma) = \frac{1}{N} \sum_{k=j-\Gamma}^{j} 1$$

Case wise, if $j < \Gamma$ then $\mathbb{P}(f_j) = j/N$ as in model I. However, if $j \geq \Gamma$ then $\mathbb{P}(f_j) = \Gamma/N$, meaning the probability stagnates. Proceeding as before, we find that in the range $1 \leq j < \Gamma$ the solution is the same as before. So we can solve that equation again subject now to the constraint $1 \leq j < \Gamma$. If the equation has no solution in that range compare the value of G at both $j = 1, N$ and choose the minimum, most likely $j^* = N$ due to the stagnation of the crash probability.

3.1.3 Model III: Uniform c_k, random partial crash propagation

Γ has to be chosen empirically for Model II, and furthermore it is not an accurate depiction of how crashes propagate. In the same physical condition, a crash will propagate more or less depending very heavily on the reaction time of riders, which is for all practical purposes random. We could draw Γ from some distribution, like a Poisson distribution. However we can also introduce randomness more easily by assuming the distribution of $f_j | c_k$ is random. It still needs to satisfy the conditions mentioned at the beginning of the present section, and it makes sense for this probability to decrease as we move further away from the crash, where riders have had more time to react and brake. Henceforth, we can model this with a geometric/discrete exponential distribution, with parameter, ω.

$$\mathbb{P}(f_j | c_k) = e^{-\omega(j-k)}, \quad k \leq j \text{ and } 0 \text{ elsewhere}$$

As a remark, a good value for ω is $1/\Gamma$, so that the conditional distribution has expectation Γ. The algebra now gets more interesting:

$$\mathbb{P}(f_j) = \frac{1}{N} \sum_{k=1}^{N} \mathbb{P}(f_j | c_k) = \frac{e^{-\omega j}}{N} \sum_{k=1}^{j} e^{\omega k} = \frac{e^{-\omega j}}{N} \frac{1 - e^{\omega j}}{1 - e^{\omega}} e^{\omega}$$

We can check that $\mathbb{P}(f_1) = 1/N$, as it should be. (First rider can only fall if they themselves fall) Now we can build G, and under the exponential assumption we get the following optimal j^* for a weighted optimization.

$$j^* = 1 + \frac{1}{\lambda - \omega} \log\left(-\frac{N \gamma \lambda w_D (1 - e^{\omega}) e^{\lambda - \omega}}{\omega w_C}\right)$$

3.1.4 Model IV: non-uniform c_k and total crash propagation

The motivation for relaxing this assumption is relaxing the fact that not all riders have the same perception of the road. The first cyclist has clear and unobstructed of the road, and for the riders behind them their visibility gradually decreases. Hence, it is reasonable to assume that the probability that they will crash first will gradually increase: if you can see the corner, you can't break for the corner! We can try several things:

$$c_k = k \quad c_k = \log k \quad c_k = \sqrt{k}$$

Say we pick the first and simplest option, $c_k = A_N k$. We need a normalization constant, namely

$$A_N = \frac{N(N+1)}{2}$$

We now proceed as before

$$\mathbb{P}(f_j) = \sum_{k=1}^{N} A_N k \mathbb{P}(f_j|c_k) = A_N \sum_{k=1}^{j} k = A_N \frac{j(j+1)}{2} = \frac{j(j+1)}{N(N+1)}$$

We can thus optimize as before, (weighted)

$$G = w_D \left(b + \gamma e^{-\lambda(j-1)}\right) + \frac{j w_C \left(j+1\right)}{N\left(N+1\right)}$$

taking a derivative,

$$-\gamma\lambda w_D e^{-\lambda(j-1)} + \frac{j w_C}{N\left(N+1\right)} + \frac{w_C\left(j+1\right)}{N\left(N+1\right)} = 0$$

We can solve this equation in terms of Lambert's $W(x)$ function, and keep the > 0 solution.

$$j^* = \frac{-\frac{\lambda}{2} + W\left(\pm\frac{N\gamma\lambda^2 w_D (N+1) e^{\frac{3\lambda}{2}}}{2w_C}\right)}{\lambda}$$

3.1.5 Model V: non-uniform c_k and deterministic partial crash propagation

Briefly, c_k as in model IV, $f_j|c_k$ as in model II. This gives us

$$\mathbb{P}(f_j) = \sum_{k=1}^{N} A_N k \mathbb{P}(f_j|c_k) = A_N \sum_{k=j-\Gamma}^{j} k$$

we can evaluate this by cases. For $j < \Gamma$, we have the same as before, namely the lower bound in the sum is 0. And for $j \geq \Gamma$, we have

$$\mathbb{P}(f_j) = \frac{(2j-\Gamma)}{N(N+1)}(\Gamma+1)$$

We can now optimize using G as we normally do. By cases, for $j^* < \Gamma$, we have the same as Model IV. For $j^* \geq \Gamma$ we have,

$$j^* = 1 + \frac{1}{\lambda}\log\left(\frac{\gamma\lambda N(N+1)w_D}{2w_C(\Gamma+1)}\right)$$

3.1.6 Model VI: non-uniform c_k and random partial crash propagation

Finally, we combine $c_k = A_N k$ and the exponential conditional probability from model III. We can get the probability that rider j falls with

$$\mathbb{P}(f_j) = A_N \sum_{k=1}^{j} k e^{-\omega(j-k)} = A_N e^{\omega j} \sum_{k=1}^{j} k e^{\omega k}$$

However, to optimize this model we must make use of numerical methods, as the above sum is intractable.

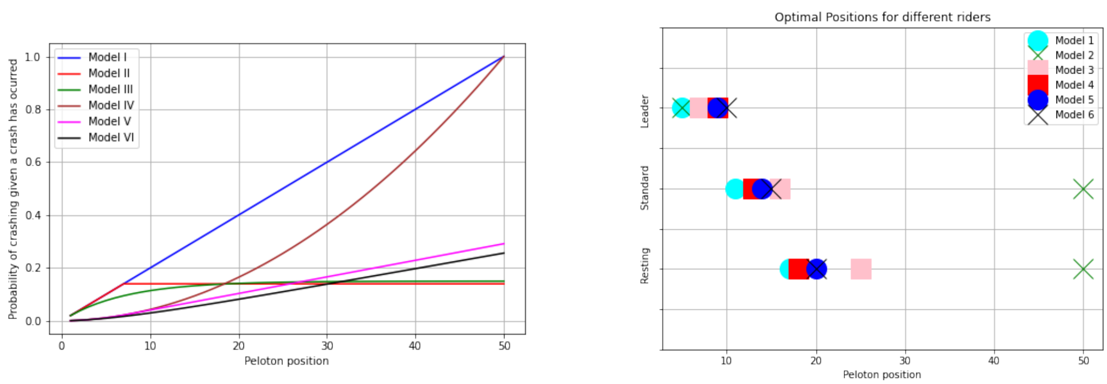
3.2 Summary Applications

In Figure 6a we plot the probability that a cyclist in position j of the peloton crashes for the different crash distribution and crash propagation of models.

Furthermore, setting some general values, we can get the optimal position according to each of the models for three different kinds of riders, with respective weights (w_C, w_D):

- Team leader/GC contender: (0.8,0.2)
- Standard Rider/Stage Hunter: (0.4,0.6)
- Domestique resting: (0.1,0.9)

The results are presented in Figure 6b.



(a) Probability of cyclist j crashing given a crash has happened

(b) Optimal positions for each rider and each model

Figure 6: Numerical Results for the Position Optimization Models

4 A peloton as a porous medium?

In this section we explore the possibility if describing the flow around and inside a peloton by modelling the pack of cyclist as a porous medium. The motivation for this procedure is remarkably simple. When modelling porosity, we have a structure which is mostly empty but has solid mass or particles scattered throughout space. We can quantify this via the void fraction ϕ, which measures how much of the space is available for the fluid [3] (air in the case of cycling).

4.1 Solution to an-isotropic permeability via diagonalozation

When the Reynolds number is low, Darcy's law holds [3]

$$\mathbf{u} = -\frac{1}{\mu}\mathbf{K}\nabla p$$

Where μ is the viscosity and $\mathbf{K}$ is a tensor that gives us the permeability of medium, which in the case of cycling is an-isotropic, for bicycles are more streamlined in the direction of travel than any other direction, and hence we expect the fluid to flow preferentially in that direction. This is a second rank symmetric tensor. The Darcy velocity further satisfies the continuity equation. Hence, assuming the permeability tensor does not vary with space, we have the following equation for the pressure instead of just $\nabla^2 p = 0$)

$$K_{ik}\frac{\partial^2 p}{\partial x_k \partial x_i} = 0$$

However, as the permeability tensor is symmetric (and positive definite) it is always diagonalizable, meaning we can work in an orthonormal basis which diagonalizes it, and reduces the above equation to a simpler form, a modified Laplace equation.

$$k_x\frac{\partial^2 p}{\partial x'^2} + k_y\frac{\partial^2 p}{\partial y'^2} + k_z\frac{\partial^2 p}{\partial z'^2}$$

Where k_x, k_y, k_z are the eigenvalues of the permeability tensor, and x', y', z' represent the principal directions of the flow. And we can transform the above into a standard Laplace Equation by re-scaling the spatial dimensions by $\sqrt{k_i}$, and then we can use vector methods as discussed in class, solve the equation for the re-scaled diagonalizing coordinates and then transform back into the original coordinate system. However, experiments show this only works for flows with Reynolds number lower than 10, which is clearly not the case for cycling.

4.2 Forchheimer correction

For flows with higher Reynolds number we can have the following quadratic correction to Darcy's law, as discussed in [3], introduced by Forchhheimer, which in one dimension takes the following form

$$\frac{\partial p}{\partial x} = -\frac{\mu}{k}u - \frac{\rho}{\kappa}u^2$$

Where κ is the *inertial permeability*. This correction takes into account inertial effects. However,the nonlinear term makes an analytical approach impractical, and it is very likely that numerical methods must be used to solve the problem.

5 Conclusion and further work

In the present paper several mathematical aspects of pelotons have been explored. In particular, we found that under strong assumptions on the drag decay in the peloton and no crosswind the optimal arrangement will be a single straight line. Furthermore, for more general cases with relaxed assumptions the introduction of the protection function could allow us to take into account the presence of more riders, not just those immediately next to the cyclist. This would give the grid description of the peloton higher accuracy and flexibility in computing the drag of each configuration.

The individual position optimization section was particularly successful, and moreover it is the most practically relevant to the professional cycling industry, as the flexibility introduced by the importance weights (w_C, w_D) and the different models introduced would allow team directors to give clear and concise instructions to team riders. This could have a profound impact on cycling strategies, which have historically relied on experience, intuition and old wise tales, instead a more rigorous and quantitative approach, which the aforementioned models could provide.

Due to time constraints the final section could not be explored in great detail, but it promises to give a rich description of the flow around and inside the peloton for idealized conditions. In particular, it would be specially interesting to further research the Forchheimer correction and attempt to give an analytical description of the flow with it, although its intrinsic non-linearity makes the use numerical methods almost inevitable.

6 Acknowledgements

The author would like to thank Prof. John Bush for his input and guidance. Furthermore, the author is grateful to Ciclismo a Fondo for generously providing all the photographs presented in this paper.

7 Bibliography

References

(1) Blocken, T., van Druenen, T., Toparlar, Y., Malizia, F., Mannion, P., Andrianne, T., Marchal, T., Maas, G.-J., Diepens, J. (2018). Aerodynamic drag in cycling pelotons: New insights by CFD simulation and wind tunnel testing. *Journal of Wind Engineering and Industrial Aerodynamics*, 179, 319–337. DOI: `https://doi.org/10.1016/j.jweia.2018.06.011`

(2) Blocken, Toparlar, Y. (2015). A following car influences cyclist drag: CFD simulations and wind tunnel measurements. *Journal of Wind Engineering and Industrial Aerodynamics*, 145, 178–186. `https://doi.org/10.1016/j.jweia.2015.06.015`

(3) Chen YF, et. al. (2015). Evaluation of Forchheimer equation coefficients for non-Darcy flow in deformable rough-walled fractures. *Journal of Hydrology*, 529(3) 993-1006. DOI: `https://doi.org/10.1016/j.jhydrol.2015.09.021`.

(4) Pinot, Grappe, F. (2011). The Record Power Profile to Assess Performance in Elite Cyclists. *International Journal of Sports Medicine*, 32(11), 839–844. DOI: `https://doi.org/10.1055/s-0031-1279773`

(5) Belden, Mansoor, M. M., Hellum, A., Rahman, S. R., Meyer, A., Pease, C., Pacheco, J., Koziol, S., Truscott, T. T. (2019). How vision governs the collective behaviour of dense cycling pelotons. *Journal of the Royal Society Interface*, 16(156), 20190197–20190197. DOI: `https://doi.org/10.1098/rsif.2019.0197`

(6) Olds T. (1998). The mathematics of breaking away and chasing in cycling. *European Journal of Applied Physiology and Occupational Physiology*, 77(6), 492–497. DOI: `https://doi.org/10.1007/s004210050365`

(7) Kyle, CR. (1979). Reduction of Wind Resistance and Power Output of Racing Cyclists and Runners Travelling in Groups. *Ergonomics*, 22(4), 387–397. DOI: `https://doi.org/10.1080/00140137908924623`

(8) Hilbert, O'Kroy, J. A., Torok, D., Campbell, K. (1999). THE EFFECT OF AERODYNAMIC POSTURE ON WORK OF BREATHING IN COMPETITIVE CYCLISTS. *Medicine and Science in Sports and Exercise*, 31(Supplement), S260–. DOI: `https://doi.org/10.1097/00005768-199905001-01253`