Exercise 1.3

(a) ~~In~~dependence assumption.

$$\prod_{n=1}^{N} p(\varepsilon_n) = \prod_{n=1}^{N} \frac{1}{\sqrt{2\pi\sigma_\varepsilon^2}} \exp\left(-\frac{1}{2\sigma_\varepsilon^2} \cdot [y_n - x\theta]^2\right) =: \mathcal{L}(\theta) \quad \text{likelihood.}$$

(b)

$$\log p(\theta | y_1, \dots, y_N) = \log p(y_1, \dots, y_N | \theta) + \log p(\theta) - \log p(y_N)$$

$$= -N\log\left(\sqrt{2\pi\sigma_\varepsilon^2}\right) - \frac{1}{2\sigma_\varepsilon^2}\sum_{n=1}^{N}(y_n - x\theta)^2 - \frac{1}{2}\log(2\pi\sigma_p^2) - \frac{1}{2\sigma_p^2}(\theta - \mu_p)^2 - \log p(y_1, \dots, y_N)$$

$$= -\frac{1}{2}N\log(2\pi\sigma_\varepsilon^2) - \frac{1}{2\sigma_\varepsilon^2}\left[\sum_{n=1}^{N} y_n^2 - 2\sum_{n=1}^{N} y_n x\theta + \sum_{n=1}^{N} x^2\theta^2\right] - \dots$$

$$= -\frac{1}{2}N\log(2\pi\sigma_\varepsilon^2) - \frac{1}{2\sigma_\varepsilon^2}\left[\sum_{n=1}^{N} y_n^2 - 2N\bar{y}x\theta + Nx^2\theta^2\right] - \frac{1}{2\sigma_p^2}\left[\theta^2 - 2\theta\mu_p + \mu_p^2\right]$$

$$- \log p(y_1, \dots, y_N)$$

(c)

$$-\frac{1}{2}\log(2\pi\sigma^2) - \frac{1}{2}\frac{(\theta-\mu)^2}{\sigma^2} \quad \text{is} \quad \log \mathcal{N}(\theta | \mu, \sigma^2)$$

both have a similar form in term of θ $\boxed{a\theta^2 + b\theta + c}$

(d)

$$-\frac{1}{2\sigma_p^2}\theta^2 - \frac{1}{2\sigma_\varepsilon^2}Nx^2\theta^2 \overset{!}{=} -\frac{1}{2\sigma^2}\theta^2 \Rightarrow \boxed{\frac{1}{\sigma^2} = \frac{1}{\sigma_p^2} + \frac{1}{\sigma_\varepsilon^2}}$$

$$\Rightarrow \boxed{\sigma^2 = \frac{\sigma_p^2\sigma_\varepsilon^2}{\sigma_p^2 + \sigma_\varepsilon^2}}$$

$$\frac{\theta\mu_p}{\sigma_p^2} + \frac{N\bar{y}x\theta}{\sigma_\varepsilon^2} = \frac{\theta\mu}{\sigma^2} \Rightarrow$$

$$\Rightarrow \frac{\mu_p}{\sigma_p^2} + \frac{N\bar{y}x}{\sigma_\varepsilon^2} = \frac{\mu}{\sigma_p^2} + \frac{\mu}{\sigma_\varepsilon^2} \Rightarrow \sigma_\varepsilon^2\mu_p + N\bar{y}x\sigma_p^2 = \mu(\sigma_\varepsilon^2 + \sigma_p^2)$$

$$\Rightarrow \boxed{\mu^* = \frac{\sigma_\varepsilon^2\mu_p + N\bar{y}x\sigma_p^2}{(\sigma_\varepsilon^2 + \sigma_p^2)}}$$