Entanglement Dominance in the Zero-Temperature Limit

A controlled Gaussian open-systems benchmark and an operational timescale criterion

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Abstract

We study the low-temperature limit of bosonic continuous-variable systems under Markovian thermal environments, with the goal of making precise when entanglement can become the operationally dominant resource shaping experimentally accessible correlations.

The main technical result is an exact entanglement threshold for symmetric two-mode squeezed thermal states (TMST), expressed in closed form through the PPT/symplectic-eigenvalue criterion. This provides a controlled benchmark: for fixed squeezing strength, cooling alone can drive a preparation from separable to entangled below a critical temperature.

Building on this benchmark, we propose an *entanglement-dominant regime* defined by a timescale inequality between entanglement generation and effective decoherence, and we outline how this criterion can be mapped to platform-level parameters (damping, dephasing floors, and thermal occupancies).

Extensions to collective modes and multipartite settings are presented as an outlook and as a numerical/experimental program, emphasizing what is rigorously proven versus what is to be tested.

Keywords: open quantum systems; Lindblad master equation; squeezed states; two-mode squeezing; Gaussian states; log-negativity; collective modes.

Contents

Chapter 1

Introduction and Scope

1.1 Problem statement

The guiding question is: *if a set of modes—or in the ideal limit, all relevant modes of a closed subsystem—is cooled toward $T \to 0$, can entanglement become the dominant structure organizing correlations and effective forces, as decoherence fades?*

1.2 Methodological choice

To ensure analytic control, we work primarily with (i) bosonic continuous-variable (CV) modes, (ii) quadratic Hamiltonians, and (iii) Markovian thermal environments leading to Lindblad master equations. This is the standard regime where entanglement criteria and steady states can be derived in closed form.

1.3 What is meant by "dominant"

We do *not* claim that entanglement is a new fundamental interaction. Instead, "dominant" is used in an operational, dynamical sense: in the low-temperature regime, the entangling part of the dynamics sets the largest relevant timescale and controls experimentally accessible observables.

Chapter 2

Postulates and Definitions

2.1 Mode decomposition and collective coordinates

Consider M bosonic modes with annihilation operators a_j satisfying $[a_j, a_k^\dagger] = \delta_{jk}$. Define quadratures $x_j = (a_j + a_j^\dagger)/\sqrt{2}$ and $p_j = (a_j - a_j^\dagger)/(\mathrm{i}\sqrt{2})$. Collect them into the phase-space vector

$$\boldsymbol{R} = (x_1, p_1, \ldots, x_M, p_M)^\top \in \mathbb{R}^{2M}. \tag{2.1}$$

2.2 Gaussian states

A state ρ is Gaussian if its characteristic function is Gaussian in phase space. It is fully specified by the displacement $\boldsymbol{d} = \mathrm{Tr}(\rho \boldsymbol{R})$ and covariance matrix

$$\sigma_{jk} = \frac{1}{2}\,\mathrm{Tr}\big(\rho\{R_j - d_j, R_k - d_k\}\big). \tag{2.2}$$

2.3 Thermal occupation

For a mode of frequency ω in a Gibbs state at temperature T,

$$\bar{n}(T) = \frac{1}{\exp(\beta\omega) - 1}, \qquad \beta \equiv \frac{1}{k_B T}. \tag{2.3}$$

In units $k_B = \hbar = 1$, $\beta = 1/T$.

2.4 Entanglement measure: log-negativity

Definition 2.1 (Log-negativity)**.** For a bipartite state ρ_{AB}, the log-negativity is

$$E_N(\rho_{AB}) \equiv \log_2 \left\|\rho_{AB}^{T_B}\right\|_1, \tag{2.4}$$

where T_B denotes partial transpose and $\|\cdot\|_1$ is the trace norm.

For two-mode Gaussian states, E_N is a function of the smallest symplectic eigenvalue $\tilde{\nu}_-$ of the partially transposed covariance matrix (Appendix A).

Chapter 3

Open-System Dynamics and Temperature

3.1 Lindblad master equation

We assume the reduced dynamics of the M-mode system is Markovian and completely positive:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\mathrm{i}[H, \rho] + \sum_{\mu} \mathcal{D}[L_\mu]\rho, \tag{3.1}$$

with dissipator $\mathcal{D}[L]\rho = L\rho L^\dagger - \frac{1}{2}\{L^\dagger L, \rho\}$.

3.2 Thermal bath for each mode

For each mode a_j coupled to a thermal bath at temperature T, one may take jump operators

$$L_{j,-} = \sqrt{\kappa_j(\bar{n}_j(T) + 1)}\, a_j, \qquad L_{j,+} = \sqrt{\kappa_j \bar{n}_j(T)}\, a_j^\dagger, \tag{3.2}$$

where κ_j is the damping rate and $\bar{n}_j(T) = \bar{n}(T; \omega_j)$. As $T \to 0$, $\bar{n}_j(T) \to 0$ and the excitation (absorption) channel $L_{j,+}$ vanishes.

3.3 Decoherence timescales

In many platforms the dephasing rate admits a decomposition

$$\Gamma_\phi(T) = \Gamma_\phi^{(0)} + \Gamma_\phi^{(\mathrm{th})}(T), \tag{3.3}$$

where $\Gamma_\phi^{(\mathrm{th})}(T) \to 0$ as $T \to 0$. The *idealized* “decoherence-free” limit corresponds to $\Gamma_\phi^{(0)} \approx 0$.

Chapter 4

Two-Mode Squeezing and an Exact Entanglement Threshold

4.1 Two-mode squeezing operator and Hamiltonian

Define two bosonic modes (a, b). The two-mode squeezing unitary is

$$S_2(r, \varphi) = \exp\Big[r\,\mathrm{e}^{\mathrm{i}\varphi} a^\dagger b^\dagger - r\,\mathrm{e}^{-\mathrm{i}\varphi} ab\Big]. \tag{4.1}$$

A standard entangling quadratic Hamiltonian generating this unitary is

$$H_{\mathrm{TMS}} = \mathrm{i}g\,(a^\dagger b^\dagger - ab), \qquad g \in \mathbb{R}. \tag{4.2}$$

4.2 Two-mode squeezed thermal state (TMST)

Let $\rho_{\mathrm{th}}(T) = \rho_{\mathrm{th}}^{(a)}(T) \otimes \rho_{\mathrm{th}}^{(b)}(T)$ be a product of single-mode thermal states with equal mean occupation $\bar{n}(T)$. Define the TMST

$$\rho_{\mathrm{TMST}}(T; r) = S_2(r, 0)\,\rho_{\mathrm{th}}(T)\,S_2^\dagger(r, 0). \tag{4.3}$$

4.3 Entanglement threshold

4.3.1 Analytic Condition for TMST Entanglement

[Exact threshold for symmetric TMST entanglement] Consider $\rho_{\mathrm{TMST}}(T; r)$ with equal thermal occupation $\bar{n}(T)$ in both modes. Then $\rho_{\mathrm{TMST}}(T; r)$ is entangled if and only if

$$\tilde{\nu}_- = (\bar{n}(T) + \tfrac{1}{2})\,\mathrm{e}^{-2r} < \tfrac{1}{2}, \tag{4.4}$$

which is equivalent to the squeezing threshold

$$r > r_c(T) \equiv \frac{1}{2}\ln\big(2\bar{n}(T) + 1\big). \tag{4.5}$$

Proof. Derivation. Using the standard PPT criterion for Gaussian states [Refs], the condition for separability becomes, separability is equivalent to positivity of partial transpose (PPT), which for Gaussian states is equivalent to $\tilde{\nu}_- \geq 1/2$. The covariance matrix of a symmetric TMST is a thermal prefactor $(\bar{n}+1/2)$ multiplying the squeezed-vacuum covariance. Under partial transpose, the smallest symplectic eigenvalue becomes $(\bar{n}+1/2)\mathrm{e}^{-2r}$. Hence entanglement occurs iff Eq. (4.4) holds, giving Eq. (4.5). A detailed covariance-matrix derivation is given in Appendix A. □

Corollary 4.1 (Zero-temperature limit)**.** *Fix any $r > 0$. As $T \to 0$, $\bar{n}(T) \to 0$ and thus $r_c(T) \to 0$; therefore $\rho_{\mathrm{TMST}}(T; r)$ is entangled for all sufficiently small temperatures.*

4.3.2 Closed-form critical temperature estimate

From Theorem 4.3.1, entanglement is guaranteed when

$$\bar{n}(T) < \bar{n}_c(r) \equiv \frac{\mathrm{e}^{2r} - 1}{2}. \tag{4.6}$$

Using $\bar{n}(T) = 1/(\exp(\beta\omega) - 1)$, a corresponding critical temperature scale is

$$T_c(r; \omega) \equiv \frac{\omega}{\ln\left(1 + \frac{1}{\bar{n}_c(r)}\right)} = \frac{\omega}{\ln\left(1 + \frac{2}{\mathrm{e}^{2r}-1}\right)}, \tag{4.7}$$

in units $k_B = \hbar = 1$. This T_c should be interpreted as a benchmark scale within the Gaussian Markovian model, not as a universal material constant.

4.4 Interpretation

The theorem gives a precise meaning to "re-entanglement near 0 K": decreasing T reduces $\bar{n}(T)$ and hence decreases the threshold $r_c(T)$. Even modest squeezing becomes sufficient to guarantee entanglement when T is low.

Chapter 5

Entanglement-Dominant Regime: A Timescale Criterion

5.1 Definition

Definition 5.1 (Entanglement-dominant regime)**.** Consider a subsystem of modes whose reduced dynamics is given by Eq. (3.1). Let τ_{ent} be a characteristic entanglement generation time (set by an entangling rate g), and let $\tau_{\mathrm{dec}}(T)$ be a decoherence time (set by rates such as $\Gamma_\phi(T)$ and thermal absorption rates $\kappa\bar{n}(T)$). We say the system is in an entanglement-dominant regime if

$$\tau_{\mathrm{ent}} \ll \tau_{\mathrm{dec}}(T) \qquad \text{equivalently} \qquad g \gg \Gamma_{\mathrm{eff}}(T), \tag{5.1}$$

where $\Gamma_{\mathrm{eff}}(T)$ is the net decoherence rate relevant to the entanglement witness used.

5.2 A model computation

For two modes with Hamiltonian H_{TMS} and symmetric thermal damping κ as in Eq. (3.2), the entanglement-building rate is of order g, while thermal noise injects excitations at rate $\kappa\bar{n}(T)$. Thus a conservative sufficient condition for robust entanglement is

$$g \gg \kappa\bar{n}(T)\,. \tag{5.2}$$

Since $\bar{n}(T) \to 0$ as $T \to 0$, this inequality becomes easier to satisfy as the system is cooled.

Chapter 6

Collective Modes and Collective Squeezing (Outlook)

6.1 Collective mode definition

Let $\{a_i\}_{i=1}^N$ be local modes ("sites", emitters, or spatial regions). Define a normalized collective mode

$$A \equiv \sum_{i=1}^{N} c_i a_i, \qquad \sum_{i=1}^{N} |c_i|^2 = 1. \tag{6.1}$$

Introduce two collective circulation channels A_R and A_L (two-mode structure) and define a collective two-mode squeezing operator

$$S_{\text{coll}}(r) = \exp\big[r\, A_R^\dagger A_L^\dagger - r\, A_R A_L\big]. \tag{6.2}$$

This matches the collective squeezing structure used in the associated framework of collective mode projection and post-selected TMSS generation.

6.2 Multipartite scaling

If the preparation protocol projects onto a collective squeezed subspace with an effective injection parameter r_{inj}, one may *expect* enhanced multipartite correlations as r_{inj} increases. In this manuscript this statement is treated as an outlook: the precise scaling is model-dependent (geometry, mode matching, loss, and witness choice) and is best established by computing Gaussian covariance steady states (Appendix B) and by platform-specific simulations/measurements. This motivates using r (or a directly measurable witness such as E_N on reduced bipartitions) as an operational order parameter for "collective re-entanglement".

Chapter 7

Discussion: What Can and Cannot Be Proven

7.1 Rigorous statements

Within the assumptions stated (Gaussianity, Markovianity, and thermal detailed balance), the TMST threshold theorem (Theorem 4.3.1) is mathematically exact. It proves that lowering T alone can convert a fixed-squeezing preparation from separable to entangled.

7.2 Physical caveats

Real systems have residual noise floors $\Gamma_\phi^{(0)}$, non-Markovianity, and additional modes. Therefore the statement "decoherence disappears" should be treated as an asymptotic idealization; experimentally one tests whether $\Gamma_{\text{eff}}(T)$ falls below the entanglement-generation scale as cooling proceeds.

Chapter 8

Experimental and Numerical Programs (Outline)

8.1 Cold-atom vortex platform

A cold-atom platform with a trapped vortex and coincidence/post-selection protocols provides a natural experimental arena in which T can be tuned to the nK regime and entanglement witnesses can be measured via correlation functions.

8.2 Rotating-frame simulations and effective rotation

In lattice or synthetic-rotation simulations, one can test whether entanglement-dependent corrections become visible as decoherence is reduced, consistent with an entanglement-dominant regime.

Appendix A

Gaussian-state technical appendix

A.1 Symplectic form and uncertainty principle

Define the symplectic form

$$\Omega = \bigoplus_{j=1}^{M} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{A.1}$$

and the uncertainty condition $\sigma + \frac{\mathrm{i}}{2}\Omega \geq 0$.

A.2 Two-mode covariance matrix and partial transpose

For a two-mode symmetric TMST with thermal prefactor $\nu \equiv \bar{n} + 1/2$ and squeezing r, one can write the covariance matrix (in the ordering (x_a, p_a, x_b, p_b)) as

$$\sigma = \nu \begin{pmatrix} \cosh 2r\, \mathbb{I}_2 & \sinh 2r\, \mathrm{diag}(1, -1) \\ \sinh 2r\, \mathrm{diag}(1, -1) & \cosh 2r\, \mathbb{I}_2 \end{pmatrix}. \tag{A.2}$$

Partial transpose corresponds to $p_b \mapsto -p_b$ at the covariance-matrix level, yielding $\tilde{\sigma}$. The smallest symplectic eigenvalue of $\tilde{\sigma}$ is

$$\tilde{\nu}_- = \nu\, \mathrm{e}^{-2r} = (\bar{n} + \tfrac{1}{2})\mathrm{e}^{-2r}. \tag{A.3}$$

The PPT criterion is $\tilde{\nu}_- \geq 1/2$ for separability.

A.3 Log-negativity for symmetric TMST

When $\tilde{\nu}_- < 1/2$, the log-negativity is

$$E_N = \max\Big\{0,\, -\log_2(2\tilde{\nu}_-)\Big\} = \max\Big\{0,\, \frac{2r - \ln(2\bar{n} + 1)}{\ln 2}\Big\}. \tag{A.4}$$

Appendix B

Optional: solving the linear open dynamics

B.1 Lyapunov equation for steady-state covariance

For quadratic Hamiltonians and linear Lindblad operators, the covariance evolves as

$$\dot{\sigma} = A\sigma + \sigma A^{\top} + D, \tag{B.1}$$

where A is the drift matrix and D is the diffusion matrix. The steady state solves the Lyapunov equation $A\sigma_\infty + \sigma_\infty A^{\top} + D = 0$. This is the natural route to compute $E_N(T)$ for multi-mode systems numerically.

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