Stanford University

EE 102B Spring-2013

Lecture 04

Sampling, Aliasing, and Reconstruction

April 8, 2013

ASSIGNMENTS

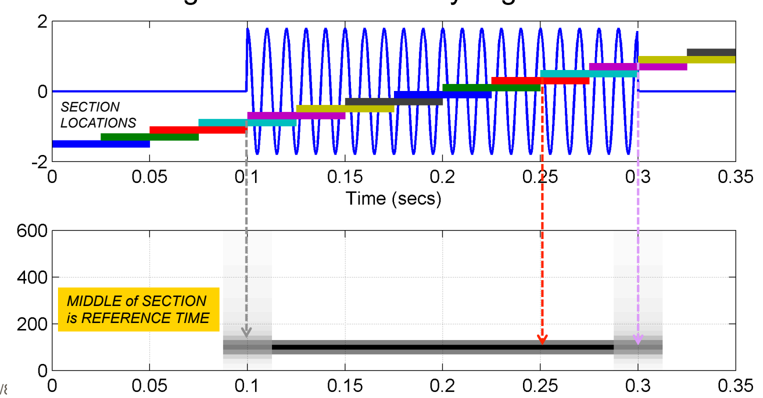
- Reading for this Lecture:
  - SPF: Chapter 4
  - S&S: The sampling discussion here will be useful when we revisit sampling in a couple of weeks.
- HW#1 and Lab #1 are posted
  - Both due on Weds. April 10 at 5pm in Packard 263.

Office Hours for Course Staff – Come see us.

- Ron Schafer: Mon. 2:00-3:00 pm and Weds. 4:00-5:00 pm in Packard 211
- Dookun Park: Tues. 1:30 ~ 3:30 pm in Packard 107
- Ruo Yu Gu (Roy): Tues. 4:00 ~ 6:00 pm in Packard 106
- https://class2go.stanford.edu/EE102B/Spring2013/pages/staff

Overlapping Sections in Spectrograms

- 50% overlap. More is required for smooth image.
- Consider edge effects when analyzing a short sinusoid

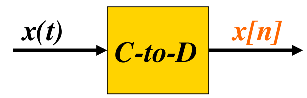


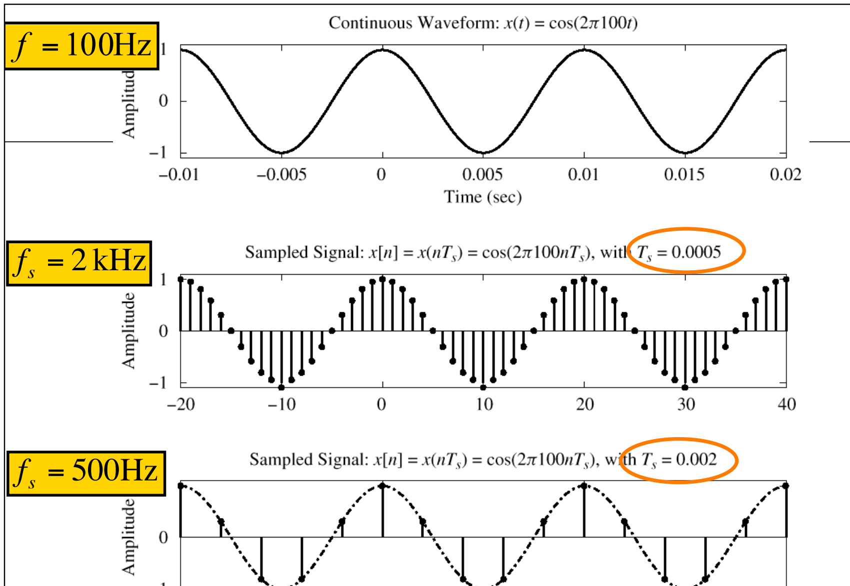
LECTURE OBJECTIVES

- SAMPLING causes ALIASING
  - **Sampling Theorem**
  - Sampling Rate > 2(Highest Frequency)
- Spectrum for digital signals, x[n]
  - Normalized Frequency in SPF $\hat{\omega} = \omega T_s$
  - In Oppenheim and Willsky, $\Omega = \omega T_s$
- Reconstruction from samples

SAMPLING x(t)

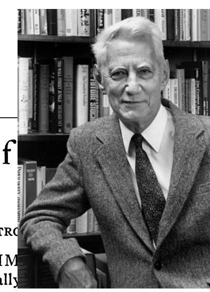
- SAMPLING PROCESS
  - Convert x(t) to numbers x[n]
  - "n" is an integer; x[n] is a sequence of values
  - Think of "n" as the storage address in memory
- UNIFORM SAMPLING at t = nT_s
  - IDEAL: $x[n] = x(nT_s)$





Either rate guarantees perfect reconstruction – theoretically.

Shannon, 1949



Communication in the Presence of Noise

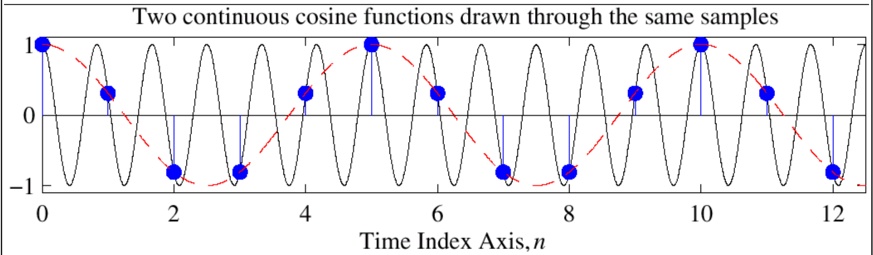
CLAUDE E. SHANNON†, MEMBER, IRE

THEOREM 1: *If a function f(t) contains no frequencies higher than W cps, it is completely determined by giving its ordinates at a series of points spaced 1/2W seconds apart.*

Proceedings of the I.R.E., January, 1949

Reconstruction? Which One?

Given the samples, draw a sinusoid through the values



$$x[n] = \cos(0.4\pi n)$$

When n is an integer
$\cos(0.4\pi n) = \cos(2.4\pi n)$

Reconstruction method picks lowest frequency sinusoid

DISCRETE-TIME SINUSOID

- Sample x(t) to obtain the sequence x[n]

$$x(t) = A\cos(\omega t + \varphi)$$

$$x[n] = x(nT_s) = A\cos(\omega n T_s + \varphi)$$

$$x[n] = A\cos((\omega T_s)n + \varphi)$$

$$x[n] = A\cos(\hat{\omega} n + \varphi)$$

$$\hat{\omega} = \omega T_s = \frac{\omega}{f_s}$$

DEFINE DIGITAL FREQUENCY

DIGITAL FREQUENCY $\hat{\omega}$

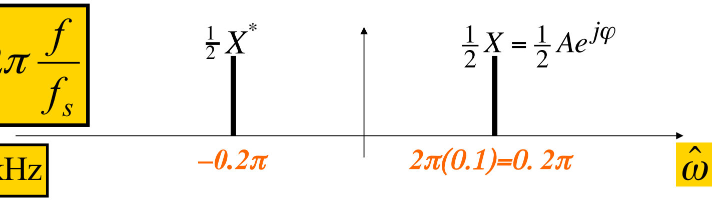
$$\boxed{\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s}}$$

- $\hat{\omega}$ VARIES from **0** to **2π**, as *f* varies from 0 to the sampling frequency f_s
- UNITS are radians, <u>**not**</u> rad/sec
  - DIGITAL FREQUENCY is <u>NORMALIZED</u>

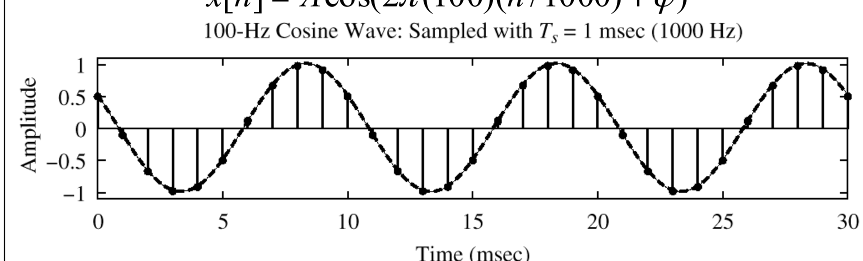
SPECTRUM (DIGITAL)

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$$f_s = 1\,\text{kHz}$$



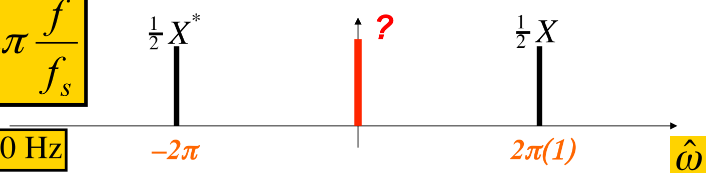
$$x[n] = A\cos(2\pi(100)(n/1000) + \varphi)$$



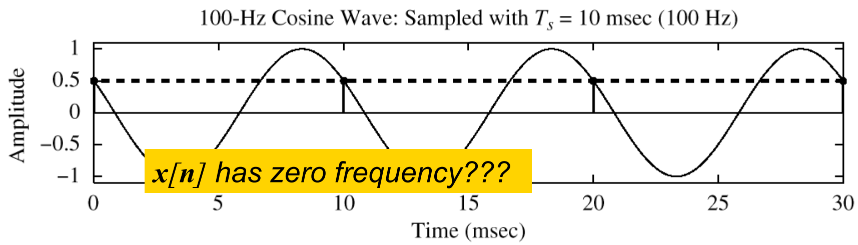
SPECTRUM (DIGITAL) ???

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$f_s = 100$ Hz



$$x[n] = A\cos(2\pi(100)(n/100) + \varphi)$$



Stroboscope Effect

- Check out this video on U-tube
- http://www.youtube.com/watch?v=YpMtanpsVeE

The REST of the STORY

- Spectrum of x[n] has more than one line for each complex exponential
  - Called **ALIASING**
  - **MANY SPECTRAL LINES**
- SPECTRUM is PERIODIC with period = **2π**
  - Because

$$\boxed{A\cos(\hat{\omega}n + \varphi) = A\cos((\hat{\omega} + 2\pi\ell)n + \varphi)}$$

ALIASING DERIVATION

- Other Frequencies give the same $\hat{\omega}$

If $x(t) = A\cos(2\pi(\underline{f + \ell f_s})t + \varphi)$ $\quad \boxed{t \leftarrow \frac{n}{f_s}}$

and we want : $x[n] = A\cos(\hat{\omega}n + \varphi)$

$$\text{then} : \hat{\omega} = \frac{2\pi(f + \ell f_s)}{f_s} = \frac{2\pi f}{f_s} + \frac{2\pi \ell f_s}{f_s}$$

$$\boxed{\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s}} \; \boxed{+2\pi\ell}$$

ALIASING CONCLUSIONS

- Adding an **INTEGER multiple** of f_s or $–f_s$ to the frequency of a sinusoid $x_c(t)$ gives **exactly the same values** for $x[n] = x_c(n/f_s)$
- **GIVEN x[n], we CAN' T KNOW whether it came from a sinusoid at f_o or $(f_o + f_s)$ or $(f_o + 2f_s)$ …**
- **This is called ALIASING**

NORMALIZED FREQUENCY

- DIGITAL FREQUENCY

Normalized Radian Frequency

$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s} \boxed{+ 2\pi\ell}$$

Normalized Cyclic Frequency

$$\hat{f} = \hat{\omega}/(2\pi) = f T_s = f/f_s$$

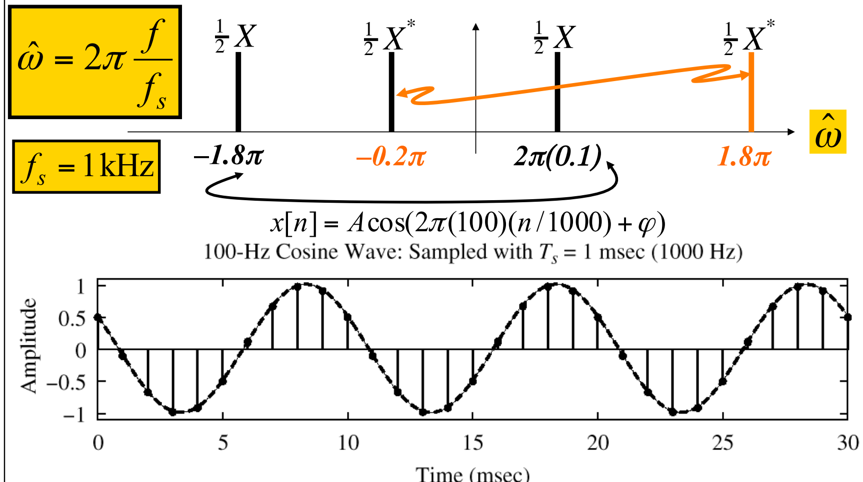
SPECTRUM for x[n]

- PLOT versus NORMALIZED FREQUENCY
- INCLUDE ALL SPECTRUM LINES
  - ALIASES
    - ADD MULTIPLES of 2π
    - SUBTRACT MULTIPLES of 2π
  - FOLDED ALIASES
    - (to be discussed later)
    - ALIASES of NEGATIVE FREQS

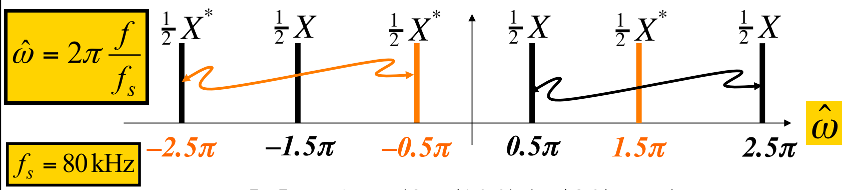
SPECTRUM (MORE LINES)

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

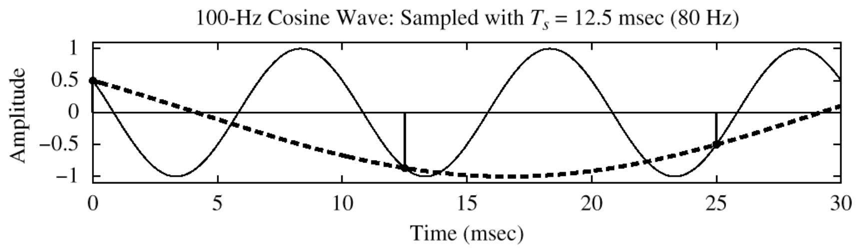
$$f_s = 1\,\text{kHz}$$



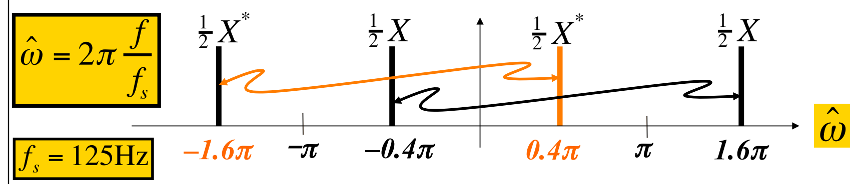
UNDERSAMPLING (ALIASING DISTORTION)



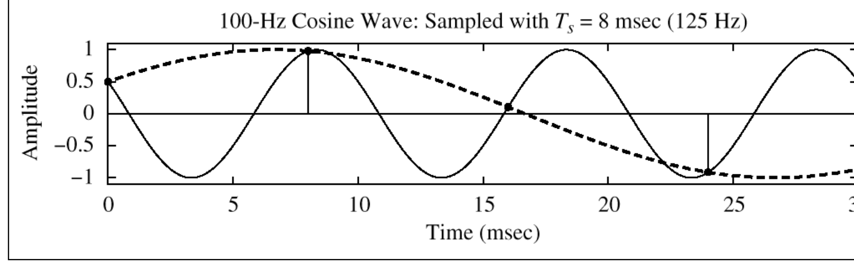
$$x[n] = A\cos(2\pi(100)(n/80)+\varphi)$$



SPECTRUM (FOLDING CASE)



$$x[n] = A\cos(2\pi(100)(n/125)+\varphi)$$



Terminology: NYQUIST RATE

- "**Nyquist Rate**" Sampling
  - f_s > **TWICE** the HIGHEST Frequency in x(t)
  - "Sampling above the Nyquist rate"
- **BANDLIMITED SIGNALS**
  - DEF: HIGHEST FREQUENCY COMPONENT in SPECTRUM of x(t) is finite
  - PERIODIC SQUARE WAVE is NOT BANDLIMITED

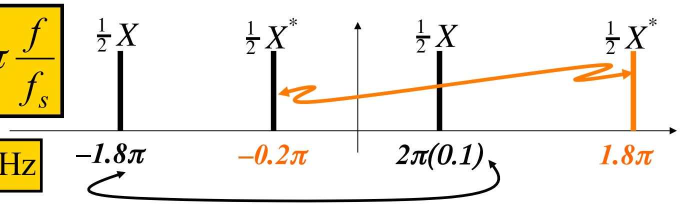
EXAMPLE: SPECTRUM

- $x[n] = A\cos(0.2\pi n+\phi)$
- FREQS @ 0.2π and -0.2π
- ALIASES:
  - $\{2.2\pi, 4.2\pi, 6.2\pi, \ldots\}$ & $\{-1.8\pi, -3.8\pi, \ldots\}$
  - EX: $x[n] = A\cos(4.2\pi n+\phi)$
- ALIASES of NEGATIVE FREQ:
  - $\{1.8\pi, 3.8\pi, 5.8\pi, \ldots\}$ & $\{-2.2\pi, -4.2\pi\ \ldots\}$

SPECTRUM (MORE LINES)

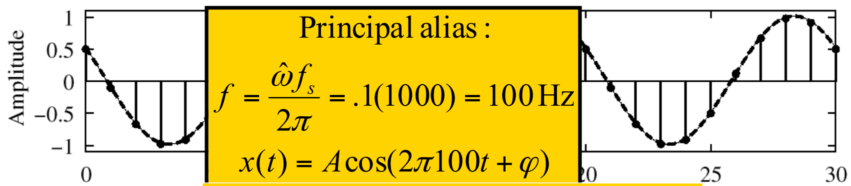
$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$f_s = 1\,\text{kHz}$



$$x[n] = A\cos(2\pi(100)(n/1000) + \varphi)$$

100-Hz Cosine Wave: Sampled with T_s = 1 msec (1000 Hz)



Principal alias :

$$f = \frac{\hat{\omega} f_s}{2\pi} = .1(1000) = 100\,\text{Hz}$$

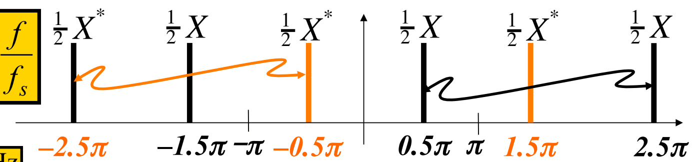
$$x(t) = A\cos(2\pi 100t + \varphi)$$

Principal alias is always between $-\pi \le \hat{\omega} \le \pi$

SPECTRUM (ALIASING CASE)

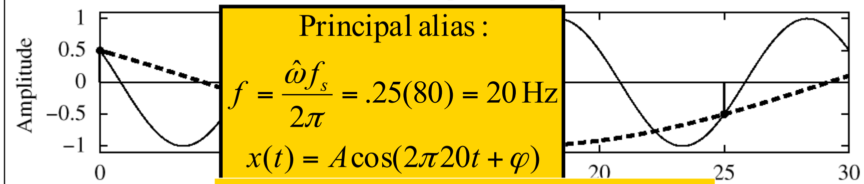
$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$f_s = 80\,\text{Hz}$



$$x[n] = A\cos(2\pi(100)(n/80) + \varphi)$$

100-Hz Cosine Wave: Sampled with T_s = 12.5 msec (80 Hz)



Principal alias :

$$f = \frac{\hat{\omega} f_s}{2\pi} = .25(80) = 20\,\text{Hz}$$

$$x(t) = A\cos(2\pi 20t + \varphi)$$

Principal alias is always between $-\pi \le \hat{\omega} \le \pi$

DIGITAL FREQ $\hat{\omega}$ AGAIN

Normalized Radian Frequency

$$\boxed{\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s}} + 2\pi\ell$$

ALIASING

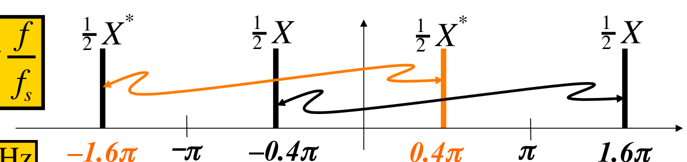
$$\hat{\omega} = \omega T_s = -\frac{2\pi f}{f_s} + 2\pi\ell$$

FOLDED ALIAS

SPECTRUM (FOLDING CASE)

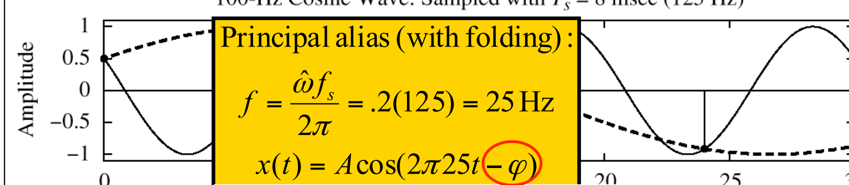
$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$f_s = 125\text{Hz}$



$$x[n] = A\cos(2\pi(100)(n/125) + \varphi)$$

100-Hz Cosine Wave: Sampled with T_s = 8 msec (125 Hz)



Principal alias (with folding) :

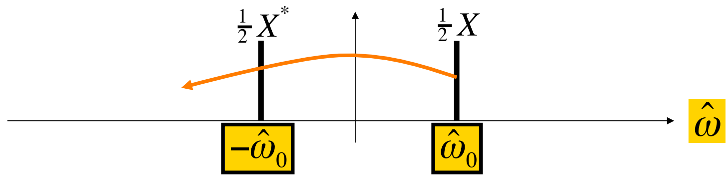
$$f = \frac{\hat{\omega} f_s}{2\pi} = .2(125) = 25\,\text{Hz}$$

$$x(t) = A\cos(2\pi 25t - \varphi)$$

Principal alias is always between $-\pi \le \hat{\omega} \le \pi$

SPECTRUM Explanation of SAMPLING THEOREM

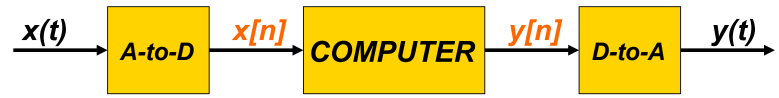
- How do we prevent aliasing?
- Guarantee original signal is principal alias:



$$\hat{\omega}_0 - 2\pi < -\hat{\omega}_0 \quad \Rightarrow \hat{\omega}_0 < \pi$$

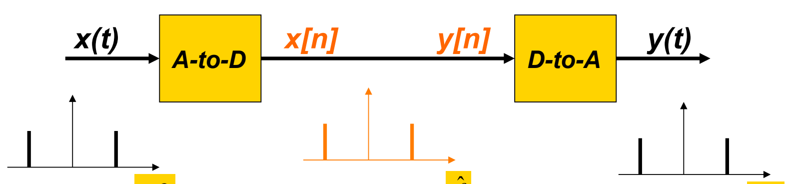
$$\hat{\omega}_0 = \frac{2\pi f_0}{f_s} < \pi \quad \Rightarrow f_0 < \frac{f_s}{2}$$

D-to-A Reconstruction



- Create continuous y(t) from y[n]
  - **IDEAL D-to-A:**
    - If you have formula for y[n]
  - Invert sampling ($t=nT_s$) by $n=f_s t$
  - $y[n] = A\cos(0.2\pi n+\phi)$ with $f_s = 8000$ Hz
  - $y(t) = A\cos(0.2\pi(8000t)+\phi) = A\cos(2\pi(\mathbf{800})t+\phi)$

FREQUENCY DOMAINS



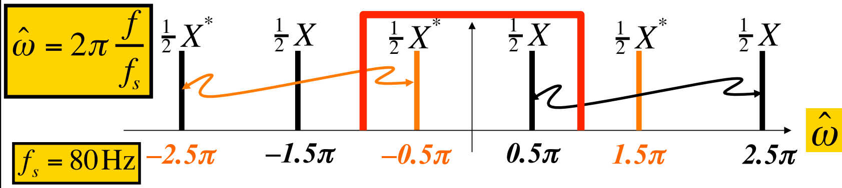
$$\hat{\omega} = 2\pi \frac{(\pm f)}{f_s} + 2\pi\ell$$

$$f = \frac{\hat{\omega}}{2\pi} f_s$$

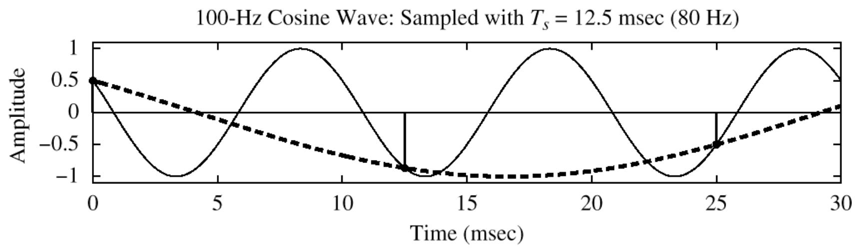
D-to-A is AMBIGUOUS !

- ALIASING
  - Given y[n], which y(t) do we pick ? ? ?
  - INFINITE NUMBER of y(t)
    - PASSING THRU THE SAMPLES, y[n]
  - D-to-A RECONSTRUCTION MUST CHOOSE ONE OUTPUT
- RECONSTRUCT THE SMOOTHEST ONE
  - THE LOWEST FREQ, if y[n] = sinusoid

SPECTRUM (ALIASING CASE)



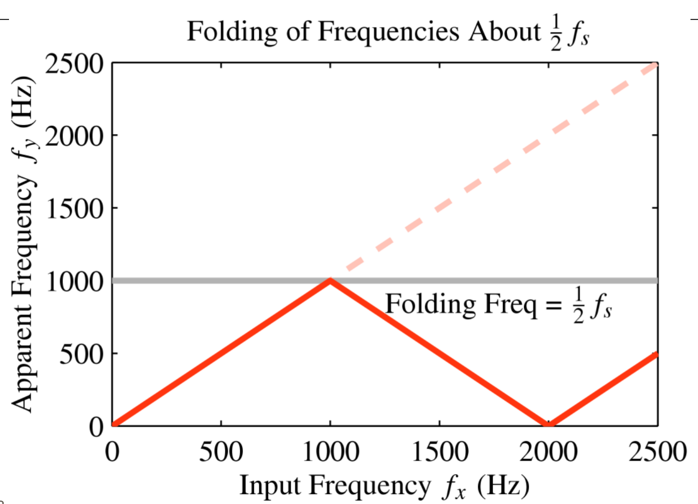
$$x[n] = A\cos(2\pi(100)(n/80) + \varphi)$$



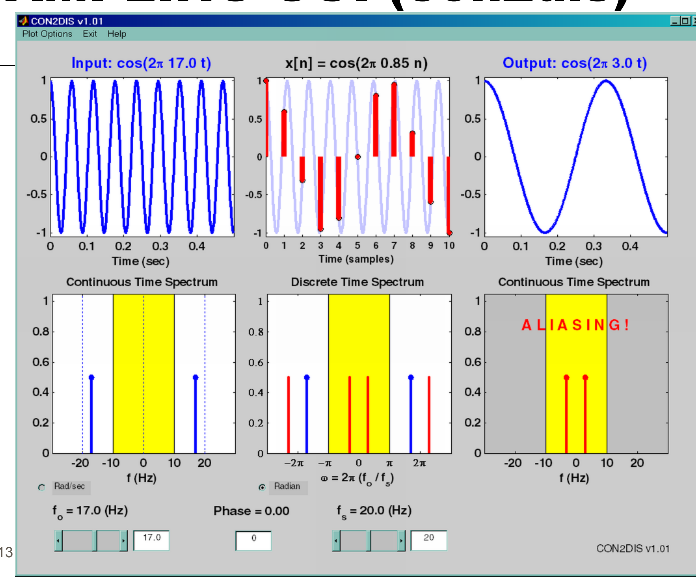
DEMOS from CHAPTER 4

- CD-ROM DEMOS
- SAMPLING DEMO (**con2dis GUI**)
  - Different Sampling Rates
    - Aliasing of a Sinusoid
- STROBE DEMO
  - Synthetic vs. Real
  - Television SAMPLES at 30 fps
- Sampling & Reconstruction

FOLDING DIAGRAM $f_s = 2000$ Hz

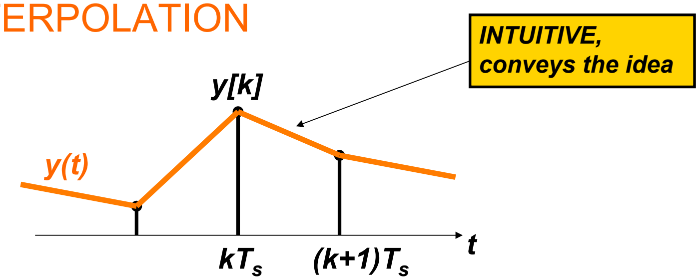


SAMPLING GUI (con2dis)



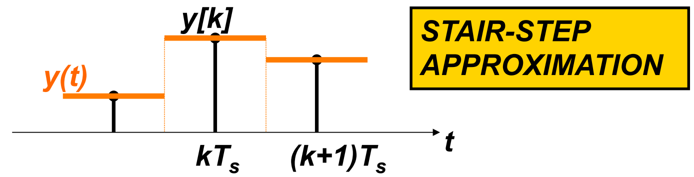
Reconstruction (D-to-A)

- CONVERT STREAM of NUMBERS to x(t)
- “CONNECT THE DOTS”
- INTERPOLATION

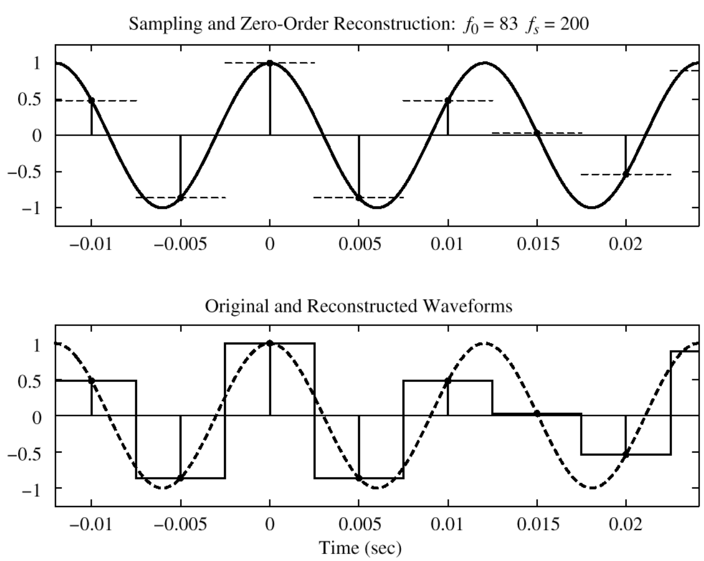


SAMPLE & HOLD DEVICE

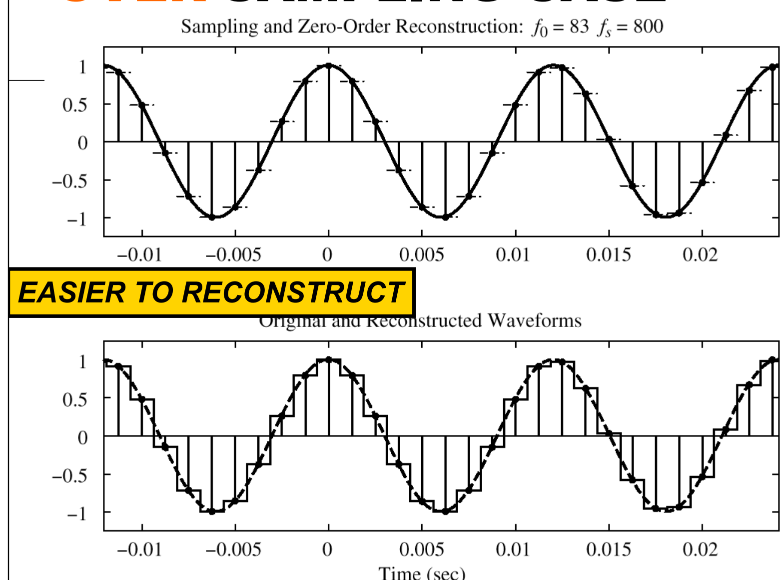
- CONVERT y[n] to y(t)
  - y[k] should be the value of y(t) at $t = kT_s$
  - Make y(t) equal to y[k] for
    - $kT_s - 0.5T_s < t < kT_s + 0.5T_s$



SQUARE PULSE CASE



OVER-SAMPLING CASE



MATH MODEL for D-to-A

$$y(t) = \sum_{n=-\infty}^{\infty} y[n]p(t - nT_s)$$

SQUARE PULSE:

$$p(t) = \begin{cases} 1 & -\frac{1}{2}T_s < t \leq \frac{1}{2}T_s \\ 0 & \text{otherwise} \end{cases}$$

EXPAND the SUMMATION

$$\sum_{n=-\infty}^{\infty} y[n]p(t - nT_s) =$$

$$\ldots + y[0]p(t) + y[1]p(t - T_s) + y[2]p(t - 2T_s) + \ldots$$

- SUM of SHIFTED PULSES p(t-nT_s)
  - "WEIGHTED" by y[n]
  - CENTERED at t=nT_s
  - SPACED by T_s
    - RESTORES "REAL TIME"

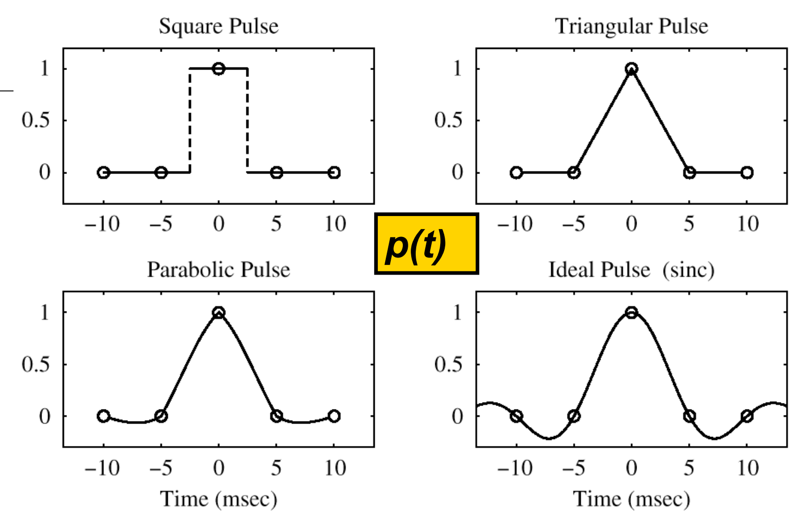
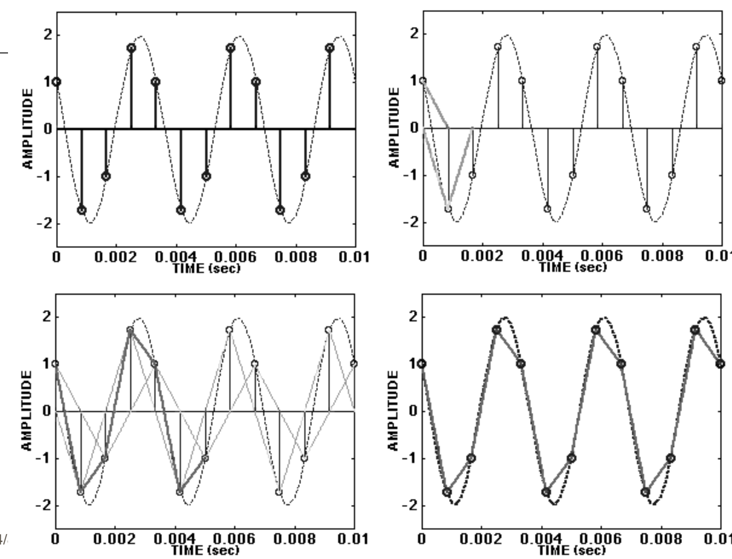


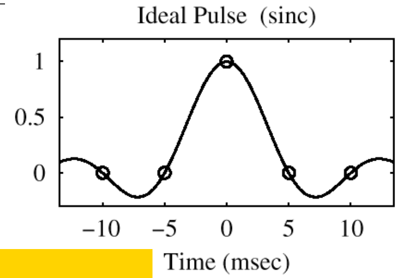
Figure 4.17 Four different pulses for D-to-C conversion. The sampling period is $T_s = 0.005$, i.e., $f_s = 200$ Hz. Note that the duration of each pulse is approximately one or two times T_s.

TRIANGULAR PULSE (2X)



OPTIMAL PULSE ?

CALLED "BANDLIMITED INTERPOLATION"



$$p(t) = \frac{\sin \frac{\pi t}{T_s}}{\frac{\pi t}{T_s}} \quad \text{for} - \infty < t < \infty$$

$$p(t) = 0 \quad \text{for} \ \ t = \pm T_s, \pm 2T_s, \ldots$$

BANDLIMITED INTERPOLATION

- Interpolation formula

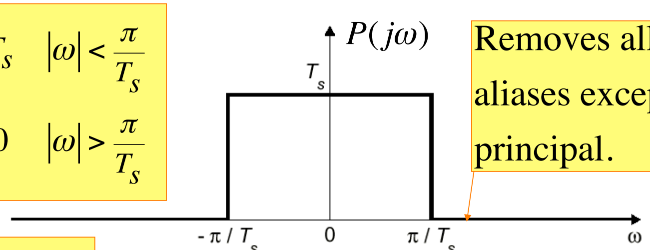
$$y(t) = \sum_{n=-\infty}^{\infty} y[n] p(t - nT_s)$$

- Corresponding Fourier transform

$$Y(j\omega) = \sum_{n=-\infty}^{\infty} y[n] e^{-j\omega n} P(j\omega) = \left(\sum_{n=-\infty}^{\infty} y[n] e^{-j\omega n} \right) P(j\omega)$$

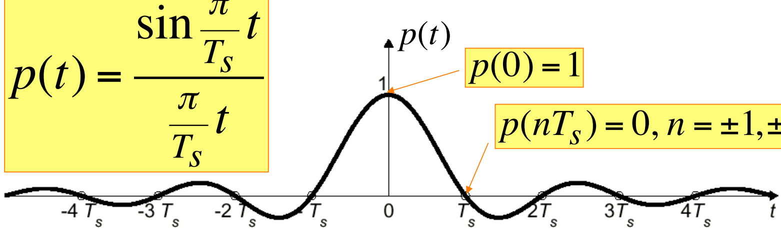
Ideal Reconstruction Filter

$$P(j\omega) = \begin{cases} T_s & |\omega| < \frac{\pi}{T_s} \\ 0 & |\omega| > \frac{\pi}{T_s} \end{cases}$$



Removes all aliases except principal.

$$p(t) = \frac{\sin \frac{\pi}{T_s} t}{\frac{\pi}{T_s} t}$$



$p(0) = 1$

$p(nT_s) = 0, n = \pm 1, \pm 2, \ldots$

Spatial Aliasing


Spatial Aliasing