Signal Processing and Linear Systems I

Lecture 10: Fourier Theorems and Generalized Fourier Transforms

February 2, 2013

Fourier Transform Notation

For convenience, we will write the Fourier transform of a signal $f(t)$ as

$$\mathcal{F}\left[f(t)\right] = F(j\omega)$$

and the inverse Fourier transform of $F(j\omega)$ as

$$\mathcal{F}^{-1}\left[F(j\omega)\right] = f(t).$$

Note that

$$\mathcal{F}^{-1}\left[\mathcal{F}\left[f(t)\right]\right] = f(t)$$

at points of continuity of $f(t)$.

Frequency Domain Convolution

There is another version of the convolution theorem that applies when the convolution is in the frequency domain.

Frequency Domain Convolution Theorem: If $f_1(t)$ and $f_2(t)$ have Fourier transforms $F_1(j\omega)$ and $F_2(j\omega)$, then the product of $f_1(t)$ and $f_2(t)$ has the Fourier transform

$$\mathcal{F}\left[f_1(t)f_2(t)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(j\theta)F_2(j(\omega-\theta))d\theta$$

This is the convolution of $F_1(j\omega)$ and $F_2(j\omega)$, considered as functions of ω. For convenience, we will write this as

$$\mathcal{F}\left[f_1(t)f_2(t)\right] = \frac{1}{2\pi}(F_1 * F_2)(j\omega)$$

while keeping in mind that the convolution is with respect to ω, *not* $j\omega$.

Multiplication in the time domain corresponds to convolution in the frequency domain.

The proof of this theorem is essentially the same as for the time domain convolution theorem.

This is a particularly useful for analyzing modulation and demodulation.

Example: What is the Fourier transform of $\text{sinc}^2(t)$?

We know the Fourier transform pair

$$\text{sinc}(t) \Leftrightarrow \text{rect}(\omega/2\pi).$$

The Fourier transform of $\text{sinc}^2(t)$ is then

$$\begin{aligned}\mathcal{F}\left[\text{sinc}^2(t)\right] &= \frac{1}{2\pi}(\text{rect}(\omega/2\pi) * \text{rect}(\omega/2\pi)) \\ &= \Delta(\omega/2\pi)\end{aligned}$$

We then have the transform pair:

$$\text{sinc}^2(t) \Leftrightarrow \Delta(\omega/2\pi)$$

Check that this is consistent with the transform pair

$$\Delta(t) \Leftrightarrow \text{sinc}^2(\omega/2\pi)$$

using the duality theorem.

Generalized Fourier Transforms: δ Functions

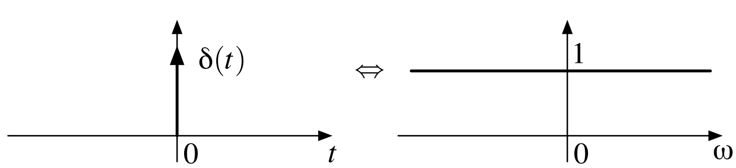
A unit impulse $\delta(t)$ is not a signal in the usual sense (it is a generalized function or distribution). However, if we proceed using the sifting property, we get a result that makes sense:

$$\mathcal{F}\left[\delta(t)\right] = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}\, dt = 1$$

so

$$\delta(t) \Leftrightarrow 1$$

This is a *generalized Fourier transform* and it behaves in most ways like an ordinary FT.



This does what you would intuitively expect from the scaling theorem example. As a time function becomes infinitely narrow, its transform becomes infinitely broad.

Example Recall that a signal can be written as a convolution of $\delta(t)$ with the signal itself

$$f(t) = f(t) * \delta(t)$$

The Fourier transform and the convolution theorem provide another perspective on why this is true

$$\begin{aligned}
\mathcal{F}\left[f(t) * \delta(t)\right] &= \mathcal{F}\left[f(t)\right]\mathcal{F}\left[\delta(t)\right] \\
&= F(j\omega) \times 1 \\
&= F(j\omega).
\end{aligned}$$

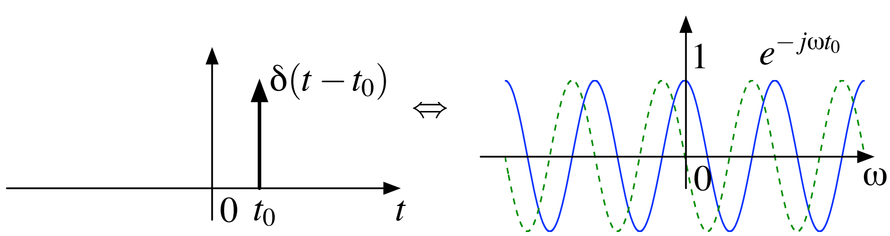
Convolving a signal with $\delta(t)$ simply multiplies the transform of the signal by 1, which leaves us with the original signal.

A shifted delta has the Fourier transform

$$\begin{aligned}
\mathcal{F}\left[\delta(t - t_0)\right] &= \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\omega t} dt \\
&= e^{-j\omega t_0}
\end{aligned}$$

so we have the transform pair

$$\delta(t - t_0) \Leftrightarrow e^{-j\omega t_0}$$



Example We can rederive the shift theorem by recalling that a shifted signal can be represented as a convolution with a shifted delta

$$f(t - t_0) = f(t) * \delta(t - t_0)$$

By the convolution theorem,

$$\begin{aligned}\mathcal{F}\left[f(t-t_0)\right] &= \mathcal{F}\left[\delta(t-t_0) * f(t)\right] \\ &= \mathcal{F}\left[\delta(t-t_0)\right]\mathcal{F}\left[f(t)\right] \\ &= e^{-j\omega t_0}F(j\omega),\end{aligned}$$

which is the shift theorem.

Next we would like to find the Fourier transform of a constant signal $f(t) = 1$. However, direct evaluation doesn't work:

$$\begin{aligned}\mathcal{F}\left[1\right] &= \int_{-\infty}^{\infty} e^{-j\omega t} dt \\ &= \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-\infty}^{\infty}\end{aligned}$$

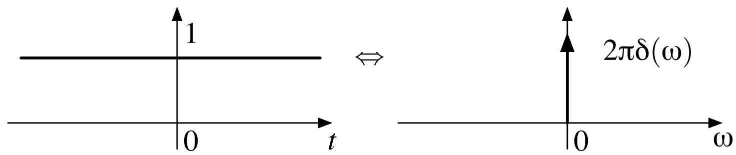
and this doesn't converge to any obvious value for a particular ω.

We instead proceed indirectly by asking what signal has a transform $\delta(\omega)$. Taking the inverse transform of $\delta(\omega)$,

$$\begin{aligned}\mathcal{F}^{-1}\left[\delta(\omega)\right] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega \\ &= \frac{1}{2\pi}\end{aligned}$$

so we have the transform pair

$$1 \Leftrightarrow 2\pi\delta(\omega)$$



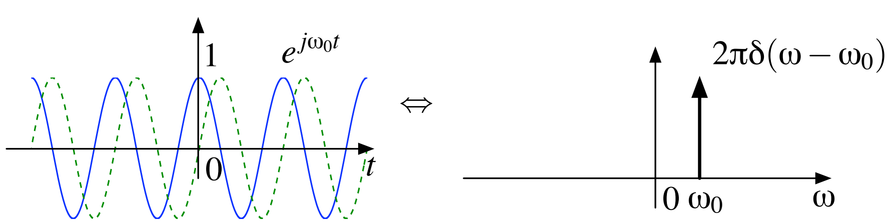
This also does what you expect, a constant signal in time corresponds to an impulse a zero frequency.

If the δ function is shifted in frequency,

$$\begin{aligned}\mathcal{F}^{-1}\left[\delta(\omega-\omega_0)\right] &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega-\omega_0)e^{j\omega t}d\omega \\ &= \frac{1}{2\pi}e^{j\omega_0 t}\end{aligned}$$

so

$$e^{j\omega_0 t} \Leftrightarrow 2\pi\delta(\omega-\omega_0)$$

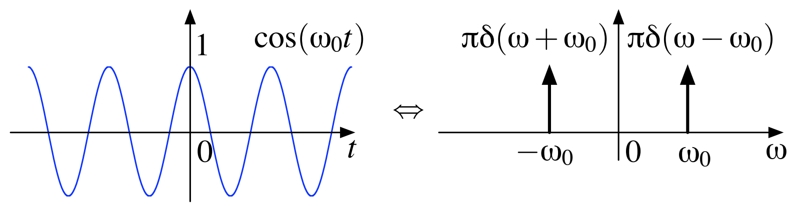


With Euler's relations we can find the Fourier transforms of sines and cosines

$$\begin{aligned}
\mathcal{F}\left[\cos(\omega_0 t)\right] &= \mathcal{F}\left[\frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)\right] \\
&= \frac{1}{2}\left(\mathcal{F}\left[e^{j\omega_0 t}\right] + \mathcal{F}\left[e^{-j\omega_0 t}\right]\right) \\
&= \frac{1}{2}\left(2\pi\delta(\omega-\omega_0) + 2\pi\delta(\omega+\omega_0)\right) \\
&= \pi\left(\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right).
\end{aligned}$$

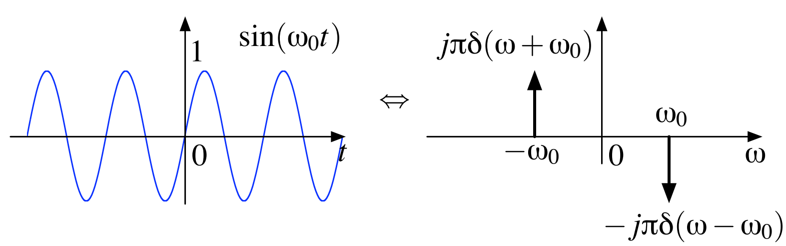
so

$$\cos(\omega_0 t) \Leftrightarrow \pi\left(\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right).$$



Similarly, since $\sin(\omega_0 t) = \frac{1}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t})$ we can show that

$$\sin(\omega_0 t) \Leftrightarrow j\pi\left(\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right).$$



The Fourier transform of a sine or cosine at a frequency ω_0 only has energy exactly at $\pm\omega_0$, which is what we would expect.

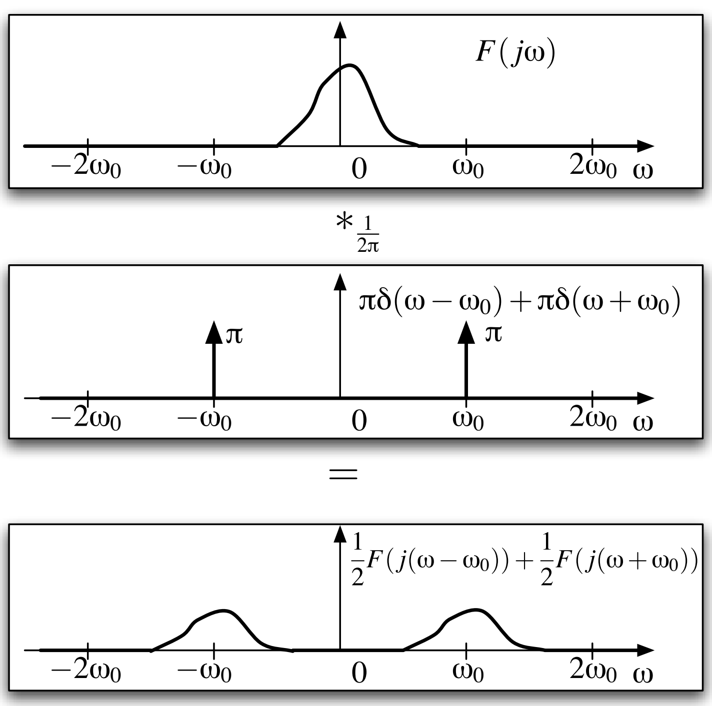
Example: The modulation theorem as frequency domain convolution.

Assume we have a signal $f(t)$ with a Fourier transform $F(j\omega)$, and that $f(t)$ is modulated by a cosine at a frequency ω_0. What is its Fourier transform?

$$\begin{aligned}
\mathcal{F}\left[f(t)\cos(\omega_0 t)\right] &= \frac{1}{2\pi}\left(\mathcal{F}\left[f(t)\right] * \mathcal{F}\left[\cos(\omega_0 t)\right]\right) \\
&= \frac{1}{2\pi}\left(F(j\omega) * \left(\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)\right)\right) \\
&= \frac{1}{2}\left(F(j\omega) * \delta(\omega-\omega_0) + F(j\omega) * \delta(\omega+\omega_0)\right) \\
&= \frac{1}{2}\left(F(j(\omega-\omega_0)) + F(j(\omega+\omega_0))\right).
\end{aligned}$$

This is the same as the result of the modulation theorem.

The frequency domain convolution is illustrated on the next page:



Limiting Transforms

Sometimes we want to find a transform for a signal for which the integral doesn't converge, and there is no obvious indirect approach.

Another alternative is to represent the signal as a limit of a sequence of signals for which the Fourier transforms do exist,

$$f_n(t) \underset{n\to\infty}{\longrightarrow} f(t)$$

Then $F(j\omega) = \lim_{n\to\infty} F_n(j\omega)$ is a reasonable definition for $F(j\omega)$ *if the limit makes sense*.

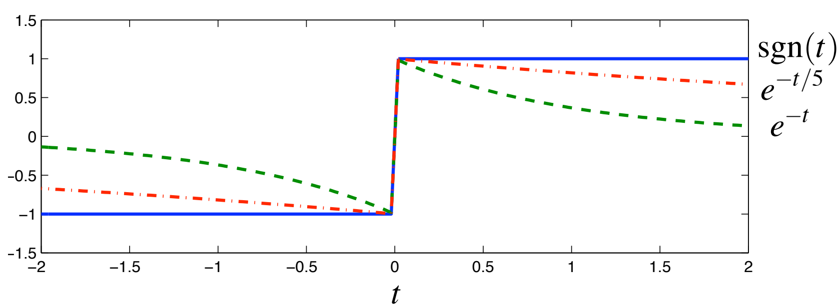
Example: find the Fourier transform of the signum or sign signal

$$f(t) = \mathrm{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}.$$

We can approximate $f(t)$ by the signal

$$f_a(t) = e^{-at}u(t) - e^{at}u(-t)$$

as $a \to 0$. This looks like



As $a \to 0$, $f_a(t) \to \mathrm{sgn}(t)$.

The Fourier transform of $f_a(t)$ is

$$\begin{aligned} F_a(j\omega) &= \mathcal{F}\left[f_a(t)\right] \\ &= \mathcal{F}\left[e^{-at}u(t) - e^{at}u(-t)\right] \\ &= \mathcal{F}\left[e^{-at}u(t)\right] - \mathcal{F}\left[e^{at}u(-t)\right] \\ &= \frac{1}{a+j\omega} - \frac{1}{a-j\omega} \\ &= \frac{-2j\omega}{a^2+\omega^2} \end{aligned}$$

If $\omega = 0$, then $F_a(j\omega) = 0$ for any $a \neq 0$. If $a > 0$, and $a \to 0$ then

$$\begin{aligned} \lim_{a\to 0} F_a(j\omega) &= \lim_{a\to 0} \frac{-2j\omega}{a^2+\omega^2} \\ &= \frac{-2j\omega}{\omega^2} \end{aligned}$$

$$= \frac{2}{j\omega}$$

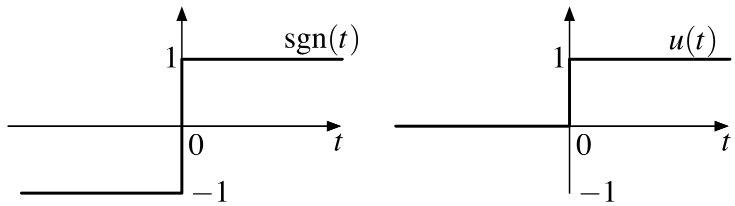
This suggests we *define* the Fourier transform of sgn(t) as

$$\text{sgn}(t) \Leftrightarrow \begin{cases} \frac{2}{j\omega} & \omega \neq 0 \\ 0 & \omega = 0 \end{cases} .$$

With this, we can find the Fourier transform of the unit step,

$$u(t) = \frac{1}{2} + \frac{1}{2}\mathsf{sgn}(t)$$

as can be seen from the plots



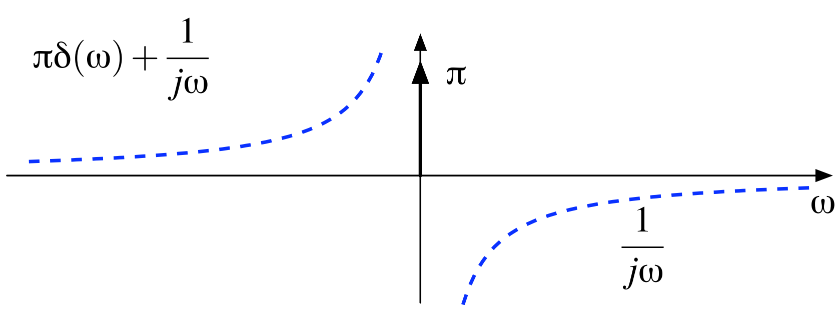
The Fourier transform of the unit step is then

$$\begin{aligned}\mathcal{F}\left[u(t)\right] &= \mathcal{F}\left[\frac{1}{2} + \frac{1}{2}\mathsf{sgn}(t)\right] \\ &= \frac{1}{2}\left(2\pi\delta(\omega)\right) + \frac{1}{2}\left(\frac{2}{j\omega}\right)\end{aligned}$$

$$= \quad \pi\delta(\omega) + \frac{1}{j\omega}$$

where the second term is replaced by zero at $\omega = 0$. The transform pair is then

$$u(t) \Leftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$$



The Integral Theorem

Recall that we can represent integration by a convolution with a unit step

$$\int_{-\infty}^{t} f(\tau)d\tau = (f * u)(t)$$

We now know the Fourier transform of the unit step, so we can solve for the Fourier transform of the integral using the convolution theorem,

$$\begin{aligned} \mathcal{F}\left[\int_{-\infty}^{t} f(\tau)d\tau\right] &= \mathcal{F}\left[f(t)\right]\mathcal{F}\left[u(t)\right] \\ &= F(j\omega)\left(\pi\delta(\omega) + \frac{1}{j\omega}\right) \\ &= \pi F(0)\delta(\omega) + \frac{F(j\omega)}{j\omega} \end{aligned}$$

The Integral Theorem: Given a continuous time integrable signal $f(t)$, then

$$\int_{-\infty}^{t} f(\tau)\,d\tau \Leftrightarrow \pi F(0)\delta(\omega) + \frac{F(j\omega)}{j\omega}$$

First term can be thought of as the transform of a contant 1, which is $2\pi\delta(t)$ multiplied by $F(0)/2$, which is half the DC component of $f(t)$ represented by its area $\int_{-\infty}^{\infty} f(t)\,dt$.

Second term shows that integration in the time domain corresponds to division by $j\omega$ in the frequency domain (except where $\omega = 0$). Integration in the inverse of differentiation, which corresponds to multiplying by $j\omega$.