# EE 102B Spring 2013

**Lecture 01**

**Review of Continuous-Time Signals and Systems -- I**

**April 1, 2013**

## Course Information -- I

- Instructor
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- Teaching Assistants
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## Course Information -- II

- Text: *Signal Processing First*, by McClellan, Schafer, and Yoder (SPF)
  - New chapter material posted on Class2Go website
- Also Recommended: *Signals and Systems*, by Oppenheim, Willsky, with Nawab (S&S)
  - Text for EE102A
- For Deeper Reading: *Discrete-Time Signal Processing*, 3E, Oppenheim and Schafer



## Course Information – III

- Website using Class2Go
  - https://class2go.stanford.edu/EE102B/Spring2013
  - Log on with SUNet
  - Homework, labs, lecture slides, videos, grades, text supplements, SP-First toolbox
  - Link to piazza for discussion forum
- Homework and Lab Assignments:
  - Assigned Wednesday and due following Weds.
  - Up to two days late (Fri at 5pm) with penalty of 10% per day late – not accepted after Friday.
  - Lowest homework/lab will be dropped

## Course Information – IV

- Course Grade
  - Homework: 15%
  - Labs 15%
  - Mid-term exam 30%
  - Final exam 40%
- Mid-term date: Friday, May 10, 2013 in class
- Final exam date: Tuesday June 11, 2013, 8:30-11:30 am room TBA



## READING ASSIGNMENT

- This Lecture:
  - SPF: Chapters 9 – 12; Review continuous-time LTI systems, convolution, Fourier series
  - S&S: Review book sections on continuous-time LTI systems, convolution, Fourier series



## LECTURE OBJECTIVES

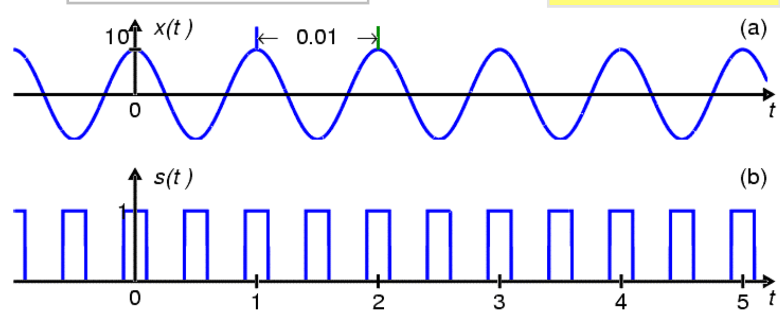
- Review: continuous-time signals and systems
  - Continuous-time signals
  - Continuous-time systems
    - Example systems
- Review: **L**inearity and **T**ime-**I**nvariance
  - Impulse response and convolution integral
- Review: frequency response of LTI systems



## CT Signals: PERIODIC

$x(t) = 10\cos(200\pi t)$

*Sinusoidal signal*




*INFINITE DURATION*

*Square Wave*

## CT Signals: ONE-SIDED

$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$

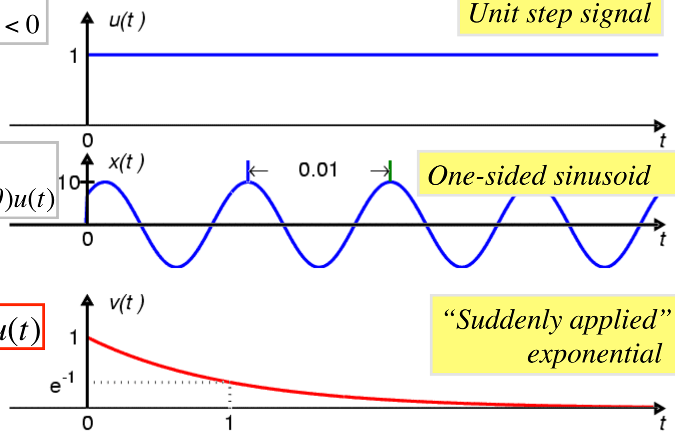
*Unit step signal*

$x(t) = 10\cos(\omega_0 t + \theta)u(t)$

*One-sided sinusoid*

$v(t) = e^{-t}u(t)$

*"Suddenly applied" exponential*





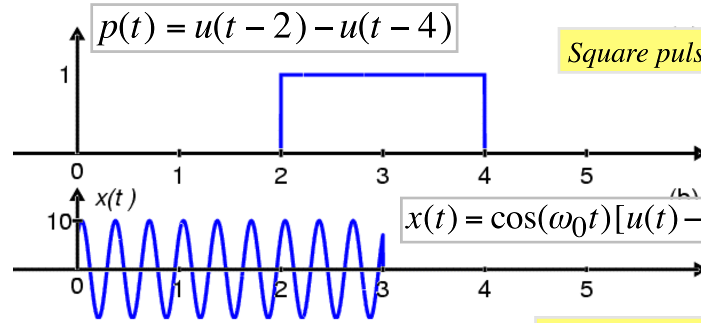

## CT Signals: FINITE LENGTH

$p(t) = u(t-2) - u(t-4)$

*Square pulse signal*

$x(t) = \cos(\omega_0 t)[u(t) - u(t-3)]$

*Sinusoid multiplied by a square pulse*



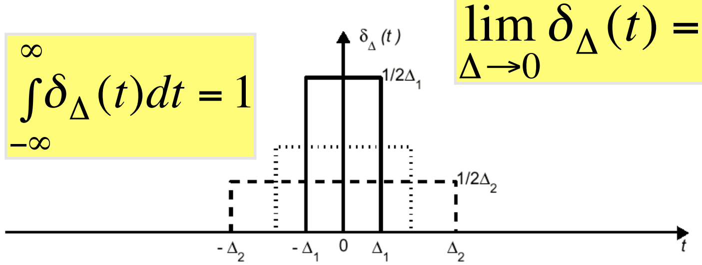



## What is an Impulse?

- A signal that is concentrated at one point.

$$\int_{-\infty}^{\infty} \delta_\Delta(t)dt = 1$$

$$\lim_{\Delta \to 0} \delta_\Delta(t) = \delta(t)$$






## Defining the Impulse

- Assume the properties apply to the limit:

$$\lim_{\Delta \to 0} \delta_\Delta(t) = \delta(t)$$

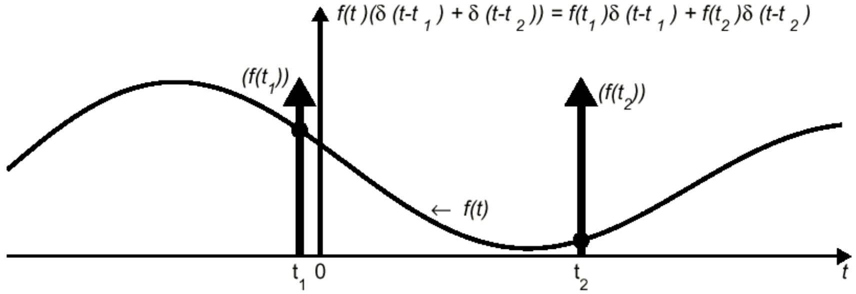
- One "INTUITIVE" definition is:

$\delta(t) = 0, \quad t \neq 0$ *Concentrated at t=0*

$\int_{-\infty}^{\infty} \delta(\tau)d\tau = 1$ *Unit area*

## General Sampling Property

$$f(t)\delta(t-t_0) = f(t_0)\delta(t-t_0)$$






## Properties of the Impulse

$$\delta(t-t_0) = 0, \quad t \neq t_0$$ *Concentrated at one time*

$$\int_{-\infty}^{\infty} \delta(t-t_0)dt = 1$$ *Unit area*

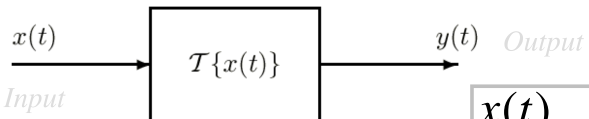
$$f(t)\delta(t-t_0) = f(t_0)\delta(t-t_0)$$ *Sampling Property*

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0)$$ *Extract one value of f(t) (Rigorous definition)*

$$\frac{du(t)}{dt} = \delta(t)$$ *Derivative of unit step*



## Continuous-Time Systems




$$x(t) \mapsto y(t)$$

- Examples:
  - Delay $\quad y(t) = x(t-t_d)$
  - Modulator $\quad y(t) = [A + x(t)]\cos\omega_c t$
  - Integrator $\quad y(t) = \int_{-\infty}^{t} x(\tau)d\tau$



## CT BUILDING BLOCKS

- INTEGRATOR (CIRCUITS)
- DIFFERENTIATOR
- DELAY by $t_o$
- MODULATOR (e.g., AM Radio)
- MULTIPLIER & ADDER

## Ideal Delay:

- Mathematical Definition:

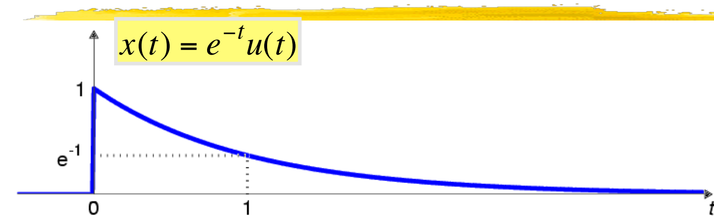
$$y(t) = x(t - t_d)$$

- To find the IMPULSE RESPONSE, *h(t),* let *x(t)* be an impulse, so

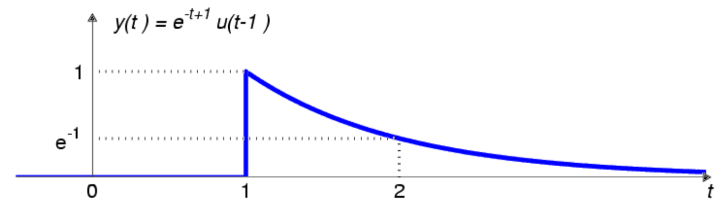
$$h(t) = \delta(t - t_d)$$



## Output of Ideal Delay of 1 sec




$$y(t) = x(t-1) = e^{-(t-1)}u(t-1)$$






## Integrator:

- Mathematical Definition:

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

*Running Integral*

- To find the IMPULSE RESPONSE, *h(t),* let *x(t)* be an impulse, so

$$h(t) = \int_{-\infty}^{t} \delta(\tau)d\tau = u(t)$$



## Integrator:

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

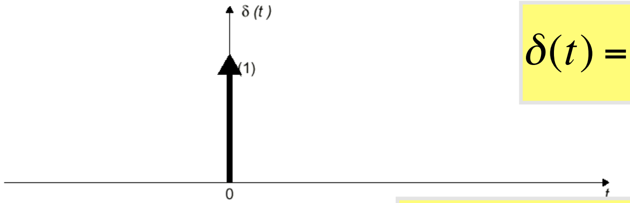
- Integrate the impulse

$$\int_{-\infty}^{t} \delta(\tau)d\tau = u(t) = h(t)$$

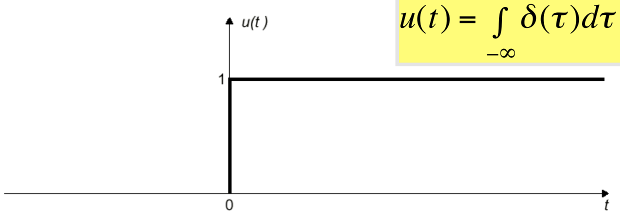
- IF t<0, we get zero
- IF t>0, we get one
  - Thus we have h(t) = u(t) for the integrator

## Graphical Representation

$$\delta(t) = \frac{du(t)}{dt}$$



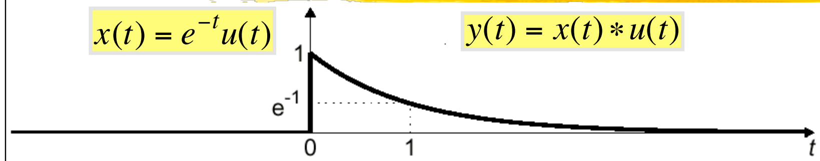

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$






## Output of Integrator

$x(t) = e^{-t}u(t)$ $y(t) = x(t) * u(t)$




$$y(t) = \int_{-\infty}^{t} e^{-\tau}u(\tau)d\tau = \begin{cases} 0 & t < 0 \\ \int_{0}^{t} e^{-\tau}d\tau & t \geq 0 \end{cases}$$

$$= (1 - e^{-t})u(t)$$



## Differentiator:

- Mathematical Definition:

$$y(t) = \frac{dx(t)}{dt}$$

- To find *h(t)*, let *x(t)* be an impulse, so

$$h(t) = \frac{d\delta(t)}{dt} = \delta^{(1)}(t)$$

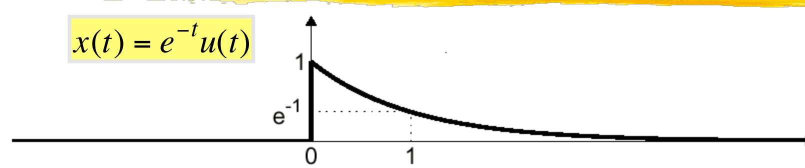
*Doublet*



## Differentiator Output:

$$y(t) = \frac{dx(t)}{dt}$$

$x(t) = e^{-t}u(t)$




$$y(t) = \frac{d}{dt}\left(e^{-t}u(t)\right) = \frac{d}{dt}\left(e^{-t}\right)u(t) + e^{-t}\frac{d}{dt}\left(u(t)\right)$$

$$= -e^{-t}u(t) + e^{-t}\delta(t)$$

$$= -e^{-t}u(t) + e^{-0}\delta(t) = -e^{-t}u(t) + \delta(t)$$

## Linear Systems

- A linear system obeys the principle of superposition $x(t) \mapsto y(t)$

$$ax(t) \mapsto ay(t) \quad \text{(homogeneous)}$$

$$x_1(t) \mapsto y_1(t) \ \text{ and } \ x_2(t) \mapsto y_2(t)$$

$$x_1(t) + x_2(t) \mapsto y_1(t) + y_2(t) \quad \text{(additive)}$$

$$ax_1(t) + bx_2(t) \mapsto ay_1(t) + by_2(t) \quad \text{(superposition)}$$



## Time-Invariant Systems

- A time-invariant system is one whose output is the same for a given input no matter when the input occurs.

$$x(t) \mapsto y(t)$$
$$x(t - t_0) \mapsto y(t - t_0)$$

- An LTI system is both linear and time-invariant.



## Linear and Time-Invariant (LTI) Systems

- If a continuous-time system is both linear and time-invariant, then the output *y(t)* is related to the input *x(t)* by a ***convolution integral***

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = x(t) * h(t) = x * h(t)$$

where *h(t)* is the ***impulse response*** of the system.



## Convolution is Commutative

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = x(t) * h(t) = x * h(t)$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau = h(t) * x(t) = h * x(t)$$

## Sampling and Convolution

- Sampling property

$$f(t)\delta(t-t_0) = f(t_0)\delta(t-t_0)$$

- Convolution Identity

$$x(t) * \delta(t) = x(t)$$



## Ideal Delay: $y(t) = x(t-t_d)$

- To find *h(t),* let *x(t)* be an impulse, so

$$h(t) = \delta(t-t_d)$$

- Therefore, convolution with an impulse simply shifts *x(t).*

$$x(t) * \delta(t-t_d) = x(t-t_d)$$



## Convolution of Impulses, etc.

- Convolution of two impulses

$$\delta(t-t_1) * \delta(t-t_2) = \delta(t-t_1-t_2)$$

- Convolution of step and shifted impulse

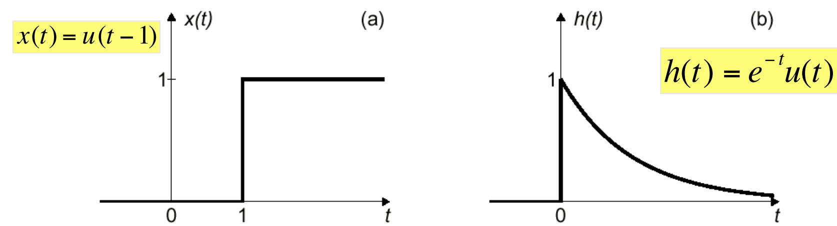
$$u(t) * \delta(t-t_0) = u(t-t_0)$$

- Convolution of step and derivative of impulse

$$u(t) * \delta^{(1)}(t) = \delta(t)$$

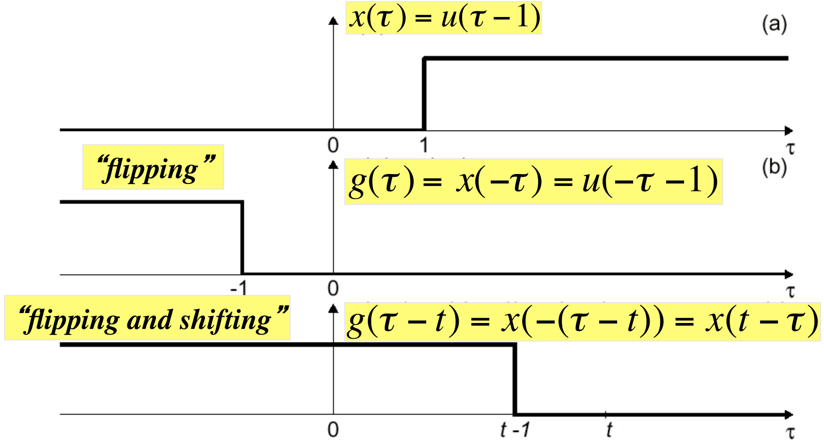


## Evaluating a Convolution



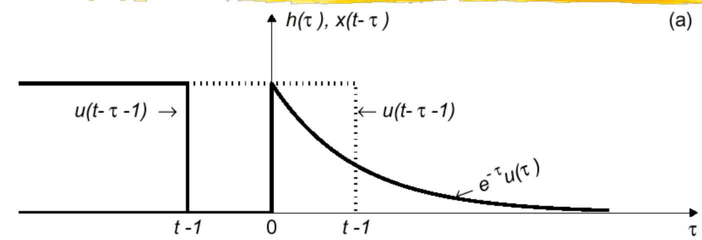

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = h(t) * x(t)$$

# "Flipping and Shifting"






# Evaluating the Integral

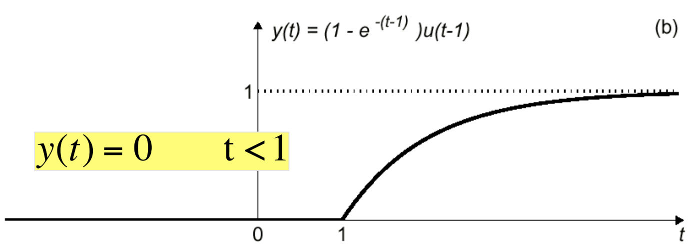



$$
\begin{aligned}
y(t) &= 0 & t-1<0 \\
&= \int_0^{t-1} e^{-\tau} d\tau & t-1 \ge 0
\end{aligned}
$$



# Solution

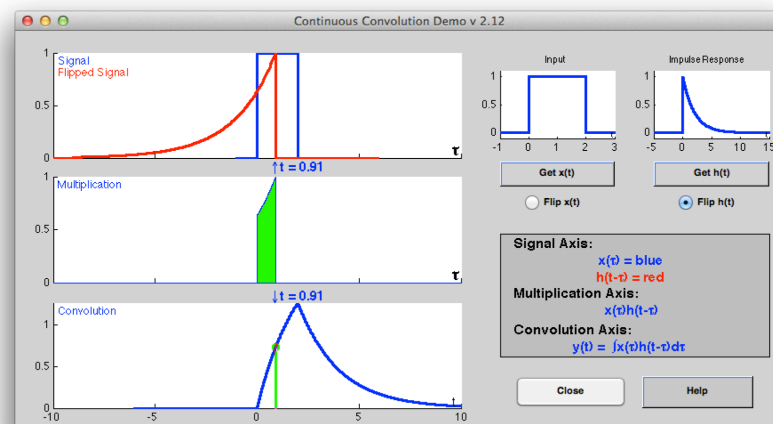
$$
\begin{aligned}
y(t) &= \int_0^{t-1} e^{-\tau} d\tau = -e^{-\tau}\Big|_0^{t-1} \\
&= 1 - e^{-(t-1)} & t \ge 1
\end{aligned}
$$




$y(t) = 0 \quad t < 1$

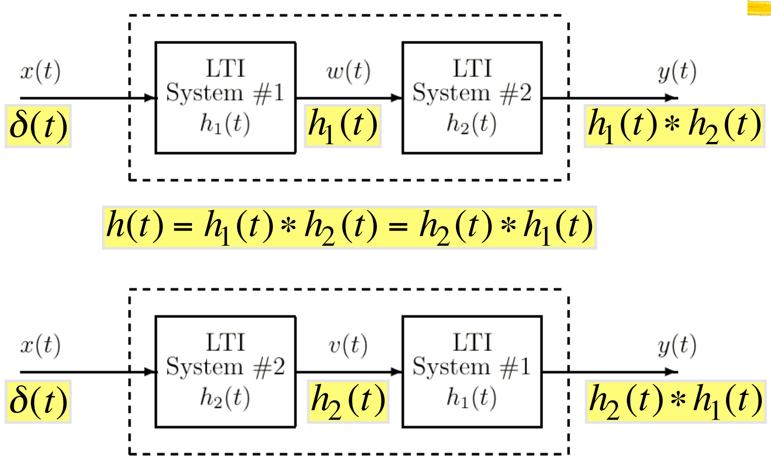


# Convolution GUI




http://users.ece.gatech.edu/mcclella/SPFirst/Updates/SPFirstMATLAB.html

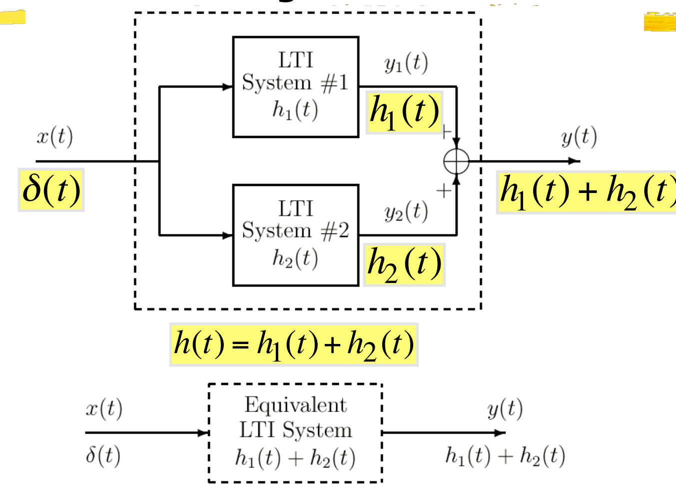
# Cascade of LTI Systems




(b)



# Parallel LTI Systems




(b)



# Stability

- A system is stable if every bounded input produces a bounded output.
- A continuous-time *LTI system* is stable if and only if

$$\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$$

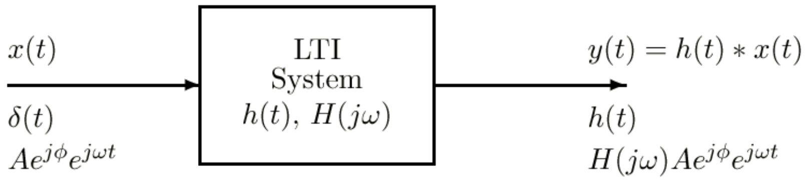


# Causal Systems

- A system is causal if and only if $y(t_0)$ depends only on $x(\tau)$ for $\tau \leq t_0$ .
- An LTI system is causal if and only if

$$h(t) = 0 \quad \text{for} \quad t < 0$$

# LTI Systems




- Convolution defines LTI system

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

- Response to a complex exponential gives frequency response *H(jω)*



# Complex Exponential Input

$$x(t) = Ae^{j\phi}e^{j\omega t} \mapsto y(t) = H(j\omega)Ae^{j\phi}e^{j\omega t}$$

$$y(t) = \int^{\infty} h(\tau)Ae^{j\phi}e^{j\omega(t-\tau)}d\tau$$

$$y(t) = \left(\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau\right)Ae^{j\phi}e^{j\omega t}$$

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau$$

***Frequency Response***



# When does *H(jω)* Exist?

- When is $|H(j\omega)| < \infty$ ?

$$|H(j\omega)| = \left|\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau\right| \le \int_{-\infty}^{\infty} |h(\tau)||e^{-j\omega\tau}|d\tau$$

$$|H(j\omega)| \le \int_{-\infty}^{\infty} |h(\tau)|d\tau < \infty$$

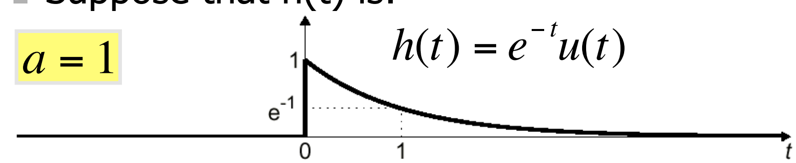
- Thus the frequency response exists if the LTI sytem is a stable system.



$$h(t) = e^{-at}u(t) \Leftrightarrow H(j\omega) = \frac{1}{a + j\omega}$$

- Suppose that h(t) is:

$a = 1$




$$H(j\omega) = \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)e^{-j\omega\tau}d\tau = \int_{0}^{\infty} e^{-(a+j\omega)\tau}d\tau$$

$a > 0$

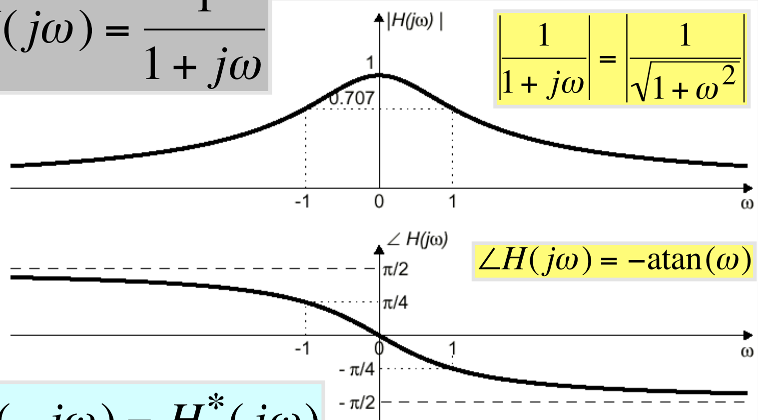
$$H(j\omega) = \left.\frac{e^{-(a+j\omega)\tau}}{-(a+j\omega)}\right|_0^{\infty} = \left.\frac{e^{-a\tau}e^{-j\omega\tau}}{-(a+j\omega)}\right|_0^{\infty} = \frac{1}{a+j\omega}$$

## Magnitude and Phase Plots

$$H(j\omega) = \frac{1}{1 + j\omega}$$

$$\left|\frac{1}{1+j\omega}\right| = \left|\frac{1}{\sqrt{1+\omega^2}}\right|$$

$$\angle H(j\omega) = -\text{atan}(\omega)$$




$$H(-j\omega) = H^*(j\omega)$$



## Freq Response of Integrator?

- Impulse Response
  - h(t) = u(t)
- NOT a Stable System
  - Frequency response *H(jω)* does NOT exist
- Leaky Integrator (a is small)
  - Cannot build a perfect Integral

$$a \to 0$$

$$h(t) = e^{-at}u(t) \Leftrightarrow H(j\omega) = \frac{1}{a + j\omega} \to \frac{1}{j\omega}?$$



## Ideal Delay: $y(t) = x(t - t_d)$

$$H(j\omega) = \int_{-\infty}^{\infty} \delta(\tau - t_d) e^{-j\omega\tau} d\tau = e^{-j\omega t_d}$$

$$H(j\omega) = e^{-j\omega t_d}$$

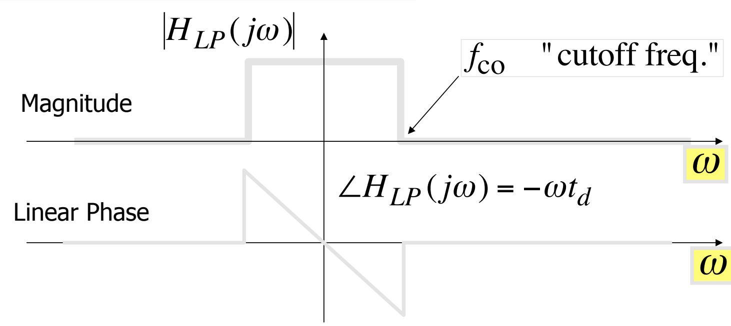
$$x(t) = e^{j\omega t} \mapsto \quad H(j\omega)$$

$$y(t) = e^{j\omega(t - t_d)} = e^{-j\omega t_d} e^{j\omega t}$$



## Ideal Lowpass Filter

$$H_{LP}(j\omega) = \begin{cases} e^{-j\omega t_d} & |\omega| < \omega_{co} \\ 0 & |\omega| > \omega_{co} \end{cases}$$



## Example: Ideal Low Pass

$$H_{LP}(j\omega) = \begin{cases} e^{-j3\omega} & |\omega| < 10 \\ 0 & |\omega| > 10 \end{cases}$$

$$x(t) = 10e^{j\pi/3}e^{j5t} \mapsto \quad y(t) = H(j5)10e^{j\pi/3}e^{j5t}$$

$$y(t) = \left(e^{-j15}\right)10e^{j\pi/3}e^{j5t} = 10e^{j\pi/3}e^{j5(t-3)}$$



## Cosine Input

$$x(t) = A\cos(\omega_0 t + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 t} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 t}$$

$$y(t) = H(j\omega_0)\frac{A}{2}e^{j\phi}e^{j\omega_0 t} + H(-j\omega_0)\frac{A}{2}e^{-j\phi}e^{-j\omega_0 t}$$

Since $H(-j\omega_0) = H^*(j\omega_0)$

$$y(t) = A|H(j\omega_0)|\cos(\omega_0 t + \phi + \angle H(j\omega_0))$$



## Review Fourier Series

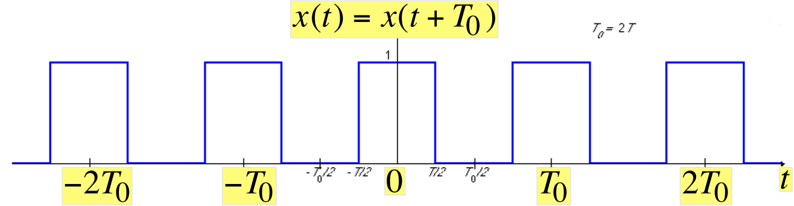
- ANALYSIS
  - Get representation from the signal
  - Works for PERIODIC Signals
- Fourier Series coefficients
  - INTEGRAL over one period

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 kt}\,dt$$



## General Periodic Signals

$x(t) = x(t + T_0)$




$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt}$$ ← ***Fourier Synthesis***

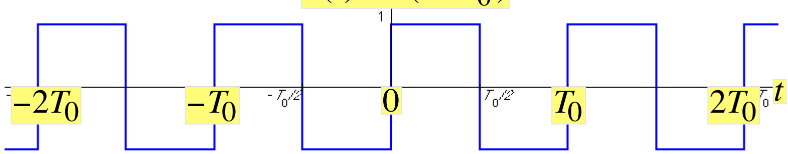
Fundamental Freq.

$\omega_0 = 2\pi / T_0 = 2\pi f_0$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 kt}\,dt$$ ← ***Fourier Analysis***

## Square Wave Signal

$x(t) = x(t + T_0)$

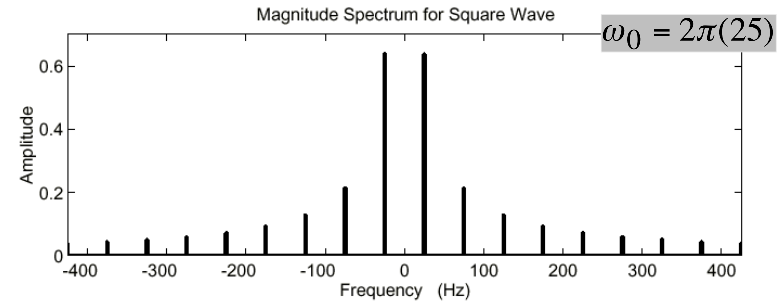



$$a_k = \frac{1}{T_0}\int_0^{T_0/2}(1)e^{-j\omega_0 kt}dt + \frac{1}{T_0}\int_{T_0/2}^{T_0}(-1)e^{-j\omega_0 kt}dt$$

$$a_k = \left.\frac{e^{-j\omega_0 kt}}{-j\omega_0 kT_0}\right|_0^{T_0/2} - \left.\frac{e^{-j\omega_0 kt}}{-j\omega_0 kT_0}\right|_{T_0/2}^{T_0} = \frac{1-e^{-j\pi k}}{j\pi k}$$

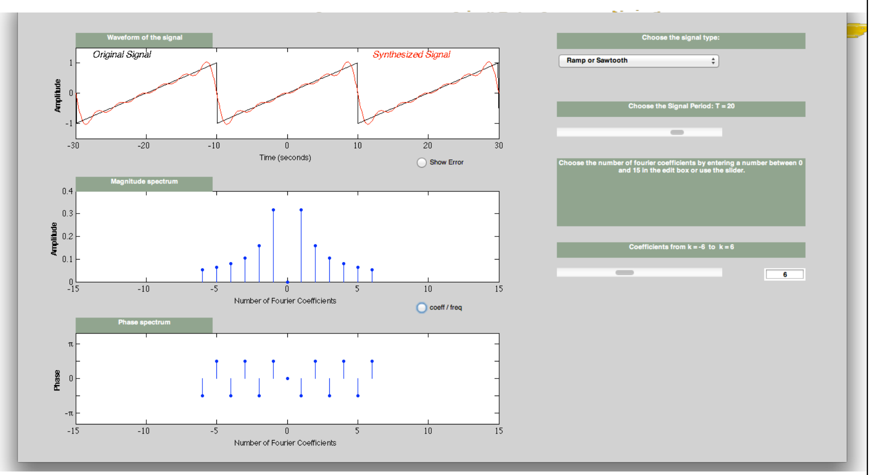
## Spectrum from Fourier Series

$$a_k = \frac{1-e^{-j\pi k}}{j\pi k} = \begin{cases} \dfrac{2}{j\pi k} & k = \pm 1, \pm 3, \ldots \\ 0 & k = 0, \pm 2, \pm 4, \ldots \end{cases}$$



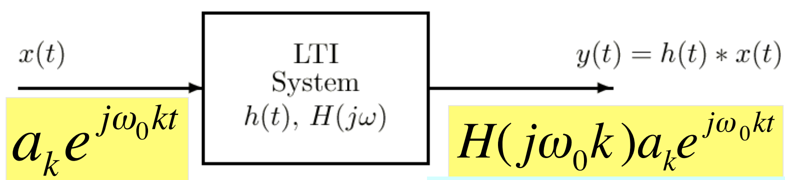



## fseriesdemo




http://users.ece.gatech.edu/mcclella/SPFirst/Updates/SPFirstMATLAB.html

## LTI Systems with Periodic Inputs




$a_k e^{j\omega_0 kt}$ $\quad H(j\omega_0 k)a_k e^{j\omega_0 kt}$

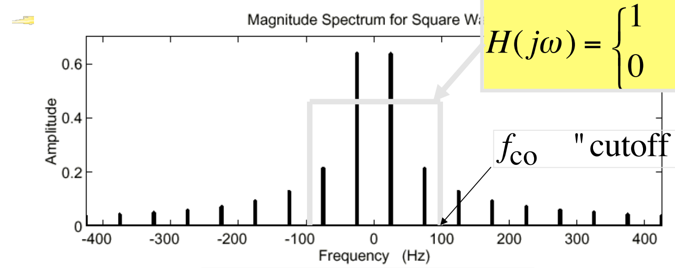
**Output has *same* frequencies**

- By superposition,

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(j\omega_0 k)e^{j\omega_0 kt} = \sum_{k=-\infty}^{\infty} b_k e^{j\omega_0 kt}$$

$$b_k = a_k H(j\omega_0 k)$$

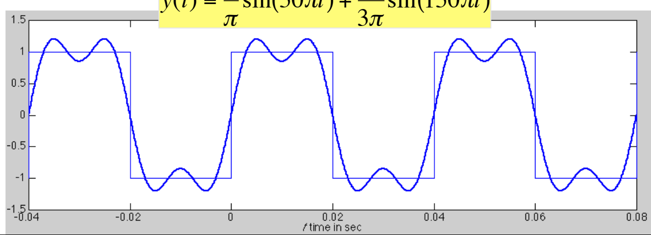
# Ideal Lowpass Filter




$$H(j\omega) = \begin{cases} 1 & |\omega| < \omega_{co} \\ 0 & |\omega| > \omega_{co} \end{cases}$$

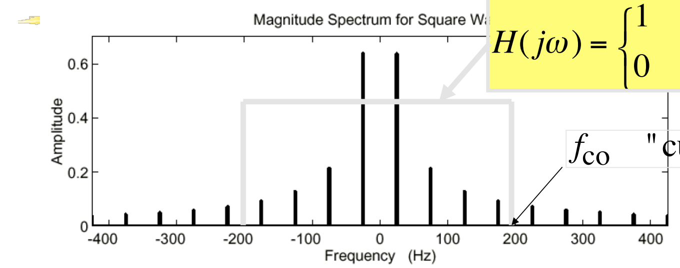
$f_{co}$ "cutoff freq."

$$y(t) = \frac{4}{\pi}\sin(50\pi t) + \frac{4}{3\pi}\sin(150\pi t)$$





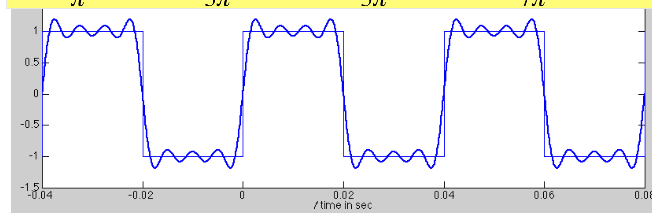

# Ideal Lowpass Filter




$$H(j\omega) = \begin{cases} 1 & |\omega| < \omega_{co} \\ 0 & |\omega| > \omega_{co} \end{cases}$$

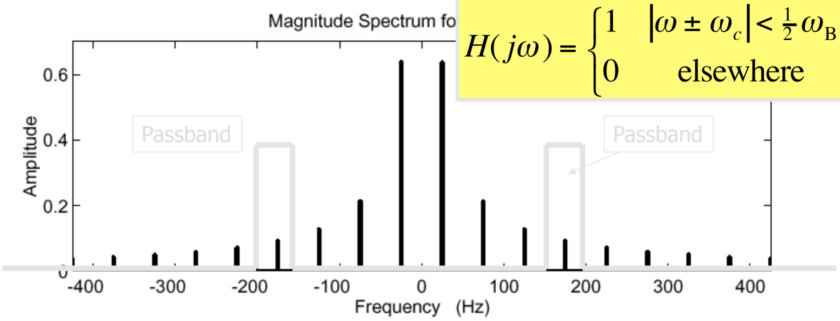
$f_{co}$ "cutoff"

$$y(t) = \frac{4}{\pi}\sin(50\pi t) + \frac{4}{3\pi}\sin(150\pi t) + \frac{4}{5\pi}\sin(250\pi t) + \frac{4}{7\pi}\sin(350\pi t)$$






# Ideal Bandpass Filter




$$H(j\omega) = \begin{cases} 1 & \left|\omega \pm \omega_c\right| < \tfrac{1}{2}\omega_B \\ 0 & \text{elsewhere} \end{cases}$$

What is the ouput signal ?

$$y(t) = \tfrac{2}{j7\pi}e^{j2\pi(175)t} - \tfrac{2}{j7\pi}e^{-j2\pi(175)t} = \frac{4}{7\pi}\cos\left(2\pi(175)t - \tfrac{1}{2}\pi\right)$$