STANFORD UNIVERSITY
EE 102B Spring-2013

Lecture 21
Introduction to the
z-Transform
May 20, 2013

ASSIGNMENTS

- Reading for this Lecture:
  - SPF: Chapters 7 and 8
  - S&S: Chapter 10
  - HW#07 is due by 5pm Wednesday, May 22, in Packard 263.
  - Lab #06 is due by 5pm, Friday, May 24, in Packard 263. Lab #06 continues Lab #05.

Office Hours for Course Staff – Come see us.

- Ron Schafer: Mon. and Weds. 2:00-4:00 pm, Packard 211.
- Dookun Park: Tues. 1:30 ~ 3:30 pm in Packard 107
- Ruo Yu Gu (Roy): Mon. 4:00 ~ 6:00 pm in Packard 106

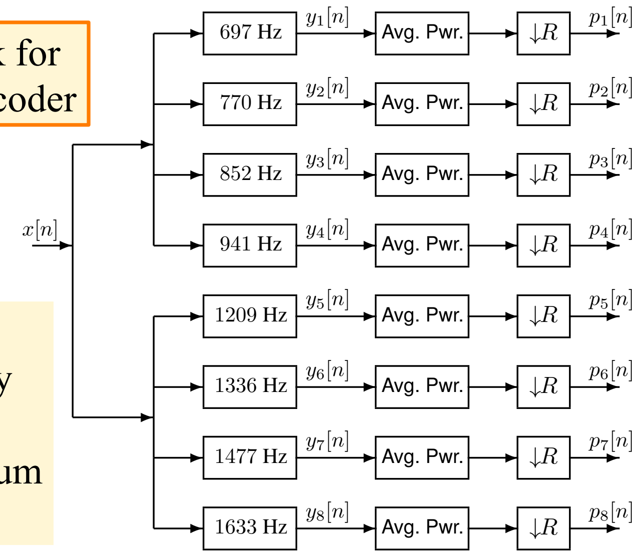
Lecture Objective

- FIR bandpass filters for DTMF decoding (Labs 05 and 06)
- The z-transform
  - Definition
  - Examples
    - Finite-length sequences
    - Right-sided exponential
    - Left-sided exponential
  - LTI systems – the system function

FIR BANDPASS FILTERS FOR DTMF DECODING

Filter bank for DTMF decoder

This is another way to do short-time spectrum analysis



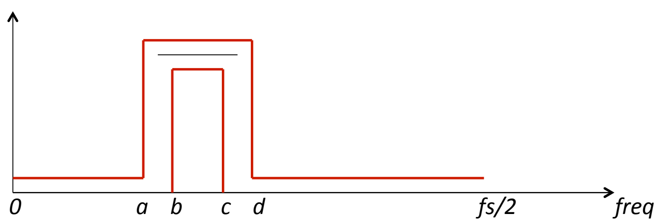
equalize delays

Bandpass Tolerance Template

- Given band edges and ripples

```
[M,f0,a0,w] = firpmord(f,a,dev);
 bk = firpm(M,f0,a0,w);
```

```
firpmord([a,b,c,d],[0,1,0],[dels1,delp,dels2],fs)
```



THE Z-TRANSFORM

The z-Transform

- The z-Transform of a sequence is defined as

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}}$$

- Since this is generally an infinite sum, we need to be concerned about "convergence"; i.e., is the sum finite? In general, the region of convergence (ROC) will depend upon z; e.g.,

$$\mathrm{ROC}_x = \{z : 0 \le r_R < |z| < r_L < \infty\}.$$

The Inverse z-Transform

- The inverse z-transform is the contour integral

$$\boxed{x[n] = \frac{1}{2\pi j}\int_C X(z)z^{n-1}\,dz}$$

$$\boxed{C \text{ in the region of convergence } 0 \le r_R < |z| < r_L < \infty.}$$

- We will not need to use this, although it is very powerful and easy to use if you invest time in learning the theory of complex variables.

Two Basic Properties of z-Transforms

- Linearity (Additivity)

$$\boxed{x[n] = ax_1[n] + bx_2[n] \Leftrightarrow X(z) = a\underbrace{X_1(z)}_{|z|\in\mathrm{ROC}_{x_1}} + b\underbrace{X_2(z)}_{|z|\in\mathrm{ROC}_{x_2}}}$$

What about the region of convergence?

$$\mathrm{ROC}_x \text{ contains } \mathrm{ROC}_{x_1} \cap \mathrm{ROC}_{x_2}$$

- Time delay

$$y[n] = x[n - n_d] \Leftrightarrow z^{-n_d}X(z) \quad \mathrm{ROC}_y = \mathrm{ROC}_x$$

z-Transform of a Finite-Length Sequence

- Consider a finite-length sequence such as the impulse response of a FIR filter

$$X(z) = \sum_{n=0}^{M} x[n]z^{-n}$$

- No problems with convergence since it is a finite sum – a polynomial in z^{-1}

$$X(z) = x[0] + x[1]z^{-1} + x[2]z^{-2} + \ldots + x[M]z^{-M}$$
$$= x[0] + x[1](z^{-1}) + x[2](z^{-1})^2 + \ldots + x[M](z^{-1})^M$$

Examples of z-Transforms of Finite-Length Sequences

- Impulse sequence $x[n] = \delta[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = 1$$

- Shifted impulse sequence $x[n] = \delta[n-n_d]$

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n-n_d] z^{-n} = z^{-n_d}$$

Table of z-Transform Pairs

- Impulse sequence

$$x[n] = \delta[n-n_d] \Leftrightarrow X(z) = z^{-n_d}$$

Inverse z-Transform of Finite-Length Sequence

- Write the z-transform in the form

$$X(z) = x[0] + x[1]z^{-1} + x[2]z^{-2} + \ldots + x[M]z^{-M}$$

- Simply pick off the sequence values as coefficients of the polynomial; e.g.

$$X(z) = x[0] + x[1]z^{-1} + \underbrace{x[2]}_{\boxed{x[2]}}z^{-2} + \ldots + x[M]z^{-M}$$

Finite-Length Example:

- Signal

$$x[n] = \delta[n] + 2\delta[n-1] - \delta[n-2] + 3\delta[n-4]$$

- z-Transform

$$X(z) = 1 + 2z^{-1} - z^{-2} + 3z^{-4}$$

- Delayed signal y[n]=x[n-2]

$$Y(z) = z^{-2}X(z) = z^{-2} + 2z^{-3} - z^{-4} + 3z^{-6}$$

$$y[n] = \delta[n-2] + 2\delta[n-3] - \delta[n-4] + 3\delta[n-6]$$

Region of Convergence for z-Transform of an Infinite Sequence

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}}$$

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x[n] z^{-n} \right| \le \sum_{n=-\infty}^{\infty} |x[n]| |z|^{-n} < \infty$$

- The region of convergence (ROC) is the set of values $\{z : 0 \le r_R < |z| < r_L < \infty\}$ such that,

$$\boxed{\sum_{n=-\infty}^{\infty} |x[n]| |z|^{-n} < \infty}$$

$|z|^{-n}$ can "tame" a growing sequence

Right-Sided Exponential Signal

$x[n] = a^n u[n]$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= \frac{1}{1-(az^{-1})} \quad \text{if } |az^{-1}| < 1$$

$$X(z) = \frac{1}{1-az^{-1}} \quad \text{if } |a| < |z|$$

$$= \frac{(z-0)}{(z-a)}$$

zero

pole

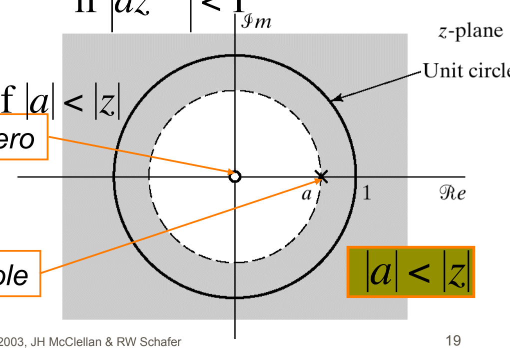


Table of z-Transform Pairs

- Impulse sequence

$$x[n] = \delta[n-n_d] \Leftrightarrow X(z) = z^{-n_d}$$

- Right-sided exponential sequence

$$x[n] = a^n u[n] \Leftrightarrow X(z) = \frac{1}{1-az^{-1}} \quad \text{if } |a| < |z|$$

- Unit step sequence (a = 1)

$$x[n] = u[n] \Leftrightarrow X(z) = \frac{1}{1-z^{-1}} \quad \text{if } 1 < |z|$$

Left-Side Exponential Signal

$x[n] = -a^n u[-n-1]$

$$X(z) = -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=1}^{\infty} a^{-n} z^n = -\sum_{n=1}^{\infty} (a^{-1}z)^n$$

$$= 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n$$

$$= 1 - \frac{1}{1-a^{-1}z} \quad \text{if } |a^{-1}z| < 1$$

$$= \frac{1-a^{-1}z-1}{1-a^{-1}z} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

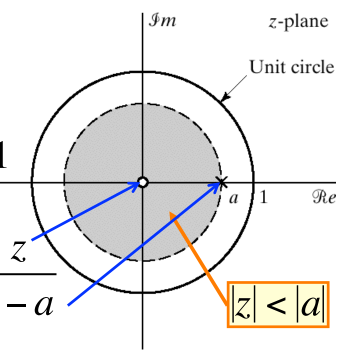


Table of z-Transform Pairs

- Impulse sequence
$$x[n] = \delta[n - n_d] \Leftrightarrow X(z) = z^{-n_d}$$
- Right-sided exponential sequence
$$x[n] = a^n u[n] \Leftrightarrow X(z) = \frac{1}{1 - az^{-1}} \quad \text{if } |a| < |z|$$
- Left-sided exponential sequence
$$x[n] = -a^n u[-n-1] \Leftrightarrow X(z) = \frac{1}{1 - az^{-1}} \quad \text{if } |z| < |a|$$

Poles and Zeros $(1 - az^{-1})$

- Numerator factor (1 zero at z = a and 1 pole at z = 0)
$$(1 - az^{-1}) = \frac{(z - a)}{z}$$
- Denominator factor (1 pole at z = a and 1 zero at z = 0)
$$\frac{1}{(1 - az^{-1})} = \frac{z}{(z - a)}$$

Two-Sided Exponential Signal

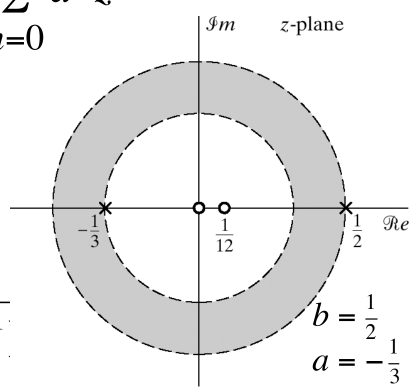
$$x[n] = -b^n u[-n-1] + a^n u[n]$$

$$X(z) = -\sum_{n=-\infty}^{-1} b^n z^{-n} + \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \underset{\text{if } |z|<|b|}{\frac{1}{1-bz^{-1}}} + \underset{\text{if } |z|>|a|}{\frac{1}{1-az^{-1}}}$$

$$= \frac{2-(a+b)z^{-1}}{(1-az^{-1})(1-bz^{-1})}$$

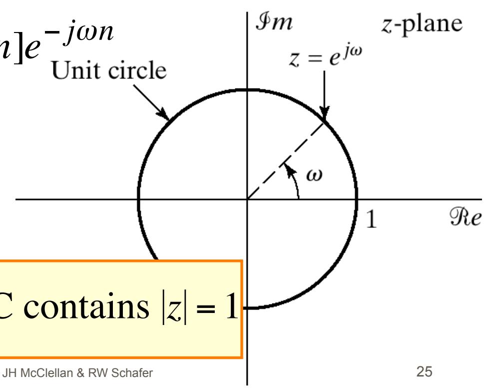
if $|a|<|z|<|b|$



Relation to DTFT

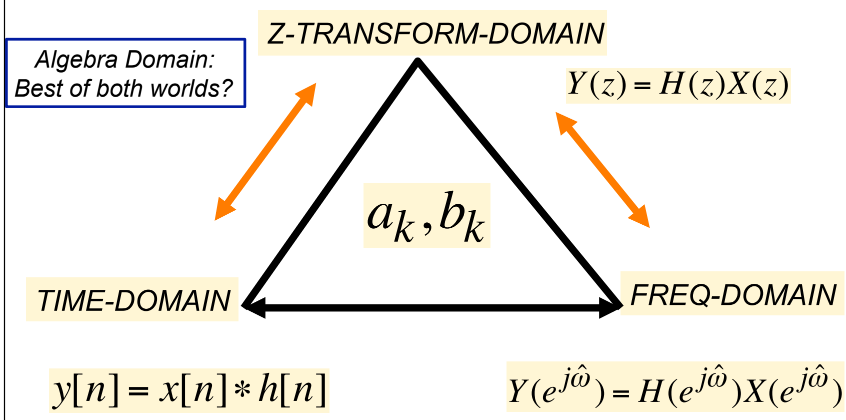
- The DTFT is equal to the z-transform evaluated on the unit circle:

$$X(z)\big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$
$$= X(e^{j\omega})$$
$$= \text{DTFT}$$



$$\boxed{\sum_{n=-\infty}^{\infty} |x[n]| < \infty \Rightarrow \text{ROC contains } |z| = 1}$$

Two, Now Three Domains and LTI Systems



Review for LTI Systems

$x[n]$ → LTI System → $y[n] = x[n] * h[n]$

$\delta[n]$ → $h[n]$

$e^{j\omega n}$ → $H(e^{j\omega})e^{j\omega n}$

$X(e^{j\omega})$ → $X(e^{j\omega})H(e^{j\omega})$

$X(z)$ → $X(z)H(z)$

$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

$$Y(z) = \sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{k=0}^{M} b_k z^{-k} X(z)$$

System Function for LTI Systems

$x[n]$ → LTI System → $y[n] = x[n] * h[n]$

$\delta[n]$ → $h[n]$

$$y[n] - \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\left(1 - \sum_{k=0}^{N} a_k z^{-k}\right) Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\left(\sum_{k=0}^{M} b_k z^{-k}\right)}{\left(1 - \sum_{k=1}^{N} a_k z^{-k}\right)}$$

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}$$

Frequency Response of a DE

$$y[n] - \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}} = \frac{\left(\sum_{k=0}^{M} b_k e^{-j\omega k}\right)}{\left(1 - \sum_{k=1}^{N} a_k e^{-j\omega k}\right)}$$

ROC must Contain the Unit circle

ROC for causal system: $|z| > \max_k \{d_k\}$ $\Longrightarrow$ Stability requires $\max_k \{d_k\} < 1$ for causal system