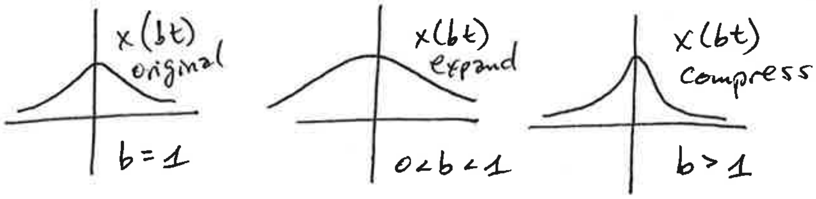


Even & Odd Components

$x_e(t) = x_e(-t)$

$x_o(t) = -x_o(-t)$

$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$

$x_o(t) = \frac{1}{2}[x(t) - x(-t)]$

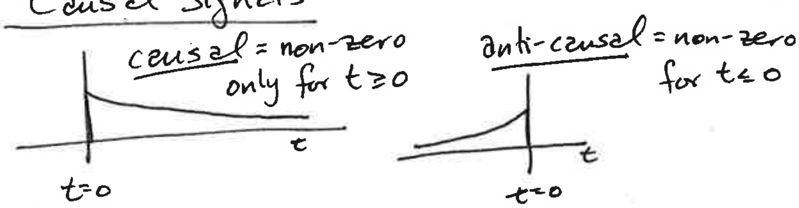
$x_e(t) + x_o(t) = x(t)$

Periodic Signals

Periodic iff $x(t+T_0) = x(t)$ for all t, $T_0 > 0$.

fundamental period = smallest T_0.

Causal Signals



Energy & Power Signals:

Energy: $0 < E_x < \infty$

Power: $0 < P_x < \infty$

$x(t) = \cos(\omega t + \theta)$

frequency $\omega = 2\pi f = \frac{2\pi}{T}$

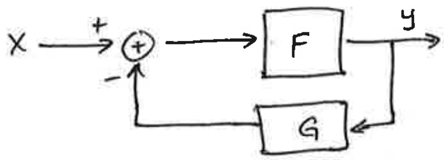
phase offset

Euler relation:

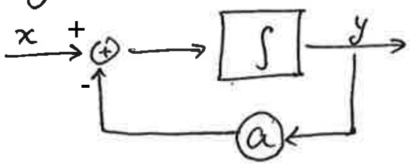
$e^{j\phi} = \cos(\phi) + j\sin(\phi)$

Systems: mostly SISO.
(single input, single output).

feedback: $y = F(x - Gy)$



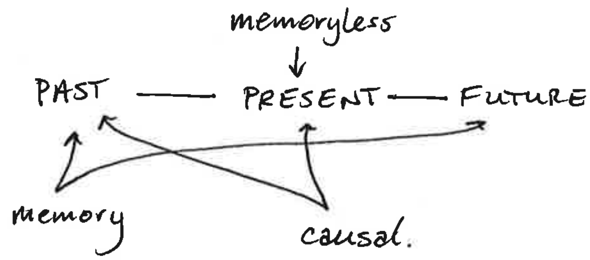
integrator w/ feedback:



$$\int^t (x(\tau) - a y(\tau))\, d\tau = y(t)$$

$$x - ay = y'$$

output depends on...



Time Invariance:

$y(t) = F\, x(t)$

$y(t-\tau) = F\, x(t-\tau)$

Invertability:

$y = Fx$

$x = F^{INV} y = F^{INV} F x$

$F^{INV} F = I$ ← identity operator

Linearity:

① Homogeneity: $F(ax) = a F(x)$
or $ay = S(ax)$ if $y = Sx$

② Superposition: $F(x + \tilde{x}) = F(x) + F(\tilde{x})$
or $y_1 + y_2 = S(x_1 + x_2)$ if $y_1 = Sx_1$, $y_2 = Sx_2$

$\Rightarrow F(ax + b\tilde{x}) = aF(x) + bF(\tilde{x})$

LCCODE:

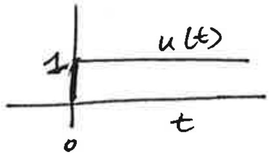
$$a_n y^{(n)}(t) + \dots + a_0 y(t) = b_m x^{(m)}(t) + \dots + b_0 x(t)$$

with given initial conditions $y^{(n-1)}(0), y'(0), \dots, y(0)$

$\Rightarrow$ If we can describe system this way, we know it is linear.

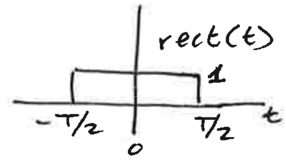
unit step

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



unit rectangle

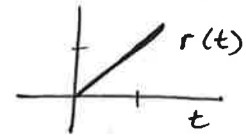
$$\text{rect}\left(\frac{t}{T}\right) = \begin{cases} 1 & |t| < \frac{T}{2} \\ 0 & \text{otherwise} \end{cases}$$



$$\text{rect}\left(\frac{t}{T}\right) = \Pi\left(\frac{t}{T}\right)$$

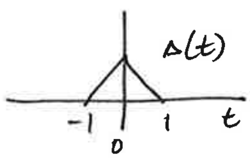
Unit ramp

$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases} = \int_{-\infty}^{t} u(\tau)\,d\tau$$

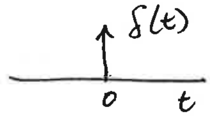


unit triangle

$$\Delta(t) = \begin{cases} 1 - |t|, & |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$



impulse signal



$$\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = f(0)$$

provided f continuous at $t=0$.

$$\delta(t-\tau)\,f(t) = \delta(t-\tau)\,f(\tau)$$

$$\phi(t)\,\delta(t) = \phi(0)\,\delta(t)$$

Sifting Property:

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-T)\,dt = f(t)\Big|_{t=T}$$

for f continuous at $t=T$.

example:

$$\int_{-\infty}^{\infty} f(t+1)\,\delta(t+1)\,dt = f(t+1)\Big|_{t=-1} = f(0)$$

Derivatives of Impulse (lec. 3)

$$\int_{-\infty}^{\infty} \delta^{(k)}(t)\,f(t)\,dt = (-1)^k f^{(k)}(0)$$

if $f^{(k)}$ continuous at $t=0$.

$\Downarrow$

from ...

$$\int_{-\infty}^{\infty} \delta'(t)\,f(t)\,dt = \delta(t)\,f(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(t)\,f'(t)\,dt = -f'(0)$$

---

zero-input response : no input

zero-state response: no initial conditions.

$$y(t) = \underbrace{\int_{0}^{t} x(\tau)\left[\frac{1}{RC}\,e^{-(t-\tau)/RC}\right]d\tau}_{\text{zero-state response.}} + \underbrace{y_0\,e^{-t/RC}}_{\text{zero-input response}}$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

$x(t)$ can be written as a weighted integral of impulse functions.

Convolution:

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau = \int_{-\infty}^{\infty} x(t-\tau) h(\tau) d\tau$$
$$= (x * h)(t)$$

Properties of convolution:

① commutative: $(f*g)(t) = (g*f)(t)$

② associative: $(f*(g*h))(t) = ((f*g)*h)(t)$

③ distributive: $f*(g+h) = f*g + f*h$

$\Longrightarrow$ Convolution systems are linear & time-invariant

$h*(\alpha x_1 + \beta x_2) = \alpha(h*x_1) + \beta(h*x_2)$

input $x_1(t) = x(t-T)$ yields output $y_1(t) = y(t-T)$.

Convolution w/ complex sinusoids:

$$y(t) = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau, \quad s = \sigma + j\omega$$
$$= e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

$$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$
$$y(t) = e^{st} H(s)$$

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau$$
$$y(t) = e^{j\omega t} H(j\omega)$$

Causal systems:

$$y(t) = \int_{-\infty}^{\infty} x(\tau) \underbrace{h(t-\tau)}_{=0 \text{ if } (t-\tau)<0.} d\tau$$

so...

$$y(t) = \int_{0}^{\infty} h(\tau) x(t-\tau) d\tau$$

equivalent

$$y(t) = \int_{-\infty}^{t} x(\tau) h(t-\tau) d\tau$$

lec 7

Fourier Series:

If $f(t)$ is periodic or time limited,

$$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

Fourier Coefficients: $D_n = \frac{1}{T_0} \int_{\tau}^{\tau+T_0} f(t) e^{-jn\omega_0 t} dt$

for all integer n.

& Parseval's theorem holds.

Fourier Transform:

If $f(t)$ is well behaved & aperiodic:

Inverse Fourier Transform $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega$

Fourier Transform $F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$

can be viewed as limit of Fourier Series as $T \to \infty$ & sum → integral.

Orthogonal Signals:

$$\int_{t_0}^{t_0+T_0} \phi_n(t) \phi_k^*(t) dt = 0 \quad \text{for } n \neq k$$

← complex conjugate

⇒ the family of complex exponentials $\{e^{-j\omega_0 k t}\}$ form an orthogonal family on any interval $[t_0, t_0+T_0)$

$$f(t) f^*(t) = |f(t)|^2$$

Parseval's Theorem:

$$\frac{1}{T_0} \int_{t_0}^{t_0+T_0} |f(t)|^2 dt = \lim_{N\to\infty} \sum_{n=-N}^{N} |D_n|^2$$

Energy of signal in time domain is same as the sum of the energies of the frequency domain components.

(comes from fact that integral square error goes to zero (i.e. truncated fourier series converges to the signal).)

Properties of Fourier Coefficients for REAL waveform...

$$\mathcal{R}(D_{-n}) = \mathcal{R}(D_n)$$
$$\mathcal{I}(D_{-n}) = -\mathcal{I}(D_n)$$
$$D_{-n} = D_n^*$$
$$|D_{-n}| = |D_n|$$
$$\angle D_{-n} = -\angle D_n$$

$$\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

$\text{rect}(t) = f(t)$

Fourier Series : $f(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \text{sinc}\left(\frac{n}{T}\right) e^{jn\omega_0 t}$

$\omega_0 = \frac{2\pi}{T_0}$

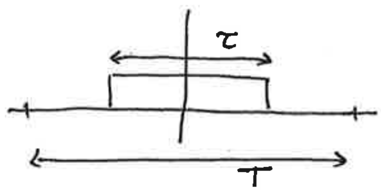
Fourier Transforms:

$\text{rect}\left(\frac{t}{T}\right) \Leftrightarrow T\,\text{sinc}\left(\omega T/2\pi\right)$

$e^{-at}u(t) \Leftrightarrow \frac{1}{a+j\omega}$

$\text{rect}\left(\frac{t}{\tau}\right) \Rightarrow D_n = \frac{\tau}{T}\,\text{sinc}\left(\frac{n\tau}{T}\right)$

pulse length. (τ) period (T)



EVEN & ODD functions:

If $e_1(t)$ and $e_2(t)$ are even functions
and $o_1(t)$ and $o_2(t)$ are odd functions,

$e_1(t) \pm e_2(t) \longrightarrow$ EVEN

$e_1(t)\, e_2(t) \longrightarrow$ EVEN

$o_1(t) \pm o_2(t) \longrightarrow$ ODD

$o_1(t)\, o_2(t) \longrightarrow$ EVEN

$e_1(t)\, o_2(t) \longrightarrow$ ODD

---

if $f(t)$ EVEN, $F(j\omega)$ EVEN

if $f(t)$ ODD, $F(j\omega)$ ODD

For real signal... Hermitian Symmetry – even real part, odd imaginary part.

$F(-j\omega) = F^*(j\omega)$

$F^*(j\omega) = R\left(F(j\omega)\right) - j\,I\left(F(j\omega)\right)$

For imaginary signal...

$F(-j\omega) = -F^*(j\omega)$ Anti-Hermitian Symmetry – even imaginary part, odd real part.

Fourier Transform

→ linear

$a f_1(t) + b f_2(t) \Leftrightarrow a F_1(j\omega) + b F_2(j\omega)$

scaling:

$f(at) \Leftrightarrow \frac{1}{|a|} F(j\frac{\omega}{a})$

$f(-t) \Leftrightarrow F(-j\omega)$

Complex conjugation:

if $f(t) \Leftrightarrow F(j\omega)$

then $f^*(t) \Leftrightarrow F^*(-j\omega)$

Duality:

if $f(t) \Leftrightarrow F(j\omega)$

then $F(j\omega)\big|_{\omega \to t} \Leftrightarrow 2\pi f(-t)\big|_{t \to \omega}$

some results: ...

$\text{sinc}\left(\frac{t}{2\pi}\right) \Leftrightarrow 2\pi \text{rect}(-\omega) = 2\pi \text{rect}(\omega)$

$\frac{1}{a+jt} \Leftrightarrow 2\pi e^{a\omega} u(-\omega)$

Shifting:

if $f_\tau \triangleq f(t-\tau)$

then $F_\tau(j\omega) = e^{-j\omega\tau} F(j\omega)$

or $f(t-\tau) \Leftrightarrow e^{-j\omega\tau} F(j\omega)$

Modulation:

$f(t)\cos(\omega_0 t) \Leftrightarrow \frac{1}{2}\left(F(j(\omega-\omega_0)) + F(j(\omega+\omega_0))\right)$

$f(t)\sin(\omega_0 t) \Leftrightarrow \frac{1}{2}\left(F(j(\omega-\omega_0)) - F(j(\omega+\omega_0))\right)$

Derivative:

if $f(t) \Leftrightarrow F(j\omega)$

$f'(t) \Leftrightarrow j\omega F(j\omega)$

$f^{(n)}(t) \Leftrightarrow (j\omega)^n F(j\omega)$

$(-jt) f(t) \Leftrightarrow F'(j\omega)$

Parseval's theorem

$\mathcal{E}_f = \int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2 d\omega$

Convolution ...

$(f_1 * f_2)(t) \Leftrightarrow F_1(j\omega) F_2(j\omega)$

$\Delta(t) \Leftrightarrow \text{sinc}^2(\omega/2\pi)$

where $\Delta t = \begin{cases} 1-|t| & |t|<1 \\ 0 & \text{o.w.} \end{cases}$

notation:

$$\mathcal{F}|f(t)| = F(j\omega)$$

$$\mathcal{F}|f_1(t) f_2(t)| = \frac{1}{2\pi}(F_1 * F_2)(j\omega)$$

multiplication in time domain corresponds to convolution in frequency domain. (convolution w/respect to ω not $j\omega$).

$$\text{sinc}^2(t) \Leftrightarrow \Delta(\omega/2\pi)$$

$$\delta(t) \Leftrightarrow 1$$

note/remember: as a function becomes infinitely narrow, its transform becomes infinitely broad.

$$\delta(t-t_0) \Leftrightarrow e^{-j\omega t_0}$$

$$1 \Leftrightarrow 2\pi\,\delta(\omega)$$

$$e^{j\omega_0 t} \Leftrightarrow 2\pi\,\delta(\omega-\omega_0)$$

$$\cos(\omega_0 t) \Leftrightarrow \pi(\delta(\omega-\omega_0) + \delta(\omega+\omega_0))$$

$$\sin(\omega_0 t) \Leftrightarrow j\pi(\delta(\omega+\omega_0) - \delta(\omega-\omega_0))$$

$$\text{sgn}(t) = \begin{cases} 1 & t>0 \\ 0 & t=0 \\ -1 & t<0 \end{cases} \Leftrightarrow \begin{cases} \frac{2}{j\omega} & \omega \neq 0 \\ 0 & \omega = 0 \end{cases}$$

taking

$$\text{sgn}(t) = e^{-at}u(t) - e^{at}u(-t)$$

with limit as $a \to 0$.

unit step: $$u(t) \Leftrightarrow \pi\,\delta(\omega) + \overbrace{\frac{1}{j\omega}}^{=0 \text{ if } \omega=0}$$

$$\int_{-\infty}^{t} f(\tau)\,d\tau \Leftrightarrow \pi F(0)\,\delta(\omega) + \frac{F(j\omega)}{j\omega}$$

integration $\Leftrightarrow$ division by $j\omega$

differentiation $\Leftrightarrow$ multiplication by $j\omega$