EE102B Signal Processing and Linear Systems II

Solutions to Problem Set One
2012-2013 Spring Quarter

Problem 1.1 $*$ (25 points)

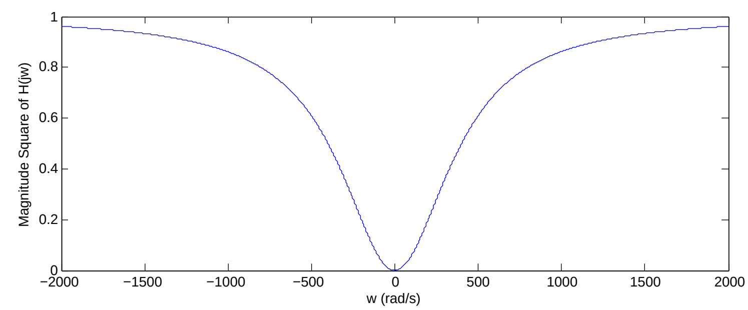
(a)

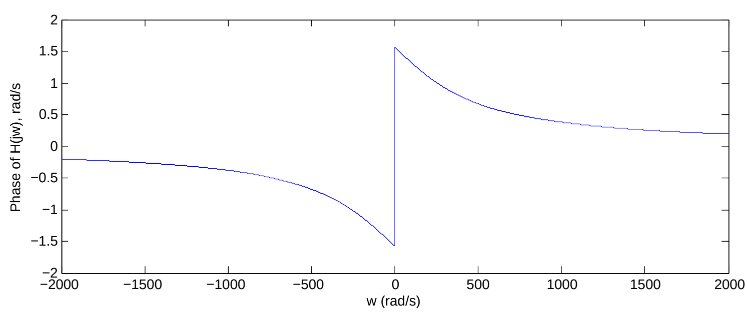
$$H(j\omega) = 1 - 400\left(\frac{1}{400 + j\omega}\right) = \frac{j\omega}{400 + j\omega}$$

(b)

$$|H(j\omega)|^2 = \frac{\omega^2}{400^2 + \omega^2}$$

$$\angle H(j\omega) = \angle\left(\frac{j\omega}{400 + j\omega}\right) = \angle\left(\frac{1}{1 - j\frac{400}{\omega}}\right) = -\arctan\left(-\frac{400}{\omega}\right) = \arctan\left(\frac{400}{\omega}\right)$$





(c) $|H(j\omega)|^2$ **is a monotonically increasing function of** ω**. Thus,**

$$\arg\sup|H(j\omega)|^2 = \infty$$

$$\sup|H(j\omega)|^2 = \lim_{\omega\to\infty}\frac{\omega^2}{400^2+\omega^2} = 1$$

By setting $|H(j\omega)|^2 = \frac{1}{2}\sup|H(j\omega)|^2 = \frac{1}{2}$**,**

$$\omega_{3dB} = 400\ (rad/s)$$

(d) Let $x(t) = x_1(t) + x_2(t) + x_3(t)$
where, $x_1(t) = 5$**,** $x_2(t) = 10\cos(200\pi t)$**, and** $x_3(t) = \delta(t-0,05)$**.**

Let $y(t) = y_1(t) + y_2(t) + y_3(t)$
where, $y_1(t) = h(t) * x_1(t)$**,** $y_2(t) = h(t) * x_2(t)$**, and** $y_3(t) = h(t) * x_3(t)$**.**

1) Getting $y_1(t)$ **using the frequency response of** $h(t)$**.**

$$Y_1(j\omega) = H(j\omega)\cdot X_1(j\omega) = H(j\omega)\cdot 10\pi\delta(\omega) = 10\pi\cdot H(j0)\cdot\delta(\omega) = 0$$
$$\therefore y_1(t) = 0$$

2) Getting $y_2(t)$ **using the frequency response of** $h(t)$**.**

$$\begin{aligned} Y_2(j\omega) &= H(j\omega)\cdot X_2(j\omega) = H(j\omega)\cdot 10\pi(\delta(\omega-200\pi)+\delta(\omega+200\pi)) \\ &= 10\pi\cdot H(j200\pi)\cdot\delta(\omega-200\pi) + 10\pi\cdot H(-j200\pi)\cdot\delta(\omega+200\pi) \end{aligned}$$

$$\begin{aligned} \therefore y_2(t) &= \frac{j5\pi}{2+j\pi}\cdot e^{j200\pi t} - \frac{j5\pi}{2-j\pi}\cdot e^{-j200\pi t} \\ &= \frac{10\pi}{2^2+\pi^2}\left[\pi\cos(200\pi t) - 2\sin(200\pi t)\right] \\ &= \frac{10\pi}{\sqrt{2^2+\pi^2}}\cos(200\pi t+\phi) \quad \textbf{where } \cos(\phi) = \frac{\pi}{\sqrt{2^2+\pi^2}},\ \sin(\phi) = \frac{2}{\sqrt{2^2+\pi^2}} \end{aligned}$$

3) Getting $y_3(t)$ **using impulse response of** $h(t)$**.**

$$\begin{aligned} \therefore y_3(t) &= h(t) * \delta(t-0.05) = h(t-0.05) \\ &= \delta(t-0.05) - 400\cdot e^{-400(t-0.05)}\cdot u(t-0.05) \end{aligned}$$

By 1) $\sim$ 3),

$$\therefore y_(t) = \frac{10\pi}{\sqrt{2^2+\pi^2}}\cos(200\pi t+\phi) + \delta(t-0.05) - 400\cdot e^{-400(t-0.05)}\cdot u(t-0.05)$$

Problem 1.2 (0 points)

(a)

$$\begin{aligned} y(t) &= -x(t) + 2x(t-T) - x(t-2T) \\ &= x(t) * [-\delta(t) + 2\delta(t-T) - \delta(t-2T)] \\ &= x(t) * h(t) \end{aligned}$$

$$\therefore h(t) = -\delta(t) + 2\delta(t-T) - \delta(t-2T)$$

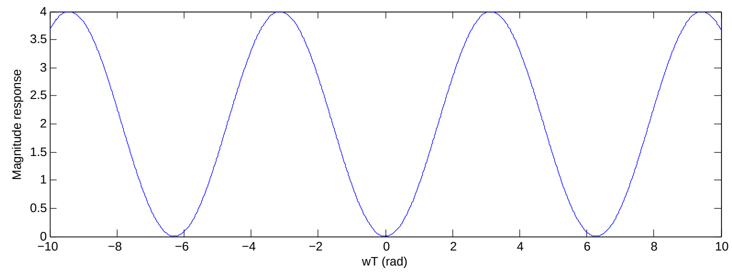
(b)

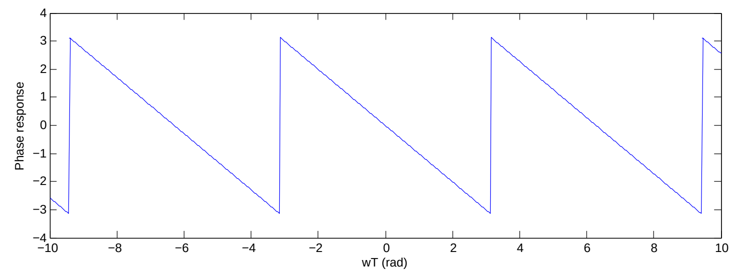
$$\begin{aligned} H(j\omega) &= \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} [-\delta(t) + 2\delta(t-T) - \delta(t-2T)] e^{-j\omega t} dt \\ &= -e^{-j\omega_0} + 2e^{-j\omega T} - e^{-j\omega 2T} \\ &= -1 + 2e^{-j\omega T} - e^{-j\omega 2T} \\ &= [-e^{j\omega T} + 2 - e^{-j\omega T}] e^{-j\omega T} \\ &= 2[1 - \cos(\omega T)] e^{-j\omega T} \end{aligned}$$

(c)

$$\begin{aligned} y(t) &= h(t) * x(t) \\ &= [-\delta(t) + 2\delta(t-T) - \delta(t-2T)] * e^{j\omega t} \\ &= -e^{j\omega t} + 2e^{j\omega(t-T)} - e^{j\omega(t-2T)} \\ &= [-e^{j\omega T} + 2 - e^{-j\omega T}] e^{-j\omega T} e^{j\omega t} \\ &= 2[1 - \cos(\omega T)] e^{-j\omega T} e^{j\omega t} = H(j\omega) e^{j\omega t} \end{aligned}$$

(d)





Problem 1.3 (0 points)

(a)

$$
\begin{aligned}
e^{-at}u(t),\ \boldsymbol{Re}(a)>0 \quad &\leftrightarrow \quad \frac{1}{a+j\omega} \\
20e^{-7t}u(t) \quad &\leftrightarrow \quad \frac{20}{7+j\omega} \\
20e^{-7(t-1)}u(t-1) \quad &\leftrightarrow \quad \frac{20}{7+j\omega}e^{-j\omega}
\end{aligned}
$$

'Time-domain shift' property was used.

(b)

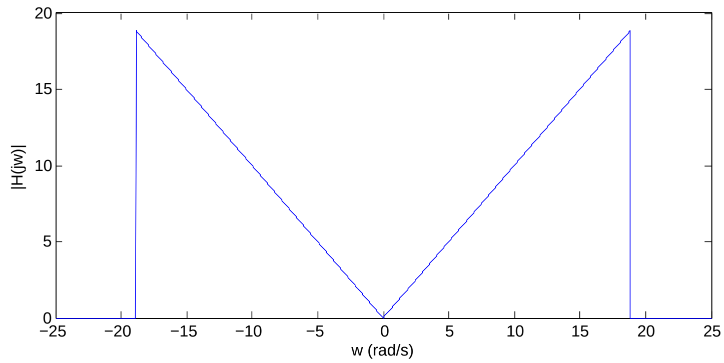
$$
\begin{aligned}
1 \quad &\leftrightarrow \quad 2\pi\delta(\omega) \\
j\frac{1}{2}(e^{-j10\pi t}-e^{j10\pi t})=\sin(10\pi t) \quad &\leftrightarrow \quad j\pi\delta(\omega+10\pi)-j\pi\delta(\omega-10\pi) \\
\sin(10\pi(t-0.5)) \quad &\leftrightarrow \quad e^{-jw/2}[j\pi\delta(\omega+10\pi)-j\pi\delta(\omega-10\pi)]
\end{aligned}
$$

'Linearity', 'Frequency-domain shift', and 'Time-domain shift' properties were used.

(c)

$$
\begin{aligned}
\frac{\sin(Wt)}{\pi t} \quad &\leftrightarrow \quad X(jw)=\begin{cases} 1, & |\omega|<W \\ 0, & |\omega|>W \end{cases} \\
\frac{\sin(6\pi t)}{\pi t} \quad &\leftrightarrow \quad \begin{cases} 1, & |\omega|<6\pi \\ 0, & |\omega|>6\pi \end{cases} \\
\frac{d}{dt}\left\{\frac{\sin(6\pi t)}{\pi t}\right\} \quad &\leftrightarrow \quad \begin{cases} jw, & |\omega|<6\pi \\ 0, & |\omega|>6\pi \end{cases}
\end{aligned}
$$

'Differentiation in Time' property was used.



(d)

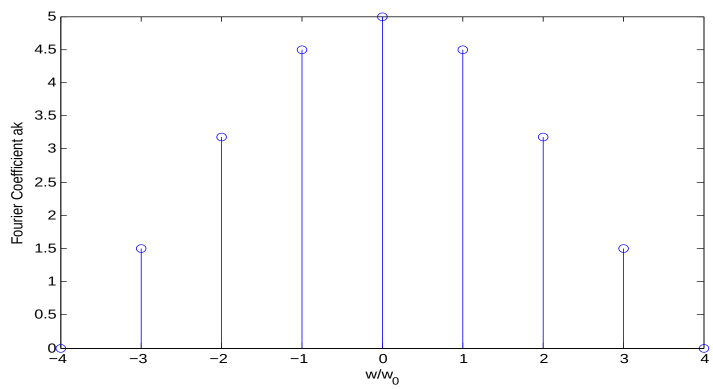
Problem 1.4 $*$ (25 points)

(a) The period is 8 seconds. Thus, $w_0 = \frac{2\pi}{8} = \frac{\pi}{4}$ (rad/s).

$$a_k = \frac{1}{8}\int_{-4}^{4} x(t)e^{-j\frac{\pi}{4}kt}dt = \frac{1}{8}\int_{-1}^{1} x(t)e^{-j\frac{\pi}{4}kt}dt$$

since, for $-4 < t < 4$,$x(t)$ is nonzero only between -1 and +1.

(b)



(c)

$$y(t) = A + B\cos(\omega_0 t + \phi) = A + \frac{B}{2}e^{j\phi}e^{j\omega_0 t} + \frac{B}{2}e^{-j\phi}e^{-j\omega_0 t}$$

Thus, what we need to pass are only the frequency components corresponding to a_{-1}, a_0, and a_1.

$$\omega_0 < \omega_c < 2\omega_0, \quad \textbf{where } \omega_0 = \frac{2\pi}{T} = \frac{\pi}{4}$$

$$\therefore \frac{\pi}{4} < \omega_c < \frac{\pi}{2}$$

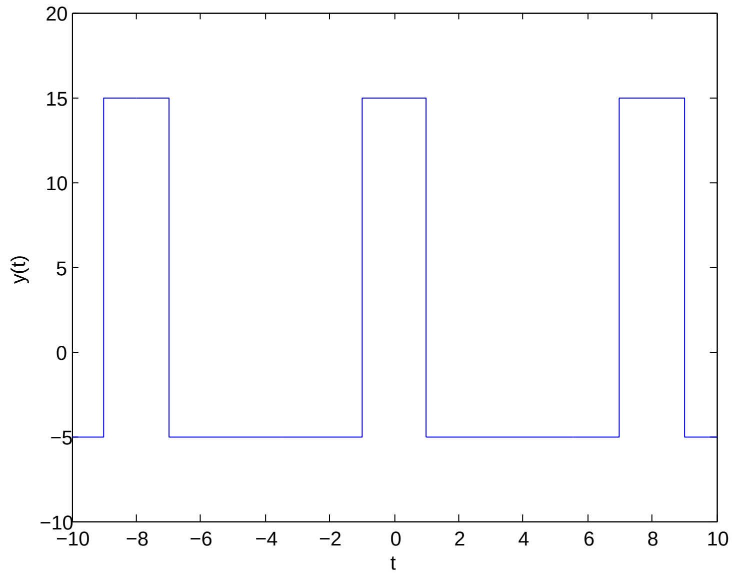
(d) This high-pass filter only removes the frequency component corresponding to a_0.

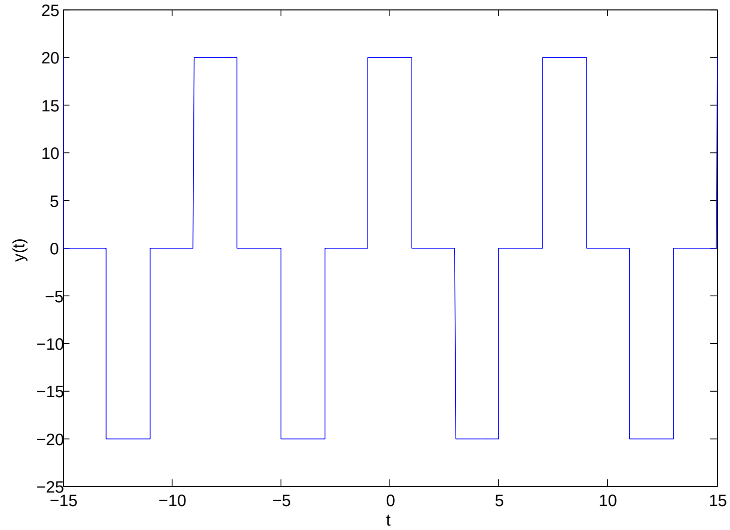
$$\therefore y(t) = x(t) - a_0 * e^{j\frac{\pi}{4}0t} = x(t) - a_0 = x(t) - 5$$

(e)

$$H(j\omega) = 1 - e^{-j4\omega} \quad \leftrightarrow \quad h(t) = \delta(t) - \delta(t-4)$$

$$\therefore y(t) = h(t) * x(t) = x(t) - x(t-4)$$



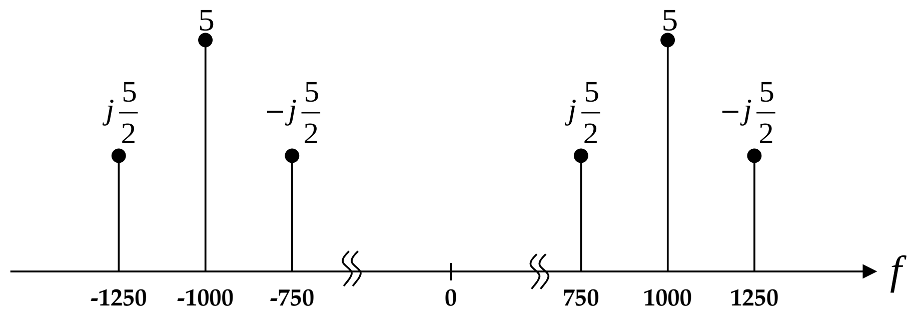


Problem 1.5 $*$ (25 points)

(a)

$$
\begin{aligned}
x(t) &= [10 + 10\cos(500\pi t - \pi/2)]\cos(2000\pi t) \\
&= [10 + 5e^{-j\pi/2}e^{j500\pi t} + 5e^{j\pi/2}e^{-j500\pi t}] \cdot \frac{1}{2}[e^{j2000\pi t} + e^{-j2000\pi t}] \\
&= [10 - 5je^{j500\pi t} + 5je^{-j500\pi t}] \cdot \frac{1}{2}[e^{j2000\pi t} + e^{-j2000\pi t}] \\
&= 5e^{j2000\pi t} + 5e^{-j2000\pi t} - j\frac{5}{2}e^{j2500\pi t} - j\frac{5}{2}e^{-j1500\pi t} + j\frac{5}{2}e^{j1500\pi t} + j\frac{5}{2}e^{-j2500\pi t}
\end{aligned}
$$

Since all frequencies are multiple of 250 Hz, $x(t)$ is periodic with $T_0 = \frac{1}{250}$ sec.

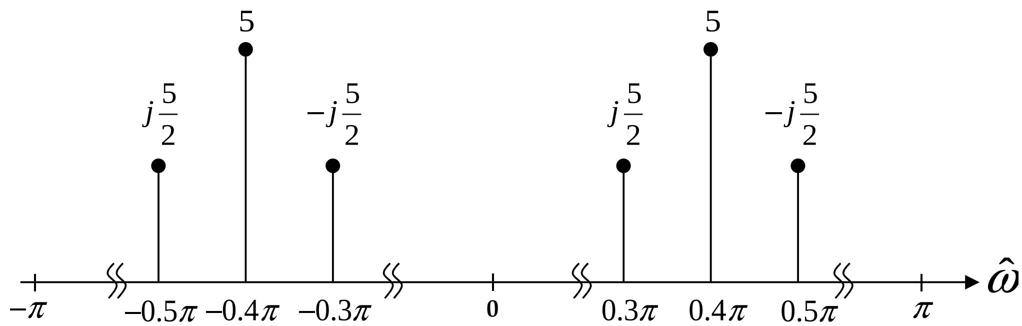


(b) For a perfect reconstruction, the sampling frequency must be higher than twice the highest frequency component of $x(t)$.

$$\therefore f_s > 2(1250) = 2500 \textbf{ samples/sec}$$

(c)

$$x[n] = x\left(\frac{n}{5000}\right) = 5e^{j0.4\pi n} + 5e^{-j0.4\pi n} - j\frac{5}{2}e^{j0.5\pi n} - j\frac{5}{2}e^{-j0.3\pi n} + j\frac{5}{2}e^{j0.3\pi t} + j\frac{5}{2}e^{-j0.5\pi t}$$



Problem 1.6 $*$ (25 points)

(a) Let us assmue that the input of the C-D converter be $x(t) = A\cos(\omega t+\phi)$. Since the sampling period is $T_s = \frac{1}{10000}$,

$$x[n] = x(T_s n) = A\cos(\omega T_s n + \phi) = A\cos\left(\frac{\omega}{10000}n + \phi\right) = 4\cos(0.3\pi n - \pi/3)$$

Since the input frequency is less than 10000 Hz, i.e. $-20000 < \omega < 20000\pi$ (rad/s),

$$\frac{\omega}{10000} = 0.3\pi + 2\pi k, \qquad -20000\pi < \omega < 20000\pi$$

where k is an integer. Thus,

$$-2\pi < 0.3\pi + 2\pi k < 2\pi$$

The possible k is either 0 or -1.

1) When k=0, $\omega_1 = 3000\pi$.

$$\begin{aligned} x_1(\frac{n}{10000}) &= A_1\cos(\frac{3000\pi}{10000}n + \phi_1) \\ &= A_1\cos(0.3\pi n + \phi_1) \\ &= 4\cos(0.3\pi n - \pi/3) \end{aligned}$$

Thus, $A_1 = 4$, $\phi_1 = -\pi/3$.

$$\therefore x_1(t) = 4\cos(3000\pi t - \pi/3)$$

2) When k=-1, $\omega_2 = -17000\pi$.

$$\begin{aligned} x_2(\frac{n}{10000}) &= A_2\cos(\frac{-17000\pi}{10000}n + \phi_2) \\ &= A_2\cos(-1.7\pi n + \phi_2) \\ &= A_2\cos(2\pi n - 1.7\pi n + \phi_2) \\ &= A_2\cos(0.3\pi n + \phi_2) \\ &= 4\cos(0.3\pi n - \pi/3) \end{aligned}$$

Thus, $A_2 = 4$, $\phi_2 = -\pi/3$.

$$\therefore x_2(t) = 4\cos(-17000\pi t - \pi/3) = 4\cos(17000\pi t + \pi/3)$$

(b) For a perfect reconstruction, the sampling rate, f_s must be greater than the twice of the highest frequency component.

$$\therefore f_s > 2 \cdot 300 = 600 \quad \textbf{(samples/sec)}$$

(c)

$$\begin{aligned} x(t) &= 10\left(e^{j\frac{3\pi}{4}}e^{-j200\pi t} + e^{-j\frac{3\pi}{4}e^{j200\pi t}}\right) + 8\left(e^{-j\pi e^{-j600\pi t}} + e^{j\pi e^{j600\pi t}}\right) \\ &= 20\cos\left(200\pi t - \frac{3\pi}{4}\right) + 16\cos\left(600\pi t + \pi\right) \end{aligned}$$

Thus,

$$\begin{aligned}
x[n] = x\left(\frac{n}{300}\right) &= 20\cos\left(\frac{2\pi}{3}n - \frac{3\pi}{4}\right) + 16\cos\left(2\pi n + \pi\right) \\
&= 20\cos\left(\frac{2\pi}{3}n - \frac{3\pi}{4}\right) - 16 \\
&= 10\left(e^{-j\frac{3\pi}{4}}e^{j\frac{2\pi}{3}n} + e^{j\frac{3\pi}{4}}e^{-j\frac{2\pi}{3}n}\right) - 16
\end{aligned}$$