3-6 Fourier Analysis of Periodic Signals

We can synthesize *any periodic signal* by using a sum of sinusoids (3.28), as long as we constrain the frequencies to be harmonically related. To demonstrate Fourier synthesis of waveshapes that do not look at all sinusoidal, we will work out the details for a square wave, a triangle wave, and rectified sinusoids in this section. The resulting formulas for their Fourier coefficients a_k are relatively compact, but the number of coefficients required for exact synthesis is infinite.

In order to synthesize the signal from its spectrum in hardware or software, a list of frequencies and a list of complex amplitudes are needed, but these lists must contain a finite number of elements. Then an approximate signal can be synthesized according to the *finite* Fourier synthesis summation formula

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{j(2\pi/T_0)kt} \tag{3.31}$$

If you have access to MATLAB, it is straightforward to write a *Fourier Synthesis Program* that implements (3.31). This MATLAB programming exercise is described in more detail in the music synthesis project of Lab #4. Figure 3-14 shows the relationship between Fourier analysis and Fourier synthesis using representative plots for the square wave case where the finite synthesis result (in the lower right) is an approximation to the original square wave.

LAB: *#4 Synthesis of Sinusoidal Signals*

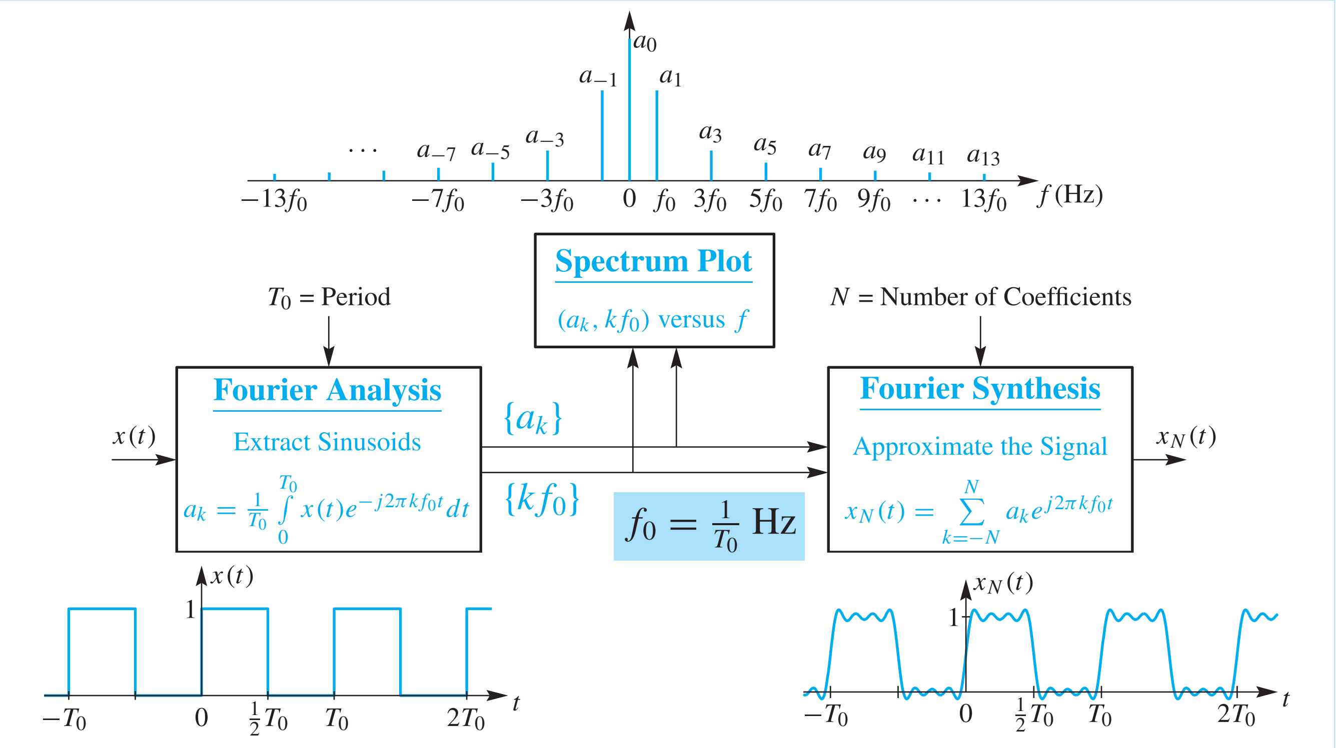

Figure 3-14: Major components of Fourier analysis and synthesis showing the relationship between the original periodic signal $x(t)$, its spectrum, and the synthesized signal $x_N(t)$ that approximates the original.

3-6.1 The Square Wave

The simplest example to consider is the periodic square wave, which is defined for one cycle by

$$x(t) = \begin{cases} 1 & \text{for } 0 \le t < \frac{1}{2}T_0 \\ 0 & \text{for } \frac{1}{2}T_0 \le t \le T_0 \end{cases} \tag{3.32}$$

Figure 3-15 shows a plot of this signal which is called a 50% duty cycle square wave because it switches between zero and one (off and on), and is on during half of its period.

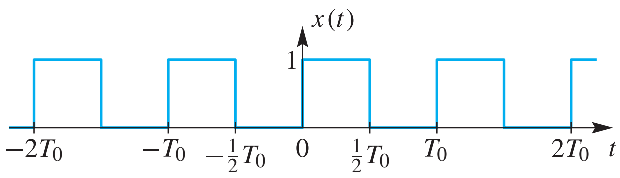

Figure 3-15: Plot of the square-wave signal whose "duty cycle" is 50%.

We will derive a formula that depends on k for the complex amplitudes a_k. First of all, we substitute the definition of $x(t)$ into the integral (3.27) and immediately recognize that the integral must be broken into two integrals to handle the two cases in (3.32)

$$a_k = \frac{1}{T_0}\int_0^{\frac{1}{2}T_0} (1)e^{-j(2\pi/T_0)kt}\,dt + \cancelto{0}{\frac{1}{T_0}\int_{\frac{1}{2}T_0}^{T_0} (0)e^{-j(2\pi/T_0)kt}\,dt}$$

The second integral drops out because the signal $x(t)$ is zero for $\frac{1}{2}T_0 \le t \le T_0$. Thus, we perform the integration from 0 to $\frac{1}{2}T_0$, and simplify as follows:

$$\begin{aligned} a_k &= \left(\frac{1}{T_0}\right) \left.\frac{e^{-j(2\pi/T_0)kt}}{-j(2\pi/T_0)k}\right|_0^{\frac{1}{2}T_0} \\ &= \left(\frac{1}{T_0}\right) \frac{e^{-j(2\pi/T_0)k(\frac{1}{2}T_0)} - e^{-j(2\pi/T_0)k(0)}}{-j2\pi/T_0k} \\ &= \frac{e^{-j\pi k} - 1}{-j2\pi k} \end{aligned}$$

Since $e^{-j\pi} = -1$, we can write the following general formula for the Fourier series coefficients of the square wave.

$$a_k = \frac{1-(-1)^k}{j2\pi k} \qquad \text{for } k \ne 0 \tag{3.33}$$

There is one shortcoming with this formula for a_k; it is not valid when $k = 0$ because k appears in the denominator. Therefore, we must evaluate the DC coefficient a_0 separately using (3.22)

$$\begin{aligned} a_0 &= \frac{1}{T_0}\int_0^{\frac{1}{2}T_0} (1)e^{-j0t}\,dt \\ &= \frac{1}{T_0}\int_0^{\frac{1}{2}T_0} (1)\,dt = \frac{1}{T_0}\left(\tfrac{1}{2}T_0\right) = \tfrac{1}{2} \end{aligned}$$

The formula (3.33) for a_k when $k \neq 0$ has a numerator that is either 0 (for k even) or 2 (for k odd), because $(-1)^k$ alternates between $+1$ and -1. Therefore, the final answer for all the Fourier series coefficients of the square wave has three cases:

$$a_k = \begin{cases} \dfrac{1}{j\pi k} & k = \pm1, \pm3, \pm5, \ldots \\ 0 & k = \pm2, \pm4, \pm6, \ldots \\ \frac{1}{2} & k = 0 \end{cases} \tag{3.34}$$

Notice that the magnitude of these coefficients decreases as $k \to \infty$, so the high frequency terms contribute less and less when synthesizing the waveform via (3.31).

3-6.1.1 DC Value of a Square Wave

The Fourier series coefficient for $k = 0$ has a special interpretation as the *average value* of the signal $x(t)$. If we repeat the analysis integral (3.27) for the case where $k = 0$, then

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt \tag{3.35}$$

The integral is the area under the function $x(t)$ for one period. If we think of the area as a sum and realize that dividing by T_0 is akin to dividing by the number of elements in the sum, we can interpret (3.35) as the average value of the signal.

In the specific case of the 50% duty-cycle square wave, the average value should be $\frac{1}{2}$ because the signal is equal to $+1$ for half the period and then 0 for the other half. In the synthesis formula (3.28), the a_0 coefficient is an additive constant, so a change in its value will move the plot of the signal up or down vertically. This is the case in the following example for another 50% duty cycle square with a different DC value.

EXERCISE 3.8: Determine the DC value of the following *bipolar* square wave that alternates between the positive and negative values $\pm\frac{1}{2}$

$$s(t) = \begin{cases} -\frac{1}{2} & \text{for } 0 \le t < \frac{1}{2}T_0 \\ \frac{1}{2} & \text{for } \frac{1}{2}T_0 \le t \le T_0 \end{cases}$$

This signal is similar to the square wave defined in (3.32).

3-6.1.2 Spectrum for a Square Wave

Figure 3-16 shows the spectrum for the 50% duty cycle square wave analyzed in (3.34) when the fundamental frequency is 25 Hz. Since $a_k = 0$ for k even and nonzero, the only frequency components present in the spectrum are DC and the odd harmonics at ± 25, ± 75, ± 125, and so on. The complex amplitudes of the odd harmonics are the Fourier series coefficients, $a_k = -j/(\pi k)$, and these are used as the labels on the spectrum lines in Fig. 3-16. Also the figure shows that the magnitude of these coefficients drops off as $1/k$.

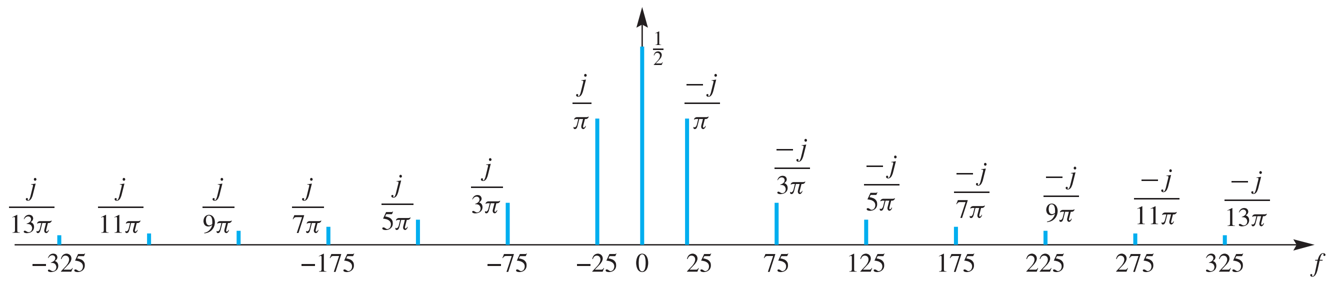

Figure 3-16: Spectrum of a square wave derived from its Fourier series coefficients. Only the range from $k = -13$ to $+13$ is shown. With a fundamental frequency of 25 Hz, this corresponds to frequencies ranging from -325 Hz to $+325$ Hz.

DEMO: *Spectrograms: Simple Sounds*

3-6.1.3 Synthesis of a Square Wave

The fact that the spectrum of a square has an infinite number of spectral lines means that it is impossible to have/use such a signal in a a real hardware system built with electronics or other physical devices. No realizable system has infinite bandwidth, so any attempt to transmit the square wave would be subject to a bandwidth limit and the higher frequency spectral components would be changed or lost. Therefore, it is quite interesting to study the approximate resynthesis of a square wave from a finite number of its spectral lines, and ask the question "How good is the approximation when using N spectral lines?"

Using a simple MATLAB M-file, a synthesis was done via (3.31) with a fundamental frequency of $f_0 = 25$ Hz, and $f_k = kf_0$. The fundamental period is $T_0 = 1/25 = 0.04$ secs. In Fig. 3-17, the plots are shown for three different cases where the number of terms in the sum is $N = 3$, 7, and 17. Notice how the period of the synthesized waveform is always the same, because it is determined by the fundamental frequency.

The synthesis formula (3.31) usually can be simplified to a cosine form. For the particular case of the square wave coefficients (3.34), when we take the DC value plus first and third harmonic terms, i.e., $N = 3$, we get the sum of two sinusoids plus the constant (DC) level derived below:

As more harmonic terms are added, the approximating formula is still a sum of sinusoids but higher frequency terms are included. In Fig. 3-17 the ripples are higher in frequency as N increases However, the synthesized waveshape does look more like a square-wave signal as N increases, and it appears to converge to the constant values $+1$ and 0. The size of the ripples gets smaller, but the convergence is not uniformly good—there are "ears" at the discontinuous steps which never go away completely. This behavior, which occurs at any discontinuity of a waveform, is called the *Gibbs phenomenon*, and it is one of the interesting subtleties of Fourier theory that is extensively studied in advanced treatments.

LAB: *#4 Synthesis of Sinusoidal Signals*

DEMO: *Fourier Series*

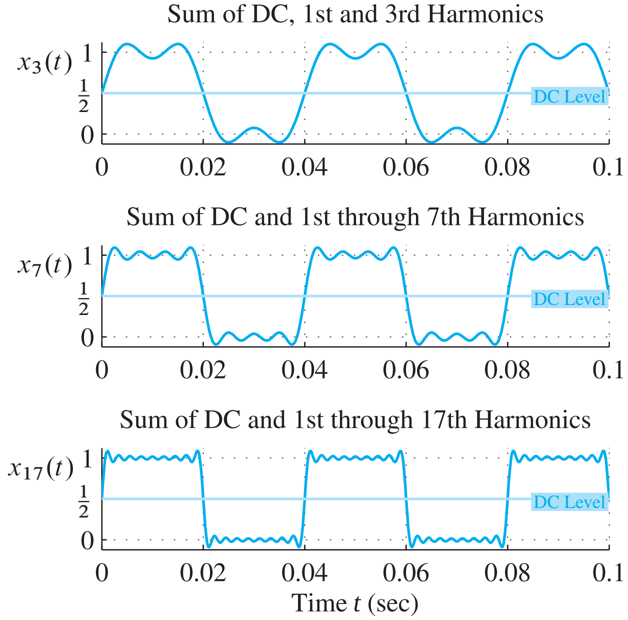

Figure 3-17: Summing harmonic components via (3.31): $N = 3$ (top panel); $N = 7$ (middle) and $N = 17$ (bottom). The DC level of $\frac{1}{2}$ can be seen in each synthesis.

$$
\begin{aligned}
x_3(t) &= a_{-3}e^{-j3\omega_0 t} + a_{-1}e^{-j\omega_0 t} + a_0 + a_1 e^{j\omega_0 t} + a_3 e^{j3\omega_0 t} \\
&= \frac{1}{-j3\pi}e^{-j3\omega_0 t} + \frac{1}{-j\pi}e^{-j\omega_0 t} + a_0 + \frac{1}{j\pi}e^{j\omega_0 t} + \frac{1}{j3\pi}e^{j3\omega_0 t} \\
&= a_0 + \frac{1}{\pi}(e^{-j\pi/2}e^{j\omega_0 t} + e^{j\pi/2}e^{-j\omega_0 t}) + \frac{1}{3\pi}(e^{-j\pi/2}e^{j3\omega_0 t} + e^{j\pi/2}e^{-j3\omega_0 t}) \\
&= \frac{1}{2} + \frac{2}{\pi}\cos(\omega_0 t - \pi/2) + \frac{2}{3\pi}\cos(3\omega_0 t - \pi/2)
\end{aligned}
\tag{3.36}
$$

3-6.2 Triangle Wave

Another interesting case that is still relatively simple is that of a triangle wave shown in Fig. 3-18. The mathe-

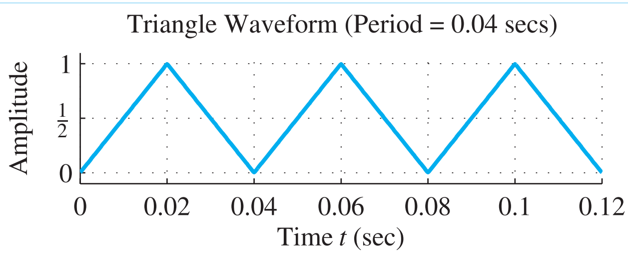

Figure 3-18: Periodic triangle wave with $T_0 = 0.04$ s.

matical formula for the triangle wave consists of two segments. To set up the Fourier analysis integral, we have to give the definition of the waveform over exactly one period, so we define two line segments:

$$x(t) = \begin{cases} 2t/T_0 & \text{for } 0 \le t < \frac{1}{2}T_0 \\ 2(T_0 - t)/T_0 & \text{for } \frac{1}{2}T_0 \le t < T_0 \end{cases} \tag{3.37}$$

The first line segment has a slope of $2/T_0$, the second, $-2/T_0$. Unlike the square wave, the triangle wave is a continuous signal.

Now we attack the Fourier integral for this case to derive a formula for the $\{a_k\}$ coefficients of the triangle wave. We might suspect from our earlier experience that the DC coefficient has to be found separately, so we do that first. Plugging into the definition with $k = 0$, we obtain

$$a_0 = \frac{1}{T_0} \int_0^{T_0} x(t)\, dt$$

If we recognize that the integral over one period is, in fact, the area under the triangle, we get

$$a_0 = \frac{1}{T_0}(\text{area}) = \frac{1}{T_0}(T_0)(\tfrac{1}{2}) = \tfrac{1}{2} \tag{3.38}$$

For the general case where $k \neq 0$, we must break the Fourier series analysis integral into two sections because the signal definition consists of two formulas:

$$\begin{aligned} a_k =& \frac{1}{T_0} \int_0^{\frac{1}{2}T_0} (2t/T_0) e^{-j(2\pi/T_0)kt}\, dt \\ &+ \frac{1}{T_0} \int_{\frac{1}{2}T_0}^{T_0} (2(T_0 - t)/T_0) e^{-j(2\pi T_0)kt}\, dt \end{aligned} \tag{3.39}$$

After integration by parts and many tedious algebraic steps, the integral for a_k can be written as

$$a_k = \frac{e^{-jk\pi} - 1}{\pi^2 k^2} \tag{3.40}$$

Since $e^{-jk\pi} = (-1)^k$, the numerator in (3.40) equals either 0 or -2, and we can write the following cases for a_k:

$$a_k = \begin{cases} \dfrac{-2}{\pi^2 k^2} & k = \pm 1, \pm 3, \pm 5, \dots \\ 0 & k = \pm 2, \pm 4, \pm 6, \dots \\ \frac{1}{2} & k = 0 \end{cases} \tag{3.41}$$

EXERCISE 3.9: Starting from (3.39), derive the formula (3.41) for the Fourier series coefficients of the triangle wave. Use integration by parts to manipulate terms of the form $te^{-j(2\pi/T_0)kt}$ which occur in the integrands.

EXERCISE 3.10: Make a plot of the spectrum for the triangle wave (similar to Fig. 3-16 for the square wave). Use the complex amplitudes from (3.41) and assume that $f_0 = 25$ Hz.

3-6.2.1 Synthesis of a Triangle Wave

The ideal triangle wave in Fig. 3-18 is a continuous signal, unlike the square wave which is discontinuous. Therefore, it is easier to approximate the triangle wave with a finite Fourier sum (3.31). Two cases are shown in Fig. 3-19, for $N = 3$ and 9. The fundamental frequency is equal to $f_0 = 25$ Hz. In the $N = 9$ case

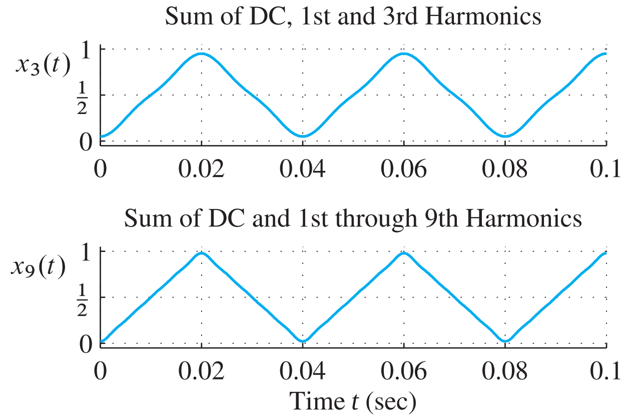

Figure 3-19: Summing harmonic components for a 25-Hz triangle wave via (3.31): first and third harmonics (top panel); up to and including the ninth harmonic (bottom).

the approximation is nearly indistinguishable from the true triangularly-shaped waveform but there is slight rounding at the top and bottom of the waveform. Adding harmonics for $N > 9$ will improve the synthesis near those points. Even the $N = 3$ case is reasonably good, despite using only DC and two sinuosoidal terms. We can see the reason for this by plotting the spectrum (as in Exercise 3.10), which will show that the high frequency components decrease in size much faster those of the square wave.

EXERCISE 3.11: For the $N = 3$ approximation of the triangle wave, derive the mathematical formula for the sinusoids; similar to what was done in (3.36) for the square wave.

3-6.3 Full-Wave Rectified Sine

Another interesting case that is still relatively simple is that of a full-wave rectified sine wave shown in Fig. 3-20. This signal is an essential part of DC power supplies as well as AC to DC power converters—the ubiquitous devices that are needed to recharge batteries in cell phones, laptops, and so on. Battery charging requires constant DC current, but plugging into a wall socket gets AC (alternating current) which, in our terms, is a pure sinusoid with zero DC value.[9] A simple rectifier circuit built with diodes will turn a sinusoid into a signal with nonzero DC component which can then be cleaned up and used as a DC power source for recharging. We will examine two rectified sinusoids: full-wave and half-wave.

The mathematical formula for the full-wave rectified sine signal is just the absolute value of a sinusoid.

$$x(t) = |\sin(2\pi t/T_1)| \tag{3.42}$$

For 50-Hz AC power the period of the sinusoid is $T_1 = 0.02$ s but after "rectification" the period is halved as shown in Fig. 3-20. The full-wave rectified sine is a continuous signal, but its first derivative is discontinuous,

[9] In most of the world, power companies distriute AC power at 50 hertz, but, in the Americas, 60 hertz is common.

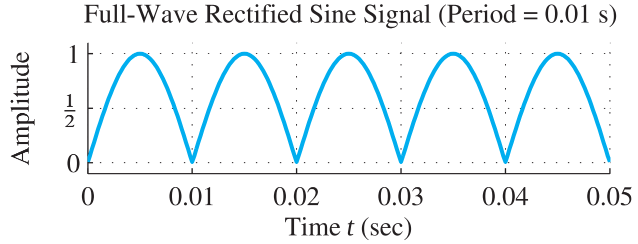

Figure 3-20: Periodic full-wave rectified sine signal.

resulting in the sharp points at locations where the signal is zero (at $t = 0, 0.01, 0.02, 0.03, \ldots$). The fundamental period (T_0) is equal to $\frac{1}{2}T_1$ because the negative lobe of the sinusoid is identical to the positive lobe after being flipped by the absolute value operator, so $f_0 = 100$ Hz in Fig. 3-20.

The Fourier integral for this case has no special cases, and requires only one integral,

$$a_k = \frac{1}{T_0}\int_0^{T_0} \sin(2\pi t/T_1)e^{-j(2\pi/T_0)kt}\,dt \tag{3.43}$$

where $T_1 = 2T_0$. The integration can be carried out efficiently by using the inverse Euler formula for sine. After simplifying many complex exponentials, the final result for a_k can be written as

$$a_k = \frac{2}{\pi(1-4k^2)} \tag{3.44}$$

Notice that the DC value is nonzero, i.e., $a_0 = 2/\pi \approx 0.6366$ which is 63.66% of the maximum value of the rectified sine as shown in Fig. 3-21.

EXERCISE 3.12: Starting from (3.43), derive the formula (3.44) for the Fourier series coefficients of the full-wave rectified sine. Exploit complex exponential simplifications such as $e^{j2\pi k} = 1$, $e^{j\pi} = -1$, and so on.

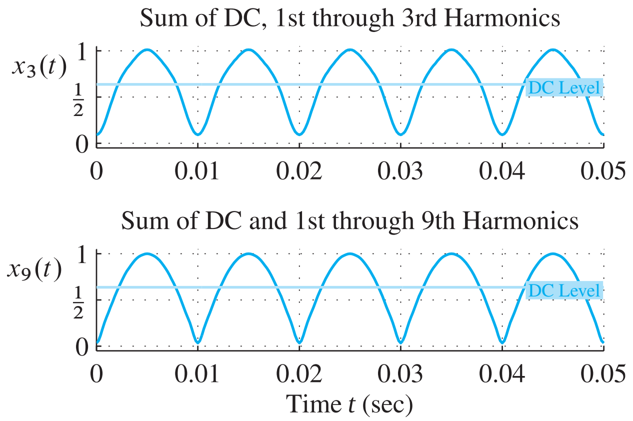

Figure 3-21: Summing harmonic components for the full-wave rectified sine signal via (3.31): DC, first and third harmonics (top panel); up to and including the ninth harmonic (bottom).

EXERCISE 3.13: Make a plot of the spectrum for the full-wave rectified sine (similar to Fig. 3-16 for the square wave). Use the complex amplitudes from (3.44) and note that $T_1 = 0.02\,\text{s}$ gives a fundamental frequency of $f_0 = 100\,\text{Hz}$.

3-6.3.1 Synthesis of a Full-Wave Rectified Sine

Since the ideal full-wave rectified sine signal in Fig. 3-20 is a continuous signal, the quality of a finite Fourier sum approximation (3.31) is better than for the square wave which is discontinuous. Two cases are shown in Fig. 3-21, for $N = 3$ and 9. In the $N = 9$ case the approximation is very close to the original waveform. Adding harmonics for $N > 9$ will improve the synthesis very little, mostly by sharpening the dips at $t = 0.01, 0.02, \ldots$. The $N = 3$ case exhibits noticeable ripples because it is only using only DC and three sinusoidal terms. It should be possible to explain these differences by plotting the spectrum (as in Exercise 3.13), and examining the relative magnitudes of the higher frequency components being dropped for $k > 3$.

3-6.4 Half-Wave Rectified Sine

Another interesting case that is still relatively simple is that of a half-wave rectified sine wave shown in Fig. 3-22, i.e., the negative lobes of a sine wave are clipped off. The mathematical formula for the half-wave rectified sine signal consists of two cases when defined over exactly one period, $0 \le t \le T_0$:

$$x(t) = \begin{cases} \sin(2\pi t/T_0) & \text{for } 0 \le t < \frac{1}{2}T_0 \\ 0 & \text{for } \frac{1}{2}T_0 \le t < T_0 \end{cases} \tag{3.45}$$

where $T_0 = 0.02$ sec in Fig. 3-22. The half-wave rectified sine is a continuous signal, but its first derivative is

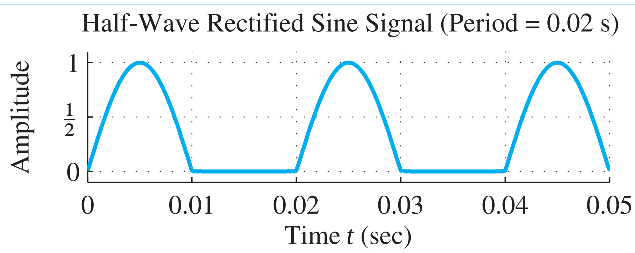

Figure 3-22: Periodic half-wave rectified sine signal.

discontinuous, resulting in the sharp points when the signal is zero at $t = 0, 0.01, 0.02, 0.03, \ldots$ in Fig. 3-22.

We must break the Fourier series analysis integral into two sections because the signal consists of two pieces:

$$\begin{aligned} a_k =& \frac{1}{T_0}\int_0^{\frac{1}{2}T_0} \sin(2\pi t/T_0)e^{-j(2\pi/T_0)kt}\,dt \\ &+ \cancelto{0}{\frac{1}{T_0}\int_{\frac{1}{2}T_0}^{T_0} (0)e^{-j(2\pi/T_0)kt}\,dt} \end{aligned} \tag{3.46}$$

For the general case where $k \neq \pm 1$, the integration can be carried out efficiently by using the inverse Euler formula for sine. After simplifying many complex exponentials, the final result for a_k can be written as

$$a_k = \begin{cases} \dfrac{1}{\pi(1-k^2)} & \text{for } k \text{ even} \\ \mp j\frac{1}{4} & \text{for } k = \pm 1 \\ 0 & \text{for } k = \pm 3, \pm 5, \ldots \end{cases} \tag{3.47}$$

The special cases of $k = \pm 1$ are handled easily because the integrand becomes (e.g., for $k = 1$)

$$\sin(2\pi t/T_0)e^{-j(2\pi/T_0)t} = -j\tfrac{1}{2} + je^{-j(4\pi/T_0)t}$$

and the two terms can be integrated directly.

EXERCISE 3.14: Starting from (3.46), derive the formula (3.47) for the Fourier series coefficients of the half-wave rectified sine. Exploit complex exponential simplifications such as $e^{j2\pi k} = 1$, $e^{j\pi} = -1$, and so on. Show how the $k = \pm 1$ cases can be treated separately.

3-6.4.1 Synthesis of a Half-Wave Rectified Sine from its Spectrum

Since the ideal full-wave rectified sine signal in Fig. 3-22 is a continuous signal, the quality of a finite Fourier sum approximation (3.31) is better than for the square wave which is discontinuous. Two cases are shown in Fig. 3-23, for $N = 2$ and 8. The fundamental frequency is equal to $f_0 = 50$ Hz. In the $N = 8$ case the approximation is very close to the original waveform, but small ripples are visible. Adding harmonics for $N > 8$ will improve the synthesis just a little bit by reducing the ripples. The $N = 2$ case exhibits noticeable ripples because it is only using DC and two sinusoidal terms. It is possible to explain these differences by examining the spectrum, which will show that the relative magnitudes of the higher frequency components being dropped for $k > 2$ are quite small. A plot of the spectrum for the half-wave rectified sine (similar to Fig. 3-16 for the square wave) is shown in Fig. 3-24 for $f_0 = 50$ Hz. The complex amplitudes given in (3.47) fall off very rapidly, and only the even-index harmonics are present for $k > 1$.

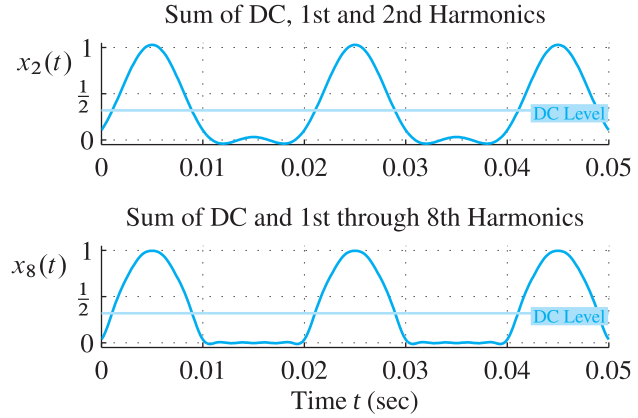

Figure 3-23: Summing harmonic components for the half-wave rectified sine signal via (3.31): DC, first and second harmonics (top panel); up to and including the eighth harmonic (bottom).

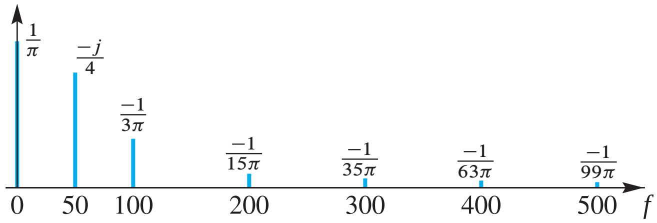

Figure 3-24: One-sided spectrum of a half-wave rectified sine signal up to the tenth harmonic, showing only the nonegative frequency region.

3-6.5 Convergence of Fourier Synthesis

We can think of the finite Fourier sum (3.31) as making an approximation to the true signal; i.e.,

$$x(t) \approx x_N(t) = \sum_{k=-N}^{N} a_k e^{j(2\pi/T_0)kt}$$

In fact, we anticipate that with enough complex exponentials the approximation will be perfect, i.e., $x_N(t) \to x(t)$, as $N \to \infty$. One way to assess the quality of the approximation is to define an error signal, $e_N(t) = x(t) - x_N(t)$, that measures the difference between the true signal and the synthesized one with N terms. Then we can quantify the error by measuring a feature of the error. For example, a commonly used feature is the maximum magnitude of the difference, which is called the *worst-case error*.

$$E_{\text{worst}} = \max_{t\in[0,T_0]} |x(t) - x_N(t)| \tag{3.48}$$

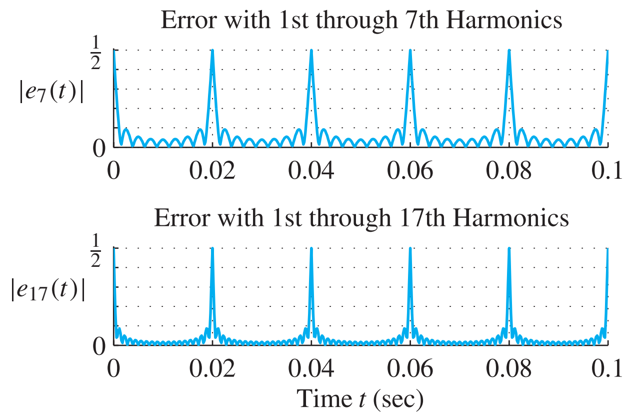

Figure 3-25: Worst-case error magnitude when approximating a square wave with a sum of harmonic components via (3.31): $N = 7$ (top panel) and $N = 17$ (bottom).

For the 50% duty cyle square wave, the worst-case error is shown in Fig. 3-25 for $N = 7$ and $N = 17$. Figure 3-25 was generated by using the Fourier coefficients from (3.34) in a MATLAB script to generate a plot of the worst-case error. Since the ideal square wave is discontinuous, the worst-case error is always half the size of the jump in the waveform right at the discontinuity point. The more interesting feature of the worst-case error comes from examining the size of the "ears" seen in Fig. 3-17. In Fig. 3-25, this *overshoot* error is found in

$e_7(t)$ at $t \approx 0.0023$ s, and is almost 0.1. In fact, it has been proven that the overshoot will be about 9% of the size of the discontinuity. The mathematical analysis of this worst-case error was first published by J. Willard Gibbs[10] in 1898, and is now referred to as the *Gibbs' Phenomenon* in Fourier Series.

If we perform a similar error analysis for the triangle wave, or the rectified sines, the worst-case error is well-behaved because these signals are continuous. The worst-case error will converge to zero as $N \to \infty$, i.e., there is no Gibbs' phenomenon. For the $N = 3$ and $N = 9$ approximations of the triangle wave a measurement of their worst-case errors can be done by making a zoomed version of Fig. 3-19. When these errors are measured, the result is 0.0497 for $N = 3$ and 0.0202 for $N = 9$. Since there is no discontinuity involved, these worst-case errors get smaller as N increases.

Worst-case error is not the only way to measure the difference; it only measures pointwise convergence of the approximation to the ideal signal. In fact, to prove convergence of the Fourier Series for signals such as the square wave, another error measure has been developed. This is the mean-square error where the measurable feature of the difference is an averaging integral of the squared error over the whole period, i.e.,

$$E_2 = \frac{1}{T_0}\int_0^{T_0} |x(t) - x_N(t)|^2 \, dt$$

Using the mean-square error measure, the theory of Fourier Series can be made mathematically rigorous.

EXERCISE 3.15: Carry out a measurement of the worst-case error for the full-wave rectified sine signal when approximating with N terms for $N = 3$ and $N = 9$.

3-6.6 Operations on Fourier Series

Scaling is a simple property because it should be obvious that multiplying a periodic signal by a scale factor, i.e., $\gamma x(t)$, will multiply all its Fourier Series coefficients by the same scale factor (γ)

$$\gamma x(t) = \sum_{k=-\infty}^{\infty} (\gamma a_k) e^{j2\pi f_0 k t}$$

Another simple operation is addition, but there are two cases: If the two periodic signals have the same fundamental frequency, then the Fourier coefficients are summed

$$x(t) + y(t) = \sum_{k=-\infty}^{\infty} (a_k + b_k) e^{j2\pi f_0 k t}$$

where the Fourier coefficients of $x(t)$ are $\{a_k\}$, and $\{b_k\}$ for $y(t)$. It the fundamental frequencies are different, the situation is much more complicated because sum signal might not even be periodic; if it is periodic, finding the new fundamental frequency involves a gcd (greatest common divisor) operation. Then using the new fundamental frequency all the Fourier coefficients must be re-indexed which makes adding the Fourier coefficients very tedious. One more thing, f_0 could change once again after the addition (see Exercise 3.17).

[10] Gibbs received the first American doctorate awarded in engineering from Yale in 1863.

Example 3-8: New Square Wave

Define a *bipolar* square wave with a 50% duty cycle as follows:

$$b(t) = \begin{cases} +1 & \text{for } 0 \le t < \frac{1}{2}T_0 \\ -1 & \text{for } \frac{1}{2}T_0 \le t < T_0 \end{cases}$$

This square can be related to the zero-one 50% duty cycle square via $b(t) = 2(x(t) - \frac{1}{2})$. Subtracting $\frac{1}{2}$ from $x(t)$ will change only the DC value, then multiplying by two will double the size of all the Fourier coefficients. Since the Fourier coefficients of the zero-one square wave are

$$a_k = \begin{cases} \frac{1}{2} & \text{for } k = 0 \\ \frac{-j}{k\pi} & \text{for } k \text{ odd} \\ 0 & \text{for } k = \pm 2, \pm 4, \ldots \end{cases}$$

the Fourier coefficients $\{b_k\}$ for $b(t)$ are

$$b_k = \begin{cases} 0 & \text{for } k = 0 \\ \frac{-2j}{k\pi} & \text{for } k \text{ odd} \\ 0 & \text{for } k = \pm 2, \pm 4, \ldots \end{cases}$$

3-6.7 Time-Shifting $x(t)$ Multiplies a_k by a Complex Exponential

If we form a new signal $y(t)$ by time shifting $x(t)$, then there is a simple change in the Fourier coefficients.

$$y(t) = x(t - \tau_d) \quad \longleftrightarrow \quad b_k = a_k e^{-j2\pi f_0 k \tau_d}$$

Because

$$y(t) = x(t - \tau_d) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi f_0 k(t-\tau_d)} = \sum_{k=-\infty}^{\infty} \underbrace{\left(a_k e^{-j2\pi f_0 k \tau_d}\right)}_{b_k} e^{j2\pi f_0 k t}$$

3-6.8 Derivative of $x(t)$ Multiplies a_k by $j\omega_0 k$

If we form a new signal $y(t)$ by taking the derivative of $x(t)$, then there is a simple change in the Fourier coefficients.

$$y(t) = \frac{d}{dt}x(t) \quad \longleftrightarrow \quad b_k = (j2\pi f_0 k)a_k$$

Since the derivative operator can be applied to each term in the sum, we get

$$y(t) = \frac{d}{dt}x(t) = \sum_{k=-\infty}^{\infty} a_k \frac{d}{dt} e^{j2\pi f_0 k t} = \sum_{k=-\infty}^{\infty} \underbrace{(j2\pi f_0 k)a_k}_{b_k} e^{j2\pi f_0 k t}$$

3-6.9 Multiply $x(t)$ by Sinusoid

If you multiply a periodic signal by a sinusoid, the Fourier Series coefficients will be changed in a simple predictable fashion. Here is the derivation for a periodic $x(t)$ multiplied by $\sin(2\pi f_0 t)$, where f_0 is the fundamental frequency of $x(t)$.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi f_0 k t} \tag{3.49a}$$

$$\sin(2\pi f_0 t) = \frac{1}{2j} e^{j2\pi f_0 t} + \frac{-1}{2j} e^{-j2\pi f_0 t} \tag{3.49b}$$

$$y(t) = x(t)\sin(2\pi f_0 t) = \left(\sum_{k=-\infty}^{\infty} a_k e^{j2\pi f_0 k t}\right)\left(\frac{1}{2j} e^{j2\pi f_0 t} + \frac{-1}{2j} e^{-j2\pi f_0 t}\right) \tag{3.49c}$$

The product signal, $y(t)$, will be periodic and it is quite likely that its fundamental frequency will also be f_0. Therefore, $y(t)$ has a Fourier Series with coefficients $\{b_k\}$.

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{j2\pi f_0 k t} \tag{3.49d}$$

$$\sum_{k=-\infty}^{\infty} b_k e^{j2\pi f_0 k t} = \sum_{k=-\infty}^{\infty} \frac{a_k}{2j} e^{j2\pi f_0 (k+1)t} + \sum_{k=-\infty}^{\infty} \frac{-a_k}{2j} e^{j2\pi f_0 (k-1)t} \tag{3.49e}$$

$$\sum_{k=-\infty}^{\infty} b_k e^{j2\pi f_0 k t} = \frac{1}{2j}\left(\sum_{\ell=-\infty}^{\infty} a_{\ell-1} e^{j2\pi f_0 (\ell)t} - \sum_{\ell=-\infty}^{\infty} a_{\ell+1} e^{j2\pi f_0 (\ell)t}\right) \tag{3.49f}$$

Finally, we see that the Fourier Series coefficients $\{b_k\}$ for $y(t)$ are

$$b_k = \frac{a_{k-1} - a_{k+1}}{2j} \qquad \text{for all } k \tag{3.50}$$

Example 3-9: Rectified Sinusoid

The product of a zero-one square wave $s_z(t)$ and a sinusoid

$$y(t) = s_z(t)\sin(2\pi f_0 t)$$

is a rectified sinusoid. When the frequency of the sinusoid equals the fundamental frequency of the square wave, make a plot of $y(t)$. Determine whether the resulting plot is a half-wave or full-wave rectified sine. ■

EXERCISE 3.16: Find the Fourier Series coefficients of the half-wave rectified sine signal by using the fact that the signal can be expressed as a product of a square wave and a sinusoid.

EXERCISE 3.17: If a periodic signal has Fourier Series coefficients where all the odd-indexed ones are zero, explain why the fundamental frequency has been chosen incorrectly. Determine the correct fundamental frequency and also redefine the Fourier coefficients for the new f_0.

EXERCISE 3.18: Find the Fourier Series coefficients of the full-wave rectified sine signal by using the fact that the signal can be expressed as the sum of the half-wave rectified sine plus a shifted version of the half-wave rectified sine, i.e.,

$$x_{\mathrm{f}}(t) = x_{\mathrm{h}}(t) + x_{\mathrm{h}}(t - T/2) \tag{3.51}$$

where T is the period of the half-wave rectified sine.

EXERCISE 3.19: Find the Fourier Series coefficients of the full-wave rectified sine signal by using the fact that the signal can be expressed as the product of a bipolar square wave and a sinusoid.

3-6.10 Parseval's Theorem

One reason that the Fourier series is so useful is that there is an equivalence between the sum of the squares of the Fourier coefficients and the signal power. In effect, a spectrum plot of the $\{|a_k|^2\}$ coefficients can be interpreted as the distribution of signal power versus frequency. Therefore, when we truncate the Fourier series and synthesize a signal from a finite Fourier sum like (3.31), we can estimate the approximation error by summing up the power in the coefficients being dropped.

The *average power* of the periodic signal $x(t)$ is the total energy of one period of $x(t)$ divided by the duration of the period.

$$E = \frac{1}{T_0}\int_0^{T_0} |x(t)|^2 dt. \tag{3.52}$$

The average power is a convenient measure of the size (or strength) of the signal.

Example 3-10: Average Power of Sinusoid

For the case where $x(t) = A\cos((2\pi/T_0)t + \varphi)$, the average power integral

$$E = \frac{1}{T_0}\int_0^{T_0} |A\cos(2\pi t/T_0 + \varphi)|^2 dt$$

can be evaluated directly. Although it is possible to use a trigonometric identity, we will use Euler's formula followed by the complex number identity $|z+w|^2 = |z|^2 + 2\Re\{zw^*\} + |w|^2$.

$$\begin{aligned}
E_T &= \frac{1}{T_0}\int_0^{T_0} \left|\tfrac{1}{2}Ae^{j((2\pi/T_0)t+\varphi)} + \tfrac{1}{2}Ae^{-j((2\pi/T_0)t+\varphi)}\right|^2 dt \\
&= \frac{1}{T_0}\int_0^{T_0} |\tfrac{1}{2}A|^2 + 2\Re\left\{\tfrac{1}{2}Ae^{j((2\pi/T_0)t+\varphi)}\tfrac{1}{2}Ae^{j((2\pi/T_0)t+\varphi)}\right\} + |\tfrac{1}{2}A|^2 dt \\
&= \frac{1}{T_0}\int_0^{T_0} \tfrac{1}{4}A^2 + 2\Re\left\{\tfrac{1}{4}A^2 e^{j((4\pi/T_0)t+2\varphi)}\right\} + \tfrac{1}{4}A^2 dt \\
&= \frac{1}{T_0}\tfrac{1}{2}A^2 T_0 + \frac{1}{T_0}\int_0^{T_0} \tfrac{1}{2}A^2\cos((4\pi/T_0)t + 2\varphi)dt \\
&= \tfrac{1}{2}A^2
\end{aligned}$$

Note that the integral of the cosine term is zero, being an integral over two complete periods. ■

The result in Example 3-10 can be related to the Fourier series coefficients if we note that Euler's formula is, in fact, the Fourier series of the sinusoid.

$$A\cos(2\pi t/T_0 + \varphi) = \tfrac{1}{2}Ae^{j\varphi}e^{j2\pi t/T_0} + \tfrac{1}{2}Ae^{-j\varphi}e^{-j2\pi t/T_0} \tag{3.53}$$

The Fourier series has only two non-zero coefficients, a_1 and a_{-1}, so an alternate way of evaluating the average power is:

$$\sum_{k=-\infty}^{\infty} |a_k|^2 = |a_1|^2 + |a_{-1}|^2 = |\tfrac{1}{2}Ae^{j\varphi}|^2 + |\tfrac{1}{2}Ae^{-j\varphi}|^2 = \tfrac{1}{2}A^2 \tag{3.54}$$

Amazingly, this result generalizes to the case where the Fourier series has more than two coefficients. The somewhat tedious proof uses the orthogonality property of complex exponential signals. The general result, known as Parseval's Theorem is

Parseval's Theorem

$$\frac{1}{T_0}\int_0^{T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2 \tag{3.55}$$

EXERCISE 3.20: Use Parseval's Theorem to complete the proof the famous formula:

$$\frac{\pi^2}{M} = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

by finding the numerical value of the integer M. *Hint*: use the square wave and its Fourier series coefficients.
