STANFORD UNIVERSITY
EE 102B Spring-2013

Lecture 24

System Function, Poles and Zeros, Implementation Structures – What can we learn from H(z)?

May 27, 2013

ASSIGNMENTS

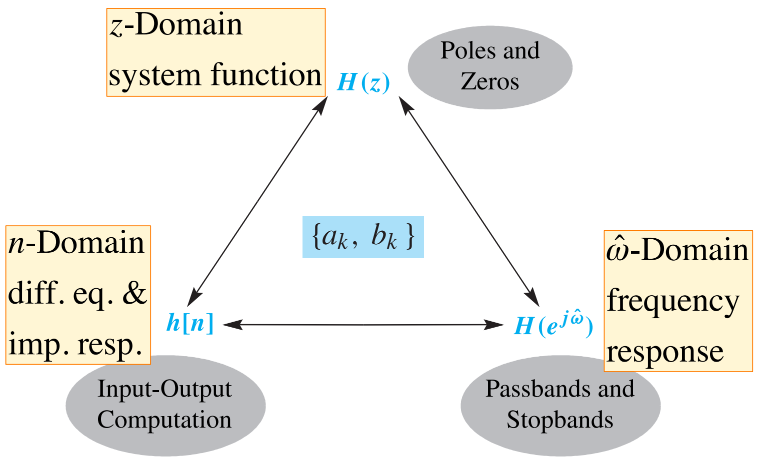
- Reading for this Lecture:
  - SPF: Chapters 7 and new notes on Chapter 8
  - S&S: Chapter 10
  - HW#08 is due by 5pm today, May 29, in Packard 263.
  - Lab #07 is due by 5pm, Friday, May 31, in Packard 263.
  - HW#09 is due by 5pm, Wednesday, June 5, in Packard 263. It is OPTIONAL to hand it in, but material on it will be covered on the final exam.

Office Hours for Course Staff – Come see us.

- Ron Schafer: Mon. and Weds. 2:00-4:00 pm, Packard 211.
- Dookun Park: Tues. 1:30 ~ 3:30 pm in Packard 107
- Ruo Yu Gu (Roy): Mon. 4:00 ~ 6:00 pm in Packard 106

LTI SYSTEMS AND THE Z-TRANSFORM

The Three Domains Again



Causal LTI Systems – 1

$x[n] \rightarrow$ **LTI System** $\rightarrow y[n] = x[n] * h[n]$

$\delta[n] \rightarrow$ **LTI System** $\rightarrow h[n]$

$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

With initial rest conditions

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}} = b_0 \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

Causal LTI Systems – II
Impulse response of DE

$$H(z) = \underbrace{\left[\sum_{r=0}^{(M-N)} B_r z^{-r} \right]}_{\text{if } M \geq N} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

ROC: $r_R = \max_k \{d_k\} < |z|$

$$h[n] = \underbrace{\left[\sum_{r=0}^{(M-N)} B_r \delta[n-r] \right]}_{\text{if } M \geq N} + \sum_{k=1}^{N} A_k d_k^n u[n]$$

Stability requires: $r_R = \max_k \{d_k\} < 1$

Frequency Response of a DE

$$y[n] - \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}} = \frac{\left(\sum_{k=0}^{M} b_k e^{-j\hat{\omega}k} \right)}{\left(1 - \sum_{k=1}^{N} a_k e^{-j\hat{\omega}k} \right)}$$

ROC must Contain the Unit circle

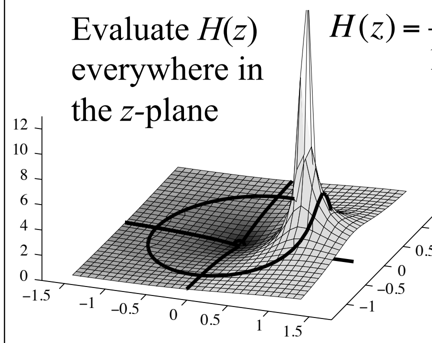
ROC for causal system: $\max_k \{d_k\} < |z|$ $\Longrightarrow$ Stability requires $\max_k \{d_k\} < 1$ for causal system

POLES AND ZEROS

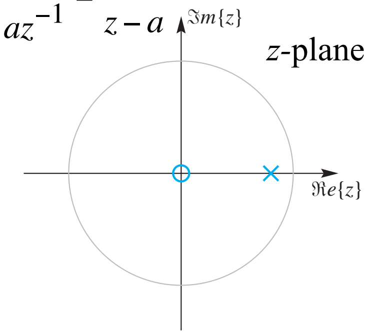
The z-Plane: Pole and Zero

Evaluate $H(z)$ everywhere in the z-plane

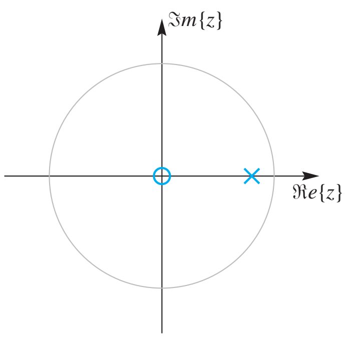
$$H(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

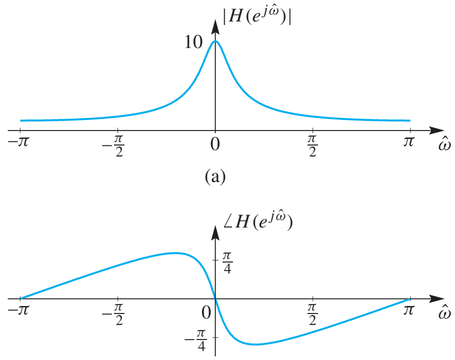


z-plane



Z-Plane and Frequency Response



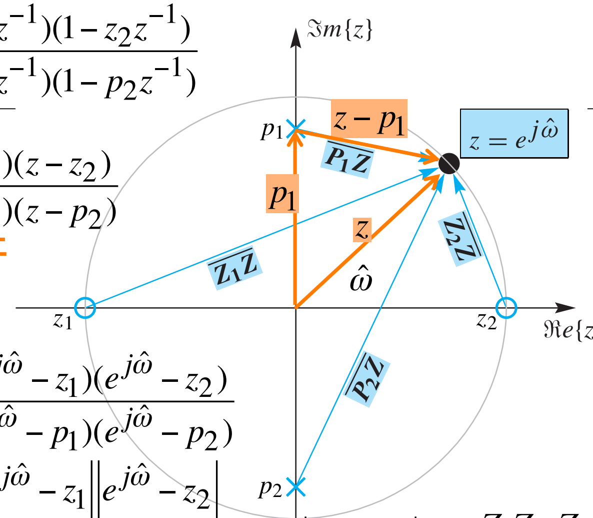


(a)

(b)

$$H(z) = G\frac{(1 - z_1 z^{-1})(1 - z_2 z^{-1})}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})}$$

$$H(z) = G\frac{(z - z_1)(z - z_2)}{(z - p_1)(z - p_2)}$$



$$H(e^{j\hat{\omega}}) = G\frac{(e^{j\hat{\omega}} - z_1)(e^{j\hat{\omega}} - z_2)}{(e^{j\hat{\omega}} - p_1)(e^{j\hat{\omega}} - p_2)}$$

$$\left|H(e^{j\hat{\omega}})\right| = G\frac{\left|e^{j\hat{\omega}} - z_1\right|\left|e^{j\hat{\omega}} - z_2\right|}{\left|e^{j\hat{\omega}} - p_1\right|\left|e^{j\hat{\omega}} - p_2\right|}$$

$$\left|H(e^{j\hat{\omega}})\right| = G\frac{Z_1Z \bullet Z_2Z}{P_1Z \bullet P_2Z}$$

PeZ DEMO

9^{th}-order Running Sum Filter

$$H(z) = \sum_{n=0}^{9} z^{-n} = \frac{1 - z^{-10}}{1 - z^{-1}}$$

Zeros: $1 - z^{-10} = 0$

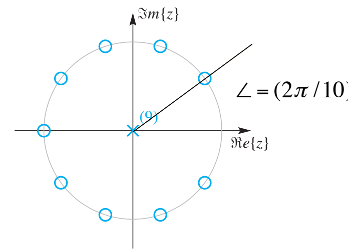
$z^{10} = 1 \Rightarrow z_k = e^{j(2\pi/10)k}$

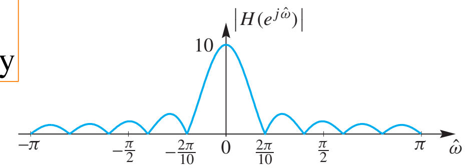
$k = 0, 1, \ldots, 9;$

i.e., the 10^{th} roots of unity

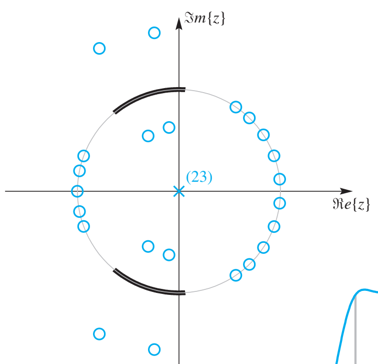
Pole: $1 - z^{-1} = 0$

$z = 1$ cancels z_0

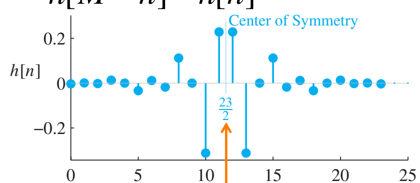




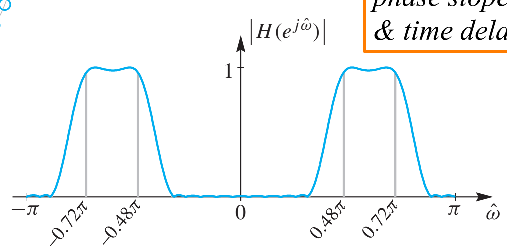
FIR Bandpass Filter (*M* = 23) with Linear Phase



$$h[M - n] = h[n]$$

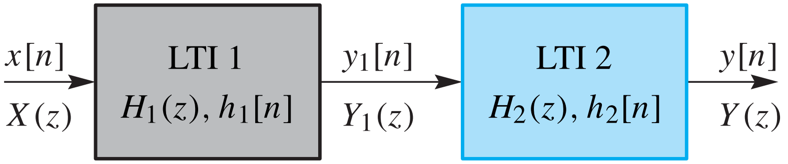


phase slope & time delay

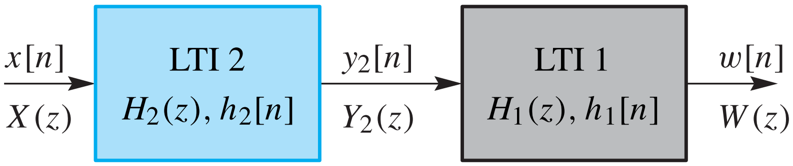


CASCADED LTI SYSTEMS

Cascaded LTI Systems

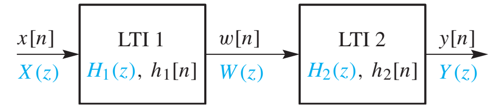


$$\boxed{Y(z) = H_1(z)H_2(z)X(z) = H_2(z)H_1(z)X(z)}$$



Inverse for Causal and Stable LTI Systems

- Inverse system undoes the effect of an LTI system



- Determine $H_2(z)$ such that y[n] = x[n]

$$\boxed{Y(z) = H_1(z)H_2(z)X(z) = X(z)}$$

$$H_2(z) = \frac{1}{H_1(z)} \quad \text{ROC}_2 \text{ must overlap ROC}_1$$

For stability, ROC_2 must also include the unit circle.

$$\boxed{\text{Therefore, for a causal and stable } H_2(z) \text{, the zeros of } H_1(z) \text{ must be inside the unit circle.}}$$

Minimum-Phase Systems

- A causal minimum-phase system has all its poles and zeros inside the unit circle.
- This implies that a causal and stable inverse system exists.

Inverse System - Example

$$H_1(z) = \frac{1-2z^{-1}}{1-0.5z^{-1}} \quad \text{ROC}_1 = \{z : 0.5 < |z|\} \Rightarrow \text{stable system}$$

$$H_2(z) = \frac{1}{H_1(z)} = \frac{1-0.5z^{-1}}{1-2z^{-1}} \quad \text{ROC}_2 = \{z : 2 < |z|\} \text{ for causality}$$

$$H_2(z) = 0.25 + \frac{0.75}{1-2z^{-1}} \quad \text{ROC}_2 = \{z : 2 < |z|\} \text{ for causality}$$

$$h_2[n] = 0.25\delta[n] + 0.75(2)^n u[n] \quad \text{causal (unstable) inverse}$$

$$H_2(z) = \frac{1-0.5z^{-1}}{1-2z^{-1}} \quad \text{ROC}_2 = \{z : |z| < 2\} \text{ not causal}$$

$$h_2[n] = 0.25\delta[n] - 0.75(2)^n u[-n-1] \quad \text{non-causal (stable) inverse}$$

Allpass Systems

- Allpass systems have constant gain with varying phase shift.

$$H(e^{j\hat{\omega}}) = Ge^{j\angle H(e^{j\hat{\omega}})}$$

$$(\text{with } \angle H(e^{j\hat{\omega}}) < 0, \ 0 < \hat{\omega} < \pi)$$

- First-order allpass system

$$H(z) = \frac{z^{-1} - a}{1 - az^{-1}} = -a\frac{1 - a^{-1}z^{-1}}{1 - az^{-1}} = z^{-1}\frac{1 - az}{1 - az^{-1}}$$

- Pole at $z = a$ and zero at $z = a^{-1}$

Allpass Systems

- First-order allpass system

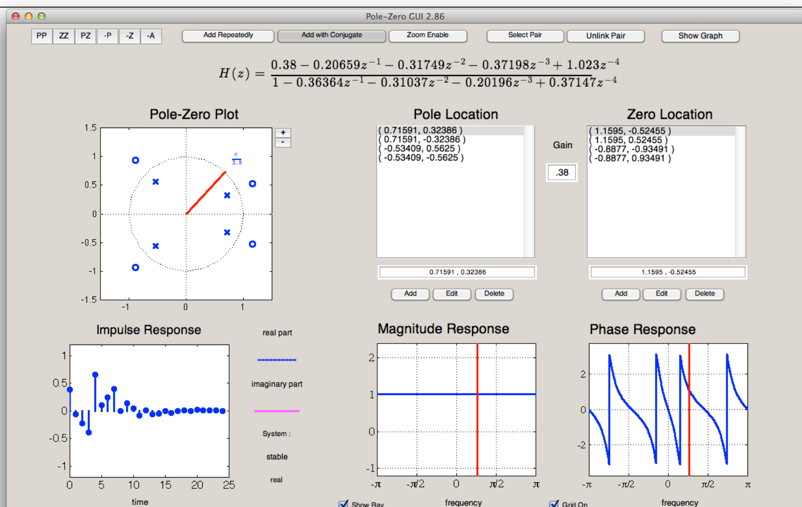
$$H(z) = \frac{z^{-1} - a}{1 - az^{-1}} = -a\frac{1 - a^{-1}z^{-1}}{1 - az^{-1}} = z^{-1}\frac{1 - az}{1 - az^{-1}}$$

$$\left|H(e^{j\hat{\omega}})\right|^2 = H(e^{j\hat{\omega}})H^*(e^{j\hat{\omega}})$$

$$\left|H(e^{j\hat{\omega}})\right|^2 = \left(\frac{e^{-j\hat{\omega}} - a}{1 - ae^{-j\hat{\omega}}}\right)\left(\frac{e^{j\hat{\omega}} - a}{1 - ae^{j\hat{\omega}}}\right)$$

$$= \frac{1 - ae^{j\hat{\omega}} - ae^{-j\hat{\omega}} - a^2}{1 - ae^{-j\hat{\omega}} - ae^{j\hat{\omega}} - a^2} = 1$$

Allpass in PeZ – Poles and Zeros in Reciprocal Pairs



A Family of Systems

- Consider a system function

$$H(z) = H_{\min}(z)H_{\text{ap}}(z)$$

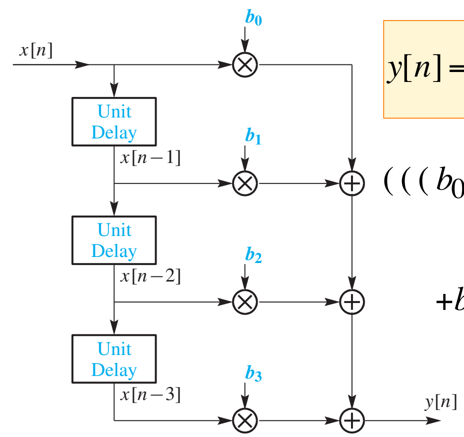
- the product of a minimum-phase and an allpass system function.
- Frequency response

$$\left|H(e^{j\hat{\omega}})\right| = \left|H_{\min}(e^{j\hat{\omega}})\right|\left|H_{\text{ap}}(e^{j\hat{\omega}})\right| = \left|H_{\min}(e^{j\hat{\omega}})\right|$$

$$-\angle H(e^{j\hat{\omega}}) = -\angle H_{\min}(e^{j\hat{\omega}}) - \angle H_{\text{ap}}(e^{j\hat{\omega}}) \quad \text{(phase lag)}$$

IMPLEMENTATION STRUCTURES FOR LTI SYSTEMS

FIR Direct Form



$$y[n] = \sum_{k=0}^{M} b_k x[n-k] \qquad (M=3)$$

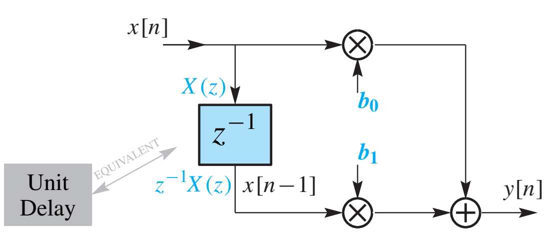
$$(((\, b_0 x[n] + b_1 x[n-1]\,)$$

$$+ b_2 x[n-2]\,)$$

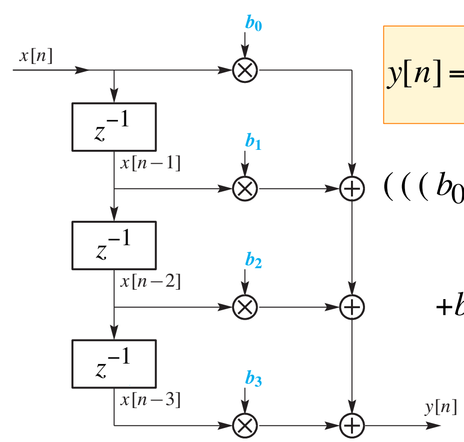
$$+ b_3 x[n-3]\,) = y[n]$$

Representing Delay in LTI Structure Diagrams

- One unit of delay: $x[n-1] \Leftrightarrow z^{-1}X(z)$
- Therefore, we can indicate delay by z^{-1}
- Example: $y[n] = b_0 x[n] + b_1 x[n-1]$



FIR Direct Form



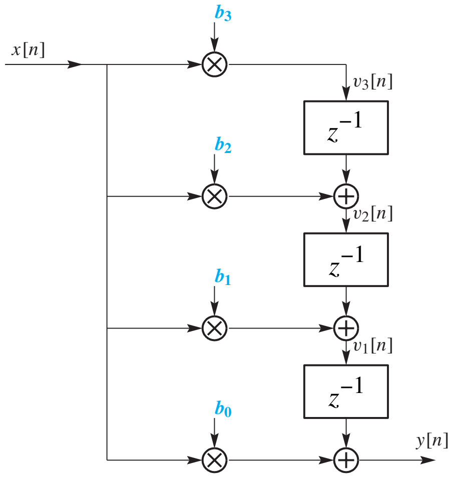
$$y[n] = \sum_{k=0}^{M} b_k x[n-k] \qquad (M=3)$$

$$(((\, b_0 x[n] + b_1 x[n-1]\,)$$

$$+ b_2 x[n-2]\,)$$

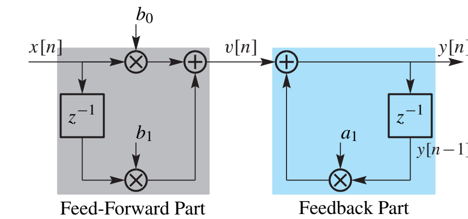
$$+ b_3 x[n-3]\,) = y[n]$$

FIR Transposed Direct Form



Direct Form – I First-Order IIR Structures

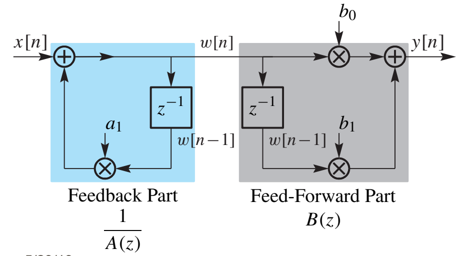
$$H(z)=\frac{b_0+b_1 z^{-1}}{1-a_1 z^{-1}}$$



$$H(z)=(b_0+b_1 z^{-1})\frac{1}{(1-a_1 z^{-1})}$$

$$v[n]=b_0 x[n]+b_1 x[n-1]$$

$$y[n]=a_1 y[n-1]+v[n]$$



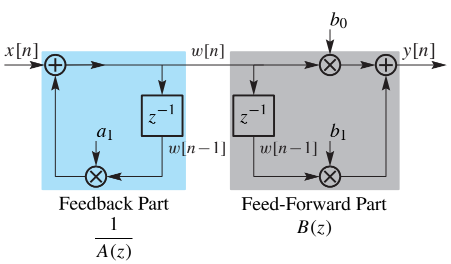
$$H(z)=\frac{1}{(1-a_1 z^{-1})}(b_0+b_1 z^{-1})$$

$$w[n]=a_1 w[n-1]+x[n]$$

$$y[n]=b_0 w[n]+b_1 w[n-1]$$

Direct Form – II First-Order IIR Structure

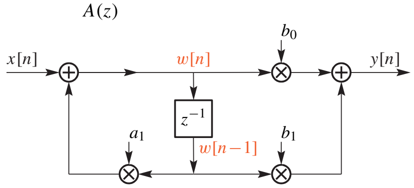
$$H(z)=\frac{b_0+b_1 z^{-1}}{1-a_1 z^{-1}}$$



$$H(z)=\frac{1}{(1-a_1 z^{-1})}(b_0+b_1 z^{-1})$$

$$w[n]=a_1 w[n-1]+x[n]$$

$$y[n]=b_0 w[n]+b_1 w[n-1]$$

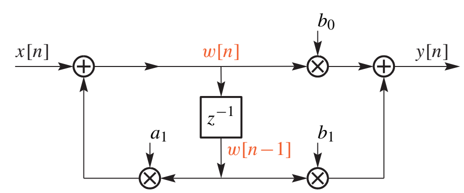


$$w[n]=a_1 w[n-1]+x[n]$$

$$y[n]=b_0 w[n]+b_1 w[n-1]$$

Direct Form – II First-Order IIR Structure

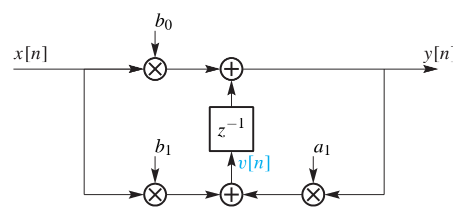
$$H(z)=\frac{b_0+b_1 z^{-1}}{1-a_1 z^{-1}}$$



$$H(z)=\frac{(b_0+b_1 z^{-1})}{(1-a_1 z^{-1})}$$

$$w[n]=a_1 w[n-1]+x[n]$$

$$y[n]=b_0 w[n]+b_1 w[n-1]$$

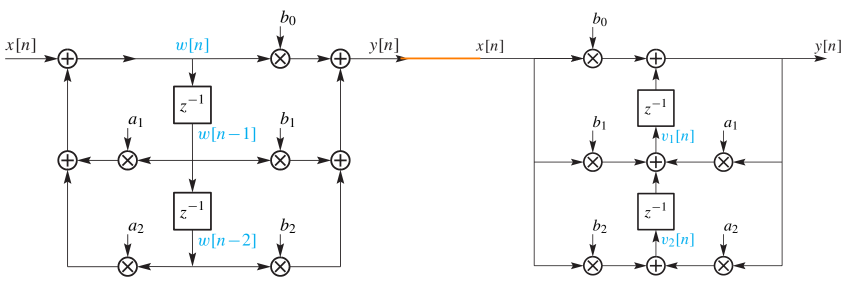


$$v[n]=a_1 y[n-1]+b_1 x[n]$$

$$y[n]=b_0 x[n]+v[n-1]$$

Transposed Direct Form II

Second-Order Direct Form Structures



Direct Form II *Transposed Direct Form II*

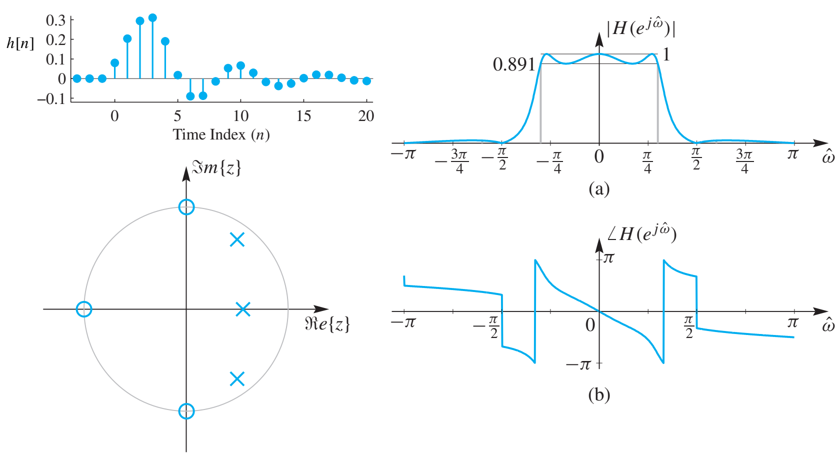
Third-Order Elliptic Filter Example

- See discussion on p. 236 of SP-First

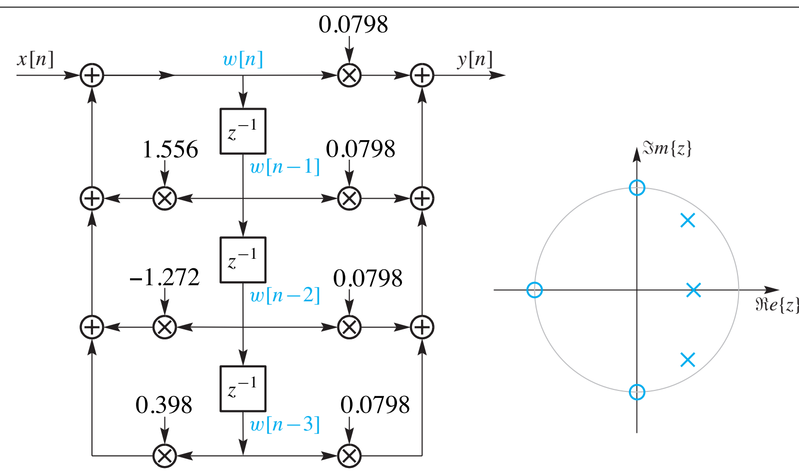
$$H(z)=\frac{0.0798(1+z^{-1}+z^{-2}+z^{-3})}{1-1.556z^{-1}+1.272z^{-2}-0.398z^{-3}}$$

- Write the difference equation that is satisfied by y[n] and x[n].

Elliptic Lowpass Filter



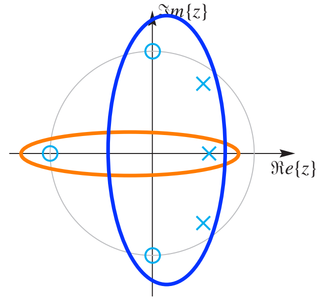
$$H(z)=\frac{0.0798(1+z^{-1}+z^{-2}+z^{-3})}{1-1.556z^{-1}+1.272z^{-2}-0.398z^{-3}}$$



$$H(z)=\frac{0.0798(1+z^{-1}+z^{-2}+z^{-3})}{1-1.556z^{-1}+1.272z^{-2}-0.398z^{-3})}$$

- Cascade form (factor num & denom)

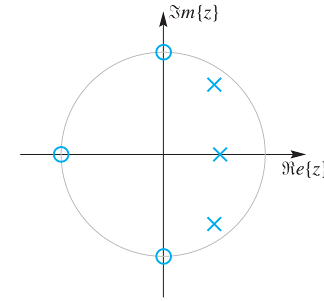
$$H(z)=\underbrace{\left(\frac{0.0798(1+z^{-1})}{1-0.556z^{-1}}\right)}\underbrace{\left(\frac{1+z^{-2}}{1-0.9945z^{-1}+0.7157z^{-2}}\right)}$$



$$H(z)=\frac{0.0798(1+z^{-1}+z^{-2}+z^{-3})}{1-1.556z^{-1}+1.272z^{-2}-0.398z^{-3}}$$

- Parallel form (partial fraction expansion)

$$H(z)=-0.2+\frac{0.62}{1-0.556z^{-1}}+\frac{0.17e^{j0.96\pi}}{1-0.846e^{j0.3\pi}z^{-1}}+\frac{0.17e^{-j0.96\pi}}{1-0.846e^{-j0.3\pi}z^{-1}}$$

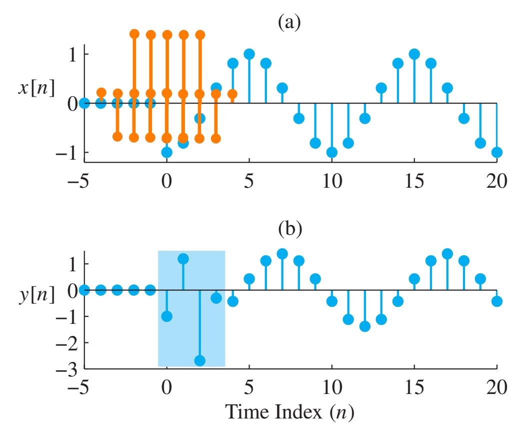


$$H(z)=-0.2+\frac{0.62}{1-0.556z^{-1}}+\frac{-0.1687-0.1386z^{-1}}{1-0.9945z^{-1}+0.7157z^{-2}}$$

STEADY-STATE RESPONSE TO A SINE WAVE

Steady-State Response - FIR

$$\begin{aligned}y[n]=x[n]&-2x[n-1]\\&+4x[n-2]\\&-2x[n-3]\\&+x[n-4]\end{aligned}$$



Steady-State Response – IIR

$$h[n] = 5(-0.8)^n u[n] \qquad x[n] = e^{j0.2\pi n}u[n]$$

$$y[n] = \sum_{m=0}^{\infty} h[m]x[n-m]$$

$$x[n] = e^{j\hat{\omega}n}u[n]$$

$$y[n] = \sum_{m=0}^{n} h[m]e^{j\hat{\omega}(n-m)}$$

$$y[n] = \left(\sum_{m=0}^{\infty} h[m]e^{-j\hat{\omega}m}\right)e^{j\hat{\omega}n} - \underbrace{\left(\sum_{m=n+1}^{\infty} h[m]e^{-j\hat{\omega}m}\right)e^{j\hat{\omega}n}}_{0}$$