Stanford University
Winter 2012-2013

Signal Processing and Linear Systems I

Lecture 5: Time Domain Analysis of Continuous Time Systems

January 14, 2013

Time Domain Analysis of Continuous Time Systems

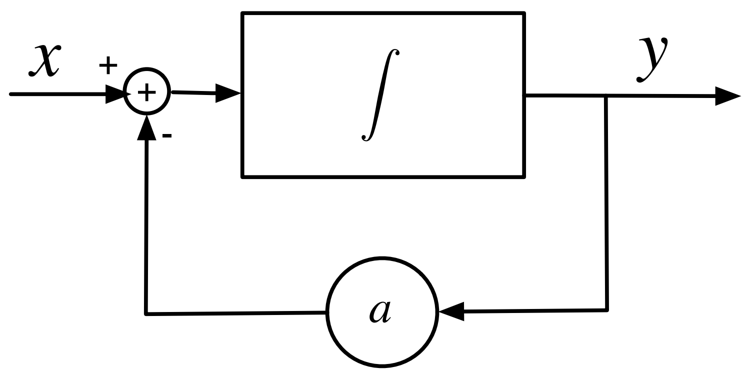
Today's topics

- Zero-input and zero-state responses of a system
- Impulse response
- Extended linearity
- Response of a linear time-invariant (LTI) system
- Superposition integral
- Convolution

System Equation

The *System Equation* relates the outputs of a system to its inputs.

Example from last time: the system described by the block diagram



has a system equation

$$y' + ay = x.$$

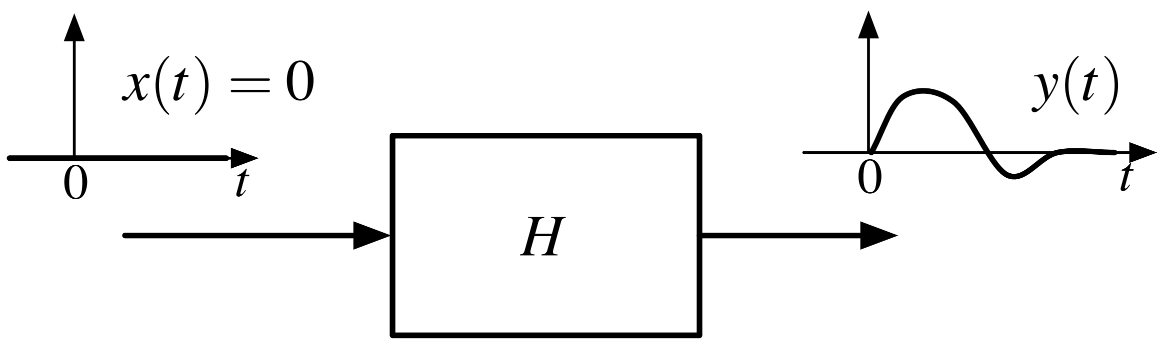
In addition, the initial conditions must be given to uniquely specifiy a solution.

Solutions for the System Equation

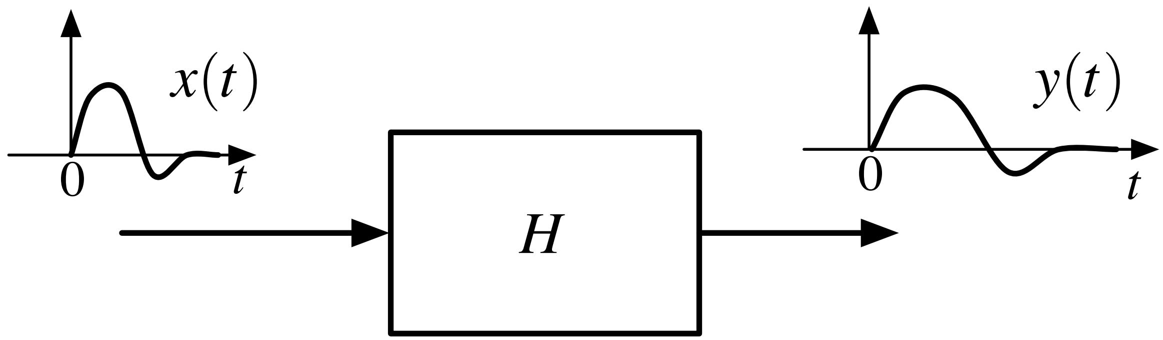
Solving the system equation tells us the output for a given input.

The output consists of two components:

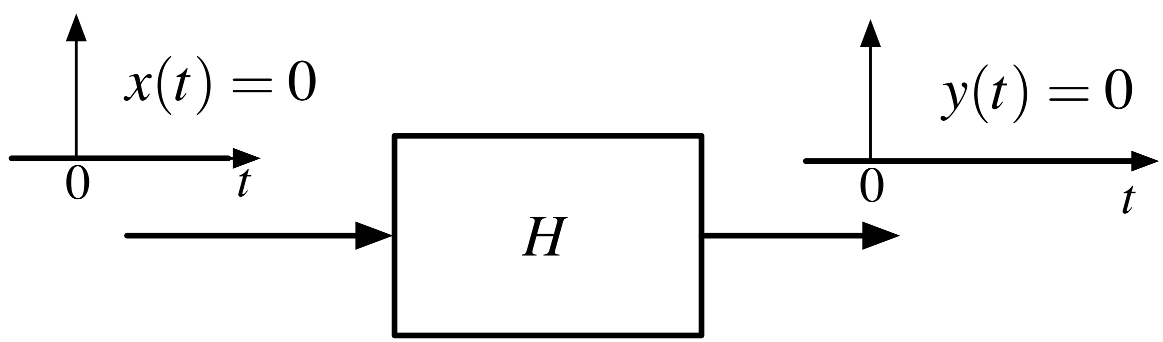
- The *zero-input* response, which is what the system does with no input at all. This is due to initial conditions, such as energy stored in capacitors and inductors.



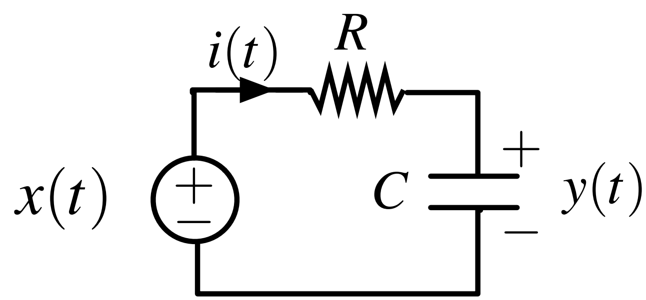
- The *zero-state* response, which is the output of the system with all initial conditions zero.



If H is a linear system, its zero-input response is zero. Homogeneity states if $y = F(ax)$, then $y = aF(x)$. If $a = 0$ then a zero input requires a zero output.



Example: Solve for the voltage across the capacitor $y(t)$ for an arbitrary input voltage $x(t)$, given an initial value $y(0) = Y_0$.



From Kirchhoff's voltage law

$$x(t) = Ri(t) + y(t)$$

Using $i(t) = Cy'(t)$

$$RCy'(t) + y(t) = x(t).$$

This is a first order LCCODE, which is linear with zero initial conditions. First we solve for the homogeneous solution by setting the right side (the input) to zero

$$RCy'(t) + y(t) = 0.$$

The solution to this is

$$y(t) = Ae^{-t/RC}$$

which can be verified by direct substitution.

To solve for the total response, we let the undetermined coefficient be a function of time

$$y(t) = A(t)e^{-t/RC}.$$

Substituting this into the differential equation

$$RC\left[A'(t)e^{-t/RC} - \frac{1}{RC}A(t)e^{-t/RC}\right] + A(t)e^{-t/RC} = x(t)$$

Simplying

$$A'(t) = x(t)\left[\frac{1}{RC}e^{t/RC}\right]$$

which can be integrated from $t = 0$ to get

$$A(t) = \int_0^t x(\tau)\left[\frac{1}{RC}e^{\tau/RC}\right]d\tau + A(0)$$

Then

$$
\begin{aligned}
y(t) &= A(t)e^{-t/RC} \\
&= e^{-t/RC}\int_0^t x(\tau)\left[\frac{1}{RC}e^{\tau/RC}\right]d\tau + A(0)e^{-t/RC} \\
&= \int_0^t x(\tau)\left[\frac{1}{RC}e^{-(t-\tau)/RC}\right]d\tau + A(0)e^{-t/RC}
\end{aligned}
$$

At $t = 0$, $y(0) = Y_0$, so this gives $A(0) = Y_0$

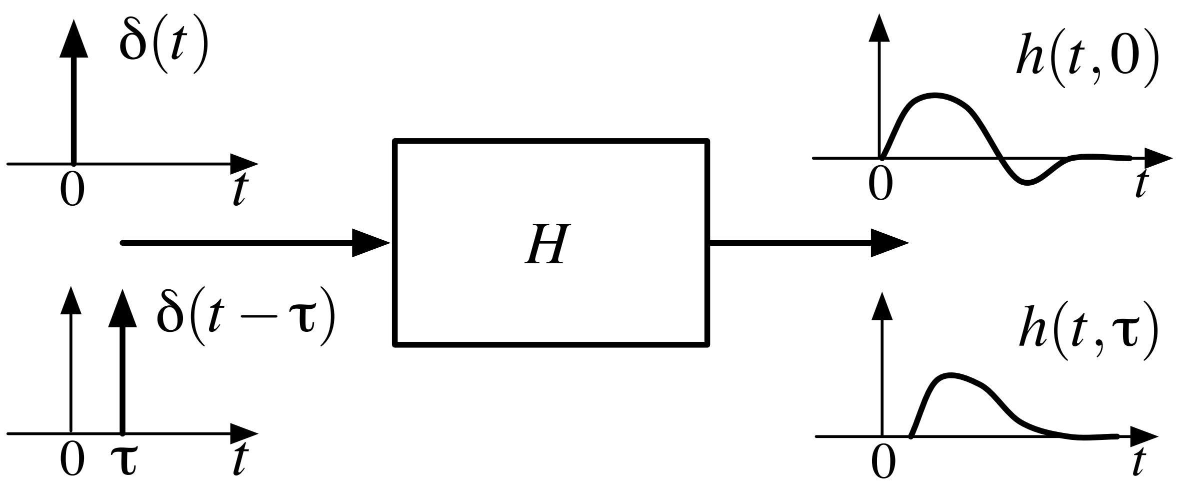
$$
y(t) = \underbrace{\int_0^t x(\tau)\left[\frac{1}{RC}e^{-(t-\tau)/RC}\right]d\tau}_{\text{zero}-\text{state response}} + \underbrace{Y_0 e^{-t/RC}}_{\text{zero}-\text{input response}} .
$$

Impulse Response

The *impulse response* of a linear system $h(t,\tau)$ is the output of the system at time t to an impulse at time τ. This can be written as

$$h(t,\tau) = H(\delta(t-\tau))$$

Care is required in interpreting this expression!



Note: Be aware of potential confusion here:

When you write

$$h(t,\tau) = H(\delta(t-\tau))$$

the variable t serves different roles on each side of the equation.

- t on the left is a specific value for time, the time at which the output is being sampled.
- t on the right is varying over all real numbers, it is not the same t as on the left.
- The output at time specific time t on the left in general depends on the input at all times t on the right (the entire input waveform).

  Alternative notation: Let $\delta_\tau = \{\delta(t-\tau);\, t \in (-\infty,\infty)\}$ denote the entire signal. Given a signal x, let $H_t(x)$ denote the output of system H at time t when the input is x. Then $h(t,\tau) = H_t(\delta_\tau)$ is unambiguous.

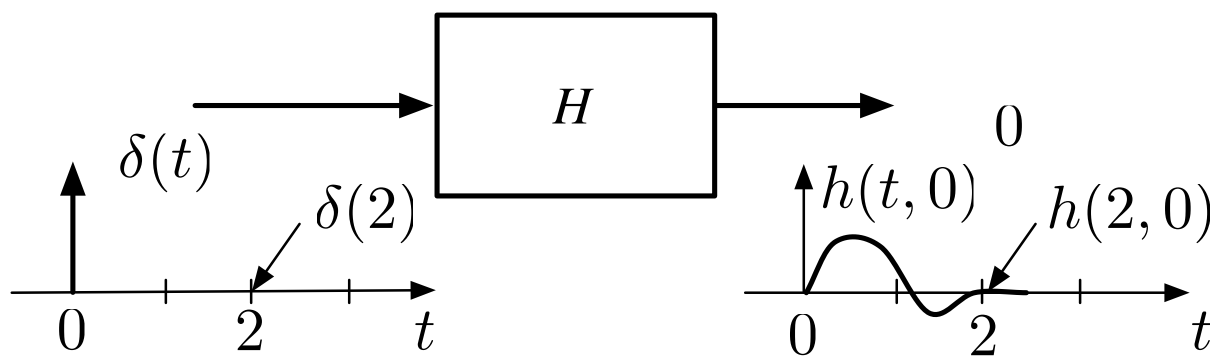
- Assume the input impulse is at $\tau = 0$,

$$h(t, 0) = H(\delta(t)).$$

We want to know the impulse response at time $t = 2$. It doesn't make any sense to set $t = 2$, and write

$$h(2, 0) = H(\delta(2)) \qquad\qquad \Leftarrow \text{No!}$$

First, $\delta(2)$ is something like zero, so $H(0)$ would be zero. Second, the value of $h(2, 0)$ depends on the entire input waveform, not just the value at $t = 2$.



- Compare to an equation such as

$$y'(t) + 2y(t) = x(t)$$

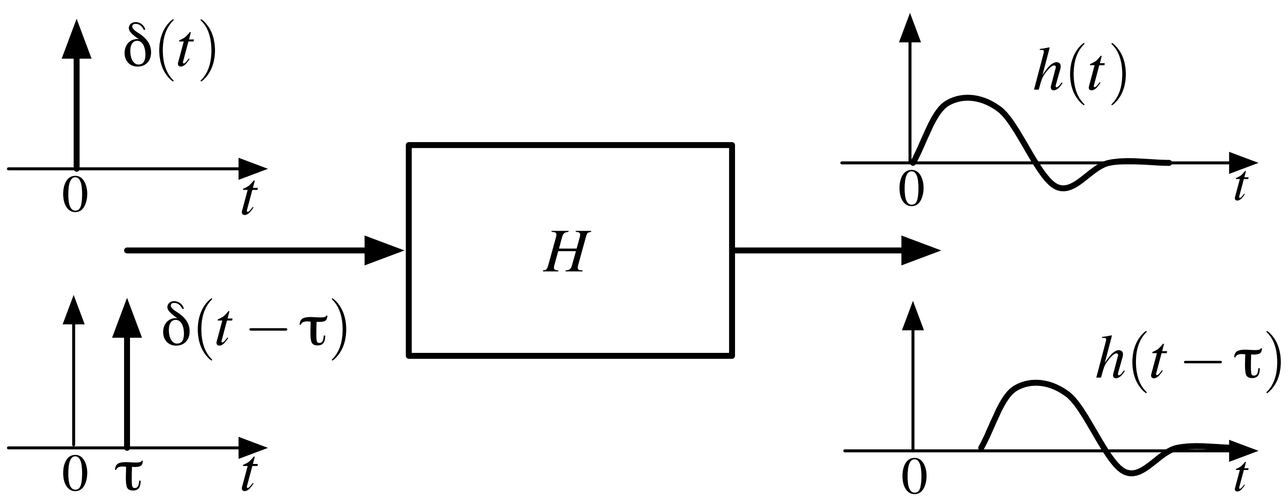
which holds for each t, so that $y'(1) + 2y(1) = x(1)$.

If H is time invariant, delaying the input and output both by a time τ should produce the same response

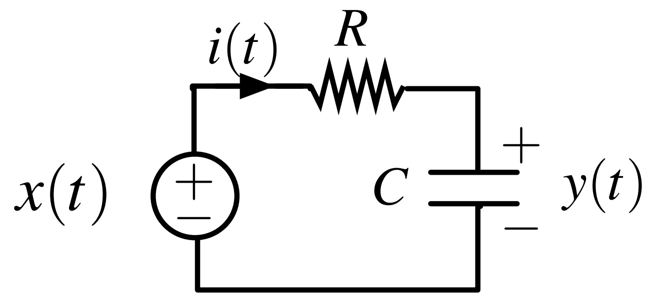
$$h(t, \tau) = h(t - \tau, \tau - \tau) = h(t - \tau, 0).$$

Hence h is only a function of $t - \tau$. We suppress the second argument, and define the impulse response of a *linear time-invariant* (LTI) system H to be

$$h(t) = H(\delta(t))$$



RC Circuit example



The solution for an input $x(t)$ and initial $y(0) = Y_0$ is

$$y(t) = \int_0^t x(\tau) \left[\frac{1}{RC} e^{-(t-\tau)/RC} \right] d\tau + Y_0 e^{-t/RC}$$

The zero-state response is ($Y_0 = 0$) is

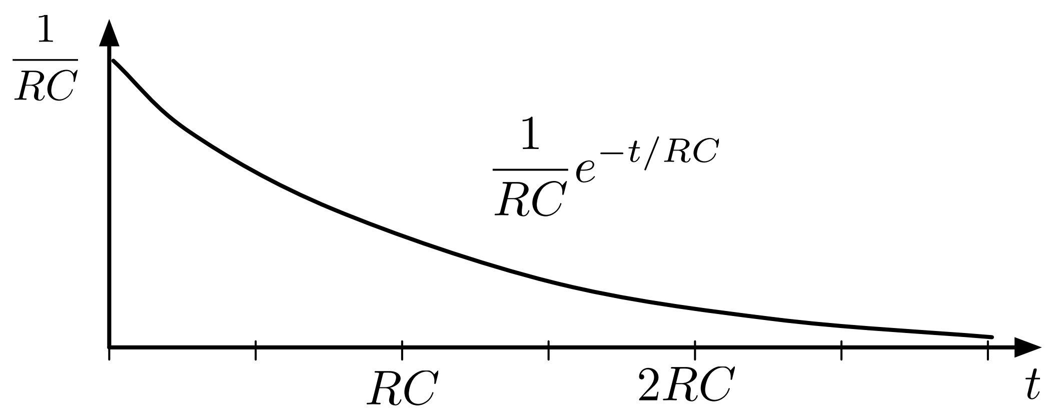
$$y(t) = \int_0^t x(\tau) \left[\frac{1}{RC} e^{-(t-\tau)/RC} \right] d\tau$$

The impulse response is then

$$
\begin{aligned}
h(t) &= \int_{0-}^{t} \delta(\tau)\left[\frac{1}{RC}e^{-(t-\tau)/RC}\right] d\tau \\
&= \frac{1}{RC}e^{-t/RC}
\end{aligned}
$$

for $t \geq 0$, and zero otherwise. We integrate from $0-$ to include the impulse.

This impulse response looks like:



Linearity and Extended Linearity

Linearity: A system S is linear if it satisfies both

- *Homogeneity:* If $y = Sx$, and a is a constant then

$$ay = S(ax).$$

- *Superposition:* If $y_1 = Sx_1$ and $y_2 = Sx_2$, then

$$y_1 + y_2 = S(x_1 + x_2).$$

Combined Homogeneity and Superposition:
If $y_1 = Sx_1$ and $y_2 = Sx_2$, and a and b are constants,

$$ay_1 + by_2 = S(ax_1 + bx_2)$$

Extended Linearity

- *Summation:* If $y_n = Sx_n$ for all n, an integer from $(-\infty < n < \infty)$, and a_n are constants

$$\sum_n a_n y_n = S\left(\sum_n a_n x_n\right)$$

Summation and the system operator commute, and can be interchanged.

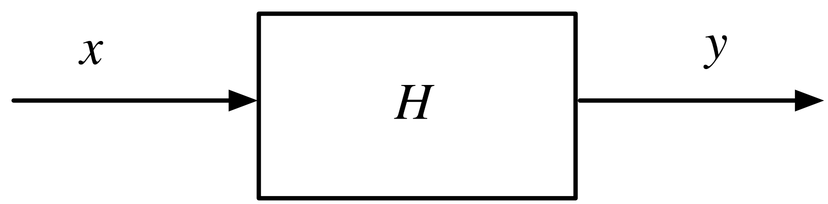
- *Integration (Simple Example)* : If $y = Sx$,

$$\int_{-\infty}^{\infty} a(\tau) y(t-\tau)\ d\tau = S\left(\int_{-\infty}^{\infty} a(\tau) x(t-\tau) d\tau\right)$$

Integration and the system operator commute, and can be interchanged.

Output of an LTI System

We would like to determine an expression for the output $y(t)$ of an linear time invariant system, given an input $x(t)$



We can write a signal $x(t)$ as a sample of itself

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\ d\tau$$

This means that $x(t)$ can be written as a weighted integral of δ functions.

Applying the system H to the input $x(t)$,

$$\begin{aligned} y(t) &= H\left(x(t)\right) \\ &= H\left(\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right) \end{aligned}$$

If the system obeys extended linearity we can interchange the order of the system operator and the integration

$$y(t) = \int_{-\infty}^{\infty} x(\tau)H\left(\delta(t-\tau)\right)d\tau.$$

The impulse response is

$$h(t,\tau) = H(\delta(t-\tau)).$$

Substituting for the impulse response gives

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t,\tau)d\tau.$$

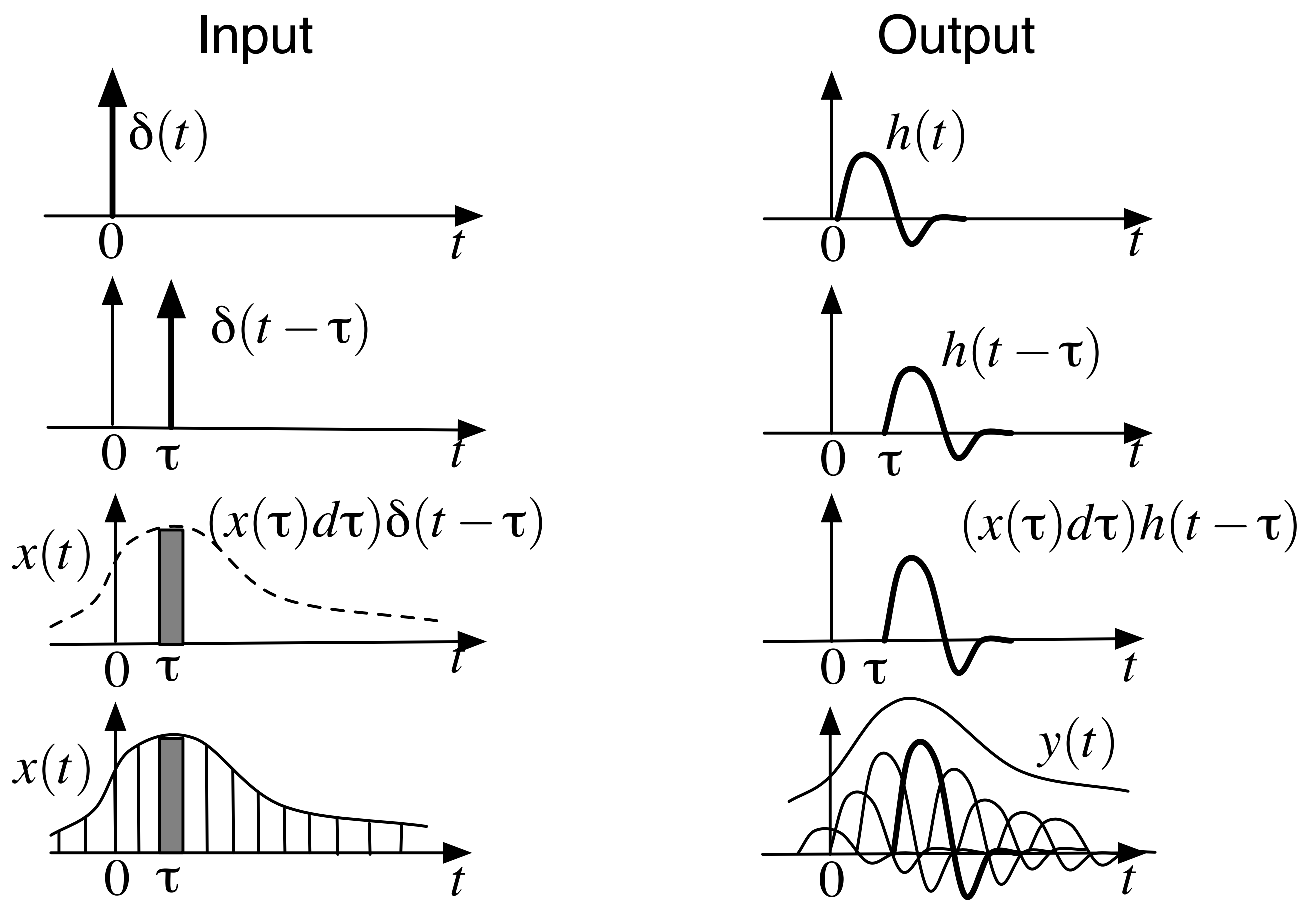
This is a *superposition integral*. The values of $x(\tau)h(t,\tau)d\tau$ are superimposed (added up) for each input time τ.

If H is time invariant, this written more simply as

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

This is in the form of a *convolution integral*, which will be the subject of the next class.

Graphically, this can be represented as:



$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \qquad y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

RC Circuit example, again

The impulse response of the RC circuit example is

$$h(t) = \frac{1}{RC} e^{-t/RC}$$

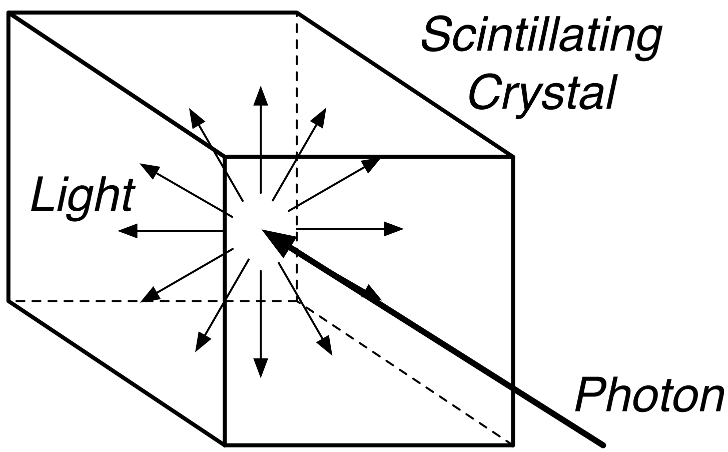
The response of this system to an input $x(t)$ is then

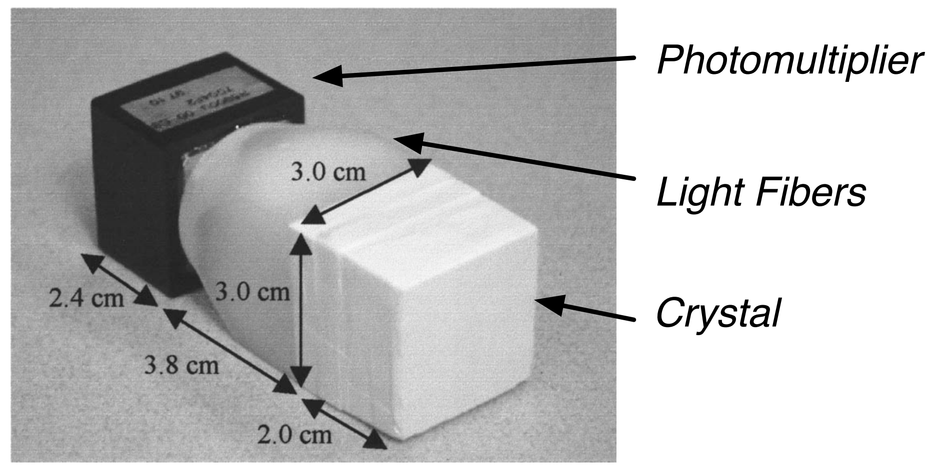
$$\begin{aligned} y(t) &= \int_0^t x(\tau) h(t-\tau) d\tau \\ &= \int_0^t x(\tau) \left[\frac{1}{RC} e^{-(t-\tau)/RC} \right] d\tau \end{aligned}$$

which is the zero state solution we found earlier.

Example:

High energy photon detectors can be modeled as having a simple exponential decay impulse response.



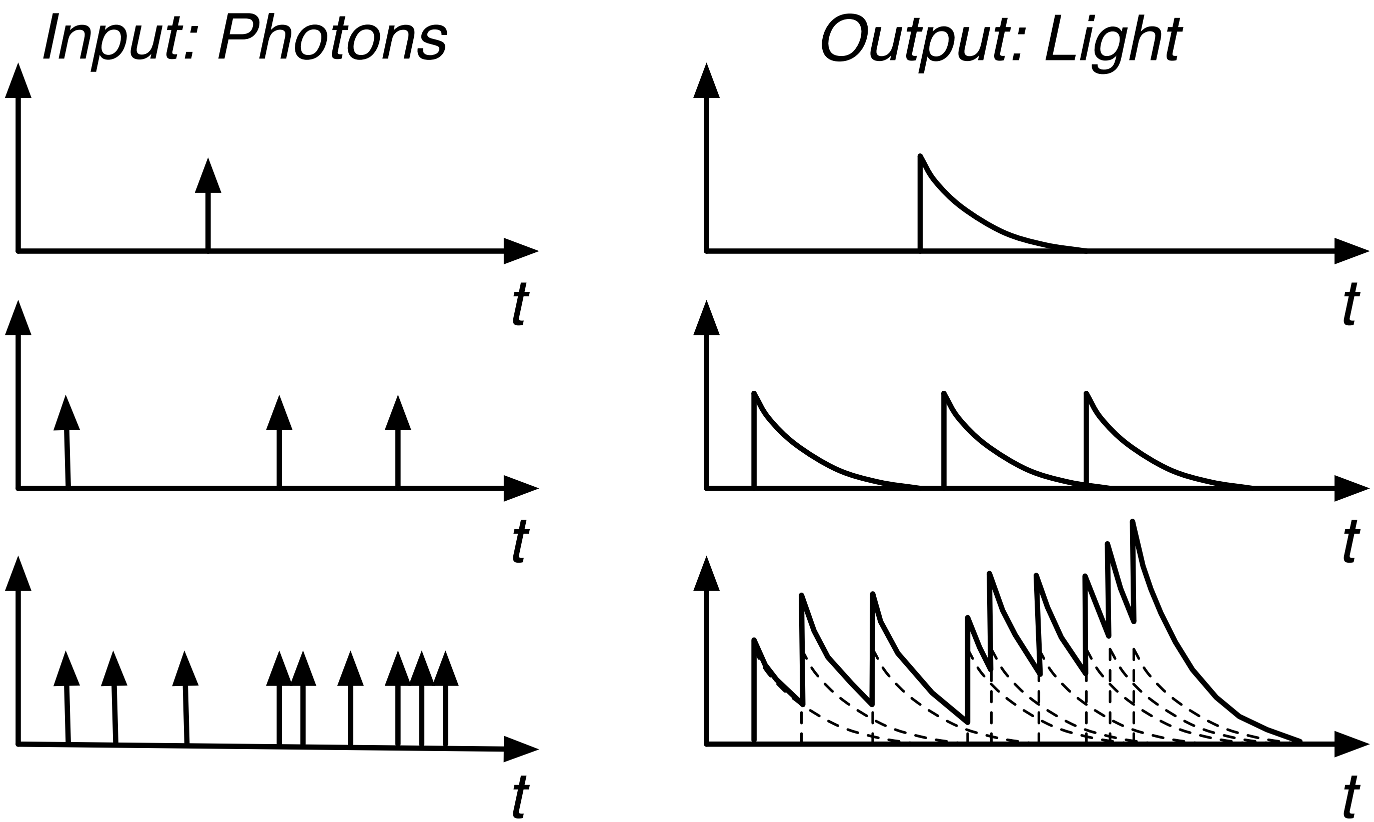


From: Doshi et al, Med Phys. 27(7), p1535 July 2000

These are used in positiron emmision tomography (PET) systems.

Input is a sequence of impulses (photons).

Output is superposition of impulse responses (light).



Summary

- For an input $x(t)$, the output of an linear system is given by the *superposition integral*

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t,\tau)\ d\tau$$

- If the system is also time invariant, the result is a *convolution integral*

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\ d\tau$$

- The response of an LTI system is completely characterized by its *impulse response* $h(t)$.