STANFORD UNIVERSITY
EE 102B Spring-2013

Lecture 14

Discrete-Time Filtering of Continuous-Time Signals

May 1, 2013

ASSIGNMENTS

- Reading for this Lecture:
  - SPF: Section 12-3
  - S&S: Chapter 5
- HW#04 is due by 5pm today, May 1, in Packard 263.
- Lab #04 is due by 5pm, Friday, May 3, in Packard 263. Contact Keith Gaul for access to EE computer cluster if you need the signal processing toolbox. gaul@ee.stanford.edu

Office Hours for Course Staff – Come see us.

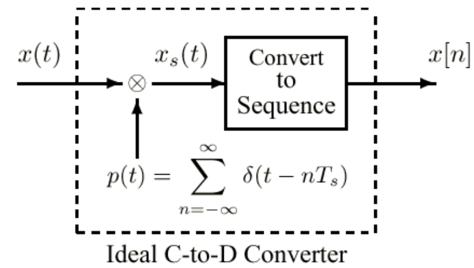
- Ron Schafer: Mon. 2:00-4:00 pm. Not available for office hours today.
- Dookun Park: Tues. 1:30 ~ 3:30 pm in Packard 107
- Ruo Yu Gu (Roy): Mon. 4:00 ~ 6:00 pm in Packard 106
- https://class2go.stanford.edu/EE102B/Spring2013/pages/staff

Lecture Objective

- Review sampling and reconstruction
  - C-to-D conversion
  - Relation of CTFT to DTFT
- A-to-D and D-to-A conversion
- Discrete-time filtering of continuous-time signals

REVIEW OF SAMPLING AND RECONSTRUCTION

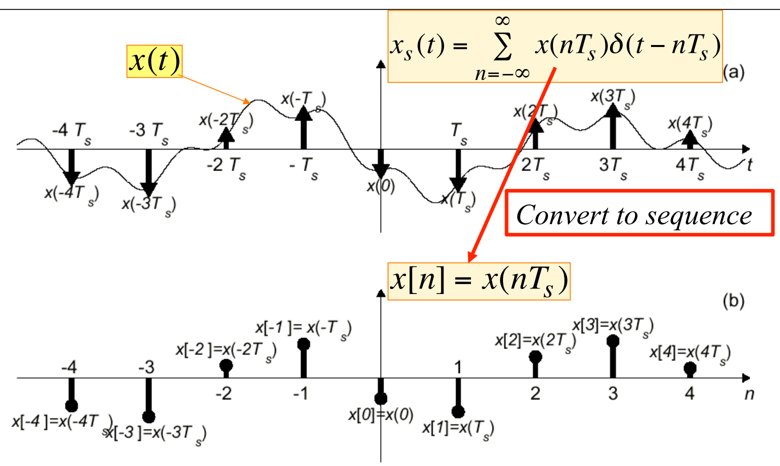
Representing Sampling by Impulse Train Modulation



$$x_s(t) = x(t) \sum_{n=-\infty} \delta(t - nT_s) = \sum_{n=-\infty} \underline{x(t)\delta(t - nT_s)}$$

$$\boxed{x_s(t) = \sum_{n=-\infty}^{\infty} \underline{x(nT_s)\delta(t - nT_s)}}$$

Illustration of Sampling



Frequency-Domain Analysis

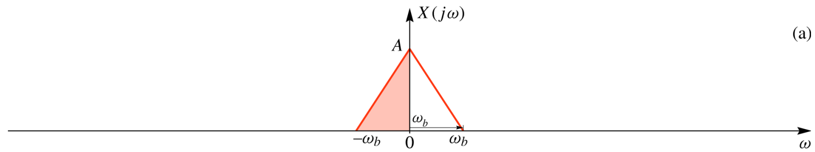
$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

$$x_s(t) = x(t) \sum_{k=-\infty}^{\infty} \frac{1}{T_s} e^{jk\omega_s t} = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} \underline{x(t)e^{jk\omega_s t}}$$

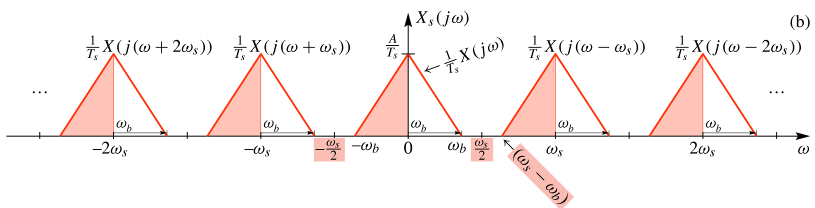
$$\boxed{X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} \underline{X(j(\omega - k\omega_s))}}$$

$$\boxed{\omega_s = \frac{2\pi}{T_s}}$$

Frequency-Domain Representation of Sampling

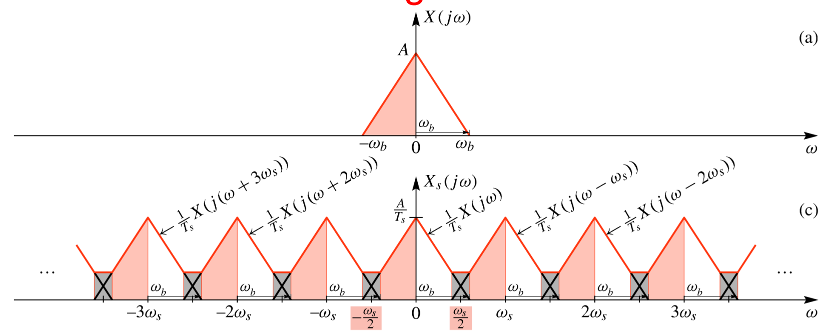


$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X\big(j(\omega - k\omega_s)\big)$$



Aliasing Distortion

- If $\omega_s < 2\omega_b$, the copies of $X(j\omega)$ overlap, and we have aliasing distortion.



Relation to the DTFT

- Look at the CTFT of $x_s(t)$ in a different way

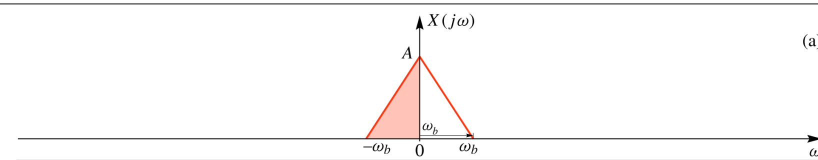
$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

$$X_s(j\omega) = \sum_{n=-\infty}^{\infty} x(nT_s)e^{-j\omega nT_s} = X(e^{j\omega T_s})$$

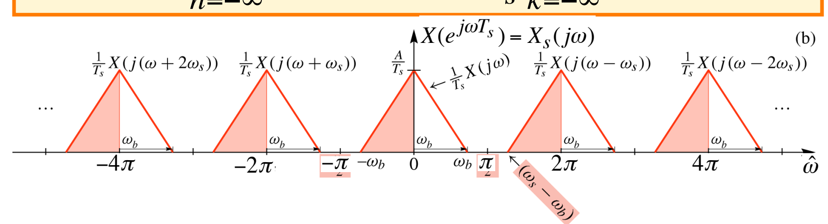
$$X(e^{j\omega T_s}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega nT_s} = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X\big(j(\omega - k\omega_s)\big)$$

Recall that: $\hat{\omega} = \omega T_s$ $\quad x[n] = x(nT_s)$ $\quad \omega_s = \frac{2\pi}{T_s}$

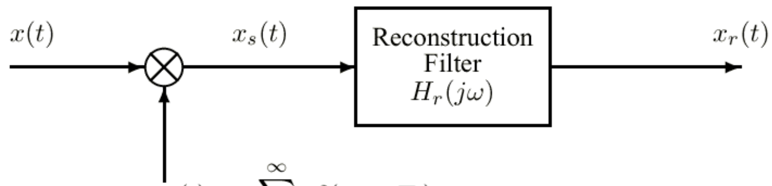
Frequency-Domain Representation of Sampling



$$X(e^{j\omega T_s}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega nT_s} = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X\big(j(\omega - k\omega_s)\big)$$



Reconstruction of $x(t)$



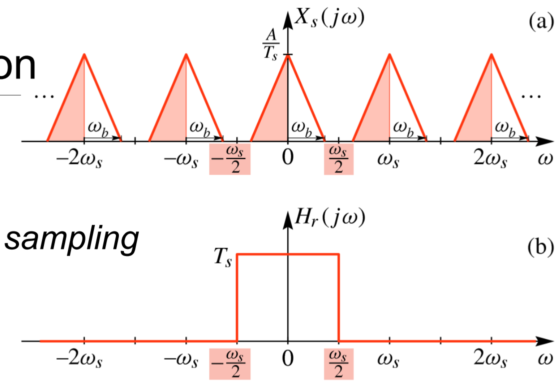
$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} x(t)e^{jk\omega_s t}$$

$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X\big(j(\omega - k\omega_s)\big)$$

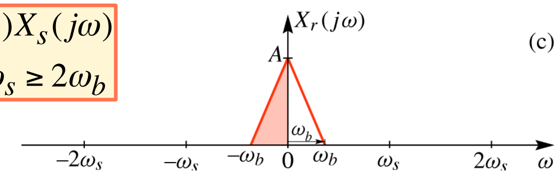
$$X_r(j\omega) = H_r(j\omega)X_s(j\omega)$$

Bandlimited Reconstruction



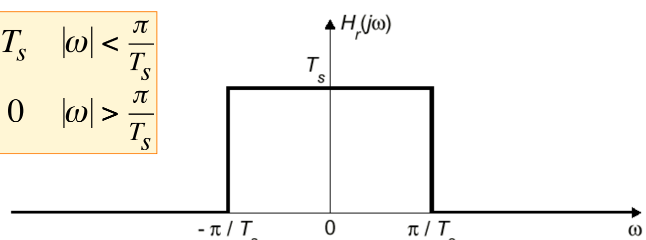
This proves the sampling theorem.

$$X_r(j\omega) = H_r(j\omega)X_s(j\omega) = X(j\omega) \text{ when } \omega_s \geq 2\omega_b$$

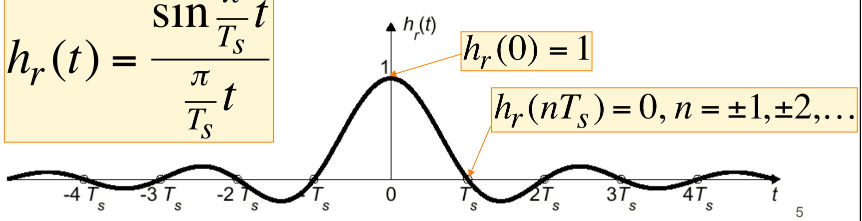


Ideal Reconstruction Filter

$$H_r(j\omega) = \begin{cases} T_s & |\omega| < \dfrac{\pi}{T_s} \\ 0 & |\omega| > \dfrac{\pi}{T_s} \end{cases}$$



$$h_r(t) = \frac{\sin\frac{\pi}{T_s}t}{\frac{\pi}{T_s}t}$$



$$h_r(0) = 1$$

$$h_r(nT_s) = 0, n = \pm 1, \pm 2, \ldots$$

Signal Reconstruction

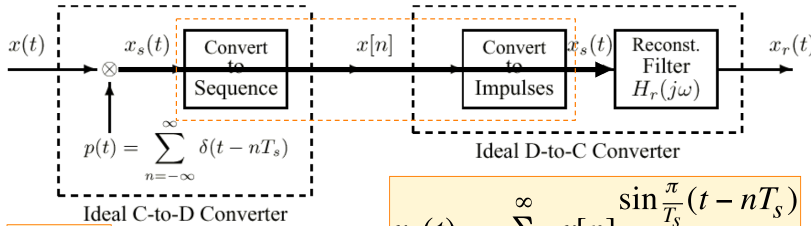
$$x_r(t) = h_r(t) * x_s(t) = h_r(t) * \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s)h_r(t - nT_s)$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\frac{\sin\frac{\pi}{T_s}(t - nT_s)}{\frac{\pi}{T_s}(t - nT_s)}$$

Ideal bandlimited interpolation formula

Ideal C-to-D and D-to-C Back-to-Back



$$\omega_s = \frac{2\pi}{T_s} \qquad x[n] = x(nT_s)$$

Ideal Sampler

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin\frac{\pi}{T_s}(t - nT_s)}{\frac{\pi}{T_s}(t - nT_s)}$$

Ideal bandlimited interpolator

$$X(e^{j\omega T_s}) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X\big(j(\omega - k\omega_s)\big)$$

$$X_r(j\omega) = H_r(j\omega) X_s(j\omega)$$

Example of Sampling a Sinusoidal Signal (Ex. 12-6)

- Consider a signal of the form:

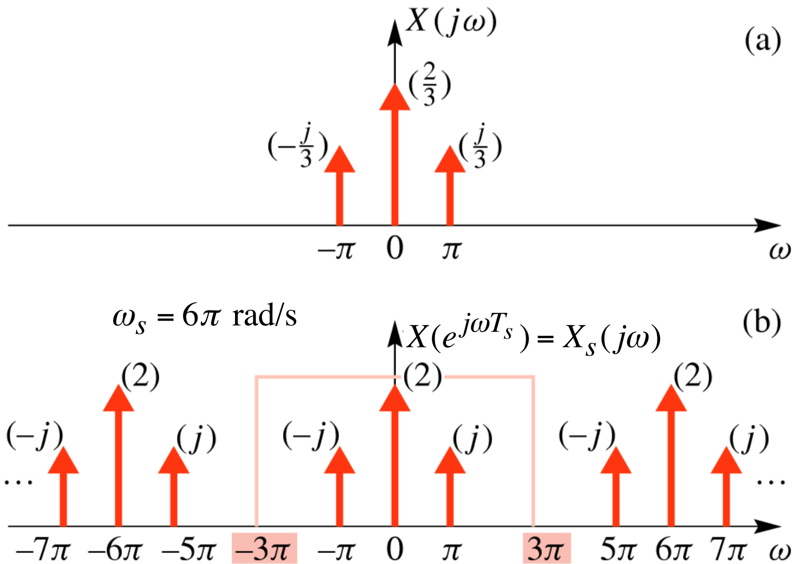
$$x(t) = \frac{1}{3\pi} + \frac{1}{3\pi}\cos(\pi t + \pi/2)$$

- Its CTFT is

$$X(j\omega) = \frac{2}{3}\delta(\omega) + \frac{j}{3}\delta(\omega - \pi) - \frac{j}{3}\delta(\omega + \pi)$$

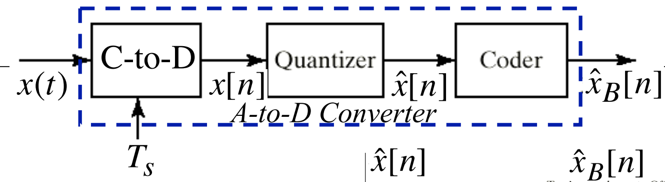
- The DTFT of x[n] with $\omega_s = 6\pi$ $(T_s = 1/3)$

$$X(e^{j\omega T_s}) = 3\left[\frac{2}{3}\delta(\omega) + \frac{j}{3}\delta(\omega - \pi) - \frac{j}{3}\delta(\omega + \pi)\right] \quad |\omega| \le \frac{\pi}{T_s}$$



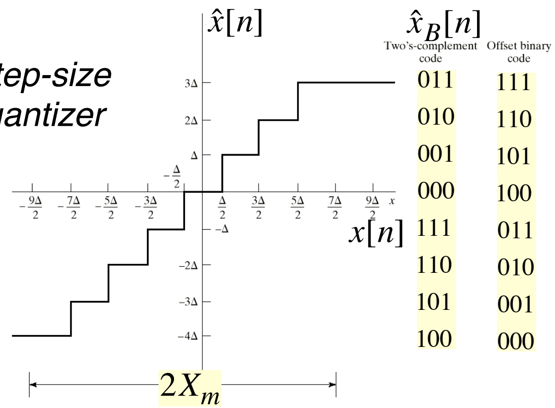
A-to-D CONVERSION

Representation of A-to-D Converter



Quantization step-size for (B+1)-bit quantizer

$$\Delta = \frac{2X_m}{2^{B+1}} = \frac{X_m}{2^B}$$



Listening Experiment on the Effects of Quantization

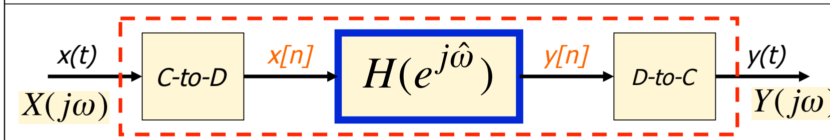
	Signal	Error
"Unquantized" speech signal		
3-bit quantized Speech signal		
8-bit quantized speech signal		X32

Bit rate = $B \bullet f_s$

Quantization is perceived as "noise" in audio signals. We study quantization noise in EE264.

DISCRETE-TIME FILTERING OF CONTINUOUS-TIME SIGNALS

DT Filtering of CT Signals

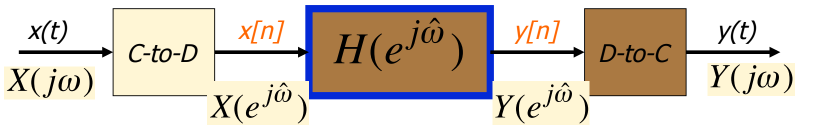


If no aliasing occurs in sampling x(t), then

$$Y(j\omega) = H_{\text{eff}}(j\omega)X(j\omega)$$

$$H_{\text{eff}}(j\omega) = \begin{cases} H(e^{j\omega T_s}) & |\omega| < \frac{1}{2}\omega_s \\ 0 & |\omega| > \frac{1}{2}\omega_s \end{cases}$$

C-to-D Converter

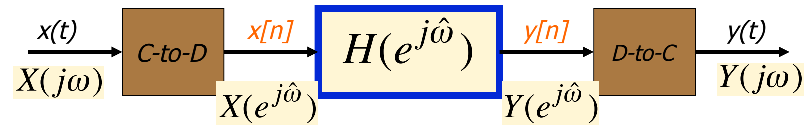


$$x[n] = x(nT_s)$$

$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$$X(e^{j\omega T_s}) = X(e^{j\hat{\omega}})\Big|_{\hat{\omega}=\omega T_s} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega T_s n} = X_s(j\omega)$$

LTI DT System

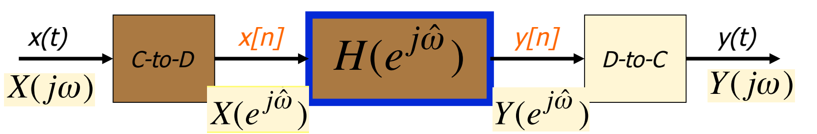


$$Y(e^{j\hat{\omega}}) = H(e^{j\hat{\omega}})X(e^{j\hat{\omega}})$$

$$\hat{\omega} = \omega T_s = \frac{\omega}{f_s}$$

$$Y(e^{j\omega T_s}) = H(e^{j\omega T_s})X(e^{j\omega T_s})$$

D-to-C Converter

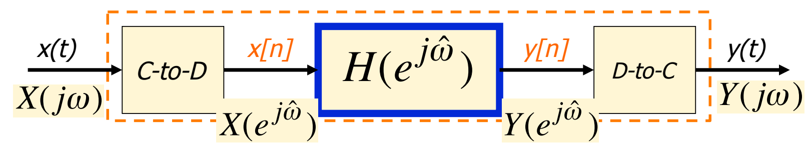


$$y(t) = \sum_{n=-\infty}^{\infty} y[n]h_r(t - nT_s)$$

$$Y(j\omega) = \sum_{n=-\infty}^{\infty} y[n]H_r(j\omega)e^{-j\omega T_s n} = H_r(j\omega)\sum_{n=-\infty}^{\infty} y[n]e^{-j\omega T_s n}$$

$$Y(j\omega) = H_r(j\omega)Y(e^{j\omega T_s})$$

Putting it All Together



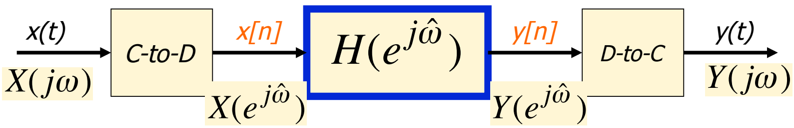
$$Y(j\omega) = H_r(j\omega)Y(e^{j\omega T_s}) = H_r(j\omega)H(e^{j\omega T_s})\underline{X(e^{j\omega T_s})}$$

$$Y(j\omega) = \overbrace{\cancel{H_r(j\omega)}}^{1 \text{ for } |\omega| \le \pi/T_s} H(e^{j\omega T_s})\underline{\frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))}$$

If no aliasing occurs in sampling x(t), then it follows that

$$Y(j\omega) = \underline{H(e^{j\omega T_s})}X(j\omega) = \underline{H_{\text{eff}}(j\omega)}X(j\omega)$$

DT Filtering of CT Signals

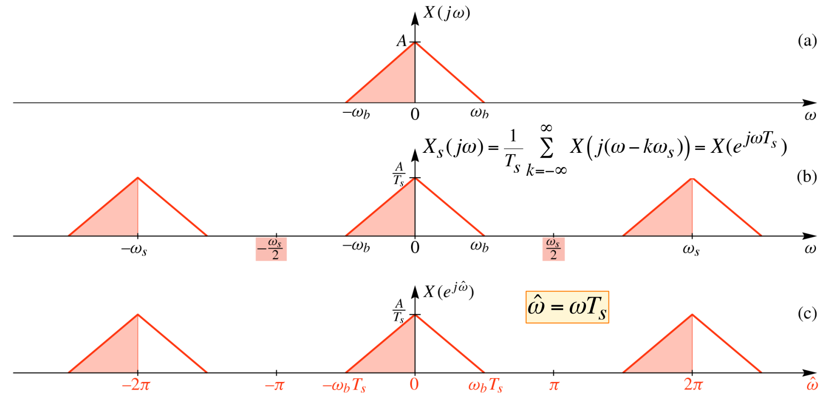


If no aliasing occurs in sampling x(t), then it follows that

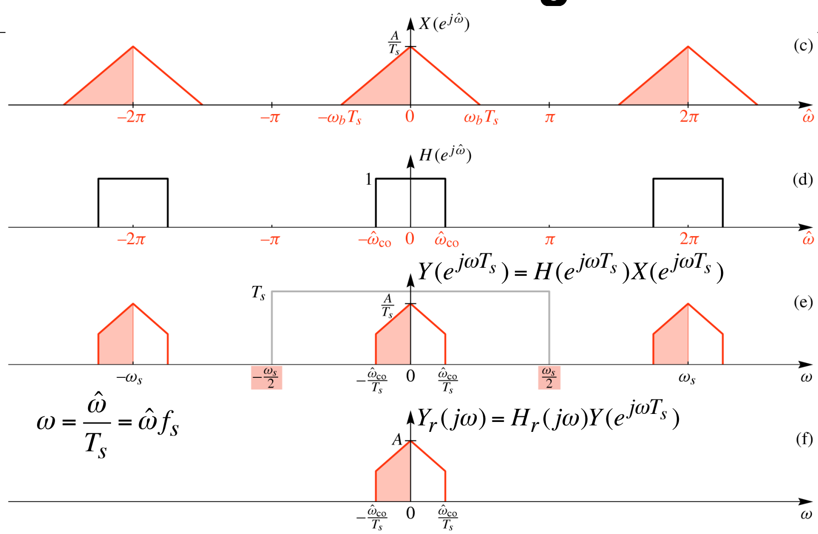
$$Y(j\omega) = H_{\text{eff}}(j\omega)X(j\omega)$$

$$H_{\text{eff}}(j\omega) = \begin{cases} H(e^{j\omega T_s}) & |\omega| < \frac{1}{2}\omega_s \\ 0 & |\omega| > \frac{1}{2}\omega_s \end{cases}$$

Sampling



Discrete-Time Filtering of Continuous-Time Signals

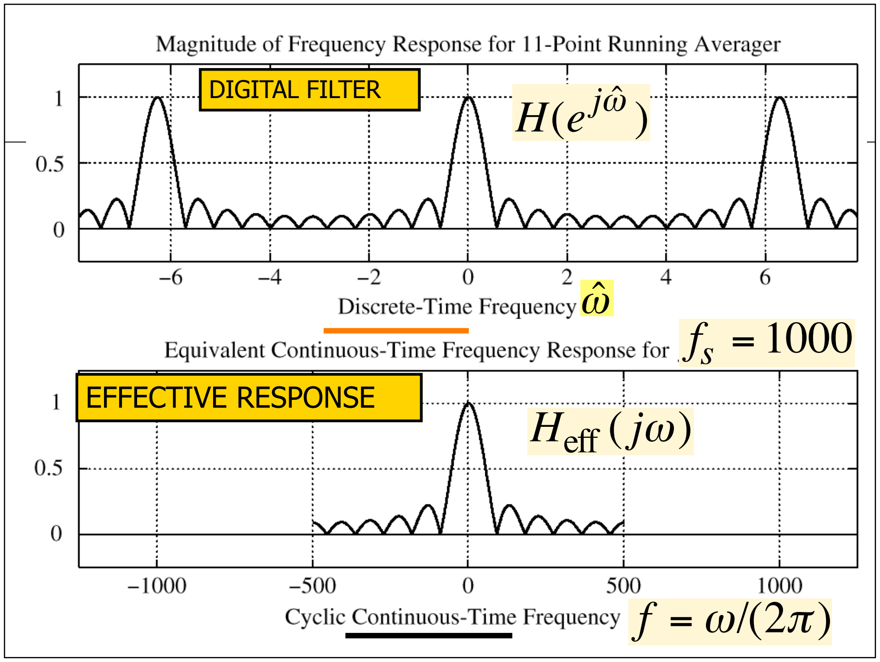


11-pt AVERAGER Example

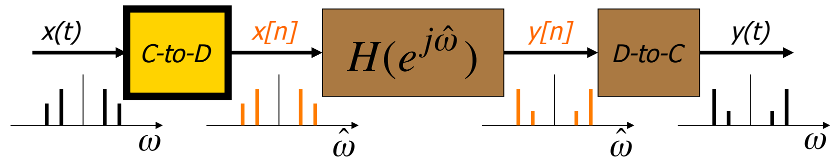
$$y[n] = \frac{1}{11}\sum_{k=0}^{10} x[n-k]$$

$$H(e^{j\hat{\omega}}) = \frac{1}{11}\sum_{k=0}^{10} e^{-j\hat{\omega}k} = \frac{\sin(\hat{\omega}11/2)}{11\sin(\hat{\omega}/2)}e^{-j\hat{\omega}5}$$

$$H_{\text{eff}}(j\omega) = \frac{\sin(\omega T_s 11/2)}{11\sin(\omega T_s/2)}e^{-j\omega T_s 5} \qquad |\omega| < \frac{\pi}{T_s}$$



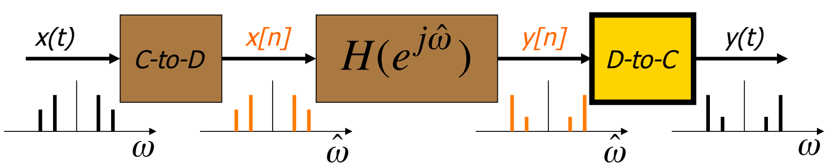
FREQUENCY SCALING



- TIME SAMPLING: $t = nT_s$
  - IF NO ALIASING:
  - FREQUENCY SCALING

$$\hat{\omega} = \omega T_s = \frac{\omega}{f_s}$$

D-A FREQUENCY SCALING



TIME SAMPLING: $t = nT_s \Rightarrow n \leftarrow t f_s$

- RECONSTRUCT up to $0.5f_s$
  - FREQUENCY SCALING
  - IDEAL LPF

$$\omega = \hat{\omega} f_s$$