STANFORD UNIVERSITY
EE 102B Spring-2013

Lecture 17
The DFT and its Properties
May 8, 2013

ASSIGNMENTS

- Reading for this Lecture:
  - SPF: Section 12-3, Chapter 66-6, 66-7
  - S&S: Chapter 5
- HW#05 is due by 5pm today, May 8, in Packard 263.
- Lab #05 is due by 5pm, Friday, May 17, in Packard 263.
- Mid-term exam on Friday, May 10, in class. Room and exam conditions next slide.

Mid-Term Exam

- Covers material through Lecture 13 (FIR filter design and intro to sampling), HWs 01-04, and LABs 01-04.
- The exam will be held in 420-041, 11am - 12:30 pm.
- You may use your textbook (either *SP-First* or Signals *and Systems*), printouts of Chapter 66, and two sheets (both sides) of notes. No computers or other materials allowed.
- Several people have conflicts that we will accommodate in 380-380D, 1 – 2:30pm. So far only three people have emailed me with their intention to take the exam at this time along with their reason for the conflict.

Office Hours for Course Staff – Come see us.

- Ron Schafer: Mon. and Weds. 2:00-4:00 pm, Packard 211.
- Dookun Park: Tues. 1:30 ~ 3:30 pm in Packard 107
- Ruo Yu Gu (Roy): Mon. 4:00 ~ 6:00 pm in Packard 106
- Mid-term exam review session: today at 3~5pm. Place : 200-203 (Main Quad)

Lecture Objective

- Questions?
- Review of the Discrete Fourier Transform
  - Definition
  - Inverse DFT
- Some DFT pairs
- Some properties of the DFT

INTRODUCTION TO THE DFT

Comparison: DFT and DTFT

DFT

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j(2\pi/N)kn} \qquad k = 0,1,\dots,N-1$$

Inverse DFT

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j(2\pi/N)kn} \qquad n = 0,1,\dots,N-1$$

DTFT

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n} \qquad -\pi \le \hat{\omega} < \pi$$

Inverse DTFT

$$x[n] = \frac{1}{2\pi}\int_{0}^{2\pi} X(e^{j\hat{\omega}})e^{j\hat{\omega}n}\,d\hat{\omega} \qquad -\infty < n < \infty$$

Review

- **D**iscrete **F**ourier **T**ransform (DFT)

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j(2\pi/N)kn} \qquad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j(2\pi/N)kn}$$

- DFT is frequency sampled DTFT
  - For finite-length signals
- DFT computation via FFT
  - FFT of zero-padded signal→more freq samples
- Transform pairs & properties (DTFT & DFT)

Sample the DTFT → DFT

- Want **computable** Fourier transform
  - Finite signal length (L)
  - Finite number of frequencies (N)

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n} \Rightarrow X(e^{j\hat{\omega}_k}) = \sum_{n=0}^{L-1} x[n]e^{-j\hat{\omega}_k n}$$

$$\hat{\omega}_k = (2\pi/N)k, \quad k = 0,1,2,\ldots N-1$$

$$X[k] = X(e^{j\hat{\omega}_k})$$

k is the frequency index

$$\boxed{\text{Periodic}: X(e^{j(\hat{\omega}+2\pi)}) = X(e^{j\hat{\omega}}) \quad \Rightarrow \quad X[k+N] = X[k]}$$

Zero-Padding → more frequency samples

- Want many samples of DTFT
  - WHY? to make a smooth plot (one reason)
  - Finite signal length (L)
  - Finite number of frequencies (N)
  - Thus, we need $L < N, \quad N \to \infty, \quad X[k] \to X(e^{j\hat{\omega}})$

$$X(e^{j\hat{\omega}_k}) = \sum_{n=0}^{L-1} x[n]e^{-j\hat{\omega}_k n}$$

$$\hat{\omega}_k = (2\pi/N)k, \quad k = 0,1,2,\ldots N-1$$

Zero-Padding with the FFT

- Get many samples of DTFT
  - Finite signal length (L)
  - Finite number of frequencies (N)

    $\hat{\omega}_k = (2\pi/N)k, \quad k = 0,1,2,\ldots N-1$
  - Thus, we need $L \le N, \quad N \to \infty, \quad X[k] \to X(e^{j\hat{\omega}})$

In MATLAB

- **Use** `X = fft(x,N)` or
- With `length(x)=L<N`
  - Then `xpadtoN = [x,zeros(1,N-L)];`
  - Take the N-pt DFT `X=fft(padtoN)`

Inherent Periodicity of the DFT and IDFT

- DFT $\boxed{X[k+N] = \sum_{n=0}^{N-1} x[n]e^{-j(2\pi/N)(k+N)n} = X[k]}$

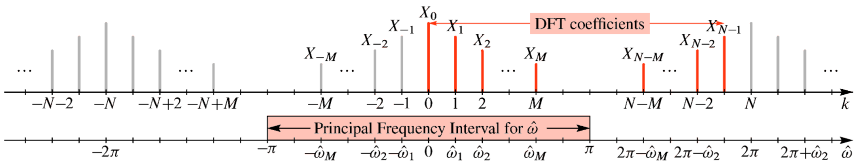
$$e^{-j(2\pi/N)(k+N)n} = e^{-j(2\pi/N)kn}\cancelto{1}{e^{-j(2\pi/N)Nn}} = e^{-j(2\pi/N)kn}$$

- IDFT $\boxed{x[n+N] = \frac{1}{N}\sum_{n=0}^{N-1} X[k]e^{j(2\pi/N)k(n+N)} = x[n]}$

$$e^{-j(2\pi/N)k(n+N)} = e^{-j(2\pi/N)kn}\cancelto{1}{e^{-j(2\pi/N)kN}} = e^{-j(2\pi/N)kn}$$

DFT Periodicity in Frequency Index

$$X[k] = X(e^{j\hat{\omega}_k}) = X(e^{j(2\pi/N)k}) = X_k$$
$$k = 0,1,2,\ldots,N-1$$



$$X[k+N] = X[k] \Leftrightarrow X(e^{j(\hat{\omega}+2\pi)}) = X(e^{j\hat{\omega}})$$

$$\Rightarrow X[N-k] = X[-k],$$
$$\text{e.g., } X[N-2] = X[-2]$$

DFT periodic in k (frequency domain)

- Since DTFT is periodic in frequency, the DFT must also be periodic in k

$$X[k] = X(e^{j(2\pi/N)k})$$
$$X[k+N] = X(e^{j(2\pi/N)(k+N)}) = X(e^{j(2\pi/N)(k)+(2\pi/N)N)}) = X(e^{j(2\pi/N)k})$$

- What about Negative indices and Conjugate Symmetry?

$$X(e^{-j(2\pi/N)k}) = X^*(e^{j(2\pi/N)k})$$
$$\Rightarrow X[-k] = X^*[k]$$
$$X[N-k] = X^*[k]$$

$$N = 32 \quad \Rightarrow$$
$$X[31] = X^*[1]$$
$$X[30] = X^*[2]$$
$$X[29] = X^*[3]$$

Some DFT Pairs - I

- Impulse: x[n] = δ[n]

$$X[k] = \sum_{n=0}^{N-1} \delta[n] e^{-j(2\pi/N)kn} = 1 \qquad k = 0,1,\ldots,N-1$$

- Shifted impulse: x[n] = δ[n-n_d]

$$X[k] = \sum_{n=0}^{N-1} \delta[n-n_d] e^{-j(2\pi/N)kn} = e^{-j(2\pi/N)kn_d}$$
$$k = 0,1,\ldots,N-1$$

Some DFT Pairs - II

- Pulse: x[n] = u[n]-u[n-L]
  - DTFT

$$X(e^{j\hat{\omega}}) = \sum_{n=0}^{L-1} e^{-j\hat{\omega}n} = \frac{\sin(\hat{\omega}L/2)}{\sin(\hat{\omega}/2)} e^{-j\hat{\omega}(L-1)/2}$$

  - DFT

$$X[k] = X(e^{j(2\pi/N)k}) = \frac{\sin((2\pi/N)kL/2)}{\sin((2\pi/N)k/2)} e^{-j(2\pi/N)k(L-1)/2}$$
$$k = 0,1,\ldots,N-1$$

DFT Properties - I

- Linearity
  - DTFT $ax_1[n]+bx_2[n] \Leftrightarrow aX_1(e^{j\hat{\omega}})+bX_2(e^{j\hat{\omega}})$
  - Sample the DTFT

$$ax_1[n]+bx_2[n] \Leftrightarrow aX_1(e^{j(2\pi/N)k})+bX_2(e^{j(2\pi/N)k})$$

  - DFT $\boxed{ax_1[n]+bx_2[n] \Leftrightarrow aX_1[k]+bX_2[k]}$

$\boxed{\text{Both } X_1[k] \text{ and } X_2[k] \text{ must be N-point DFTs}}$

DFT Properties - II

- Shift (of implicit periodic sequence)
  - DTFT $x[n-n_d] \Leftrightarrow e^{-j\hat{\omega}n_d}X(e^{j\hat{\omega}})$
  - Sample the DTFT

$$x[n-n_d] \Leftrightarrow e^{-j(2\pi/N)kn_d}X(e^{j(2\pi/N)k})$$

  - DFT $\boxed{x[n-n_d] \Leftrightarrow e^{-j(2\pi/N)kn_d}X[k]}$

DFT Properties - III

- Convolution (of finite-length sequences):
  - DTFT $\sum_{m=0}^{N-1} x_1[m]x_2[n-m] \Leftrightarrow X_1(e^{j\hat{\omega}})X_2(e^{j\hat{\omega}})$
  - Sample the DTFT

$$\sum_{m=0}^{N-1} x_1[m]x_2[n-m] \Leftrightarrow X_1(e^{j(2\pi/N)k})X_2(e^{j(2\pi/N)k})$$

  - DFT (implicitly periodic convolution)

$$\sum_{m=0}^{N-1} x_1[m]x_2[n-m] \Leftrightarrow X_1[k]X_2[k]$$

DTFT of <u>Finite</u> complex exponential (1)

- We know DTFT of finite rectangular pulse
  - Dirichlet form and a linear phase term

$$\boxed{x[n] = \begin{cases} 1 & 0 \le n < L \\ 0 & \text{otherwise} \end{cases} \Leftrightarrow X(e^{j\hat{\omega}}) = \underbrace{\frac{\sin(\frac{1}{2}L\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})}}_{D_L(\hat{\omega}) = \frac{\sin(\frac{1}{2}L\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})}} e^{-j\hat{\omega}(L-1)/2}}$$

- Use frequency-shift property

$$\boxed{y[n] = \begin{cases} e^{j\hat{\omega}_0 n} & 0 \le n < L \\ 0 & \text{otherwise} \end{cases} \Leftrightarrow Y(e^{j\hat{\omega}}) = \frac{\sin(\frac{1}{2}L(\hat{\omega}-\hat{\omega}_0))}{\sin(\frac{1}{2}(\hat{\omega}-\hat{\omega}_0))} e^{-j(\hat{\omega}-\hat{\omega}_0)(L-1)/2}}$$

DFT of <u>Finite</u> complex exponential (2)

- Know DTFT, sample in frequency

$$y[n] = \begin{cases} e^{j\hat{\omega}_0 n} & 0 \le n < L \\ 0 & \text{otherwise} \end{cases} \Leftrightarrow Y(e^{j\hat{\omega}}) = \frac{\sin(\frac{1}{2}L(\hat{\omega}-\hat{\omega}_0))}{\sin(\frac{1}{2}(\hat{\omega}-\hat{\omega}_0))} e^{-j(\hat{\omega}-\hat{\omega}_0)(L-1)/2}$$

- Take N-point DFT

$$y[n] = \begin{cases} e^{j\hat{\omega}_0 n} & 0 \le n < L \\ 0 & L \le n < N \end{cases} \Leftrightarrow$$

$$Y[k] = Y(e^{j\hat{\omega}}) \quad \text{at } \hat{\omega} = \tfrac{2\pi k}{N}$$

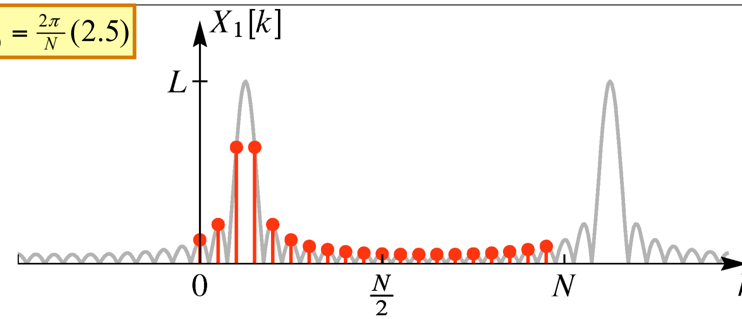
$$Y[k] = \frac{\sin(\frac{1}{2}L(\frac{2\pi k}{N}-\hat{\omega}_0))}{\sin(\frac{1}{2}(\frac{2\pi k}{N}-\hat{\omega}_0))} e^{-j(\frac{2\pi k}{N}-\hat{\omega}_0)(L-1)/2}$$

Dirichlet Function

$$D_L(\hat{\omega}) = \frac{\sin(\frac{1}{2}L\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})}$$

20-pt DFT of Complex Exponential

$$\hat{\omega}_0 = \tfrac{2\pi}{N}(2.5)$$

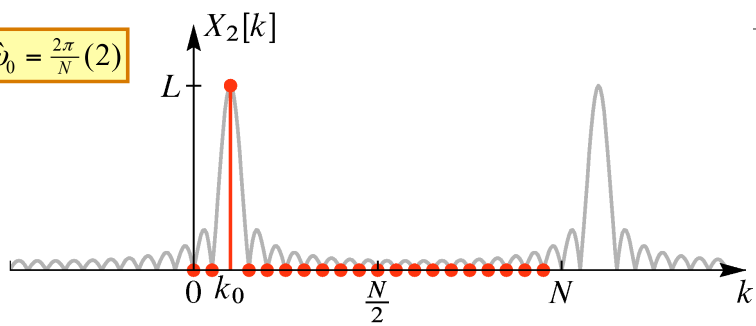


$$x_1[n] = \begin{cases} e^{j\hat{\omega}_0 n} & 0 \le n < L \\ 0 & \text{otherwise} \end{cases} \Leftrightarrow X_1[k] = \underbrace{\frac{\sin(\frac{1}{2}L(\frac{2\pi k}{N}-\hat{\omega}_0))}{\sin(\frac{1}{2}(\frac{2\pi k}{N}-\hat{\omega}_0))}} e^{-j(\frac{2\pi k}{N}-\hat{\omega}_0)(L-1)/2}$$

$$D_L(\hat{\omega}) = \frac{\sin(\frac{1}{2}L\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})} \quad \text{so this is a shifted Dirichlet form}$$

20-pt DFT of Complex Exp: "on the grid"

$$\hat{\omega}_0 = \tfrac{2\pi}{N}(2)$$

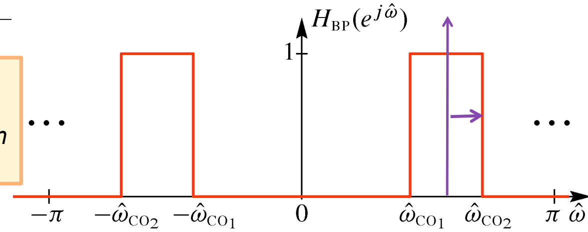


$$x_2[n] = \begin{cases} e^{j\hat{\omega}_0 n} & 0 \le n < L \\ 0 & \text{otherwise} \end{cases} \Leftrightarrow X_2[k] = \underbrace{\frac{\sin(\frac{1}{2}L(\frac{2\pi k}{N}-\hat{\omega}_0))}{\sin(\frac{1}{2}(\frac{2\pi k}{N}-\hat{\omega}_0))}} e^{-j(\frac{2\pi k}{N}-\hat{\omega}_0)(L-1)/2}$$

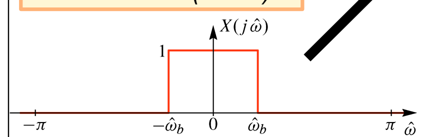
$$D_L(\hat{\omega}) = \frac{\sin(\frac{1}{2}L\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})} \quad \text{so this is a shifted Dirichlet form}$$

RECALL: BandPass Filter (BPF)

Frequency shifting up and down *is done by cosine multiplication in the time domain*



BPF is frequency shifted version of LPF (below)



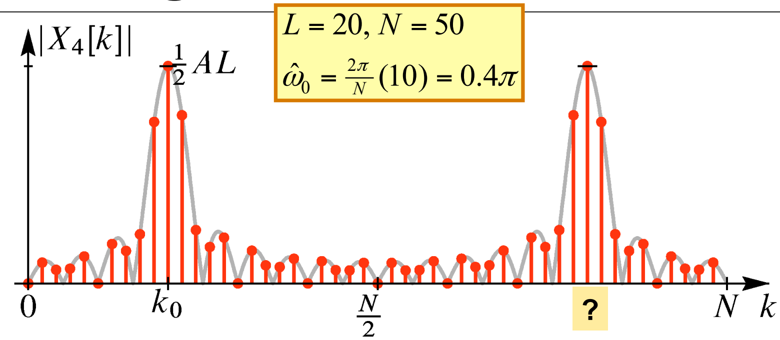
$$h_{\text{BP}}[n] = 2\cos(\hat{\omega}_{\text{mid}} n)\frac{\sin(\frac{1}{2}\hat{\omega}_{\text{diff}} n)}{\pi n}$$

$$\overset{DTFT}{\Leftrightarrow} H_{\text{BP}}(e^{j\hat{\omega}}) = \begin{cases} 0 & |\hat{\omega}| \le \hat{\omega}_{co_1} \\ 1 & \hat{\omega}_{co_1} < |\hat{\omega}| \le \hat{\omega}_{co_2} \\ 0 & \hat{\omega}_{co_2} < |\hat{\omega}| \le \pi \end{cases}$$

50-pt DFT of Sinusoid: zero padding

$L = 20,\ N = 50$

$\hat{\omega}_0 = \frac{2\pi}{N}(10) = 0.4\pi$



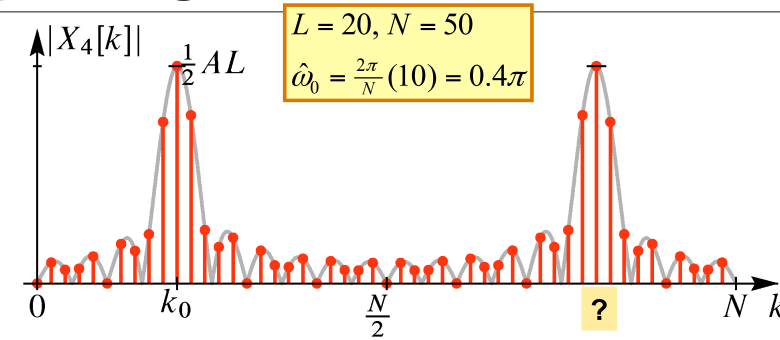
$$x_4[n] = \begin{cases} A\cos(\hat{\omega}_0 n) & 0 \le n < L \\ 0 & \text{otherwise} \end{cases} \Leftrightarrow$$

$$X_4[k] = \tfrac{1}{2} A D_L(\tfrac{2\pi k}{N} - \hat{\omega}_0)\, e^{-j(\frac{2\pi k}{N} - \hat{\omega}_0)(L-1)/2} + \tfrac{1}{2} A D_L(\tfrac{2\pi k}{N} + \hat{\omega}_0)\, e^{-j(\frac{2\pi k}{N} + \hat{\omega}_0)(L-1)/2}$$

50-pt DFT of Sinusoid: zero padding

$L = 20,\ N = 50$

$\hat{\omega}_0 = \frac{2\pi}{N}(10) = 0.4\pi$



Zero-crossings of Dirichlet ?

Width of Dirichlet ?

Density of frequency samples?