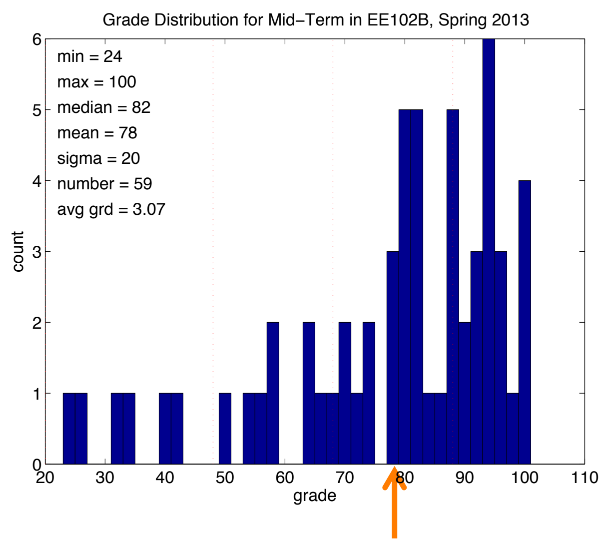
STANFORD UNIVERSITY
EE 102B Spring-2013

Lecture 18
The DFT as a Sampled DTFT and Block Processing
May 13, 2013

ASSIGNMENTS

- Reading for this Lecture:
  - SPF: Section 12-3, Chapter 66-6 thru 66-9
  - S&S:
  - HW#06 is due by 5pm Wednesday, May 15, in Packard 263. No late penalty if handed in by 5pm Friday, May 17.
- Lab #05 is due by 5pm, Friday, May 17, in Packard 263.



Office Hours for Course Staff – Come see us.

- Ron Schafer: Mon. and Weds. 2:00-4:00 pm, Packard 211. (Available today only 4:00-5:00 pm.)
- Dookun Park: Tues. 1:30 ~ 3:30 pm in Packard 107
- Ruo Yu Gu (Roy): Mon. 4:00 ~ 6:00 pm in Packard 106

Lecture Objective

- A DFT example
- Sampling theorem for the DTFT
  - The DFT is a sampled DTFT
  - Derivation of result for recovery of x[n]
- Block processing with the DFT
  - Segmenting a signal
  - Convolution with a segmented signal

DFT Is sampled version of DTFT

Comparison: DFT and DTFT

DFT
$$X[k]=\sum_{n=0}^{N-1} x[n]e^{-j(2\pi/N)kn} \qquad k=0,1,\ldots,N-1$$

Inverse DFT
$$x[n]=\frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j(2\pi/N)kn} \qquad n=0,1,\ldots,N-1$$

DTFT
$$X(e^{j\hat{\omega}})=\sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n} \qquad -\pi\le\hat{\omega}<\pi$$

Inverse DTFT
$$x[n]=\frac{1}{2\pi}\int_{0}^{2\pi} X(e^{j\hat{\omega}})e^{j\hat{\omega}n}\,d\hat{\omega} \qquad -\infty<n<\infty$$

A DFT EXAMPLE

A DFT Example

- The DFT of an impulse is a constant

$$P[k]=\sum_{n=0}^{N-1}\delta[n]e^{-j(2\pi/N)kn}=1, \quad k=0,1,\ldots,N-1$$

$$p[n]=\frac{1}{N}\sum_{k=0}^{N-1}e^{j(2\pi/N)kn}=\begin{cases}1 & n=rN\\ 0 & \text{otherwise}\end{cases}$$

- Showing the implicit periodicity explicitly:

$$p[n]=\sum_{r=-\infty}^{\infty}\delta[n-rN]$$

A SAMPLING THEOREM FOR THE DTFT

Sampling the DTFT

- The DTFT of finite-length x[n]

$$X(e^{j\hat{\omega}}) = \sum_{n=0}^{N-1} x[n]e^{-j\hat{\omega}n} \qquad 0 \le \hat{\omega} < 2\pi$$

- The DFT is the sampled DTFT

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j(2\pi/N)kn} = X(e^{j\omega_k}) \qquad k = 0,1,\ldots,N-1$$

- DFT samples the DTFT at frequencies

$$\omega_k = (2\pi / N)k \qquad k = 0,1,\ldots,N-1$$

Sample the DTFT and then reconstruct x[n] by inverse - I

$$X[k] = X(e^{j\hat{\omega}})\Big|_{\hat{\omega}=(2\pi/N)k} = \sum_{m=0}^{N-1} x[m]e^{-j(2\pi/N)km}$$

Inverse DFT of sampled DTFT

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j(2\pi/N)kn}$$

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{m=0}^{N-1} x[m]e^{-j(2\pi/N)km}\right)e^{j(2\pi/N)kn}$$

$$\tilde{x}[n] = \sum_{m=0}^{N-1} x[m]\left(\frac{1}{N}\sum_{k=0}^{N-1} e^{j(2\pi/N)k(n-m)}\right) = \sum_{m=0}^{N-1} x[m]p[n-m]$$

Sample the DTFT and then reconstruct - II

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j(2\pi/N)kn} = \sum_{m=0}^{N-1} x[m]p[n-m]$$

$$\tilde{x}[n] = \sum_{m=0}^{N-1} x[m]\left(\sum_{r=-\infty}^{\infty} \delta[n-m-rN]\right)$$

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty}\left(\sum_{m=0}^{N-1} x[m]\delta[n-m-rN]\right)$$

$$\boxed{\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j(2\pi/N)kn} = \sum_{r=-\infty}^{\infty} x[n-rN]}$$

Time-aliased signal $x[n]$

Summary of DTFT Sampling Theorem

- Sample DTFT to get DFT

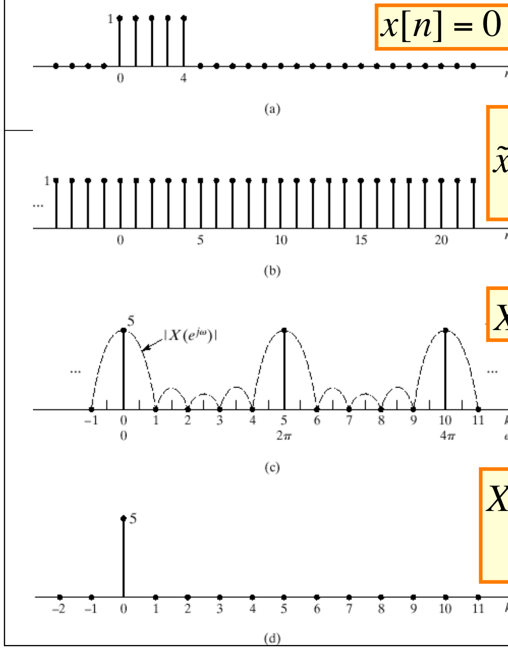
$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j(2\pi/N)kn} = X(e^{j\omega_k}) \quad k = 0,1,\ldots,N-1$$

- Reconstruction by IDFT

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j(2\pi/N)kn} = \sum_{r=-\infty}^{\infty} x[n-rN]$$

- Exact reconstruction if no overlap; i.e.,

$$\tilde{x}[n] = x[n] \quad 0 \le n \le N-1, \text{ if } x[n] \ne 0 \text{ only for } 0 \le n \le N-1$$



$x[n] = 0 \quad n < 0 \text{ and } n > M$

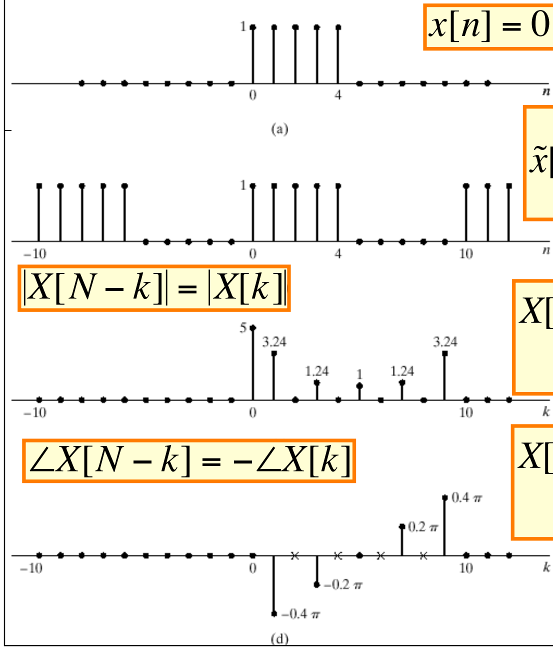
$M = 4$

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n-rN]$$

$N = 5$

$X[k] = X(e^{j(2\pi/N)k})$

$X[k] = X(e^{j(2\pi/N)k}),$
$k = 0,1,\ldots,N-1$



$x[n] = 0 \quad n < 0 \text{ and } n > M$

$M = 4$

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n-rN]$$

$N = 10$

$|X[N-k]| = |X[k]|$

$X[k] = X(e^{j(2\pi/N)k}),$
$k = 0,1,\ldots,N-1$

$\angle X[N-k] = -\angle X[k]$

$X[N-k] = X^*[k],$
$k = 0,1,\ldots,N-1$

USING THE DTFT SAMPLING THEOREM FOR TIME-DOMAIN CONVOLUTION

Convolution of Finite-Length Sequences

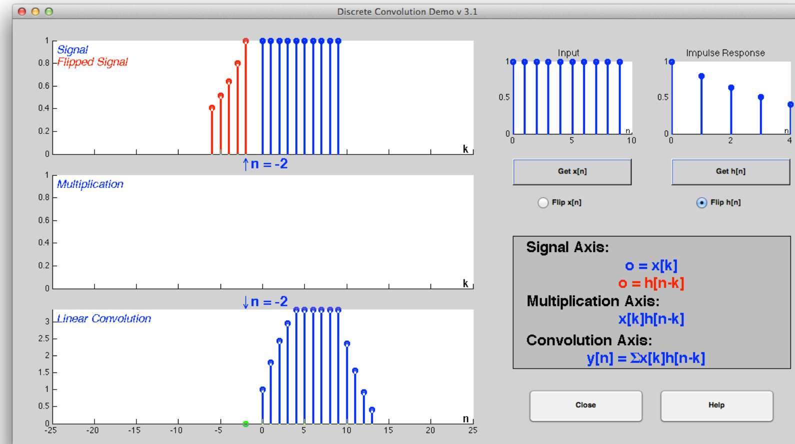
- Let h[n] be of length P and x[n] of length L

$$y[n] = x[n] * h[n] = \sum_{m=0}^{L-1} x[m]h[n-m]$$

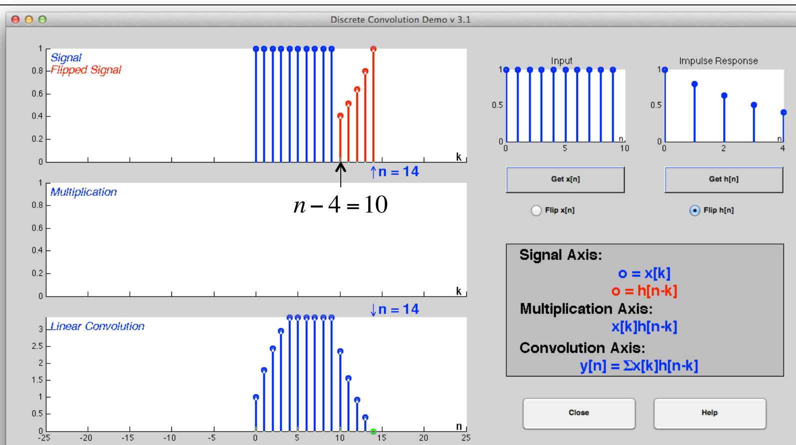
- y[n] also has finite length, L+P-1 samples

$$y[n] = 0 \quad \text{for } n < 0 \text{ and for } n > L + P - 2$$

n < 0
L = 10, P = 5



n > L + P – 2
L = 10, P = 5



Frequency Domain

- DTFT of a convolution

$$y[n] = h[n] * x[n] \Leftrightarrow Y(e^{j\hat{\omega}}) = H(e^{j\hat{\omega}})X(e^{j\hat{\omega}})$$

- Sample the DTFT

$$Y(e^{j(2\pi/N)k}) = H(e^{j(2\pi/N)k})X(e^{j(2\pi/N)k})$$

$$Y[k] = H[k]X[k]$$

Sampling the DTFT of a Convolution

$$y[n] = h[n] * x[n] \Leftrightarrow Y(e^{j\hat{\omega}}) = H(e^{j\hat{\omega}})X(e^{j\hat{\omega}})$$

$$Y(e^{j(2\pi/N)k}) = H(e^{j(2\pi/N)k})X(e^{j(2\pi/N)k})$$

Compute the sampled DTFT by computing the DFTs

$$X[k] = \sum_{n=0}^{L-1} x[n]e^{-j(2\pi/N)kn} \qquad H[k] = \sum_{n=0}^{P-1} h[n]e^{-j(2\pi/N)kn}$$

$$\tilde{y}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \underbrace{H[k]X[k]}_{Y[k]} e^{j(2\pi/N)kn} = \sum_{r=-\infty}^{\infty} y[n-rN]$$

$$\boxed{\tilde{y}[n] = y[n] \quad 0 \le n \le N-1 \quad \text{if } N \ge L+P-1}$$

BLOCK PROCESSING OF LONG SIGNALS

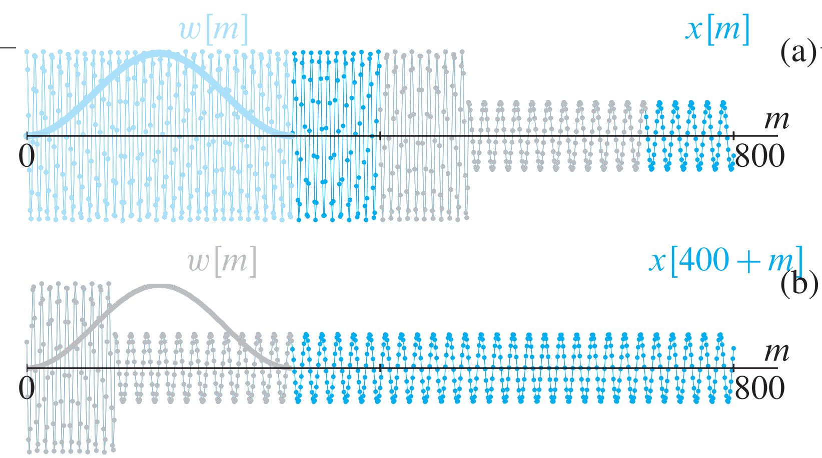
Time-Dependent (Short Time) DTFT and DFT

- Definition: short-time DTFT

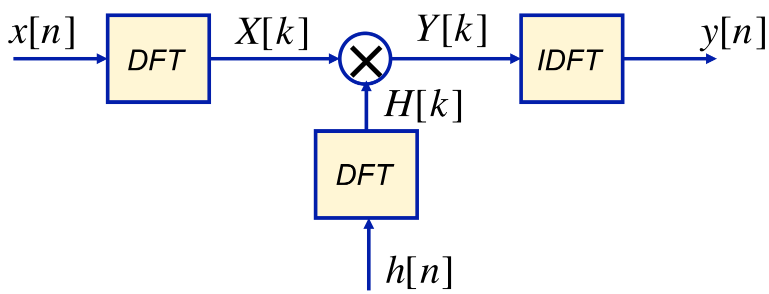
$$X(e^{j\hat{\omega}}, n) = \sum_{m=0}^{L-1} w[m]x[m+n]e^{-j\hat{\omega}n} \qquad 0 \le \hat{\omega} < 2\pi$$

- Definition: short-time DFT

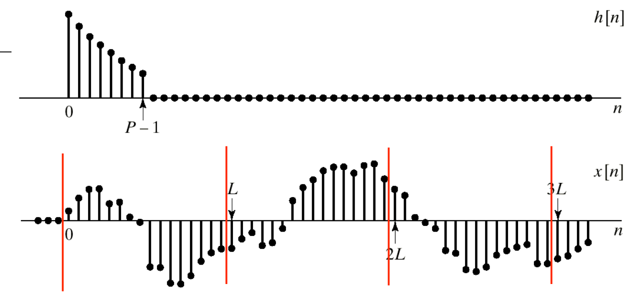
$$X[k,n] = \sum_{m=0}^{L-1} w[m]x[m+n]e^{-j(2\pi/N)kn} \qquad k = 0,1,\ldots,N-1$$



Convolution Using the DFT



Block Convolution



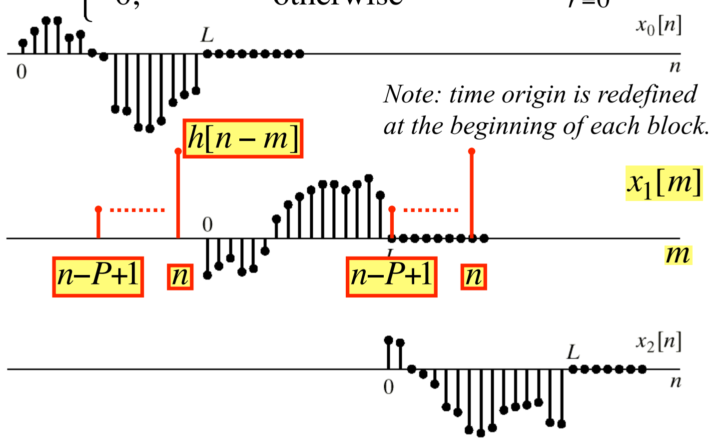
$$x[n] = \sum_{r=0}^{\infty} x_r[n-rL] \rightarrow y[n] = x[n] * h[n] = \sum_{r=0}^{\infty} y_r[n-rL]$$

$$x_r[m] = \begin{cases} x[m+rL], & 0 \le m \le L\text{-}1 \\ 0, & \text{otherwise} \end{cases} \rightarrow y_r[n] = x_r[n] * h[n]$$

Rectangular window

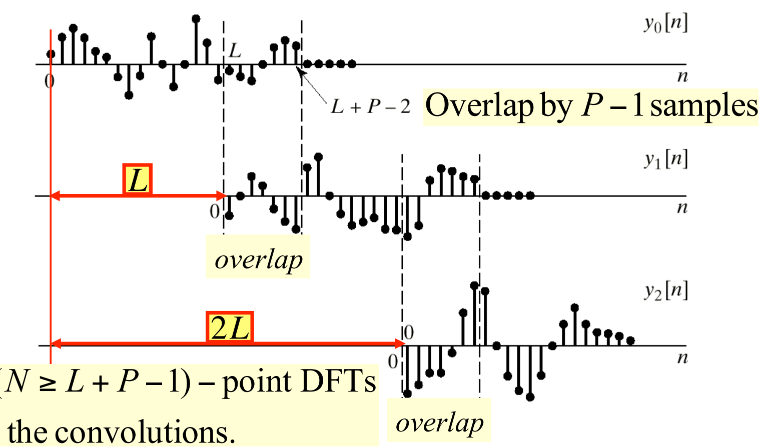
Segmenting the Input

$$x_r[n] = \begin{cases} x[n+rL], & 0 \le \text{n} \le L\text{-}1 \\ 0, & \text{otherwise} \end{cases} \qquad x[n] = \sum_{r=0}^{\infty} x_r[n-rL]$$



Putting the Output Pieces Together

$$y[n] = \sum_{r=0}^{\infty} y_r[n-rL] \qquad y_r[n] = x_r[n] * h[n]$$



Use $(N \ge L+P-1)$ – point DFTs to do the convolutions.