STANFORD UNIVERSITY
EE 102B Spring-2013

Lecture 03
Periodic Signals, Harmonics & Time-Varying Sinusoids
April 5, 2013

ASSIGNMENTS

- Reading for this Lecture:
  - SPF: Chapter 3 and .pdf of new Section 3.6
  - S&S: Review book sections on continuous-time Fourier series
- HW#1 and Lab #1 are posted
  - Both due on April 10

LECTURE OBJECTIVES

- Signals with HARMONIC Frequencies
  - Add Sinusoids with $f_k = kf_0$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

***Second Topic**: FREQUENCY can change vs. TIME*

Introduce Spectrogram Visualization

`(spectrogram.m)` **`(plotspec.m)`**

Chirps: $x(t) = \cos(\alpha t^2)$

Lab #01

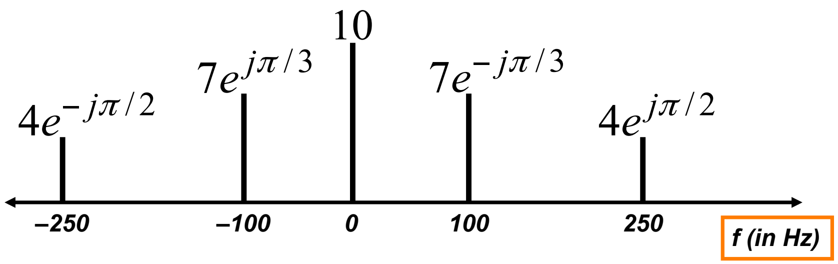
$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

```
function    xx = syn_fourier(tt, ak, fk)
%SYN_FOURIER  Function to synthesize a sum of complex
%         exponentials over the time range given by tt
% usage:
%   xx = syn_fourier( tt, ak, fk )
%     tt = vector of times, for the time axis
%     ak = vector of complex Fourier coefficients
%     fk = vector of frequencies
%           (usually contains both negative and positive freqs)
%     xx = vector of synthesized waveform values
%
%   Note: fk and ak must be the same length.
%         ak(1) corresponds to frequency fk(1),
%         ak(2) corresponds to frequency fk(2), etc.
%
% Note: the output might have a tiny imaginary part even if it
%     is supposed to be purely real.  If so, take the real part.
```

SPECTRUM PLOT

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

- Complex Amplitude vs. Frequency

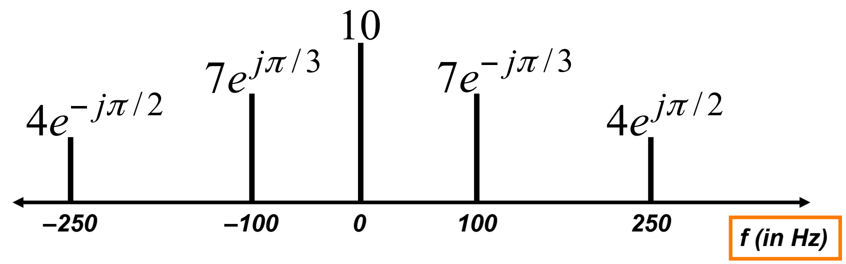


$$x(t) = 10 + 7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t} + 4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

SPECTRUM PLOT

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

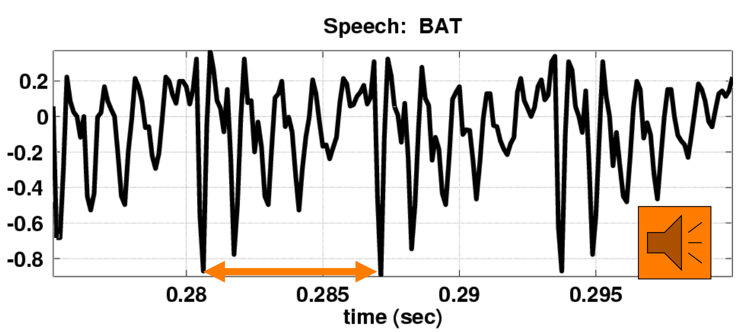
- Complex Amplitude vs. Frequency



$$x(t) = 10 + 14\cos(2\pi(100)t - \pi/3) + 8\cos(2\pi(250)t + \pi/2)$$

Example of a Periodic Signal

- Nearly Periodic in the Vowel Region
  - Period is (Approximately) T = 0.0065 sec



Harmonic Signal

Periodic signal : $x(t) = x(t+T)$

Can only have ***harmonic*** freqs : $f_k = k f_0$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$x(t)$ will be periodic if $\boxed{f_0 T = 1}$

$$\cos(2\pi k f_0 (t+T) + \varphi_k) = \cos(2\pi k f_0 t + 2\pi k f_0 T + \varphi_k)$$

$\boldsymbol{\mathit{Harmonic}}$ freqs : $f_k = k f_0$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

Phasor: $X_k = A_k e^{j\varphi_k}$

$$f_0 = \frac{1}{T}$$

$$x(t) = X_0 + \sum_{k=1}^{N} \left\{ \tfrac{1}{2} X_k e^{j2\pi k f_0 t} + \tfrac{1}{2} X_k^* e^{-j2\pi k f_0 t} \right\}$$

$$x(t) = a_0 + \sum_{k=1}^{N} \left\{ a_k e^{j2\pi k f_0 t} + a_{-k} e^{-j2\pi k f_0 t} \right\}$$

Define FUNDAMENTAL FREQ

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \phi_k) = \sum_{k=-N}^{N} a_k e^{j2\pi k f_0 t}$$

Largest f_0 such that

$f_k = k f_0 \qquad (\omega_0 = 2\pi f_0)$

$$f_0 = \frac{1}{T_0}$$

f_0 = fundamental Frequency

f_k / f_0 = integer, for all k

T_0 = fundamental Period

__Main point__:
for periodic signals, all spectral components are __integer__ multiples of the fundamental frequency

General Periodic Signals

- Synthesis $\quad x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k t}$
- Analysis $\quad a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j\omega_0 k t}\, dt$
- Fourier transform

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

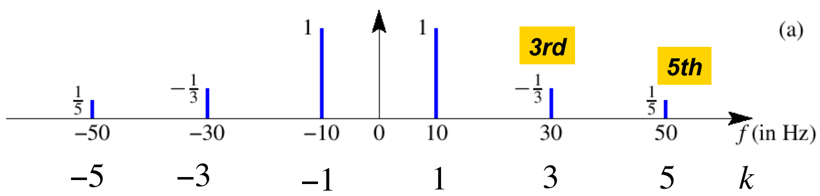
General Non-Periodic Signals

- Synthesis $\quad x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_k t}$
- Analysis
- Fourier transform

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - \omega_k)$$

Harmonic Spectrum (3 Freqs)

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi k f_0 t}$$

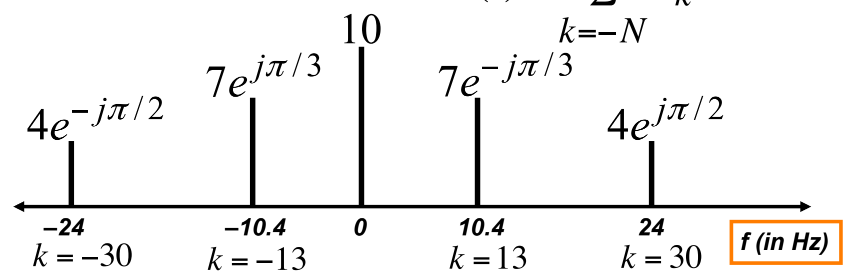


What is the fundamental frequency?

10 Hz

Another Example

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi k f_0 t}$$



What is the fundamental frequency?

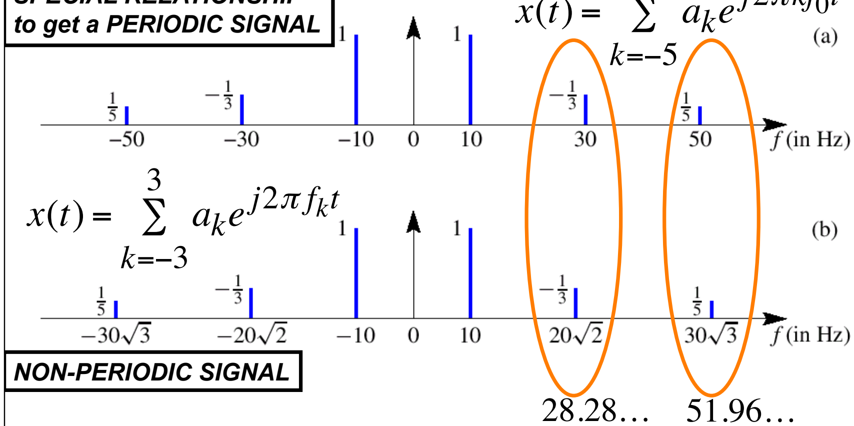
(0.1)GCD(104,240) = (0.1)(8)=0.8 Hz

Irrational Frequencies

SPECIAL RELATIONSHIP to get a PERIODIC SIGNAL

$$x(t) = \sum_{k=-5}^{5} a_k e^{j2\pi k f_0 t}$$

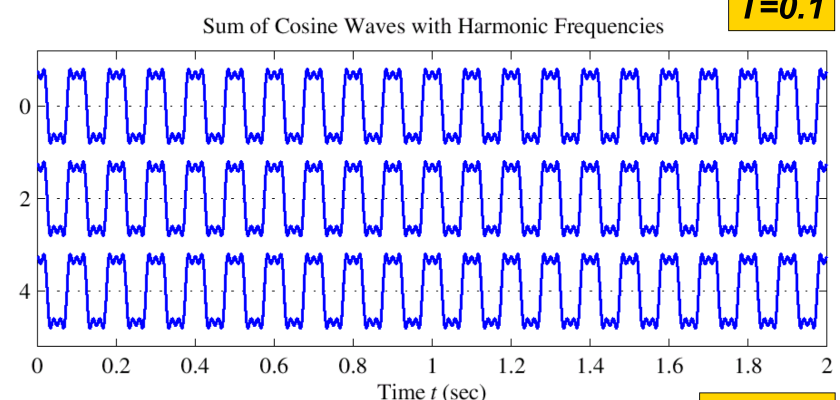
$$x(t) = \sum_{k=-3}^{3} a_k e^{j2\pi f_k t}$$



NON-PERIODIC SIGNAL

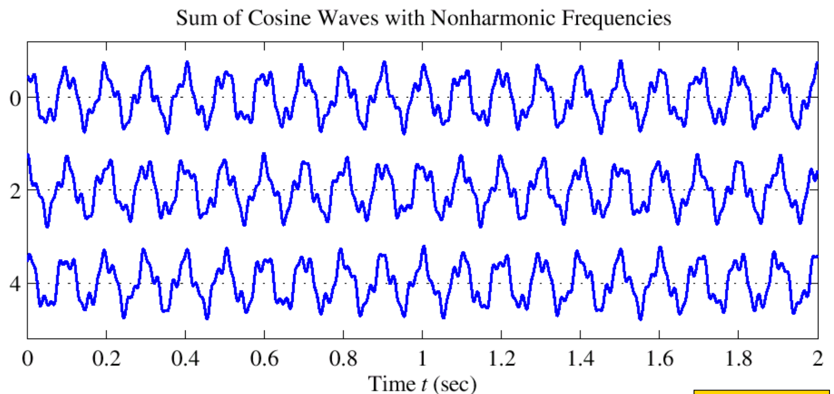
Harmonic Signal (3 Freqs)

T=0.1



PERIODIC

NON-Harmonic Signal



NOT PERIODIC

TIME-VARYING SPECTRUM

Time-Varying FREQUENCIES Diagram

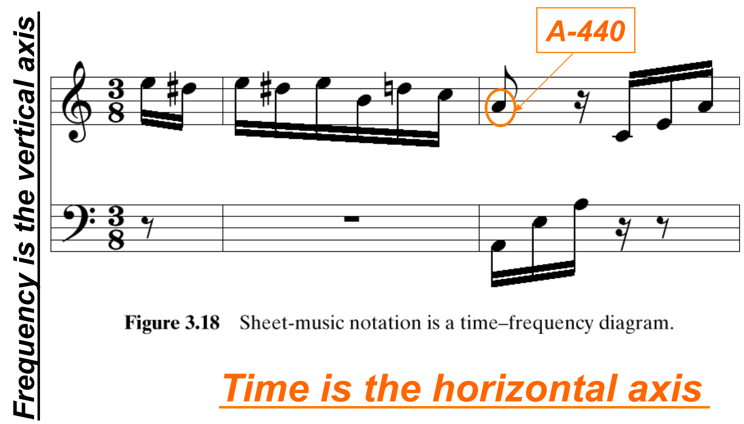
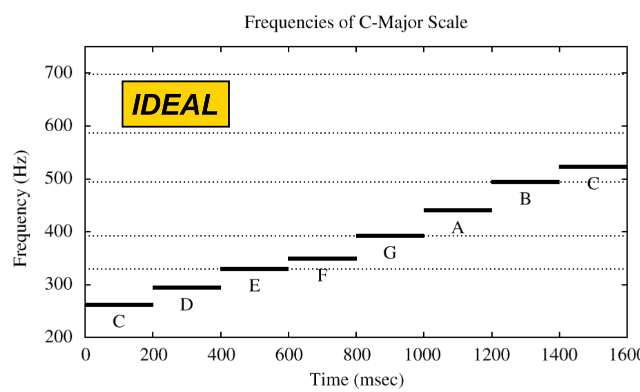


Figure 3.18 Sheet-music notation is a time–frequency diagram.

Time is the horizontal axis

A Simple Time-Varying Signal

- C-major SCALE: stepped frequencies
  - Frequency is constant for each note

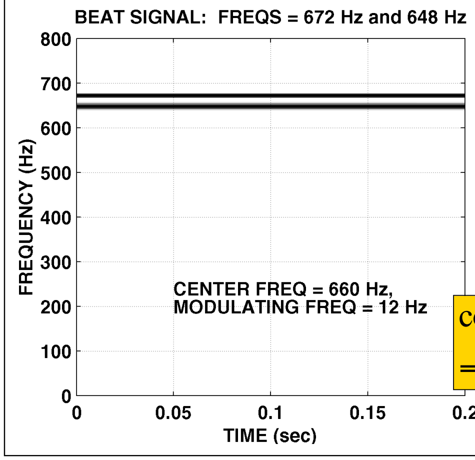


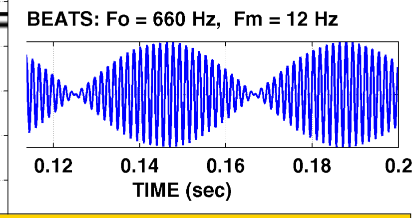
SPECTROGRAM

- SPECTROGRAM Tool
  - MATLAB function is `spectrogram.m`
  - SP-First has `plotspec.m` & `spectgr.m`
- **ANALYSIS** program
  - Takes x(t) as input
  - Produces spectrum values X_k
  - Breaks x(t) into SHORT TIME SEGMENTS
    - Then uses the FFT (Fast Fourier Transform)

SPECTROGRAM EXAMPLE

- Two **Constant** Frequencies: Beats





$$\cos(2\pi(672)t) + \cos(2\pi(648)t)$$
$$= 2\cos(2\pi(12)t)\cos(2\pi(660)t)$$

Like a AM DSB-SC Radio Signal

- Same as BEAT Notes

$$\cos(2\pi(660)t)\sin(2\pi(12)t)$$

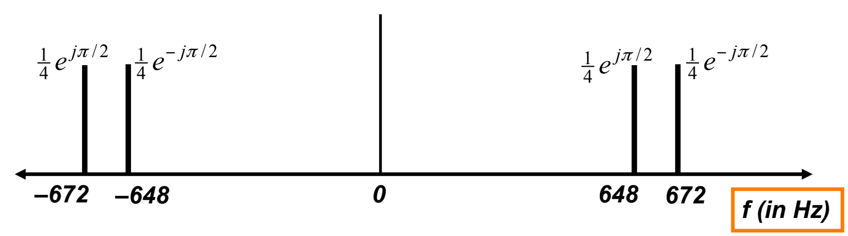
$$\tfrac{1}{2}\left(e^{j2\pi(660)t} + e^{-j2\pi(660)t}\right)\tfrac{1}{2j}\left(e^{j2\pi(12)t} - e^{-j2\pi(12)t}\right)$$

$$\tfrac{1}{4j}\left(e^{j2\pi(672)t} - e^{-j2\pi(672)t} - e^{j2\pi(648)t} + e^{-j2\pi(648)t}\right)$$

$$\tfrac{1}{2}\cos(2\pi(672)t - \tfrac{\pi}{2}) + \tfrac{1}{2}\cos(2\pi(648)t + \tfrac{\pi}{2})$$

SPECTRUM of AM (Amplitude Modulation)

- **SUM** of 4 complex exponentials:

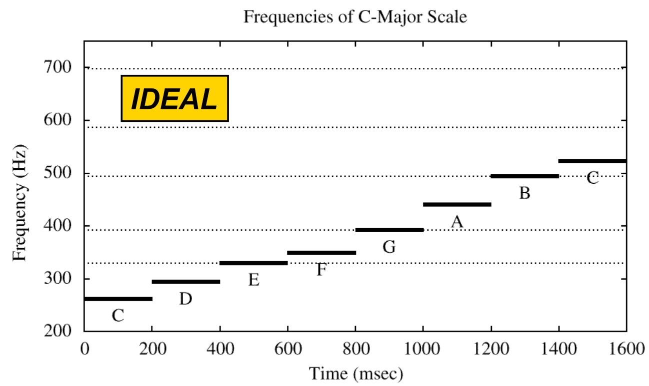


What is the fundamental frequency?

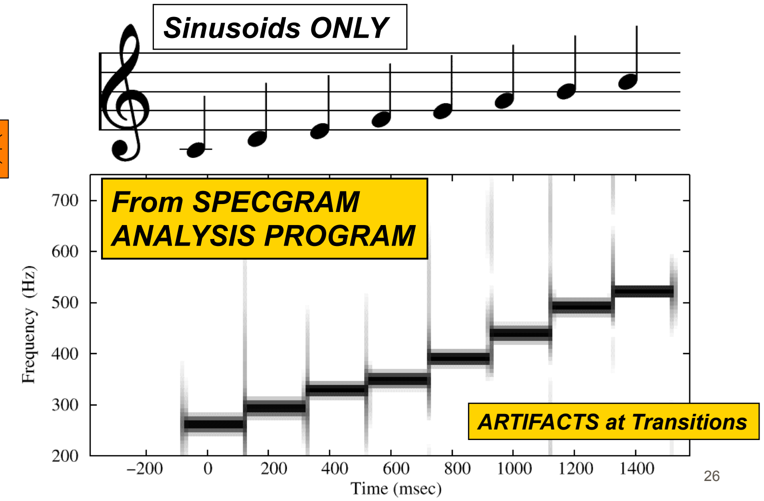
648 Hz ? ***24 Hz ?***

STEPPED FREQUENCIES

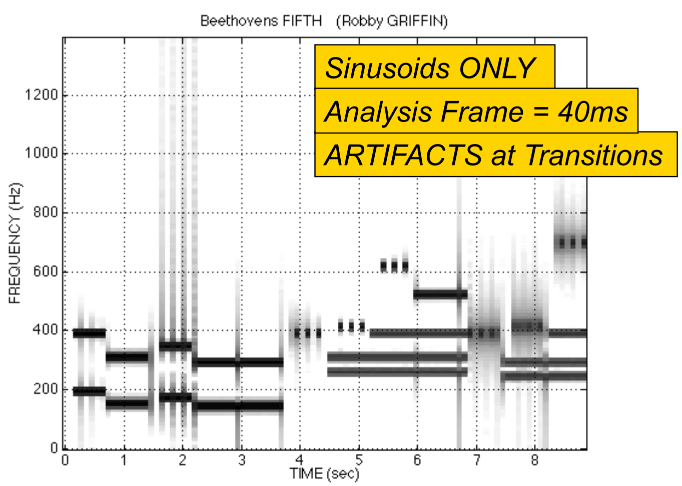
- C-major SCALE: successive sinusoids
  - Frequency is constant for each note



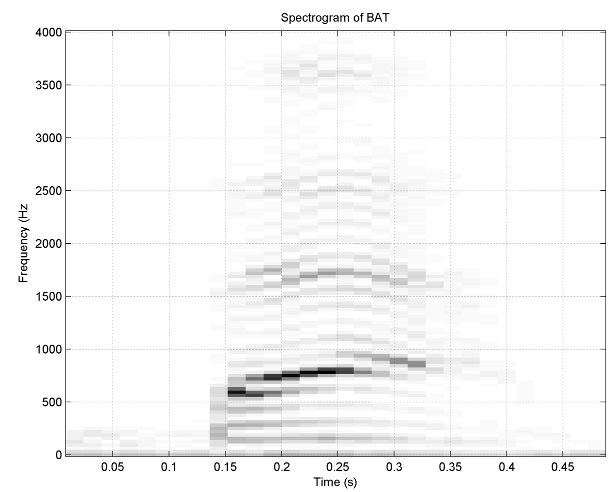
SPECTROGRAM of C-Scale



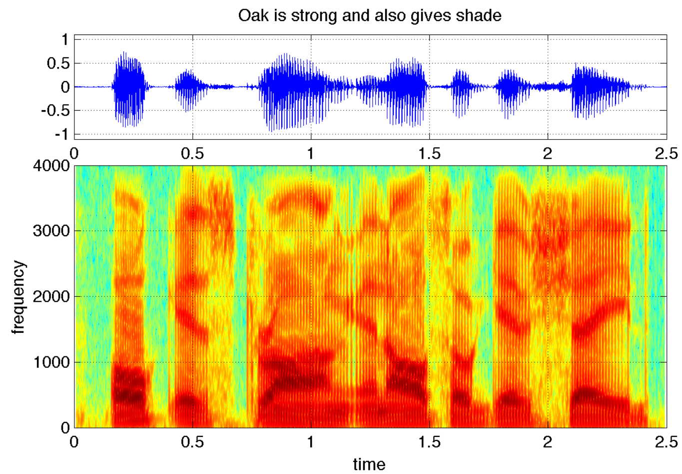
Spectrogram of LAB SONG



Spectrogram of BAT (plotspec)



Speech Spectrogram



Time-Varying Frequency

- Frequency can change vs. time
  - Continuously, not stepped
- **<u>FREQUENCY MODULATION (FM)</u>**

$$\boxed{x(t) = \cos(2\pi f_c t + v(t))}$$

VOICE

- CHIRP SIGNALS
  - Linear Frequency Modulation (LFM)

New Signal: Linear FM

- Called Chirp Signals (LFM)
  - Quadratic phase

QUADRATIC

$$\boxed{x(t) = A\cos(\alpha t^2 + 2\pi f_0 t + \varphi)}$$

- Freq will change LINEARLY vs. time
  - Example of Frequency Modulation (FM)
  - Define "instantaneous frequency"

INSTANTANEOUS FREQ

- Definition

$$x(t) = A\cos(\psi(t))$$
$$\Rightarrow \omega_i(t) = \frac{d}{dt}\psi(t)$$

Derivative of the "Angle"

- For Sinusoid:

$$x(t) = A\cos(2\pi f_0 t + \varphi)$$
$$\psi(t) = 2\pi f_0 t + \varphi$$

Makes sense

$$\Rightarrow \omega_i(t) = \frac{d}{dt}\psi(t) = 2\pi f_0$$

INSTANTANEOUS FREQ of the Chirp

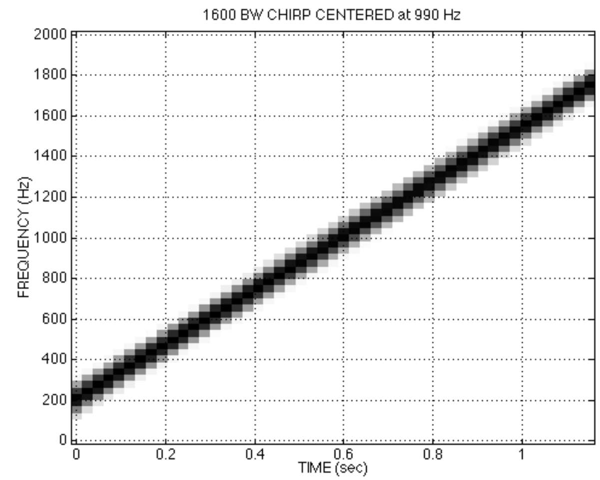
- Chirp Signals have Quadratic phase
- Freq will change LINEARLY vs. time

$$x(t) = A\cos(\alpha t^2 + \beta t + \varphi)$$

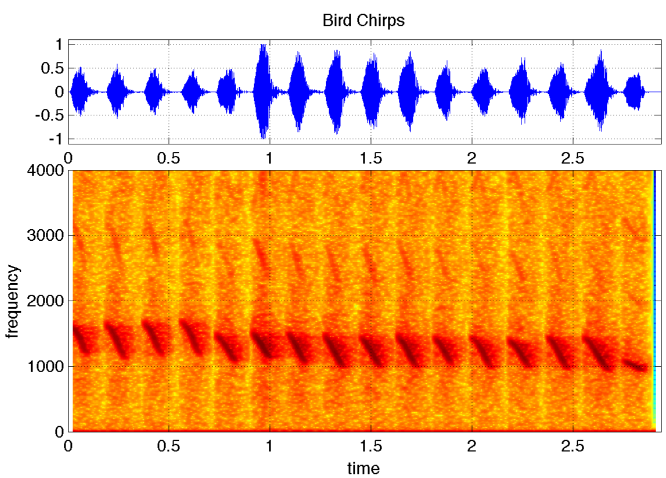
$$\Rightarrow \psi(t) = \alpha t^2 + \beta t + \varphi$$

$$\Rightarrow \omega_i(t) = \frac{d}{dt}\psi(t) = 2\alpha t + \beta$$

CHIRP SPECTROGRAM



Bird Chirp



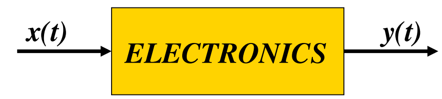
SAMPLING CONTINUOUS-TIME SIGNALS

SYSTEMS Process Signals

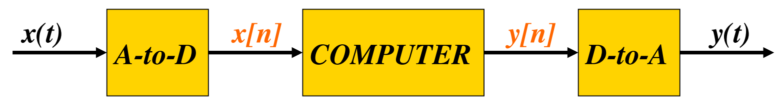
- PROCESSING GOALS:
  - Change x(t) into y(t)
    - For example, more BASS, pitch shifting
  - Improve x(t), e.g., image deblurring
  - Extract Information from x(t)
  - Digital code for transmission and storage

System IMPLEMENTATION

- *ANALOG/ELECTRONIC:*
  - *Circuits: resistors, capacitors, op-amps*

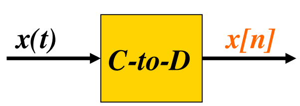


- DIGITAL/MICROPROCESSOR
  - Convert x(t) to numbers stored in memory



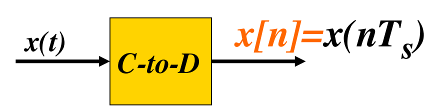
SAMPLING x(t)

- SAMPLING PROCESS
  - Convert x(t) to numbers x[n]
  - "n" is an integer; x[n] is a sequence of values
  - Think of "n" as the storage address in memory
- UNIFORM SAMPLING at $t = nT_s$
  - IDEAL: $x[n] = x(nT_s)$



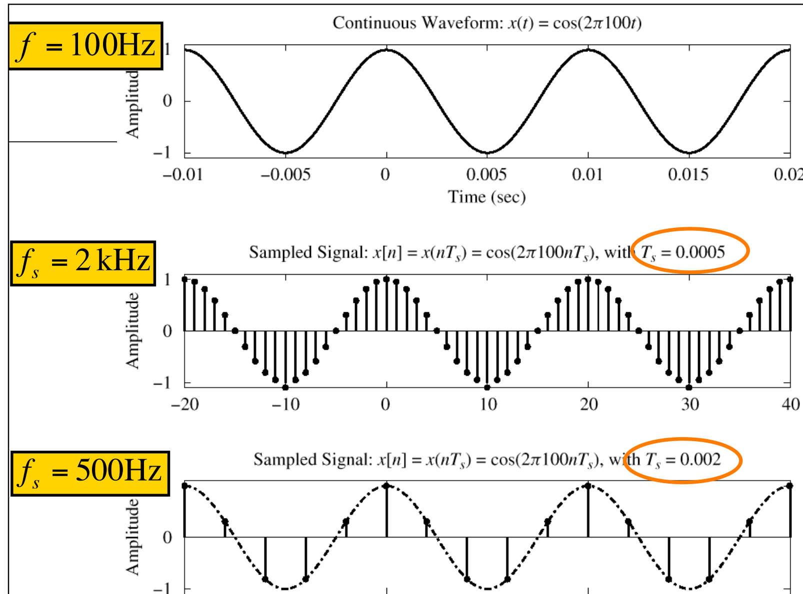
SAMPLING RATE, f_s

- SAMPLING RATE (f_s)
  - $f_s = 1/T_s$
    - NUMBER of SAMPLES PER SECOND
  - T_s = 125 microsec → f_s = 8000 samples/sec
    - UNITS ARE HERTZ: 8000 Hz
- UNIFORM SAMPLING at $t = nT_s = n/f_s$
  - IDEAL: $x[n] = x(nT_s) = x(n/f_s)$



STORING DIGITAL SOUND

- $x[n]$ is a SAMPLED SINUSOID
  - A list of numbers stored in memory
- EXAMPLE: audio CD
- CD rate is 44,100 samples per second
  - 16-bit samples
  - Stereo uses 2 channels
- Number of bytes for 1 minute is
  - 2 X (16/8) X 60 X 44100 = 10.584 Mbytes



Which one provides most accurate representation of x(t)?

SAMPLING THEOREM

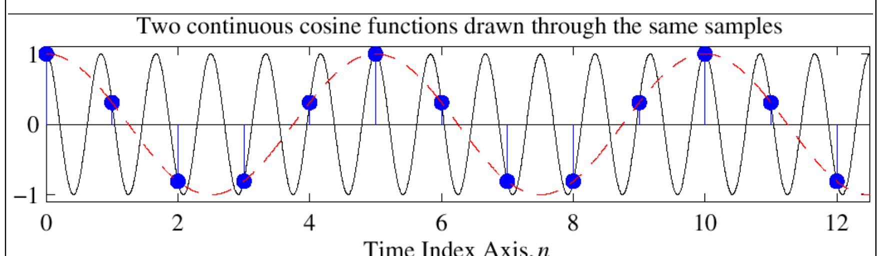
- HOW OFTEN DO WE NEED TO SAMPLE?
  - DEPENDS on FREQUENCY of SINUSOID
  - ANSWERED by SHANNON/NYQUIST Theorem
  - ALSO DEPENDS on "**RECONSTRUCTION**"

Shannon Sampling Theorem

A continuous-time signal $x(t)$ with frequencies no higher than f_{max} can be reconstructed exactly from its samples $x[n] = x(nT_s)$, if the samples are taken at a rate $f_s = 1/T_s$ that is greater than $2f_{max}$.

Reconstruction? Which One?

Given the samples, draw a sinusoid through the values



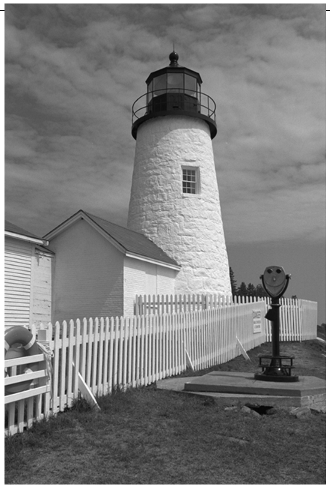
$$x[n] = \cos(0.4\pi n)$$

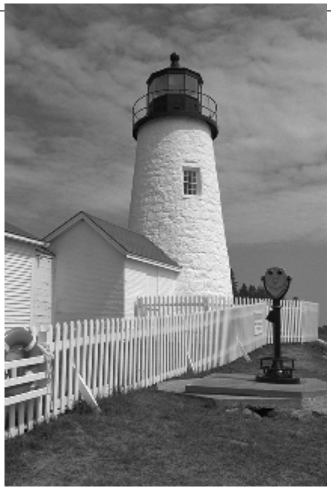
When n is an integer

$\cos(0.4\pi n) = \cos(2.4\pi n)$

Reconstruction picks lowest frequency sinusoid

Spatial Aliasing





Spatial Aliasing

DISCRETE-TIME SINUSOID

- Change x(t) into x[n]

$$x(t) = A\cos(\omega t + \varphi)$$

$$x[n] = x(nT_s) = A\cos(\omega n T_s + \varphi)$$

$$x[n] = A\cos((\omega T_s)n + \varphi)$$

$$x[n] = A\cos(\hat{\omega} n + \varphi)$$

$$\hat{\omega} = \omega T_s = \frac{\omega}{f_s}$$

DEFINE DIGITAL FREQUENCY

DIGITAL FREQUENCY $\hat{\omega}$

$$\boxed{\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s}}$$

- $\hat{\omega}$ VARIES from **0** to **2π**, as f varies from 0 to the sampling frequency
- UNITS are radians, <u>**not**</u> rad/sec
  - DIGITAL FREQUENCY is <u>NORMALIZED</u>