STANFORD UNIVERSITY
EE 102B Spring-2013

Lecture 08

Discrete-Time Filtering of Continuous-Time Signals

April 17, 2013

ASSIGNMENTS

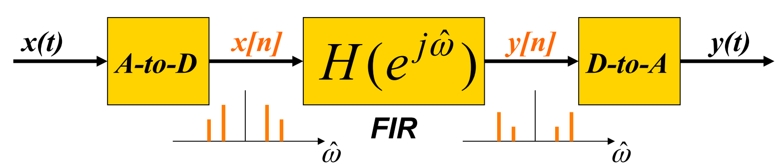
- Reading for this Lecture:
  - SPF: Sections 6-6, 6-7, & 6-8
  - S&S: Sections 2.1 and 3.2
- HW#02 is due by 5pm today, April 17 in Packard 263.
- HW#03 will be posted today.
- Lab #02 is posted. It is due by 5pm, Friday, April 19, in Packard 263

Office Hours for Course Staff – Come see us.

- Ron Schafer: Mon. 2:00-3:00 pm and Weds. 4:00-5:00 pm in Packard 211
- Dookun Park: Tues. 1:30 ~ 3:30 pm in Packard 107
- Ruo Yu Gu (Roy): Mon. 4:00 ~ 6:00 pm in Packard 106
- https://class2go.stanford.edu/EE102B/Spring2013/pages/staff

LECTURE OBJECTIVES

- Two Domains: Time & Frequency
- Track the spectrum of x[n] thru an FIR Filter: **Sinusoid-IN gives Sinusoid-OUT**
- UNIFICATION: How does the Frequency Response affect x(t) to produce y(t) ?



TIME & FREQUENCY

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{M} h[k]x[n-k]$$

FIR DIFFERENCE EQUATION is the TIME-DOMAIN

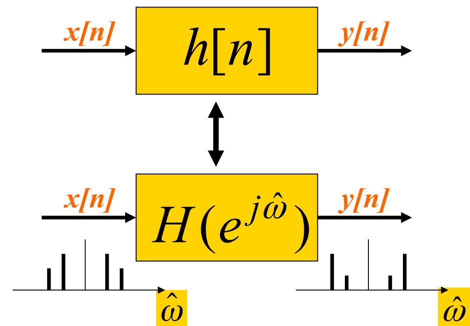
$$\boxed{H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} h[k]e^{-j\hat{\omega}k}}$$

$$H(e^{j\hat{\omega}}) = h[0] + h[1]e^{-j\hat{\omega}} + h[2]e^{-j2\hat{\omega}} + h[3]e^{-j3\hat{\omega}} + \cdots$$

$$= \left|H(e^{j\hat{\omega}})\right| e^{j\angle H(e^{j\hat{\omega}})}$$

BLOCK DIAGRAMS

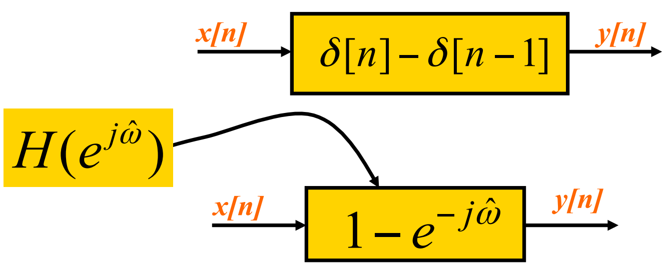
- Equivalent Representations



FIRST DIFFERENCE SYSTEM

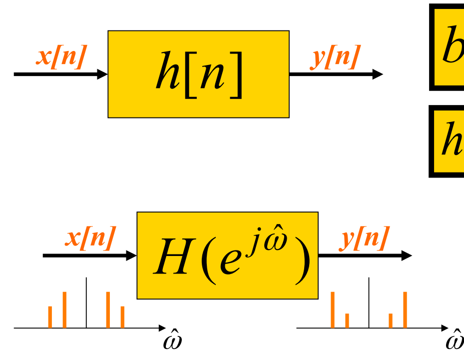
Find $h[n]$ and $H(e^{j\hat{\omega}})$ for the Difference

Equation: $y[n] = x[n] - x[n-1]$



Ex: DELAY by 2 SYSTEM

Find $h[n]$ and $H(e^{j\hat{\omega}})$ for $y[n] = x[n-2]$

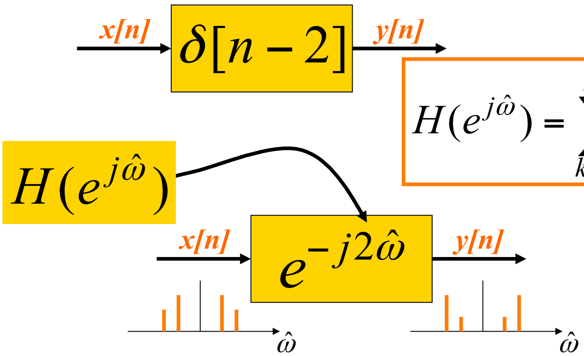


$$\boxed{b_k = \{0, 0, 1\}}$$

$$\boxed{h[n] = \delta[n-2]}$$

DELAY by 2 SYSTEM

Find $h[n]$ and $H(e^{j\hat{\omega}})$ for $y[n] = x[n-2]$



$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} \delta[k-2] e^{-j\hat{\omega}k}$$

***k* = 2 ONLY**

GENERAL DELAY PROPERTY

Find $h[n]$ and $H(e^{j\hat{\omega}})$ for $y[n] = x[n-n_d]$

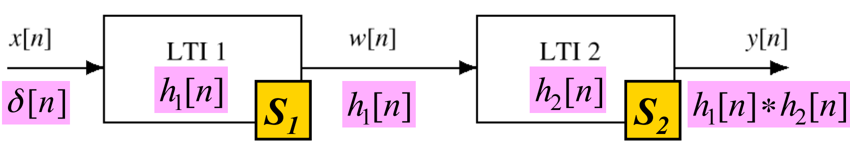
$$h[n] = \delta[n-n_d]$$

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} \delta[k-n_d] e^{-j\hat{\omega}k} = e^{-j\hat{\omega}n_d}$$

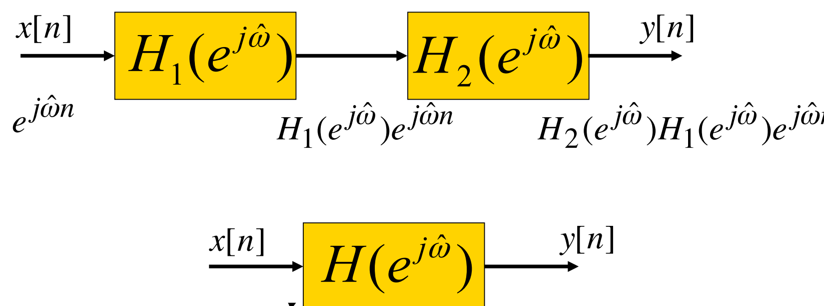
ONLY ONE non-ZERO TERM for k at k = n_d

CASCADE SYSTEMS

- Does the order of S_1 & S_2 matter?
  - WHAT ARE THE FILTER COEFFS? $\{b_k\}$
  - WHAT is the overall FREQUENCY RESPONSE ?



CASCADE EQUIVALENT

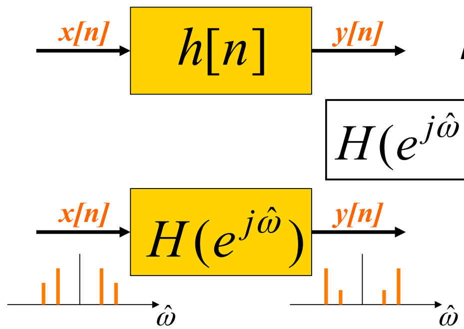


EQUIVALENT SYSTEM

$$H(e^{j\hat{\omega}}) = H_1(e^{j\hat{\omega}})H_2(e^{j\hat{\omega}})$$

FREQ DOMAIN --> TIME ??

Start with $H(e^{j\hat{\omega}})$ and find $h[n]$ or b_k



$$h[n] = ?$$

$$\boxed{H(e^{j\hat{\omega}}) = 7e^{-j2\hat{\omega}}\cos(\hat{\omega})}$$

FREQ DOMAIN --> TIME

$$H(e^{j\hat{\omega}}) = 7e^{-j2\hat{\omega}}\cos(\hat{\omega})$$

EULER's Formula

$$= 7e^{-j2\hat{\omega}}(0.5e^{j\hat{\omega}} + 0.5e^{-j\hat{\omega}})$$

$$= (3.5e^{-j\hat{\omega}} + 3.5e^{-j3\hat{\omega}})$$

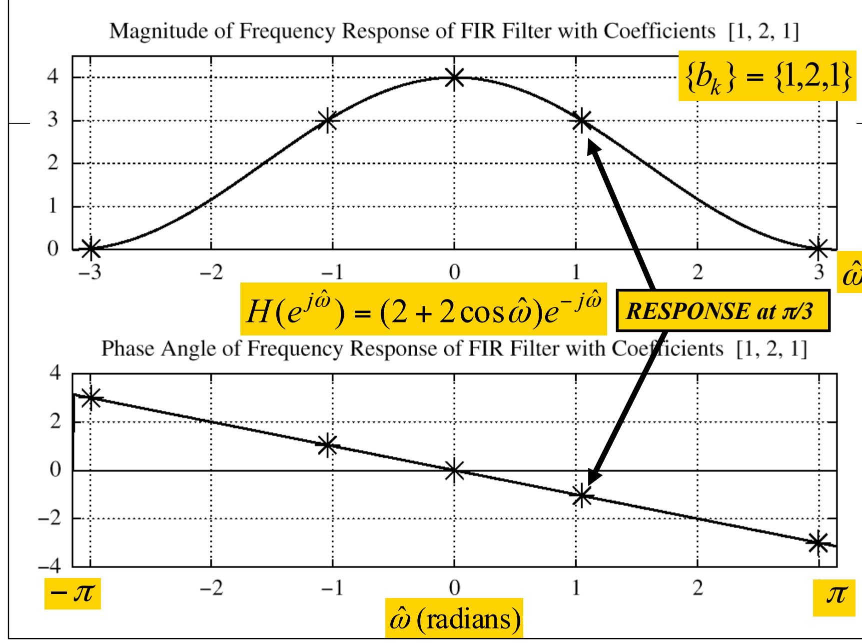
$$h[n] = 3.5\delta[n-1] + 3.5\delta[n-3]$$

$$b_k = \{\,0, 3.5, 0, 3.5\,\}$$

FREQ. RESPONSE PLOTS

- DENSE GRID **(ww)** from -π to +π
  - `ww = -pi:(pi/100):pi;`
- `HH = freqz(bb,1,ww)`
  - VECTOR **bb** contains Filter Coefficients
  - SP-First: `HH = freekz(bb,1,ww)`

$$\boxed{H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}}$$



EX: COSINE INPUT (ans-2)

Find $y[n]$ when $x[n] = 2\cos(\frac{\pi}{3}n + \frac{\pi}{4})$

$$H(e^{j\hat{\omega}}) = (2 + 2\cos\hat{\omega})e^{-j\hat{\omega}}$$

$$y_1[n] = H(e^{j\pi/3})e^{j(\pi n/3+\pi/4)} = 3e^{-j(\pi/3)}e^{j(\pi n/3+\pi/4)}$$

$$y_2[n] = H(e^{-j\pi/3})e^{-j(\pi n/3+\pi/4)} = 3e^{j(\pi/3)}e^{-j(\pi n/3+\pi/4)}$$

$$y[n] = 3e^{j(\pi n/3-\pi/12)} + 3e^{-j(\pi n/3-\pi/12)}$$

$$\Rightarrow y[n] = 6\cos(\tfrac{\pi}{3}n - \tfrac{\pi}{12})$$

General Result for Sinusoidal Input - I

$$x[n] = A\cos(\hat{\omega}_0 n + \phi)$$

$$= \frac{A}{2}e^{j\phi}e^{j\hat{\omega}_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\hat{\omega}_0 n}$$

$$y[n] = \frac{A}{2}e^{j\phi}\left|H(e^{j\hat{\omega}_0})\right|e^{j\angle H(e^{j\hat{\omega}_0})}e^{j\hat{\omega}_0 n}$$

$$+ \frac{A}{2}e^{-j\phi}\left|H(e^{-j\hat{\omega}_0})\right|e^{j\angle H(e^{-j\hat{\omega}_0})}e^{-j\hat{\omega}_0 n}$$

General Result for Sinusoidal Input - II

$$y[n] = \frac{A}{2}e^{j\phi}\left|H(e^{j\hat{\omega}_0})\right|e^{j\angle H(e^{j\hat{\omega}_0})}e^{j\hat{\omega}_0 n}$$

$$+ \frac{A}{2}e^{-j\phi}\left|H(e^{-j\hat{\omega}_0})\right|e^{j\angle H(e^{-j\hat{\omega}_0})}e^{-j\hat{\omega}_0 n}$$

If $h[n]$ is real, $H(e^{-j\hat{\omega}}) = H^*(e^{j\hat{\omega}})$, or

$$\left|H(e^{-j\hat{\omega}})\right| = \left|H(e^{j\hat{\omega}})\right| \text{ and } \angle H(e^{-j\hat{\omega}}) = -\angle H(e^{j\hat{\omega}})$$

General Result for Sinusoidal Input - III

$$y[n] = \frac{A}{2}e^{j\phi}\left|H(e^{j\hat{\omega}_0})\right|e^{j\angle H(e^{j\hat{\omega}_0})}e^{j\hat{\omega}_0 n}$$

$$+ \frac{A}{2}e^{-j\phi}\left|H(e^{j\hat{\omega}_0})\right|e^{-j\angle H(e^{-j\hat{\omega}_0})}e^{-j\hat{\omega}_0 n}$$

$$y[n] =$$

$$A\left|H(e^{j\hat{\omega}_0})\right|\cos\left[\hat{\omega}_0 n + \phi + \angle H(e^{j\omega_0})\right]$$

Running Average Filters

- Difference equation

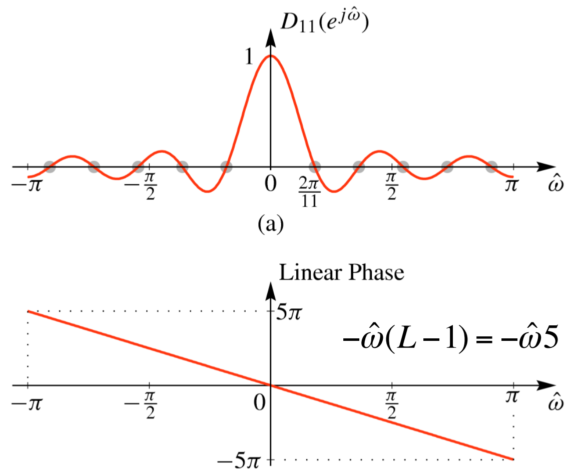
$$y[n] = \frac{1}{L}\sum_{k=0}^{L-1} x[n-k]$$

- Frequency response

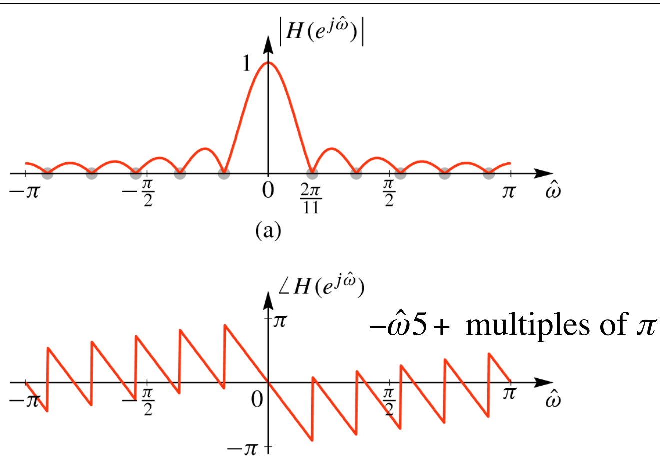
$$H(e^{j\hat{\omega}}) = \frac{1}{L}\sum_{k=0}^{L-1} e^{-j\hat{\omega}k} = D_L(e^{j\hat{\omega}})e^{-j\hat{\omega}(L-1)/2}$$

where $D_L(e^{j\hat{\omega}}) = \dfrac{\sin(\hat{\omega}L/2)}{L\sin(\hat{\omega}/2)}$ (Dirichlet function)

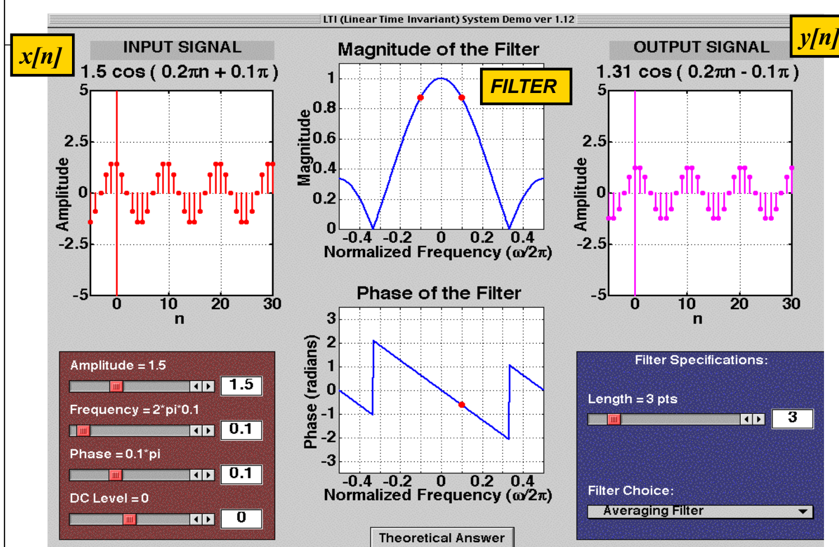
Plot of Frequency Response - I



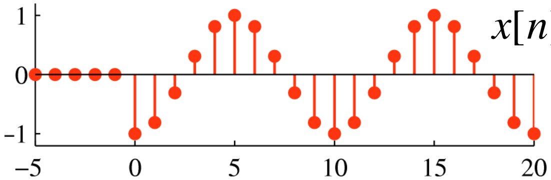
Plot of Frequency Response - II



LTI Demo with Sinusoids

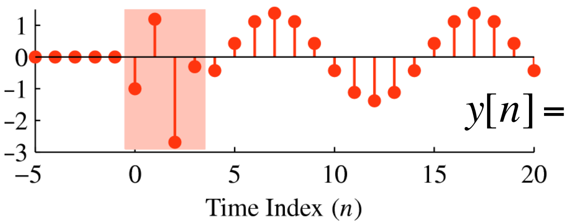


Sinusoidal Steady-State



$$x[n] = \cos(\omega_0 n)u[n]$$

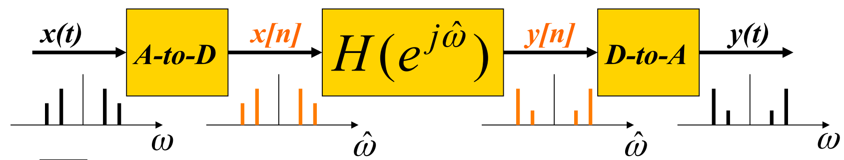
(a)



(b)

$$y[n] = \sum_{k=0}^{M} h[k]x[n-k]$$

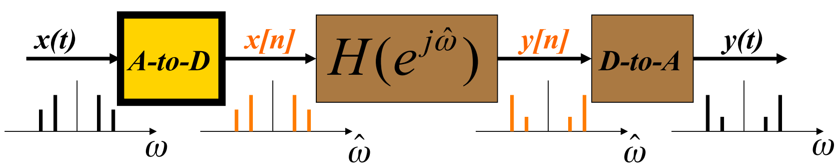
DIGITAL "FILTERING"



- ω SPECTRUM of x(t) (SUM of SINUSOIDS)
- $\hat{\omega}$ SPECTRUM of x[n] (ALIASING a PROBLEM?)
- SPECTRUM y[n] (FIR Gain or Nulls)
- ω Then, OUTPUT y(t) = SUM of SINUSOIDS

Main question: how to characterize the effects of a digital filter on the analog signals?

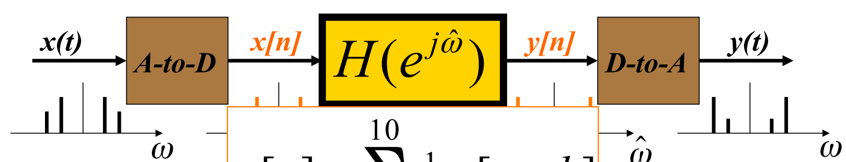
FREQUENCY SCALING



- TIME SAMPLING: $t = nT_s$
  - IF NO ALIASING:
  - FREQUENCY SCALING

$$\hat{\omega} = \omega T_s = \frac{\omega}{f_s}$$

11-pt AVERAGER Example



$$y[n] = \sum_{k=0}^{10} \tfrac{1}{11} x[n-k]$$

250 Hz

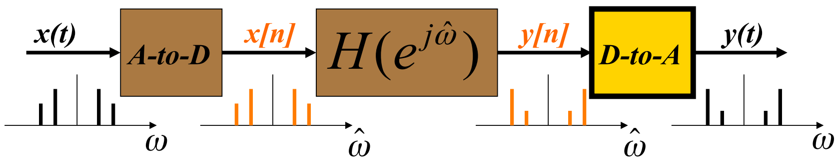
25 Hz

$$H(e^{j\hat{\omega}}) = \frac{\sin(\frac{11}{2}\hat{\omega})}{11\sin(\frac{1}{2}\hat{\omega})} e^{-j5\hat{\omega}}$$

?

$$x(t) = \cos(2\pi(25)t) + \cos(2\pi(250)t - \tfrac{1}{2}\pi)$$

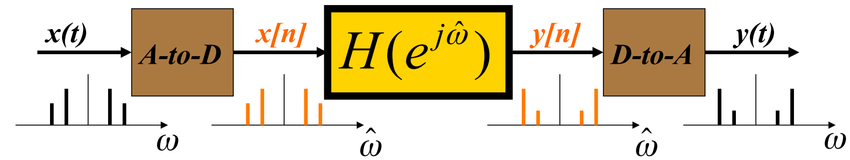
D-A FREQUENCY SCALING



- *TIME SAMPLING:* $t = nT_s \Rightarrow n \leftarrow t f_s$
- RECONSTRUCT up to $0.5f_s$
  - FREQUENCY SCALING

$$\omega = \hat{\omega} f_s$$

TRACK the FREQUENCIES

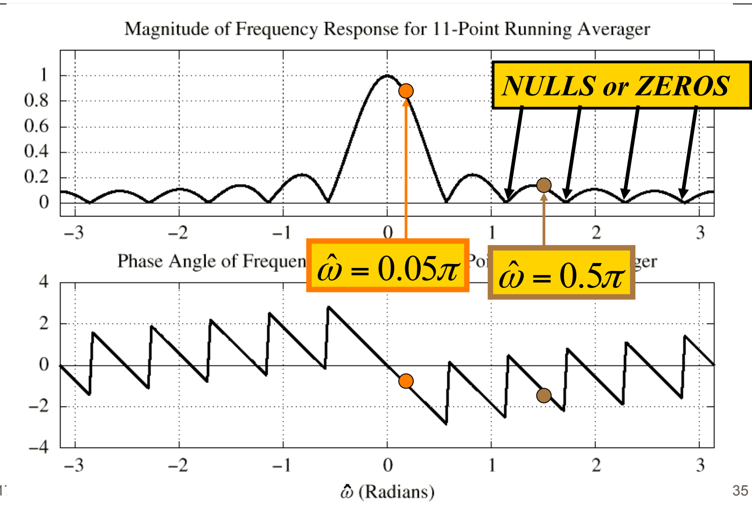


250 Hz	0.5π	$H(e^{j0.5\pi})$	0.5π	250 Hz
25 Hz	$.05\pi$	$H(e^{j0.05\pi})$	$.05\pi$	25 Hz

Fs = 1000 Hz

NO new freqs

11-pt AVERAGER



EVALUATE Freq. Response

$$H(e^{j\hat{\omega}}) = \frac{\sin(\frac{11}{2}\hat{\omega})}{11\sin(\frac{1}{2}\hat{\omega})} e^{-j5\hat{\omega}}$$

At $\hat{\omega} = 0.5\pi$

$$H(e^{j\hat{\omega}}) = \frac{\sin(\frac{11}{2}(0.5\pi))}{11\sin(\frac{1}{2}(0.5\pi))} e^{-j5(0.5\pi)}$$

$$= \frac{\sin(2.75\pi)}{11\sin(0.25\pi)} e^{-j2.5\pi}$$

$$= 0.0909 e^{-j0.5\pi}$$

EVALUATE Freq. Response

$$x(t) = \cos(2\pi(25)t) + \sin(2\pi(250)t)$$

evaluating at 25 and 250 Hz.

$f_s = 1000$

$$H(e^{j2\pi(25)/1000}) = \frac{\sin(\pi(25)(11)/1000)}{11\sin(\pi(25)/1000)} e^{-j2\pi(25)(5)/1000}$$

MAG SCALE

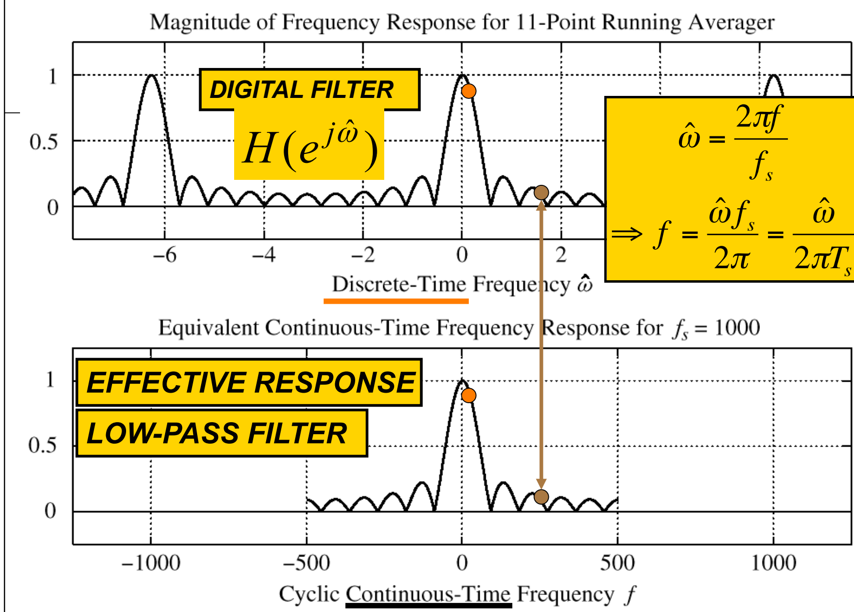
$$= 0.8811 e^{-j\pi/4}$$

PHASE CHANGE

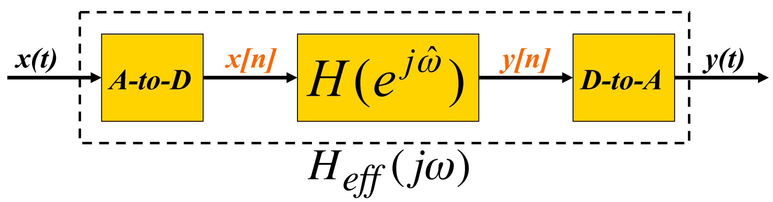
$$H(e^{j2\pi(250)/1000}) = \frac{\sin(\pi(250)(11)/1000)}{11\sin(\pi(250)/1000)} e^{-j2\pi(250)(5)/1000}$$

$$= 0.0909 e^{-j\pi/2}$$

$$y(t) = \underline{0.8811}\cos(2\pi(25)t \underline{- \pi/4}) + \underline{0.0909}\sin(2\pi(250)t \underline{- \pi/2})$$



DIGITAL "FILTERING"

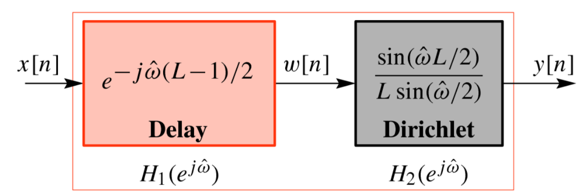


If $x(t) = e^{j\omega_0 t}$ and $2\pi f_s > 2\omega_0$,

then $y(t) = H_{eff}(j\omega_0)e^{j\omega_0 t}$, where

$$H_{eff}(j\omega) = H(e^{j\omega T_s})$$

Interpretation of Time-Delay for Sampled Signals



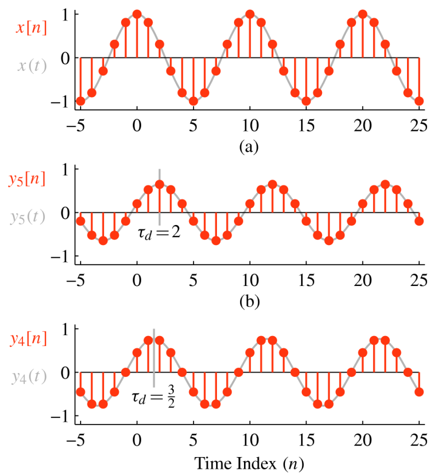
$x(t) = e^{j\omega t}$ (continuous-time signal)

$x[n] = e^{j\omega n T_s} = e^{j\hat{\omega}n}$ (sampled signal)

$$y[n] = D_L(e^{j\hat{\omega}})e^{-j\hat{\omega}(L-1)/2}e^{j\hat{\omega}n} = D_L(e^{j\hat{\omega}})e^{j\hat{\omega}(n-(L-1)/2)}$$

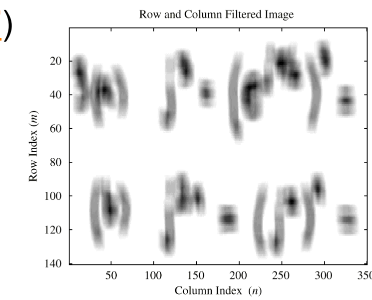
$$y(t) = D_L(e^{j\omega T_s})e^{j\omega(t-T_s(L-1)/2)}$$

Time Delay for Sampled and Reconstructed Sinusoids

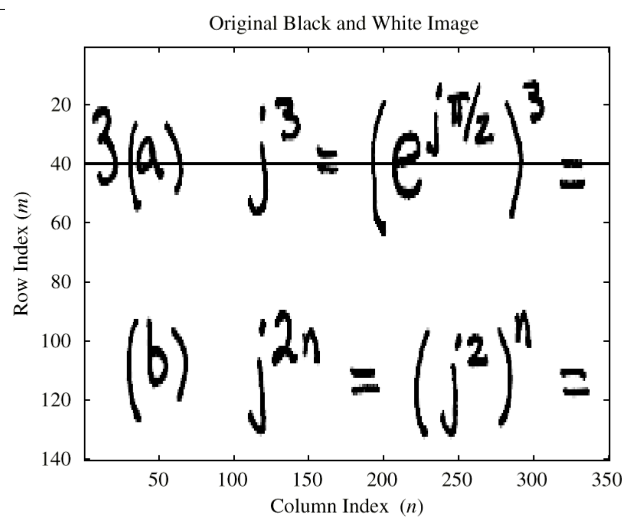


FILTER TYPES

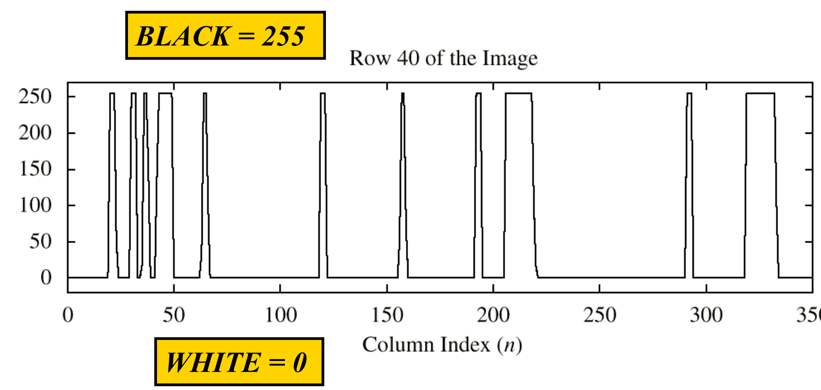
- LOW-PASS FILTER (LPF)
  - BLURRING
  - ATTENUATES HIGH FREQUENCIES
- HIGH-PASS FILTER (HPF)
  - SHARPENING for IMAGES
  - BOOSTS THE HIGHS
  - REMOVES DC
- BAND-PASS FILTER (BPF)



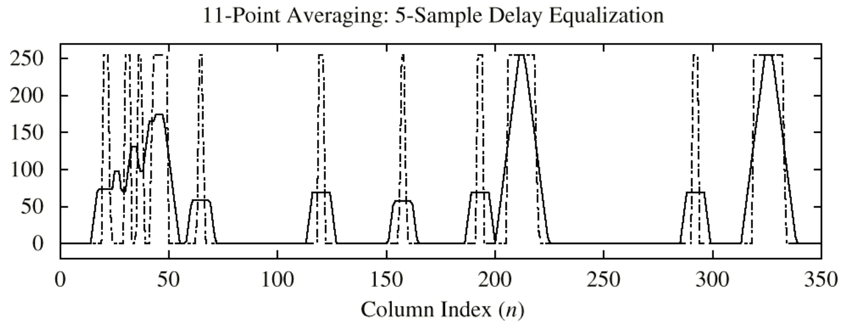
B & W IMAGE



ROW of B&W IMAGE



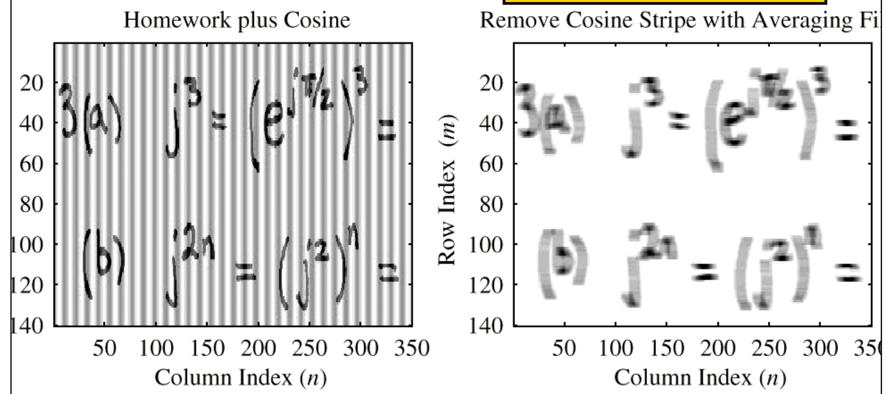
FILTERED ROW of IMAGE



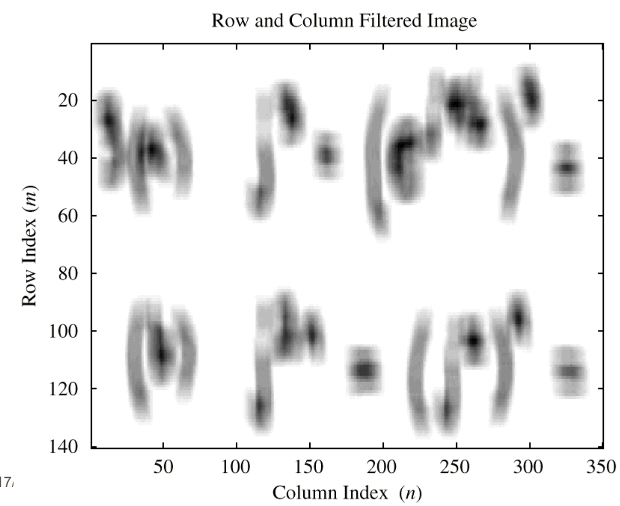
ADJUSTED DELAY by 5 samples

B&W IMAGE with COSINE

FILTERED: 11-pt AVG



FILTERED B&W IMAGE



LPF: BLUR

Simple Nulling Filter - I

- How could we get rid of a sinusoidal component?
- Make the frequency response zero at the frequency of the sinusoid.

$$x[n] = \cos(\omega_0 n) = 0.5e^{j\omega_0 n} + 0.5e^{-j\omega_0 n}$$

$$H(e^{j\hat{\omega}}) = (1 - e^{j\hat{\omega}_0} e^{-j\hat{\omega}})(1 - e^{-j\hat{\omega}_0} e^{-j\hat{\omega}})$$

$$= 1 - 2\cos(\omega_0)e^{-j\hat{\omega}} + e^{-j\hat{\omega}2}$$

Simple Nulling Filter – II

- ww = (-200:200)*pi/200;
  H = freqz([1, -2*cos(1), 1],1,ww);
  plot(ww,abs(H))