STANFORD UNIVERSITY
EE 102B Spring-2013

Lecture 11

Theorems, Properties and Applications of the DTFT

April 24, 2013

ASSIGNMENTS

- Reading for this Lecture:
  - SPF: Sections 66-1, 66-2, 66-3 & 66-4 (notes posted on Course2Go website)
  - S&S: Chapter 5
- HW#03 is due by 5pm today, April 24 in Packard 263.
- Lab #03 is due by 5pm, Friday, April 26, in Packard 263.

Office Hours for Course Staff – Come see us.

- Ron Schafer: Mon. 2:00-3:00 pm and Weds. 2:00-4:00 pm in Packard 211
- Dookun Park: Tues. 1:30 ~ 3:30 pm in Packard 107
- Ruo Yu Gu (Roy): Mon. 4:00 ~ 6:00 pm in Packard 106
- https://class2go.stanford.edu/EE102B/Spring2013/pages/staff

Lecture Objective

- Review the definition and properties of the DTFT
- The autocorrelation function and its properties
  - Illustrative examples
- IIR difference equations
  - Examples
- Time-domain multiplication property
  - Illustrative examples

Discrete-Time Fourier Transform

- **Definition of the DTFT:**

Discrete-time Fourier Transform

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n}$$

Inverse Discrete-time Fourier Transform

$$x[n] = \frac{1}{2\pi}\int_0^{2\pi} X(e^{j\hat{\omega}})e^{j\hat{\omega}n}\,d\hat{\omega}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\hat{\omega}})e^{j\hat{\omega}n}\,d\hat{\omega}$$

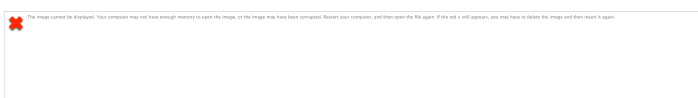
$$\boxed{\text{Periodic}: X(e^{j(\hat{\omega}+2\pi)}) = X(e^{j\hat{\omega}})}$$

Table of DTFT Pairs

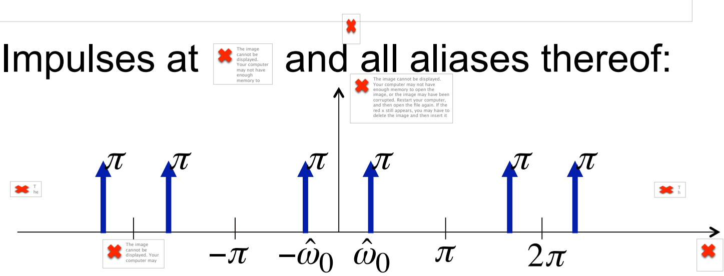
Time-Domain: $x[n]$		*Frequency-Domain:* $X(e^{j\hat{\omega}})$	
$\delta[n]$	$\Leftrightarrow$	1	
$\delta[n-n_d]$	$\Leftrightarrow$	$e^{-j\hat{\omega}n_d}$	
$u[n]-u[n-L]$	$\Leftrightarrow$	$\dfrac{\sin(\frac{1}{2}L\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})}e^{-j\hat{\omega}(L-1)/2}$	
$\dfrac{\sin(\hat{\omega}_b n)}{\pi n}$	$\Leftrightarrow$	$u(\hat{\omega}+\hat{\omega}_b)-u(\hat{\omega}-\hat{\omega}_b) = \begin{cases}1 & \lvert\hat{\omega}\rvert \le \hat{\omega}_b \\ 0 & \hat{\omega}_b < \lvert\hat{\omega}\rvert \le \pi\end{cases}$	
$a^n u[n] \quad (\lvert a\rvert < 1)$	$\Leftrightarrow$	$\dfrac{1}{1-ae^{-j\hat{\omega}}}$	*right-sided exponential*
$b^n u[-n] \quad (\lvert b\rvert > 1)$	$\Leftrightarrow$	$\boxed{\dfrac{-be^{-j\hat{\omega}}}{1-be^{-j\hat{\omega}}}}$ $\cancel{\dfrac{1}{1-be^{-j\hat{\omega}}}}$	*left-sided exponential*

DTFT of a Sinusoidal Signal - I

- Define the DTFT of a signal as



- Impulses at and all aliases thereof:



DTFT of a Sinusoidal Signal - II

- What is the signal corresponding to this DTFT? Plug into the inverse DTFT

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\hat{\omega}})e^{j\hat{\omega}n}d\hat{\omega}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}[\pi\delta(\hat{\omega}+\hat{\omega}_0)+\pi\delta(\hat{\omega}-\hat{\omega}_0)]e^{j\hat{\omega}n}d\hat{\omega}$$

$$= \frac{1}{2}e^{j\hat{\omega}_0 n} + \frac{1}{2}e^{-j\hat{\omega}_0 n} = \cos(\hat{\omega}_0 n)$$

Table of DTFT Properties		
Property Name	*Time-Domain:* $x[n]$	*Frequency-Domain:* $X(e^{j\hat{\omega}})$
Periodic in $\hat{\omega}$		$X(e^{j(\hat{\omega}+2\pi)}) = X(e^{j\hat{\omega}})$
Linearity	$ax_1[n] + bx_2[n]$ $\Leftrightarrow$	$aX_1(e^{j\hat{\omega}}) + bX_2(e^{j\hat{\omega}})$
Conjugate Symmetry	$x[n]$ is real	$X(e^{-j\hat{\omega}}) = X^*(e^{j\hat{\omega}})$
Conjugation	$x^*[n]$ $\Leftrightarrow$	$X^*(e^{-j\hat{\omega}})$
Time-Reversal	$x[-n]$ $\Leftrightarrow$	$X(e^{-j\hat{\omega}})$
Delay	$x[n-n_d]$ $\Leftrightarrow$	$e^{-j\hat{\omega}n_d}X(e^{j\hat{\omega}})$
Frequency Shift	$x[n]e^{j\hat{\omega}_0 n}$ $\Leftrightarrow$	$X(e^{j(\hat{\omega}-\hat{\omega}_0)})$
Modulation	$x[n]\cos(\hat{\omega}_0 n)$ $\Leftrightarrow$	$\frac{1}{2}X(e^{j(\hat{\omega}-\hat{\omega}_0)}) + \frac{1}{2}X(e^{j(\hat{\omega}+\hat{\omega}_0)})$
Convolution	$x[n] * h[n]$ $\Leftrightarrow$	$X(e^{j\hat{\omega}})H(e^{j\hat{\omega}})$
Parseval's Theorem	$\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2$ $=$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\hat{\omega}})\rvert^2 d\hat{\omega}$

Autocorrelation Function of a Signal

- Definition: "deterministic" autocorrelation

$$c_{xx}[n] = x[-n] * x[n] = \sum_{k=-\infty}^{\infty} x[-k]x[n-k]$$

- Change summation to $-k = m$

$$\boxed{c_{xx}[n] = \sum_{m=-\infty}^{\infty} x[m]x[n+m]}$$

- DTFT when $x[n]$ is real:

$$C_{xx}(e^{j\hat{\omega}}) = X(e^{-j\hat{\omega}})X(e^{j\hat{\omega}}) = X^*(e^{j\hat{\omega}})X(e^{j\hat{\omega}}) = \left|X(e^{j\hat{\omega}})\right|^2$$

Energy Spectrum and Parseval's Theorem

$$c_{xx}[n] = \sum_{m=-\infty}^{\infty} x[m]x[n+m] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X(e^{j\hat{\omega}})\right|^2 e^{j\hat{\omega}n} d\hat{\omega}$$

- Energy definition and Parseval's Theorem:

$$\boxed{E = c_{xx}[0] = \sum_{k=-\infty}^{\infty} \left|x[k]\right|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X(e^{j\hat{\omega}})\right|^2 d\hat{\omega}}$$

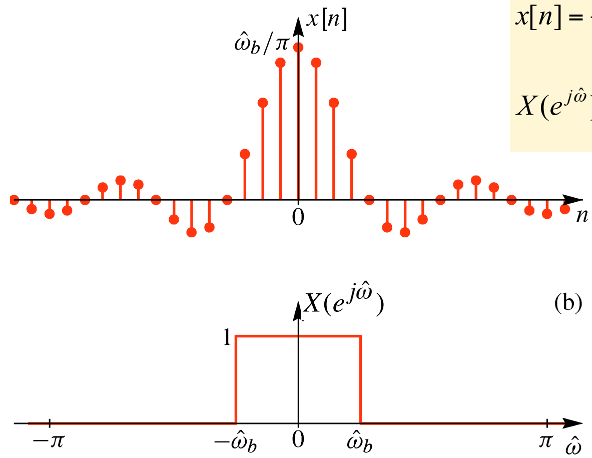
- Energy density spectrum

$$\boxed{C_{xx}(e^{j\hat{\omega}}) = \left|X(e^{j\hat{\omega}})\right|^2}$$

SINC Function – Rectangle DTFT pair

$$x[n] = \frac{\sin(\hat{\omega}_b n)}{\pi n} \quad \overset{DTFT}{\Longleftrightarrow}$$

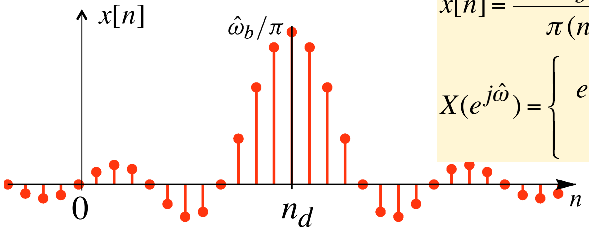
$$X(e^{j\hat{\omega}}) = \begin{cases} 1, & |\hat{\omega}| \le \hat{\omega}_b \\ 0, & \hat{\omega}_b < |\hat{\omega}| \le \pi \end{cases}$$

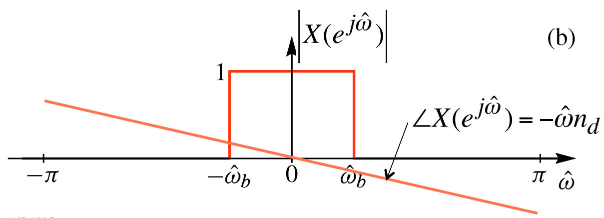


Delayed SINC Function – Rectangle DTFT pair

$$x[n] = \frac{\sin[\hat{\omega}_b(n-n_d)]}{\pi(n-n_d)} \quad \overset{DTFT}{\Leftrightarrow}$$

$$X(e^{j\hat{\omega}}) = \begin{cases} e^{-j\hat{\omega}n_d}, & |\hat{\omega}| \le \hat{\omega}_b \\ 0, & \hat{\omega}_b < |\hat{\omega}| \le \pi \end{cases}$$





Delayed sinc Signal

- Energy density spectrum

Note that phase cancels

$$C_{xx}(e^{j\hat{\omega}}) = X^*(e^{j\hat{\omega}})X(e^{j\hat{\omega}}) = \left|X(e^{j\hat{\omega}})\right|^2 = \left|X(e^{j\hat{\omega}})\right|$$

- Autocorrelation function

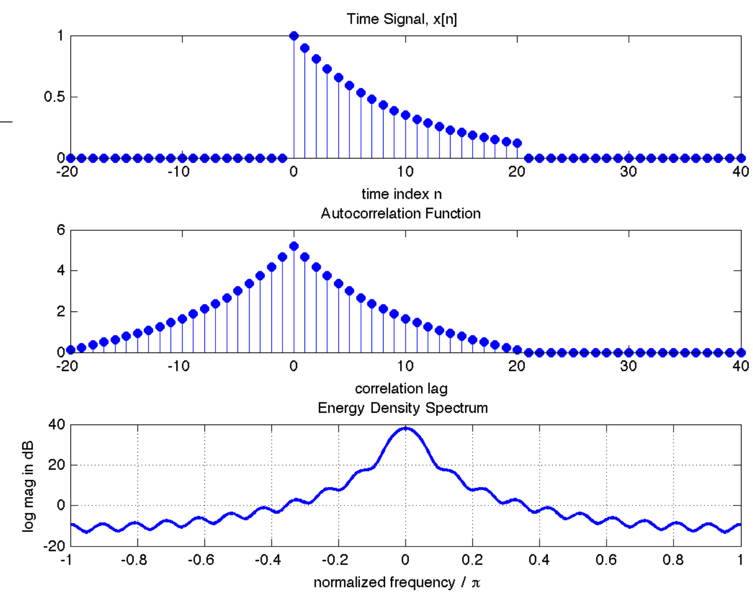
$$c_{xx}[n] = x[-n] * x[n] = \frac{\sin(\hat{\omega}_b n)}{\pi n}$$

Note that time-delay cancels

- Total energy

$$\boxed{E = c_{xx}[0] = \sum_{k=-\infty}^{\infty}\left|\frac{\sin[\hat{\omega}_b(k-n_d)]}{\pi(k-n_d)}\right|^2 = \frac{1}{2\pi}\int_{-\omega_b}^{\omega_b} d\hat{\omega} = \frac{\omega_b}{\pi}}$$

Discussion



Properties of the Autocorrelation Function

- Even function of n

$$c_{xx}[-n] = c_{xx}[n]$$

- Maximum value at $n = 0$

$$\max_n \{c_{xx}[n]\} = c_{xx}[0]$$

Difference Equations for an IIR System - I

- Consider an LTI system with impulse response and frequency response

$$h[n] = a^n u[n] \Leftrightarrow H(e^{j\hat{\omega}}) = \frac{1}{1 - ae^{-j\hat{\omega}}}$$

infinite-duration (IIR)

$$Y(e^{j\hat{\omega}}) = H(e^{j\hat{\omega}})X(e^{j\hat{\omega}}) = \frac{1}{1 - ae^{-j\hat{\omega}}} X(e^{j\hat{\omega}})$$

$$(1 - ae^{-j\hat{\omega}})Y(e^{j\hat{\omega}}) = \boxed{Y(e^{j\hat{\omega}}) - ae^{-j\hat{\omega}}Y(e^{j\hat{\omega}}) = X(e^{j\hat{\omega}})}$$

$$\text{IDTFT gives:} \quad y[n] - ay[n-1] = x[n]$$

$$\boxed{\text{Recurrence formula:} \quad y[n] = ay[n-1] + x[n]}$$

Difference Equations for an IIR System - II

- The recurrence formula can be used to compute the output from the input.

$$\boxed{\text{Recurrence formula:} \quad y[n] = ay[n-1] + x[n]}$$

- For example, compute the impulse resp.

$$h[n] = ah[n-1] + \delta[n]$$

n	-2	-1	0	1	2	3	4	5	6
δ[n]	0	0	1	0	0	0	0	0	0
h[n]	0	0	1	a	a^2	a^3	a^4	a^5	a^6

General IIR Systems

- Frequency response:

$$h[n] = a^n u[n] \Leftrightarrow H(e^{j\hat{\omega}}) = \frac{\sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}}{1 - \sum_{k=1}^{N} a_k e^{-j\hat{\omega}k}}$$

- Difference equation:

$$y[n] - \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- Iterate with initial rest conditions:

$$\boxed{y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]}$$

$$\boxed{x[n] = 0 \text{ for } n < 0 \text{ and } y[-1] = y[-2] = \ldots = y[-N] = 0}$$

General IIR Systems in MATLAB

- Difference equation:

$$y[n] = 0.8y[n-1] + x[n] + 2x[n-1] + x[n-2]$$

- Frequency response:

$$H(e^{j\hat{\omega}}) = \frac{1 + 2e^{-j\hat{\omega}} + e^{-j\hat{\omega}2}}{1 - 0.8e^{-j\hat{\omega}}}$$

```
bb = [ 1,  2,  1];
aa = [1, -0.8];
y = filter( bb , aa , xx)
om = (0:200)*pi/200;
H = freqz( bb , aa , om);
```


Examples

- Find difference equation from frequency response $H(e^{j\hat{\omega}}) = \dfrac{1 + e^{-j\hat{\omega}}}{1 + 0.9e^{-j\hat{\omega}}}$
- Find frequency response directly from difference equation.

$$y[n] = y[n-1] + 0.8y[n-2] + x[n] - x[n-2]$$

Another Property of the DTFT

- Multiplication in the time domain corresponds to (periodic) convolution in the frequency domain.

$$y[n] = w[n]x[n] \Leftrightarrow Y(e^{j\hat{\omega}}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})W(e^{j(\hat{\omega}-\theta)})\,d\theta$$

Derivation

$$Y(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} w[n]x[n]e^{-j\hat{\omega}n}$$

$$= \sum_{n=-\infty}^{\infty} w[n]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})e^{j\theta n}\,d\theta\right)e^{-j\hat{\omega}n}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})\left(\sum_{n=-\infty}^{\infty} w[n]e^{-j(\hat{\omega}-\theta)n}\right)d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})W(e^{j(\hat{\omega}-\theta)})\,d\theta$$

Simple Example of Periodic Convolution

$$y[n] = w[n]x[n] \Leftrightarrow Y(e^{j\hat{\omega}}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\hat{\omega}-\theta)})d\theta$$

- Trivial example illustrates general case

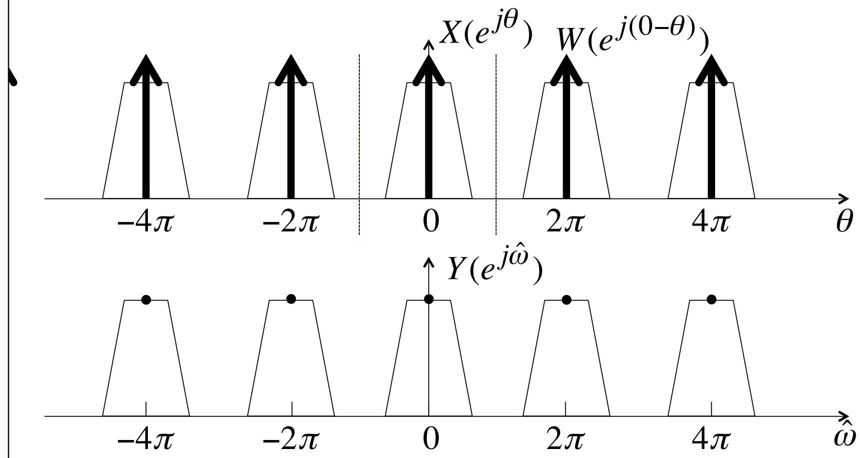
$$w[n] = 1 \Leftrightarrow W(e^{j\hat{\omega}}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\hat{\omega} + 2\pi r)$$

$$y[n] = w[n]x[n] = x[n] \Rightarrow Y(e^{j\hat{\omega}}) = X(e^{j\hat{\omega}})$$

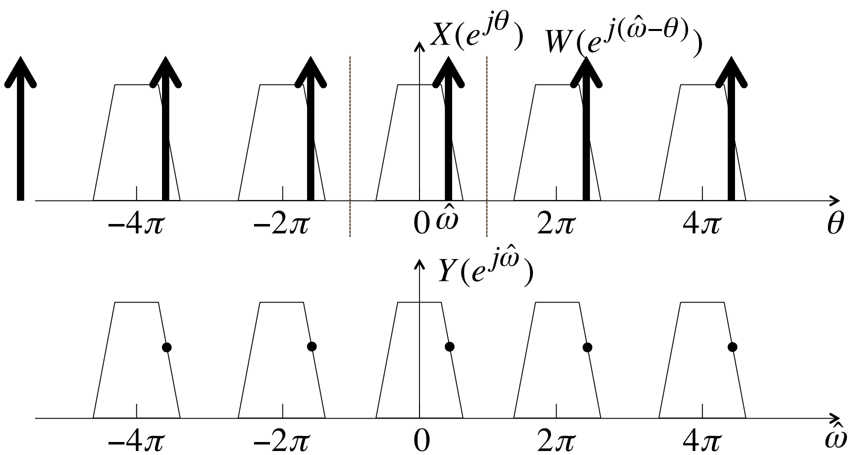
- Convolution integral over –π to π

$$y[n] = 1x[n] \Leftrightarrow Y(e^{j\hat{\omega}}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})2\pi\delta(\hat{\omega}-\theta)d\theta$$

Periodic Convolution



Periodic Convolution



Discussion

Another Example of Periodic Convolution

$$y[n] = w[n]\cos(\hat{\omega}_0 n)$$

$$\cos(\hat{\omega}_0 n) \Leftrightarrow \sum_{r=-\infty}^{\infty} [\pi\delta(\hat{\omega} - \hat{\omega}_0 + 2\pi r) + \pi\delta(\hat{\omega} + \hat{\omega}_0 + 2\pi r)]$$

- Convolution integral over –π to π

$$Y(e^{j\hat{\omega}}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})[\pi\delta(\hat{\omega} - \theta - \hat{\omega}_0) + \pi\delta(\hat{\omega} - \theta + \hat{\omega}_0)]d\theta$$

$$Y(e^{j\hat{\omega}}) = \frac{1}{2}X(e^{j(\hat{\omega}-\hat{\omega}_0)}) + \frac{1}{2}X(e^{j(\hat{\omega}+\hat{\omega}_0)})$$