STANFORD UNIVERSITY
EE 102B Spring-2013

Lecture 12
FIR Filter Design
April 26, 2013

ASSIGNMENTS

- Reading for this Lecture:
  - SPF: Sections 66-1, 66-2, 66-3 & 66-4 (notes posted on Course2Go website)
  - Lab 04 Warm-Up section
  - S&S: Chapter 5
- HW#04 is due by 5pm Wednesday, May 1, in Packard 263.
- Lab #03 is due by 5pm, Friday, April 26, in Packard 263.

Office Hours for Course Staff – Come see us.

- Ron Schafer: Mon. 2:00-3:00 pm and Weds. 2:00-4:00 pm in Packard 211
- Dookun Park: Tues. 1:30 ~ 3:30 pm in Packard 107
- Ruo Yu Gu (Roy): Mon. 4:00 ~ 6:00 pm in Packard 106
- https://class2go.stanford.edu/EE102B/Spring2013/pages/staff

Lecture Objective

- Time-domain multiplication property
  - Illustrative examples
- Filter design
  - Linear phase condition
  - Window design
  - filterdesign.m demonstration and discussion

Another Property of the DTFT

- Multiplication in the time domain corresponds to (periodic) convolution in the frequency domain.

$$\boxed{y[n] = w[n]x[n] \Leftrightarrow Y(e^{j\hat{\omega}}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})W(e^{j(\hat{\omega}-\theta)})d\theta}$$

Derivation

$$Y(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} w[n]x[n]e^{-j\hat{\omega}n}$$

$$= \sum_{n=-\infty}^{\infty} w[n]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})e^{j\theta n}\,d\theta\right)e^{-j\hat{\omega}n}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})\left(\sum_{n=-\infty}^{\infty} w[n]e^{-j(\hat{\omega}-\theta)n}\right)d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})W(e^{j(\hat{\omega}-\theta)})d\theta$$

Simple Example of Periodic Convolution

$$\boxed{y[n] = w[n]x[n] \Leftrightarrow Y(e^{j\hat{\omega}}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\hat{\omega}-\theta)})d\theta}$$

- Trivial example illustrates general case

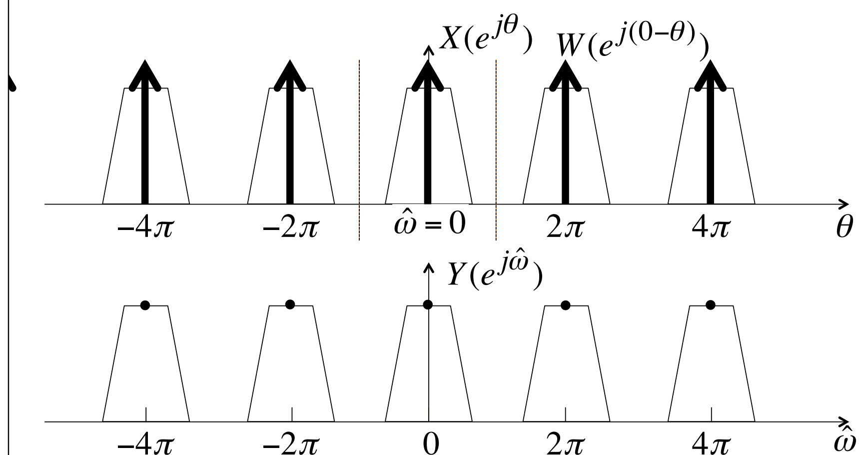
$$w[n] = 1 \Leftrightarrow W(e^{j\hat{\omega}}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\hat{\omega}+2\pi r)$$

$$y[n] = w[n]x[n] = x[n] \Rightarrow Y(e^{j\hat{\omega}}) = X(e^{j\hat{\omega}})$$

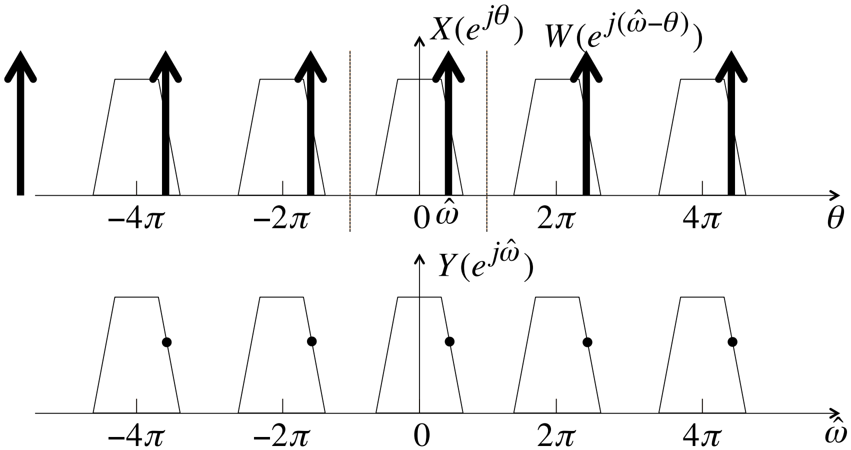
- Convolution integral over $-\pi$ to π

$$Y(e^{j\hat{\omega}}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})2\pi\delta(\hat{\omega}-\theta)d\theta = X(e^{j\hat{\omega}})$$

Periodic Convolution



Periodic Convolution



Discussion

Another Example of Periodic Convolution

$$y[n] = w[n]\cos(\hat{\omega}_0 n)$$

$$\cos(\hat{\omega}_0 n) \Leftrightarrow \sum_{r=-\infty}^{\infty} [\pi\delta(\hat{\omega}-\hat{\omega}_0+2\pi r)+\pi\delta(\hat{\omega}+\hat{\omega}_0+2\pi r)]$$

- Convolution integral over –π to π

$$Y(e^{j\hat{\omega}}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})[\pi\delta(\hat{\omega}-\theta-\hat{\omega}_0)+\pi\delta(\hat{\omega}-\theta+\hat{\omega}_0)]d\theta$$

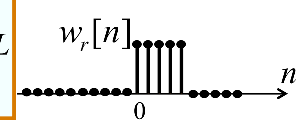
$$Y(e^{j\hat{\omega}}) = \frac{1}{2}X(e^{j(\hat{\omega}-\hat{\omega}_0)})+\frac{1}{2}X(e^{j(\hat{\omega}+\hat{\omega}_0)})$$

Windows

- Finite-Length signal (L) with positive values
  - Extractor
  - Truncator

Rectangular Window

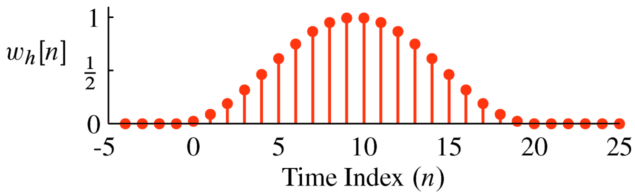
$$w_r[n] = \begin{cases} 0 & n < 0 \\ 1 & 0 \le n < L \\ 0 & n \ge L \end{cases}$$



$$w_r[n]x[n+n_0] = \begin{cases} 0 & n < 0 \\ x[n+n_0] & 0 \le n < L \\ 0 & n \ge L \end{cases}$$

Von Hann Window (Time Domain)

- Plot of Length-20 von Hann window

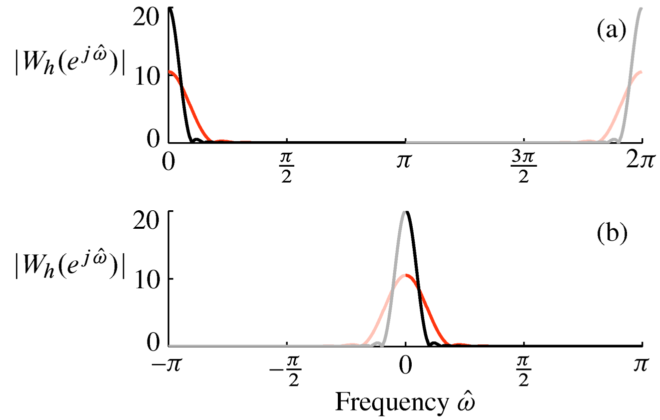


von Hann Window (Length L)

$$w_h[n] = \begin{cases} 0 & n < 0 \\ \frac{1}{2} - \frac{1}{2}\cos(2\pi(n+1)/(L+1)) & 0 \le n < L \\ 0 & n \ge L \end{cases}$$

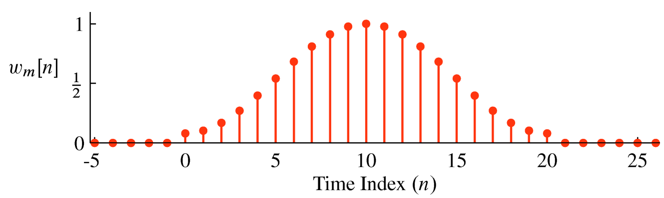
Hann Window (Frequency Domain)

- DTFTs (magnitude) of Hann windows (L=20 & 40)



Hamming Window (Time Domain)

- Plot of Length-21 Hamming window



Hamming Window

$$w_m[n] = \begin{cases} 0 & n < 0 \\ 0.54 - 0.46\cos(2\pi(n)/(L-1)) & 0 \le n < L \\ 0 & n \ge L \end{cases}$$

Windowed section of sinusoid

- Multiply the very long sinusoid by window

$$x[n] = A\cos(\hat{\omega}_0 n + \varphi) \quad -\infty < n < \infty$$

- DTFT of L-point windowed signal $\hat{\omega}_0 = 0.4\pi$

$$X_L(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} w_L[n]x[n]e^{-j\hat{\omega}n} = \sum_{n=0}^{L-1} w_L[n]x[n]e^{-j\hat{\omega}n}$$

$$X_L(e^{j\hat{\omega}}) = \frac{Ae^{j\varphi}}{2} W_L\left(e^{j(\hat{\omega}-\hat{\omega}_0)}\right) + \frac{Ae^{-j\varphi}}{2} W_L\left(e^{j(\hat{\omega}+\hat{\omega}_0)}\right)$$

Expectation: 2 narrow spectrum lines

Compute values of the DTFT using the DFT - I

- DTFT of L-point windowed signal

$$X_L(e^{j\hat{\omega}}) = \sum_{n=0}^{L-1} w_L[n]x[n]e^{-j\hat{\omega}n}$$

- The DFT is a sampled version of DTFT

$$x_L[n] = w_L[n]x[n]$$

$$X_L(e^{j\hat{\omega}_k}) = X_L[k] = \sum_{n=0}^{L-1} x_L[n]e^{-j\hat{\omega}_k n}$$

$$\hat{\omega}_k = (2\pi k/N), \quad k = 0,1,\ldots,N-1$$

Compute values of the DTFT using the DFT - II

- The DFT is a sampled version of DTFT

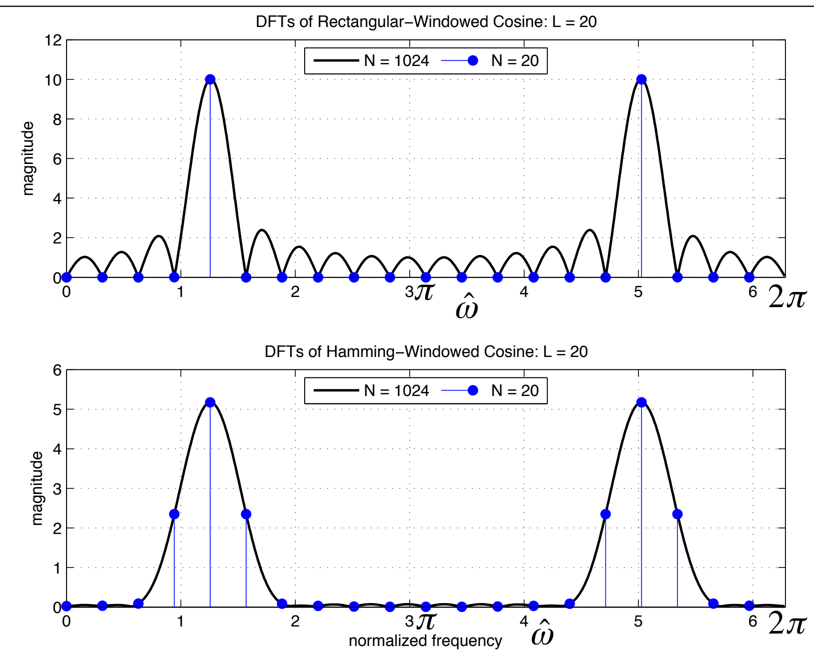
$$x_L[n] = w_L[n]x[n]$$

$$X_L(e^{j\hat{\omega}_k}) = X_L[k] = \sum_{n=0}^{L-1} x_L[n]e^{-j\hat{\omega}_k n}$$

$$\hat{\omega}_k = (2\pi k/N), \quad k = 0,1,\ldots,N-1$$

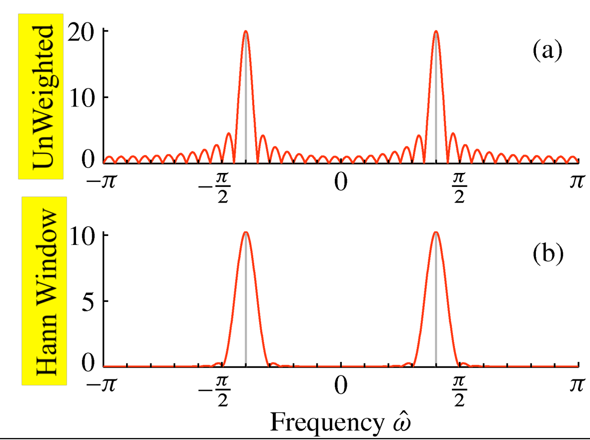
- Large N is essentially same as the DTFT
- In MATLAB:

```
xL = wL .* A * cos(om0*(0:L-1)+phi);
XL = fft( xL , N );
```

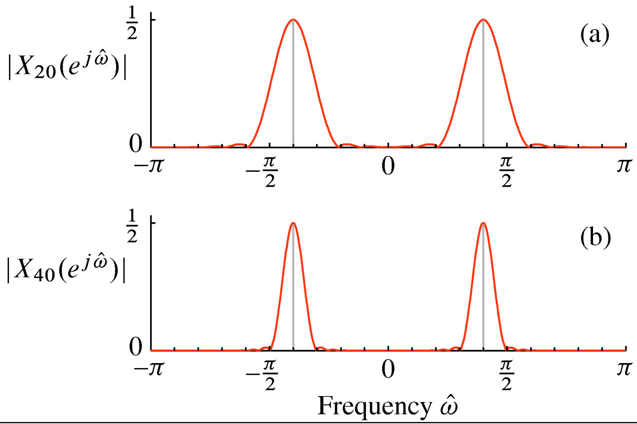
Windowed Sinusoid (in Freq Domain)

- DTFT (magnitude) of windowed sinusoid
  - Length-40 Hann window vs Length-40 Rectangular window



Change Window Length

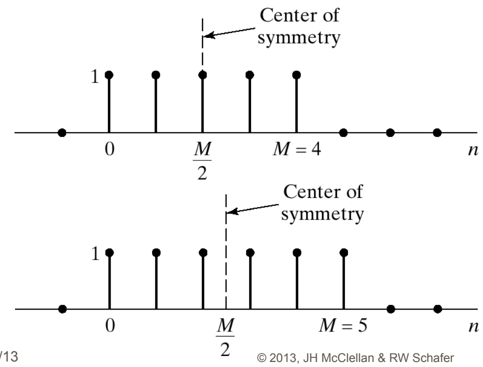
- DTFT (magnitude) of windowed sinusoid.
  - Length-20 Hann window vs. Length-40 Hann window



Generalized Linear Phase FIR Systems - I

- Types I & II: $h[M-n] = h[n] \quad 0 \le n \le M$

$$H(e^{j\hat{\omega}}) = A_e(e^{j\hat{\omega}})e^{-j\hat{\omega}M/2}, \quad A_e(e^{j\hat{\omega}}) = A_e(e^{-j\hat{\omega}})$$

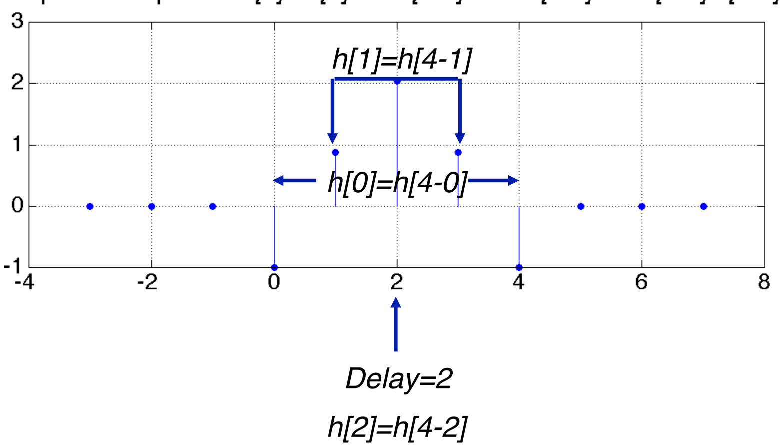


Type I:
M even
integer delay

Type II:
M odd
half sample delay

Example of Linear Phase Impulse Response: h[4-n] = h[n]

Impulse Response: *h[n]=-δ[n]+.88δ[n-1]+2.04δ[n-2]+.88δ[n-3]-δ[n-4]*



Show Linear Phase when h[4-n] = h[n]

- Frequency response: $H(e^{j\hat{\omega}}) = \sum_{n=0}^{4} h[n]e^{-j\hat{\omega}n}$

$$H(e^{j\hat{\omega}}) = h[0] + h[1]e^{-j\hat{\omega}} + h[2]e^{-j\hat{\omega}2} + h[3]e^{-j\hat{\omega}3} + h[4]e^{-j\hat{\omega}4}$$

$$H(e^{j\hat{\omega}}) = \left(h[0] + h[4]e^{-j\hat{\omega}4}\right)$$

$$+\left(h[1]e^{-j\hat{\omega}} + h[3]e^{-j\hat{\omega}3}\right) + h[2]e^{-j\hat{\omega}2}$$

Since $h[4] = h[0]$ and $h[3] = h[1]$,

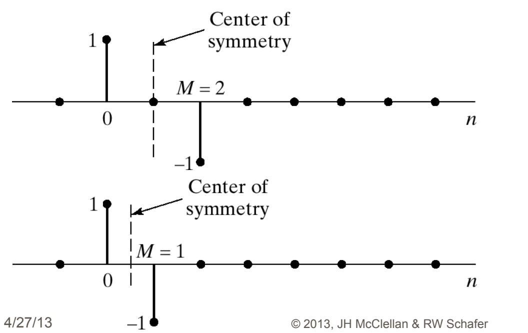
$$H(e^{j\hat{\omega}}) = 2h[0]\cos(\hat{\omega}2)e^{-j\hat{\omega}2}$$

$$+ 2h[1]\cos(\hat{\omega})e^{-j\hat{\omega}2} + h[2]e^{-j\hat{\omega}2}$$

Generalized Linear Phase FIR Systems - II

- Types III & IV: $h[M-n] = -h[n] \quad 0 \le n \le M$

$$H(e^{j\hat{\omega}}) = jA_o(e^{j\hat{\omega}})e^{-j\hat{\omega}M/2}, \quad A_o(e^{j\hat{\omega}}) = -A_o(e^{-j\hat{\omega}})$$



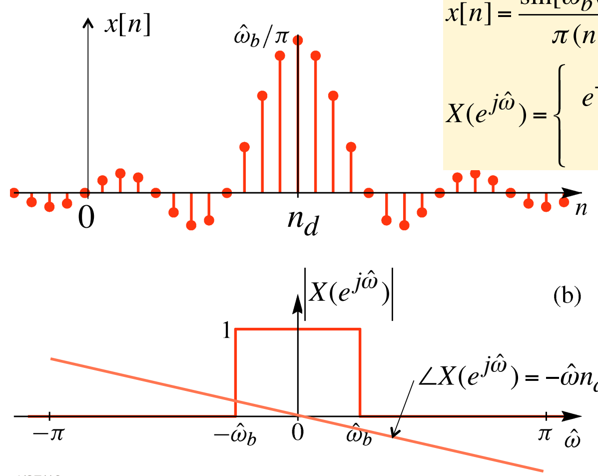
Type III:
M even
integer delay

Type IV:
M odd
half sample
delay

Delayed SINC Function – Rectangle DTFT pair

$$x[n] = \frac{\sin[\hat{\omega}_b(n-n_d)]}{\pi(n-n_d)} \overset{DTFT}{\Leftrightarrow}$$

$$X(e^{j\hat{\omega}}) = \begin{cases} e^{-j\hat{\omega}n_d}, & |\hat{\omega}| \le \hat{\omega}_b \\ 0, & \hat{\omega}_b < |\hat{\omega}| \le \pi \end{cases}$$



Ideal Lowpass Filter with Delay

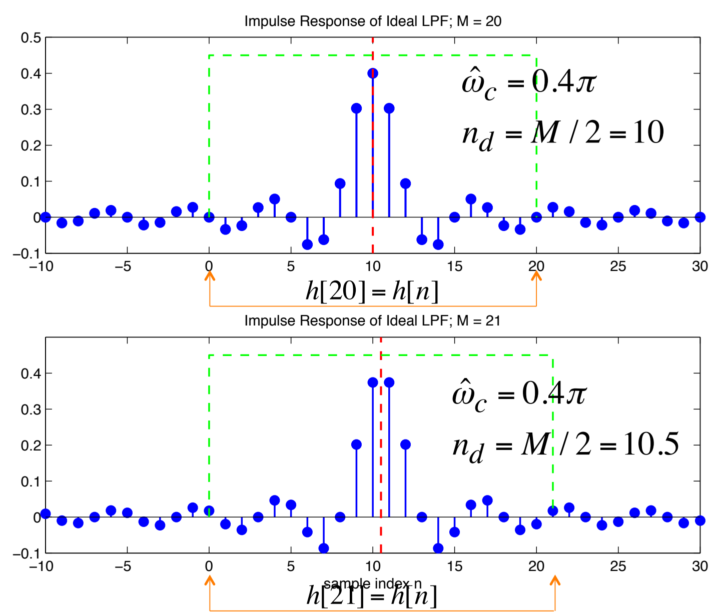
- Ideal frequency response:

$$H(e^{j\hat{\omega}}) = \begin{cases} e^{-j\hat{\omega}n_d}, & |\hat{\omega}| \le \hat{\omega}_c \\ 0, & \hat{\omega}_c < |\hat{\omega}| \le \pi \end{cases} \qquad \begin{matrix} \hat{\omega}_c = \text{ cutoff freq.} \\ n_d \text{ any real num.} \end{matrix}$$

- Impulse response:

$$h[n] = \frac{1}{2\pi}\int_{-\hat{\omega}_c}^{\hat{\omega}_c} e^{-j\hat{\omega}n_d} e^{j\hat{\omega}n}\, d\hat{\omega} = \frac{e^{j\hat{\omega}(n-n_d)}}{2\pi j(n-n_d)}\Bigg|_{-\hat{\omega}_c}^{\hat{\omega}_c}$$

$$= \frac{\sin[\hat{\omega}_c(n-n_d)]}{\pi(n-n_d)} \qquad -\infty < n < \infty$$



Windowed Filter Design

- sinc is inverse DTFT of ideal LPF

$$h[n] = \frac{\sin[\hat{\omega}_c(n - M/2)]}{\pi(n - M/2)} \qquad -\infty < n < \infty$$

  - Truncate: Multiply sinc by a window
  - Finite h[n] of length L = M+1 = window length

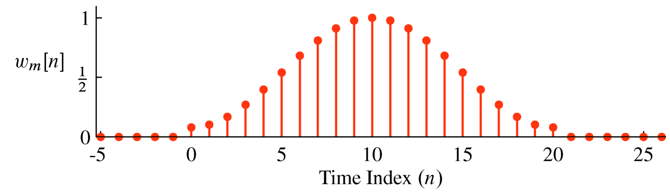
$$h[n] = w_L[n]h[n]$$

$$H_L(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} h_L[n]e^{-j\hat{\omega}n} \rightarrow \sum_{n=0}^{L-1} h_L[n]e^{-j\hat{\omega}_k n}$$

$$\hat{\omega}_k = (2\pi/N)k, \quad k = 0,1,2,\ldots N-1$$

Window for Filter Design

- Plot of Length-21 Hamming window



Hamming Window

$$w_m[n] = \begin{cases} 0 & n < 0 \\ 0.54 - 0.46\cos(2\pi(n)/(L-1)) & 0 \le n < L \\ 0 & n \ge L \end{cases}$$

Demonstrate filterdesign GUI

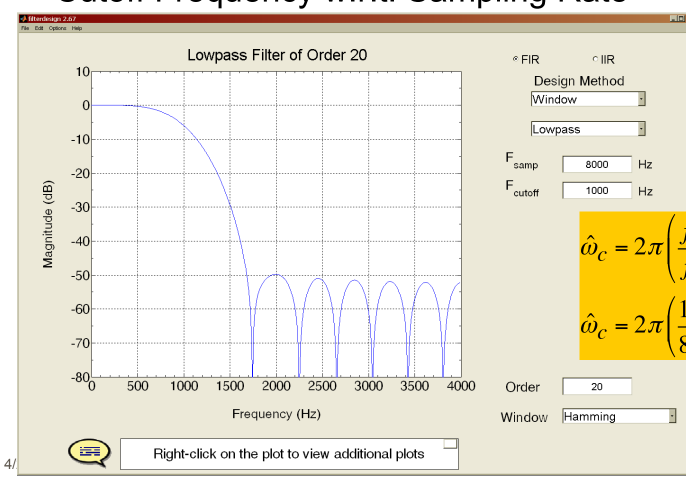
$$\hat{\omega}_c = 2\pi\left(\frac{f_c}{f_s}\right)$$

$$\hat{\omega}_c = 2\pi\left(\frac{0.4}{2}\right) = 0.4\pi$$

- Show filter designs in the following order:
  - Set fs=2, and cutoff freq = 0.4
  - Rectangular Window: M=20, M=40, M=200
  - Show Slide to define passband & stopband
  - Show Slide with Template for Filter Design Specs
  - Hamming Window: M=20, M=40
    - Need to reset cutoff when Window Type is changed.
  - Hamming Window for L=40 in dB (click Magnitude)
  - Hamming Window for L=40, zoom in on passband
  - Hamming Window: M=200
  - Same for Hann?

Filter Design GUI

- Cutoff Frequency w.r.t. Sampling Rate

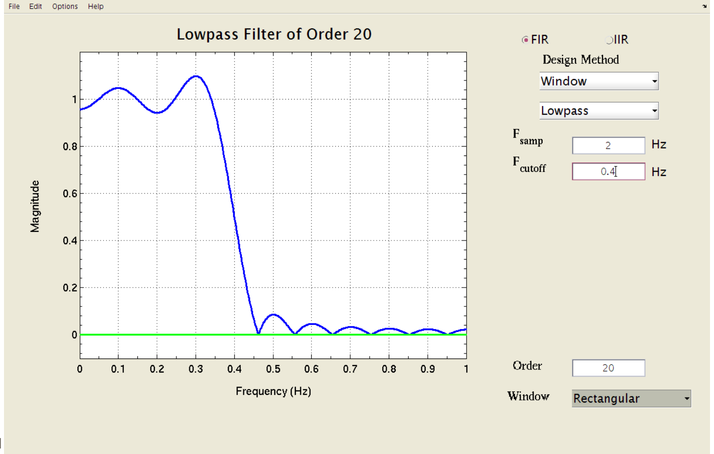


$$\hat{\omega}_c = 2\pi\left(\frac{f_c}{f_s}\right)$$

$$\hat{\omega}_c = 2\pi\left(\frac{1000}{8000}\right) = 0.25\pi$$

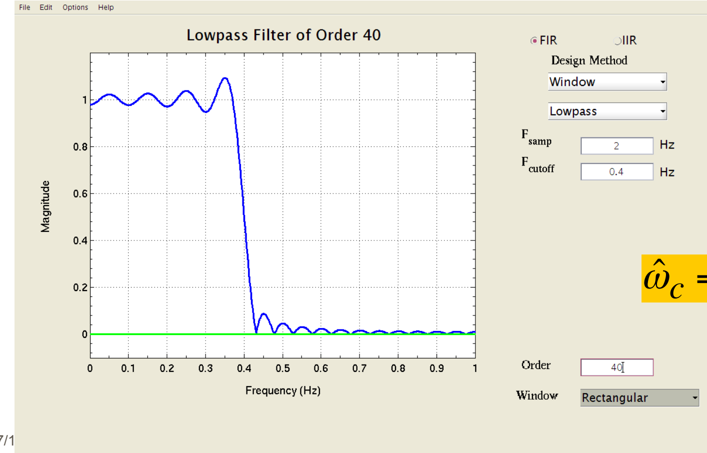
Filter Design via Rectangular Windowing

- Rectangular Window, L=21 (order M=20) $\hat{\omega}_c = 0.4\pi$



Filter Design via Rectangular Windowing

- Rectangular Window, L=41 (order M=40)

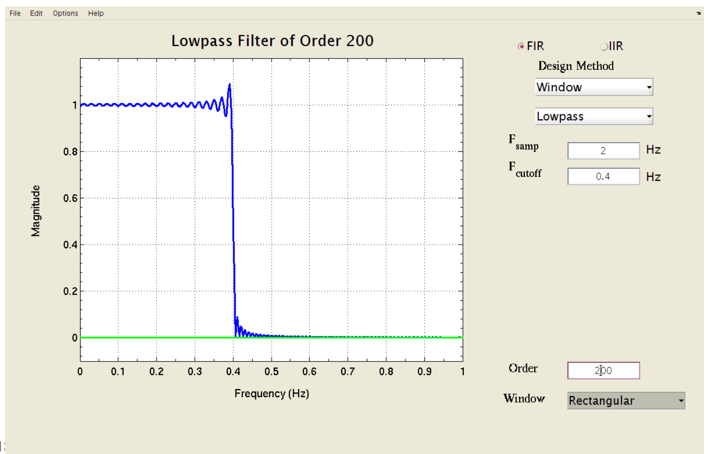


$\hat{\omega}_c = 0.4\pi$

Filter Design via Rectangular Windowing

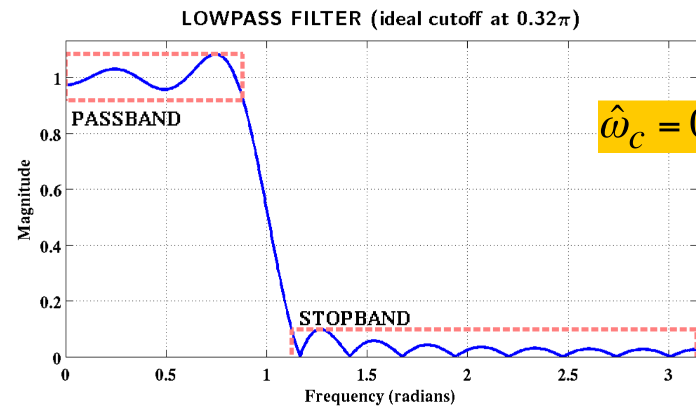
$\hat{\omega}_c = 0.4\pi$

- Rectangular Window, L=201 (order M=200)



Filter Design: Passband & Stopband

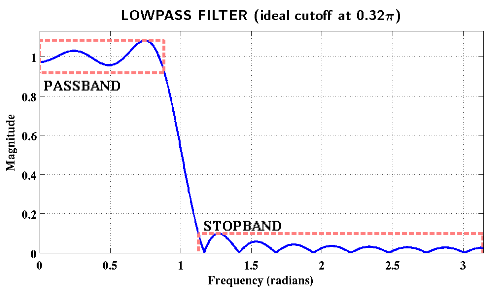
- Rectangular Window, L=41 (order M=40)



$\hat{\omega}_c = 0.32\pi$

Ripples, Band edges, & Transition Width

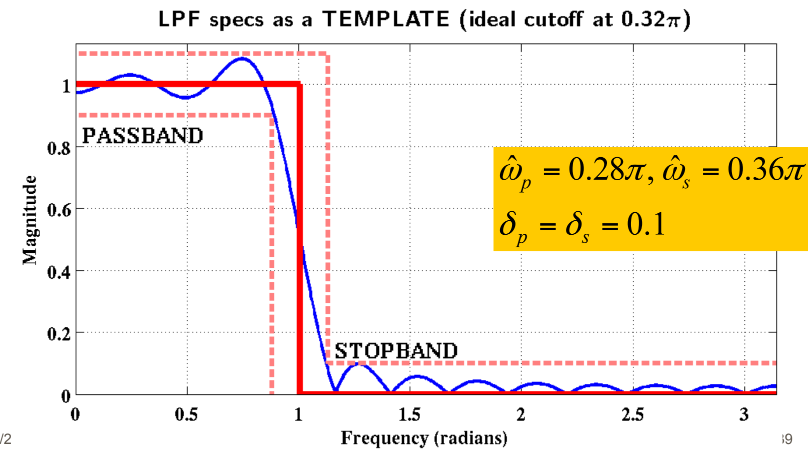
- Passband Ripple is one plus or minus δ_p
- Stopband Ripple is less than δ_s
- Band edges are $\hat{\omega}_p, \hat{\omega}_s$
- Transition Width is $\hat{\omega}_s - \hat{\omega}_p$



***Can't have it all:* small transition width, small ripples, and lowest possible order (M)**

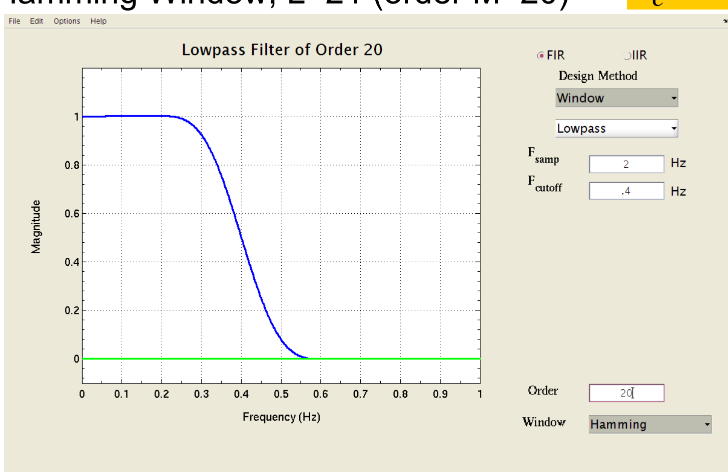
Filter Design: Tolerance Template

- Want the actual response inside the template



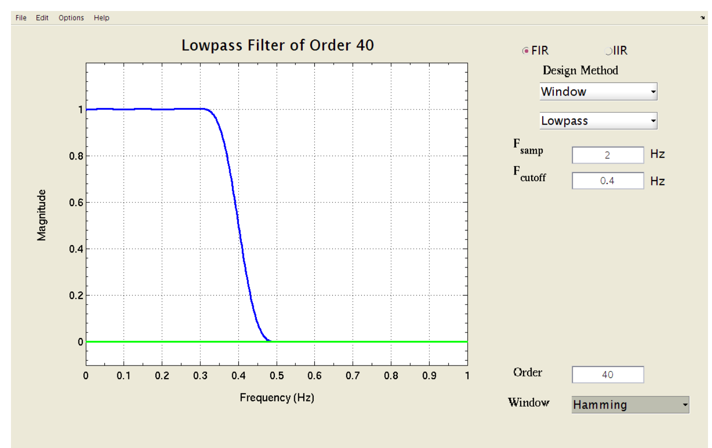
Filter Design with Hamming Window

- Hamming Window, L=21 (order M=20) $\hat{\omega}_c = 0.4\pi$



Filter Design with Hamming Window

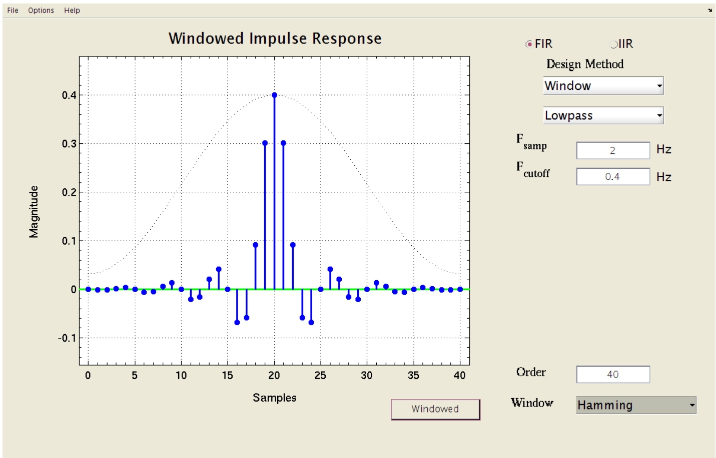
- Hamming Window, L=41 (order M=40) $\hat{\omega}_c = 0.4\pi$



Filter Design with Hamming Window

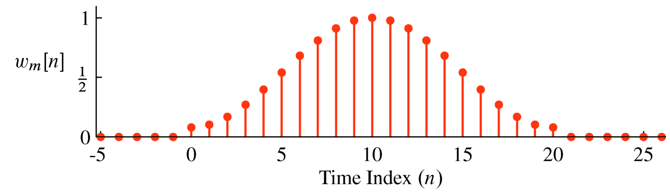
- Hamming Window, L=41 (order M=40)

$\hat{\omega}_c = 0.4\pi$



Hamming Window (Time Domain)

- Plot of Length-21 Hamming window



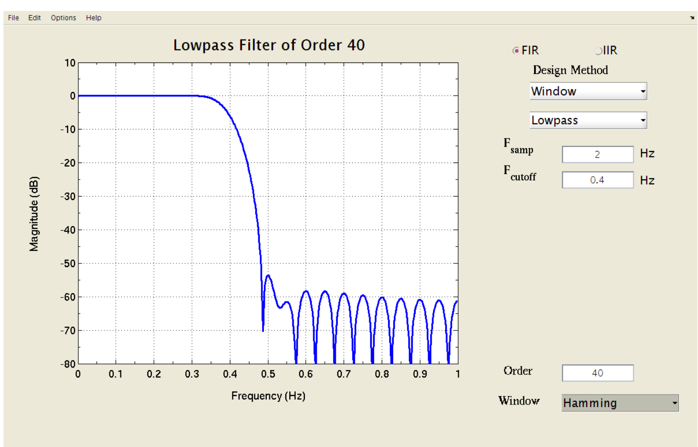
Hamming Window

$$w_m[n] = \begin{cases} 0 & n < 0 \\ 0.54 - 0.46\cos(2\pi(n)/(L-1)) & 0 \le n < L \\ 0 & n \ge L \end{cases}$$

Filter Design with Hamming Window

- Hamming Window, L=41 (order M=40)

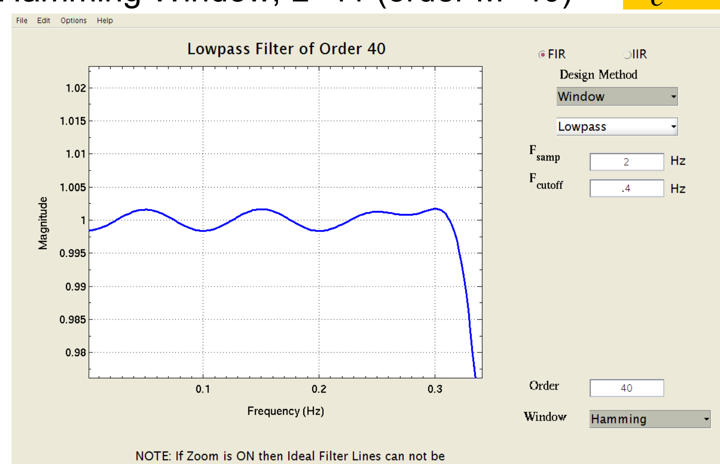
$\hat{\omega}_c = 0.4\pi$



Filter Design: zoom on passband

- Hamming Window, L=41 (order M=40)

$\hat{\omega}_c = 0.4\pi$



Filter Design with Hamming Window

- Hamming Window, L=201 (order M=200) $\hat{\omega}_c = 0.4\pi$