# STANFORD UNIVERSITY
# EE 102B Spring-2013

## Lecture 16

## Decimation/Interpolation and Introduction the DFT

## May 6, 2013

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# ASSIGNMENTS

- Reading for this Lecture:
  - SPF: Section 12-3, Chapter 66-6, 66-7
  - S&S: Chapter 5
- HW#05 is due by 5pm Wednesday, May 8, in Packard 263.
- Lab #05 is due by 5pm, Friday, May 17, in Packard 263.
- Mid-term exam on Friday, May 10, in class. Room and exam conditions next slide.




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# Mid-Term Exam

- Covers material through Lecture 13 (FIR filter design and intro to sampling), HWs 01-04, and LABs 01-04.
- The exam will be held in 420-041, 11am - 12:30 pm.
- You may use your textbook (either *SP-First* or Signals *and Systems*) and two sheets (both sides) of notes. No computers or other materials allowed.
- Several people have conflicts that we will accommodate in 380-380D, 1 – 2:30pm. So far only three people have emailed me with their intention to take the exam at this time along with their reason for the conflict.




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# Office Hours for Course Staff – Come see us.

- Ron Schafer: Mon. and Weds. 2:00-4:00 pm, Packard 211.
- Dookun Park: Tues. 1:30 ~ 3:30 pm in Packard 107
- Ruo Yu Gu (Roy): Mon. 4:00 ~ 6:00 pm in Packard 106
- https://class2go.stanford.edu/EE102B/Spring2013/pages/staff

## Lecture Objective

- Sampling rate changing by discrete-time filtering
  - Review decimation
  - Interpolation
- The Discrete Fourier Transform
  - Definition
  - Inverse DFT



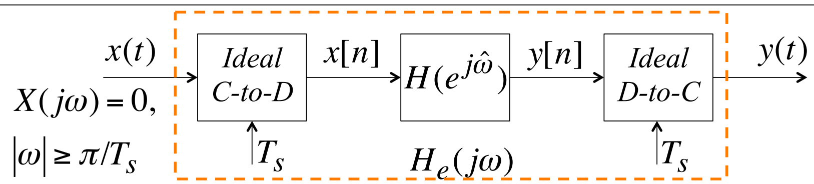


## REVIEW OF DISCRETE-TIME FILTERING OF CONTINUOUS-TIME SIGNALS



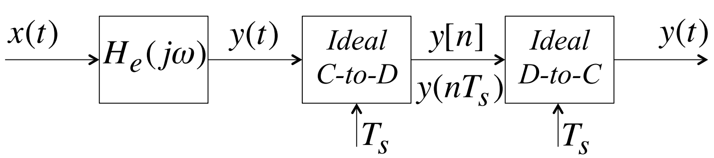


## Effective Filter Equivalent System




$$H_e(j\omega)=\begin{cases} H(e^{j\omega T_s}) & |\omega|<\pi/T_s \\ 0 & |\omega|>\pi/T_s \end{cases}$$

$$y(t)=\sum_{n=-\infty}^{\infty} y[n]\frac{\sin\left[\frac{\pi}{T_s}(t-nT_s)\right]}{\frac{\pi}{T_s}(t-nT_s)}$$








## Example

- Difference equation:

$$y[n]=ay[n-1]+bx[n]$$

- Frequency response:

$$H(e^{j\hat{\omega}})=\frac{b}{1-ae^{-j\hat{\omega}}}$$

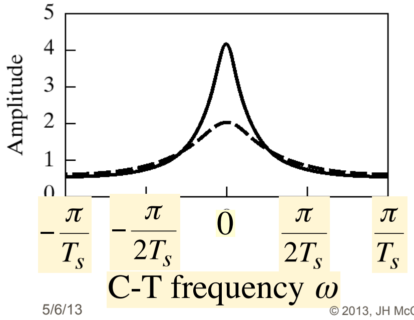
- Overall frequency response

$$H_{\text{eff}}(j\omega)=H(e^{j\omega T_s})=\frac{b}{1-ae^{-j\omega T_s}} \qquad |\omega|<\frac{\pi}{T}$$

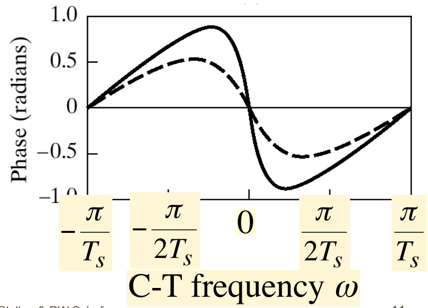
$$|H_{\text{eff}}(j\omega)| = |H(e^{j\omega T_s})| = \frac{b}{(1 + a^2 - 2a\cos(\omega T_s))^{1/2}}$$

$$\angle H_{eff}(j\omega) = \angle X(e^{j\omega T_s}) = \arctan\left(\frac{-a\sin(\omega T_s)}{1 - a\cos(\omega T_s)}\right)$$

$|H_{\text{eff}}(j\omega)|$




$\angle H_{\text{eff}}(j\omega)$





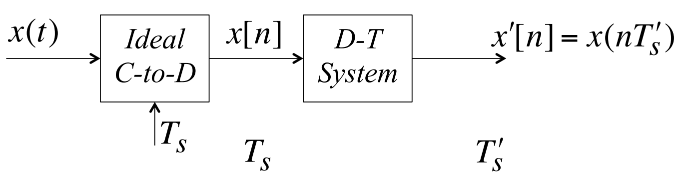



# CHANGING THE SAMPLING RATE USING DISCRETE-TIME FILTERING





# Sampling Rate Changing by Discrete-Time Processing

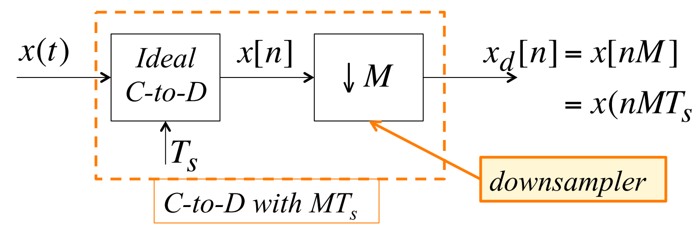



$$X'(e^{j\omega T_s'}) = \frac{1}{T_s'} \sum_{k=-\infty}^{\infty} X\left(j(\omega - k\tfrac{2\pi}{T_s'})\right)$$





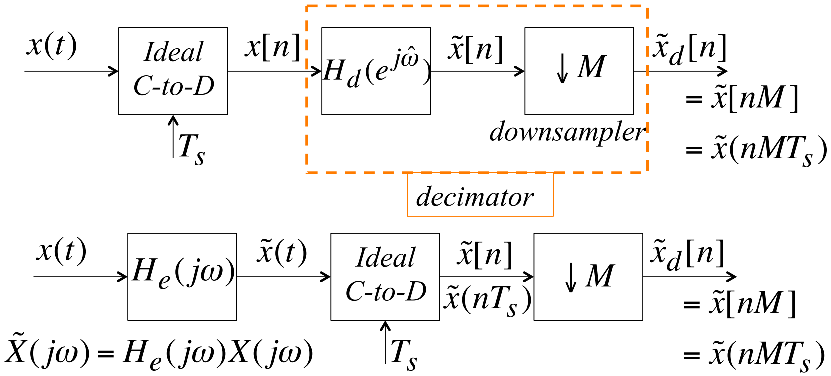
# Sampling Rate Reduction by *downsampling*




$$X_d(e^{j\omega MT_s}) = \frac{1}{MT_s} \sum_{k=-\infty}^{\infty} X\left(j(\omega - k\tfrac{2\pi}{MT_s})\right)$$

*We will have aliasing distortion unless the input is M-times over-sampled; i.e,* $X(j\omega) = 0, \quad |\omega| \geq \pi/(MT_s)$

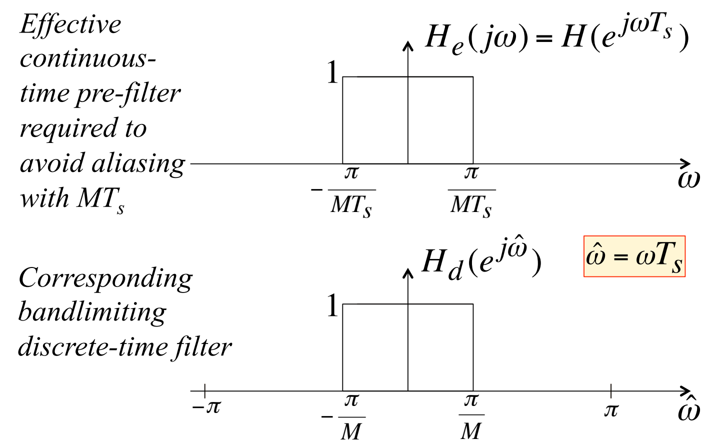
# Sampling Rate Reduction by Decimation



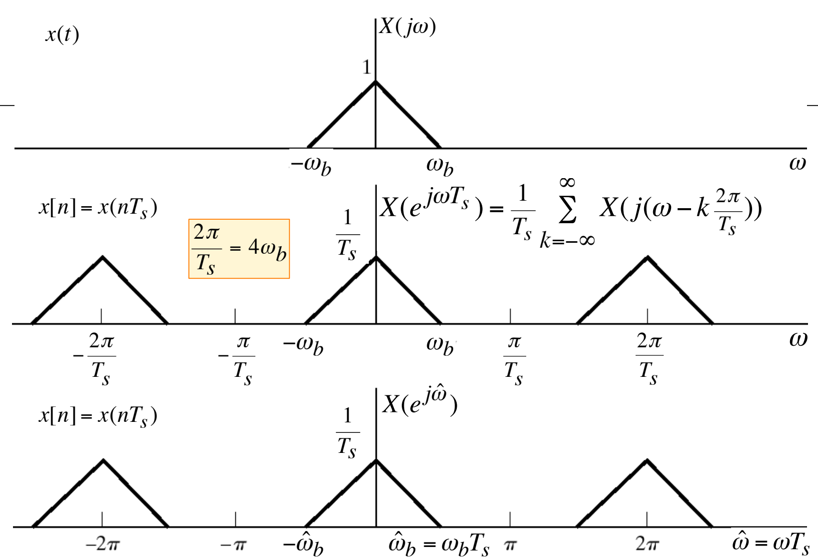

$$\tilde{X}_d(e^{j\omega MT_s}) = \frac{1}{MT_s}\sum_{k=-\infty}^{\infty} \tilde{X}\left(j(\omega - k\tfrac{2\pi}{MT_s})\right)$$



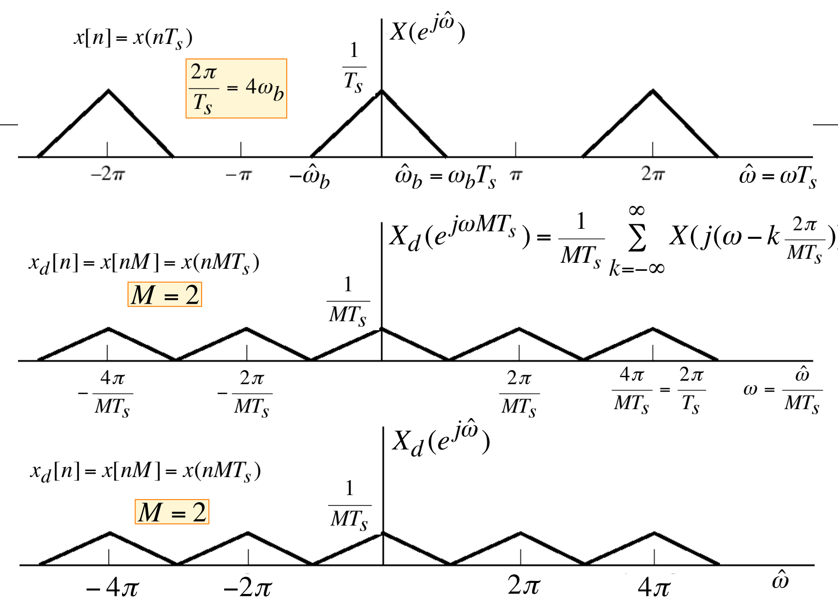
# Ideal Filters for Decimation

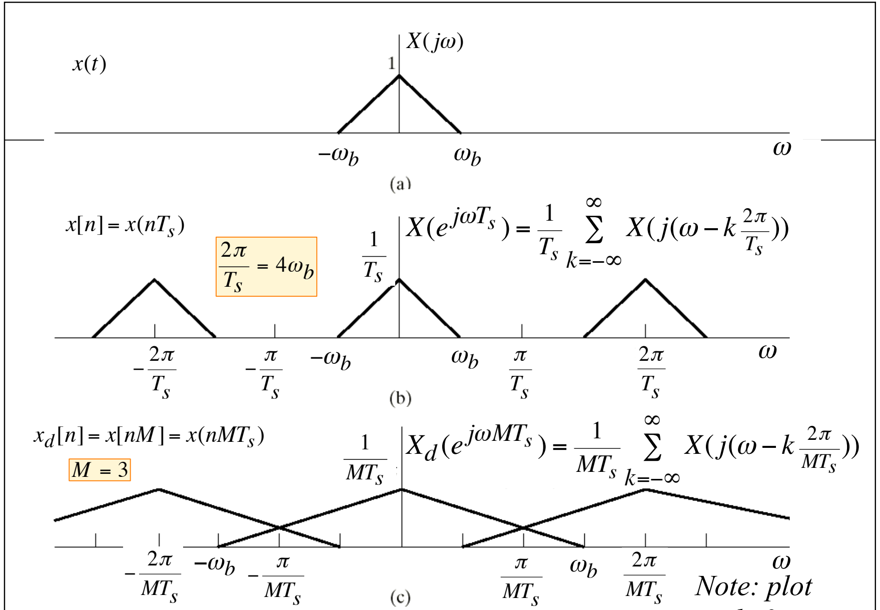




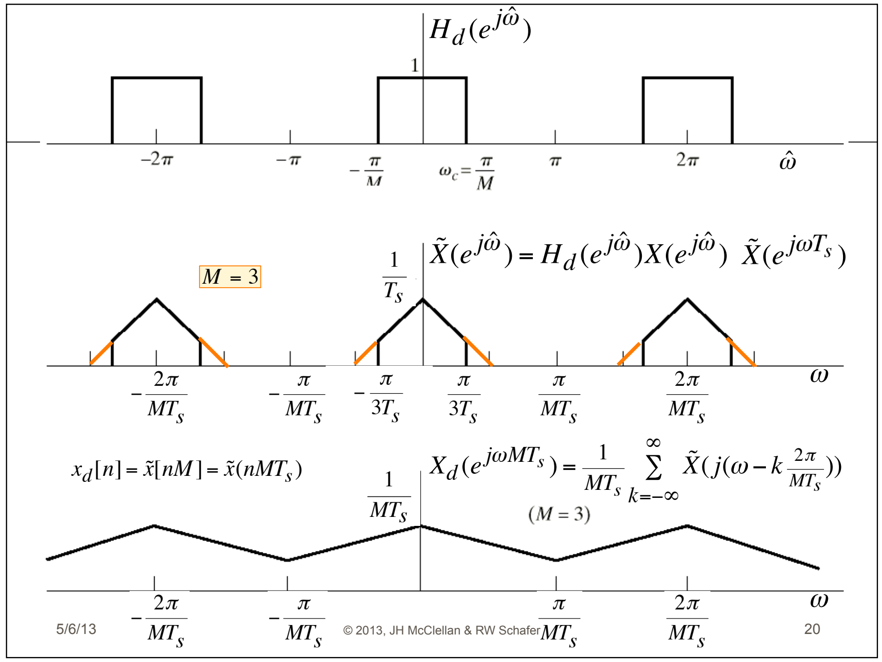





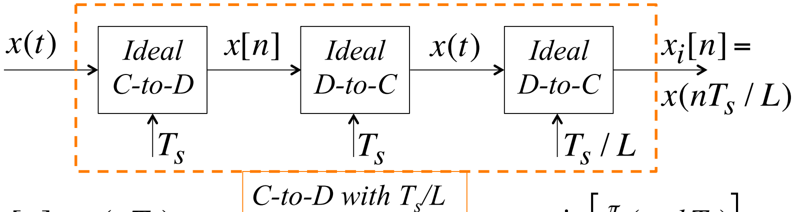







# Increasing Sampling Rate by Interpolation - I




$$x[n] = x(nT_s)$$

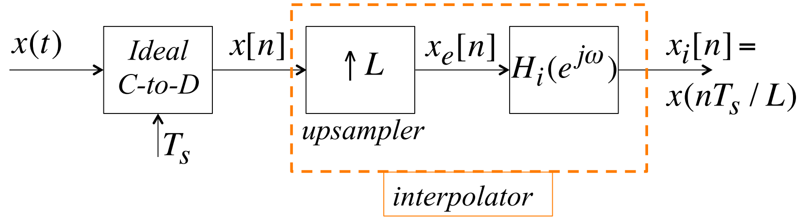
$$X(j\omega) = 0, \quad |\omega| \ge \pi / T_s$$

$$x(t) = \sum_{k=-\infty}^{\infty} x[k] \frac{\sin\left[\frac{\pi}{T_s}(t - kT_s)\right]}{\frac{\pi}{T_s}(t - kT_s)}$$

$$x_i[n] = \sum_{k=-\infty}^{\infty} x[k] \frac{\sin\left[\frac{\pi}{T_s}(nT_s/L - kT_s)\right]}{\frac{\pi}{T_s}(nT_s/L - kT_s)} = \sum_{k=-\infty}^{\infty} x[k] \frac{\sin\left[\frac{\pi}{L}(n - kL)\right]}{\frac{\pi}{L}(n - kL)}$$



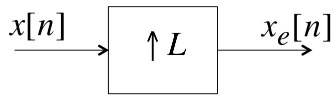
# Increasing Sampling Rate by Interpolation - II




$$x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n - kL] = \begin{cases} x[n/L] & n = 0, \pm 1, \pm 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$x_i[n] = x_e[n] * h_i[n] = \sum_{k=-\infty}^{\infty} x[k] h_i[n - kL] = \sum_{k=-\infty}^{\infty} x[k] \frac{\sin\left[\frac{\pi}{L}(n - kL)\right]}{\frac{\pi}{L}(n - kL)}$$

# Upsampling in the Frequency Domain




$$x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-kL] = \begin{cases} x[n/L] & n = 0,\pm1,\pm2,\ldots \\ 0 & \text{otherwise} \end{cases}$$

$$X_e(e^{j\hat{\omega}}) = \sum_{k=-\infty}^{\infty} x[k]\delta[n-kL]e^{-j\hat{\omega}k}$$

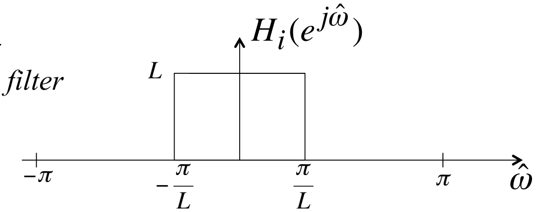
$$= \sum_{k=-\infty}^{\infty} x[k]e^{-j\hat{\omega}kL} = X(e^{j\hat{\omega}L})$$





# Ideal Filter for Interpolation

*Bandlimiting discrete-time filter*



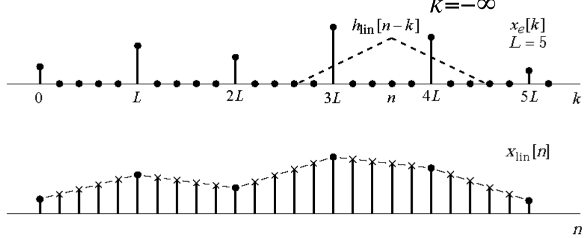

$$h_i[n] = L\frac{\sin\left[\frac{\pi}{L}n\right]}{\pi n} = \frac{\sin\left[\frac{\pi}{L}n\right]}{\frac{\pi}{L}n}$$



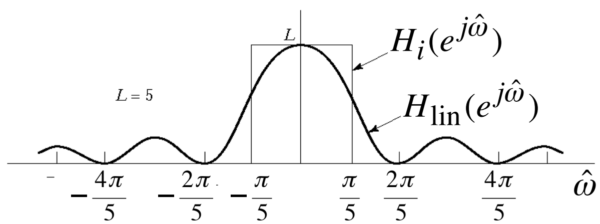


# Linear Interpolation

$$x_{\text{lin}}[n] = \sum_{k=-\infty}^{\infty} x_e[k]h_{\text{lin}}[n-k]$$

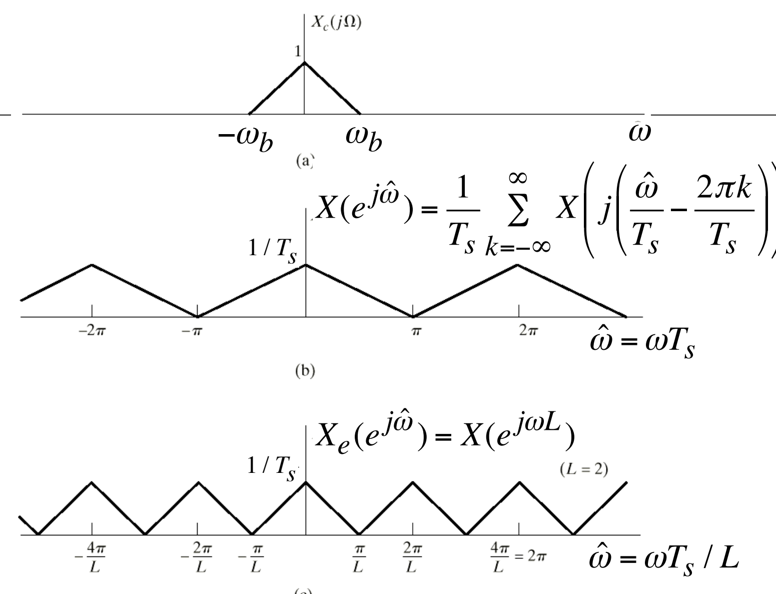


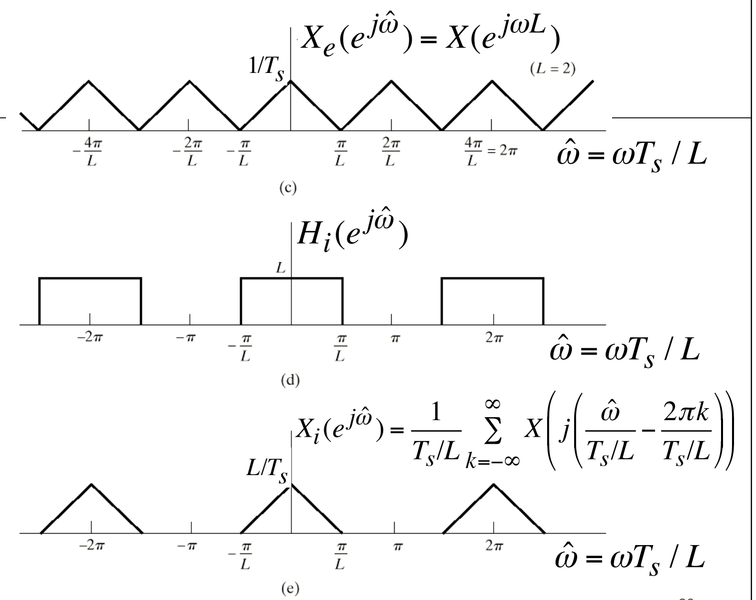

(a)














# INTRODUCTION TO THE DFT





# Sample the DTFT → DFT

- Want **<u>computable</u>** Fourier transform
  - <u>Finite</u> signal length (L)
  - Finite number of frequencies (N)

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n} \Rightarrow X(e^{j\hat{\omega}_k}) = \sum_{n=0}^{L-1} x[n]e^{-j\hat{\omega}_k n}$$

$$\hat{\omega}_k = (2\pi/N)k, \quad k = 0,1,2,\dots N-1$$

$$X[k] = X(e^{j\hat{\omega}_k})$$

*k is the frequency index*

$$\boxed{\text{Periodic}: X(e^{j(\hat{\omega}+2\pi)}) = X(e^{j\hat{\omega}}) \quad \Rightarrow \quad X[k+N] = X[k]}$$





# The Discrete Fourier Transform

- Direct transform

$$\boxed{X[k] = \sum_{n=0}^{N-1} x[n]e^{-j(2\pi/N)kn} \qquad k = 0,1,\dots,N-1}$$

- Inverse transform

$$\boxed{x[n] = \frac{1}{N}\sum_{n=0}^{N-1} X[k]e^{j(2\pi/N)kn} \qquad n = 0,1,\dots,N-1}$$

## Inverse DTFT when L=N (proof)

- Complex exponentials are <u>ORTHOGONAL</u>

$$x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]e^{j(2\pi/N)kn}$$
$$=\frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{m=0}^{N-1}x[m]e^{-j(2\pi/N)km}\right)e^{j(2\pi/N)kn}$$
$$=\frac{1}{N}\sum_{m=0}^{N-1}x[m]\left(\sum_{k=0}^{N-1}e^{-j(2\pi/N)km}e^{j(2\pi/N)kn}\right)$$
$$=N\delta[n-m]=\begin{cases}N & n=m\\ 0 & n\neq m\end{cases}$$
$$=\frac{1}{N}\sum_{m=0}^{N-1}x[m]\left(\sum_{k=0}^{N-1}e^{j(2\pi/N)k(n-m)}\right)=x[n]$$





## Orthogonality of Complex Exponentials

***The sequence set:*** $\left\{e^{j(2\pi/N)kn}\right\}_{k=0}^{N-1}$ for $n=0,1,\cdots,N-1$

$$\frac{1}{N}\sum_{n=0}^{N-1}e^{j(2\pi/N)kn}e^{-j(2\pi/N)mn}=\frac{1}{N}\sum_{n=0}^{N-1}e^{j(2\pi/N)n(k-m)}$$
$$=\frac{1}{N}\frac{1-e^{j2\pi(k-m)}}{1-e^{j(2\pi/N)(k-m)}}=\begin{cases}1, & k=m\\ 0, & \text{otherwise}\end{cases}$$

***because*** $0\le k,m<N$, $|k-m|<N$

***and***

$$\lim_{l\to 0}\frac{1-e^{j2\pi l}}{1-e^{j(2\pi/N)l}}=N$$





## 4-pt DFT: Numerical Example

- Take the 4-pt DFT of the following signal

$x[n]=\delta[n]+\delta[n-1]$ $\quad \{x[n]\}=[1,1,0,0]$

$$X[0]=x[0]e^{-j0}+x[1]e^{-j0}+x[2]e^{-j0}+x[3]e^{-j0}=1+1+0+0=2$$
$$X[1]=x[0]e^{-j0}+x[1]e^{-j\pi/2}+x[2]e^{-j2\pi/2}+x[3]e^{-j3\pi/2}$$
$$=1-j=\sqrt{2}e^{-j\pi/4}$$
$$X[2]=x[0]e^{-j0}+x[1]e^{-j\pi}+x[2]e^{-j2\pi}+x[3]e^{-j3\pi}=1-1+0+0=0$$
$$X[3]=x[0]e^{-j0}+x[1]e^{-j3\pi/2}+x[2]e^{-j3\pi}+x[3]e^{-j9\pi/2}$$
$$=1+j=\sqrt{2}e^{j\pi/4}$$





## N-pt DFT: Numerical Example

- “Take” the N-pt DFT of the impulse

$x[n]=\delta[n]$ $\quad \{x[n]\}=[1,0,0,\ldots 0]$

$$X[k]=\sum_{n=0}^{N-1}\delta[n]e^{-j(2\pi/N)kn}$$
$$=\sum_{n=0}^{0}\delta[n]e^{-j(2\pi/N)kn}=1$$

$$\{X[k]\}=[1,1,1,\ldots,1]$$

# 4-pt iDFT: Numerical Example

- Take the 4-pt inverse DFT of the following

**Example 66-8: Short-Length IDFT**

The 4-point DFT in Example 66-7 is $X[k] = \{2,\ \sqrt{2}e^{-j\pi/4},\ 0,\ \sqrt{2}e^{j\pi/4}\}$. If we compute the 4-point IDFT of the sequence $X[k]$, we should recover $x[n]$ when we apply the IDFT summation (66.52) for each value of $n = 0, 1, 2, 3$. As before, the exponents in (66.52) will all be integer multiples of $\pi/2$ when $N = 4$.

$$\begin{aligned}
x[0] &= \tfrac{1}{4}\left(X[0]e^{j0} + X[1]e^{j0} + X[2]e^{j0} + X[3]e^{j0}\right)\\
&= \tfrac{1}{4}\left(2 + \sqrt{2}e^{-j\pi/4} + 0 + \sqrt{2}e^{j\pi/4}\right) = 1\\
x[1] &= \tfrac{1}{4}\left(X[0]e^{j0} + X[1]e^{j\pi/2} + X[2]e^{j\pi} + X[3]e^{j3\pi/2}\right)\\
&= \tfrac{1}{4}\left(2 + \sqrt{2}e^{j(-\pi/4+\pi/2)} + 0 + \sqrt{2}e^{j(\pi/4+3\pi/2)}\right) = \tfrac{1}{4}(2 + (1 + j) + (1 - j)) = 1\\
x[2] &= \tfrac{1}{4}\left(X[0]e^{j0} + X[1]e^{j\pi} + X[2]e^{j2\pi} + X[3]e^{j3\pi}\right)\\
&= \tfrac{1}{4}\left(2 + \sqrt{2}e^{j(-\pi/4+\pi)} + 0 + \sqrt{2}e^{j(\pi/4+3\pi)}\right) = \tfrac{1}{4}(2 + (-1 + j) + (-1 - j)) = 0\\
x[3] &= \tfrac{1}{4}\left(X[0]e^{j0} + X[1]e^{j3\pi/2} + X[2]e^{j3\pi} + X[3]e^{j9\pi/2}\right)\\
&= \tfrac{1}{4}\left(2 + \sqrt{2}e^{j(-\pi/4+3\pi/2)} + 0 + \sqrt{2}e^{j(\pi/4+9\pi/2)}\right) = \tfrac{1}{4}(2 + (-1 - j) + (-1 + j)) = 0
\end{aligned}$$

Thus we recover the signal $x[n] = \{1,\ 1,\ 0,\ 0\}$ from its DFT coefficients, $X[k] = \{2,\ \sqrt{2}e^{-j\pi/4},\ 0,\ \sqrt{2}e^{j\pi/4}\}$.



# Matrix Form for N-pt DFT

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-j2\pi/N} & e^{-j4\pi/N} & \cdots & e^{-j2(N-1)\pi/N} \\ 1 & e^{-j4\pi/N} & e^{-j8\pi/N} & \cdots & e^{-j4(N-1)\pi/N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j2(N-1)\pi/N} & e^{-j4(N-1)\pi/N} & \cdots & e^{-j2(N-1)(N-1)\pi/N} \end{bmatrix}}_{\text{DFT matrix}} \underbrace{\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix}}_{\text{Signal vector}}$$

**In MATLAB**

- **DFT matrix is** `dftmtx(N)` **for an NxN matrix;**
- **Obtain DFT by** `X = dftmtx(N)*x`
- ***Fast Fourier transform (FFT) algorithm***
  `fft(x,N)`





# FFT: Fast Fourier Transform

- FFT is an **algorithm** for computing the DFT
- N $\log_2$N versus $N^2$ operations
  - Count multiplications (and additions)
  - N = 1024 = $2^{10}$
  - ≈10,000 ops vs. ≈1,000,000 operations
  - ≈1000 times faster
- What about N=256, how much faster?





# Zero-Padding→ more frequency samples

- Want many samples of DTFT
  - WHY? to make a smooth plot (one reason)
  - Finite signal length (L)
  - Finite number of frequencies (N)
  - Thus, we need $L < N, \quad N \to \infty, \quad X[k] \to X(e^{j\hat{\omega}})$

$$X(e^{j\hat{\omega}_k}) = \sum_{n=0}^{L-1} x[n] e^{-j\hat{\omega}_k n}$$

$$\hat{\omega}_k = (2\pi/N)k, \quad k = 0,1,2,\ldots N-1$$

## Zero-Padding with the FFT

- Get many samples of DTFT
  - Finite signal length (L)
  - Finite number of frequencies (N)

    $\hat{\omega}_k = (2\pi / N)k, \quad k = 0,1,2,\ldots N-1$

  - Thus, we need $L < N, \quad N \to \infty, \quad X[k] \to X(e^{j\hat{\omega}})$

**In MATLAB**

- **Use** `X = fft(x,N)` **or**
- With `length(x)=L<N`
  - Then `xpadtoN = [x,zeros(1,N-L)];`
  - Take the N-pt DFT `X=fft(padtoN)`





## DFT periodic in k (frequency domain)

- Since DTFT is periodic in frequency, the DFT must also be periodic in k

$$X[k] = X(e^{j(2\pi/N)k})$$

$$X[k+N] = X(e^{j(2\pi/N)(k+N)}) = X(e^{j(2\pi/N)(k)+(2\pi/N)N)}) = X(e^{j(2\pi/N)k})$$

- What about Negative indices and Conjugate Symmetry?

$$X(e^{-j(2\pi/N)k}) = X^*(e^{j(2\pi/N)k})$$

$$\Rightarrow X[-k] = X^*[k]$$

$$X[N-k] = X^*[k]$$

$N = 32 \quad \Rightarrow$

$X[31] = X^*[1]$

$X[30] = X^*[2]$

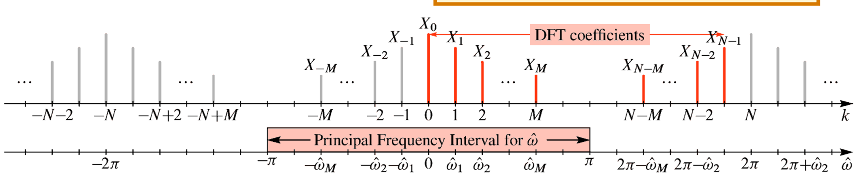
$X[29] = X^*[3]$





## DFT Periodicity in Frequency Index

$$\boxed{\begin{aligned} &X[k] = X(e^{j\hat{\omega}_k}) = X(e^{j(2\pi/N)k}) \\ &k = 0,1,2,\ldots N-1 \end{aligned}}$$




$$\boxed{X[k+N] = X[k] \Leftrightarrow X(e^{j(\hat{\omega}+2\pi)}) = X(e^{j\hat{\omega}})}$$

$$\Rightarrow X[N-k] = X[-k],$$

e.g., $X[N-2] = X[-2]$