STANFORD UNIVERSITY
EE 102B Spring-2013

Lecture 23

The z-Transform and LTI Systems – Poles and Zeros

May 24, 2013

ASSIGNMENTS

- Reading for this Lecture:
  - SPF: Chapters 7 and 8
  - S&S: Chapter 10
  - HW#08 is due by 5pm Wednesday, May 29, in Packard 263.
  - Lab #06 is due by 5pm, today, May 24, in Packard 263. Lab #07 is posted. It is due on May 31.

Office Hours for Course Staff – Come see us.

- Ron Schafer: Mon. and Weds. 2:00-4:00 pm, Packard 211.
- Dookun Park: Tues. 1:30 ~ 3:30 pm in Packard 107
- Ruo Yu Gu (Roy): Mon. 4:00 ~ 6:00 pm in Packard 106

THE Z-TRANSFORM

The z-Transform

- The z-Transform of a sequence is defined as

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}}$$

- Since this is generally an infinite sum, we need to be concerned about "convergence"; i.e., is the sum finite? In general, the region of convergence (ROC) will depend upon z; e.g.,

$$\mathrm{ROC}_x = \{z : 0 \le r_R < |z| < r_L < \infty\}.$$

Basic Properties of z-Transforms

- Linearity (Additivity)

$$x[n] = ax_1[n] + bx_2[n] \Leftrightarrow X(z) = a \underbrace{X_1(z)}_{|z|\in \mathrm{ROC}_{x_1}} + b \underbrace{X_2(z)}_{|z|\in \mathrm{ROC}_{x_2}}$$

$$\mathrm{ROC}_x \text{ contains } \mathrm{ROC}_{x_1} \cap \mathrm{ROC}_{x_2}$$

- Time delay

$$y[n] = x[n-n_d] \Leftrightarrow z^{-n_d}X(z) \quad \mathrm{ROC}_y = \mathrm{ROC}_x$$

- Convolution

$$y[n] = x[n] * h[n] \Leftrightarrow Y(z) = X(z)H(z)$$

$$\mathrm{ROC}_y = \mathrm{ROC}_x \cap \mathrm{ROC}_x$$

Table of z-Transform Pairs

- Impulse sequence

$$x[n] = \delta[n-n_d] \Leftrightarrow X(z) = z^{-n_d}$$

- Right-sided exponential sequence

$$x[n] = a^n u[n] \Leftrightarrow X(z) = \frac{1}{1-az^{-1}} \quad \text{if } |a| < |z|$$

- Left-sided exponential sequence

$$x[n] = -a^n u[-n-1] \Leftrightarrow X(z) = \frac{1}{1-az^{-1}} \quad \text{if } |z| < |a|$$

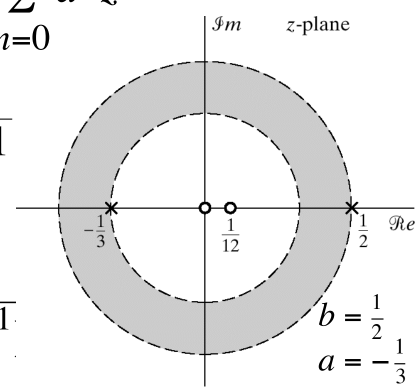
Two-Sided Exponential Signal

$$x[n] = -b^n u[-n-1] + a^n u[n]$$

$$X(z) = -\sum_{n=-\infty}^{-1} b^n z^{-n} + \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \underset{\text{if } |z|<|b|}{\frac{1}{1-bz^{-1}}} + \underset{\text{if } |a|<|z|}{\frac{1}{1-az^{-1}}}$$

$$= \underset{\text{if } |a|<|z|<|b|}{\frac{2-(a+b)z^{-1}}{(1-az^{-1})(1-bz^{-1})}}$$



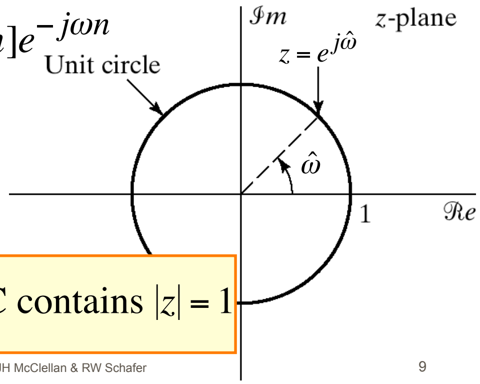
Relation to DTFT

- The DTFT is equal to the z-transform evaluated on the unit circle:

$$X(z)\big|_{z=e^{j\hat{\omega}}} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$= X(e^{j\hat{\omega}})$$

$$= \text{DTFT}$$



$$\boxed{\sum_{n=-\infty}^{\infty} |x[n]| < \infty \Rightarrow \text{ ROC contains } |z| = 1}$$

THE INVERSE Z-TRANSFORM BY PARTIAL FRACTIONS

Partial Fraction Expansion - I

- Consider a general rational z-transform

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}} = b_0 \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

- We can find a partial fraction expansion in the form

$$X(z) = \underbrace{\left[\sum_{r=0}^{(M-N)} B_r z^{-r}\right]}_{\text{only if } M \geq N} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

Partial Fraction Expansion - II

$$X(z) = \underbrace{\left[\sum_{r=0}^{(M-N)} B_r z^{-r}\right]}_{\text{long division gets us this}} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

$$X(z)(1 - d_i z^{-1}) = \left[\sum_{r=0}^{(M-N)} B_r z^{-r}\right](1 - d_i z^{-1})$$

$$+ \sum_{k=1}^{N} \frac{A_k (1 - d_i z^{-1})}{1 - d_k z^{-1}}$$

$$\boxed{A_i = (1 - d_i z^{-1})X(z)\big|_{z=d_i}}$$

Called the residue at $z = d_i$

Partial Fraction Expansion - III

$$X(z)=\underbrace{\left[\sum_{r=0}^{(M-N)} B_r z^{-r}\right]}_{\text{if } M\ge N}+\sum_{k=1}^{N}\frac{A_k}{1-d_k z^{-1}}$$

$$\text{ROC}:\ r_R<|z|<r_L$$

$$x[n]=\underbrace{\left[\sum_{r=0}^{(M-N)} B_r \delta[n-r]\right]}_{\text{if } M\ge N}$$

$$+\underbrace{\sum_k A_k d_k^n u[n]}_{\text{when } |d_k|<r_R}-\underbrace{\sum_k A_k d_k^n u[-n-1]}_{\text{when } |d_k|>r_L}$$

Inverse by Partial Fractions

1. Use polynomial long division to write H(z) in form $X(z)=\underbrace{\left[\sum_{r=0}^{(M-N)} B_r z^{-r}\right]}_{\text{if } M\ge N}+X_r(z)$
2. Factor denominator of X(z) or $X_r(z)$
3. Compute residues using $A_i=(1-d_i z^{-1})X(z)\big|_{z=d_i}$
4. Sort poles based on ROC
5. Write down the answer using table

Example

$$H(z)=\frac{1+3z^{-1}+3z^{-2}+z^{-3}}{1-3z^{-1}+2z^{-2}}$$

$$H(z)=\frac{(1+z)^3}{(1-z^{-1})(1-2z^{-1})}$$

- What are the possible regions of convergence?

$2<\lvert z\rvert$	$1<\lvert z\rvert<2$	$\lvert z\rvert<1$
causal	*two-sided*	*anti-causal*

Example Step - 1

- Divide to get remainder of degree < N

			.5	2.25		
2	-3	1	1	3	3	1
			1	-1.5	.5	
				4.5	2.5	1
				4.5	-6.75	2.25
					9.25	-1.25

Example Steps – 2 & 3

- Compute residues

$$H(z) = 2.25 + 0.5z^{-1} + \frac{-1.25 + 9.25z^{-1}}{(1-z^{-1})(1-2z^{-1})}$$

$$= 2.25 + 0.5z^{-1} + \frac{A_1}{1-z^{-1}} + \frac{A_2}{1-2z^{-1}}$$

$$A_1 = H(z)(1-z^{-1})\Big|_{z=1} = \frac{-1.25+9.25}{1-2} = -8$$

$$A_2 = H(z)(1-2z^{-1})\Big|_{z=2} = \frac{-1.25+9.25/2}{1-1/2} = 2(3.375) = 6.75$$

Example Steps – 4 & 5

- Look up inverse transforms

$$H(z) = 2.25 + 0.5z^{-1} + \frac{-8}{1-z^{-1}} + \frac{6.75}{1-2z^{-1}}$$

ROC: $|z| > 2$

$$h[n] = 2.25\,\delta[n] + 0.5\,\delta[n-1] - 8\,u[n] + 6.75(2)^n u[n]$$

ROC: $1 < |z| < 2$

$$h[n] = 2.25\,\delta[n] + 0.5\,\delta[n-1] - 8\,u[n] - 6.75(2)^n u[-n-1]$$

Partial Fractions in MATLAB: residuez( )

```
function [ r, p, k ] = residuez( b, a )
% RESIDUEZ   ------- Z-transform partial-fraction expansion
%    [R,P,K] = RESIDUEZ(B,A)
%      finds the residues, poles and direct terms of a
%      partial-fraction expansion of B(z)/A(z)
%
%    B(z)        r(1)              r(n)
%    ---- =  ----------  +... + ----------   + k(1) + k(2)z^-1 ...
%    A(z)    1-p(1)z^-1         1-p(n)z^-1
%
%       B: numerator polynomial coefficients
%       A: denominator coeffs (in ascending powers of z^-1)
%       R: the residues   (in a column vector)
%       P: the poles  (column vector)
%       K: the direct terms  (ROW vector)
%    [B,A] = RESIDUEZ(R,P,K)
%      convert partial-fraction expansion back to B/A form.
%    MULTIPLE POLES (order of residues):
%      residue for 1st power pole, then 2nd power, etc.
```


Partial Fraction Expansion in MATLAB

$$H(z) = \frac{1+3z^{-1}+3z^{-2}+z^{-3}}{1-3z^{-1}+2z^{-2}} = 2.25 + 0.5z^{-1} + \frac{-8}{1-z^{-1}} + \frac{6.75}{1-2z^{-1}}$$

```
» [r,p,k]=residuez([1,3,3,1],[1,-3,2])
r =
   6.75000000000000
  -8.00000000000000
p =
     2
     1
k =
   2.25000000000000   0.50000000000000
```

$$h[n] = 2.25\delta[n] + 0.5\delta[n-1] - 8u[n] + 6.75(2)^n u[n]$$

$$\text{or:}\quad h[n] = 2.25\delta[n] + 0.5\delta[n-1] - 8u[n] - 6.75(2)^n u[-n-1]$$

LTI SYSTEMS AND THE Z-TRANSFORM

Review for LTI Systems

$$x[n] \longrightarrow \boxed{\text{LTI System}} \longrightarrow y[n] = x[n]*h[n]$$

$$\delta[n] \longrightarrow \boxed{\text{LTI System}} \longrightarrow h[n]$$

$$e^{j\hat{\omega}n} \qquad H(e^{j\hat{\omega}})e^{j\hat{\omega}n}$$

$$X(e^{j\hat{\omega}}) \qquad X(e^{j\hat{\omega}})H(e^{j\hat{\omega}})$$

$$X(z) \qquad X(z)H(z)$$

$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

$$Y(z) = \sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{k=0}^{M} b_k z^{-k} X(z)$$

System Function for LTI Systems

$$x[n] \longrightarrow \boxed{\text{LTI System}} \longrightarrow y[n] = x[n]*h[n]$$

$$\delta[n] \longrightarrow \boxed{\text{LTI System}} \longrightarrow h[n]$$

$$y[n] - \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\left(1 - \sum_{k=0}^{N} a_k z^{-k}\right) Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\boxed{H(z) = \frac{Y(z)}{X(z)} = \frac{\left(\sum_{k=0}^{M} b_k z^{-k}\right)}{\left(1 - \sum_{k=1}^{N} a_k z^{-k}\right)}}$$

$$\boxed{H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}}}$$

Causal LTI Systems – 1

$$x[n] \longrightarrow \boxed{\text{LTI System}} \longrightarrow y[n] = x[n]*h[n]$$

$$\delta[n] \longrightarrow \boxed{\text{LTI System}} \longrightarrow h[n]$$

$$\boxed{y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]}$$

With initial rest conditions

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}} = b_0 \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

Causal LTI Systems – II
Impulse response of DE

$$H(z) = \underbrace{\left[\sum_{r=0}^{(M-N)} B_r z^{-r}\right]}_{\text{if } M \geq N} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

ROC: $r_R = \max_k \{d_k\} < |z|$

$$h[n] = \underbrace{\left[\sum_{r=0}^{(M-N)} B_r \delta[n-r]\right]}_{\text{if } M \geq N} + \sum_{k=1}^{N} A_k d_k^n u[n]$$

Stability requires: $r_R = \max_k \{d_k\} < 1$

Frequency Response of a DE

$$y[n] - \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}} = \frac{\left(\sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}\right)}{\left(1 - \sum_{k=1}^{N} a_k e^{-j\hat{\omega}k}\right)}$$

ROC must Contain the Unit circle

ROC for causal system: $\max_k \{d_k\} < |z|$ $\Longrightarrow$ Stability requires $\max_k \{d_k\} < 1$ for causal system

POLES AND ZEROS

Pole and Zero Factor

- The basic pole/zero factor is:

$$(1 - az^{-1}) = (1 - re^{j\theta} z^{-1}) \qquad a = re^{j\theta}$$

- Evaluate on unit circle

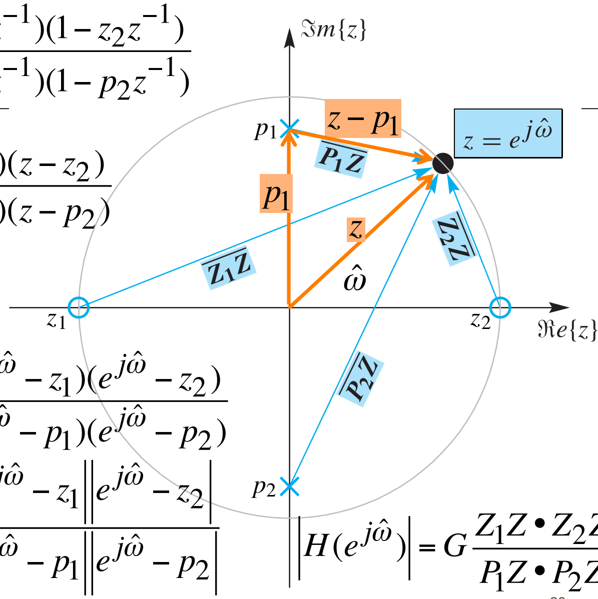
$$(1 - re^{j\theta} z^{-1})\Big|_{z=e^{j\hat{\omega}}} = (1 - re^{j\theta} e^{-j\hat{\omega}})$$

- For $\hat{\omega}$ around θ

$$(1 - re^{j\theta} e^{-j\hat{\omega}})\Big|_{\hat{\omega}=\theta} = (1 - r) \xrightarrow[r \to 1]{} 0$$

$$H(z) = G\frac{(1-z_1z^{-1})(1-z_2z^{-1})}{(1-p_1z^{-1})(1-p_2z^{-1})}$$

$$H(z) = G\frac{(z-z_1)(z-z_2)}{(z-p_1)(z-p_2)}$$



$$H(e^{j\hat{\omega}}) = G\frac{(e^{j\hat{\omega}}-z_1)(e^{j\hat{\omega}}-z_2)}{(e^{j\hat{\omega}}-p_1)(e^{j\hat{\omega}}-p_2)}$$

$$\left|H(e^{j\hat{\omega}})\right| = G\frac{\left|e^{j\hat{\omega}}-z_1\right|\left|e^{j\hat{\omega}}-z_2\right|}{\left|e^{j\hat{\omega}}-p_1\right|\left|e^{j\hat{\omega}}-p_2\right|}$$

$$\left|H(e^{j\hat{\omega}})\right| = G\frac{Z_1Z \bullet Z_2Z}{P_1Z \bullet P_2Z}$$

THREE DOMAINS DEMOS

9th-order Running Sum Filter

$$H(z) = \sum_{n=0}^{9} z^{-n} = \frac{1-z^{-10}}{1-z^{-1}}$$

Zeros: $1 - z^{-10} = 0$

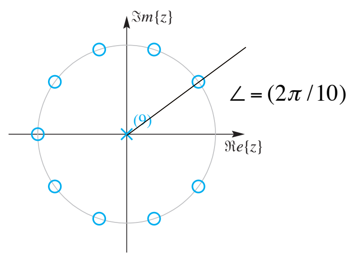
$z^{10} = 1 \Rightarrow z_k = e^{j(2\pi/10)k}$

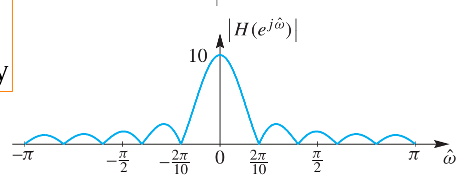
$k = 0, 1, \ldots, 9;$

i.e., the 10^{th} roots of unity

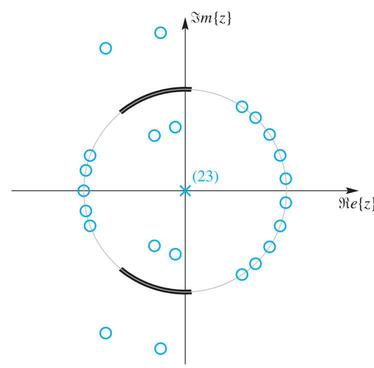
Pole: $1 - z^{-1} = 0$

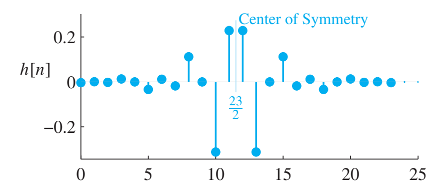
$z = 1$ cancels z_0

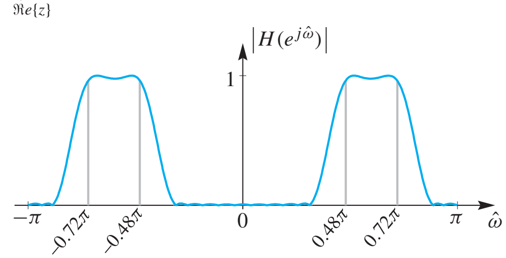




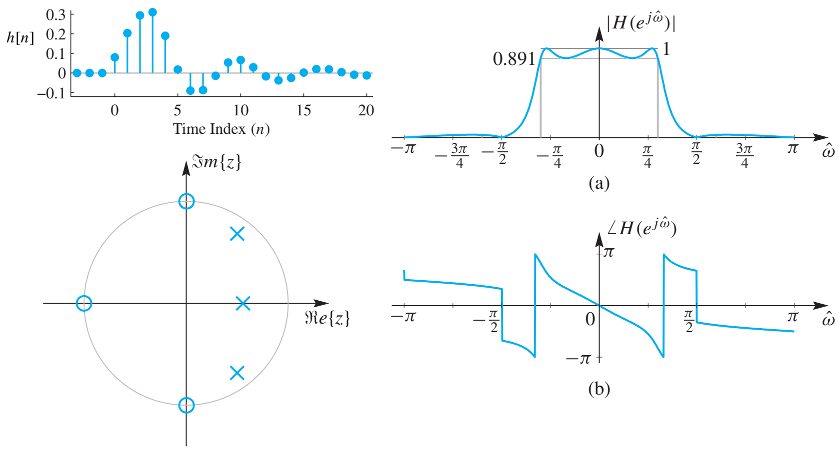
FIR Bandpass Filter M = 23





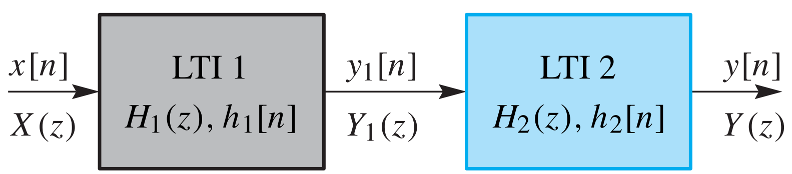


Elliptic Lowpass Filter

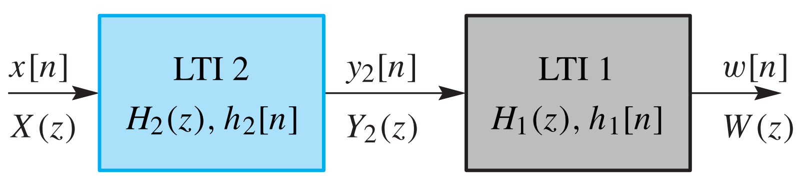


CASCADED LTI SYSTEMS

Cascaded LTI Systems

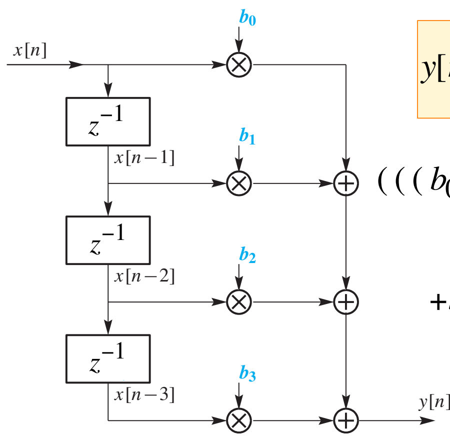


$$Y(z) = H_1(z)H_2(z) = H_2(z)H_1(z)$$



IMPLEMENTATION STRUCTURES FOR LTI SYSTEMS

FIR Direct Form



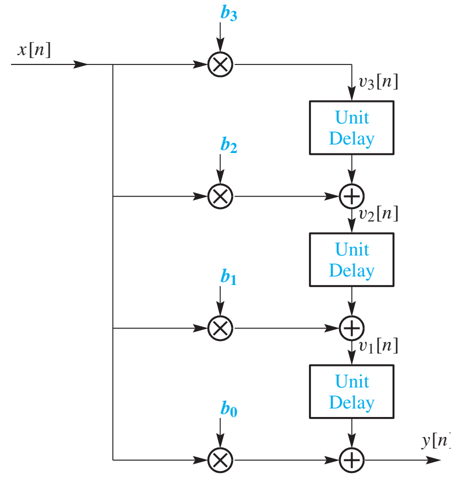
$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

$$(((\,b_0 x[n] + b_1 x[n-1]\,)$$

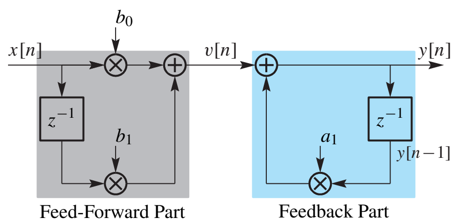
$$+ b_2 x[n-2]\,)$$

$$+ b_3 x[n-3]\,) = y[n]$$

FIR Transposed Direct Form



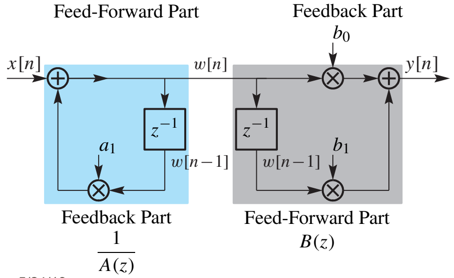
Direct Form – I First-Order IIR Structures



$$H(z) = (b_0 + b_1 z^{-1}) \frac{1}{(1 - a_1 z^{-1})}$$

$$v[n] = b_0 x[n] + b_1 x[n-1]$$

$$y[n] = a_1 y[n-1] + v[n]$$

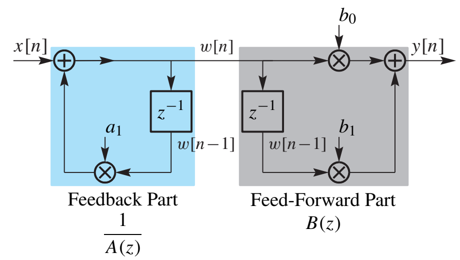


$$H(z) = \frac{1}{(1 - a_1 z^{-1})} (b_0 + b_1 z^{-1})$$

$$w[n] = a_1 w[n-1] + x[n]$$

$$y[n] = b_0 w[n] + b_1 w[n-1]$$

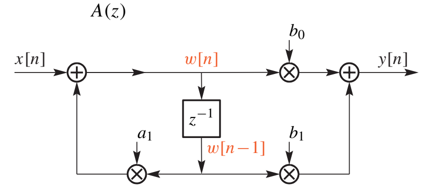
Direct Form – II First-Order IIR Structure



$$H(z) = \frac{1}{(1 - a_1 z^{-1})} (b_0 + b_1 z^{-1})$$

$$w[n] = a_1 w[n-1] + x[n]$$

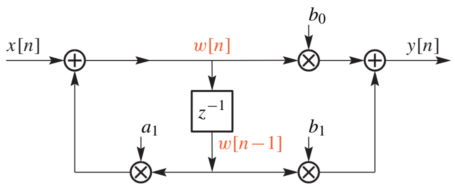
$$y[n] = b_0 w[n] + b_1 w[n-1]$$



$$w[n] = a_1 w[n-1] + x[n]$$

$$y[n] = b_0 w[n] + b_1 w[n-1]$$

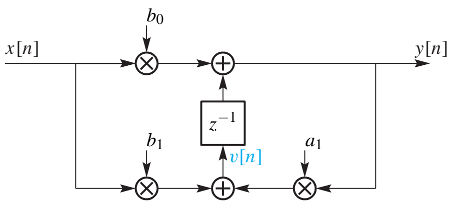
Direct Form – II First-Order IIR Structure



$$H(z) = \frac{(b_0 + b_1 z^{-1})}{(1 - a_1 z^{-1})}$$

$$w[n] = a_1 w[n-1] + x[n]$$

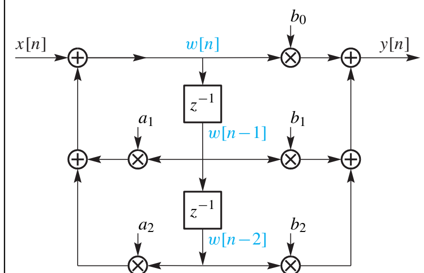
$$y[n] = b_0 w[n] + b_1 w[n-1]$$



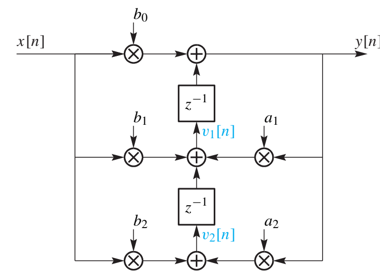
$$v[n] = a_1 y[n-1] + b_1 x[n]$$

$$y[n] = b_0 x[n] + v[n-1]$$

Second-Order Direct Form Structures



Direct Form II



Transposed Direct Form II