STANFORD UNIVERSITY
EE 102B Spring-2013

Lecture 22

Inverse z-Transform and Properties

May 22, 2013

ASSIGNMENTS

- Reading for this Lecture:
  - SPF: Chapters 7 and 8
  - S&S: Chapter 10
  - HW#07 is due by 5pm today, May 22, in Packard 263.
  - Lab #06 is due by 5pm, Friday, May 24, in Packard 263. Lab #06 continues Lab #05.

Office Hours for Course Staff – Come see us.

- Ron Schafer: Mon. and Weds. 2:00-4:00 pm, Packard 211.
- Dookun Park: Tues. 1:30 ~ 3:30 pm in Packard 107
- Ruo Yu Gu (Roy): Mon. 4:00 ~ 6:00 pm in Packard 106

Lecture Objective

- Review of the z-transform
  - Definition
  - Examples
    - Two-sided exponential
  - Inverse z-transform by partial fractions
  - LTI systems – the system function

THE Z-TRANSFORM

The *z*-Transform

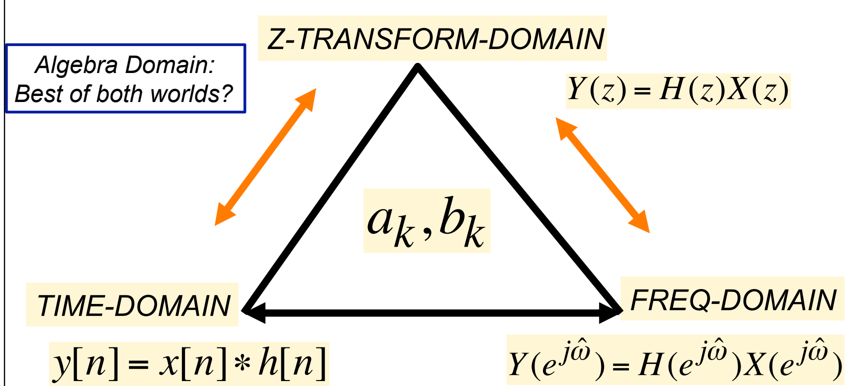
- The *z*-Transform of a sequence is defined as

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}}$$

- Since this is generally an infinite sum, we need to be concerned about "convergence"; i.e., is the sum finite? In general, the region of convergence (ROC) will depend upon *z*; e.g.,

$$\mathrm{ROC}_x = \{z : 0 \le r_R < |z| < r_L < \infty\}.$$

Two, Now Three Domains and LTI Systems



$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

Basic Properties of z-Transforms

- Linearity (Additivity)

$$x[n] = ax_1[n] + bx_2[n] \Leftrightarrow X(z) = a\underbrace{X_1(z)}_{|z|\in \mathrm{ROC}_{x_1}} + b\underbrace{X_2(z)}_{|z|\in \mathrm{ROC}_{x_2}}$$

$$\mathrm{ROC}_x \text{ contains } \mathrm{ROC}_{x_1} \bigcap \mathrm{ROC}_{x_2}$$

- Time delay

$$y[n] = x[n-n_d] \Leftrightarrow z^{-n_d}X(z) \quad \mathrm{ROC}_y = \mathrm{ROC}_x$$

- Convolution

$$y[n] = x[n] * h[n] \Leftrightarrow Y(z) = X(z)H(z)$$

$$\mathrm{ROC}_y = \mathrm{ROC}_x \cap \mathrm{ROC}_x$$

Table of z-Transform Pairs

- Impulse sequence

$$x[n] = \delta[n - n_d] \Leftrightarrow X(z) = z^{-n_d}$$

- Right-sided exponential sequence

$$x[n] = a^n u[n] \Leftrightarrow X(z) = \frac{1}{1 - az^{-1}} \quad \text{if } |a| < |z|$$

- Left-sided exponential sequence

$$x[n] = -a^n u[-n-1] \Leftrightarrow X(z) = \frac{1}{1 - az^{-1}} \quad \text{if } |z| < |a|$$

Two-Sided Exponential Signal

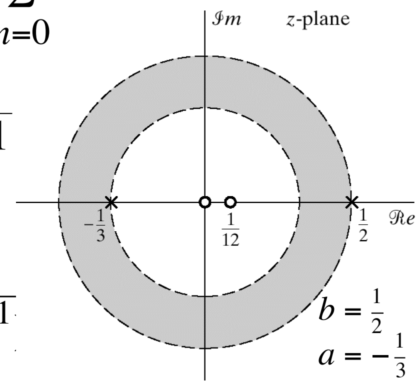
$$x[n] = -b^n u[-n-1] + a^n u[n]$$

$$X(z) = -\sum_{n=-\infty}^{-1} b^n z^{-n} + \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \underset{\text{if } |z|<|b|}{\frac{1}{1-bz^{-1}}} + \underset{\text{if } |a|<|z|}{\frac{1}{1-az^{-1}}}$$

$$= \frac{2-(a+b)z^{-1}}{(1-az^{-1})(1-bz^{-1})}$$

if $|a|<|z|<|b|$



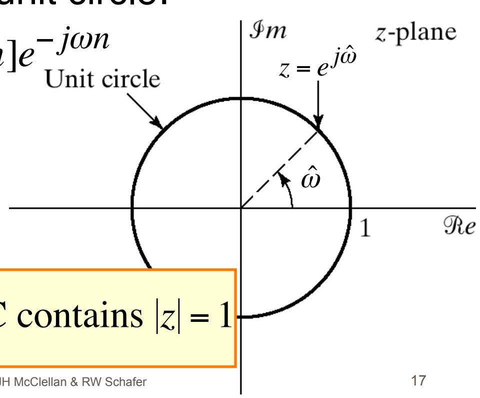
Relation to DTFT

- The DTFT is equal to the z-transform evaluated on the unit circle:

$$X(z)\Big|_{z=e^{j\hat{\omega}}} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$= X(e^{j\hat{\omega}})$$

$$= \text{DTFT}$$



$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \Rightarrow \text{ROC contains } |z| = 1$$

EXAMPLES OF FINDING Z-TRANSFORMS

Table of z-Transform Pairs

- Impulse sequence

$$x[n] = \delta[n - n_d] \Leftrightarrow X(z) = z^{-n_d}$$

- Right-sided exponential sequence

$$x[n] = a^n u[n] \Leftrightarrow X(z) = \frac{1}{1 - az^{-1}} \quad \text{if } |a| < |z|$$

- Left-sided exponential sequence

$$x[n] = -a^n u[-n-1] \Leftrightarrow X(z) = \frac{1}{1 - az^{-1}} \quad \text{if } |z| < |a|$$

Example 1
Finding a z-Transform

$$h[n] = 2.25\delta[n] + 0.5\delta[n-1] - 8u[n] - 6.75(2)^n u[-n-1]$$

$$H(z) = 2.25 + 0.5z^{-1} + \underbrace{\frac{-8}{1 - z^{-1}}}_{1<|z|} + \underbrace{\frac{6.75}{1 - 2z^{-1}}}_{|z|<2}$$

$$H(z) = \frac{(2.25 + 0.5z^{-1})(1 - z^{-1})(1 - 2z^{-1}) - 8(1 - 2z^{-1}) + 6.75(1 - z^{-1})}{(1 - z^{-1})(1 - 2z^{-1})}$$

$$H(z) = \frac{1 + 3z^{-1} + 3z^{-2} + z^{-3}}{1 - 3z^{-1} + 2z^{-2}} = \frac{(1 + z^{-1})^3}{(1 - z^{-1})(1 - 2z^{-1})}$$

$$\text{ROC} = \{z : 1 < |z| < 2\}$$

Example 2
Finding a z-Transform

$$x[n] = u[-n-1] + a^n u[n]$$

$$X(z) = \underbrace{\frac{-1}{1 - z^{-1}}}_{|z|<1} + \underbrace{\frac{1}{1 - az^{-1}}}_{|a|<|z|} = \underbrace{\frac{(a-1)z^{-1}}{(1 - az^{-1})(1 - z^{-1})}}_{|a|<|z|<1}$$

Example 3
Finding a z-Transform

$$y[n] = u[-n] + a^{n-1} u[n-1] = x[n-1]$$

$$Y(z) = z^{-1} X(z) = \underbrace{\frac{(a-1)z^{-2}}{(1 - az^{-1})(1 - z^{-1})}}_{|a|<|z|<1}$$

THE INVERSE Z-TRANSFORM BY PARTIAL FRACTIONS

Partial Fraction Expansion - I

- Consider a general rational z-transform

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}} = b_0 \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

- We can find a partial fraction expansion in the form

$$X(z) = \underbrace{\left[\sum_{r=0}^{(M-N)} B_r z^{-r} \right]}_{\text{only if } M \geq N} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

Partial Fraction Expansion - II

$$X(z) = \underbrace{\left[\sum_{r=0}^{(M-N)} B_r z^{-r} \right]}_{\text{long division gets us this}} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

$$X(z)(1 - d_i z^{-1}) = \left[\sum_{r=0}^{(M-N)} B_r z^{-r} \right] (1 - d_i z^{-1})$$

$$+ \sum_{k=1}^{N} \frac{A_k (1 - d_i z^{-1})}{1 - d_k z^{-1}}$$

$$\boxed{A_i = (1 - d_i z^{-1}) X(z) \Big|_{z = d_i}}$$

Called the residue at $z = d_i$

Partial Fraction Expansion - III

$$X(z) = \underbrace{\left[\sum_{r=0}^{(M-N)} B_r z^{-r} \right]}_{\text{if } M \geq N} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

ROC: $r_R < |z| < r_L$

$$x[n] = \underbrace{\left[\sum_{r=0}^{(M-N)} B_r \delta[n-r] \right]}_{\text{if } M \geq N}$$

$$+ \underbrace{\sum_{k} A_k d_k^n u[n]}_{\text{when } |d_k| < r_R} - \underbrace{\sum_{k} A_k d_k^n u[-n-1]}_{\text{when } |d_k| > r_L}$$

Inverse by Partial Fractions

1. Use polynomial long division to write H(z) in form $X(z) = \underbrace{\left[\sum_{r=0}^{(M-N)} B_r z^{-r}\right]}_{\text{if } M \geq N} + X_r(z)$
2. Factor denominator of X(z) or $X_r(z)$
3. Compute residues using $A_i = (1 - d_i z^{-1}) X(z)\Big|_{z=d_i}$
4. Sort poles based on ROC
5. Write down the answer using table

Example

$$H(z) = \frac{1 + 3z^{-1} + 3z^{-2} + z^{-3}}{1 - 3z^{-1} + 2z^{-2}}$$

$$H(z) = \frac{(1+z)^3}{(1 - z^{-1})(1 - 2z^{-1})}$$

- What are the possible regions of convergence?

Partial Fractions in MATLAB: residuez( )

```
function [ r, p, k ] = residuez( b, a )
% RESIDUEZ   ------- Z-transform partial-fraction expansion
%    [R,P,K] = RESIDUEZ(B,A)
%      finds the residues, poles and direct terms of a
%      partial-fraction expansion of B(z)/A(z)
%
%    B(z)        r(1)                r(n)
%    ----  =   ----------    +... + ----------     + k(1) + k(2)z^-1 ...
%    A(z)    1-p(1)z^-1           1-p(n)z^-1
%
%        B: numerator polynomial coefficients
%        A: denominator coeffs (in ascending powers of z^-1)
%        R: the residues   (in a column vector)
%        P: the poles  (column vector)
%        K: the direct terms  (ROW vector)
%    [B,A] = RESIDUEZ(R,P,K)
%      convert partial-fraction expansion back to B/A form.
%    MULTIPLE POLES (order of residues):
%      residue for 1st power pole, then 2nd power, etc.
```


Partial Fraction Expansion in MATLAB

$$H(z) = \frac{1 + 3z^{-1} + 3z^{-2} + z^{-3}}{1 - 3z^{-1} + 2z^{-2}} = 2.25 + 0.5z^{-1} + \frac{-8}{1 - z^{-1}} + \frac{6.75}{1 - 2z^{-1}}$$

```
» [r,p,k]=residuez([1,3,3,1],[1,-3,2])
r =
   6.75000000000000
  -8.00000000000000
p =
     2
     1
k =
   2.25000000000000   0.50000000000000
```

$$h[n] = 2.25\delta[n] + 0.5\delta[n-1] - 8u[n] + 6.75(2)^n u[n]$$

$$\text{or:}\quad h[n] = 2.25\delta[n] + 0.5\delta[n-1] - 8u[n] - 6.75(2)^n u[-n-1]$$

LTI SYSTEMS AGAIN

Review for LTI Systems

$$x[n] \;\;\delta[n] \longrightarrow \boxed{\textit{LTI System}} \longrightarrow y[n] = x[n]*h[n] \;\; h[n]$$

$$e^{j\hat{\omega}n} \qquad H(e^{j\hat{\omega}})e^{j\hat{\omega}n}$$

$$X(e^{j\hat{\omega}}) \qquad X(e^{j\hat{\omega}})H(e^{j\hat{\omega}})$$

$$X(z) \qquad X(z)H(z)$$

$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

$$Y(z) = \sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{k=0}^{M} b_k z^{-k} X(z)$$

System Function for LTI Systems

$$x[n] \;\;\delta[n] \longrightarrow \boxed{\textit{LTI System}} \longrightarrow y[n] = x[n]*h[n] \;\; h[n]$$

$$y[n] - \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\left(1 - \sum_{k=0}^{N} a_k z^{-k}\right) Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\boxed{H(z) = \frac{Y(z)}{X(z)} = \frac{\left(\sum_{k=0}^{M} b_k z^{-k}\right)}{\left(1 - \sum_{k=1}^{N} a_k z^{-k}\right)}}$$

$$\boxed{H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}}}$$

Causal LTI Systems – 1

$$x[n] \;\;\delta[n] \longrightarrow \boxed{\textit{LTI System}} \longrightarrow y[n] = x[n]*h[n] \;\; h[n]$$

$$\boxed{y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]}$$

With initial rest conditions

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}} = b_0 \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

Causal LTI Systems – II
Impulse response of DE

$$H(z) = \underbrace{\left[\sum_{r=0}^{(M-N)} B_r z^{-r}\right]}_{\text{if } M \geq N} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

ROC: $r_R = \max_k |d_k| < |z|$

$$h[n] = \underbrace{\left[\sum_{r=0}^{(M-N)} B_r \delta[n-r]\right]}_{\text{if } M \geq N} + \sum_{k=1}^{N} A_k d_k^n u[n]$$

Frequency Response of a DE

$$y[n] - \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}} = \frac{\left(\sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}\right)}{\left(1 - \sum_{k=1}^{N} a_k e^{-j\hat{\omega}k}\right)}$$

ROC must Contain the Unit circle

ROC for causal system: $|z| > \max_k \{d_k\}$

$\Longrightarrow$

Stability requires $\max_k \{d_k\} < 1$ for causal system