Stanford University
Winter 2012-2013

Signal Processing and Linear Systems I

Lecture 6: Convolution

January 16, 2013

Convolution Evaluation and Properties

Today's topics:

- Review: response of an LTI system
- Representation of convolution
- Graphical interpretation
- Examples
- Properties of convolution

Response of LTI System

- A linear system is completely characterized by its *impulse response* $h(t, \tau)$. If the system is also time invariant, the impulse response is only a function of t, and is written $h(t)$.
- For a linear system with an input signal $x(t)$, the output is given by the *superposition integral*

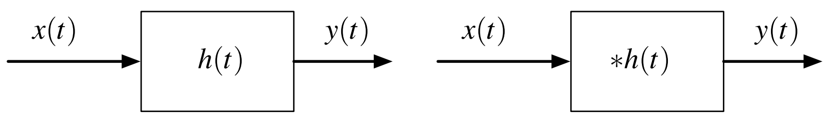
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t, \tau)\ d\tau$$

- If the system is also time invariant, the superposition integral simplifies to

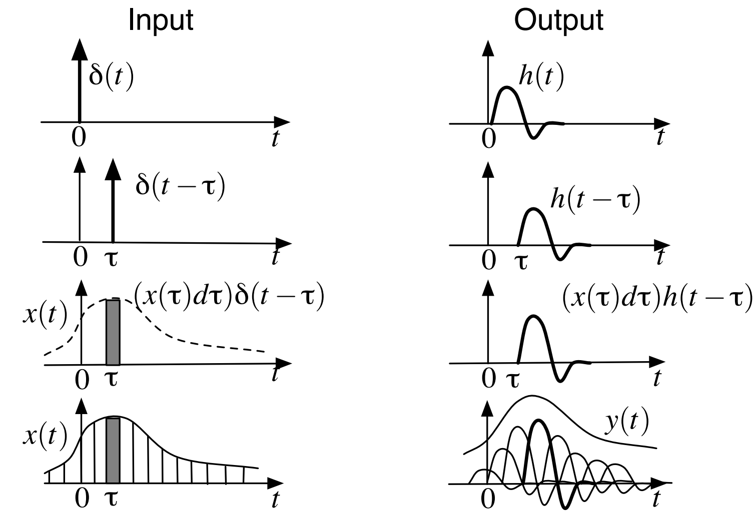
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)\ d\tau = \int_{-\infty}^{\infty} x(t - \tau)h(\tau)\, d\tau$$

which is in the form of a *convolution integral*.

The block diagrams for a convolution system with an impulse response $h(t)$:



From last time, this was illustrated:



$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \qquad y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Convolution Integral

The convolution of an input signal $x(t)$ with and impulse response $h(t)$ is

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\ d\tau \\ &= (x * h)(t) \end{aligned}$$

or

$$y = x * h.$$

This is also often written as

$$y(t) = x(t) * h(t)$$

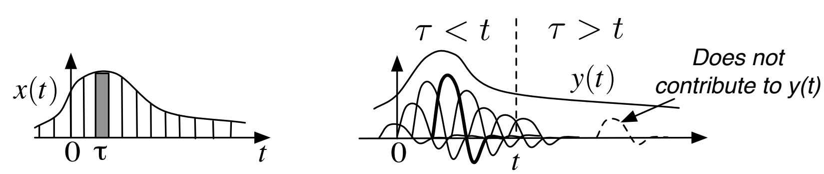
which is potentially confusing, since the t's have different interpretations on the left and right sides of the equation.

Convolution Integral for Causal Systems

For a causal system $h(t) = 0$ for $t < 0$,

$$y(t) = \int_{-\infty}^{\infty} x(\tau) \underbrace{h(t-\tau)}_{=0 \text{ if } t-\tau<0} \, d\tau = y(t) = \int_{-\infty}^{t} x(\tau) h(t-\tau) \, d\tau$$

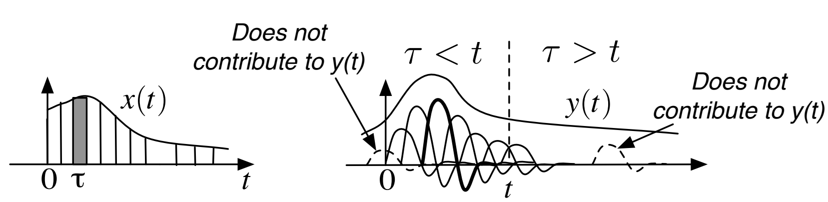
Only past and present values of $x(\tau)$ contribute to $y(t)$.



$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \qquad y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau$$

If $x(t)$ is also causal, $x(t) = 0$ for $t < 0$, and the integral further simplifies

$$y(t) = \int_{0}^{t} x(\tau) h(t-\tau) \, d\tau.$$



$$x(t) = \int_{0}^{t} x(\tau)\delta(t-\tau)d\tau \qquad y(t) = \int_{0}^{t} x(\tau)h(t-\tau)d\tau$$

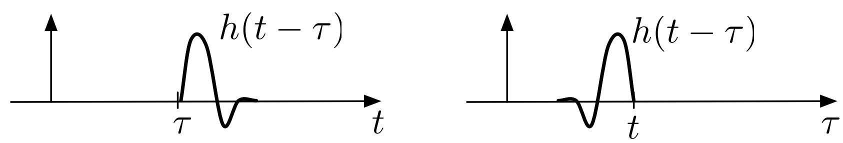
Graphical Interpretation

An increment in input $x(\tau)\delta(t-\tau)d\tau$ produces an impulse response $x(\tau)h(t-\tau)d\tau$. The output is the integral of all of these responses

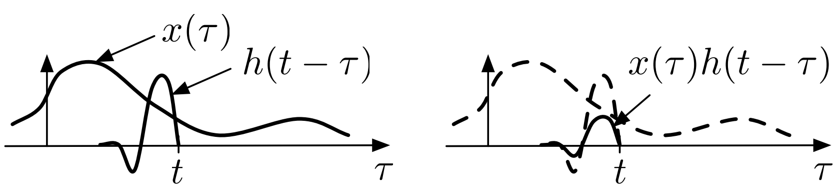
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau$$

Another perspective is just to look at the integral.

- $h(t-\tau)$ is the impulse response delayed to time τ
- If we consider $h(t-\tau)$ to be a function of τ, then $h(t-\tau)$ is delayed to time t, and *reversed*.



- This is multiplied point by point with the input,

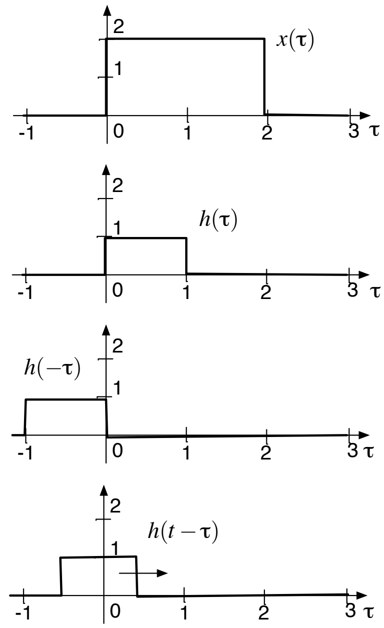


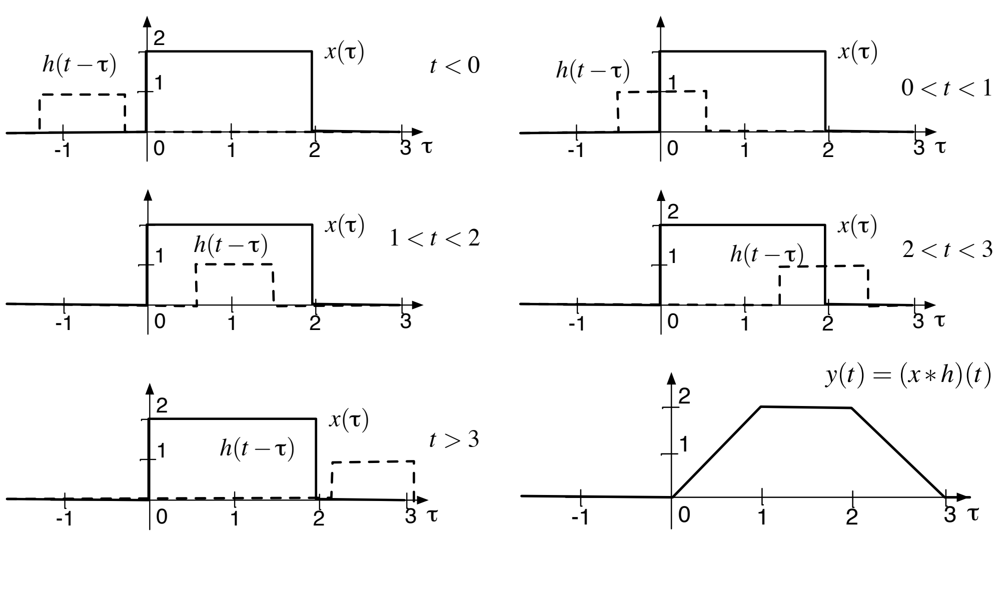
- Then integrate over τ to find $y(t)$ for this t.

Graphically, to find $y(t)$:

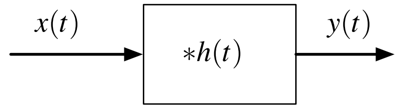
- flip impulse response $h(\tau)$ backwards in time (yields $h(-\tau)$)
- drag to the right over t (yields $h(t-\tau)$)
- multiply pointwise by x (yields $x(\tau)h(t-\tau)$)
- integrate over τ to get $y(t) = \displaystyle\int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau$

Simple Example

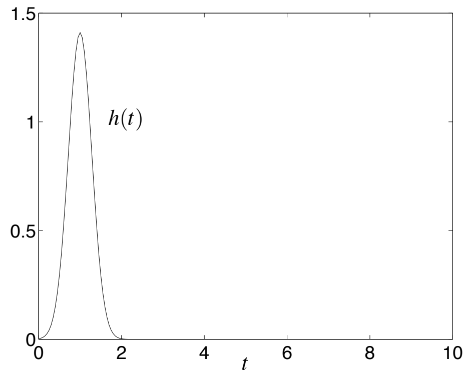




Communication channel, *e.g.*, twisted pair cable

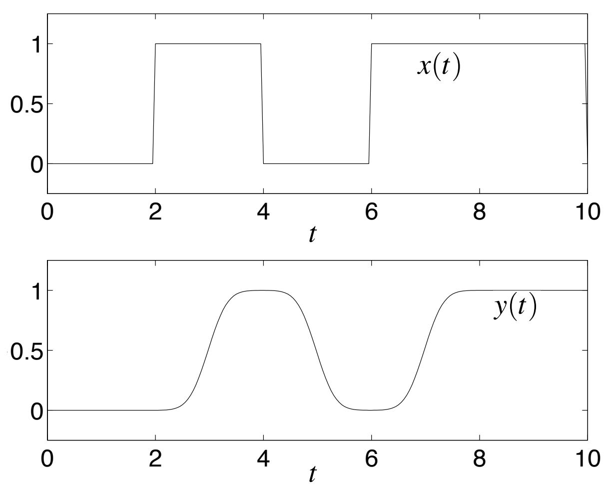


Impulse response:



This is a delay ≈ 1, plus smoothing.

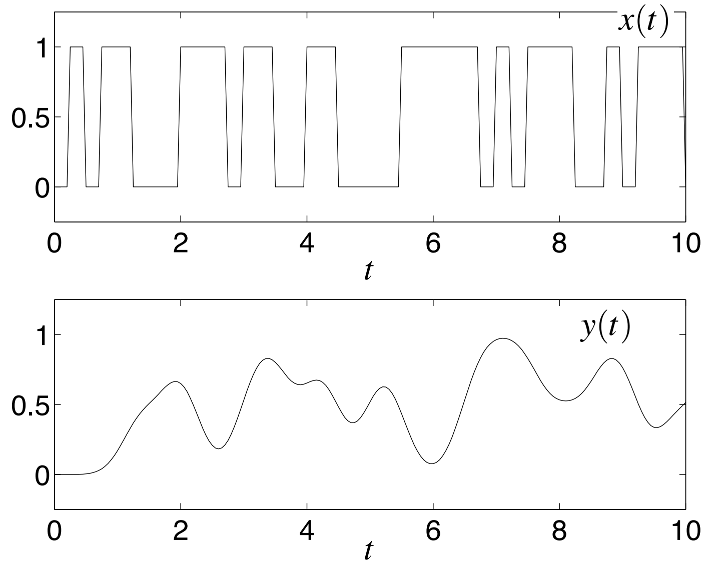
Simple signaling at 0.5 bit/sec; Boolean signal $0, 1, 0, 1, 1, \ldots$



Output is delayed, smoothed version of input.

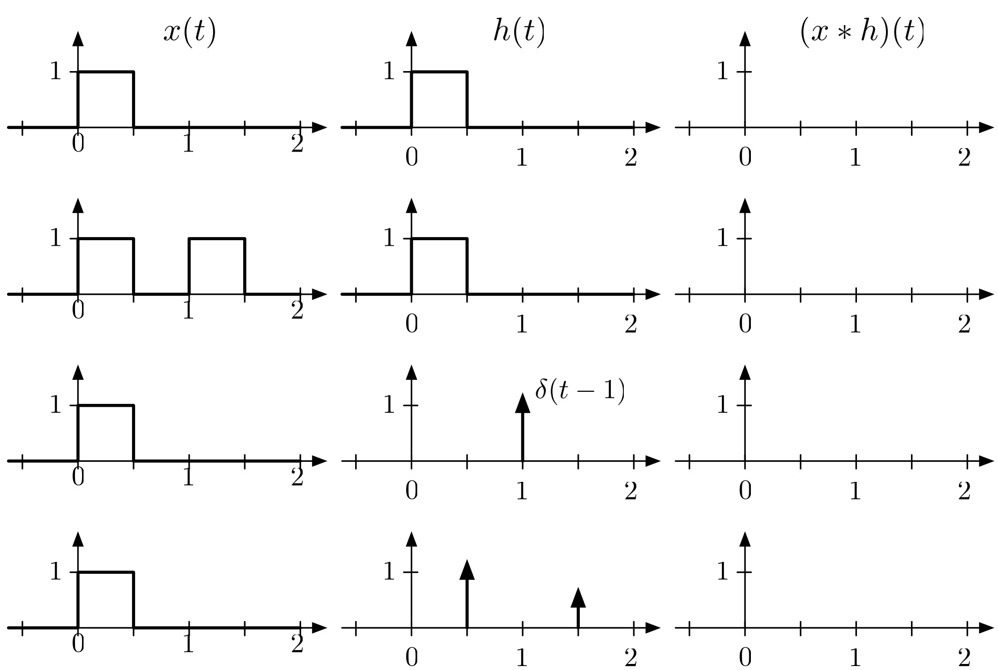
1's & 0's easily distinguished in y

Simple signalling at 4 bit/sec; same Boolean signal



Smoothing makes 1's & 0's very hard to distinguish in y.

Examples: Try these:



Properties of Convolution

For any two functions f and g the convolution is

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\, d\tau$$

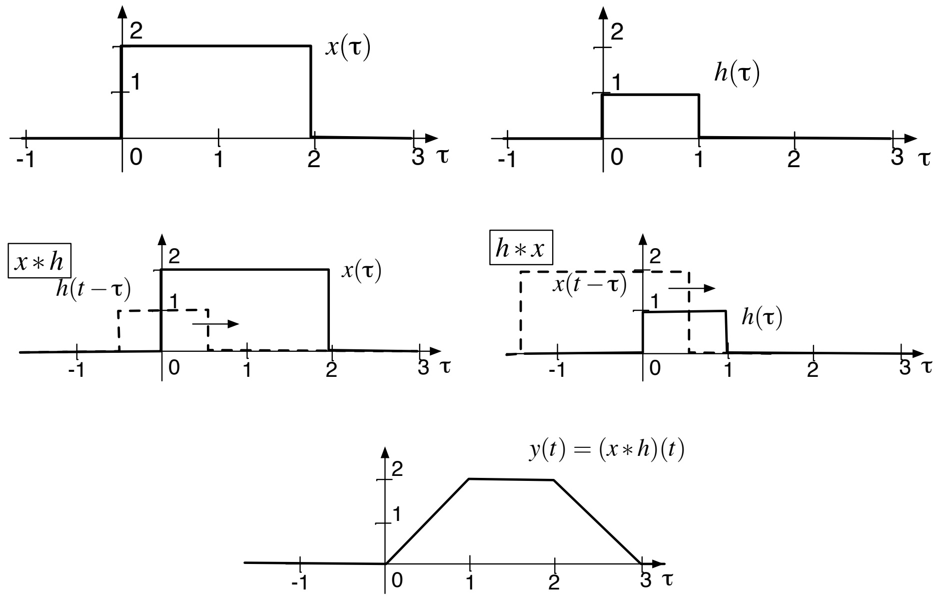
If we make the substitution $\tau_1 = t - \tau$, then $\tau = t - \tau_1$, and $d\tau = -d\tau_1$.

$$\begin{aligned}
(f * g)(t) &= \int_{\infty}^{-\infty} f(t-\tau_1)g(\tau_1)\, (-d\tau_1) \\
&= \int_{-\infty}^{\infty} g(\tau)f(t-\tau_1)\, d\tau_1 \\
&= (g * f)(t)
\end{aligned}$$

This means that convolution is *commutative*.

Practically, If we have two signals to convolve, we can choose either to be the signal we hold constant, and which to "flip and drag." One will generally be easier than the other.

Simple Example (x*h)



If we convolve three functions f, g, and h

$$(f * (g * h))(t) = ((f * g) * h)(t)$$

which means that convolution is *associative*.

To show this, we write out the integrals

$$\begin{aligned}(f * (g * h))(t) &= \int_{-\infty}^{\infty} f(\tau_1)\left[(g * h)(t-\tau_1)\right]\, d\tau_1 \\ &= \int_{-\infty}^{\infty} f(\tau_1)\left[\int_{-\infty}^{\infty} g(\tau_2) h(t-\tau_1-\tau_2)\, d\tau_2\right] d\tau_1\end{aligned}$$

We let $\tau_3 = \tau_1 + \tau_2$. $d\tau_3 = d\tau_2$ in the inner integral,

$$(f * (g * h))(t) = \int_{-\infty}^{\infty} f(\tau_1)\left[\int_{-\infty}^{\infty} g(\tau_3-\tau_1) h(t-\tau_3)\, d\tau_3\right] d\tau_1$$

Interchanging the order of integration,

$$
\begin{aligned}
(f * (g * h))(t) &= \int_{-\infty}^{\infty} \underbrace{\left[\int_{-\infty}^{\infty} f(\tau_1) g(\tau_3 - \tau_1)\, d\tau_1\right]}_{(f*g)(\tau_3)} h(t - \tau_3)\, d\tau_3 \\
&= ((f * g) * h)(t)
\end{aligned}
$$

Combining the commutative and associate properties,

$$f * g * h = f * h * g = \cdots = h * g * f$$

We can perform the convolutions in any order.

Convolution is also *distributive*,

$$f * (g + h) = f * g + f * h$$

which is easily shown by writing out the convolution integral,

$$
\begin{aligned}
(f * (g + h))(t) &= \int_{-\infty}^{\infty} f(\tau)\left[g(t - \tau) + h(t - \tau)\right]\, d\tau \\
&= \int_{-\infty}^{\infty} f(\tau) g(t - \tau)\, d\tau + \int_{-\infty}^{\infty} f(\tau) h(t - \tau)\, d\tau \\
&= (f * g)(t) + (f * h)(t)
\end{aligned}
$$

Together, the commutative, associative, and distributive properties mean that there is an "algebra of signals" where

- addition is like arithmetic or ordinary algebra, and
- multiplication is replaced by convolution.

Properties of Convolution Systems

The properties of the convolution integral have important consequences for systems described by convolution:

- Convolution systems are **linear**: for all signals x_1, x_2 and all α, $\beta \in \Re$,

$$h * (\alpha x_1 + \beta x_2) = \alpha(h * x_1) + \beta(h * x_2)$$

- Convolution systems are **time-invariant**: if we shift the input signal x by T, *i.e.*, apply the input

$$x_1(t) = x(t - T)$$

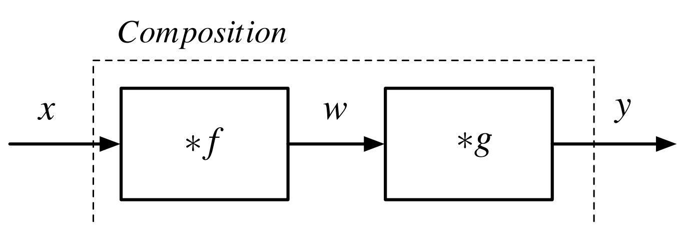
  to the system, the output is

$$y_1(t) = y(t - T).$$

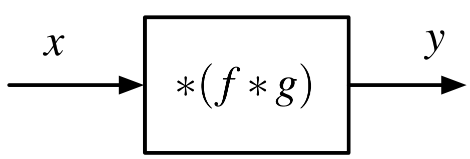
  In other words: convolution systems commute with delay.

- **Composition** of convolution systems corresponds to convolution of impulse responses.

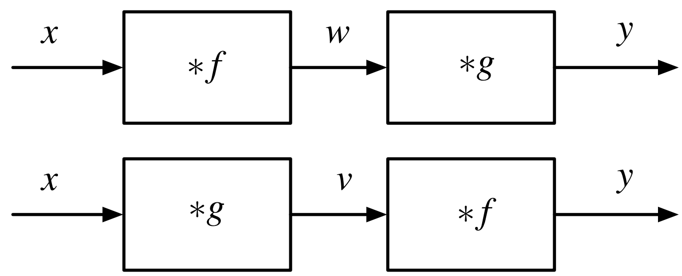
  The cascade connection of two convolution systems $y = (x * f) * g$



  is the same as a single system with an impulse response $h = f * g$



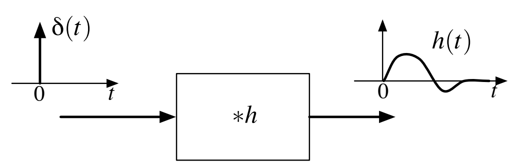
Since convolution is commutative, the convolution systems are also commutative. These two cascade connections have the same response



Many operations can be written as convolutions, and these all commute (integration, differentiation, delay, ...)

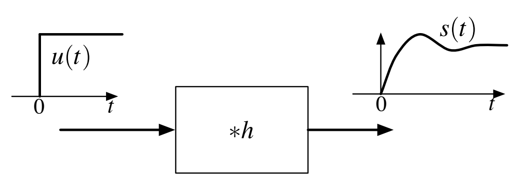
Example: Measuring the impulse response of an LTI system.

We would like to measure the impulse response of an LTI system, described by the impulse response $h(t)$

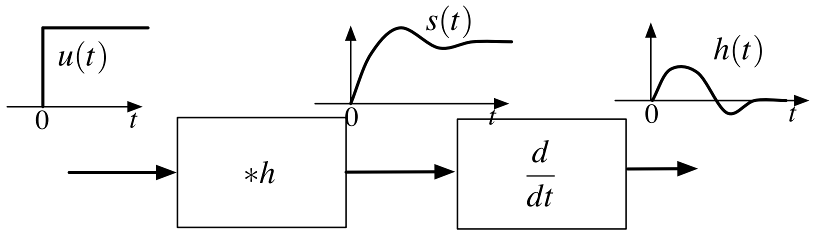


This can be practically difficult because input amplitude is often limited. A very short pulse then has very little energy.

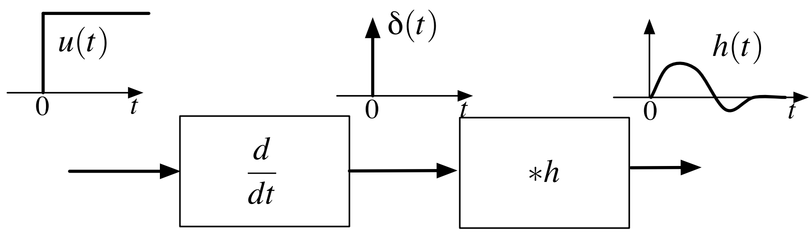
A common alternative is to measure the *step response* $s(t)$, the response to a unit step input $u(t)$



The impulse response is determined by differentiating the step response,



To show this, commute the convolution system and the differentiator to produce a system with the same overall impulse response



Convolution Systems with Complex Exponential Inputs

- If we have a convolution system with an impulse response $h(t)$, and and input e^{st} where $s = \sigma + j\omega$

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)}\, d\tau \\ &= e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau}\, d\tau \end{aligned}$$

- We get the complex exponential back, with a complex constant multiplier

$$\begin{aligned} H(s) &= \int_{-\infty}^{\infty} h(\tau) e^{-s\tau}\, d\tau \\ y(t) &= e^{st} H(s) \end{aligned}$$

provided the integral converges.

- $H(s)$ is the *transfer function* of the system.
- Putting a complex exponential into an LTI system results in a complex exponential output with the same frequency multiplied by a complex constant. The complex exponential is said to be an *eigenfunction* of the LTI described by h or H and $H(s)$ is the corresponding *eigenvalue*.
- If the input is a complex sinusoid $e^{j\omega t}$,

$$\begin{aligned} H(j\omega) &= \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}\, d\tau \\ y(t) &= e^{j\omega t}H(j\omega) \end{aligned}$$

Summary

- LTI systems can be represented as a the convolution of the input with an impulse response.
- Convolution has many useful properties (associative, commutative, etc).
- These carry over to LTI systems
  - Composition of system blocks
  - Order of system blocks

  Useful both practically, and for understanding.
- While convolution is conceptually simple, it can be practically difficult. It can be tedious to convolve your way through a complex system.
- There has to be a better way . . .