Stanford University
Winter 2012-2013

Signal Processing and Linear Systems I

Lecture 3: Signal Models

January 7, 2013

Models of Continuous Time Signals

Today's topics:

- Sinuoidal signals
- Exponential signals
- Complex exponential signals
- Unit step and unit ramp
- Impulse functions

Sinusoidal Signals

- A sinusoidal signal is of the form

$$x(t) = \cos(\omega t + \theta).$$

where the *radian frequency* is ω, which has the units of radians/s.

- Also very commonly written as

$$x(t) = A\cos(2\pi f t + \theta).$$

where f is the frequency in Hertz.

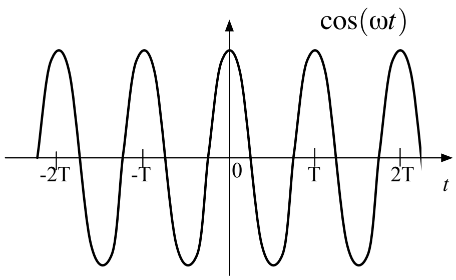
- We will often refer to ω as the frequency, but it must be kept in mind that it is really the *radian frequency*, and the *frequency* is actually f.

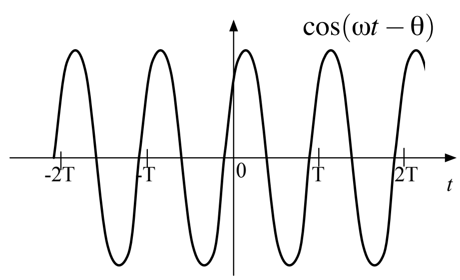
- The (fundamental) period of the sinuoid is

$$T = \frac{1}{f} = \frac{2\pi}{\omega}$$

with the units of seconds.

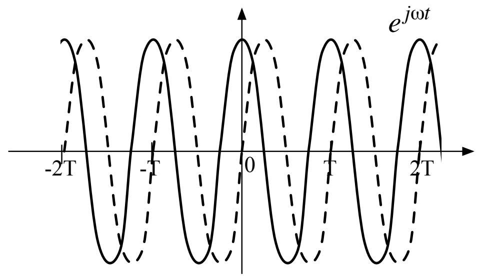
- The *phase* or *phase angle* of the signal is θ, given in radians.





Complex Sinusoids

- The Euler relation defines $e^{j\phi} = \underbrace{\cos\phi}_{\Re(e^{j\phi})} + j \underbrace{\sin\phi}_{\Im(e^{j\phi})}$.
- A complex sinusoid is $Ae^{j(\omega t+\theta)} = A\cos(\omega t+\theta) + jA\sin(\omega t+\theta)$.



- Real sinusoid can be represented as the real part of a complex sinusoid

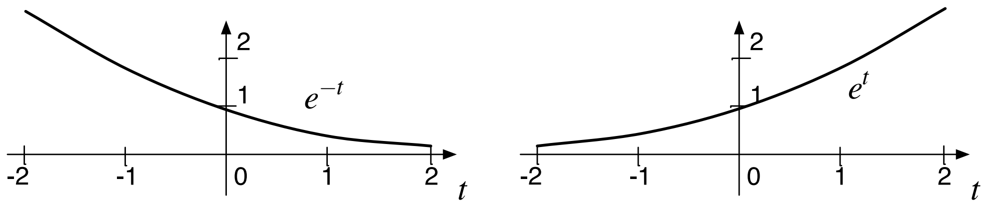
$$\Re\{Ae^{j(\omega t+\theta)}\} = A\cos(\omega t+\theta)$$

Exponential Signals

- An exponential signal is given by

$$x(t) = e^{\sigma t}$$

- If $\sigma < 0$ this is *exponential decay*.
- If $\sigma > 0$ this is *exponential growth*.

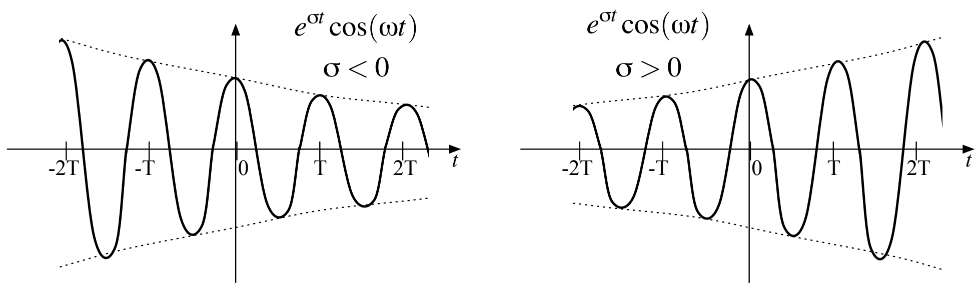


Damped or Growing Sinusoids

- A damped or growing sinusoid is given by

$$x(t) = e^{\sigma t} \cos(\omega t + \theta)$$

- Exponential growth ($\sigma > 0$) or decay ($\sigma < 0$), modulated by a sinusoid.

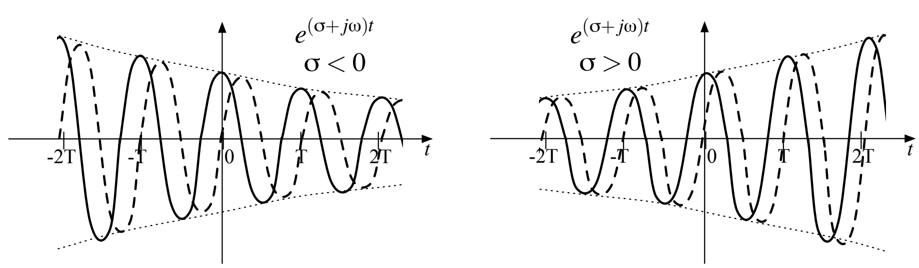


Complex Exponential Signals

- A complex exponential signal is given by

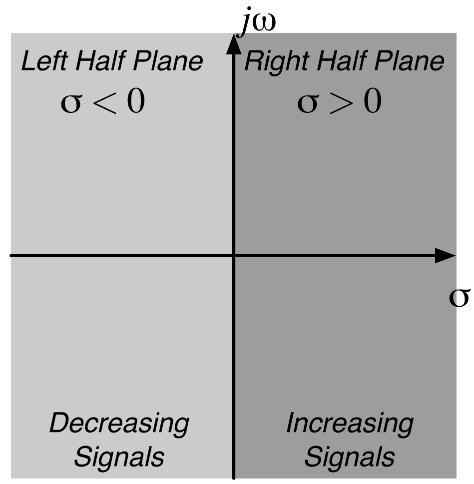
$$e^{(\sigma + j\omega)t + j\theta} = e^{\sigma t}(\cos(\omega t + \theta) + j \sin(\omega t + \theta))$$

- A exponential growth or decay, modulated by a complex sinusoid.
- Includes all of the previous signals as special cases.



Complex Plane

Each complex frequency $s = \sigma + j\omega$ corresponds to a position in the complex plane.

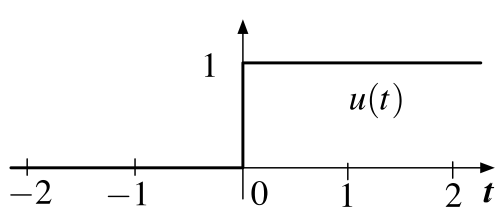


Unit Step Functions

- The *unit step function* $u(t)$ (or $u_{-1}(t)$) is defined as

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

- Also known as the *Heaviside step function*.
- Alternate definitions of value exactly at zero, such as 1/2.



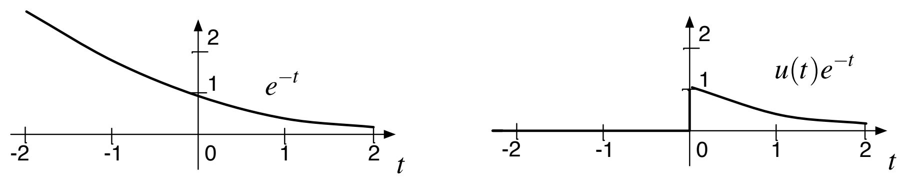
Uses for the unit step:

- Extracting part of another signal. For example, the piecewise-defined signal

$$x(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

can be written as

$$x(t) = u(t)e^{-t}$$

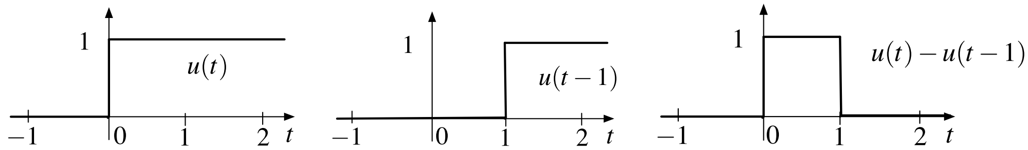


- Combinations of unit steps to create other signals. The offset rectangular signal

$$x(t) = \begin{cases} 0, & t \geq 1 \\ 1, & 0 \leq t < 1 \\ 0, & t < 0 \end{cases}$$

can be written as

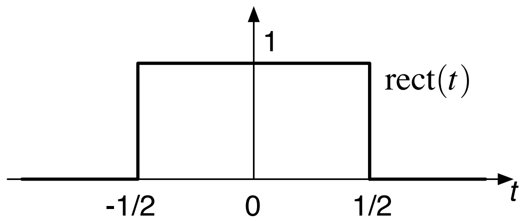
$$x(t) = u(t) - u(t-1).$$



Unit Rectangle

Unit rectangle signal:

$$\text{rect}(t) = \begin{cases} 1 & \text{if } |t| \le 1/2 \\ 0 & \text{otherwise.} \end{cases}$$



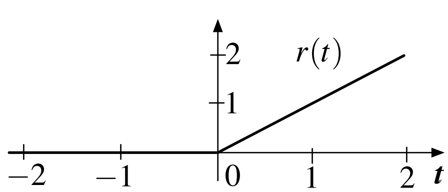
Unit Ramp

- The *unit ramp* is defined as

$$r(t) = \begin{cases} t, & t \ge 0 \\ 0, & t < 0 \end{cases}$$

- The unit ramp is the integral of the unit step,

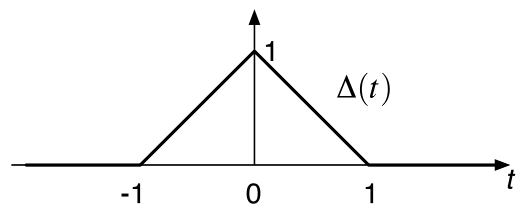
$$r(t) = \int_{-\infty}^{t} u(\tau) d\tau$$



Unit Triangle

Unit Triangle Signal

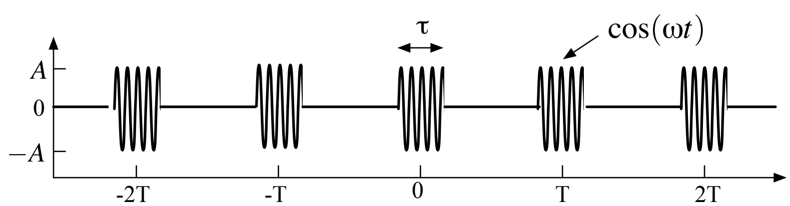
$$\Delta(t) = \begin{cases} 1 - |t| & \text{if } |t| < 1 \\ 0 & \text{otherwise.} \end{cases}$$



More Complex Signals

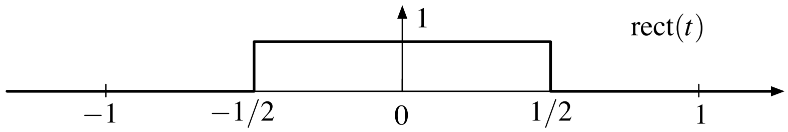
Many more interesting signals can be made up by combining these elements.

Example: Pulsed Doppler RF Waveform (we'll talk about this later!)

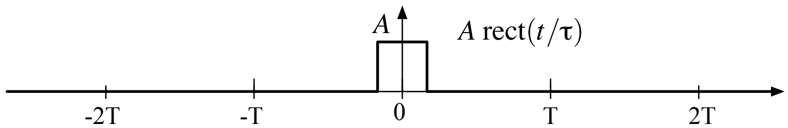


RF cosine gated on for τ μs, repeated every T μs, for a total of N pulses.

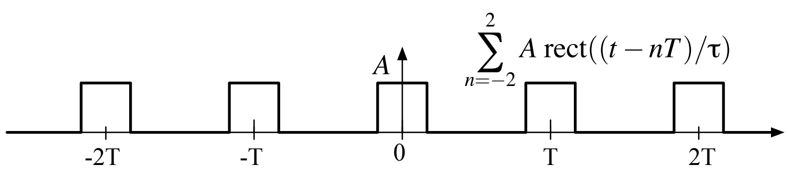
Start with a simple rect(t) pulse



Scale to the correct duration and amplitude for one subpulse

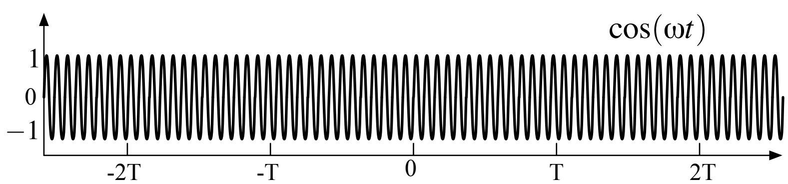


Combine shifted replicas

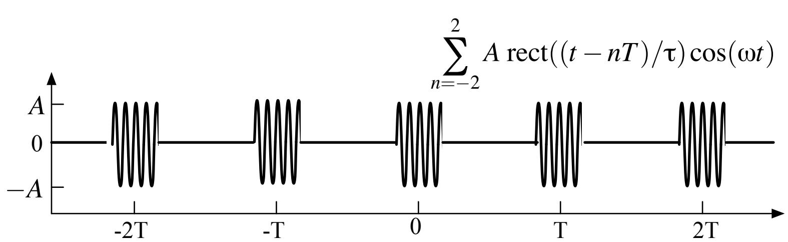


This is the *envelope* of the signal.

Then multiply by the RF carrier, shown below



to produce the pulsed Doppler waveform

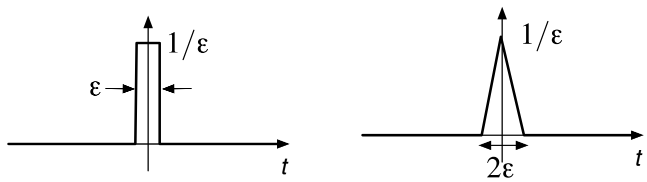


Impulsive signals

(Dirac's) **delta function** or **unit impulse** δ is an *idealization* of a signal that

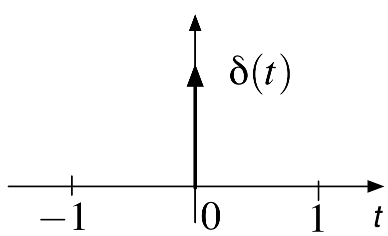
- is very large near $t = 0$
- is very small away from $t = 0$
- has integral 1

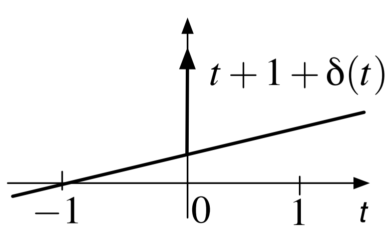
for example:



- the exact shape of the function doesn't matter
- ϵ is small (which depends on context)

On plots δ is shown as a solid arrow:





Formal properties

Formally we **define** δ by the property that

$$\int_{-\infty}^{\infty} f(t)\delta(t)\ dt = f(0)$$

provided f is continuous at $t = 0$

Basic Idea: δ acts over a time interval very small, over which $f(t) \approx f(0)$

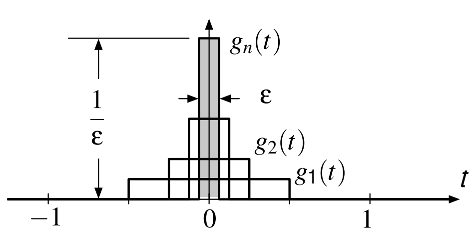
- $\delta(t)$ is not really defined for any t, only its behavior in an integral.
- Often described (incorrectly) as $\delta(t) = 0$ for $t \neq 0$, infinite at $t = 0$. This is a valid (if bizarre) function, but it does *not* have unit integral! Its Riemann integral is 0!

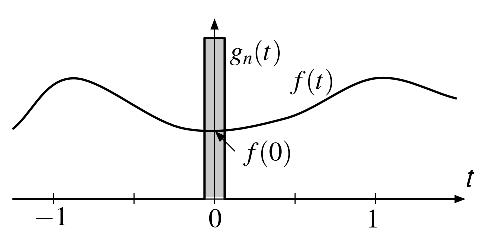
Example: Model $\delta(t)$ as

$$g_n(t) = n\ \mathrm{rect}(nt)$$

as $n \to \infty$. This has an area $(n)(1/n) = 1$. If $f(t)$ is continuous at $t = 0$, then

$$\int_{-\infty}^{\infty} f(t)\delta(t)\ dt = \lim_{n\to\infty} \int_{-\infty}^{\infty} f(t)g_n(t)\ dt = f(0)\int_{-\infty}^{\infty} g_n(t)\ dt = f(0)$$





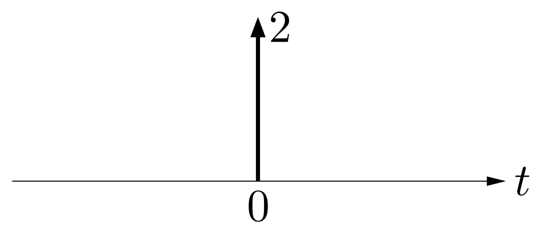
Not true that $\lim_{n\to\infty} g_n(t) = \delta(t)$!! Limit is *outside* the integral.

Scaled impulses

$\alpha\delta(t)$ is an impulse at time 0, with *magnitude* or *strength* or *area* α

We have
$$\int_{-\infty}^{\infty} \alpha\delta(t)f(t)\,dt = \alpha f(0)$$
provided f is continuous at 0

On plots: write area next to the arrow, *e.g.*, for $2\delta(t)$,



Multiplication of a Function by an Impulse

- Consider a function $\phi(t)$ multiplied by an impulse $\delta(t)$,
$$\phi(t)\delta(t)$$
If $\phi(t)$ is continuous at $t = 0$, can this be simplified?
- Substitute into the formal definition with a continuous $f(t)$ and evaluate,
$$\begin{aligned}\int_{-\infty}^{\infty} f(t)\left[\phi(t)\delta(t)\right]\,dt &= \int_{-\infty}^{\infty} \left[f(t)\phi(t)\right]\delta(t)\,dt \\ &= f(0)\phi(0)\end{aligned}$$
- Hence
$$\phi(t)\delta(t) = \phi(0)\delta(t)$$
is a scaled impulse, with strength $\phi(0)$.

Sifting property

- The signal $x(t) = \delta(t-T)$ is an impulse function with impulse at $t = T$.
- For f continuous at $t = T$,

$$\int_{-\infty}^{\infty} f(t)\delta(t-T)\,dt = f(T)$$

- Multiplying a function $f(t)$ by an impulse at time T and integrating, extracts the value of $f(T)$.
- This will be important in modeling sampling later in the course.

Limits of Integration

The integral of a δ is non-zero only if it is in the integration interval:

- If $a < 0$ and $b > 0$ then

$$\int_a^b \delta(t)\,dt = 1$$

  because the δ is within the limits.
- If $a > 0$ or $b < 0$, and $a < b$ then

$$\int_a^b \delta(t)\,dt = 0$$

  because the δ is outside the integration interval.
- **Ambiguous** if $a = 0$ or $b = 0$

Our convention: to avoid confusion we use limits such as $a-$ or $b+$ to denote whether we include the impulse or not.

$$\int_{0+}^{1} \delta(t)\, dt = 0, \quad \int_{0-}^{1} \delta(t)\, dt = 1, \quad \int_{-1}^{0-} \delta(t)\, dt = 0, \quad \int_{-1}^{0+} \delta(t)\, dt = 1$$

Example:

$$\begin{aligned}
&\int_{-2}^{3} f(t)(2 + \delta(t+1) - 3\delta(t-1) + 2\delta(t+3))\, dt \\
&= 2\int_{-2}^{3} f(t)\, dt + \int_{-2}^{3} f(t)\delta(t+1)\, dt - 3\int_{-2}^{3} f(t)\delta(t-1)\, dt \\
&\quad + 2\int_{-2}^{3} f(t)\delta(t+3))\, dt \\
&= 2\int_{-2}^{3} f(t)\, dt + f(-1) - 3f(1)
\end{aligned}$$

Physical interpretation

Impulse functions are used to model physical signals

- that act over short time intervals
- whose effect depends on integral of signal

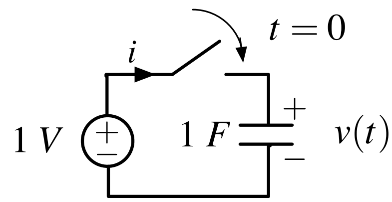
Example: hammer blow, or bat hitting ball, at $t = 2$

- force f acts on mass m between $t = 1.999$ sec and $t = 2.001$ sec
- $\int_{1.999}^{2.001} f(t)\, dt = I$ (mechanical impulse, $\mathrm{N} \cdot \mathrm{sec}$)
- blow induces change in velocity of

$$v(2.001) - v(1.999) = \frac{1}{m}\int_{1.999}^{2.001} f(\tau)\, d\tau = I/m$$

For most applications, model force as impulse at $t = 2$, with magnitude I.

example: rapid charging of capacitor

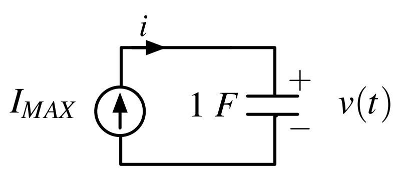


assuming $v(0) = 0$, what is $v(t)$, $i(t)$ for $t > 0$?

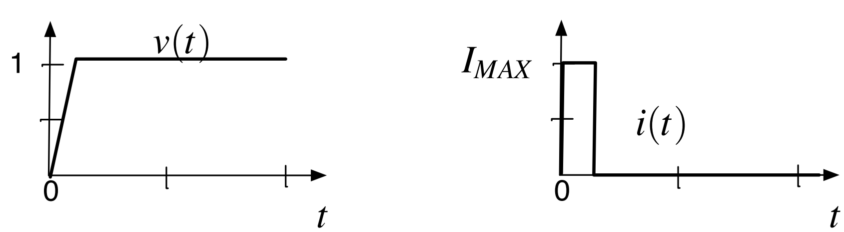
- $i(t)$ is very large, for a very short time
- a unit charge is transferred to the capacitor 'almost instantaneously'
- $v(t)$ increases to $v(t) = 1$ 'almost instantaneously'

To calculate i, v, we need a more detailed model.

For example, assume the current delivered by the source is limited: if $v(t) < 1$, the source acts as a current source $i(t) = I_{\max}$



$$i(t) = \frac{dv(t)}{dt} = I_{\max}, \quad v(0) = 0$$



As $I_{\max} \to \infty$, i approaches an impulse, v approaches a unit step

In conclusion,

- large current i acts over very short time between $t = 0$ and ϵ
- total charge transfer is $\int_0^{\epsilon} i(t)\, dt = 1$
- resulting change in $v(t)$ is $v(\epsilon) - v(0) = 1$
- can approximate i as impulse at $t = 0$ with magnitude 1

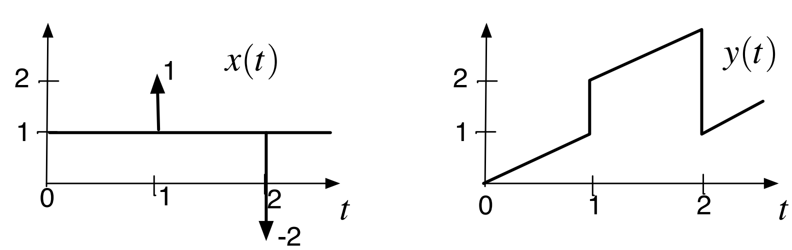
Modeling current as impulse

- obscures details of current signal
- obscures details of voltage change during the rapid charging
- preserves total change in charge, voltage
- is reasonable model for time scales $\gg \epsilon$

Integrals of impulsive functions

Integral of a function with impulses has jump at each impulse, equal to the magnitude of impulse

example: $x(t) = 1 + \delta(t-1) - 2\delta(t-2)$; define $y(t) = \int_0^t x(\tau)\, d\tau$

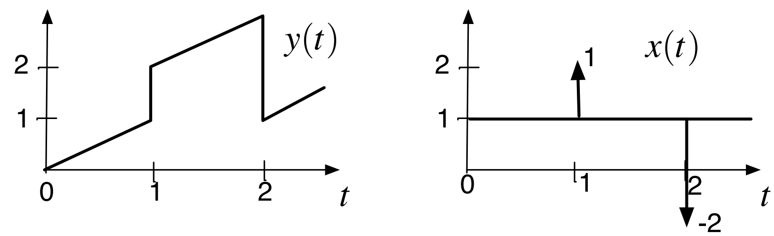


Derivatives of discontinuous functions

Conversely, derivative of function with discontinuities has impulse at each jump in function

- Derivative of unit step function $u(t)$ is $\delta(t)$
- Signal y of previous page

$$y'(t) = 1 + \delta(t-1) - 2\delta(t-2)$$



Derivatives of impulse functions

Integration by parts suggests we define

$$\int_{-\infty}^{\infty} \delta'(t) f(t)\, dt = \delta(t) f(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(t) f'(t)\, dt = -f'(0)$$

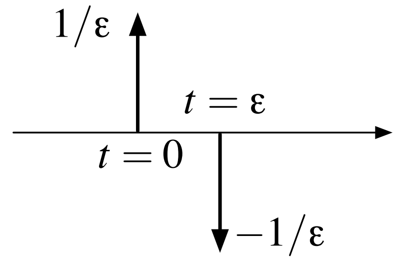
provided f' continuous at $t = 0$

- δ' is called *doublet*
- δ', δ'', etc. are called *higher-order impulses* (or singularity functions)
- Similar rules for higher-order impulses:

$$\int_{-\infty}^{\infty} \delta^{(k)}(t) f(t)\, dt = (-1)^k f^{(k)}(0)$$

  if $f^{(k)}$ continuous at $t = 0$

Interpretation of doublet δ': take two impulses with magnitude $\pm 1/\epsilon$, a distance ϵ apart, and let $\epsilon \to 0$



Then

$$\int_{-\infty}^{\infty} f(t) \left(\frac{\delta(t)}{\epsilon} - \frac{\delta(t-\epsilon)}{\epsilon} \right) \, dt = \frac{f(0) - f(\epsilon)}{\epsilon}$$

converges to $-f'(0)$ if $\epsilon \to 0$

Caveat

$\delta(t)$ is not a signal or function in the ordinary sense, it only makes mathematical sense when inside an integral sign

- We manipulate impulsive functions as if they were real functions, which they aren't
- It is safe to use impulsive functions in expressions like

  $$\int_{-\infty}^{\infty} f(t)\delta(t-T) \, dt, \quad \int_{-\infty}^{\infty} f(t)\delta'(t-T) \, dt$$

  provided f (resp, f') is continuous at $t = T$.
- Some innocent looking expressions don't make any sense at all (*e.g.*, $\delta(t)^2$ or $\delta(t^2)$)