STANFORD UNIVERSITY
EE 102B Spring-2013

Lecture 25

Second-Order Systems, Steady-State Response, Stability

May 31, 2013

ASSIGNMENTS

- Reading for this Lecture:
  - SPF: Chapters 7 and new notes on Chapter 8
  - S&S: Chapter 10
  - Lab #07 is due by 5pm, today, May 31, in Packard 263.
  - HW#09 is due by 5pm, Wednesday, June 5, in Packard 263. It is OPTIONAL to hand it in, but material on it will be covered on the final exam.
  - Please complete course evaluation on axess.

Office Hours for Course Staff – Come see us.

- Ron Schafer: Mon. and Weds. 2:00-4:00 pm, Packard 211.
- Dookun Park: Tues. 1:30 ~ 3:30 pm in Packard 107
- Ruo Yu Gu (Roy): Mon. 4:00 ~ 6:00 pm in Packard 106

LTI SYSTEMS AND THE Z-TRANSFORM

Causal LTI Systems – 1

$$x[n] \longrightarrow \boxed{\text{LTI System}} \longrightarrow y[n] = x[n] * h[n]$$

$$\delta[n] \longrightarrow \boxed{\text{LTI System}} \longrightarrow h[n]$$

$$\boxed{y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]}$$

With initial rest conditions

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}} = b_0 \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

Causal LTI Systems – II
Impulse response of DE

$$H(z) = \underbrace{\left[\sum_{r=0}^{(M-N)} B_r z^{-r}\right]}_{\text{if } M \ge N} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

ROC: $r_R = \max_k \{d_k\} < |z|$

$$\boxed{h[n] = \underbrace{\left[\sum_{r=0}^{(M-N)} B_r \delta[n-r]\right]}_{\text{if } M \ge N} + \sum_{k=1}^{N} A_k d_k^n u[n]}$$

Stability requires: $r_R = \max_k \{d_k\} < 1$

Frequency Response of a DE

$$\boxed{y[n] - \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]}$$

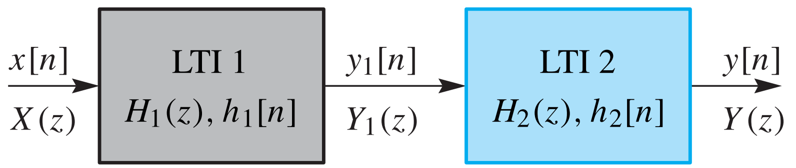
$$\boxed{H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}} = \frac{\left(\sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}\right)}{\left(1 - \sum_{k=1}^{N} a_k e^{-j\hat{\omega}k}\right)}}$$

ROC must Contain the Unit circle

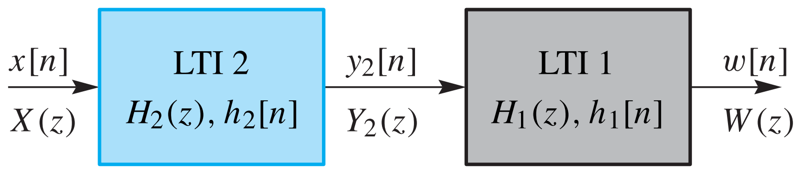
ROC for causal system: $\max_k \{d_k\} < |z|$ $\Longrightarrow$ Stability requires $\max_k \{d_k\} < 1$ for causal system

CASCADED LTI SYSTEMS

Cascaded LTI Systems



$$\boxed{Y(z) = H_1(z)H_2(z)X(z) = H_2(z)H_1(z)X(z)}$$



Classes of LTI Systems Based on Pole-Zero Locations

- Stable and causal system
- Minimum-phase systems $H_{min}(z)$
- Allpass systems $H_{ap}(z)$
- Causal and stable inverse system for H(z)

Allpass Systems

- Allpass systems have constant gain with varying phase shift.

$$H_{ap}(z) = Ge^{j\angle H_{ap}(z)}$$

$$(\text{with } \angle H_{ap}(z) < 0,\ 0 < \hat{\omega} < \pi)$$

- $H_{ap}(z)$ composed of factors of the form:

$$H_{ap}(z) = \frac{z^{-1} - a}{1 - az^{-1}} = -a\frac{1 - a^{-1}z^{-1}}{1 - az^{-1}} = z^{-1}\frac{1 - az}{1 - az^{-1}}$$

- That is, a pole at $z = a$ and zero at $z = a^{-1}$

General Representation of Systems - I

- Any system with *no zeros on the unit circle* can be represented as

$$H(z) = H_{min}(z)H_{ap}(z)$$

- Example

$$H(z) = \frac{1 - 2z^{-1}}{1 + 0.8z^{-1}} = -2\frac{z^{-1} - 0.5}{1 + 0.8z^{-1}} = \left(-2\frac{z^{-1} - 0.5}{1 - 0.5z^{-1}}\right)\left(\frac{1 - 0.5z^{-1}}{1 + 0.8z^{-1}}\right)$$

$$H_{min}(z) = \frac{1 - 0.5z^{-1}}{1 + 0.8z^{-1}} \qquad H_{ap}(z) = -2\frac{z^{-1} - 0.5}{1 - 0.5z^{-1}}$$

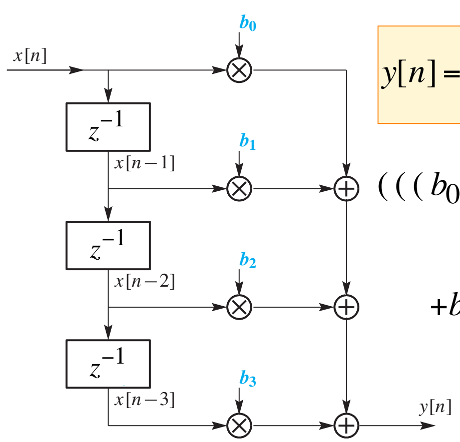
General Representation of Systems - II

- If the system has zeros on the unit circle it can be represented as

$$H(z) = H_{\text{uc}}(z)H_{\min}(z)H_{\text{ap}}(z)$$

IMPLEMENTATION STRUCTURES FOR LTI SYSTEMS

FIR Direct Form



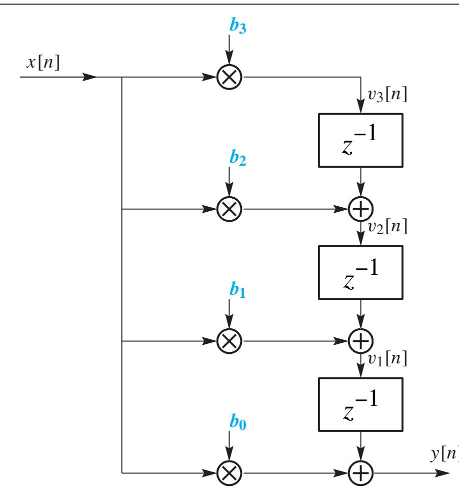
$$y[n] = \sum_{k=0}^{M} b_k x[n-k] \qquad (M = 3)$$

$$(((\, b_0 x[n] + b_1 x[n-1]\,)$$

$$+ b_2 x[n-2]\,)$$

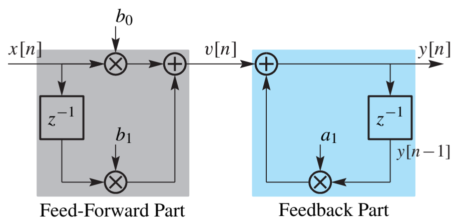
$$+ b_3 x[n-3]\,) = y[n]$$

FIR Transposed Direct Form



Direct Form First-Order IIR Structures

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}$$

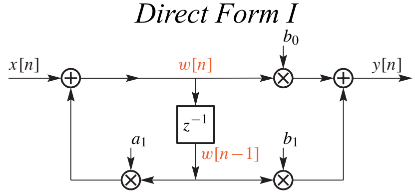


Direct Form I

$$H(z) = (b_0 + b_1 z^{-1}) \frac{1}{(1 - a_1 z^{-1})}$$

$$v[n] = b_0 x[n] + b_1 x[n-1]$$

$$y[n] = a_1 y[n-1] + v[n]$$



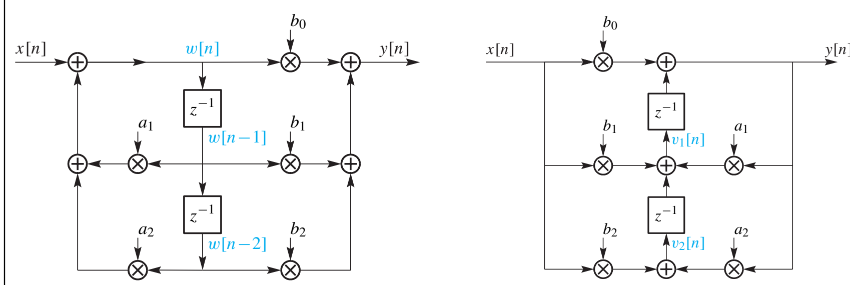
Direct Form II

$$H(z) = \frac{1}{(1 - a_1 z^{-1})}(b_0 + b_1 z^{-1})$$

$$w[n] = a_1 w[n-1] + x[n]$$

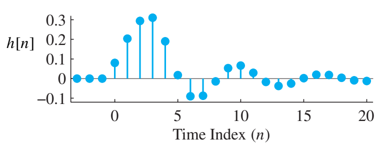
$$y[n] = b_0 w[n] + b_1 w[n-1]$$

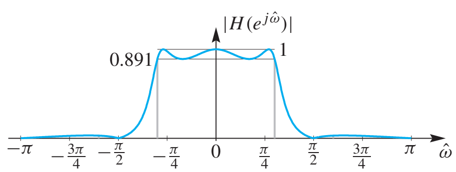
Second-Order Direct Form Structures



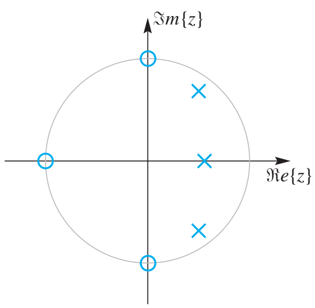
Direct Form II *Transposed Direct Form II*

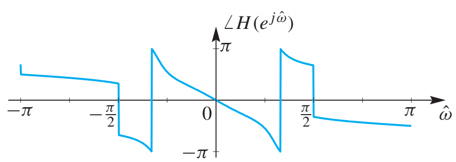
Third-Order Elliptic Filter Example





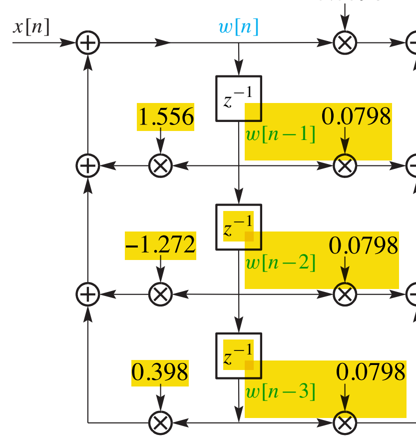
(a)



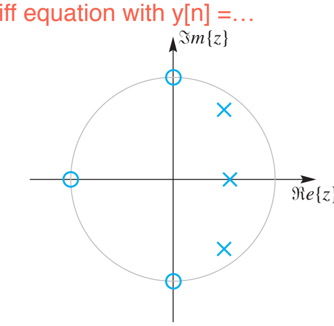


(b)

$$H(z) = \frac{0.0798(1 + z^{-1} + z^{-2} + z^{-3})}{1 - 1.556 z^{-1} + 1.272 z^{-2} - 0.398 z^{-3}}$$



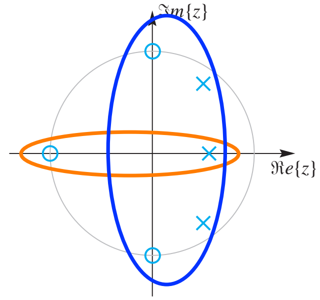
coefficients are same as
diff equation with y[n] =…



$$H(z)=\frac{0.0798(1+z^{-1}+z^{-2}+z^{-3})}{1-1.556z^{-1}+1.272z^{-2}-0.398z^{-3})}$$

- Cascade form (factor num & denom)

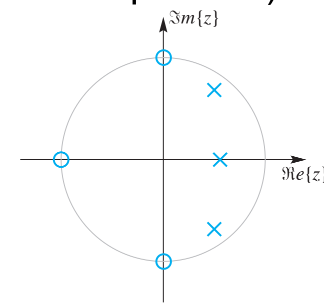
$$H(z)=\underline{\left(\frac{0.0798(1+z^{-1})}{1-0.556z^{-1}}\right)}\underline{\left(\frac{1+z^{-2}}{1-0.9945z^{-1}+0.7157z^{-2}}\right)}$$



$$H(z)=\frac{0.0798(1+z^{-1}+z^{-2}+z^{-3})}{1-1.556z^{-1}+1.272z^{-2}-0.398z^{-3}}$$

- Parallel form (partial fraction expansion)

$$H(z)=-0.2+\frac{0.62}{1-0.556z^{-1}}+\frac{0.17e^{j0.96\pi}}{1-0.846e^{j0.3\pi}z^{-1}}+\frac{0.17e^{-j0.96\pi}}{1-0.846e^{-j0.3\pi}z^{-1}}$$



$$H(z)=-0.2+\frac{0.62}{1-0.556z^{-1}}+\frac{-0.1687-0.1386z^{-1}}{1-0.9945z^{-1}+0.7157z^{-2}}$$

STEADY-STATE RESPONSE TO A COMPLEX EXPONENTIAL

Frequency Response and Complex Exponential Signals

$x[n]$	*LTI System*	$y[n]=x[n]*h[n]$
$\delta[n]$		$h[n]$
$e^{j\hat{\omega}n}$		$H(e^{j\hat{\omega}})e^{j\hat{\omega}n}$
$e^{j\hat{\omega}n}u[n]$		???

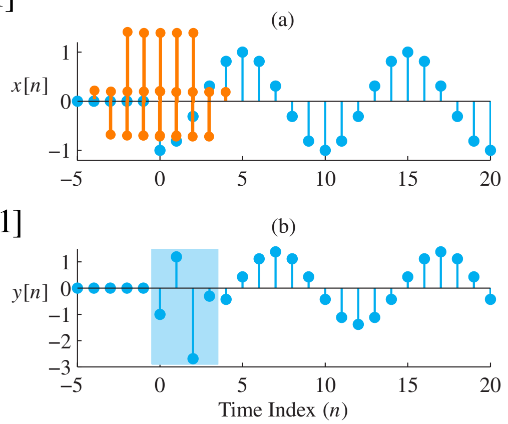
- What if the complex exponential is "suddenly applied"?

Steady-State Response – FIR

$$x[n]=\cos(0.2\pi n-\pi)u[n]=\Re\left\{-e^{j0.2\pi n}u[n]\right\}$$

$$h[n]=\delta[n]-2\delta[n-1]+4\delta[n-2]-2\delta[n-3]+\delta[n-4]$$

$$y[n]=x[n]-2x[n-1]+4x[n-2]-2x[n-3]+x[n-4]$$



IIR Steady-State Example

$$h[n]=5(-0.8)^n u[n] \qquad x[n]=e^{j0.2\pi n}u[n]$$

$$H(z)=\frac{5}{1+0.8z^{-1}} \quad 0.8<|z|, \qquad X(z)=\frac{1}{1-e^{j0.2\pi}z^{-1}} \quad 1<|z|$$

$$Y(z)=H(z)X(z)=\left(\frac{5}{1+0.8z^{-1}}\right)\left(\frac{1}{1-e^{j0.2\pi}z^{-1}}\right) \quad 1<|z|$$

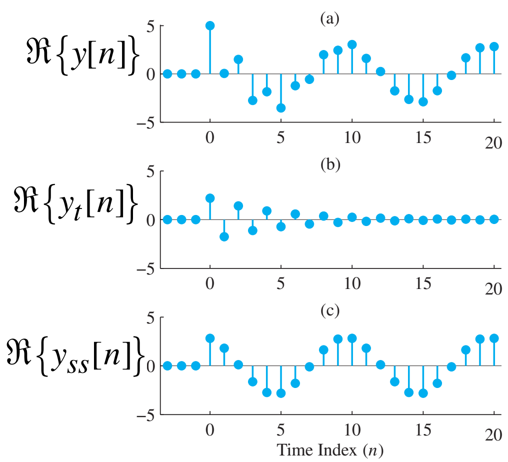
$$Y(z)=\frac{\left(\frac{5}{1+1.25e^{j0.2\pi}}\right)}{1+0.8z^{-1}}+\frac{\left(H(e^{j0.2\pi})\right)}{1-e^{j0.2\pi}z^{-1}} \quad 1<|z|$$

$$y[n]=\left(\frac{5}{1+1.25e^{j0.2\pi}}\right)(-0.8)^n u[n]+\left(H(e^{j0.2\pi})\right)e^{j0.2\pi n}u[n]$$

$$y[n]=\quad y_t[n] \quad + \quad y_{ss}[n]$$

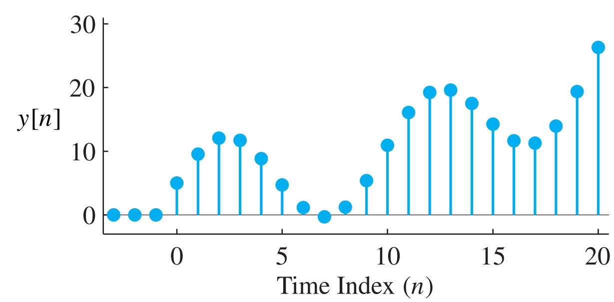
Steady-State Response – IIR

$$h[n]=5(-0.8)^n u[n] \qquad x[n]=e^{j0.2\pi n}u[n]$$



Unstable Example

$$h[n]=5(1.1)^n u[n] \qquad x[n]=e^{j0.2\pi n}u[n]$$



Steady-State Response of IIR Systems - I

- Suddenly applied complex exponential

$$x[n] = e^{j\hat{\omega}_0 n}u[n] \Leftrightarrow X(z) = \frac{1}{1 - e^{j\hat{\omega}_0}z^{-1}} \qquad 1 < |z|$$

- z-transform of output

$$Y(z) = H(z)X(z) = H(z)\frac{1}{1 - e^{j\hat{\omega}_0}z^{-1}} \qquad 1 < |z|$$

Steady-State Response of IIR Systems - II

- Make a partial fraction expansion

$$Y(z) = H(z)\frac{1}{1 - e^{j\hat{\omega}_0}z^{-1}} = \sum_{k=1}^{N}\frac{A_k}{1 - d_k z^{-1}} + \frac{B}{1 - e^{j\hat{\omega}_0}z^{-1}} \qquad 1 < |z|$$

$$A_k = H(z)\frac{1 - d_k z^{-1}}{1 - e^{j\hat{\omega}_0}z^{-1}}\bigg|_{z=d_k}$$

$$\boxed{\begin{array}{c}(1 - d_k z^{-1}) \text{ cancels in } H(z) \\ \text{before the substitution } z = d_k\end{array}}$$

$$B = H(z)\cancelto{1}{\frac{1 - e^{j\hat{\omega}_0}z^{-1}}{1 - e^{j\hat{\omega}_0}z^{-1}}}\bigg|_{z=e^{j\hat{\omega}_0}} = H(e^{j\hat{\omega}_0})$$

Steady-State Response of IIR Systems - III

- Partial fraction expansion

$$Y(z) = H(z)\frac{1}{1 - e^{j\hat{\omega}_0}z^{-1}} = \sum_{k=1}^{N}\frac{A_k}{1 - d_k z^{-1}} + \frac{H(e^{j\hat{\omega}_0})}{1 - e^{j\hat{\omega}_0}z^{-1}} \qquad 1 < |z|$$

- Inverse z-transform

$$y[n] = \cancelto{0}{\sum_{k=1}^{N} A_k d_k^n u[n]} + H(e^{j\hat{\omega}_0})e^{j\hat{\omega}_0 n}u[n]$$

$$y[n] = \quad y_t[n] \quad + \quad y_{ss}[n]$$