STANFORD UNIVERSITY
EE 102B Spring-2013

Lecture 10

Theorems and Properties of the DTFT

April 22, 2013

ASSIGNMENTS

- Reading for this Lecture:
  - SPF: Sections 66-1, 66-2, 66-3 & 66-4 (notes posted on Course2Go website)
  - S&S: Chapter 5
- HW#03 is due by 5pm Wednesday, April 24 in Packard 263.
- Lab #03 is due by 5pm, Friday, April 26, in Packard 263.

Office Hours for Course Staff – Come see us.

- Ron Schafer: Mon. 2:00-3:00 pm and Weds. 2:00-4:00 pm in Packard 211
- Dookun Park: Tues. 1:30 ~ 3:30 pm in Packard 107
- Ruo Yu Gu (Roy): Mon. 4:00 ~ 6:00 pm in Packard 106
- https://class2go.stanford.edu/EE102B/Spring2013/pages/staff

Lecture Objective

- Derive and summarize the important properties of the DTFT
- **Convolution is mapped to Multiplication**
- We will see that:
  - DTFT is the math behind the general concept of "**frequency domain**" representations
  - The **spectrum** is now a **continuous** function of (normalized) frequency – not just a line spectrum

Discrete-Time Fourier Transform

- **Definition of the <u>DTFT</u>:**

Discrete-time Fourier Transform

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n}$$

<u>Inverse</u> Discrete-time Fourier Transform

$$x[n] = \frac{1}{2\pi}\int_{0}^{2\pi} X(e^{j\hat{\omega}})e^{j\hat{\omega}n}d\hat{\omega}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\hat{\omega}})e^{j\hat{\omega}n}d\hat{\omega}$$

$$\text{Periodic}: X(e^{j(\hat{\omega}+2\pi)}) = X(e^{j\hat{\omega}})$$

Table of DTFT Pairs

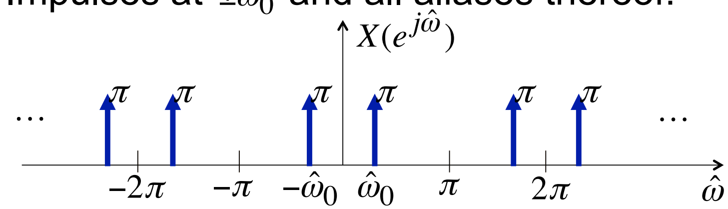
Time-Domain: $x[n]$		*Frequency-Domain:* $X(e^{j\hat{\omega}})$	
$\delta[n]$	$\Leftrightarrow$	1	
$\delta[n-n_d]$	$\Leftrightarrow$	$e^{-j\hat{\omega}n_d}$	
$u[n]-u[n-L]$	$\Leftrightarrow$	$\dfrac{\sin(\frac{1}{2}L\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})}e^{-j\hat{\omega}(L-1)/2}$	
$\dfrac{\sin(\hat{\omega}_b n)}{\pi n}$	$\Leftrightarrow$	$u(\hat{\omega}+\hat{\omega}_b)-u(\hat{\omega}-\hat{\omega}_b) = \begin{cases}1 & \lvert\hat{\omega}\rvert \le \hat{\omega}_b \\ 0 & \hat{\omega}_b < \lvert\hat{\omega}\rvert \le \pi\end{cases}$	
$a^n u[n] \quad (\lvert a\rvert < 1)$	$\Leftrightarrow$	$\dfrac{1}{1-ae^{-j\hat{\omega}}}$	*right-sided exponential*
$b^n u[-n] \quad (\lvert b\rvert > 1)$	$\Leftrightarrow$	$\dfrac{-be^{-j\hat{\omega}}}{1-be^{-j\hat{\omega}}}$ $\quad \cancel{\dfrac{1}{1-be^{-j\hat{\omega}}}}$	*left-sided exponential*

DTFT of a Sinusoidal Signal - I

- Define the DTFT of a signal as

$$X(e^{j\hat{\omega}}) = \sum_{r=-\infty}^{\infty} \pi\delta(\hat{\omega}+\hat{\omega}_0+2\pi r) + \pi\delta(\hat{\omega}-\hat{\omega}_0+2\pi r)$$

- Impulses at $\pm\hat{\omega}_0$ and all aliases thereof:



DTFT of a Sinusoidal Signal - II

- What is the signal corresponding to this DTFT? Plug into the inverse DTFT

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\hat{\omega}})e^{j\hat{\omega}n}d\hat{\omega}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} [\pi\delta(\hat{\omega}+\hat{\omega}_0)+\pi\delta(\hat{\omega}-\hat{\omega}_0)]e^{j\hat{\omega}n}d\hat{\omega}$$

$$= \frac{1}{2}e^{j\hat{\omega}_0 n} + \frac{1}{2}e^{-j\hat{\omega}_0 n} = \cos(\hat{\omega}_0 n)$$

Table of DTFT Properties		
Property Name	***Time-Domain:*** $x[n]$	***Frequency-Domain:*** $X(e^{j\hat{\omega}})$
Periodic in $\hat{\omega}$		$X(e^{j(\hat{\omega}+2\pi)}) = X(e^{j\hat{\omega}})$
Linearity	$ax_1[n] + bx_2[n]$ $\Leftrightarrow$	$aX_1(e^{j\hat{\omega}}) + bX_2(e^{j\hat{\omega}})$
Conjugate Symmetry	$x[n]$ is real	$X(e^{-j\hat{\omega}}) = X^*(e^{j\hat{\omega}})$
Conjugation	$x^*[n]$ $\Leftrightarrow$	$X^*(e^{-j\hat{\omega}})$
Time-Reversal	$x[-n]$ $\Leftrightarrow$	$X(e^{-j\hat{\omega}})$
Delay	$x[n-n_d]$ $\Leftrightarrow$	$e^{-j\hat{\omega}n_d}X(e^{j\hat{\omega}})$
Frequency Shift	$x[n]e^{j\hat{\omega}_0 n}$ $\Leftrightarrow$	$X(e^{j(\hat{\omega}-\hat{\omega}_0)})$
Modulation	$x[n]\cos(\hat{\omega}_0 n)$ $\Leftrightarrow$	$\frac{1}{2}X(e^{j(\hat{\omega}-\hat{\omega}_0)}) + \frac{1}{2}X(e^{j(\hat{\omega}+\hat{\omega}_0)})$
Convolution	$x[n] * h[n]$ $\Leftrightarrow$	$X(e^{j\hat{\omega}})H(e^{j\hat{\omega}})$
Parseval's Theorem	$\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2$ $=$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\hat{\omega}})\rvert^2 d\hat{\omega}$

Linearity

$$x[n] = ax_1[n] + bx_2[n] \Leftrightarrow X(e^{j\hat{\omega}}) = aX_1(e^{j\hat{\omega}}) + bX_2(e^{j\hat{\omega}})$$

$$X_1(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x_1[n]e^{-j\hat{\omega}n} \qquad X_2(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x_2[n]e^{-j\hat{\omega}n}$$

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} (ax_1[n] + bx_2[n])e^{-j\hat{\omega}n}$$

$$= \sum_{n=-\infty}^{\infty} (ax_1[n])e^{-j\hat{\omega}n} + \sum_{n=-\infty}^{\infty} (bx_2[n])e^{-j\hat{\omega}n}$$

$$= aX_1(e^{j\hat{\omega}}) + bX_2(e^{j\hat{\omega}})$$

Conjugate Symmetry

$$x[n] = x^*[n] \quad \text{(real-valued)}$$

$$\Rightarrow X(e^{j\hat{\omega}}) = X^*(e^{-j\hat{\omega}}) \ \text{ or, } X(e^{-j\hat{\omega}}) = X^*(e^{j\hat{\omega}})$$

$$y[n] = x^*[n]$$

$$Y(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x^*[n]e^{-j\hat{\omega}n}$$

$$= \left(\sum_{n=-\infty}^{\infty} x[n]e^{-j(-\hat{\omega})n}\right)^* = X^*(e^{-j\hat{\omega}})$$

$$x[n] = a^n u[n] \Leftrightarrow X(e^{j\hat{\omega}}) = \frac{1}{1 - ae^{-j\hat{\omega}}}$$

EVEN Function

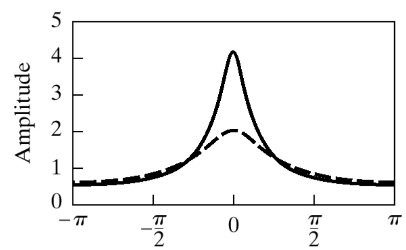
$$\lvert X(e^{j\hat{\omega}})\rvert = \frac{1}{(1 + a^2 - 2a\cos(\hat{\omega}))^{1/2}}$$

$$\lvert X(e^{-j\hat{\omega}})\rvert = \lvert X(e^{j\hat{\omega}})\rvert$$

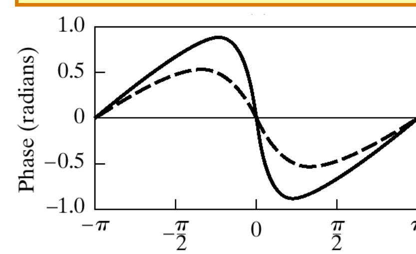
ODD Function

$$\angle X(e^{j\hat{\omega}}) = \arctan\left(\frac{-a\sin(\hat{\omega})}{1 - a\cos(\hat{\omega})}\right)$$

$$\angle X(e^{-j\hat{\omega}}) = -\angle X(e^{j\hat{\omega}})$$



Radian Frequency $(\hat{\omega})$



Radian Frequency $(\hat{\omega})$

DTFT of a Left-Sided Exponential Signal

- Turn the exponential signal around

$$y[n] = x[-n] = a^{-n}u[-n], \quad |a|<1$$

- Apply the time-reversal property

$$Y(e^{j\hat{\omega}}) = X(e^{-j\hat{\omega}}) = \frac{1}{1-ae^{j\hat{\omega}}} = \frac{-a^{-1}e^{-j\hat{\omega}}}{1-a^{-1}e^{-j\hat{\omega}}} \quad \text{if } |a|<1$$

$$= \frac{-be^{-j\hat{\omega}}}{1-be^{-j\hat{\omega}}} \quad \text{if } |b|>1 \qquad (b=a^{-1})$$

Time-Delay Property

$$y[n] = x[n-n_d] \Leftrightarrow Y(e^{j\hat{\omega}}) = X(e^{j\hat{\omega}})e^{-j\hat{\omega}n_d}$$

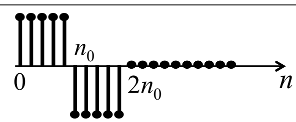
$$Y(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n-n_d]e^{-j\hat{\omega}n}$$

$$= e^{-j\hat{\omega}n_d}\left(\sum_{n=-\infty}^{\infty} x[n-n_d]e^{-j\hat{\omega}(n-n_d)}\right)$$

$$= e^{-j\hat{\omega}n_d}\left(\sum_{m=-\infty}^{\infty} x[m]e^{-j\hat{\omega}(m)}\right) = X(e^{j\hat{\omega}})e^{-j\hat{\omega}n_d}$$

Use Properties to Find DTFT

$$y[n] = \begin{cases} 1, & 0 \le n \le n_0 - 1 \\ -1, & n_0 \le n \le 2n_0 - 1 \\ 0, & \text{elsewhere} \end{cases}$$



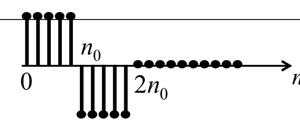
$$y[n] = x[n] - x[n-n_0]$$

Strategy: Exploit Known Transform Pair

$$x[n] = \begin{cases} 1, & 0 \le n \le n_0 - 1 \\ 0, & \text{elsewhere} \end{cases} \Rightarrow X(e^{j\hat{\omega}}) = \frac{e^{-j(n_0-1)\hat{\omega}/2}\left(\sin\frac{n_0\hat{\omega}}{2}\right)}{\left(\sin\frac{\hat{\omega}}{2}\right)}$$

Use Properties to Find DTFT

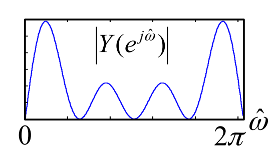
$$y[n] = \begin{cases} 1, & 0 \le n \le n_0 - 1 \\ -1, & n_0 \le n \le 2n_0 - 1 \\ 0, & \text{elsewhere} \end{cases}$$



$$x[n] = \begin{cases} 1, & 0 \le n \le n_0 - 1 \\ 0, & \text{elsewhere} \end{cases} \quad X(e^{j\hat{\omega}}) = \frac{e^{-j(n_0-1)\hat{\omega}/2}\left(\sin\frac{n_0\hat{\omega}}{2}\right)}{\left(\sin\frac{\hat{\omega}}{2}\right)}$$

$$y[n] = x[n] - x[n-n_0] \quad \Leftrightarrow \quad Y(e^{j\hat{\omega}}) = X(e^{j\hat{\omega}}) - e^{-j\hat{\omega}n_0}X(e^{j\hat{\omega}})$$

$$Y(e^{j\hat{\omega}}) = \frac{e^{-j(n_0-1)\hat{\omega}/2}\left(\sin\frac{n_0\hat{\omega}}{2}\right)\left(1-e^{-jn_0\hat{\omega}}\right)}{\left(\sin\frac{\hat{\omega}}{2}\right)}$$



Frequency Shift

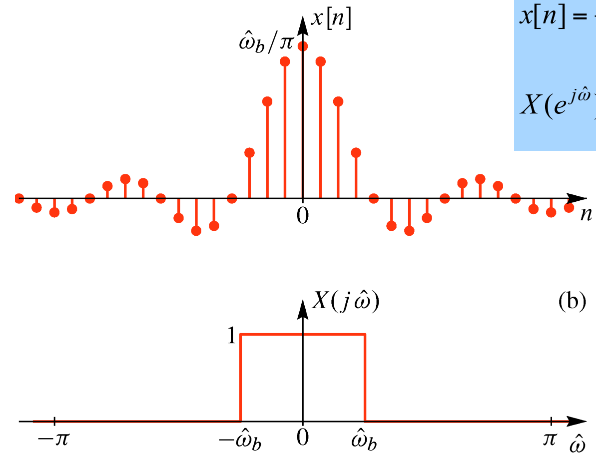
$$y[n] = e^{j\hat{\omega}_c n}x[n] \Leftrightarrow Y(e^{j\hat{\omega}}) = X(e^{j(\hat{\omega}-\hat{\omega}_c)})$$

$$\begin{aligned}Y(e^{j\hat{\omega}}) &= \sum_{n=-\infty}^{\infty} e^{j\hat{\omega}_c n}x[n]e^{-j\hat{\omega}n} \\ &= \sum_{n=-\infty}^{\infty} x[n]e^{-j(\hat{\omega}-\hat{\omega}_c)n} \\ &= X(e^{j(\hat{\omega}-\hat{\omega}_c)})\end{aligned}$$

SINC Function – Rectangle DTFT pair

$$x[n] = \frac{\sin(\hat{\omega}_b n)}{\pi n} \overset{DTFT}{\Leftrightarrow}$$

$$X(e^{j\hat{\omega}}) = \begin{cases} 1, & |\hat{\omega}| \le \hat{\omega}_b \\ 0, & \hat{\omega}_b < |\hat{\omega}| \le \pi \end{cases}$$

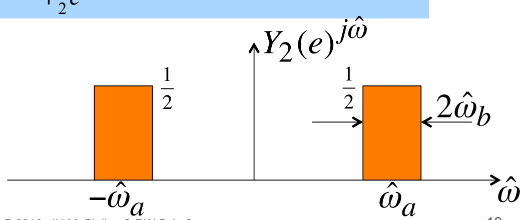


Sinc times sinusoid: find DTFT

$$y[n] = \frac{\sin(\hat{\omega}_b n)}{\pi n} e^{j\hat{\omega}_a n} \quad \Leftrightarrow \quad Y(e^{j\hat{\omega}}) = ?$$

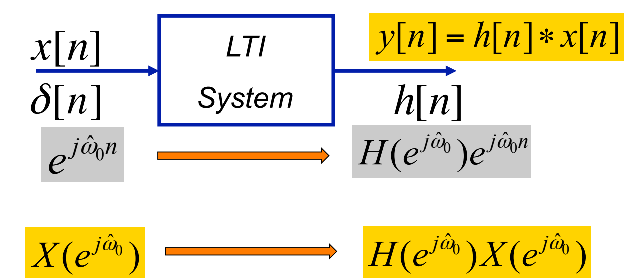
$$y_2[n] = \frac{\sin(\hat{\omega}_b n)}{\pi n} \underbrace{\cos(\hat{\omega}_a n)}_{\frac{1}{2}e^{j\hat{\omega}_a n} + \frac{1}{2}e^{-j\hat{\omega}_a n}} \quad \Leftrightarrow \quad Y_2(e^{j\hat{\omega}}) = ?$$

*Frequency shifting **up and down** is done by cosine multiplication in the time domain*



DTFT maps Convolution to Multiplication

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \quad \Leftrightarrow \quad Y(e^{j\hat{\omega}}) = H(e^{j\hat{\omega}})X(e^{j\hat{\omega}})$$



Stable LTI Systems have Frequency Response - I

- A stable system is one for which every bounded input produces a bounded output.

$$|y[n]| = \left|\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right| \le \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]|$$

$$\le \sum_{k=-\infty}^{\infty} |h[k]| B_x \quad \text{where} \quad |x[n]| < B_x < \infty \qquad \forall n$$

$$\boxed{\sum_{k=-\infty}^{\infty} |h[k]| < \infty \qquad \text{(sufficient for stability)}}$$

Stable LTI Systems have Frequency Response - II

$$\boxed{\sum_{k=-\infty}^{\infty} |h[k]| < \infty \qquad \text{(sufficient for stability)}}$$

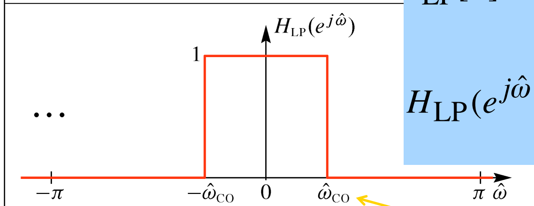
- Also a *necessary* condition.
- This is also the sufficient condition for the existence of the DTFT of the impulse response, i.e., the frequency response, of the system.

$$\boxed{H(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\hat{\omega}n}}$$

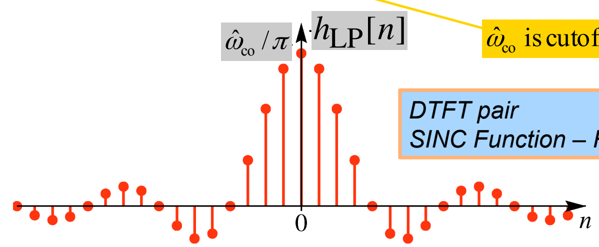
IDEAL LowPass Filter (LPF)

$$h_{\mathrm{LP}}[n] = \frac{\sin(\hat{\omega}_{\mathrm{co}} n)}{\pi n} \quad \overset{DTFT}{\Leftrightarrow}$$

$$H_{\mathrm{LP}}(e^{j\hat{\omega}}) = \begin{cases} 1, & |\hat{\omega}| \le \hat{\omega}_{\mathrm{co}} \\ 0, & \hat{\omega}_{\mathrm{co}} < |\hat{\omega}| \le \pi \end{cases}$$



$\hat{\omega}_{co}$ is cutoff frequency

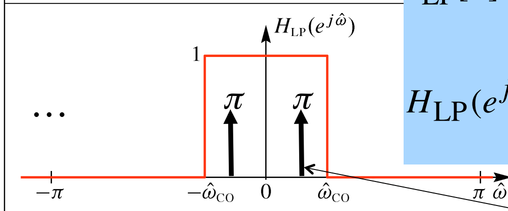


DTFT pair
SINC Function – Rectangle

IDEAL LowPass Filter (LPF)

$$h_{\mathrm{LP}}[n] = \frac{\sin(\hat{\omega}_{\mathrm{co}} n)}{\pi n} \quad \overset{DTFT}{\Leftrightarrow}$$

$$H_{\mathrm{LP}}(e^{j\hat{\omega}}) = \begin{cases} 1, & |\hat{\omega}| \le \hat{\omega}_{\mathrm{co}} \\ 0, & \hat{\omega}_{\mathrm{co}} < |\hat{\omega}| \le \pi \end{cases}$$



Find the output when the input is a sinusoid: $x[n] = \cos(\hat{\omega}_0 n)$

$$y[n] = h[n] * x[n] = \frac{\sin(\hat{\omega}_{co} n)}{\pi n} * \cos(\hat{\omega}_0 n) \quad ??$$

Multiply *the spectrum of the input times the DTFT of the filter to get*

$$y[n] = \begin{cases} \cos(\hat{\omega}_0 n) & \hat{\omega}_0 \le \hat{\omega}_{co} \\ 0 & \hat{\omega}_0 > \hat{\omega}_{co} \end{cases}$$

Response to a Sinusoid using the DTFT

- DTFT of cosine signal (without aliases)

$$X(e^{j\hat{\omega}}) = \pi\delta(\hat{\omega}+\hat{\omega}_0) + \pi\delta(\hat{\omega}-\hat{\omega}_0) \quad -\pi \le \hat{\omega} < \pi$$

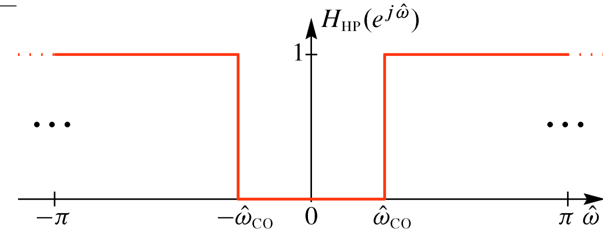
- DTFT of output of LTI system

$$Y(e^{j\hat{\omega}}) = \pi H(e^{-j\hat{\omega}_0})\delta(\hat{\omega}+\hat{\omega}_0) + \pi H(e^{j\hat{\omega}_0})\delta(\hat{\omega}-\hat{\omega}_0) \quad -\pi \le \hat{\omega} < \pi$$

- Inverse DTFT

$$y[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\hat{\omega}})e^{j\hat{\omega}n}\,d\hat{\omega} = \frac{1}{2}H(e^{j\hat{\omega}_0})e^{j\hat{\omega}_0 n} + \frac{1}{2}H(e^{-j\hat{\omega}_0})e^{-j\hat{\omega}_0 n}$$

IDEAL HighPass Filter (HPF)



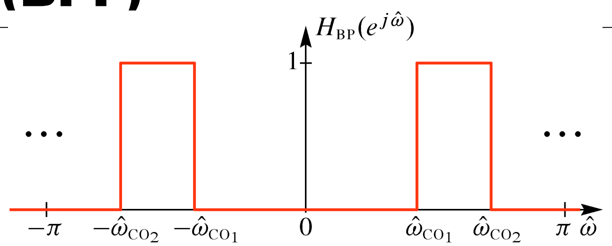
HPF is 1 minus LPF

Inverse DTFT of 1 is a delta

$$h_{HP}[n] = \delta[n] - h_{LP}[n] = \delta[n] - \frac{\sin(\hat{\omega}_{co}n)}{\pi n} \quad \overset{DTFT}{\Leftrightarrow}$$

$$H_{HP}(e^{j\hat{\omega}}) = 1 - H_{LP}(e^{j\hat{\omega}}) = \begin{cases} 0, & |\hat{\omega}| \le \hat{\omega}_{co} \\ 1, & \hat{\omega}_{co} < |\hat{\omega}| \le \pi \end{cases}$$

IDEAL BandPass Filter (BPF)

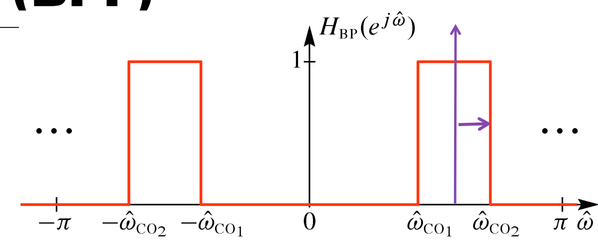


BPF is has two stopbands

Band Reject Filter has one stopband and two passbands. It is one minus BPF

$$H_{BP}(e^{j\hat{\omega}}) = \begin{cases} 0 & |\hat{\omega}| \le \hat{\omega}_{co_1} \\ 1 & \hat{\omega}_{co_1} < |\hat{\omega}| \le \hat{\omega}_{co_2} \\ 0 & \hat{\omega}_{co_2} < |\hat{\omega}| \le \pi \end{cases}$$

IDEAL BandPass Filter (BPF)



BPF is frequency shifted version of LPF

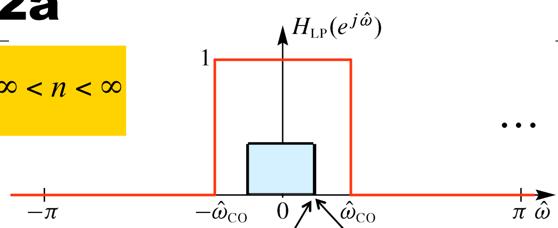
Frequency shifting ***up and down*** is done by cosine multiplication in the time domain

$$h_{BP}[n] = 2\cos(\hat{\omega}_{mid}n)\frac{\sin(\frac{1}{2}\hat{\omega}_{diff}n)}{\pi n}$$

$$\overset{DTFT}{\Leftrightarrow} H_{BP}(e^{j\hat{\omega}}) = \begin{cases} 0 & |\hat{\omega}| \le \hat{\omega}_{co_1} \\ 1 & \hat{\omega}_{co_1} < |\hat{\omega}| \le \hat{\omega}_{co_2} \\ 0 & \hat{\omega}_{co_2} < |\hat{\omega}| \le \pi \end{cases}$$

Example 2a

$$h[n]=\frac{\sin(\hat{\omega}_{co}n)}{\pi n}, \quad -\infty<n<\infty$$



- Find the output when the input is $|\hat{\omega}_b|<|\hat{\omega}_{co}|$

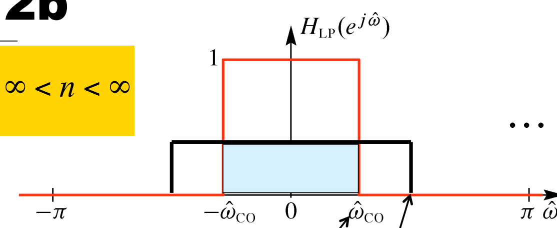
Input bandwidth Less than Filter's passband

$$x[n]=\frac{\sin(\hat{\omega}_b n)}{3\pi n}, \quad -\infty<n<\infty$$

$$y[n]=\frac{\sin(\hat{\omega}_b n)}{3\pi n}, \quad \text{for } |\hat{\omega}_b|<|\hat{\omega}_{co}|$$

Example 2b

$$h[n]=\frac{\sin(\hat{\omega}_{co}n)}{\pi n}, \quad -\infty<n<\infty$$



- Find the output when the input is $|\hat{\omega}_b|>|\hat{\omega}_{co}|$

Filter chops off Signal bandwidth Outside of passband

$$x[n]=\frac{\sin(\hat{\omega}_b n)}{3\pi n}, \quad -\infty<n<\infty$$

$$y[n]=\frac{\sin(\hat{\omega}_{co} n)}{3\pi n}, \quad \text{for } |\hat{\omega}_b|>|\hat{\omega}_{co}|$$

The Autocorrelation Function of a Signal

- Definition:

$$c_{xx}[n]=x[-n]*x[n]=\sum_{k=-\infty}^{\infty}x[-k]x[n-k]$$

- Change summation to $-k=m$

$$\boxed{c_{xx}[n]=\sum_{k=-\infty}^{\infty}x[m]x[n+m]}$$

- DTFT when $x[n]$ is real:

$$C_{xx}(e^{j\hat{\omega}})=X(e^{-j\hat{\omega}})X(e^{j\hat{\omega}})=X^*(e^{j\hat{\omega}})X(e^{j\hat{\omega}})=\left|X(e^{j\hat{\omega}})\right|^2$$

Energy Spectrum and Parseval's Theorem

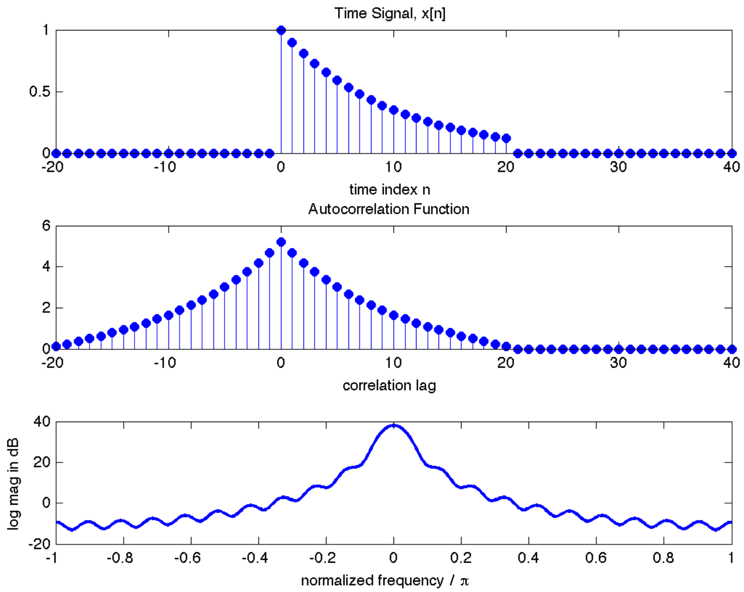
$$\boxed{c_{xx}[n]=\sum_{k=-\infty}^{\infty}x[m]x[n+m]=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|X(e^{j\hat{\omega}})\right|^2e^{j\hat{\omega}n}d\hat{\omega}}$$

- Energy definition and Parseval's Theorem:

$$\boxed{E=c_{xx}[0]=\sum_{k=-\infty}^{\infty}|x[m]|^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|X(e^{j\hat{\omega}})\right|^2d\hat{\omega}}$$

- Energy density spectrum

$$C_{xx}(e^{j\hat{\omega}})=\left|X(e^{j\hat{\omega}})\right|^2$$


Time Signal, x[n]
time index n
Autocorrelation Function
correlation lag
log mag in dB
normalized frequency / π