STANFORD UNIVERSITY
EE 102B Spring-2013

Lecture 13

FIR Filter Design and Sampling

April 26, 2013

ASSIGNMENTS

- Reading for this Lecture:
  - SPF: Section 12-3
  - Lab 04 Warm-Up section
  - S&S: Chapter 5
- HW#04 is due by 5pm Wednesday, May 1, in Packard 263.
- Lab #04 is due by 5pm, Friday, May 3, in Packard 263.

Office Hours for Course Staff – Come see us.

- Ron Schafer: Mon. 2:00-4:00 pm. Not available for Weds. office hours this week.
- Dookun Park: Tues. 1:30 ~ 3:30 pm in Packard 107
- Ruo Yu Gu (Roy): Mon. 4:00 ~ 6:00 pm in Packard 106
- https://class2go.stanford.edu/EE102B/Spring2013/pages/staff

Lecture Objective

- Review FIR Filter design
  - Linear phase condition
  - Window design
  - filterdesign.m demonstration and discussion
- The sampling theorem revisited with the CTFT and DTFT
  - Sampling
  - Reconstruction

Time-domain Multiplication DTFT Property

- Multiplication in the time domain corresponds to (periodic) convolution in the frequency domain.

$$\boxed{y[n] = w[n]x[n] \Leftrightarrow \quad Y(e^{j\hat{\omega}}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})W(e^{j(\hat{\omega}-\theta)})d\theta}$$

FIR Filter Design by Windowing

- Define ideal frequency response. $H_{\text{ideal}}(e^{j\hat{\omega}})$ Determine the impulse response of the delayed ideal filter so that $h_{\text{ideal}}[M-n] = h_{\text{ideal}}[n]$

$$h_{\text{ideal}}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|H_{\text{ideal}}(e^{j\hat{\omega}})\right| e^{-j\hat{\omega}M/2} e^{j\hat{\omega}n}\, d\hat{\omega}$$

- Apply the symmetric window $w_L[M-n] = w_L[n]$

$$h_L[n] = w_L[n]h_{\text{ideal}}[n] = \begin{cases} w_L[n]h_{\text{ideal}}[n] & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

Windowed Lowpass Filter Design

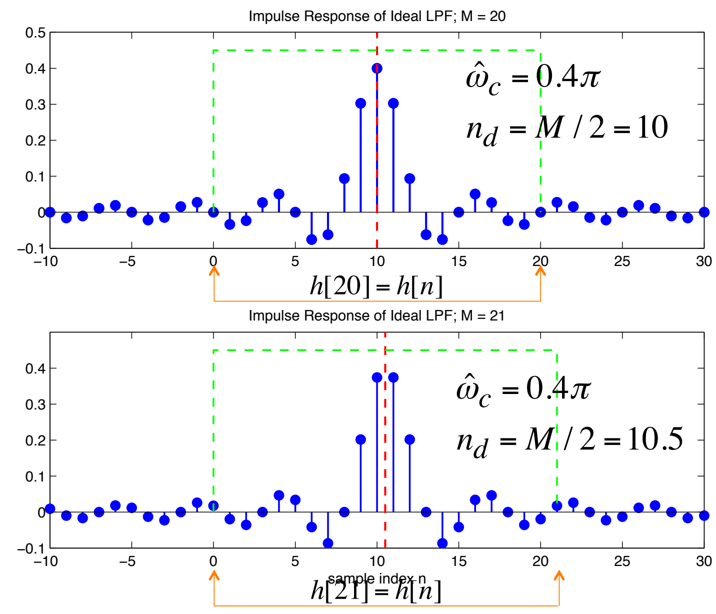
- sinc is inverse DTFT of ideal LPF

$$h_{\text{ideal}}[n] = \frac{\sin[\hat{\omega}_c(n - M/2)]}{\pi(n - M/2)} \qquad -\infty < n < \infty$$

  - Truncate: Multiply sinc by a window
  - Finite h[n] of length L = M+1 = window length

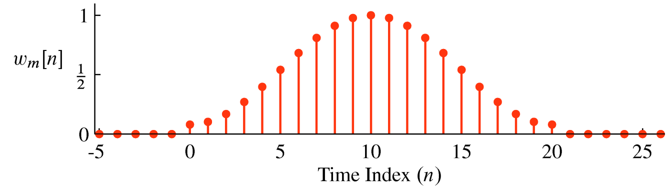
$$H_L(e^{j\hat{\omega}}) = \sum_{n=0}^{M} w_L[n]h_{\text{ideal}}[n]e^{-j\hat{\omega}n}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{\text{ideal}}(e^{j\theta})W_L(e^{j(\hat{\omega}-\theta)})d\theta$$



Window for Filter Design

- Plot of Length-21 Hamming window

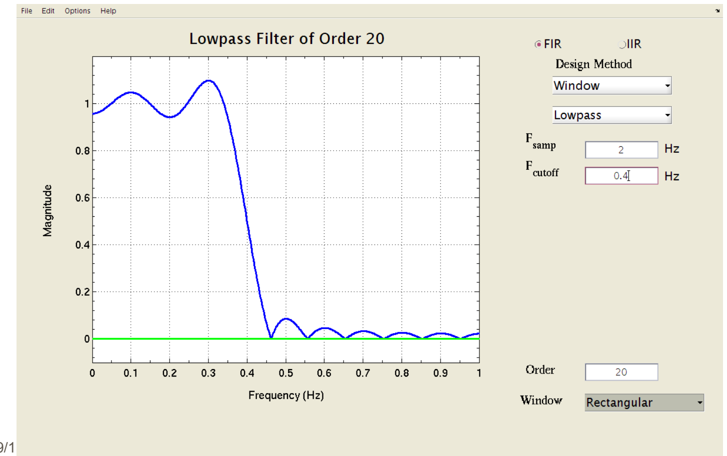


Hamming Window

$$w_m[n] = \begin{cases} 0 & n < 0 \\ 0.54 - 0.46\cos(2\pi(n)/(L-1)) & 0 \le n < L \\ 0 & n \ge L \end{cases}$$

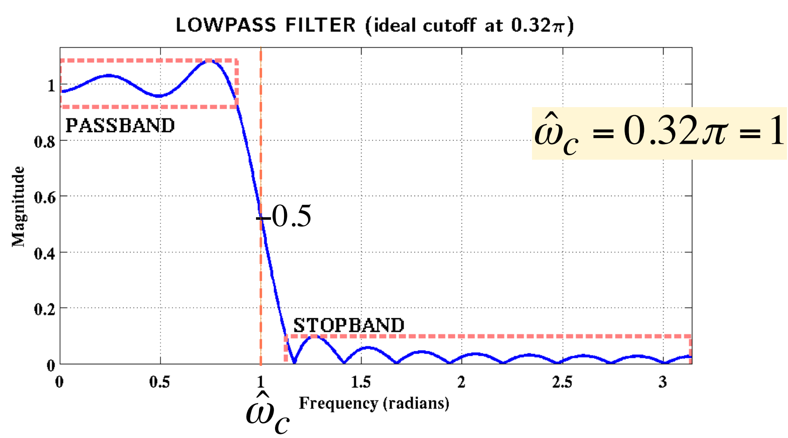
Filter Design via Rectangular Windowing

- Rectangular Window, L=21 (order M=20) $\hat{\omega}_c = 0.4\pi$



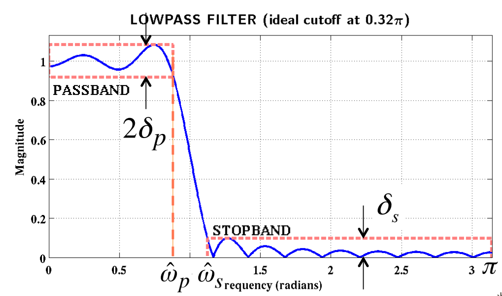
Filter Design: Passband & Stopband

- Rectangular Window, L=41 (order M=40)



Ripples, Band edges, & Transition Width

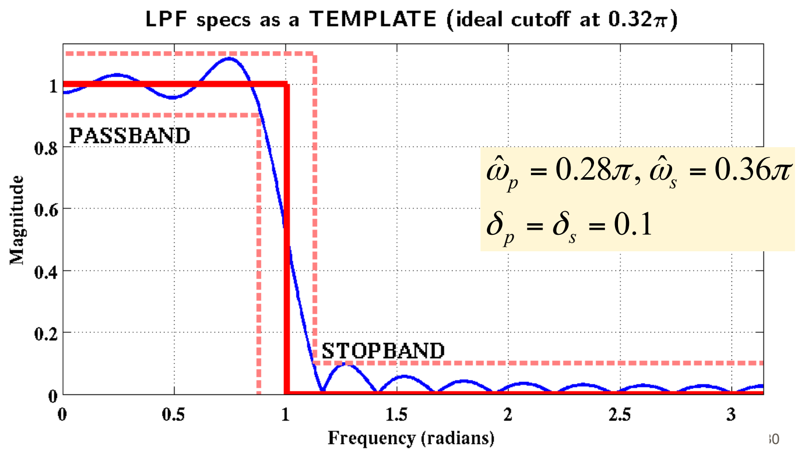
- Passband Ripple is one plus or minus δ_p
- Stopband Ripple is less than δ_s
- Band edges are $\hat{\omega}_p, \hat{\omega}_s$
- Transition Width is $\hat{\omega}_s - \hat{\omega}_p$



***Can't have it all:* small transition width, small ripples, and lowest possible order (M)**

Filter Design: Tolerance Template

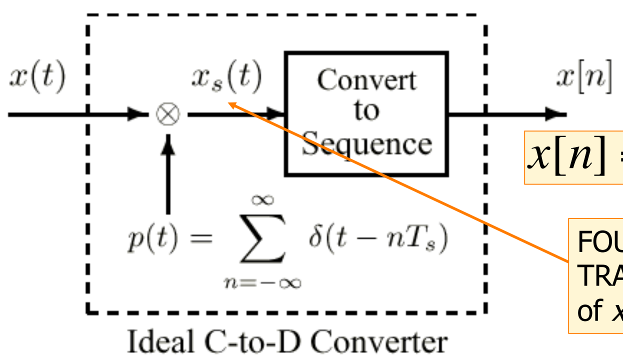
- Want the actual response inside the template



RE-DERIVATION OF THE SAMPLING THEOREM

Ideal C-to-D Converter

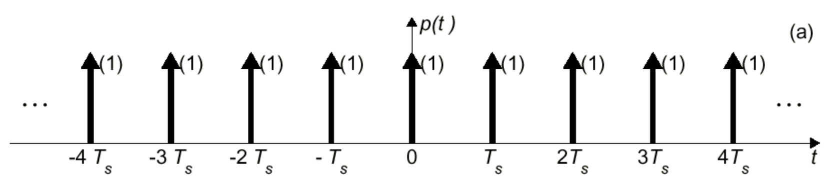
- **Mathematical Model for Sampling**



$x[n] = x(nT_s)$

FOURIER TRANSFORM of $x_s(t)$???

Periodic Impulse Train

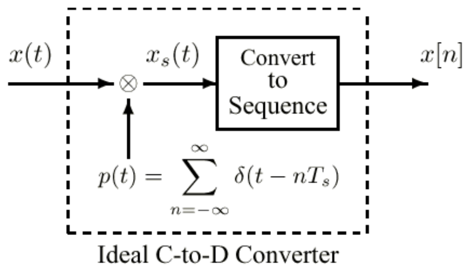


$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$$

$$a_k = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \delta(t) e^{-jk\omega_s t} dt = \frac{1}{T_s}$$

$$\omega_s = \frac{2\pi}{T_s}$$

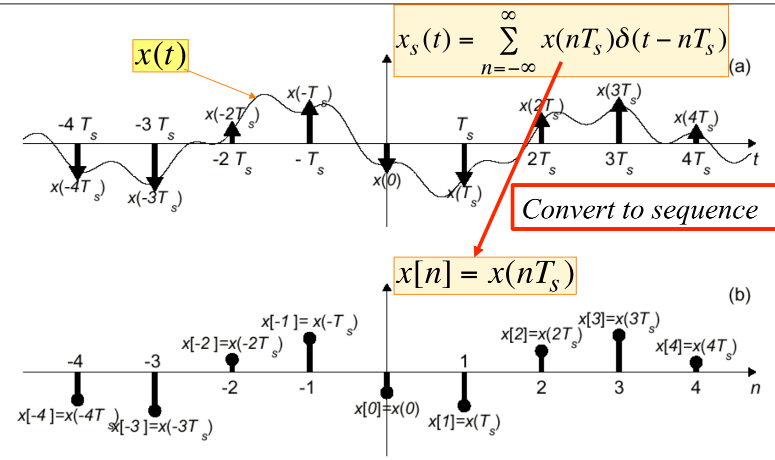
Representing Sampling by Impulse Train Modulation



$$x_s(t) = x(t) \sum_{n=-\infty} \delta(t - nT_s) = \sum_{n=-\infty} \underline{x(t)\delta(t - nT_s)}$$

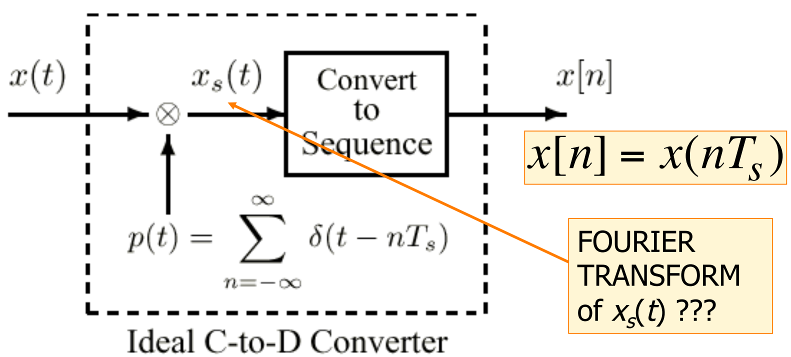
$$x_s(t) = \sum_{n=-\infty}^{\infty} \underline{x(nT_s)\delta(t - nT_s)}$$

Illustration of Sampling



Ideal C-to-D Converter

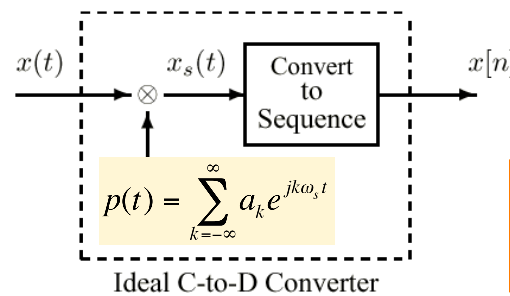
- **Mathematical Model for A-to-D**



$$x[n] = x(nT_s)$$

FOURIER TRANSFORM of $x_s(t)$???

Sampling: Freq. Domain



EXPECT A LOT OF FREQUENCY SHIFTING !!!

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$$

Frequency-Domain Analysis

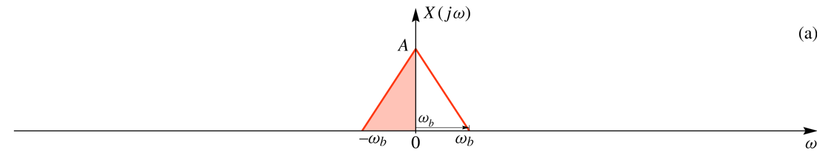
$$x_s(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t-nT_s)$$

$$x_s(t) = x(t)\sum_{k=-\infty}^{\infty}\frac{1}{T_s}e^{jk\omega_s t} = \frac{1}{T_s}\sum_{k=-\infty}^{\infty}\underline{x(t)e^{jk\omega_s t}}$$

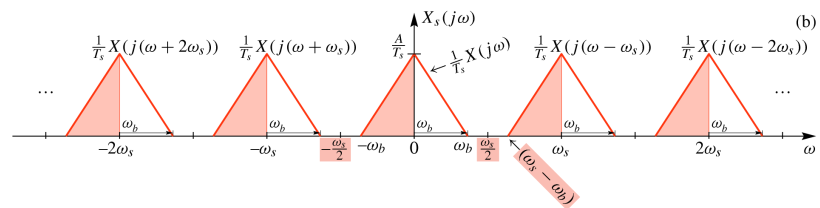
$$\boxed{X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty}\underline{X(j(\omega - k\omega_s))}}$$

$$\boxed{\omega_s = \frac{2\pi}{T_s}}$$

Frequency-Domain Representation of Sampling

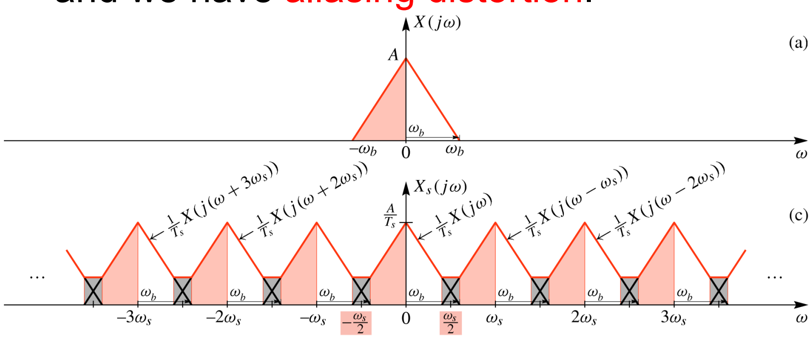


$$\boxed{X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X\big(j(\omega - k\omega_s)\big)}$$



Aliasing Distortion

- If $\omega_s < 2\omega_b$, the copies of $X(j\omega)$ overlap, and we have aliasing distortion.



Relation to the DTFT

- Look at the CTFT of $x_s(t)$ in a different way

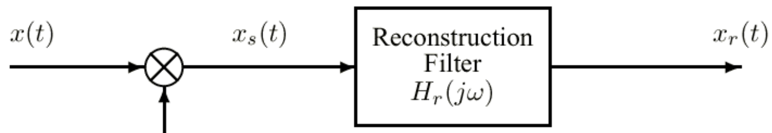
$$x_s(t) = \sum_{n=-\infty}^{\infty}\underline{x(nT_s)\delta(t-nT_s)}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{\infty}\underline{x(nT_s)e^{-j\omega nT_s}} = X(e^{j\omega T_s})$$

$$\boxed{X(e^{j\omega T_s}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega nT_s} = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X\big(j(\omega - k\omega_s)\big)}$$

Recall that: $\hat{\omega} = \omega T_s$ $\quad x[n] = x(nT_s)$ $\quad \boxed{\omega_s = \frac{2\pi}{T_s}}$

Reconstruction of *x*(*t*)



$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} x(t)e^{jk\omega_s t}$$

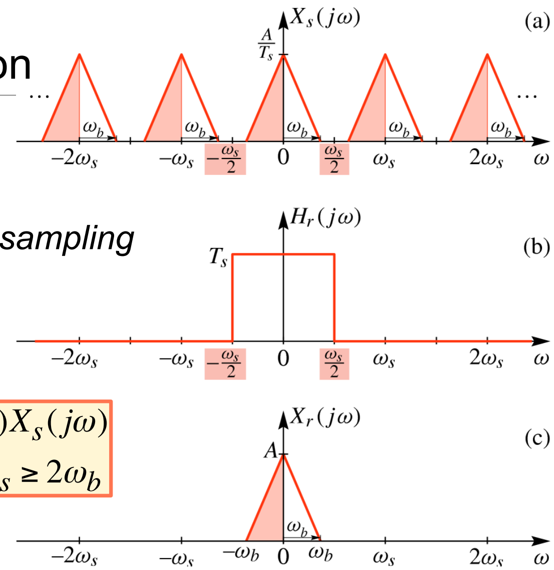
$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X\left(j(\omega - k\omega_s)\right)$$

$$\boxed{X_r(j\omega) = H_r(j\omega)X_s(j\omega)}$$

Bandlimited Reconstruction

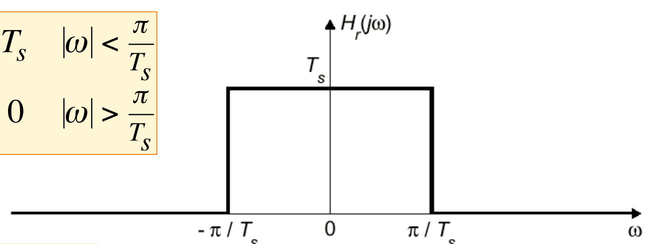
This proves the sampling theorem.

$$\boxed{\begin{aligned} &X_r(j\omega) = H_r(j\omega)X_s(j\omega) \\ &= X(j\omega) \text{ when } \omega_s \geq 2\omega_b \end{aligned}}$$

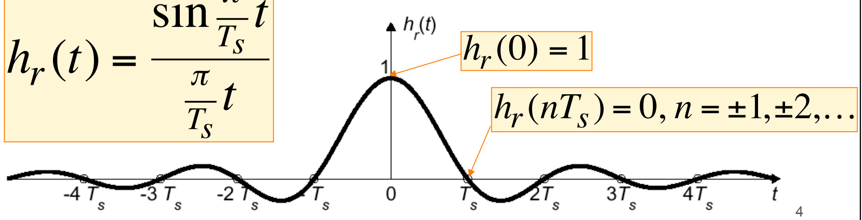


Ideal Reconstruction Filter

$$H_r(j\omega) = \begin{cases} T_s & |\omega| < \frac{\pi}{T_s} \\ 0 & |\omega| > \frac{\pi}{T_s} \end{cases}$$



$$h_r(t) = \frac{\sin\frac{\pi}{T_s}t}{\frac{\pi}{T_s}t}$$



$h_r(0) = 1$

$h_r(nT_s) = 0, n = \pm1, \pm2, \ldots$

Signal Reconstruction

$$x_r(t) = h_r(t) * x_s(t) = h_r(t) * \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s)h_r(t - nT_s)$$

$$\boxed{x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\frac{\sin\frac{\pi}{T_s}(t - nT_s)}{\frac{\pi}{T_s}(t - nT_s)}}$$

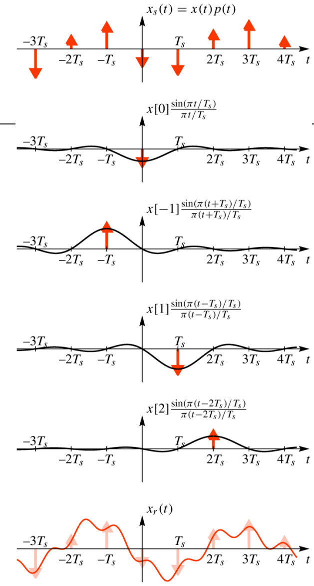
Ideal bandlimited interpolation formula

Shannon Sampling Theorem

- **"SINC" Interpolation** is the ideal
  - PERFECT RECONSTRUCTION
  - of BANDLIMITED SIGNALS

A signal $x(t)$ with bandlimited Fourier transform such that $X(j\omega) = 0$ for $|\omega| \geq \omega_b$ can be reconstructed exactly from samples taken with sampling rate $\omega_s = 2\pi/T_s \geq 2\omega_b$ using the following bandlimited interpolation formula:

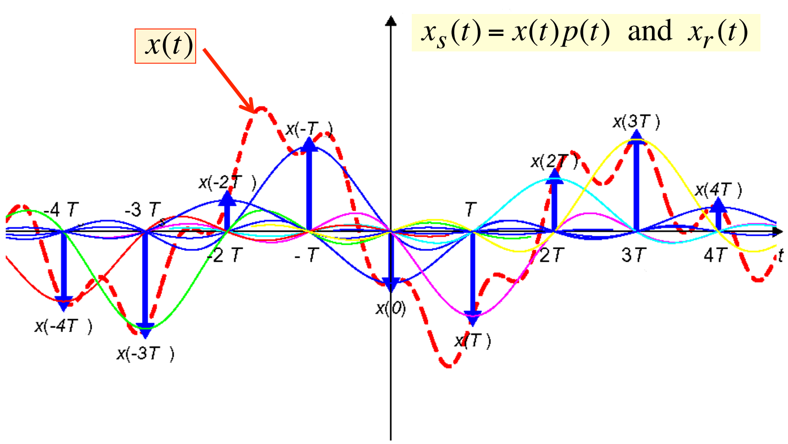
$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin\left[\frac{\pi}{T_s}(t - nT_s)\right]}{\frac{\pi}{T_s}(t - nT_s)}.$$



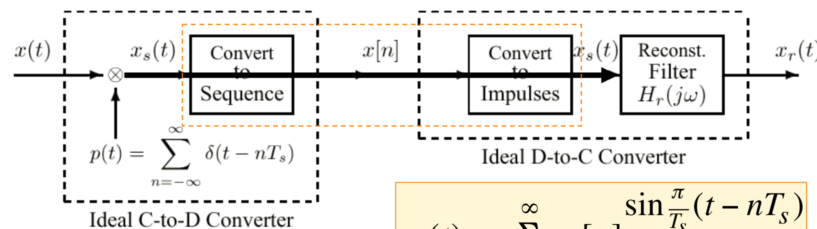
Reconstruction in Time-Domain

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)h_r(t-nT) = \sum_{n=-\infty}^{\infty} x_c(nT)\frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

$x_s(t) = x(t)p(t)$ and $x_r(t)$



Ideal C-to-D and D-to-C Back-to-Back



$x[n] = x(nT_s)$

Ideal Sampler

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]\frac{\sin\frac{\pi}{T_s}(t - nT_s)}{\frac{\pi}{T_s}(t - nT_s)}$$

Ideal bandlimited interpolator

$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$$X_r(j\omega) = H_r(j\omega)X_s(j\omega)$$