Stanford University
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Signal Processing and Linear Systems I

Lecture 9: Fourier Transform Theorems

February 2, 2013

Linearity

Linear combination of two signals $f_1(t)$ and $f_2(t)$ is a signal of the form $af_1(t) + bf_2(t)$.

Linearity Theorem: The Fourier transform is linear; that is, given two signals $f_1(t)$ and $f_2(t)$ and two complex numbers a and b, then

$$af_1(t) + bf_2(t) \Leftrightarrow aF_1(j\omega) + bF_2(j\omega).$$

This follows from linearity of integrals:

$$\begin{aligned}
&\int_{-\infty}^{\infty} (af_1(t) + bf_2(t))e^{-j\omega t}\, dt \\
&= a\int_{-\infty}^{\infty} f_1(t)e^{-j\omega t}\, dt + b\int_{-\infty}^{\infty} f_2(t)e^{-j\omega t}\, dt \\
&= aF_1(j\omega) + bF_2(j\omega)
\end{aligned}$$

This easily extends to finite combinations. Given signals $f_k(t)$ with Fourier transforms $F_k(j\omega)$ and complex constants a_k, $k = 1, 2, \ldots K$, then

$$\sum_{k=1}^{K} a_k f_k(t) \Leftrightarrow \sum_{k=1}^{K} a_k F_k(j\omega).$$

If you consider a system which has a signal $f(t)$ as its input and the Fourier transform $F(j\omega)$ as its output, the system is linear!

Linearity Example

Find the Fourier transform of the signal

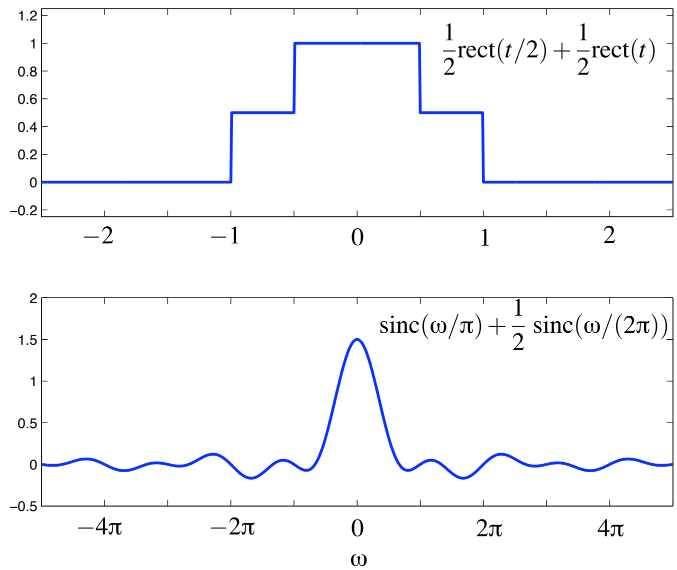
$$f(t) = \begin{cases} \frac{1}{2} & \frac{1}{2} \le |t| < 1 \\ 1 & |t| \le \frac{1}{2} \end{cases}$$

This signal can be recognized as

$$f(t) = \frac{1}{2}\,\mathrm{rect}\left(\frac{t}{2}\right) + \frac{1}{2}\,\mathrm{rect}\left(t\right).$$

From linearity and the fact that the transform of $\mathrm{rect}(t/T)$ is $T\,\mathrm{sinc}(T\omega/(2\pi))$, we have

$$F(\omega) = \left(\frac{1}{2}\right) 2\,\mathrm{sinc}(2\omega/(2\pi)) + \frac{1}{2}\,\mathrm{sinc}(\omega/(2\pi)) = \mathrm{sinc}(\omega/\pi) + \frac{1}{2}\,\mathrm{sinc}(\omega/(2\pi))$$



Linearity Example

Scaling Theorem

Stretch (Scaling) Theorem: Given a transform pair $f(t) \Leftrightarrow F(j\omega)$, and a real-valued nonzero constant a,

$$f(at) \Leftrightarrow \frac{1}{|a|} F\left(j\frac{\omega}{a}\right)$$

Proof: Here consider only $a > 0$. Negative a left as an exercise. Change variables $\tau = at$

$$\int_{-\infty}^{\infty} f(at)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} f(\tau)e^{-j\omega\tau/a}\frac{d\tau}{a} = \frac{1}{a}F\left(j\frac{\omega}{a}\right).$$

If $a = -1 \Rightarrow$ "time reversal theorem:"

$$f(-t) \Leftrightarrow F(-j\omega)$$

Scaling Examples

We have already seen that

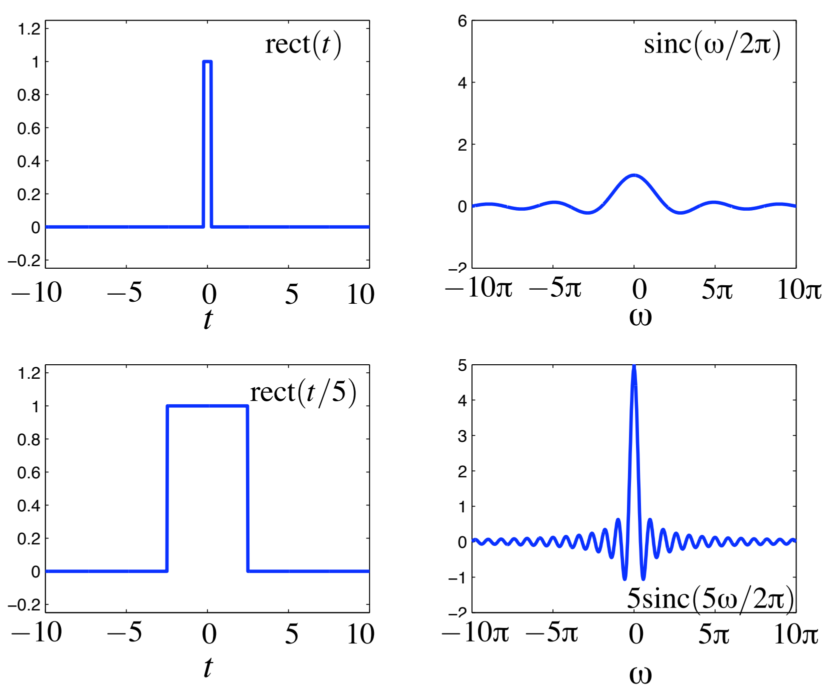
$$\mathrm{rect}(t/T) \Leftrightarrow T\,\mathrm{sinc}(T\omega/2\pi)$$

by brute force integration. The scaling theorem provides a shortcut proof given the simpler result

$$\mathrm{rect}(t) \Leftrightarrow \mathrm{sinc}(\omega/2\pi).$$

This is a good point to illustrate a property of transform pairs.

Consider this Fourier transform pair for a small T and large T, say $T = 1$ and $T = 5$. The resulting transform pairs are shown below to a common horizontal scale:



The narrow pulse yields a wide transform and a wide pulse yields a narrow spectrum!

This example shows that the shorter the pulse (and hence the more pulses we could cram into a transmission channel), the greater the bandwidth required by the transform!!

Imagine each pulse being ± 1, carrying one "bit" of information.

The more pulses per second, the more information, and the greater the bandwidth required.

Example: Find the transform of the time-reversed exponential

$$f_1(t) = e^{at}u(-t).$$

This is the exponential signal

$$f_2(t) = e^{-at}u(t)$$

with time scaled by -1. The Fourier transform is then

$$F_1(j\omega) = F_2(-j\omega) = \frac{1}{a - j\omega}.$$

by the time reversal theorem.

Example: Find the transform of

$$f(t) = e^{-a|t|} \qquad \text{all real } t.$$

This signal can be written as

$$f(t) = e^{-at}u(t) + e^{at}u(-t).$$

Combining the original transform of the exponential signal, the time reversal result above, and linearity yields

$$\begin{aligned} F(j\omega) &= \frac{1}{a+j\omega} + \frac{1}{a-j\omega} \\ &= \frac{2a}{a^2-(j\omega)^2} \\ &= \frac{2a}{a^2+\omega^2} \end{aligned}$$

Much easier than direct integration!

Complex Conjugation Theorem

Complex Conjugation Theorem: If $f(t) \Leftrightarrow F(j\omega)$, then

$$f^*(t) \Leftrightarrow F^*(-j\omega)$$

Proof: The Fourier transform of $f^*(t)$ is

$$\begin{aligned} \int_{-\infty}^{\infty} f^*(t)e^{-j\omega t}\,dt &= \left(\int_{-\infty}^{\infty} f(t)e^{j\omega t}\,dt\right)^* \\ &= \left(\int_{-\infty}^{\infty} f(t)e^{-(-j\omega)t}\,dt\right)^* = F^*(-j\omega) \end{aligned}$$

Duality Theorem

This theorem is complicated by the book's use of the notation $F(j\omega)$ for the Fourier transform. However, if we consider the Fourier transform to be a function of ω instead of $j\omega$, the forward and inverse Fourier transforms are almost symmetric, and we can use a transform in one direction to solve for a transform in the other direction.

Duality Theorem: If $f(t) \Leftrightarrow F(j\omega)$, then

$$F(j\omega)|_{\omega \to t} \Leftrightarrow 2\pi f(-t)|_{t \to \omega}$$

The left hand side is a time signal created by taking $F(j\omega)$ and replacing all ω's by t's. The right had side is the the Fourier transform of this function, formed by time reversing $f(t)$, multiplying by 2π, and the replacing all t's by ω's.

To see why this is true, note that the inverse Fourier transform of $F(j\omega)$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} \, d\omega$$

so

$$2\pi f(-t) = \int_{-\infty}^{\infty} F(j\omega) e^{-j\omega t} \, d\omega$$

The right had side is now the *forward* Fourier transform of $F(j\omega)$, where the roles of ω and t have been reversed.

Hence, if we create a time signal from $F(j\omega)$ by replacing ω with t, then it's Fourier transform is found by taking $2\pi f(-t)$ and replacing t with ω.

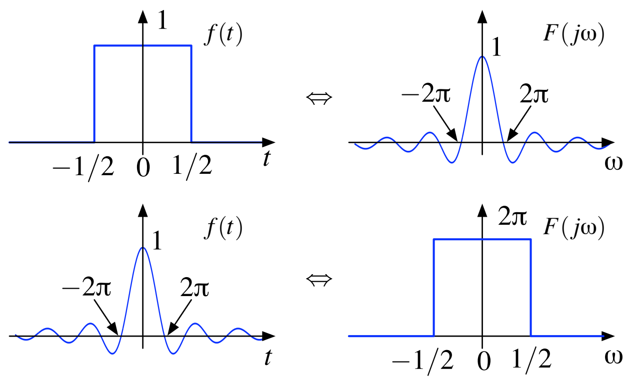
This result effectively gives us two transform pairs for every transform we find.

Examples of Duality

- Since $\mathrm{rect}(t) \Leftrightarrow \mathrm{sinc}(\omega/2\pi)$ then

$$\mathrm{sinc}(t/2\pi) \Leftrightarrow 2\pi\,\mathrm{rect}(-\omega) = 2\pi\,\mathrm{rect}(\omega)$$

  since $\mathrm{rect}()$ is an even function of its argument.



- Another example: since for $a > 0$

$$e^{-at}u(t) \Leftrightarrow \frac{1}{a + j\omega}$$

  then

$$\frac{1}{a + jt} \Leftrightarrow 2\pi e^{a\omega}u(-\omega)$$

- *Exercise* What signal $f(t)$ has a Fourier transform $e^{-|\omega|}$?

Shift Theorem

The Shift Theorem: Given a signal $f(t)$ with Fourier transform $F(j\omega)$, define for a fixed τ the shifted (or delayed) signal $f_\tau(t)$ by

$$f_\tau(t) \triangleq f(t-\tau)$$

for all t. Then the Fourier transform $F_\tau(j\omega)$ of $f_\tau(t)$ is given by

$$F_\tau(j\omega) = e^{-j\omega\tau} F(j\omega)$$

or

$$f(t-\tau) \Leftrightarrow e^{-j\omega\tau} F(j\omega)$$

Proof: Change variables $\alpha = t - \tau$ to find

$$\int_{-\infty}^{\infty} f(t-\tau) e^{-j\omega t}\, dt \quad = \quad \int_{-\infty}^{\infty} f(\alpha) e^{-j\omega(\alpha+\tau)}\, d\alpha$$

$$\begin{aligned} &= e^{-j\omega\tau} \int_{-\infty}^{\infty} f(\alpha) e^{-j\omega\alpha}\, d\alpha \\ &= e^{-j\omega\tau} F(j\omega) \end{aligned}$$

Example: square pulse $p(t) = 1$ for $t \in [0, T)$ and 0 otherwise.

$$p(t) = \mathrm{rect}\left(\frac{t - \frac{T}{2}}{T}\right)$$

From shift and scaling theorems

$$P(j\omega) = e^{-j\omega T/2} T \,\mathrm{sinc}(\omega T/2\pi).$$

Modulation

For $f(t)$ with Fourier transform $F(j\omega)$, define

$$f_m(t) = f(t)e^{j\omega_0 t}$$

where ω_0 is a fixed frequency

Modulation of complex exponential (carrier) by signal $f(t)$

Amplitude Modulation (AM)

Variations

- $f_c(t) = f(t)\cos(\omega_0 t)$ (DSB-SC)
- $f_s(t) = f(t)\sin(\omega_0 t)$ (DSB-SC)
- $f_a(t) = A[1 + mf(t)]\cos(\omega_0 t)$ (DSB, commercial AM radio)

  - m is the *modulation index*
  - Typically m and $f(t)$ are chosen so that $|mf(t)| < 1$ for all t

Amplitude modulation without the carrier term ($f_c(t)$ or $f_s(t)$) is called $linear$ modulation since operation linear (but time varying!)

Corresponding system is linear but not time-invariant.

The Modulation Theorem: Given a signal $f(t)$ with spectrum $F(j\omega)$, then

$$f(t)e^{j\omega_0 t} \Leftrightarrow F(j(\omega - \omega_0))$$

$$f(t)\cos(\omega_0 t) \Leftrightarrow \frac{1}{2}\left(F(j(\omega-\omega_0)) + F(j(\omega+\omega_0))\right)$$
$$f(t)\sin(\omega_0 t) \Leftrightarrow \frac{1}{2j}\left(F(j(\omega-\omega_0)) - F(j(\omega+\omega_0))\right)$$

Modulating a signal by an exponential shifts the spectrum in the frequency domain. This is a *dual* to the shift theorem, it results from interchanging the roles of t and ω.

Modulation by a cosine causes replicas of $F(j\omega)$ to be placed at plus and minus the carrier frequency.

Replicas are called $sidebands$.

Proof: The Fourier transform of $f_m(t) = f(t)e^{j\omega_0 t}$ is

$$\begin{aligned}
F_m(j\omega) &= \int_{-\infty}^{\infty} f_m(t)e^{-j\omega t}\,dt \\
&= \int_{-\infty}^{\infty} (f(t)e^{j\omega_0 t})e^{-j\omega t}\,dt \\
&= \int_{-\infty}^{\infty} f(t)e^{-j(\omega-\omega_0)t}\,dt \\
&= F(j(\omega-\omega_0)).
\end{aligned}$$

The results for cosine and sine modulation then follow via Euler's relations. For example

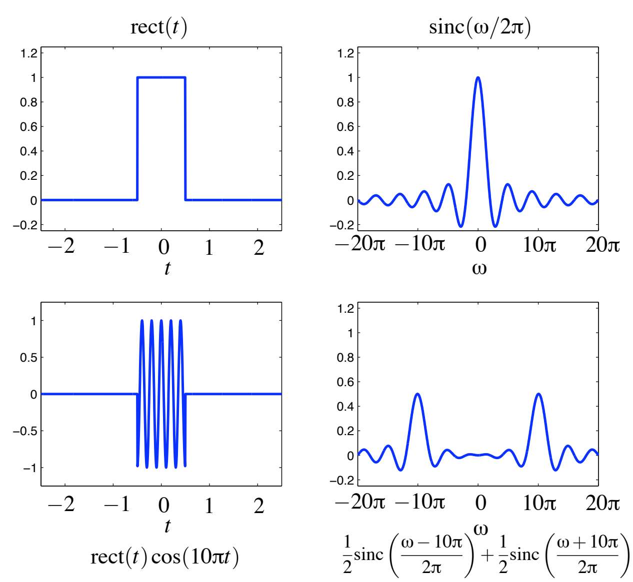
$$\begin{aligned}
f_c(t) &= f(t)\cos(\omega_0 t) \\
&= f(t)\left[\frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)\right]
\end{aligned}$$

$$= \frac{1}{2}\left[f(t)e^{j\omega_0 t} + f(t)e^{-j\omega_0 t}\right]$$

Hence, by linearity and modulation theorems

$$F_c(j\omega) = \frac{1}{2}\left[F(j(\omega-\omega_0)) + F(j(\omega+\omega_0))\right]$$

Examples of Modulation Theorem



The Derivative Theorem

Given a transform pair $f(t) \Leftrightarrow F(j\omega)$ with $f(t)$ everywhere differentiable with respect to t, define signal f' by

$$f'(t) = \frac{df(t)}{dt}.$$

What is transform of $f'(t)$?

Indirect derivation: differentiate both sides of inversion formula

$$\begin{aligned} f'(t) &= \frac{d}{dt}f(t) = \frac{d}{dt}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}\,d\omega\right) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\frac{d}{dt}e^{j\omega t}\,d\omega \end{aligned}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)(j\omega)e^{j\omega t}\,d\omega.$$

From the inversion formula can identify $j\omega F(j\omega)$ as transform of $f'(t)$.

The Derivative Theorem: Given an everywhere differentiable signal $f(t)$ with Fourier transform $F(j\omega)$, then

$$f'(t) \Leftrightarrow j\omega F(j\omega)$$

Similarly, if $f(t)$ is n times differentiable and $f^{(n)}(t) = \frac{d^n f(t)}{dt^n}$, then

$$f^{(n)}(t) \Leftrightarrow (j\omega)^n F(j\omega)$$

There is a dual to the derivative theorem, i.e., a result interchanging the role of t and ω. Differentiate the spectrum with respect to ω

$$\begin{aligned} F'(j\omega) &= \frac{d}{d\omega}\left(\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt\right) \\ &= \int_{-\infty}^{\infty} (-jt)f(t)e^{-j\omega t}\,dt. \end{aligned}$$

so that

$$(-jt)f(t) \Leftrightarrow F'(j\omega)$$

Parseval's Theorem

(Parseval proved for Fourier series, Rayleigh for Fourier transforms. Also called Plancherel's theorem.)

Recall signal *energy* of $f(t)$ is

$$\mathcal{E}_f = \int_{-\infty}^{\infty} |f(t)|^2\,dt$$

Interpretation: energy dissipated in a one ohm resistor if $f(t)$ is a voltage. Can also be viewed as a measure of the size of a signal.

The energy $\mathcal{E}_f$ of a signal $f(t)$ can be expressed as

$$\begin{aligned}
\mathcal{E}_f &= \int_{-\infty}^{\infty} f(t) f^*(t)\, dt \\
&= \int_{-\infty}^{\infty} f(t) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t}\, d\omega \right)^* dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(j\omega) \left(\int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt \right) d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(j\omega) F(j\omega)\, d\omega
\end{aligned}$$

Thus

$$\mathcal{E}_f = \int_{-\infty}^{\infty} |f(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2\, d\omega$$

Example of Parseval's Theorem

Parseval's theorem provides many simple integral evaluations. For example, evaluate

$$\int_{-\infty}^{\infty} \mathrm{sinc}^2(t)\, dt$$

We can show $\mathrm{sinc}(t) \Leftrightarrow \mathrm{rect}(\omega/2\pi)$ using duality and scaling.

Parseval's Theorem yields

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{sinc}^2(t)\, dt &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{rect}^2(\omega/2\pi)\, d\omega \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} 1\, d\omega = \frac{2\pi}{2\pi} = 1.
\end{aligned}$$

Try to evaluate this integral directly and you will appreciate Parseval's shortcut.

⋆ The Convolution Theorem ⋆

Convolution in the time domain $\Leftrightarrow$ multiplication in the frequency domain

This can simplify evaluating convolutions, especially when cascaded.

This is how most simulation programs (e.g., Matlab) compute convolutions, using FFTs.

The Convolution Theorem: Given two signals $f_1(t)$ and $f_2(t)$ with Fourier transforms $F_1(j\omega)$ and $F_2(j\omega)$, then

$$(f_1 * f_2)(t) \Leftrightarrow F_1(j\omega)F_2(j\omega)$$

Proof: The Fourier transform of $(f_1 * f_2)(t)$ is

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\, d\tau \right) e^{-j\omega t}\, dt \\
&= \int_{-\infty}^{\infty} f_1(\tau) \left(\int_{-\infty}^{\infty} f_2(t-\tau) e^{-j\omega t}\, dt \right) d\tau.
\end{aligned}$$

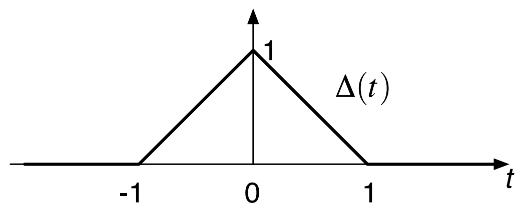
Using the shift theorem from page 16, this is

$$\begin{aligned}
&= \int_{-\infty}^{\infty} f_1(\tau) \left(e^{-j\omega\tau} F_2(j\omega) \right) d\tau \\
&= F_2(j\omega) \int_{-\infty}^{\infty} f_1(\tau) e^{-j\omega\tau}\, d\tau \\
&= F_2(j\omega) F_1(j\omega).
\end{aligned}$$

Examples of Convolution Theorem

Unit Triangle Signal $\Delta(t)$

$$\begin{cases} 1-|t| & \text{if } |t|<1 \\ 0 & \text{otherwise.} \end{cases}$$



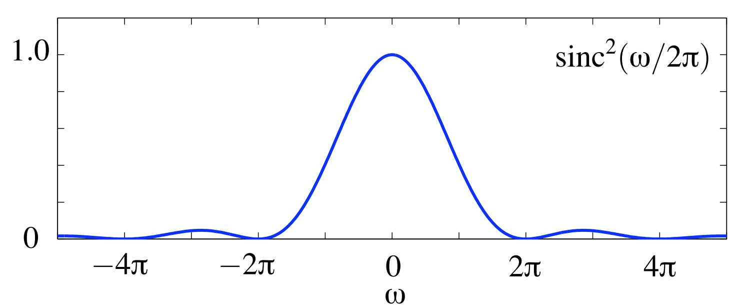
Easy to show $\Delta(t) = \mathrm{rect}(t) * \mathrm{rect}(t)$.

Since

$$\mathrm{rect}(t) \Leftrightarrow \mathrm{sinc}(\omega/2\pi)$$

then

$$\Delta(t) \Leftrightarrow \mathrm{sinc}^2(\omega/2\pi)$$



Transform of Unit Triangle Signal