EE 102B Spring 2013

Lecture 02

Review of the Fourier Transform

April 3, 2013

Course Information -- I

- Instructor
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- Teaching Assistants
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Course Information -- II

- Text: *Signal Processing First*, by McClellan, Schafer, and Yoder (SPF)
  - New chapter material posted on Class2Go website
- Also Recommended: *Signals and Systems*, by Oppenheim, Willsky, with Nawab (text for EE102A) (S&S)
- For Deeper Reading: *Discrete-Time Signal Processing*, 3E, Oppenheim and Schafer

Course Information – III

- Website on Class2Go
  - https://class2go.stanford.edu/EE102B/Spring2013
  - Log on with SUNet
  - Homework, labs, lecture slides, videos, grades, text supplements, SP-First toolbox
  - Link to piazza for discussion forum
- Homework and Lab Assignments:
  - Assigned Wednesday and due following Weds.
  - Up to two days late (Fri at 5pm) with penalty of 10% per day late – not accepted after Friday.
  - Lowest homework/lab will be dropped

Course Information – IV

- Course Grade
  - Homework: 15%
  - Labs 15%
  - Mid-term exam 30%
  - Final exam 40%
- Mid-term date: Friday, May 10, 2013 in class
- Final exam date: Tuesday June 11, 2013, 8:30-11:30 am room TBA

ASSIGNMENTS

- Reading for this Lecture:
  - SPF: Chapters 9 – 12; Review continuous-time Fourier series, Fourier transforms
  - S&S: Review book sections on continuous-time Fourier series, Fourier transforms
- HW#1 and Lab #1 posted later today

LECTURE OBJECTIVES

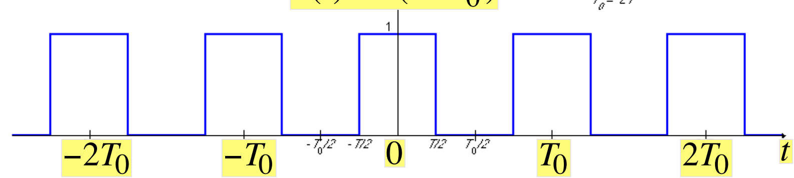
- Review
  - Frequency Response
  - Fourier Series
- Review of Fourier transform

$$\boxed{X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt}$$

  - Relation to Fourier Series
  - Examples of Fourier transform pairs
- Solving signals and systems problems

Fourier Series: Periodic *x(t)*

$x(t) = x(t + T_0)$



$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k t}$$

Fourier Synthesis

Fundamental Freq.

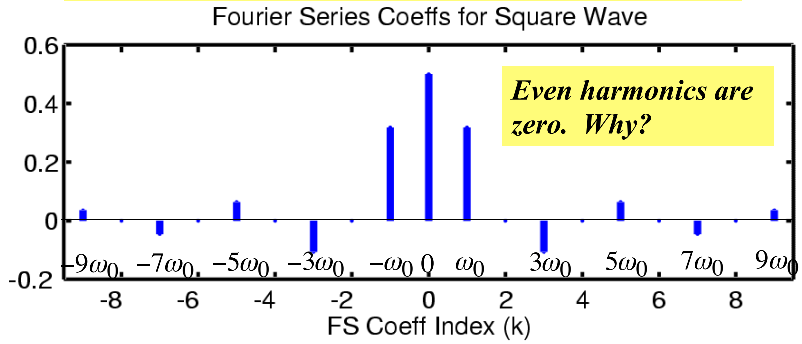
$\omega_0 = 2\pi / T_0 = 2\pi f_0$

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-j\omega_0 k t}\, dt$$

Fourier Analysis

Spectrum from Fourier Series

$$a_k = \frac{\sin(\pi k/2)}{\pi k} = \begin{cases} \neq 0 & k = 0, \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4, \ldots \end{cases}$$



Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Fourier Synthesis (Inverse Transform)

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Analysis (Forward Transform)

$$\text{Time-domain} \Leftrightarrow \text{Frequency-domain}$$

$$x(t) \Leftrightarrow X(j\omega)$$

When does *X(jω)* Exist?

- When is $\left|X(j\omega)\right| < \infty$?

$$\left|X(j\omega)\right| = \left| \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau \right| \leq \int_{-\infty}^{\infty} \left|x(\tau)\right| \left|e^{-j\omega\tau}\right| d\tau$$

$$\left|X(j\omega)\right| \leq \int_{-\infty}^{\infty} \left|x(\tau)\right| d\tau < \infty$$

- Thus a ***sufficient*** condition for the Fourier transform to exist is that the signal waveform is absolutely integrable.

Example 1: $x(t) = e^{-at}u(t)$

$$X(j\omega) = \int_0^{\infty} e^{-at} e^{-j\omega t} dt = \int_0^{\infty} e^{-(a+j\omega)t} dt$$

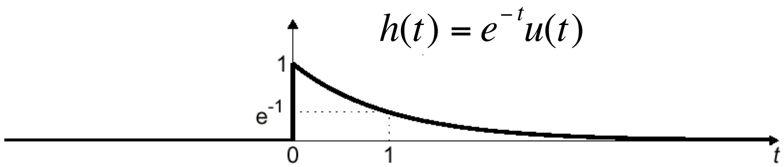
$a > 0$

$$X(j\omega) = -\frac{e^{-at} e^{-j\omega t}}{a + j\omega} \Bigg|_0^{\infty} = \frac{1}{a + j\omega}$$

$$x(t) = e^{-at}u(t) \Leftrightarrow X(j\omega) = \frac{1}{a + j\omega}$$

Frequency Response

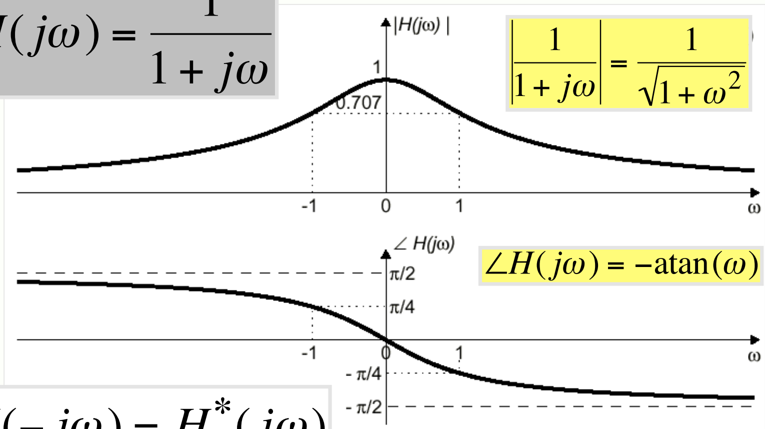
- Fourier Transform of h(t) is the Frequency Response



$$h(t) = e^{-t}u(t) \Leftrightarrow H(j\omega) = \frac{1}{1 + j\omega}$$

Magnitude and Phase Plots

$$H(j\omega) = \frac{1}{1 + j\omega}$$



$$\left|\frac{1}{1 + j\omega}\right| = \frac{1}{\sqrt{1 + \omega^2}}$$

$$\angle H(j\omega) = -\mathrm{atan}(\omega)$$

$$H(-j\omega) = H^*(j\omega)$$

Example 2:

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases}$$

$$X(j\omega) = \int_{-T/2}^{T/2} (1)e^{-j\omega t}dt = \int_{-T/2}^{T/2} e^{-j\omega t}dt$$

$$X(j\omega) = \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-T/2}^{T/2} = \frac{e^{-j\omega T/2} - e^{j\omega T/2}}{-j\omega}$$

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \Leftrightarrow X(j\omega) = \frac{\sin(\omega T/2)}{(\omega/2)}$$

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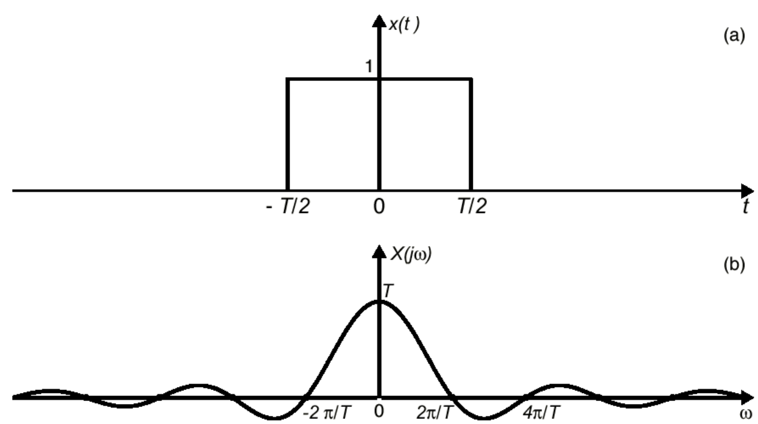

Example 3:

$$X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{j\omega t}d\omega$$

$$x(t) = \frac{1}{2\pi}\left.\frac{e^{j\omega t}}{jt}\right|_{-\omega_0}^{\omega_0} = \frac{1}{2\pi}\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{jt}$$

$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)} \Leftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$

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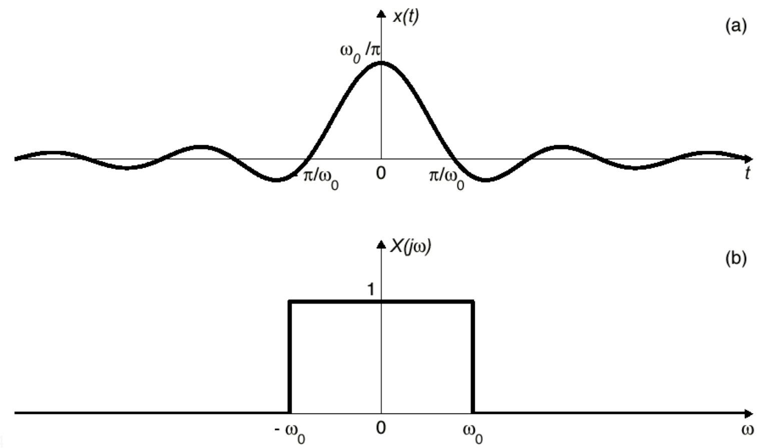


Uncertainty Principle

- Try to make x(t) shorter
  - Then X(jω) will get wider
  - Narrow pulses have wide bandwidth
- Try to make X(jω) narrower
  - Then x(t) will have longer duration
- **Cannot simultaneously reduce time duration and bandwidth**

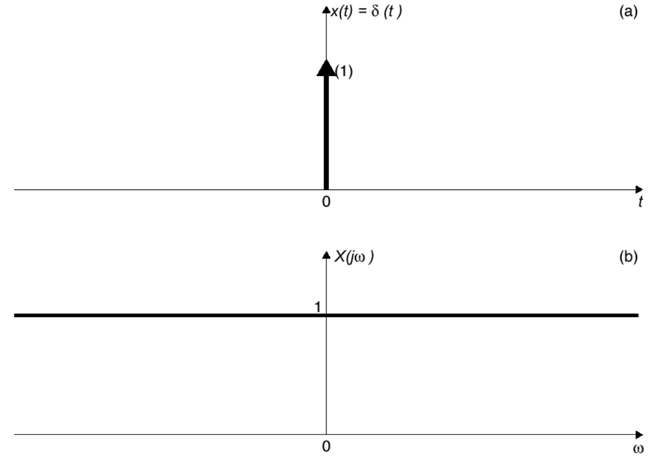
Example 4: $x(t) = \delta(t)$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt = 1$$

Shifting Property of the Impulse

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t - t_0)e^{-j\omega t}dt = e^{-j\omega t_0}$$

$x(t) = \delta(t) \Leftrightarrow X(j\omega) = 1$



Example 5: $X(j\omega) = 2\pi\delta(\omega - \omega_0)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}\,d\omega = e^{j\omega_0 t}$$

$$x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

$$x(t) = 1 \Leftrightarrow X(j\omega) = 2\pi\delta(\omega)$$

$$x(t) = \cos(\omega_0 t) \Leftrightarrow$$
$$X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

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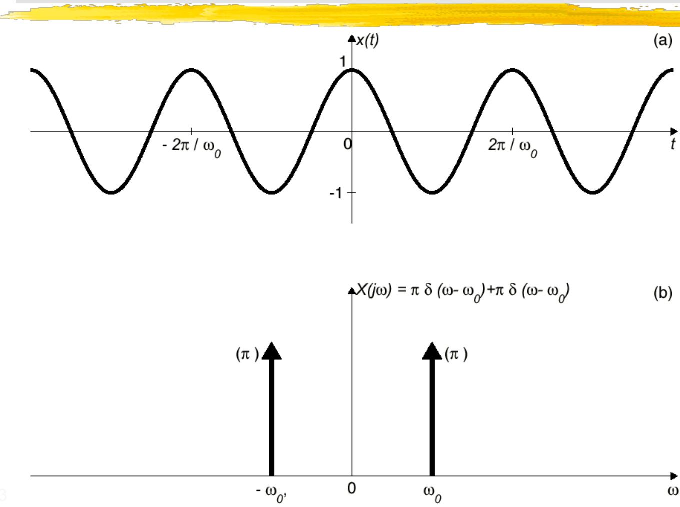


Strategy for using the FT

- Develop a set of known Fourier transform pairs.
- Develop a set of "theorems" or properties of the Fourier transform.
- Develop skill in formulating the problem in either the time-domain or the frequency-domain, *which ever leads to the simplest solution*.

Table of Fourier Transforms

$$x(t) = e^{-at}u(t) \Leftrightarrow X(j\omega) = \frac{1}{a + j\omega}$$

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \Leftrightarrow X(j\omega) = \frac{\sin(\omega T/2)}{(\omega/2)}$$

$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)} \Leftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$

$$x(t) = \delta(t - t_0) \Leftrightarrow X(j\omega) = e^{-j\omega t_0}$$

$$x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

Fourier Transform of a General Periodic Signal

- If $x(t)$ is periodic with period T_0,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}dt$$

$$\text{Therefore, since } e^{jk\omega_0 t} \Leftrightarrow 2\pi\delta(\omega - k\omega_0)$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

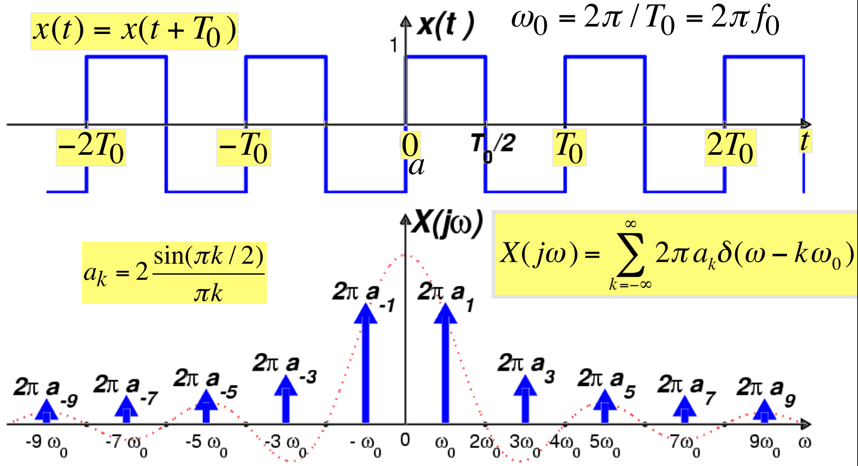
Square Wave Fourier Transform

$x(t) = x(t + T_0)$

$\omega_0 = 2\pi/T_0 = 2\pi f_0$

$a_k = 2\frac{\sin(\pi k/2)}{\pi k}$

$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$



FT of Impulse Train

- The periodic impulse train is

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0) = \sum_{n=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\omega_0 = 2\pi / T_0$$

$$a_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)e^{-j\omega_0 t}dt = \frac{1}{T_0} \quad \text{for all } k$$

$$\therefore \quad P(j\omega) = \left(\frac{2\pi}{T_0}\right) \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0)$$

Table of Easy FT Properties

Linearity Property

$$ax_1(t) + bx_2(t) \Leftrightarrow aX_1(j\omega) + bX_2(j\omega)$$

Delay Property

$$x(t - t_d) \Leftrightarrow e^{-j\omega t_d} X(j\omega)$$

Frequency Shifting

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

Scaling

$$x(at) \Leftrightarrow \frac{1}{|a|} X(j(\tfrac{\omega}{a}))$$

Significant FT Properties

$$x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

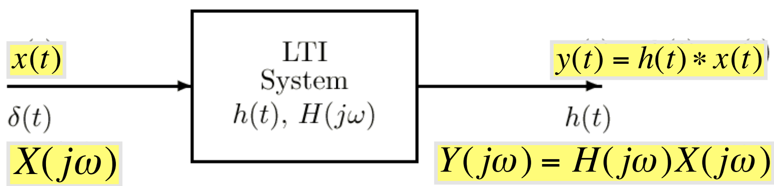
$$x(t)p(t) \Leftrightarrow \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

Differentiation Property

$$\frac{dx(t)}{dt} \Leftrightarrow (j\omega)X(j\omega)$$

Convolution Property

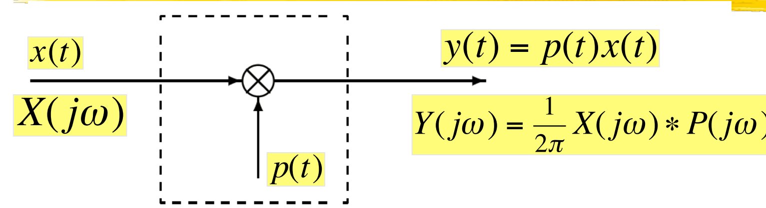


- Convolution in the time-domain

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

corresponds to MULTIPLICATION in the frequency-domain $Y(j\omega) = H(j\omega)X(j\omega)$

Signal Multiplier (Modulator)



- Multiplication in the time-domain corresponds to convolution in the frequency-domain.

$$Y(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta)P(j(\omega - \theta))d\theta$$

Frequency Shifting Property

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega-\omega_0))$$

$$\int_{-\infty}^{\infty} e^{j\omega_0 t}x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}dt$$

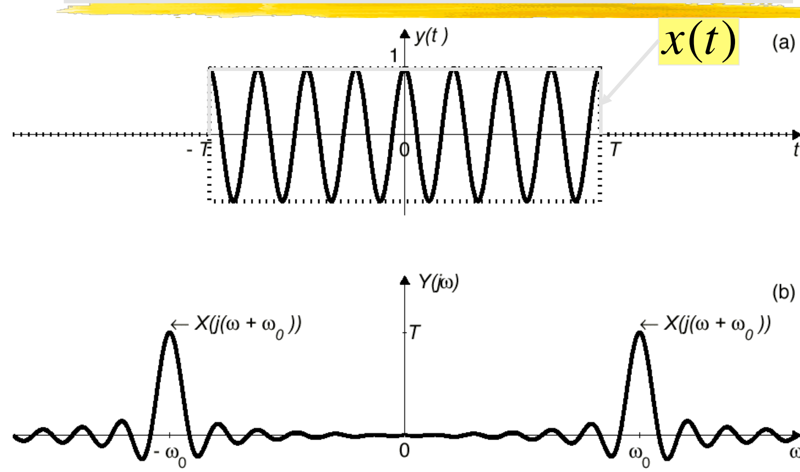
$$= X(j(\omega-\omega_0))$$

Also $\frac{1}{2\pi}X(j\omega)*2\pi\delta(\omega-\omega_0) = X(j(\omega-\omega_0))$

$$y(t) = \frac{\sin 7t}{\pi t}e^{j\omega_0 t} \Leftrightarrow Y(j\omega) = \begin{cases} 1 & \omega_0-7<\omega<\omega_0+7 \\ 0 & elsewhere \end{cases}$$

$$y(t) = x(t)\cos(\omega_0 t) \Leftrightarrow$$

$$Y(j\omega) = \frac{1}{2}X(j(\omega-\omega_0)) + \frac{1}{2}X(j(\omega+\omega_0))$$



Differentiation Property

$$\frac{dx(t)}{dt} = \frac{d}{dt}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega\right)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}(j\omega)X(j\omega)e^{j\omega t}d\omega$$

$$\frac{d}{dt}\left(e^{-at}u(t)\right) = -ae^{-at}u(t) + e^{-at}\delta(t)$$

$$= \delta(t) - ae^{-at}u(t)$$

$$\Leftrightarrow \frac{j\omega}{a+j\omega}$$