Stanford University
Spring 2011-2012

Signal Processing and Linear Systems I

Lecture 7: Continuous-Time Fourier Series

January 16, 2013

Introduction

Freqency domain representation of continuous time signals in general means a Fourier series or Fourier transform.

- Fourier series: time limited signals and periodic signals.
- Fourier transforms: any energy signal, many power signals.

Applications of Fourier transforms

- Decomposes signals into fundamental or "primitive" components
- Shortcuts to the computation of sums and integrals,
- Reveals hidden structure in systems or signals
- Sparser representation of many signals (speech, images) which is useful for compression.

Bottom line: Will see that

If $f(t)$ is a well-behaved signal which is either (1) periodic with period T_0, or (2) defined only on an interval $\{\tau \le t < \tau + T_0\}$ of length T_0 (finite-duration), then $f(t)$ can be written as a Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

where $\omega_0 = 2\pi/T_0$, and

$$D_n = \frac{1}{T_0}\int_{\tau}^{\tau+T_0} f(t)e^{-jn\omega_0 t}\,dt$$

for all integer n. The sequence $\{D_n\}$ constitutes the *Fourier coefficients* of $f(t)$.

If $f(t)$ is a well-behaved aperiodic signal, then $f(t)$ can be written as

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}\,d\omega,$$

where

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

The function $F(j\omega)$ is called the *Fourier transform* of $f(t)$.

In either case $f(t)$ is represented as a *weighted average* or *linear combination* of complex exponentials.

$F(j\omega)$ and $\{D_n\}$ are *frequency domain representations* of $f(t)$.

Fourier Series

Special case: Single sinusoid

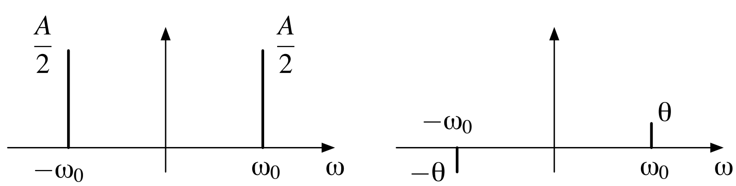
$$\begin{aligned} f(t) &= A\cos(\omega_0 t + \theta) = A\left(\frac{e^{j(\omega_0 t+\theta)} + e^{-j(\omega_0 t+\theta)}}{2}\right) \\ &= \frac{Ae^{j\theta}}{2}e^{j\omega_0 t} + \frac{Ae^{-j\theta}}{2}e^{-j\omega_0 t} \end{aligned}$$

Characterized by:

1. magnitude $A > 0$
2. phase θ
3. frequency $\omega_0 > 0$

These provide a complete description of signal of this form.

Frequency domain representation is the *spectrum* of the signal



Amplitude Spectrum

Phase Spectrum

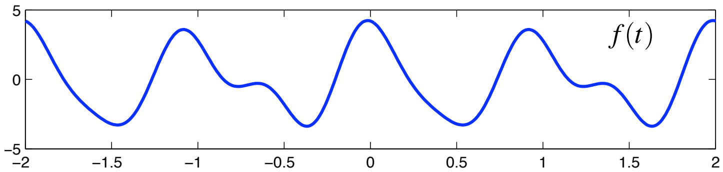
Given the parameters A, θ, and ω_0 we can reconstruct the sinusoidal signal:

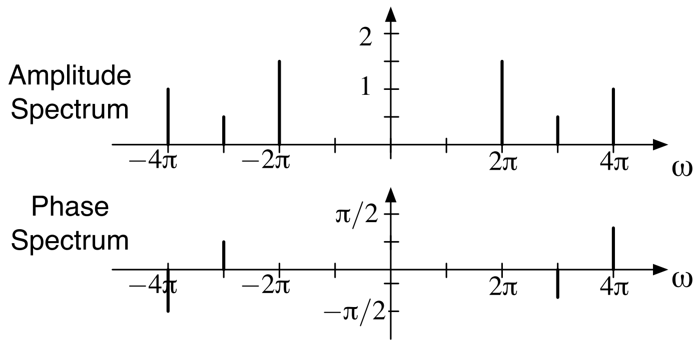
- As a cosine with the given amplitude, frequency, and phase.
- As a sum of two complex exponentials, each with 1/2 magnitude. One has phase θ and frequency f_0, the other has the corresponding negative frequency and negative phase.

Signal $\Leftrightarrow$ frequency domain representation

Add more frequencies in spectrum, more complicated shape

$$f(t) = 3\cos(2\pi t) + \cos(3\pi t - \pi/4) + 2\cos(4\pi t + \pi/3)$$





Signals as Sums of Complex Exponentials

Summing complex exponentials provides a variety of signal shapes.

How general is a representation of this form?

$$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

First note that $f(t)$ is periodic with period $T_0 = 2\pi/\omega_0$,

$$\begin{aligned} f(t+T_0) &= \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0(t+2\pi/\omega_0)} \\ &= \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} e^{j2\pi n} = f(t) \end{aligned}$$

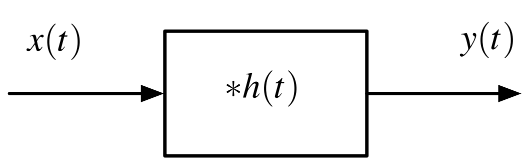
Will see that

- Almost any signal with period $T_0 = 2\pi/\omega_0$ can be expressed this way.
- Almost any signal defined on a time interval of length $T_0 = 2\pi/\omega_0$ can be expressed this way.

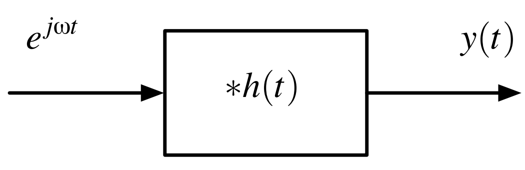
Complex Exponentials and LTI systems

For an LTI system with impulse response $h(t)$, output is the convolution of input and impulse response:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$$



If the input is a complex exponential $x(t) = e^{j\omega t}$



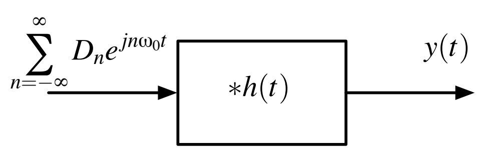
$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} h(\tau) e^{j\omega(t-\tau)}\, d\tau \\
&= e^{j\omega t} \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau}\, d\tau \triangleq H(j\omega) e^{j\omega t}
\end{aligned}
$$

$H(j\omega)$ is the continuous time *Fourier transform* of the time function h.

Note that

- Complex exponential in $\Rightarrow$
  same complex exponential $\times$ complex constant out,
- Complex exponential is *eigenfunction* of LTI system with *eigenvalue* $H(j\omega)$

If the input is a sum of complex exponentials, $x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$,



$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} h(\tau) \left(\sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0(t-\tau)} \right) d\tau \\
&= \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \int_{-\infty}^{\infty} h(\tau) e^{-jn\omega_0\tau}\, d\tau \\
&= \sum_{n=-\infty}^{\infty} \left(D_n H(jn\omega_0) \right) e^{jn\omega_0 t}
\end{aligned}
$$

Same frequencies as input, different complex multipliers.
Input Fourier series + Fourier transform of LTI $\Rightarrow$ Fourier series of output!

Representation of signals over an interval

Given a signal $f(t)$, when can be it written in the form

$$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \ ?$$

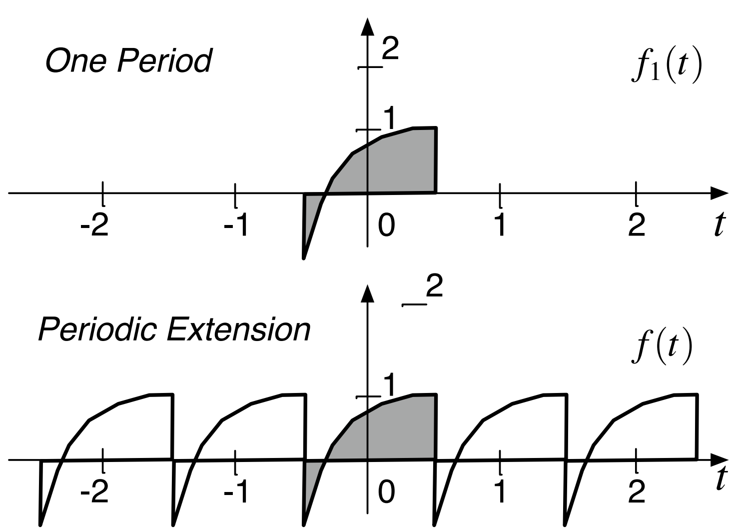
If we can do this, say f has a Fourier series. D_n are called the Fourier coefficients.

Consider two cases, which are equivalent:

- We only wish to represent $f(t)$ on a finite interval $\{t_0 \leq t < t_0 + T_0\}$
- $f(t)$ is periodic with period T_0

Equivalent because if we can represent $f(t)$ on a finite interval, then we

need only repeat $f(t)$ every T_0 seconds to get a periodic function with period T_0 and vice versa



Finding the Fourier Coefficients

How do we find the D_n?. First, assume that the signal $f(t)$ can be written as a Fourier series. Consider this integral for some fixed integer k:

$$\begin{aligned}\int_{t_0}^{t_0+T_0} f(t)e^{-jk\omega_0 t}\,dt &= \int_{t_0}^{t_0+T_0}\left(\sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}\right)e^{-jk\omega_0 t}\,dt \\ &= \sum_{n=-\infty}^{\infty} D_n \int_{t_0}^{t_0+T_0} e^{jn\omega_0 t}e^{-jk\omega_0 t}\,dt\end{aligned}$$

where $\omega_0 = 2\pi/T_0$. To evaluate the integral

$$\int_{t_0}^{t_0+T_0} e^{jn\omega_0 t}e^{-jk\omega_0 t}\,dt = \int_{t_0}^{t_0+T_0} e^{j\frac{2\pi(n-k)}{T_0}t}\,dt$$

there are two very different cases:

- If $n = k$, then

$$\int_{t_0}^{t_0+T_0} e^{j\frac{2\pi(n-k)}{T_0}t}\,dt = \int_{t_0}^{t_0+T_0} dt = T_0$$

- If $n \neq k$, let $K = n - k$. We are integrating a complex exponential with frequency K/T_0 and period T_0/K over K periods. Writing the complex exponential in terms of sines and cosines

$$\int_{t_0}^{t_0+T_0} e^{j\frac{2\pi K}{T_0}t}\,dt = \int_{t_0}^{t_0+T_0}\cos\left(\frac{2\pi K}{T_0}t\right)\,dt + j\int_{t_0}^{t_0+T_0}\sin\left(\frac{2\pi K}{T_0}t\right)\,dt$$

This is the integral of a sine or cosine over an integral number of periods, which is 0!

Thus,

$$\int_{t_0}^{t_0+T_0} e^{jn\omega_0 t} e^{-jk\omega_0 t}\, dt = \int_{t_0}^{t_0+T_0} e^{j\frac{2\pi(n-k)}{T_0}t}\, dt = \begin{cases} T_0 & \text{of } n = k \\ 0 & \text{otherwise} \end{cases}$$

Returning to the equation two pages back

$$\begin{aligned} \int_{t_0}^{t_0+T_0} f(t) e^{-jk\omega_0 t}\, dt &= \sum_{n=-\infty}^{\infty} D_n \int_{t_0}^{t_0+T_0} e^{jn\omega_0 t} e^{-jk\omega_0 t}\, dt \\ &= D_k T_0 \end{aligned}$$

Dividing by T_0, we get an explicit expression for the Fourier coefficients D_k,

$$D_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} f(t) e^{-jk\omega_0 t}\, dt$$

Orthogonal Signals

In general, given a family of signals $\phi_n(t)$ defined on an interval $[t_0, t_0 + T_0)$ we say the signals are *orthogonal* if

$$\int_{t_0}^{t_0+T_0} \phi_n(t) \phi_k^*(t)\, dt = 0 \qquad \text{for } n \neq k$$

The family of complex exponentials $\{e^{-j\frac{2\pi k}{T_0}t}\}$ form an orthogonal family on any interval $[t_0, t_0 + T_0)$.

There are lots of different families of orthogonal signals. We'll see a set of orthogonal binary signals when we talk about how cell telephones work.

Does the Fourier Series Converge to the Function?

Given a signal $f(t)$ and the coefficients D_n defined by

$$D_n = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} f(t) e^{-jn\omega_0 t}\, dt$$

does it follow that

$$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \, ??$$

To answer this consider the truncated Fourier series:

$$\hat{f}_N(t) = \sum_{n=-N}^{N} D_n e^{jn\omega_0 t}$$

and the resulting instantaneous approximation error

$$\epsilon_N(t) \triangleq \hat{f}_N(t) - f(t)$$

and the *integral square error* or *mean squared error*

$$\mathcal{E}_N \triangleq \int_{t_0}^{t_0+T_0} |\epsilon_N(t)|^2\, dt = \int_{t_0}^{t_0+T_0} |\hat{f}_N(t) - f(t)|^2\, dt$$

We say that the truncated Fourier series converges to the signal if the integral square error $\mathcal{E}_N \to 0$ as $N \to \infty$.

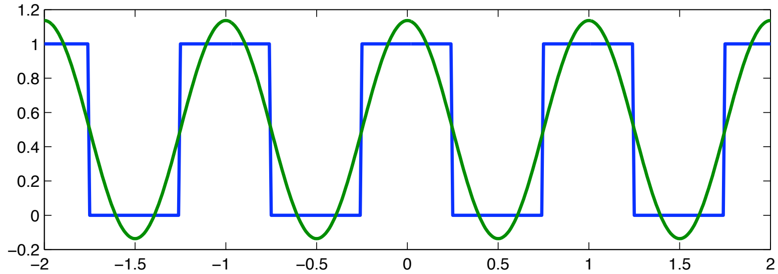
Often abbreviate this as

$$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t},$$

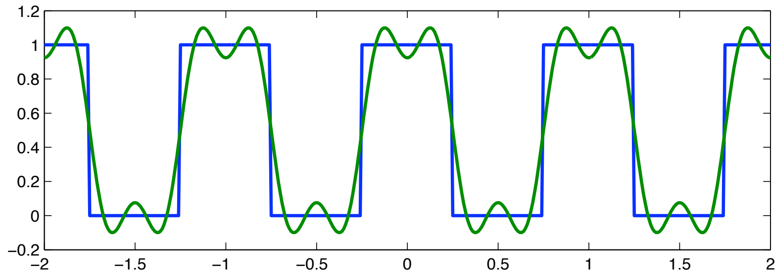
but this does not mean that this relation necessarily holds for all t, only in the integral average sense is it true.

Example Truncated Fourier Series approximation to a square wave:

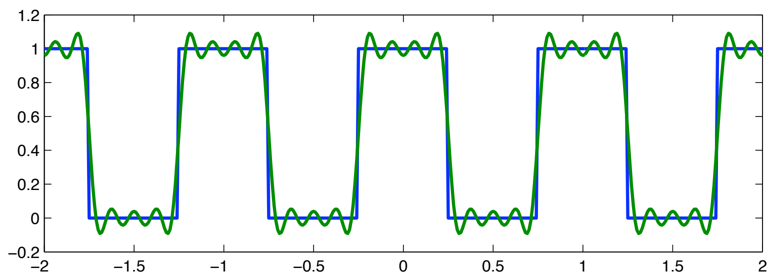
Constant + 2 Terms



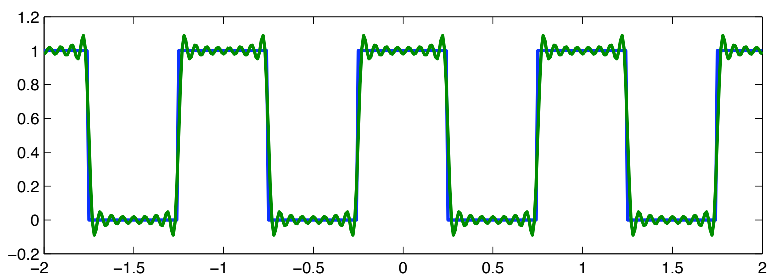
Constant + 4 Terms



Constant + 8 Terms



Constant + 16 Terms



In general this looks good, but there are several odd things,

- Converges to the midpoint of a discontinuity
- Oscillates at either side of discontinuity (Gibbs effect)
- Increasing number of terms compresses ringing, but doesn't reduce it's amplitude!

What conditions must we impose on f to guarantee this approximation holds?

The book describes the *Dirichlet conditions*: the signal has finite absolute integral, has at most a finite number of maxima and minima in the interval, and has at most a finite number of discontinuities. Essentially a "smoothness" requirement, the signal must be reasonably well behaved.

Parseval's Theorem for Continuous Time Fourier Series

The fact that the integral square error goes to zero allows us to relate the signal power in the time and frequency domains.

Expanding the integral square error:

$$\begin{aligned}
\mathcal{E}_N &= \int_{t_0}^{t_0+T_0} \left| \sum_{n=-N}^{N} D_n e^{jn\omega_0 t} - f(t) \right|^2 dt \\
&= \int_{t_0}^{t_0+T_0} \left(\sum_{n=-N}^{N} D_n e^{jn\omega_0 t} - f(t) \right) \left(\sum_{k=-N}^{N} D_k e^{jk\omega_0 t} - f(t) \right)^* dt \\
&= \int_{t_0}^{t_0+T_0} \left(\sum_{n=-N}^{N} D_n e^{jn\omega_0 t} - f(t) \right) \left(\sum_{k=-N}^{N} D_k^* e^{-jk\omega_0 t} - f^*(t) \right) dt
\end{aligned}$$

Multiplying all the terms out

$$
\begin{aligned}
\mathcal{E}_N &= \int_{t_0}^{t_0+T_0} \left[\left(\sum_{n=-N}^{N} D_n e^{jn\omega_0 t} \right) \left(\sum_{k=-N}^{N} D_k^* e^{-jk\omega_0 t} \right) + f(t) f^*(t) \right. \\
&\quad \left. - f(t) \left(\sum_{k=-N}^{N} D_k^* e^{-jk\omega_0 t} \right) - \left(\sum_{n=-N}^{N} D_n e^{jn\omega_0 t} \right) f^*(t) \right] dt \\
&= \sum_{n=-N}^{N} \sum_{k=-N}^{N} D_n D_k^* \int_{t_0}^{t_0+T_0} e^{jn\omega_0 t} e^{-jk\omega_0 t}\, dt + \int_{t_0}^{t_0+T_0} |f(t)|^2\, dt \\
&\quad - \sum_{k=-N}^{N} D_k^* \int_{t_0}^{t_0+T_0} f(t) e^{-jk\omega_0 t}\, dt - \sum_{n=-N}^{N} D_n \int_{t_0}^{t_0+T_0} f^*(t) e^{jn\omega_0 t}\, dt \\
&= T_0 \sum_{n=-N}^{N} |D_n|^2 - T_0 \sum_{k=-N}^{N} |D_k|^2 - T_0 \sum_{n=-N}^{N} |D_n|^2 \\
&\quad + \int_{t_0}^{t_0+T_0} |f(t)|^2\, dt
\end{aligned}
$$

Simplifying, we get

$$
\mathcal{E}_N = \int_{t_0}^{t_0+T_0} |f(t)|^2\, dt - T_0 \sum_{n=-N}^{N} |D_n|^2
$$

If $\mathcal{E}_N$ goes to zero, $f(t)$ has a Fourier series representation *if and only if*

$$
\lim_{N\to\infty} \sum_{n=-N}^{N} |D_n|^2 = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} |f(t)|^2\, dt
$$

i.e., the *energy* of the signal is the time domain is same as the sum of the energies of the frequency domain components.

Fundamental result of Fourier series:

- If a signal $f(t)$ is well behaved, then it has a Fourier series, and Parseval's theorem holds.

A similar derivation works for other collections of orthogonal signals.

- If the limiting integral error is 0, the collection of signals is *complete* and forms a *basis*.
- A family of orthogonal signals is complete if and only if Parseval's theorem holds.

In this class we focus on the basis of complex exponentials. Other bases are useful in signal processing: Haar functions, Gabor functions, wavelets, Walsh functions.

Other Forms of Fourier Series and Symmetry Properties

Suppose that $f(t)$ has a Fourier Series on $[t_0, t_0 + T_0)$

$$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{j\frac{2\pi n}{T_0}t},$$

where

$$D_n = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)e^{-jn\omega_0 t}\, dt$$

The zero frequency term corresponding to $n = 0$,

$$D_0 = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)\, dt$$

is often called the *dc coefficient* or the *time average mean* of the signal.

In general the Fourier coefficients D_n are complex numbers, so can express in real/imaginary or magnitude/phase form:

$$D_n = \Re(D_n) + j\Im(D_n) = |D_n|e^{j\angle D_n}$$

From Euler's formula,

$$\begin{aligned}
D_n &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)e^{-j\frac{2\pi n}{T_0}t}\,dt \\
&= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)\left[\cos\left(\frac{2\pi n}{T_0}t\right) - j\sin\left(\frac{2\pi n}{T_0}t\right)\right]\,dt \\
&= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)\cos\left(\frac{2\pi n}{T_0}t\right)\,dt - \frac{j}{T_0}\int_{t_0}^{t_0+T_0} f(t)\sin\left(\frac{2\pi n}{T_0}t\right)\,dt
\end{aligned}$$

If $f(t)$ is a real signal, then

$$\frac{1}{T_0}\int_{t_0}^{t_0+T} f(t)\cos\left(\frac{2\pi n}{T_0}t\right)\,dt$$

and

$$-\frac{1}{T_0}\int_{t_0}^{t_0+T} f(t)\sin\left(\frac{2\pi n}{T}t\right)\,dt$$

are both real. These are called the cosine and sine transforms of $f(t)$. These are the real and imaginary components of D_n,

$$\begin{aligned}
\Re(D_n) &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)\cos\left(\frac{2\pi n}{T}t\right)\,dt \\
\Im(D_n) &= -\frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)\sin\left(\frac{2\pi n}{T}t\right)\,dt
\end{aligned}$$

Since $\cos$ is an even and $\sin$ an odd function of its argument, Fourier coefficients for a real waveform have the following symmetry properties:

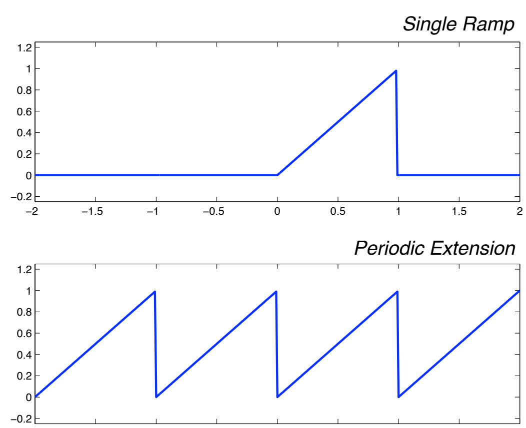
$$\begin{aligned}
\Re(D_{-n}) &= \Re(D_n) \\
\Im(D_{-n}) &= -\Im(D_n) \\
D_{-n} &= D_n^* \\
|D_{-n}| &= |D_n| \\
\angle D_{-n} &= -\angle D_n
\end{aligned}$$

The Fourier series as developed above is by far the most commonly used form.

Other forms use cosines with phases, or cosines and sines. See the book for examples.

Example: Ramp and Sawtooth Signal

Ramp: Describe signal in interval $[0,1)$: $f(t) = t$ for $t \in [0,1)$. This signal and its periodic extention are plotted below:



The periodic extension is the sawtooth signal $f(t) = t \bmod (1)$

Fourier series representation in interval $[0, 1)$

$$f(t) = \sum_{k=-\infty}^{\infty} D_n e^{j2\pi\frac{t}{T_0}n}$$

with $T_0 = 1$. The Fourier series coefficients are given by

$$\begin{aligned} D_n &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)e^{-j\frac{2\pi n}{T_0}t}dt \\ &= \int_0^1 te^{-j2\pi nt}dt \end{aligned}$$

For any real ω

$$\int_0^1 te^{-j\omega t}dt = \begin{cases} \frac{1}{2} & \text{if } \omega = 0 \\ \frac{je^{-j\omega}}{\omega} + \frac{e^{-j\omega}-1}{\omega^2} & \text{if } \omega \neq 0 \end{cases}.$$

Thus for $\omega = 2\pi n$, $D_0 = 1/2$ and $D_n = j/2\pi n$ for $k \neq 0$. Thus

$$t \bmod 1 = \frac{1}{2} + \sum_{n\neq 0} \frac{j}{2\pi n} e^{j2\pi tn}$$

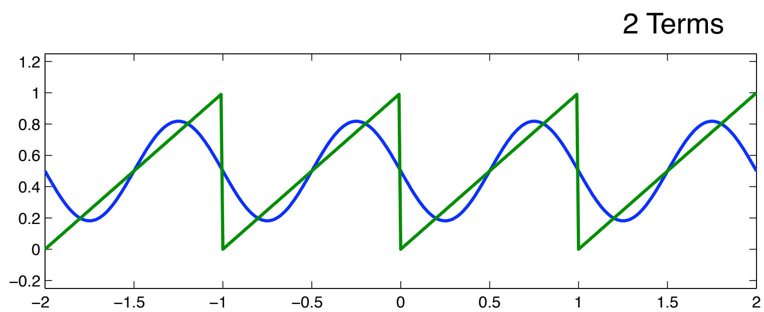
Get simpler form: rewrite sum as

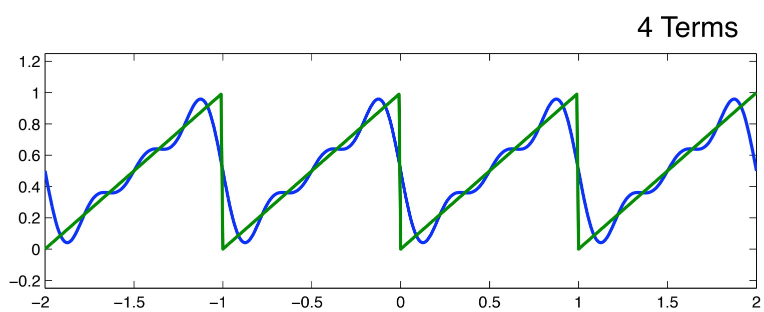
$$\sum_{n=1}^{\infty} \frac{j}{2\pi n}(e^{j2\pi tn} - e^{-j2\pi tn}) = -\sum_{n=1}^{\infty} \frac{\sin(2\pi nt)}{\pi n}$$

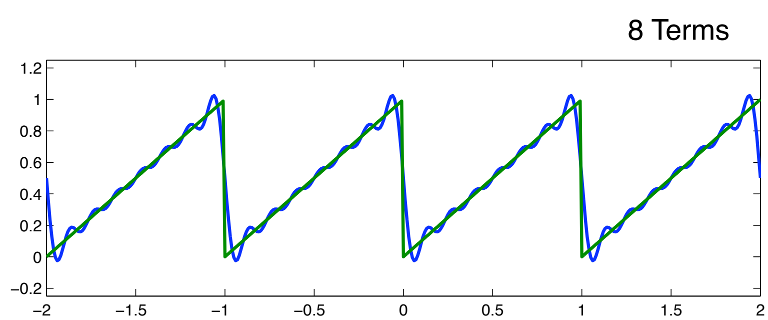
yielding the trigonometric Fourier series

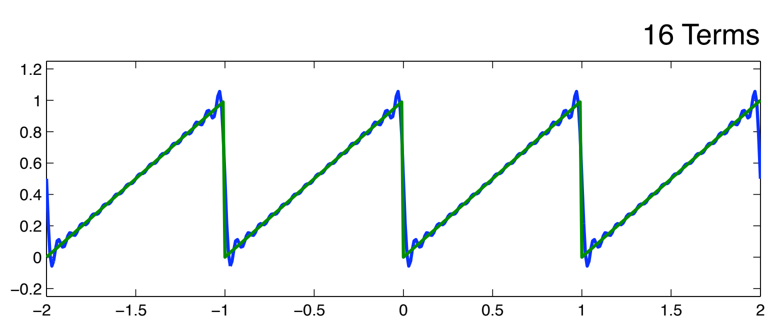
$$t \bmod 1 = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{\sin(2\pi nt)}{\pi n}.$$

Consider successive truncated Fourier series: N=2,4,8, and 16 terms:





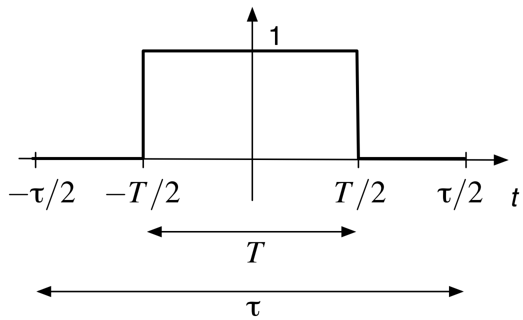




Square Pulse Example

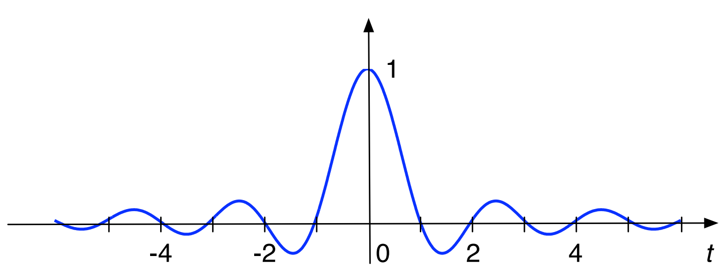
Consider the square pulse of width T defined on $[-\tau/2, \tau/2)$ $(\tau > T)$ by

$$f(t) = \begin{cases} 1 & |t| \leq \frac{T}{2} \\ 0 & \text{otherwise} \end{cases} \triangleq \mathrm{rect}\left(\frac{t}{T}\right) \text{ or } \Pi\left(\frac{t}{T}\right)$$



$$D_n = \frac{1}{\tau}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-j\frac{2\pi n}{\tau}t}\,dt = \left.\frac{e^{-j\frac{2\pi n}{\tau}t}}{-j2\pi n}\right|_{-\frac{T}{2}}^{\frac{T}{2}} = \frac{\sin(\pi\frac{T}{\tau}n)}{\pi n} = \frac{T}{\tau}\,\mathrm{sinc}\left(n\frac{T}{\tau}\right),$$

where $\mathrm{sinc}\, t \triangleq \frac{\sin \pi t}{\pi t}$ is purely real:



Note that sinc is often also defined as $\frac{\sin(t)}{t}$.

Fourier Series:

$$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{j2\pi \frac{t}{\tau} n} = \sum_{n=-\infty}^{\infty} \frac{T}{\tau} \operatorname{sinc}\left(n\frac{T}{\tau}\right) e^{j2\pi \frac{t}{\tau} n}$$

Since sinc is an even function, this becomes a cosine series

$$x(t) = \frac{T}{\tau}\left(1 + 2\sum_{n=1}^{\infty} \operatorname{sinc}\left(n\frac{T}{\tau}\right)\cos\left(2\pi\frac{t}{\tau}n\right)\right)$$

This was plotted on pages 22-23 for 2,4, 8, and 16 terms, for the case where $T = 0.5$, and $\tau = 1$.