Stanford University
Winter 2012-2013

Signal Processing and Linear Systems I

Lecture 11: Frequency Response of LTI Systems

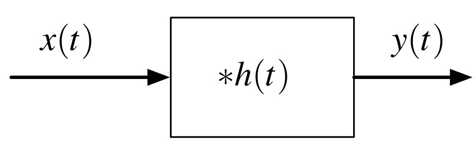
February 9, 2013

Linear Time-Invariant Systems, Revisited

- A linear time-invariant system is completely characterized by its *impulse response* $h(t)$.

- For a linear system with an input signal $x(t)$, the output is given by the convolution

$$y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau$$



- The Fourier transform of the convolution is

$$Y(j\omega) = H(j\omega)X(j\omega)$$

where $X(j\omega)$ is the input spectrum, $Y(j\omega)$ is the output spectrum, and $H(j\omega)$ is the Fourier transform of the impulse response $h(t)$.

- $H(j\omega)$ is called the *frequency response* or *transfer function* of the system. Each frequency in the input spectrum $X(j\omega)$ is
  - Scaled by the *system amplitude response* $|H(j\omega)|$,

$$|Y(j\omega)| = |H(j\omega)||X(j\omega)|$$

  - Phase shifted by the *system phase response* $\angle H(j\omega)$,

$$\angle Y(j\omega) = \angle H(j\omega) + \angle X(j\omega)$$

to produce the output spectrum $Y(j\omega)$.

- If the input is to a system is a complex exponential $e^{j\omega_0 t}$, the input spectrum is

$$\begin{aligned} X(j\omega) &= \mathcal{F}\left[e^{j\omega_0 t}\right] \\ &= 2\pi\delta(\omega - \omega_0). \end{aligned}$$

The output spectrum is

$$\begin{aligned} Y(j\omega) &= H(j\omega)(2\pi\delta(\omega - \omega_0)) \\ &= H(j\omega_0)(2\pi\delta(\omega - \omega_0)). \end{aligned}$$

The ouput signal is

$$\begin{aligned} y(t) &= \mathcal{F}^{-1}\left[Y(j\omega)\right] \\ &= \mathcal{F}^{-1}\left[H(j\omega_0)(2\pi\delta(\omega - \omega_0))\right] \end{aligned}$$

$$\begin{aligned} &= H(j\omega_0)e^{j\omega_0 t} \\ &= |H(j\omega_0)|e^{j(\omega_0 t + \angle H(j\omega_0))} \end{aligned}$$

A sinusoidal input $e^{j\omega_0 t}$ to an LTI system produces a sinusoidal output at the

– Same frequency,
– Scaled in amplitude, and
– Phase shifted.

This corresponds to multiplication by a complex number $H(j\omega_0)$.

Frequency Response Example

An input signal

$$x(t) = 2\cos(t) + 3\cos(3t/2) + \cos(2t)$$

is applied to a system with an impulse response $h(t)$

$$h(t) = \frac{2}{\pi}\mathrm{sinc}^2(t/\pi)$$

Find the output signal $(x * h)(t)$.

First, the frequency response or transfer function of the system is

$$\mathcal{F}\left[\frac{2}{\pi}\mathrm{sinc}^2(t/\pi)\right] = \left(\frac{2}{\pi}\right)\pi\Delta(\pi\omega/2\pi) = 2\Delta(\omega/2)$$

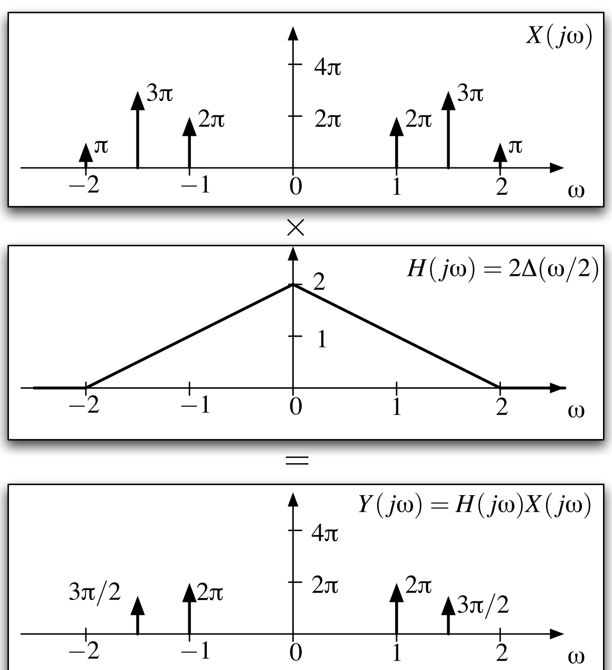
so

$$H(j\omega) = 2\Delta(\omega/2)$$

The input spectrum is

$$\begin{aligned} X(j\omega) &= 2\pi\left[\delta(\omega-1)+\delta(\omega+1)\right] + 3\pi\left[\delta(\omega-3/2)+\delta(\omega+3/2)\right] \\ &\quad +\pi\left[\delta(\omega-2)+\delta(\omega+2)\right] \end{aligned}$$

The output spectrum is the product of the input spectrum, and the transfer function, as shown on the next page:



The output signal spectrum is then

$$Y(j\omega) = 2\pi\left[\delta(\omega-1)+\delta(\omega+1)\right] + \frac{3\pi}{2}\left[\delta(\omega-3/2)+\delta(\omega+3/2)\right]$$

and the output signal is

$$y(t) = 2\cos(t) + \frac{3}{2}\cos(3t/2).$$

Another Example

A signal

$$x(t) = e^{-t}u(t)$$

is applied to a zero-state system with an impulse response

$$h(t) = 2e^{-2t}u(t)$$

What is the spectrum of the output $Y(j\omega) = H(j\omega)X(j\omega)$, and the output signal $y(t)$?

The Fourier transform of the input is

$$X(j\omega) = \mathcal{F}\left[e^{-t}u(t)\right] = \frac{1}{1+j\omega}.$$

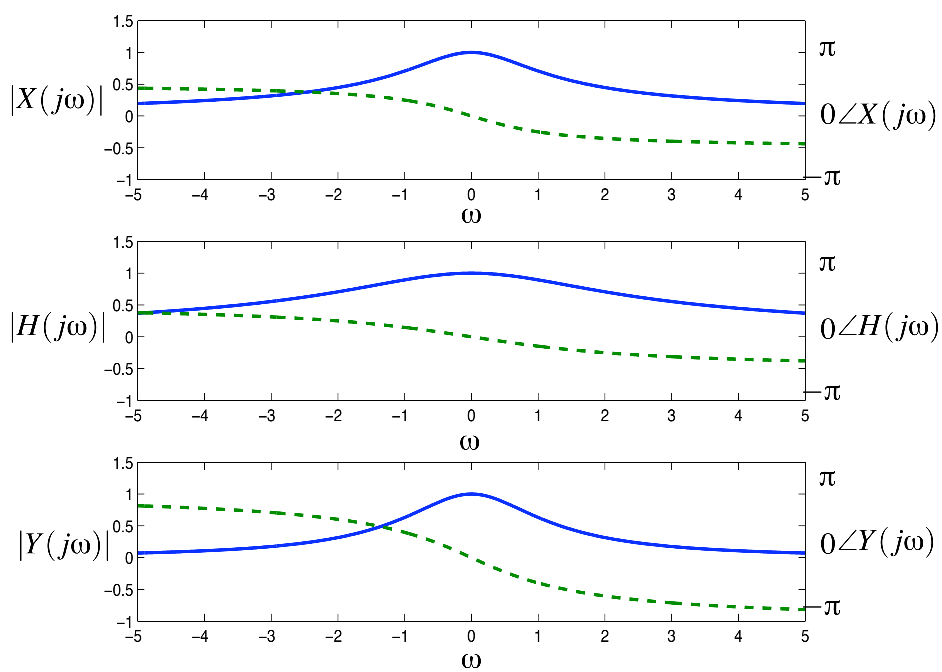
The transfer function or frequency response is the Fourier transform of the impulse response

$$H(j\omega) = \mathcal{F}\left[2e^{-2t}u(t)\right] = \frac{2}{2+j\omega}.$$

The Fourier transform of the output is then

$$Y(j\omega) = H(j\omega)X(j\omega) = \frac{2}{(1+j\omega)(2+j\omega)}$$

This is illustrated on the next page:



Note that the magnitude profiles are multiplied ($Y(j\omega)$ is narrower than either $X(j\omega)$ or $H(j\omega)$) and that the phase profiles add.

We can find the time signal $y(t)$ by noting that

$$Y(j\omega) = \frac{2}{(1+j\omega)(2+j\omega)} = \frac{2}{1+j\omega} - \frac{2}{2+j\omega}$$

which you can check. Then

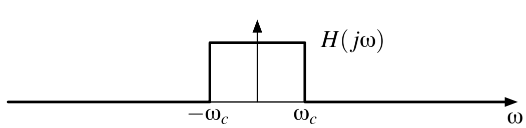
$$y(t) = 2(e^{-t} - e^{-2t})u(t).$$

The last two steps we'll cover in the section on Laplace transforms, later in the quarter.

Ideal Filters

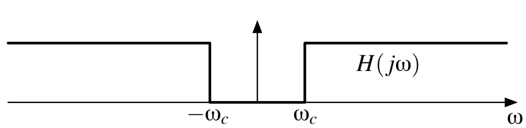
There are several basic filter types we will encounter. Some of these are:

Ideal Lowpass:



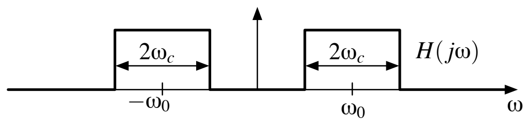
This suppresses all frequencies about a cutoff frequency ω_c. It will be important for reconstructing a continuous waveform from its samples.

Ideal Highpass:



This suppresses all frequencies below ω_c.

Ideal Bandpass

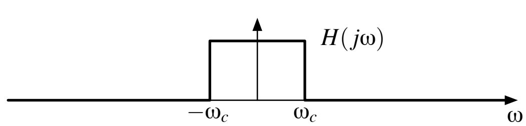


This passes a band of frequencies. The bands are of width $2\omega_c$, and are centered at $\pm\omega_0$. This is useful in communications, were we want to select for a specific frequency range.

Each of these can be implemented as a convolution. The impulse response of the filter is the inverse Fourier transform of the frequency response.

Ideal Lowpass

The ideal lowpass



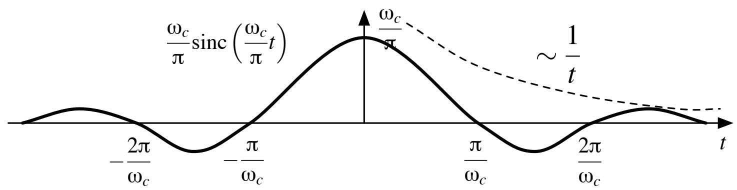
can be written as

$$H(j\omega) = \text{rect}(\omega/(2\omega_c))$$

The impulse response is then

$$\begin{aligned}\mathcal{F}^{-1}\left\{\text{rect}(\omega/(2\omega_c))\right\} &= \mathcal{F}^{-1}\left\{\left(\frac{2\omega_c}{2\pi}\right)\left(\frac{2\pi}{2\omega_c}\right)\text{rect}\left(\frac{2\pi}{2\omega_c}\left(\frac{\omega}{2\pi}\right)\right)\right\} \\ &= \left(\frac{\omega_c}{\pi}\right)\text{sinc}\left(\frac{\omega_c}{\pi}t\right)\end{aligned}$$

This is plotted below:

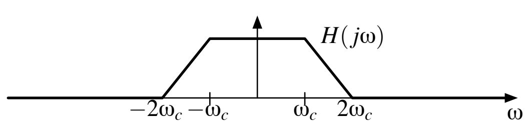


This has several practical problems. The impulse response is

- Non-causal: we'll address this in a few slides.
- Infinite in duration: must be truncated somewhere.
- Decays very slowly, as $1/t$.

Since we are going to have to truncate the impulse response, we'd like it to decay as fast as possible, so that we minimize its length. We can do this by making the response smoother.

For example, this more practical filter



has the same passband, but allows for a transition band of width ω_c. This can be written as a convolution of two rect's,

$$H(j\omega) = \text{rect}(\omega/3\omega_c) * \text{rect}(\omega/\omega_c)$$

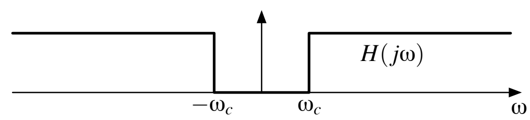
(Convince yourself this is true!) The impulse response is then

$$h(t) = \left(\frac{3\omega_c}{2\pi}\text{sinc}\left(\frac{3\omega_c}{2\pi}t\right)\right)\left(\frac{\omega_c}{2\pi}\text{sinc}\left(\frac{\omega_c}{2\pi}t\right)\right)$$

This will decay as $1/t^2$, which is much faster. Smoother transition bands will result in faster decay, and even shorter impulse responses for a given truncation error.

Ideal Highpass Filter

The ideal highpass



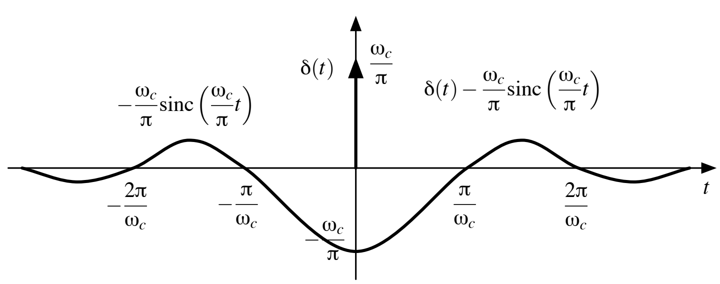
can be written as 1 minus the ideal lowpass

$$H(j\omega) = 1 - \text{rect}(\omega/(2\omega_c))$$

The impulse response is then

$$h(t) = \delta(t) - \left(\frac{\omega_c}{\pi}\right) \text{sinc}\left(\frac{\omega_c}{\pi} t\right)$$

This is of course also not practical. It is plotted below:

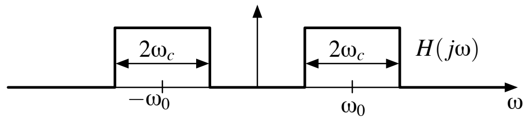


The comments about the ideal lowpass apply.

In addition, the impulse must be approximated. If we do this by limiting the range of frequencies, we get the next filter.

Ideal Bandpass

The ideal bandpass



can be considered an ideal lowpass filter that has been modulated to ω_0.

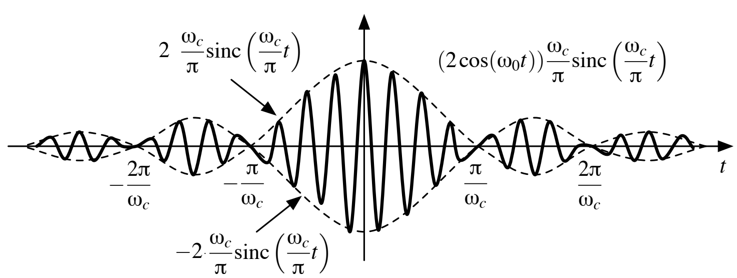
The frequency response can be written

$$H(j\omega) = \text{rect}\left(\frac{\omega + \omega_0}{2\omega_c}\right) + \text{rect}\left(\frac{\omega - \omega_0}{2\omega_c}\right)$$

By the modulation theorem, the impulse response is

$$h(t) = (2\cos(\omega_0 t))\left(\frac{\omega_c}{\pi}\right)\text{sinc}\left(\frac{\omega_c}{\pi}t\right)$$

The impulse response of a bandpass filter is simply a lowpass filter that has been multiplied by a cosine!



Again, the bandpass filter would be easier to implement if it were based on a more practical lowpass filter.

Distortionless LTI Systems

The frequency response of a system includes

- Amplitude scaling by $|H(j\omega)|$
- Phase shift by $\angle H(j\omega)$

Often we would like a system to pass a signal without distortion

$$y(t) = Kx(t - t_d)$$

where K is some constant gain, and t_d is a constant delay. We get the same signal out, amplified and delayed.

The Fourier transform of such a system is

$$Y(j\omega) = Ke^{-j\omega t_d}X(j\omega)$$

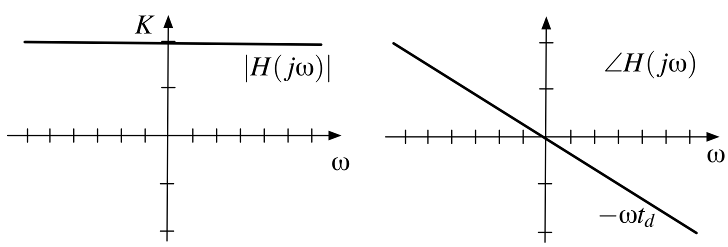
so the frequency response is

$$H(j\omega) = Ke^{-j\omega t_d}$$

or

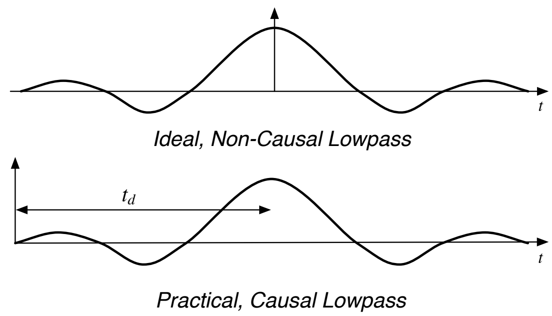
$$\begin{aligned} |H(j\omega)| &= K \\ \angle H(j\omega) &= -\omega t_d \end{aligned}$$

Hence, a distortionless system has a negative, linear phase as a function of frequency.



Practical Implementation of Non-Causal Filters

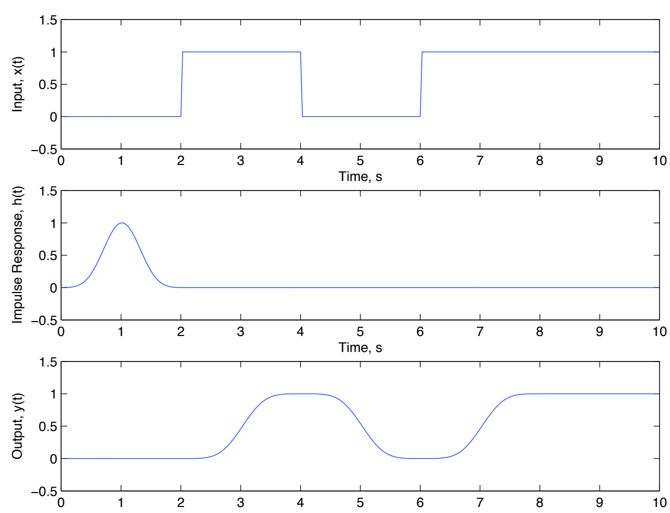
The ideal filters are not causal, so they can't be implemented. In practice they must be truncated and delayed. An ideal lowpass, and a truncated and delayed lowpass are plotted below:



The result is a filter with the ideal frequency response multiplied by a negative linear phase. This approximates a distortionless system over the passband of the filter.

Amplitude and Phase Distortion

So far we have mostly been concerned with the system amplitude response, as in the communication example (Lecture 6, page 11):



The impulse response of this system is a shifted real, symmetric function, so the phase is linear. Only amplitude scaling of the input spectrum is occurring here.

The frequency response of a system can affect both the amplitude and phase of the output signal.

Both of these are important, depending on the application.

Phase Distortion and Group Delay

A distortionless system has a negative linear phase

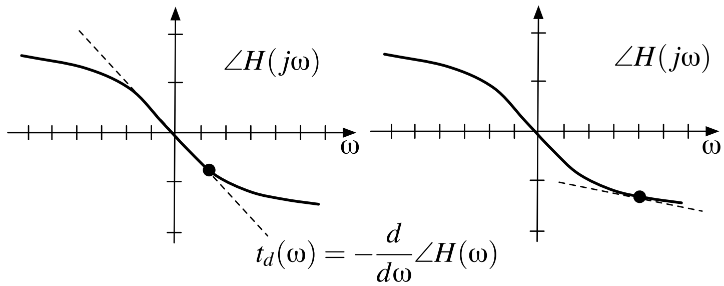
$$\angle H(j\omega) = -\omega t_d$$

If the phase is not linear, the time delay is no longer a constant, and is a function of frequency.

The time delay is the negative slope of $\angle H(j\omega)$,

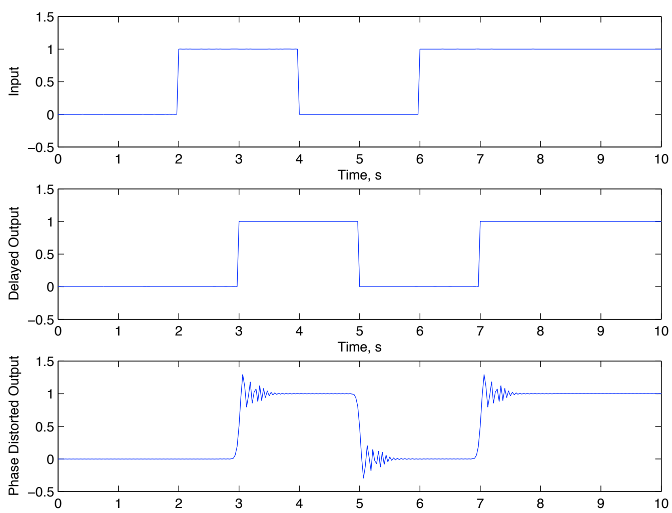
$$t_d(\omega) = -\frac{d}{d\omega}\angle H(j\omega).$$

This frequency dependent delay is the *group delay*.



Here high frequencies have a smaller delay.

The effect of frequency dependent delays are shown below. The same input (top) is applied to a linear phase system (middle), and system with linear phase plus a small third order term (lower).



High and low frequencies aren't aligned, and the transition gets broader.

Importance of amplitude and phase distortion depends on application.

For audio or speech:

- Amplitude distortion is very is important.
- Humans are relatively insensitive to phase distortion.

For images or video:

- Amplitude distortion is relatively unimportant, as long as it is slowly varying.
- Phase distortion is very important. Small amounts of non-linear phase result in very blurry looking images.

Example from MRI

Small additional quadratic phase produces substantial blurring. These have zero, 1/2, and 1 cycle of additional quadratic phase. This is less than 1% of the linear phase.