# STANFORD UNIVERSITY
# EE 102B Spring-2013

**Lecture 05**

**FIR Filtering-Introduction**

**April 10, 2013**

## ASSIGNMENTS

- Reading for this Lecture:
  - SPF: Chapter 5
  - S&S:
- HW#01 and Lab #01 are due at 5pm in Packard 263.
- HW#02 and Lab #02 will be posted later today. They are due by 5pm, April 17, in Packard 263



## Office Hours for Course Staff – Come see us.

- Ron Schafer: Mon. 2:00-3:00 pm and Weds. 4:00-5:00 pm in Packard 211
- Dookun Park: Tues. 1:30 ~ 3:30 pm in Packard 107
- Ruo Yu Gu (Roy): Mon. 4:00 ~ 6:00 pm in Packard 106
- https://class2go.stanford.edu/EE102B/Spring2013/pages/staff



## LECTURE OBJECTIVES

- REVIEW THE SPECTRUM CONCEPT
- INTRODUCE FILTERING CONCEPT
  - Weighted Average
  - Running Average
- FINITE IMPULSE RESPONSE FILTERS
  - **FIR** Filters
- Discrete convolution

## Bandlimited Continuous-Time Signals

- Bandlimited continuous-time signal

$$x(t) = \sum_{k=-N}^{N} a_k e^{j\omega_k t}$$

- Periodic signals: $\omega_k = k\omega_0 = (2\pi / T_0)k$

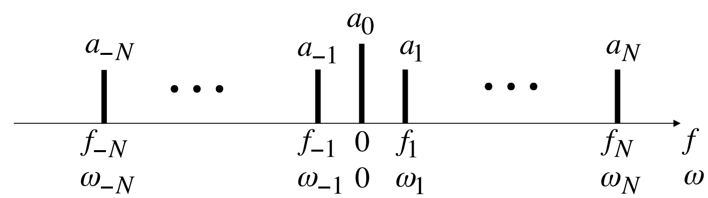
$$x(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t} \Leftrightarrow a_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-j\omega_0 t} dt$$



## Spectrum of a Bandlimited Continuous-Time Signal

- Spectrum: the collection of information required to construct $x(t) = \sum_{k=-N}^{N} a_k e^{j\omega_k t}$

$$\text{spectrum:} \quad \{a_k, f_k\} \text{ or } \{a_k, \omega_k\}$$






## Bandlimited Discrete-Time Signals

- Bandlimited discrete-time signal

$$x[n] = x(nT_s) = \sum_{k=-N}^{N} a_k e^{j\omega_k n T_s}$$

- Normalized frequency: $\hat{\omega} = \omega T_s$

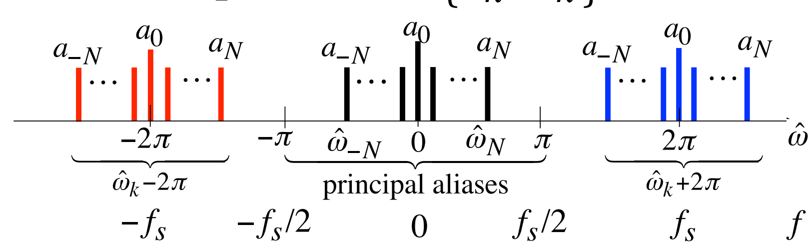
$$x[n] = \sum_{k=-N}^{N} a_k e^{j\hat{\omega}_k n}$$



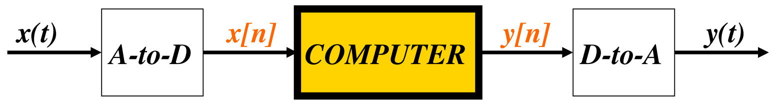
## Spectrum of a Bandlimited Discrete-Time Signal

- Spectrum: the collection of information required to construct $x[n] = \sum_{k=-N}^{N} a_k e^{j\hat{\omega}_k n}$

$$\text{spectrum:} \quad \{a_k, \hat{\omega}_k\}$$



# DIGITAL FILTERING




- Characterized SIGNALS (Fourier series)
- Converted to DIGITAL (sampling)
- Today: How to PROCESS them (DSP)?
- CONCENTRATE on the COMPUTER
  - ALGORITHMS, SOFTWARE (MATLAB) and HARDWARE (DSP chips, VLSI)



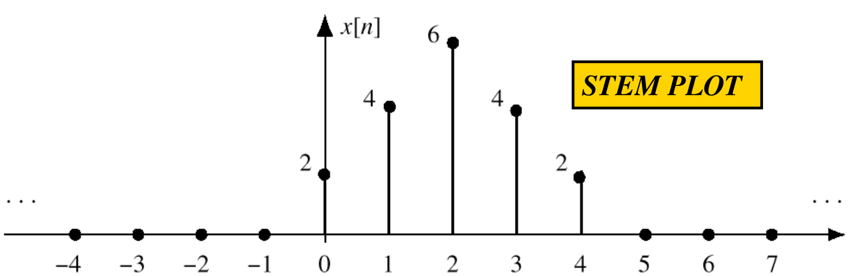
# DISCRETE-TIME SYSTEM


- OPERATE on x[n] to get y[n]
- WANT a **GENERAL** CLASS of SYSTEMS
  - **ANALYZE** the SYSTEM
    - TOOLS: TIME-DOMAIN & FREQUENCY-DOMAIN
  - **SYNTHESIZE** the SYSTEM



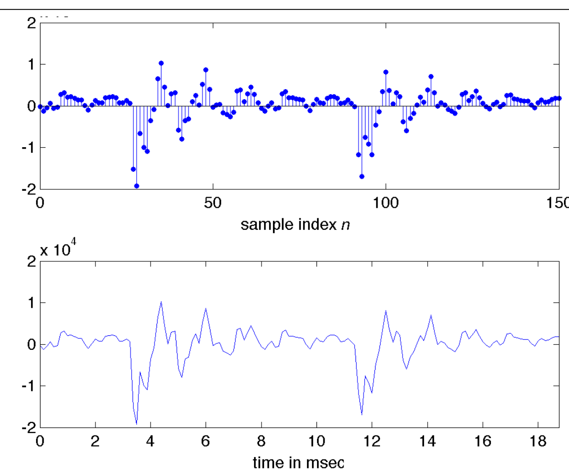
# DISCRETE-TIME SIGNAL

- x[n] is a LIST of NUMBERS
  - INDEXED by "$n$"
  - Just numbers $\{\ldots,0,2,4,6,4,2,0,\ldots\}$






# Sampled Speech Signal




$f_s = 8000$ sample/s

Note the straight-line connections between points.

# Some Discrete-Time Signals with Formulas

- Discrete-time signals are just sequences of numbers.
- Unit step
- Real exponential
- Complex exponential
- Impulse
- Can a discrete-time signal be discontinuous?



# D-T SYSTEM EXAMPLES


$$x[n] \mapsto y[n]$$

- SYSTEMS ARE OPERATORS:
  - POINTWISE OPERATORS
    - SQUARING: $y[n] = (x[n])^2$
    - MODULATION: $y[n]=x[n]\cos(\omega_c n)$
  - RUNNING AVERAGE
    - RULE: "the output at time n is the average of three consecutive input values"



# 3-PT AVERAGE SYSTEM

- ADD 3 CONSECUTIVE NUMBERS
  - Do this for each "$n$"

the following input–output equation

**Make a TABLE**

$$y[n] = \tfrac{1}{3}(x[n] + x[n+1] + x[n+2])$$

| $n$ | $n < -2$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ | $n > 5$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $x[n]$ | 0 | 0 | 0 | 2 | 4 | **6** | **4** | **2** | 0 | 0 |
| $y[n]$ | 0 | $\frac{2}{3}$ | 2 | 4 | $\frac{14}{3}$ | **4** | 2 | $\frac{2}{3}$ | 0 | 0 |

**n=0** $y[0] = \frac{1}{3}(x[0] + x[1] + x[2])$

**n=1** $y[1] = \frac{1}{3}(x[1] + x[2] + x[3])$

**INPUT SIGNAL**

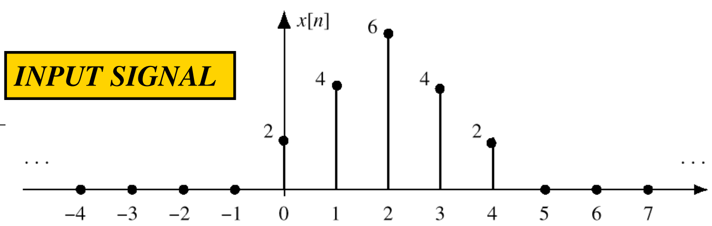



**Figure 5.2** Finite-length input signal, $x[n]$.

$$y[n] = \tfrac{1}{3}(x[n] + x[n+1] + x[n+2])$$

**OUTPUT SIGNAL**

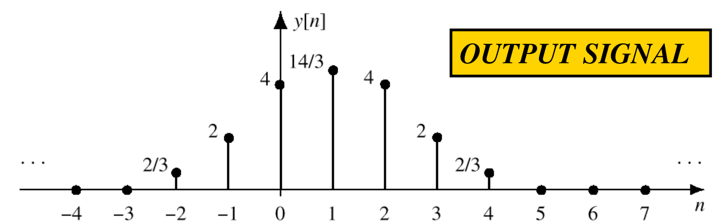



**Figure 5.3** Output of running average, $y[n]$.

# PAST, PRESENT, FUTURE



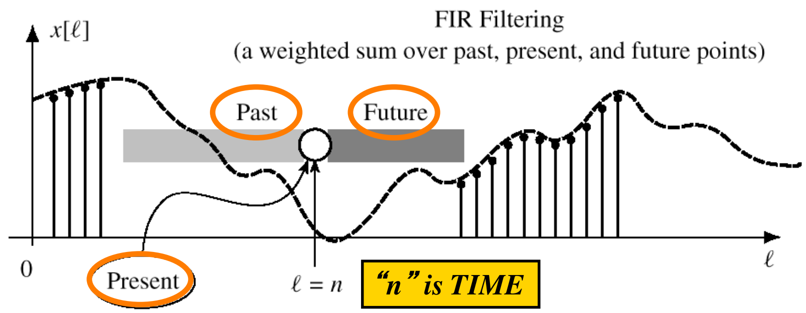



**Figure 5.4** The running-average filter calculation at time index $n$ uses values within a sliding window (shaded). Dark shading indicates the future ($\ell > n$); light shading, the past ($\ell < n$).

# ANOTHER 3-pt AVERAGER

- Uses "PAST" VALUES of x[n]
  - IMPORTANT IF "n" represents REAL TIME
    - WHEN x[n] & y[n] ARE STREAMS

$$y[n] = \tfrac{1}{3}(x[n] + x[n-1] + x[n-2])$$

| $n$ | $n < -2$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | $n > 7$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $x[n]$ | 0 | 0 | 0 | 2 | 4 | **6** | **4** | **2** | 0 | 0 | 0 | 0 |
| $y[n]$ | 0 | 0 | 0 | $\frac{2}{3}$ | 2 | 4 | $\frac{14}{3}$ | **4** | 2 | $\frac{2}{3}$ | 0 | 0 |



# GENERAL CAUSAL FIR FILTER

- FILTER COEFFICIENTS $\{b_k\}$
  - DEFINE THE FILTER $y[n] = \sum_{k=0}^{M} b_k x[n-k]$
  - For example, $b_k = \{3, -1, 2, 1\}$

$$y[n] = \sum_{k=0}^{3} b_k x[n-k] = 3x[n] - x[n-1] + 2x[n-2] + x[n-3]$$



# GENERAL CAUSAL FIR FILTER

- FILTER COEFFICIENTS $\{b_k\}$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

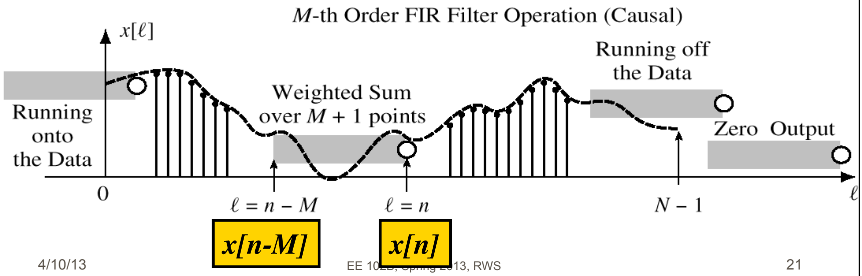
- FILTER ORDER is M
- FILTER "LENGTH" is L = M+1
  - NUMBER of FILTER COEFFS is L

# GENERAL CAUSAL FIR FILTER

- SLIDE a WINDOW across x[n]

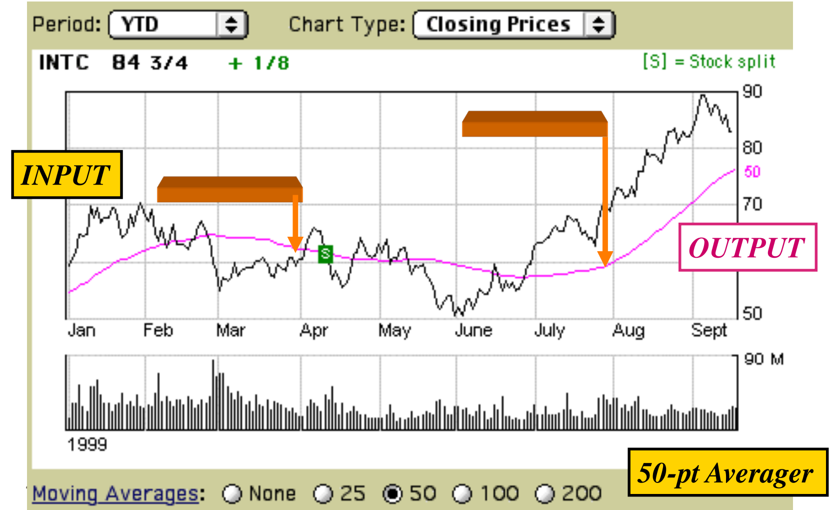
$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

*Causal because y[n] depends only on x[n] and past samples*






# FILTERED STOCK SIGNAL

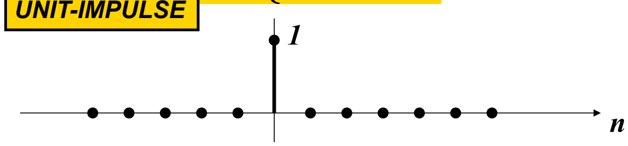



# SPECIAL INPUT SIGNALS

- x[n] = SINUSOID **FREQUENCY RESPONSE (LATER)**
- x[n] has only one NON-ZERO VALUE

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

**UNIT-IMPULSE**






# UNIT IMPULSE SIGNAL δ[n]

| $n$ | ... | −2 | −1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | ... |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\delta[n]$ | 0 | 0 | 0 | $\boxed{1}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\delta[n-3]$ | 0 | 0 | 0 | 0 | 0 | 0 | $\boxed{1}$ | 0 | 0 | 0 | 0 |

***δ[n] is NON-ZERO When its argument is equal to ZERO***

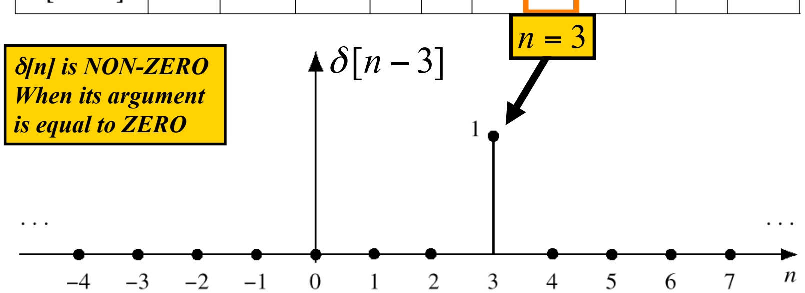
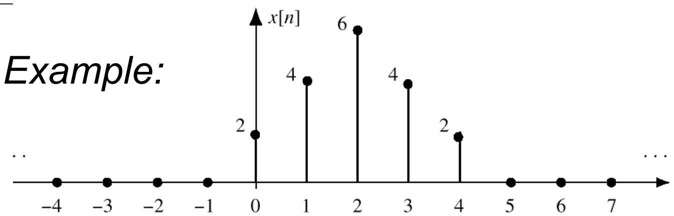



**Figure 5.7** Shifted impulse sequence, $\delta[n-3]$.

## Sequence Representation

*Example:*




$$x[n=0]=x[0]=2 \qquad x[n=1]=x[1]=4$$

$$x[n=2]=x[2]=6 \qquad x[n=3]=x[3]=4$$

$$x[n]=\cdots+\ \mathbf{0}\ \delta[n+1]+\ \mathbf{2}\ \delta[n]+\ \mathbf{4}\ \delta[n-1]$$

$$+\ \mathbf{6}\ \delta[n-2]+\ \mathbf{4}\ \delta[n-3]+\cdots$$



## UNIT IMPULSE RESPONSE

- FIR filter description usually given in terms of coefficients $b_k$

$$\boxed{y[n]=\sum_{k=0}^{M} b_k x[n-k]}$$

- What happens if input is a unit impulse?



## Example: 4-pt AVERAGER

- CAUSAL SYSTEM: USE PAST VALUES

$$y[n]=\tfrac{1}{4}(x[n]+x[n-1]+x[n-2]+x[n-3])$$

- *INPUT = UNIT IMPULSE SIGNAL = δ[n]*

$$x[n]=\delta[n]$$

$$y[n]=\tfrac{1}{4}\delta[n]+\tfrac{1}{4}\delta[n-1]+\tfrac{1}{4}\delta[n-2]+\tfrac{1}{4}\delta[n-3]$$

- *OUTPUT is “IMPULSE RESPONSE”*
    - *y[n]=h[n] when x[n]=δ[n]*



## Unit Impulse Response

$$y[n]=\tfrac{1}{4}x[n]+\tfrac{1}{4}x[n-1]+\tfrac{1}{4}x[n-2]+\tfrac{1}{4}x[n-3]$$

| $n$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ |
|---|---|---|---|---|---|---|---|---|---|
| $x[n]$ | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| $y[n]$ | 0 | 0 | 0 | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | 0 | 0 |

$$x[n]=\delta[n]$$

$$y[n]=\tfrac{1}{4}\delta[n]+\tfrac{1}{4}\delta[n-1]+\tfrac{1}{4}\delta[n-2]+\tfrac{1}{4}\delta[n-3]$$

$$=h[n]$$

## SUM of Shifted Impulses

$$h[n] = \tfrac{1}{4}\delta[n] + \tfrac{1}{4}\delta[n-1] + \tfrac{1}{4}\delta[n-2] + \tfrac{1}{4}\delta[n-3] + 0\delta[n-4]$$

| $n$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ |
|---|---|---|---|---|---|---|---|---|---|
| $h[n]$ | $0$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $0$ | $0$ | $0$ | $0$ |
| $h[0]\delta[n]$ | $0$ | $\frac{1}{4}$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $h[1]\delta[n-1]$ | $0$ | $0$ | $\frac{1}{4}$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $h[2]\delta[n-2]$ | $0$ | $0$ | $0$ | $\frac{1}{4}$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $h[3]\delta[n-3]$ | $0$ | $0$ | $0$ | $0$ | $\frac{1}{4}$ | $0$ | $0$ | $0$ | $0$ |
| $h[4]\delta[n-4]$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |



## FIR IMPULSE RESPONSE

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] \qquad h[n] = \sum_{k=0}^{M} b_k \delta[n-k]$$

FIR means Finite-duration Impulse Response

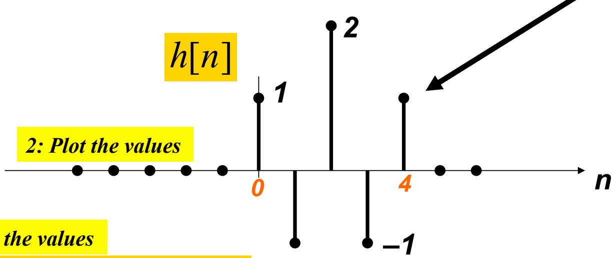
| $n$ | $n<0$ | $0$ | $1$ | $2$ | $3$ | $\dots$ | $M$ | $M+1$ | $n > M+1$ |
|---|---|---|---|---|---|---|---|---|---|
| $x[n] = \delta[n]$ | $0$ | $1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $y[n] = h[n]$ | $0$ | $b_0$ | $b_1$ | $b_2$ | $b_3$ | $\dots$ | $b_M$ | $0$ | $0$ |



## Three Representations for a FIR filter

*1* Use SHIFTED IMPULSES to write $h[n]$

$$h[n] = \delta[n] - \delta[n-1] + 2\delta[n-2] - \delta[n-3] + \delta[n-4]$$




*3: List the values*

$$b_k = \{1, -1, 2, -1, 1\}$$

*True for any signal, x[n]*



## FILTERING EXAMPLE

- 7-point AVERAGER
  - Removes cosine
    - By making its amplitude (A) smaller

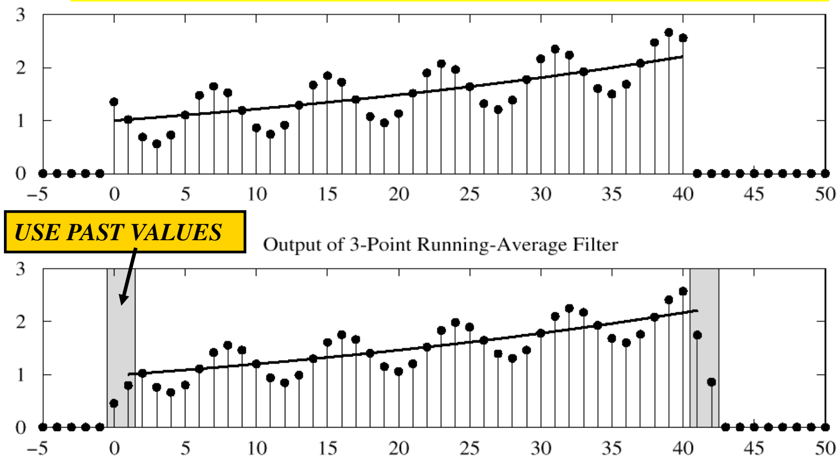
$$y_7[n] = \sum_{k=0}^{6} \left(\tfrac{1}{7}\right) x[n-k]$$

- *3-point AVERAGER*
  - *Changes A slightly*

$$y_3[n] = \sum_{k=0}^{2} \left(\tfrac{1}{3}\right) x[n-k]$$

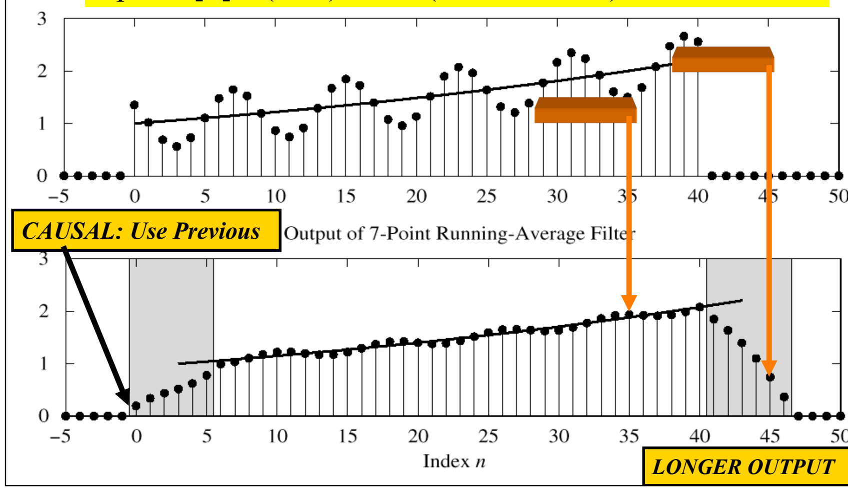
# 3-pt AVG EXAMPLE

Input : $x[n] = (1.02)^n + \cos(2\pi n/8 + \pi/4) \quad \text{for } 0 \le n \le 40$




# 7-pt FIR EXAMPLE (AVG)

Input : $x[n] = (1.02)^n + \cos(2\pi n/8 + \pi/4) \quad \text{for } 0 \le n \le 40$




# POP QUIZ

- FIR Filter is "FIRST DIFFERENCE"
  - $y[n] = x[n] - x[n-1]$
- What are filter coefficients?

$$b_k = \{1, -1\}$$

- Find $h[n]$

$$h[n] = \delta[n] - \delta[n-1]$$



# MATLAB for FIR FILTER

- `yy = conv(bb,xx)`
  - VECTOR `bb` contains Filter Coefficients
  - DSP-First: `yy = firfilt(bb,xx)`
- FILTER COEFFICIENTS $\{b_k\}$

**conv2()** *for images*

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

# What system does this implement?

- `yy = conv([1,2,1],xx)`
- `How about this one?`

`yy =conv([1,2,1],conv([1,-1],xx))`



# Discrete-Time Linear Systems

- A linear system obeys the principle of superposition

$$x[n] \mapsto y[n]$$

$$\alpha x[n] \mapsto \alpha y[n] \quad \text{(homogeneous)}$$

$$x_1[n] \mapsto y_1[n] \text{ and } x_2[n] \mapsto y_2[n]$$

$$x_1[n] + x_2[n] \mapsto y_1[n] + y_2[n] \quad \text{(additive)}$$

$$\alpha x_1[n] + \beta x_2[n] \mapsto \alpha y_1[n] + \beta y_2[n] \quad \text{(superposition)}$$



# TESTING LINEARITY

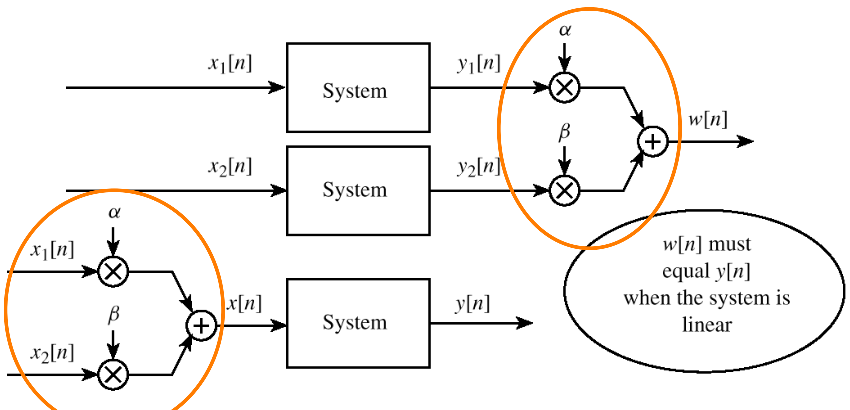



**Figure 5.17** Testing linearity by checking the interchange of operations.



# Time-Invariant Systems

- A time-invariant system is one whose output is the same for a given input no matter when the input occurs.

$$x[n] \mapsto y[n]$$

$$x[n-n_0] \mapsto y[n-n_0]$$

- An LTI discrete-time system is both linear and time-invariant.

# TESTING Time-Invariance

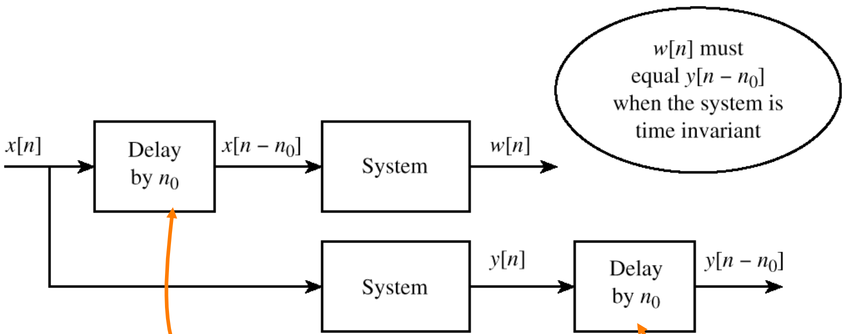



**Figure 5.16** Testing time-invariance property by checking the interchange of operations.



# Derivation of Discrete Convolution

- Represent x[n] in terms of impulses

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

- Assume LTI system

$$\delta[n] \mapsto h[n]$$

$$\delta[n-k] \mapsto h[n-k]$$

$$x[k]\delta[n-k] \mapsto x[k]h[n-k]$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \mapsto \sum_{k=-\infty}^{\infty} x[k]h[n-k] = y[n]$$



# Discrete Convolution

- LTI systems defined by convolution

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n]*h[n] = (x*h)[n]$$

- Discrete convolution is commutative

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = h[n]*x[n] = (h*x)[n]$$



# Discrete Convolution and FIR Systems

- LTI systems defined by convolution

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = h[n]*x[n] = (h*x)[n]$$

- FIR system has finite-duration impulse response $h[n] = \begin{cases} b_n & n = 0,1,\ldots,M \\ 0 & \text{otherwise} \end{cases}$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] = h[n]*x[n] = \sum_{k=n-M}^{n} x[k]b_{n-k}$$

# CONVOLUTION Exa

$$y[n] = \sum_{k=0}^{M} h[k]x[n-k]$$

$$h[n] = \delta[n] - \delta[n-1] + 2\delta[n-2] - \delta[n-3] + \delta[n-4]$$

$$x[n] = u[n]$$

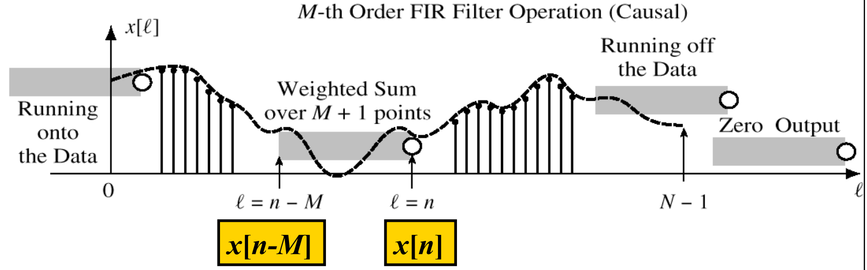
| $n$ | $-1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|---|
| $x[n]$ | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | ... |
| $h[n]$ | 0 | 1 | $-1$ | 2 | $-1$ | 1 | 0 | 0 | 0 |
| $h[0]x[n]$ | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $h[1]x[n-1]$ | 0 | 0 | $-1$ | $-1$ | $-1$ | $-1$ | $-1$ | $-1$ | $-1$ |
| $h[2]x[n-2]$ | 0 | 0 | 0 | 2 | 2 | 2 | 2 | 2 | 2 |
| $h[3]x[n-3]$ | 0 | 0 | 0 | 0 | $-1$ | $-1$ | $-1$ | $-1$ | $-1$ |
| $h[4]x[n-4]$ | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| $y[n]$ | 0 | 1 | 0 | 2 | 1 | 2 | 2 | 2 | ... |



# GENERAL FIR FILTER

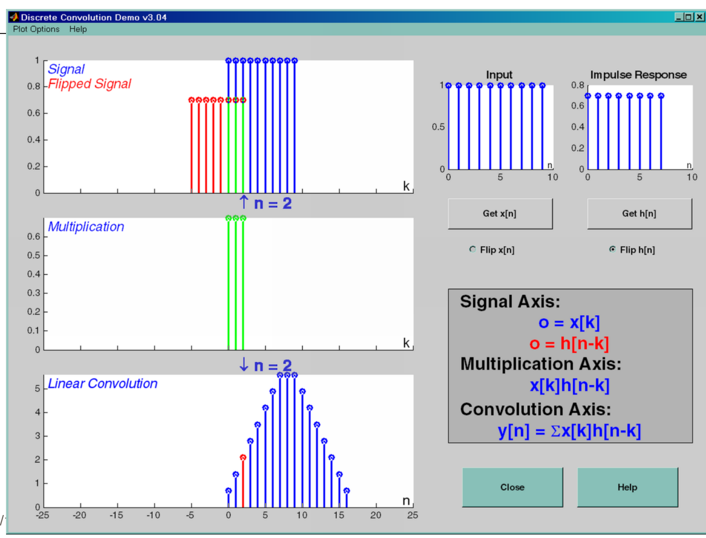
- SLIDE a Length-L WINDOW over x[n]

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$






# DCONVDEMO: MATLAB GUI






# HARDWARE STRUCTURES

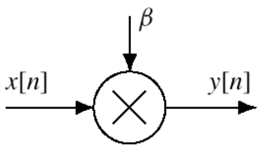

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

- INTERNAL STRUCTURE of “FILTER”
  - WHAT COMPONENTS ARE NEEDED?
  - HOW DO WE “HOOK” THEM TOGETHER?
- SIGNAL FLOW GRAPH NOTATION

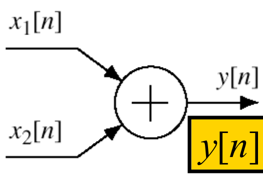
# HARDWARE ATOMS

- Add, Multiply & Store

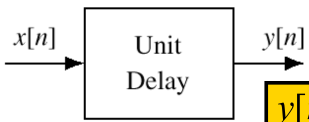
$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$




$$y[n] = \beta x[n]$$




$$y[n] = x_1[n] + x_2[n]$$




$$y[n] = x[n-1]$$



# FIR STRUCTURE

- Direct Form

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

***SIGNAL FLOW GRAPH***

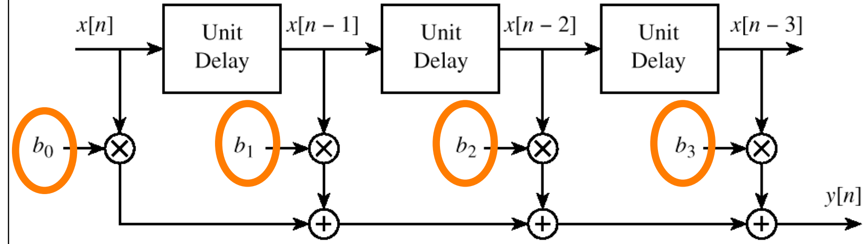



**Figure 5.13** Block-diagram structure for the $M$th order FIR filter.