Stanford University
Winter 2012-2013

# Signal Processing and Linear Systems I

Lecture 4: Systems Characteristics and Models

January 11, 2013



## System Characteristics and Models

Today's topics:

- What are systems?
- Block diagrams
- Interconnected systems
- Linearity
- Differential equations
- System characteristics

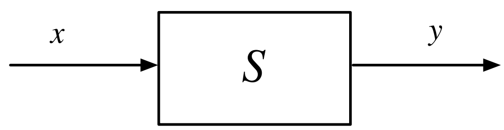
## Systems

- A system transforms *input signals* into *output signals*.
- A system is a *function* mapping input signals into output signals.
- We will concentrate on systems with one input and one output *i.e. single-input, single-output* (SISO) systems.
- Notation:
  - $y = Sx$ or $y = S(x)$, meaning the system $S$ acts on an input signal $x$ to produce output signal $y$.
  - $y = Sx$ does not (in general) mean multiplication!



## Block diagrams

Systems often denoted by *block diagram*:



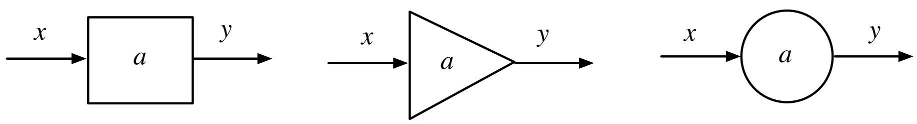

- Lines with arrows denote signals (*not* wires).
- Boxes denote systems; arrows show inputs & outputs.
- Special symbols for some systems.

# Examples

(with input signal $x$ and output signal $y$)

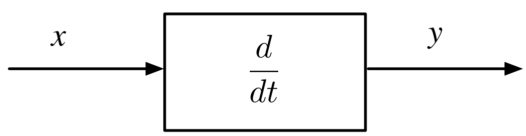
**Scaling system:** $y(t) = ax(t)$

- Called an *amplifier* if $|a| > 1$.
- Called an *attenuator* if $|a| < 1$.
- Called *inverting* if $a < 0$.
- $a$ is called the *gain* or *scale factor*.
- Sometimes denoted by triangle or circle in block diagram:



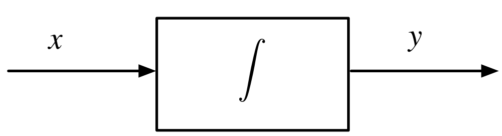



**Differentiator:** $y(t) = x'(t)$




**Integrator:** $y(t) = \int_a^t x(\tau)\, d\tau$ ($a$ is often $0$ or $-\infty$)

Common notation for integrator:




**time shift system:** $y(t) = x(t - T)$

- called a *delay system* if $T > 0$
- called a *predictor system* if $T < 0$

**convolution system:**

$$y(t) = \int x(t - \tau) h(\tau) \, d\tau,$$

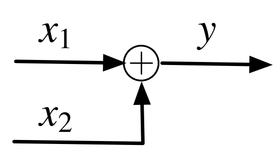
where $h$ is a given function (you'll be hearing much more about this!)



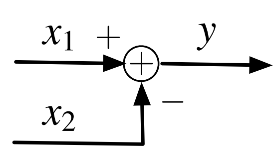
# Examples with multiple inputs

Inputs $x_1(t)$, $x_2(t)$, and Output $y(t)$)

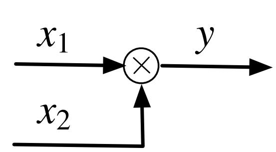
- **summing system:** $y(t) = x_1(t) + x_2(t)$




- **difference system:** $y(t) = x_1(t) - x_2(t)$



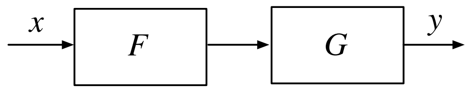

- **multiplier system:** $y(t) = x_1(t)x_2(t)$



# Interconnection of Systems

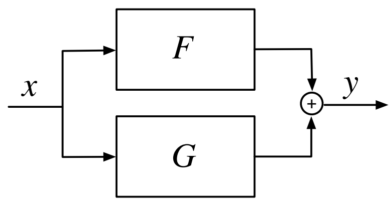
We can interconnect systems to form new systems,

- **cascade (or series):** $y = G(F(x)) = GFx$



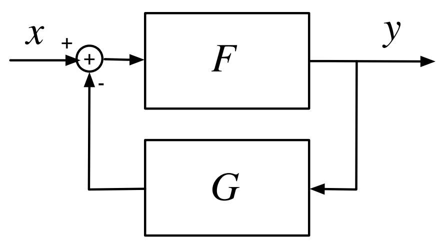

(note that block diagrams and algebra are *reversed*)

- **sum (or parallel):** $y = Fx + Gx$





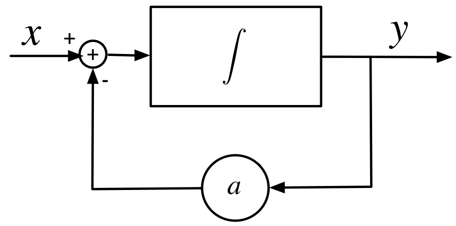

- **feedback:** $y = F(x - Gy)$




In general,

- Block diagrams are a symbolic way to describe a connection of systems.
- We can just as well write out the equations relating the signals.
- We can go back and forth between the system block diagram and the system equations.

## Example: Integrator with feedback




Input to integrator is $x - ay$, so

$$\int^t (x(\tau) - ay(\tau))\, d\tau = y(t)$$

Another useful method: the *input* to an integrator is the derivative of its output, so we have

$$x - ay = y'$$

(of course, same as above)



## Linearity

A system $F$ is **linear** if the following two properties hold:

1. **homogeneity:** if $x$ is any signal and $a$ is any scalar,

$$F(ax) = aF(x)$$

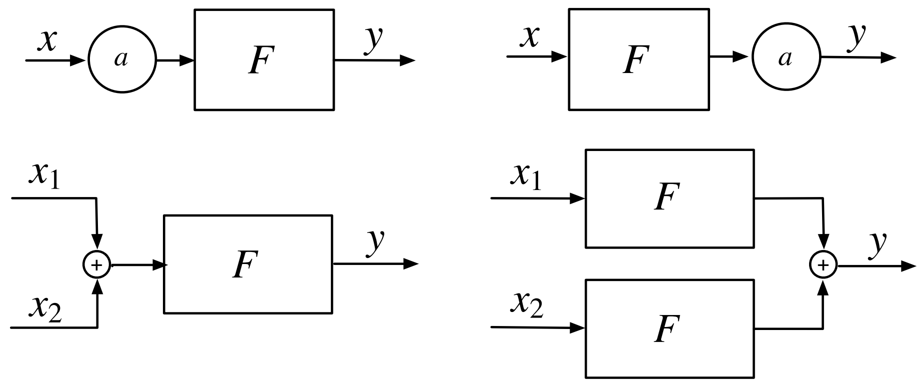
2. **superposition:** if $x$ and $\tilde{x}$ are any two signals,

$$F(x + \tilde{x}) = F(x) + F(\tilde{x})$$

In words, linearity means:

- Scaling before or after the system is the same.
- Summing before or after the system is the same.

Linearity means the following pairs of block diagrams are equivalent, *i.e.*, have the same output for any input(s)




**Equivalent Definition of Linearity:** Superposition and homogeneity can be combined. If $x$ and $\tilde{x}$ are any two signals, and $a$ and $b$ are constants, a system is linear if

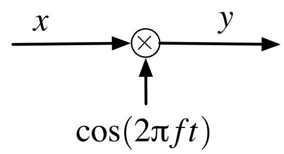
$$F(ax + b\tilde{x}) = aF(x) + bF(\tilde{x})$$



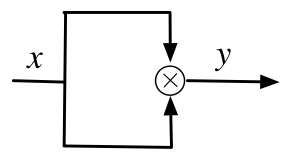
**Examples of linear systems:** scaling system, differentiator, integrator, running average, time shift, convolution, summer, difference systems.

**Examples of nonlinear systems:** sign detector, multiplier (sometimes), comparator

- Multiplier as a modulator, $y(t) = x(t)\cos(2\pi f t)$, is *linear*.

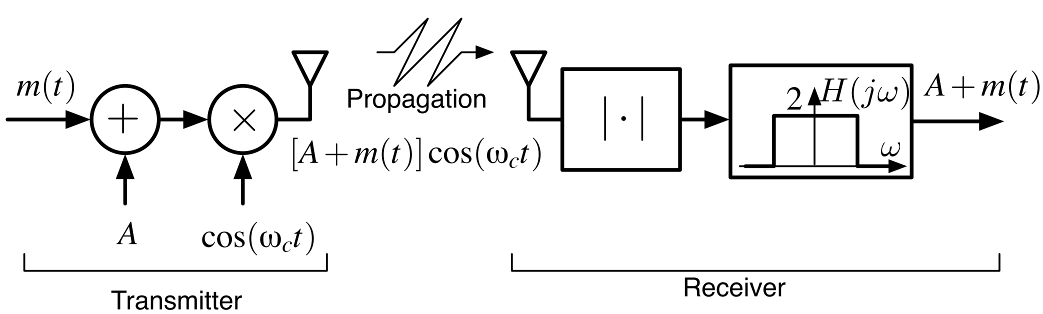



- Multiplier as a squaring system, $y(t) = x^2(t)$ is *nonlinear*.






## Example: AM Radio Transmitter and receiver




- Multiple input systems
- Linear and non-linear systems

# Systems Described by Differential Equations

Many systems are described by a *linear constant coefficient ordinary differential equation* (LCCODE):

$$a_n y^{(n)}(t) + \cdots + a_1 y'(t) + a_0 y(t) = b_m x^{(m)}(t) + \cdots + b_1 x'(t) + b_0 x(t)$$

with given *initial conditions*

$$y^{(n-1)}(0), \quad \ldots \quad , y'(0), \quad y(0)$$

(which fixes $y(t)$, given $x(t)$)

- $n$ is called the *order* of the system
- $b_0, \ldots, b_m, a_0, \ldots, a_n$ are the *coefficients* of the system



This is important because LCCODE systems are **linear** when initial conditions are all zero.

- Many systems can be described this way
- If we can describe a system this way, we know it is linear

Note that an LCCODE gives an *implicit* description of a system.

- It describes how $x(t)$, $y(t)$, and their derivatives interrelate
- It doesn't give you an explicit solution for $y(t)$ in terms of $x(t)$

Soon we'll be able to *explicitly* express $y(t)$ in terms of $x(t)$

## Examples

**Simple examples**

- scaling system ($a_0 = 1,\ b_0 = a$)

$$y = ax$$

- integrator ($a_1 = 1,\ b_0 = 1$)

$$y' = x$$

- differentiator ($a_0 = 1,\ b_1 = 1$)

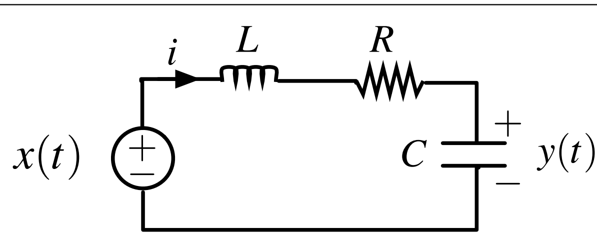
$$y = x'$$

- integrator with feedback (a few slides back, $a_1 = 1, a_0 = a, b_0 = 1$)

$$y' + ay = x$$



## 2nd Order Circuit Example




By Kirchoff's voltage law

$$x - Li' - Ri - y = 0$$

Using $i = Cy'$,

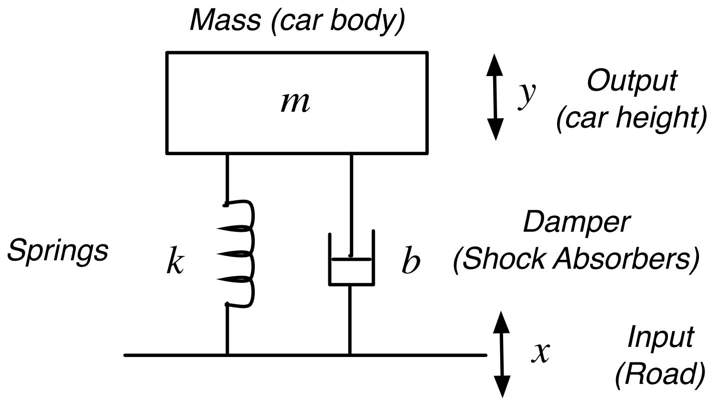
$$x - LCy'' - RCy' - y = 0$$

or

$$LCy'' + RCy' + y = x$$

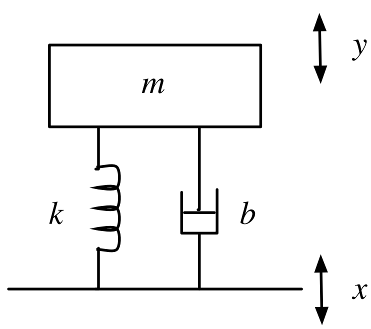
which is an LCCODE. This is a linear system.

## Mechanical System




This can represent suspension system, or building during earthquake, . . .






- $x(t)$ is displacement of base; $y(t)$ is displacement of mass
- spring force is $k(x-y)$; damping force is $b(x-y)'$
- Newton's equation is $my'' = b(x-y)' + k(x-y)$

Rewrite as second-order LCCODE

$$my'' + by' + ky = bx' + kx$$

This is a linear system.

## System Memory

- A system is *memoryless* if the output depends only on the present input.
  - Ideal amplifier
  - Ideal gear, transmission, or lever in a mechanical system
- A *system with memory* has an output signal that depends on inputs in the past or future.
  - Energy storage circuit elements such as capacitors and inductors
  - Springs or moving masses in mechanical systems
- A *causal* system has an output that depends only on past or present inputs.
  - Any real physical circuit, or mechanical system.



## Time-Invariance

- A system is time-invariant if a time shift in the input only produces the same time shift in the output.
- For a system $F$,

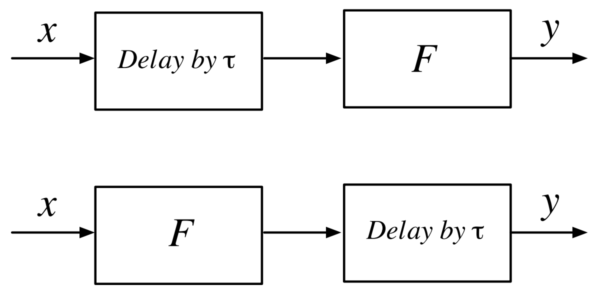
$$y(t) = Fx(t)$$

implies that

$$y(t - \tau) = Fx(t - \tau)$$

for any time shift $\tau$.

- Implies that delay and the system $F$ commute. These block diagrams are equivalent:



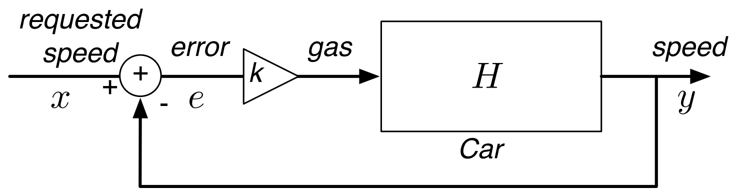

- Time invariance is an important system property, it greatly simplifies the analysis in the next class!



## System Stability

- Stability important for most engineering applications.
- Many definitions
- If a bounded input
$$|x(t)| \leq M_x < \infty$$
always results in a bounded output
$$|y(t)| \leq M_y < \infty,$$
where $M_x$ and $M_y$ are finite positive numbers, the system is *Bounded Input Bounded Output (BIBO) stable*.

**Example:** Cruise control, from introduction,




The output $y$ is

$$y = H(k(x - y))$$

We'll see later that this system can become unstable if $k$ is too large (depending on $H$)

- Postive error adds bolus of gas
- Delay car velocity change, speed overshoots
- Negative error cuts gas off
- Delay in velocity change, speed undershoots
- Repeat!



## System Invertability

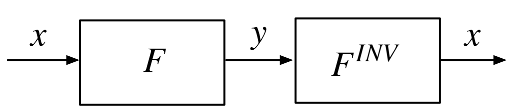
- A system is invertable if the input signal can be recovered from the output signal.
- If $F$ is an invertable system, and

  $$y = Fx$$

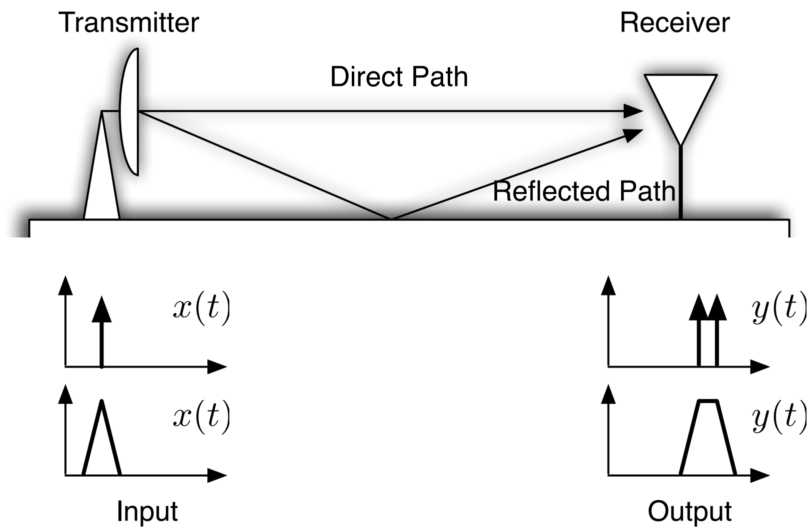
  then there is an inverse system $F^{INV}$ such that

  $$x = F^{INV}y = F^{INV}Fx$$

  so $F^{INV}F = I$, the identity operator.



**Example:** Multipath echo cancellation




Important problem in communications, radar, radio, cell phones.



Generally there will be multiple echoes.

Multipath can be described by a system $y = Fx$

- If we transmit an impulse, we receive multiple delayed impulses.
- One transmitted message gives multiple overlapping messages

We want to find a system $F^{INV}$ that takes the multipath corrupted signal $y$ and recovers $x$

$$\begin{aligned} F^{INV}y &= F^{INV}(Fx) \\ &= (F^{INV}F)\,x \\ &= x \end{aligned}$$

Often possible it we allow a delay in the output.

# Questions

Are these systems linear? Time invariant?

- $y(t) = \sqrt{x(t)}$
- $y(t) = x(t)z(t)$, where $z(t)$ is a known, non-zero signal
- $y(t) = x(at)$
- $y(t) = 0$
- $y(t) = x(T - t)$

A linear system $F$ has an inverse system $F^{inv}$. Is $F^{inv}$ linear?