EE102B Signal Processing and Linear Systems II

Solutions to Lab One
2012-2013 Spring Quarter

1. **(15 points)** ***Derivation of the Fourier Coefficients for the periodic pulse of Problem 1.4 and a full-wave rectified sine wave.***

   (a) **The periodic pulse of Problem 1.4 :**

$$\begin{aligned} a_k &= \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j\frac{2\pi}{T}kt}dt \\ &= \frac{1}{8}\int_{-4}^{4} x(t)e^{-j\frac{\pi}{4}kt}dt \\ &= \frac{1}{8}\int_{-1}^{1} x(t)e^{-j\frac{\pi}{4}kt}dt \end{aligned}$$

   (i) **When k = 0,**

$$a_k = \frac{20\cdot 2}{8} = 5$$

   (ii) **When k $\neq$ 0,**

$$a_k = \frac{5}{2}\left[\frac{1}{-j\frac{\pi}{4}k}e^{-j\frac{\pi}{4}kt}\right]_{-1}^{1} = \frac{20\sin(\frac{\pi}{4}k)}{\pi k}$$

(b) A full-wave rectified sine wave :

$$
\begin{aligned}
a_k &= \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j\frac{2\pi}{T}kt}dt \\
&= 120\int_{-\frac{1}{240}}^{\frac{1}{240}} |120\sqrt{2}\sin(120\pi t)|e^{-j240\pi kt}dt \\
&= 120\int_{0}^{\frac{1}{120}} 120\sqrt{2}\sin(120\pi t)e^{-j240\pi kt}dt \\
&= 120^2\sqrt{2}\int_{0}^{\frac{1}{120}} \frac{1}{j2}\left(e^{j120\pi t} - e^{-j120\pi t}\right)e^{-j240\pi kt}dt \\
&= \frac{120^2\sqrt{2}}{j2}\left[\frac{e^{-j120\pi(2k-1)t}}{-j120\pi(2k-1)} - \frac{e^{-j120\pi(2k+1)t}}{-j120\pi(2k+1)}\right]_0^{\frac{1}{120}} \\
&= \frac{60\sqrt{2}}{\pi(4k^2-1)}\left[(2k+1)e^{-j120\pi(2k-1)t} - (2k-1)e^{-j120\pi(2k+1)t}\right]_0^{\frac{1}{120}} \\
&= \frac{60\sqrt{2}}{\pi(4k^2-1)}\left[(2k+1)e^{-j(2k-1)\pi} - (2k-1)e^{-j(2k+1)\pi} - (2k+1) + (2k-1)\right] \\
&= \frac{60\sqrt{2}}{\pi(4k^2-1)}\left[(2k+1)\cdot(-1) - (2k-1)\cdot(-1) - 2\right] \\
&= \frac{60\sqrt{2}}{\pi(4k^2-1)}\left[-4\right] \\
&= \frac{240\sqrt{2}}{\pi(1-4k^2)}
\end{aligned}
$$

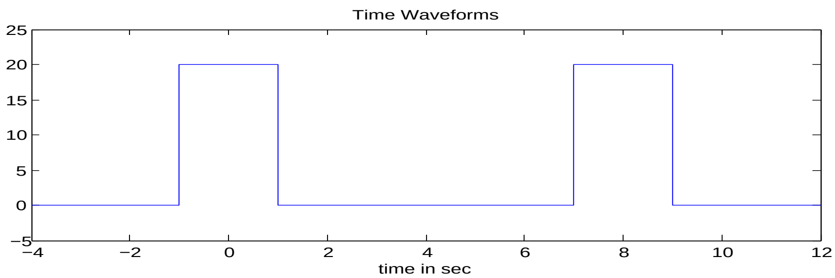
2. (20 points) ***Implement your M-file, syn_Fourier().***

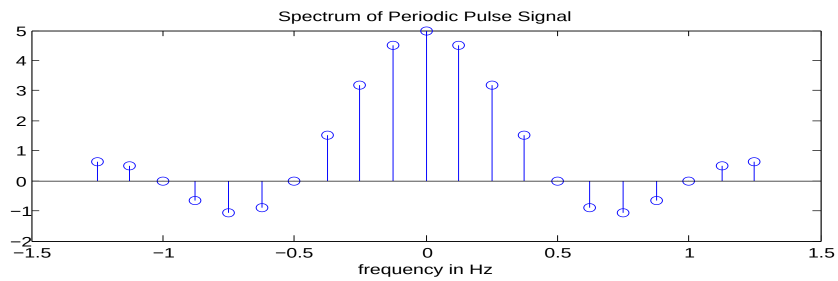
```
function        xx = syn_fourier(tt, ak, fk)
%SYN_FOURIER  Function to synthesize a sum of complex
%             exponentials over the time range given by tt
%  usage:
%    xx = syn_fourier(tt, ak, fk)
%      tt = vector of times, for the time axis
%      ak = vector of complex Fourier coefficients
%      fk = vector of frequencies
%              (usually contains both negative and positive freqs)
%      xx = vector of synthesized waveform values
%
%    Note: fk and ak must be the same length.
%             ak(1) corresponds to frequency fk(1),
%             ak(2) corresponds to frequency fk(2), etc.
%
% Note: the output might have a tiny imaginary part even if it
%       is supposed to be purely real.  If so, take the real part.
%
xx = exp( tt(:)*(2i*pi*fk(:)') )* ak(:);
%set imaginary part to zero if it is small
if( max(abs(imag(xx)))<1e-6 ), xx = real(xx); end
```

3. **(50 points)** ***Experiments and Annotated Plots.***

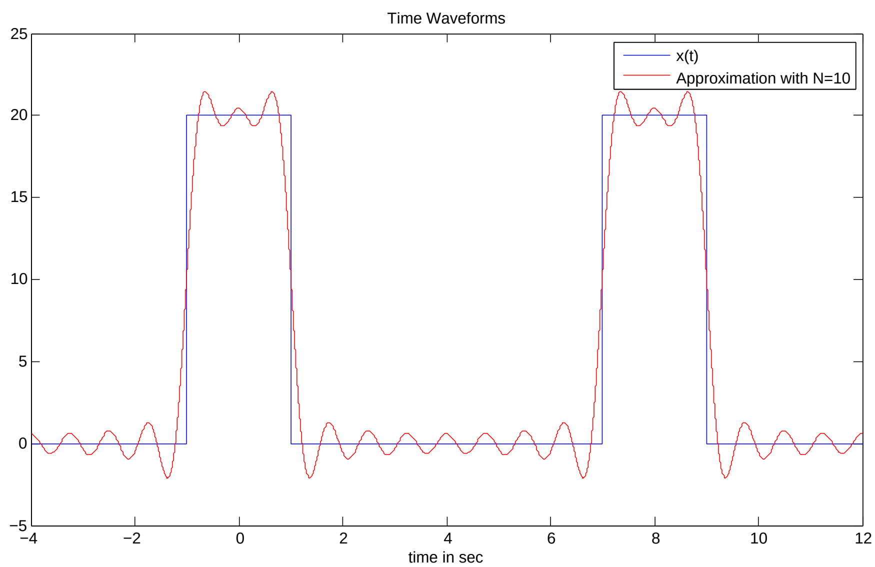
(a) **3.3 Synthesis of the Periodic Pulse Signal of Problem 1.4**

(i) **The periodic pulse signal and its Fourier coefficients.**





(ii) **Fourier Synthesis with N=10.**

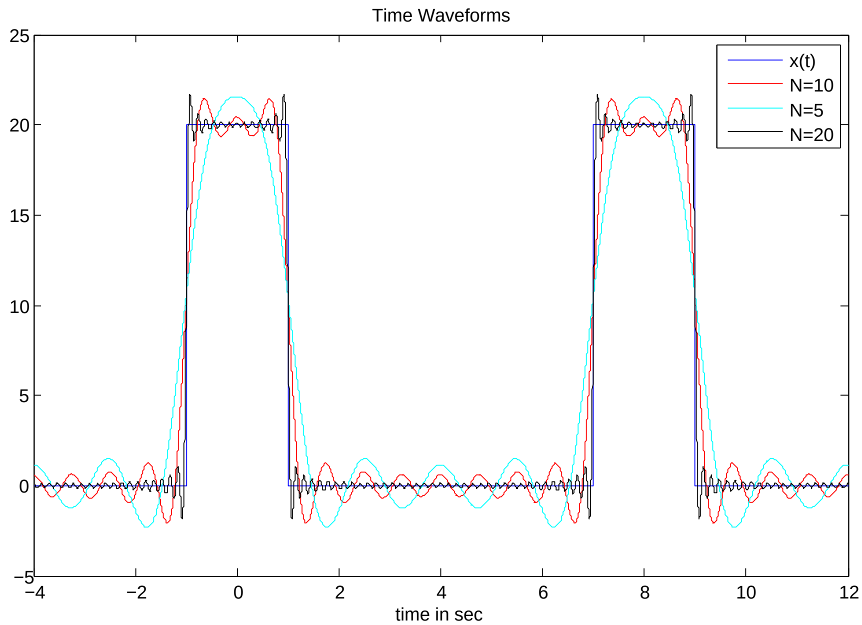


(iii) **What is the equivalent ideal low-pass filter's cutoff frequency?**
The filtered signal must have the f_{-10} and f_{10} components, and should not have any components for $|k| > 10$.

$$\therefore \frac{1}{8} \cdot 10 \ \textbf{(Hz)} \ < f_{cutoff} < \frac{1}{8} \cdot 11 \ \textbf{(Hz)}$$

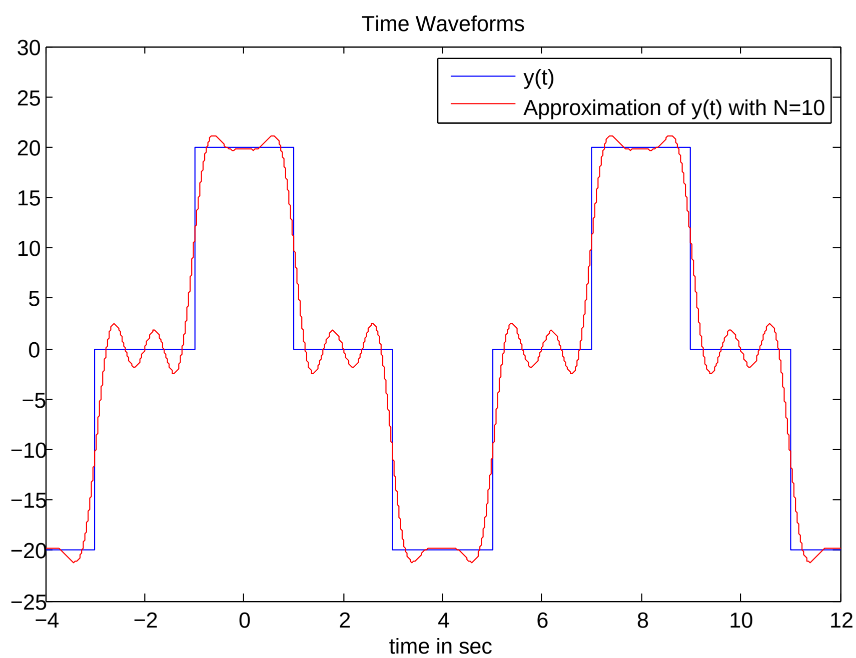
The answer is Not unique.

(iv) **Add two more examples, N>10 and N<10, and report what you observe.**



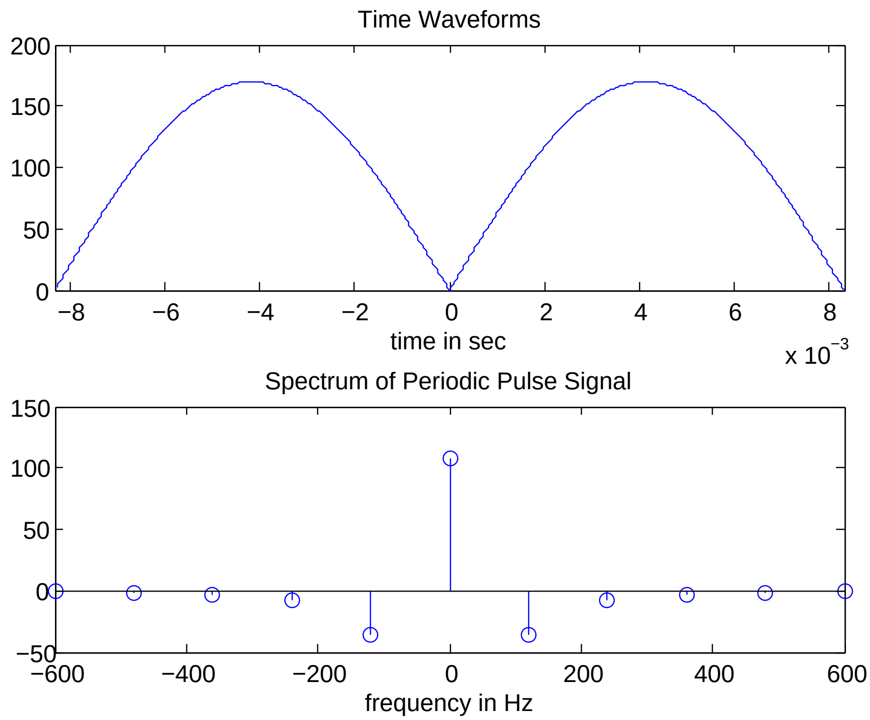
In the N>10 case, we can observe the Gibbs phenomenon.

(v) **Approximate the periodic output of the LTI system,** $h(t) = \delta(t) - \delta(t-4)$**.**

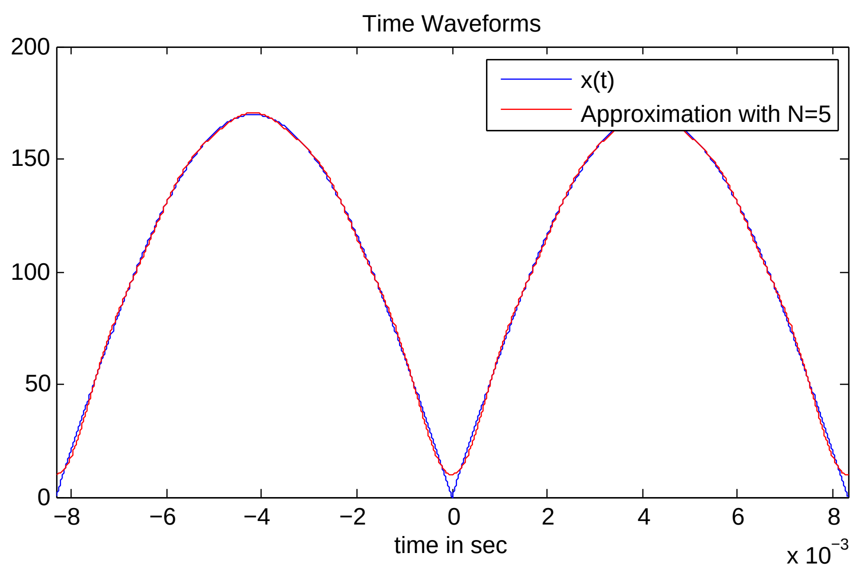


(b) **3.4 Synthesis of a Full-Wave Rectified Sine Wave**

(i) **The periodic pulse signal and its Fourier coefficients.**



(ii) Fourier Synthesis with N=5.



(iii) We want to make the output signal of a low-pass filer a constant DC value of the input signal. What is the equivalent ideal low-pass filter's cutoff frequency?
The filtered signal must have the f_0 components, and should not have

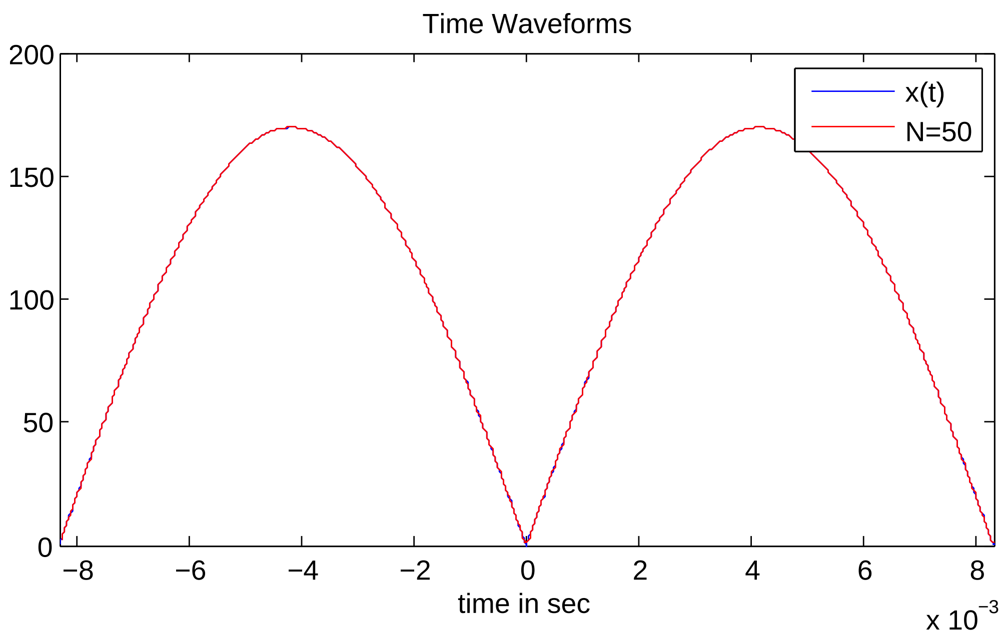
any components for $|k| > 0$.

$$\frac{1}{1/120}\cdot 0\ \textbf{(Hz)}\ < f_{cutoff} < \frac{1}{1/120}\cdot 1\ \textbf{(Hz)}$$

$$\therefore 0\ \textbf{(Hz)}\ < f_{cutoff} < 120\ \textbf{(Hz)}$$

The answer is Not unique and the constant output of the low-pass filter is 108.04 which is the a_0 coefficient value.

(iv) Increase N and report what you observe.
Here we chose N=50.



Even though the N value is large enough and the difference between the original the synthesized signals is negligible, we can still find relatively high error around the sharp zero-touching regions of the original signal. The reason is that the Fourier synthesis is basically a sum of continuous/differentiable functions. Since the components are all continuous/differentiable, so the summation is. Unless we choose $N = \infty$, we will still see the 'relatively' high errors around those regions.

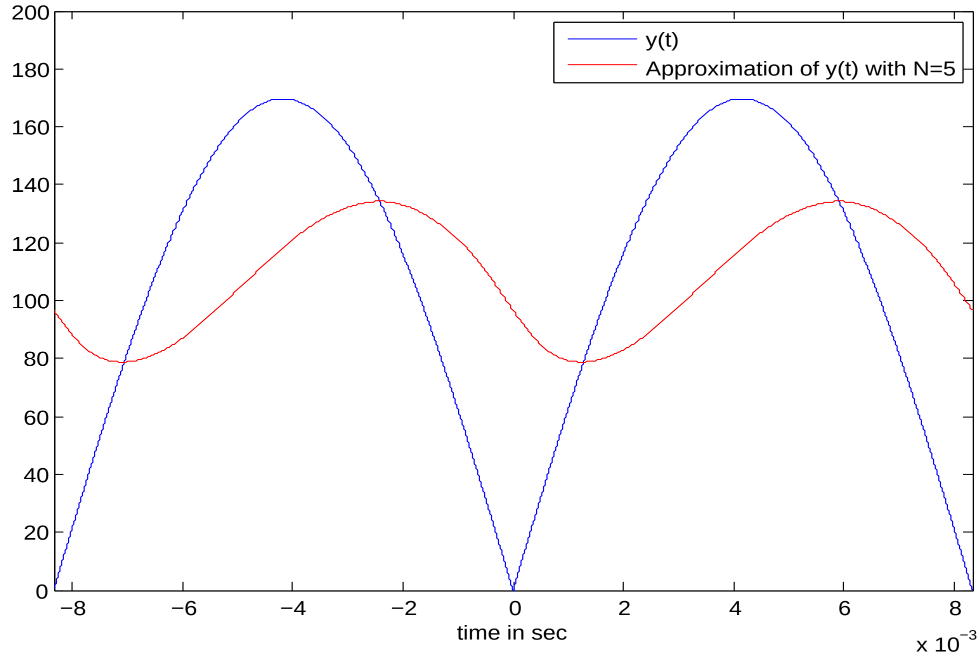
(v) Approximate the periodic output of the LTI system, $h(t) = \alpha e^{-\alpha t}u(t)$.
The output of the filter for the $2N + 1$ terms approximation is

$$y(t) = \sum_{k=-N}^{N} b_k e^{j\omega_0 kt} \quad \textbf{where} \quad b_k = a_k H(j\omega_0 k)$$

$$\omega_0 = \frac{2\pi}{T}$$

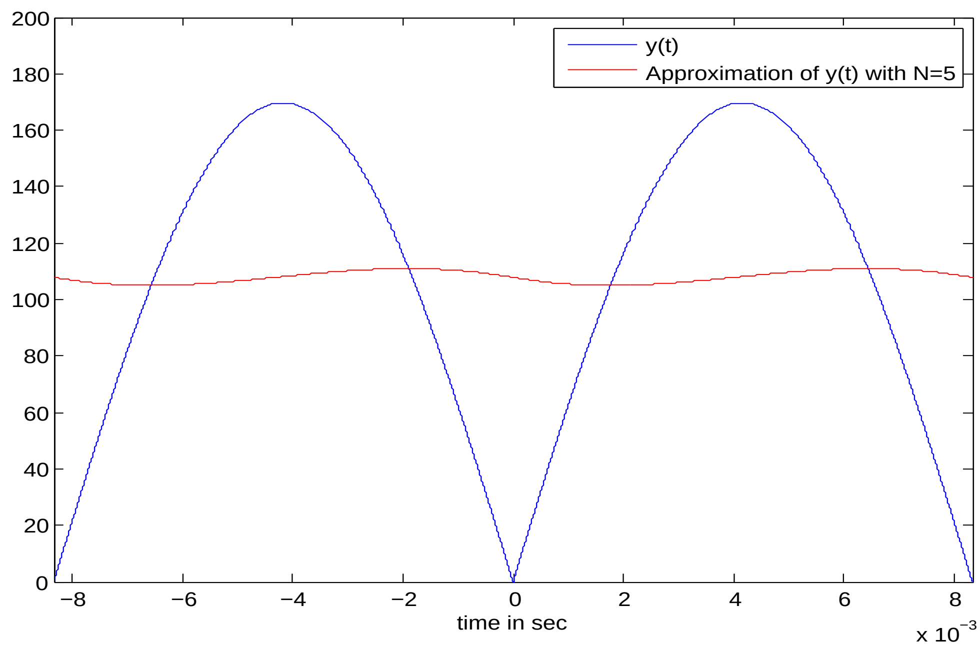
$$H(j\omega) = \frac{\alpha}{\alpha + j\omega}$$

We have 2 figures below. The top one is when $\alpha = 100\pi$ which is quite a large number. Large N implies a fast decay in h(t). Reminding that what



a low-pass filter does is basically to add many consecutive samples, fast decaying h(t) roughly corresponds to a weighted sum of a small number of samples. This is why the filtered signal follows the original signal and doesn't look like a DC component.

When we decrease the α to 10π, we can observe the bottom one.



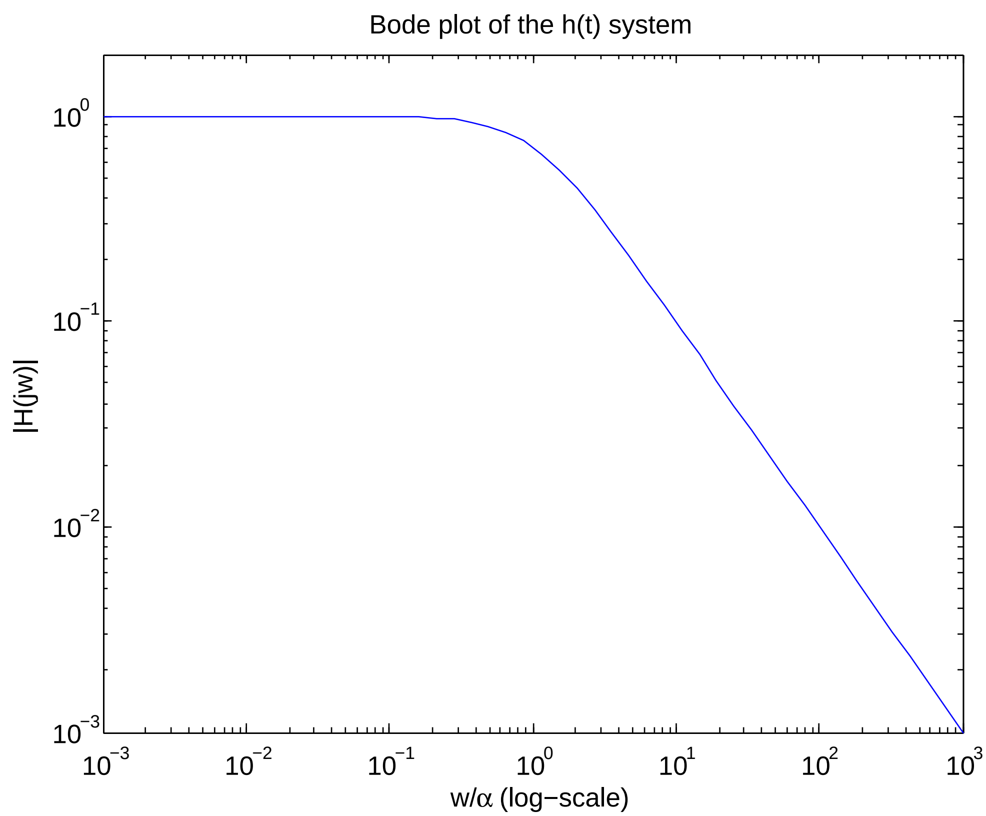
In this case, the filtered signal more looks like a DC component (i.e. less

ripple), and we can say the filter does a good job as a low-pass filter.

4. **(15 points)** ***Formula for the peak-to-peak ripple remaining in the filtered output of the full-wave rectified sine wave.***

$$h(t) = \alpha e^{-\alpha t} u(t) \quad \leftrightarrow \quad H(jw) = \frac{\alpha}{a + jw}$$

The below is a Bode plot of the $h(t)$ system. As you can see, it is a simple RC low-pass filter. The cutoff frequency of the filter is $w_{cutoff} = \alpha$. Since the



problem's assumption says to only consider the fundamental frequency, i.e. higher frequencies are negligible for the value of α to be used. To analyze the out ripple's peak-to-peak value, let's start from the corresponding output signal component

at the fundamental frequency, $w_0 = \frac{2\pi}{T} = 240\pi$**.**

$$\begin{aligned}
\textbf{(output signal component at } w_0\textbf{)} &= a_1 H(jw_0)e^{jw_0t} + a_{-1}H(-jw_0)e^{-jw_0t} \\
&= a_1 \left[H(jw_0)e^{jw_0t} + H(-jw_0)e^{-jw_0t}\right] \\
&= a_1 \left[\frac{\alpha}{\alpha + jw_0}e^{jw_0t} + \frac{\alpha}{\alpha - jw_0}e^{-jw_0t}\right] \\
&= a_1\alpha \left[\frac{(\alpha - jw_0)e^{jw_0t} + (\alpha + jw_0)e^{-jw_0t}}{(\alpha + jw_0)(\alpha + jw_0)}\right] \\
&= a_1\alpha \left[\frac{2\alpha\cos(w_0t) + 2\alpha\sin(w_0t)}{\alpha^2 + w_0^2}\right] \\
&= \frac{2a_1\alpha\sqrt{\alpha^2 + w_0^2}\sin(w_0 + \phi)}{\alpha^2 + w_0^2} \\
&= \frac{2a_1\alpha\sin(w_0 + \phi)}{\sqrt{\alpha^2 + w_0^2}}
\end{aligned}$$

$$\therefore \textbf{(peak-to-peak ripple size)} = \frac{4|a_1\alpha|}{\sqrt{\alpha^2 + w_0^2}} \approx \frac{144|\alpha|}{\sqrt{\alpha^2 + (240\pi)^2}}$$

Here is a comparison between the theoretical and experimental peak-to-peak ripple values for various α **values.**

α	2.4π	24π	240π	2400π	24000π
Experimental value	**1.43**	**14.21**	**104.50**	**159.07**	**160.74**
Theoretical value	**1.44**	**14.32**	**101.82**	**143.29**	**144.00**

We can find that the above theoretical analysis works well.