Stanford University
Spring 2012-2013

Signal Processing and Linear Systems I

Lecture 8: The Continuous Time Fourier Transform

January 28, 2013

Introduction to Fourier Transforms

- Fourier transform as a limit of Fourier series
- Inverse Fourier transform: The Fourier integral theorem
- Examples: the rect function, one-sided exponential
- Symmetry properties

Fourier Series

We can expand $f(t)$ as a Fourier series in interval $[-T/2, T/2)$:

$$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

where

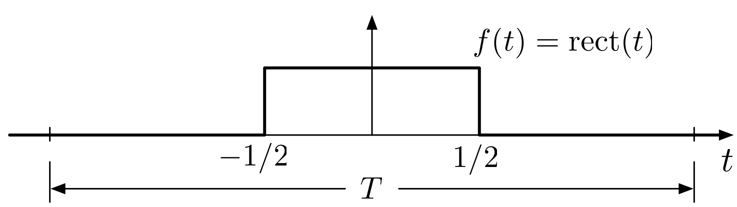
$$D_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0}\,dt,$$

and

$$\omega_0 = \frac{2\pi}{T}.$$

What happens if we let T increase?

For example, assume $y(t) = \mathrm{rect}(t)$, and that we are computing the Fourier series over an interval T,

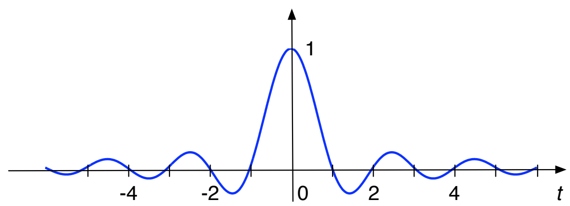


The fundamental period for the Fourier series in T, and the fundamental frequency is $\omega_0 = 2\pi/T$.

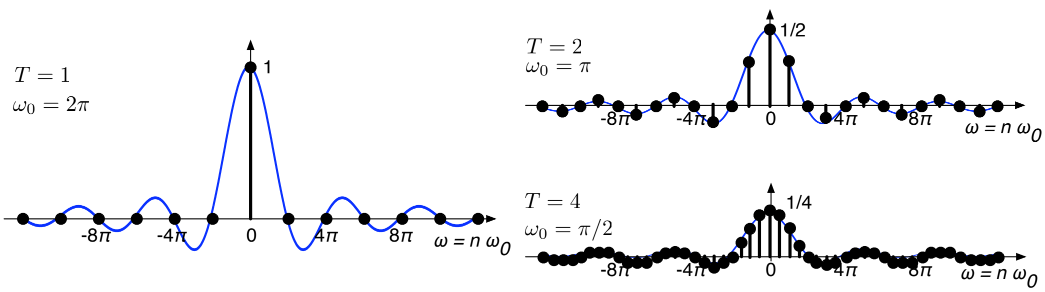
From Lecture 7, the Fourier series is

$$f(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} \mathrm{sinc}\left(\frac{n}{T}\right) e^{jn\omega_0 t}$$

where $\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$ is plotted below



If we plot the Fourier series coefficients as a function of $\omega = n\omega_0$ for T $=$ 1, 2, and 4,



More densely sampled, same sinc() envelope, decreased amplitude.

Fourier Transforms

Given a continuous time signal $f(t)$, define its *Fourier transform* as the function of a real ω:

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\, dt$$

if the integral makes sense.

This is similar to the expression for the Fourier series coefficients.

We can interpret this as the result of expanding $f(t)$ as a Fourier series in an interval $[-T/2, T/2)$, and then letting $T \to \infty$.

The Fourier series for $f(t)$ in interval $[-T/2, T/2)$:

$$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

where

$$D_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}\,dt.$$

Define the truncated Fourier transform:

$$F_T(j\omega) = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-j\omega t}\,dt$$

so that

$$D_n = \frac{1}{T}F_T(jn\omega_0).$$

The Fourier series is then

$$f_T(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T}F_T(jn\omega_0)e^{jn\omega_0 t}$$

As $T \to \infty$, then $\frac{2\pi}{T} = \omega_0 \to 0$. Suppose that n increases with T so that $n\omega_0 \to \omega$ where ω is fixed (this is only approximate except as limit).

The limit of the truncated Fourier transform is

$$F(j\omega) = \lim_{T\to\infty} F_T(j\omega) = \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

Taking limit in Fourier series for $f(t)$ using Riemann approximations to integral with $\Delta\omega = 2\pi/T$ and $\omega = 2\pi n/T = n\Delta\omega$

$$\begin{aligned}
f(t) &= \lim_{T\to\infty} f_T(t) \\
&= \lim_{\Delta\omega\to 0} \sum_{n=-\infty}^{\infty} \frac{1}{T}F_T(jn\Delta\omega)e^{jn\Delta\omega t} \\
&= \lim_{\Delta\omega\to 0} \sum_{n=-\infty}^{\infty} F_T(jn\Delta\omega)e^{jn\Delta\omega t}\frac{\Delta\omega}{2\pi} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}\,d\omega
\end{aligned}$$

Which yields the *inversion formula* for the Fourier transform, the *Fourier integral theorem*:

$$\boxed{\begin{aligned} F(j\omega) &= \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \\ f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}\,d\omega \end{aligned}}$$

Comments:

- There are usually technical conditions which must be satisfied for the integrals to make sense in the mean square or ordinary improper Riemann integral sense. Form of smoothness or Dirichlet conditions.
- As with Fourier series, inverse transform formula generally gives $f(t)$ accurately at points where $f(t)$ is continuous, but yields the midpoint when $f(t)$ has jumps.

- The intuition is that Fourier transforms can be viewed as a limit of Fourier series as the period grows to infinity and the sum becomes an integral.
- $\frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}\,d\omega$ is called the *inverse Fourier transform* of $F(j\omega)$. Note it is identical in form to the Fourier transform except for the sign in the complex exponential, and the factor of $1/2\pi$.

Uniqueness of a Signal Recovered From its Fourier Transform

- The signal recovered from its Fourier transform is not necessarily unique. For example, consider the Fourier transform of (one definition of) the rect signal

$$f(t) = \mathrm{rect}(t/T) = \begin{cases} 1 & |t| \le \frac{T}{2} \\ 0 & \text{otherwise} \end{cases}$$

where

$$\mathrm{rect}(t) = \begin{cases} 1 & |t| \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

This is the same signal as considered in Fourier series, but now it is defined for all real t and not just in a finite interval.

The Fourier transform is

$$\begin{aligned} F(j\omega) &= \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \\ &= \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-j\omega t}\, dt \\ &= \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-\frac{T}{2}}^{\frac{T}{2}} \\ &= \frac{1}{-j\omega}\left(e^{-j\omega T/2} - e^{j\omega T/2}\right) \\ &= \frac{1}{-j\omega}\left(-2j\sin(\omega T/2)\right) \\ &= \frac{2\sin(\omega T/2)}{\omega} \end{aligned}$$

$$
\begin{aligned}
&= T\frac{\sin(\pi(\omega T/2\pi))}{\pi(\omega T/2\pi)} \\
&= T\,\mathrm{sinc}(\omega T/2\pi)
\end{aligned}
$$

where $\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$. In particular

$$\mathrm{rect}(t/T) \Leftrightarrow T\,\mathrm{sinc}(\omega T/2\pi)$$

and

$$\mathrm{rect}(t) \Leftrightarrow \mathrm{sinc}(\omega/2\pi).$$

If we use any of the alternative definitions for rect which define the value at the jump points $\pm T/2$ to be 1 or 0 or 1/2, we get the same transform.

The inverse transform of $F(j\omega)$ will be 1/2 at $t = \pm T/2$, so the signal is *not* uniquely recoverable (unless its values were 1/2 at the jumps).

It is common to define $\mathrm{rect}(1/2) = 1/2$ so that the inverse FT agrees with the original signal.

Another Perspective on the Signal Recovered from its Fourier Transform

Assume $f(t)$ has a Fourier transform

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

for all real ω.

Define truncated inverse transform

$$f_a(t) = \frac{1}{2\pi}\int_{-a}^{a} F(j\omega)e^{j\omega t}\,d\omega$$

Will argue that

$$\lim_{a\to\infty} f_a(t) = f(t)$$

at points of continuity of f; that is, that the integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}\,d\omega$$

exists and has the desired value.

Substituting the expression for the Fourier transform into the expression for the truncated inverse Fourier transform

$$f_a(t) = \frac{1}{2\pi}\int_{-a}^{a}\left(\int_{-\infty}^{\infty} f(s)e^{-j\omega s}\,ds\right)e^{j\omega t}\,d\omega.$$

Assume we can change the order of integration (important detail)

$$f_a(t) = \int_{-\infty}^{\infty} f(s)\left(\frac{1}{2\pi}\int_{-a}^{a} e^{j\omega(t-s)}\,d\omega\right)\,ds$$

Evaluating the integral in parentheses:

$$\begin{aligned}\left.\frac{e^{j\omega(t-s)}}{j2\pi(t-s)}\right|_{-a}^{a} &= \frac{\sin(a(t-s))}{\pi(t-s)} = \left(\frac{a}{\pi}\right)\frac{\sin(\pi\frac{a}{\pi}(t-s))}{\pi\frac{a}{\pi}(t-s)} \\ &= \frac{a}{\pi}\operatorname{sinc}\left(\frac{a}{\pi}(t-s)\right) \triangleq F_a(t-s),\end{aligned}$$

where

$$F_a(t) = \frac{a}{\pi}\operatorname{sinc}\left(\frac{a}{\pi}t\right)$$

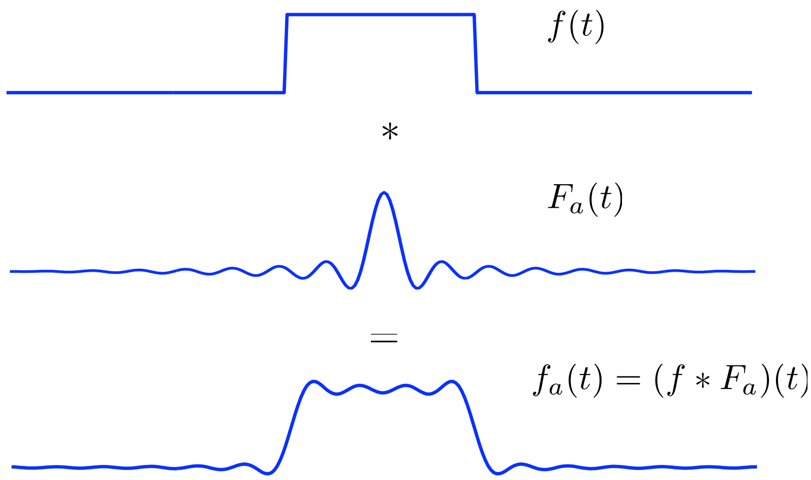
is called the *Fourier integral kernel*

$F_a(t)$ is symmetric about the origin and has unit integral. Thus

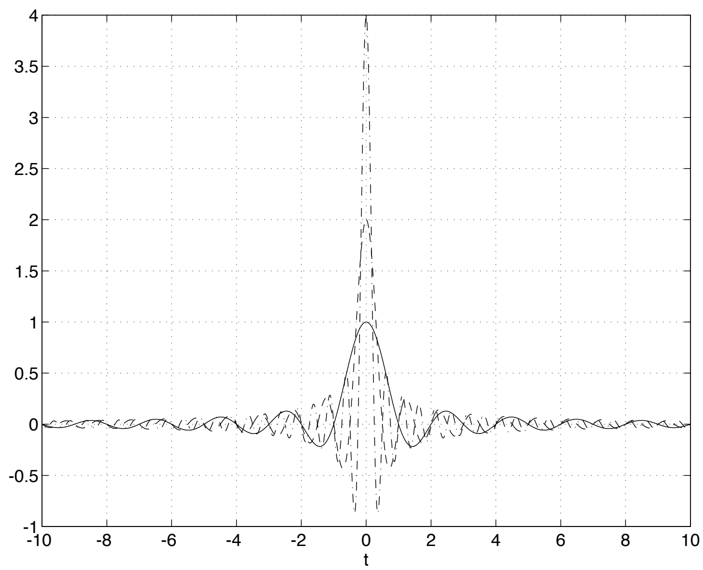
$$f_a(t) = \int_{-\infty}^{\infty} f(s)F_a(t-s)\,ds,$$

which is a convolution integral!

Example: If $f(t) = \operatorname{rect} t$, then $f_a(t)$ is:



As a increases, $F_a(t)$ gets narrower and taller,



The Fourier kernel $F_a(t)$:
$a = \pi$ (solid line), 2π (dashed line),4π (dash-dot line)

As $a \to \infty$ the kernel behaves as an impulse function:

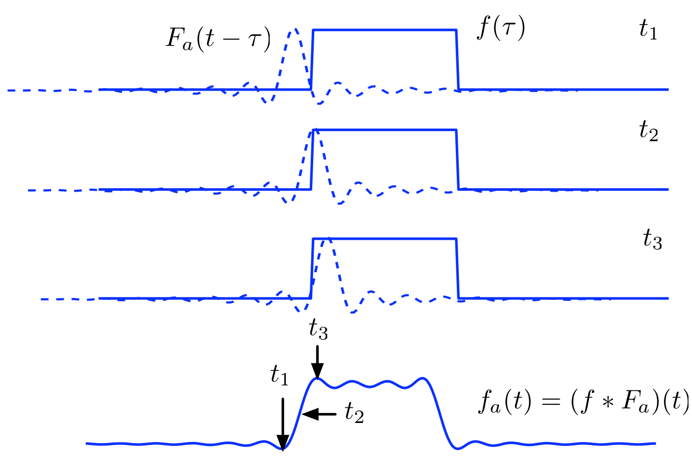
- kernel is zero except near origin
- always has area 1

Thus

$$\begin{aligned}\lim_{a\to\infty} f_a(t) &= \lim_{a\to\infty}\int_{-\infty}^{\infty} f(s)F_a(t-s)\,ds. \\ &= f(t)\end{aligned}$$

so $f_a(t) \to f(t)$ as $a \to \infty$.

This argument explains the behavior at discontinuities. If $f(t) = \mathrm{rect}(t)$,



At t_2, half of the sinc() overlaps the rect(), so the value of the convolution is $1/2$.

Suppose that $f(t)$ has a simple jump from $f(t^-)$ on the left to $f(t^+)$ on the right. If $f(t)$ is continuous near t except at the jump, for very large a and very small ϵ

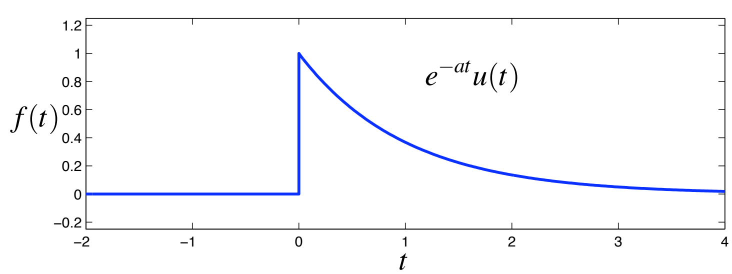
$$\begin{aligned} f_a(t) &= \int_{-\infty}^{\infty} f(s)F_a(t-s)\,ds \\ &\approx \int_{t-\epsilon}^{t} f(s)F_a(t-s)\,ds + \int_{t}^{t+\epsilon} f(s)F_a(t-s)\,ds \\ &\approx f(t^-)\int_{t-\epsilon}^{t} F_a(t-s)\,ds + f(t^+)\int_{t}^{t+\epsilon} F_a(t-s)\,ds \\ &\approx \frac{f(t^-)}{2} + \frac{f(t^+)}{2} \end{aligned}$$

Which explains why the value of inverse Fourier transforms (and for similar reasons, Fourier series) yield midpoints at jumps.

Fourier Transform Example: Exponential Signal

$$\begin{aligned} f(t) &= \begin{cases} e^{-at} & t \geq 0 \\ 0 & \text{otherwise} \end{cases} \\ &= e^{-at}u(t); \quad \text{all } t \end{aligned}$$

where $a > 0$.



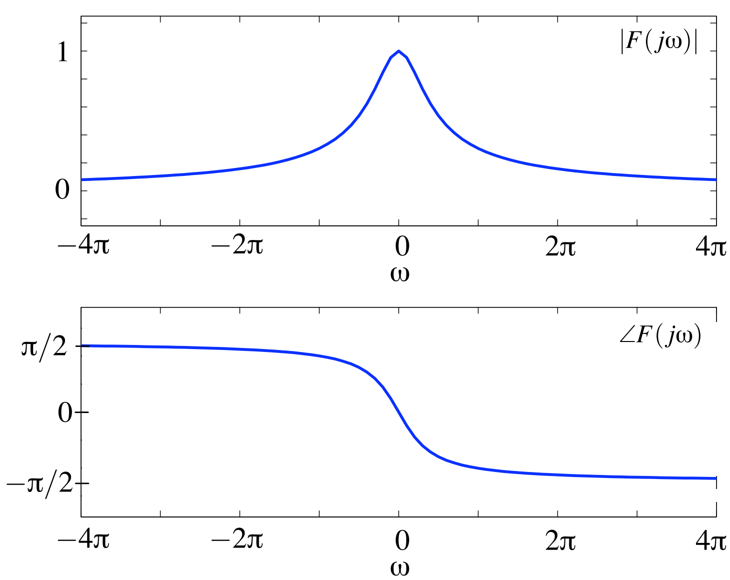
By the definition of the Fourier transform

$$\begin{aligned} F(j\omega) &= \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \\ &= \int_{0}^{\infty} e^{-at}e^{-j\omega t}\,dt \\ &= \int_{0}^{\infty} e^{-(a+j\omega)t}\,dt \\ &= \left.\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right|_0^\infty \\ &= \frac{1}{a+j\omega} \end{aligned}$$

Thus

$$e^{-at}u(t) \Leftrightarrow \frac{1}{a+j\omega}$$

For $a = 1$:



Notes:

- The inverse Fourier transform of $F(j\omega)$ will equal $f(t)$ at all values of t except possibly at the origin, where the inverse Fourier transform evaluates to $1/2$. Whether or not this equals $f(t)$ for $t = 0$ depends on the definition of $u(t)$.
- The magnitude of the transform is symmetric about the origin,

$$|F(-j\omega)| = |F(j\omega)|; \text{ all } \omega$$

- The phase of the transform is antisymmetric about the origin,

$$\angle F(-j\omega) = -\angle F(j\omega); \text{ all } \omega$$

  Will see Fourier transforms have many such symmetry properties
- As we progress, will compute and collect Fourier transforms of many common signals.

- So far have the FT for a rect signal and an exponential signal.
- In the real world, usually look transforms up in book or compute numerically.

Symmetry Properties

Useful for checks on results, finding computational shortcuts, or for signal encoding and reconstruction.

Recall: Even and Odd Functions

Any signal $f(t)$ can be uniquely decomposed into an even part and an odd part; that is,

$$f(t) = f_e(t) + f_o(t)$$

where $f_e(t)$ is even and $f_o(t)$ is odd.

Construction

$$f_e(t) = \frac{1}{2}(f(t) + f(-t))$$
$$f_o(t) = \frac{1}{2}(f(t) - f(-t))$$

If $e_1(t)$ and $e_2(t)$ are even functions and $o_1(t)$ and $o_2(t)$ are odd functions, then

- $e_1(t) \pm e_2(t)$ is even
- $e_1(t)e_2(t)$ is even
- $o_1(t) \pm o_2(t)$ is odd
- $o_1(t)o_2(t)$ is even
- and $e_1(t)o_2(t)$ is odd

Fourier Transforms of Even and Odd Functions.

Assume $f(t)$ is a possibly complex function, $f(t) = f_e(t) + f_o(t)$, where $f_e(t)$ and $f_o(t)$ are the even and odd components of $f(t)$.

$$
\begin{aligned}
F(j\omega) &= \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \\
&= \int_{-\infty}^{\infty} \left(f_e(t) + f_o(t)\right)\left(\cos(\omega t) - j\sin(\omega t)\right)\, dt \\
&= \int_{-\infty}^{\infty} f_e(t)\cos(\omega t) dt - j\int_{-\infty}^{\infty} f_e(t)\sin(\omega t)\, dt \\
&\qquad + \int_{-\infty}^{\infty} f_o(t)\cos(\omega t)\, dt - j\int_{-\infty}^{\infty} f_o(t)\sin(\omega t)\, dt \\
&= \int_{-\infty}^{\infty} f_e(t)\cos(\omega t)\, dt - j\int_{-\infty}^{\infty} f_o(t)\sin(\omega t)\, dt
\end{aligned}
$$

$$
= \quad F_e(j\omega) + F_o(j\omega)
$$

where $F_e(j\omega)$ is the cosine transform of the even part of $f(t)$ and $F_o(j\omega)$ is $-j$ times the sine transform of the odd part.

Note:

- if $f(t)$ is an *even* function of t, then $F(j\omega)$ is an *even* function of ω.
- If $f(t)$ is an *odd* function of t, then $F(j\omega)$ is an *odd* function of ω.

Here $f(t)$ could be real, imaginary, or complex.

Fourier Transform of Real Functions

If $f(t)$ is further restricted to be real-valued, then $f_e(t)$ and $f_o(t)$ are also both real. Then

$$F(j\omega) = \underbrace{\int_{-\infty}^{\infty} f_e(t)\cos(\omega t)\,dt}_{real} - j\underbrace{\int_{-\infty}^{\infty} f_o(t)\sin(\omega t)\,dt}_{imaginary}$$

and

$$\begin{aligned}\Re(F(j\omega)) &= \int_{-\infty}^{\infty} f_e(t)\cos(\omega t)\,dt \\ \Im(F(j\omega)) &= -\int_{-\infty}^{\infty} f_o(t)\sin(\omega t)\,dt.\end{aligned}$$

The real part of $F(j\omega)$ is even in ω, and the imaginary part of is odd in ω.

- If $f(t)$ is *real* and *even* in t, then $F(j\omega)$ is *real* and *even* in ω.
- If $f(t)$ is *real* and *odd* in t, then $F(j\omega)$ is *imaginary* and *odd* in ω.

Combining these two facts,

$$\begin{aligned}F(-j\omega) &= \Re(F(-j\omega)) + j\Im(F(-j\omega)) \\ &= \Re(F(j\omega)) - j\Im(F(j\omega)) \\ &= F^*(j\omega),\end{aligned}$$

The Fourier transform of a *real-valued* signal is *Hermitian*.

Fourier Transform of Imaginary Functions

If $f(t)$ is purely imaginary then $f_e(t)$ and $f_o(t)$ are also imaginary, and

$$F(j\omega) = \underbrace{\int_{-\infty}^{\infty} f_e(t)\cos(\omega t)\,dt}_{imaginary} - \underbrace{j\int_{-\infty}^{\infty} f_o(t)\sin(\omega t)\,dt}_{real}.$$

Then

$$\Re(F(j\omega)) = -j\int_{-\infty}^{\infty} f_o(t)\sin(\omega t)\,dt$$

is odd in ω, and

$$\Im(F(j\omega)) = -j\int_{-\infty}^{\infty} f_e(t)\cos(\omega t)\,dt$$

is even in ω. Combining these

$$\begin{aligned} F(-j\omega) &= \Re(F(-j\omega)) + j\Im(F(-j\omega)) \\ &= -\Re(F(j\omega)) + j\Im(F(j\omega)) \\ &= -\left(\Re(F(j\omega)) - j\Im(F(j\omega))\right) \\ &= -F^*(j\omega); \end{aligned}$$

The Fourier transform of an *imaginary-valued* signal is *anti-Hermitian*.

Summary of Symmetry Properties

- For any real, imaginary, or complex signal,
  - An *even* signal has an *even* transform, and
  - An *odd* signal has an *odd* transform.
- A *real* signal has a *Hermitian* symmetric transform,

$$F(-j\omega) = F^*(j\omega).$$

- An *imaginary* signal has an *anti-Hermitian* symmetric transform,

$$F(-j\omega) = -F^*(j\omega).$$