STANFORD UNIVERSITY
EE 102B Spring-2013

Lecture 09

The Discrete-Time Fourier Transform (DTFT)

April 19, 2013

ASSIGNMENTS

- Reading for this Lecture:
  - SPF: Sections 66-1, 66-2, & 66-3 (notes posted on Course2Go website)
  - S&S: Chapter 5
- HW#03 is due by 5pm Wednesday, April 24 in Packard 263.
- Lab #03 to be posted later today. It is due by 5pm, Friday, April 26, in Packard 263.

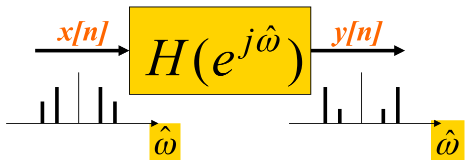
Office Hours for Course Staff – Come see us.

- Ron Schafer: Mon. 2:00-3:00 pm and Weds. 2:00-4:00 pm in Packard 211
- Dookun Park: Tues. 1:30 ~ 3:30 pm in Packard 107
- Ruo Yu Gu (Roy): Mon. 4:00 ~ 6:00 pm in Packard 106
- https://class2go.stanford.edu/EE102B/Spring2013/pages/staff

Lecture Objective

- Review a few things about the Frequency Response
- Introduce the discrete-time Fourier transform, called the **DTFT**, for discrete time sequences that may not be finite or periodic.
- Establish general concept of "**frequency domain**" representations and **spectrum** that is a **continuous** function of (normalized) frequency – not necessarily just the line spectrum that we have been using so far.

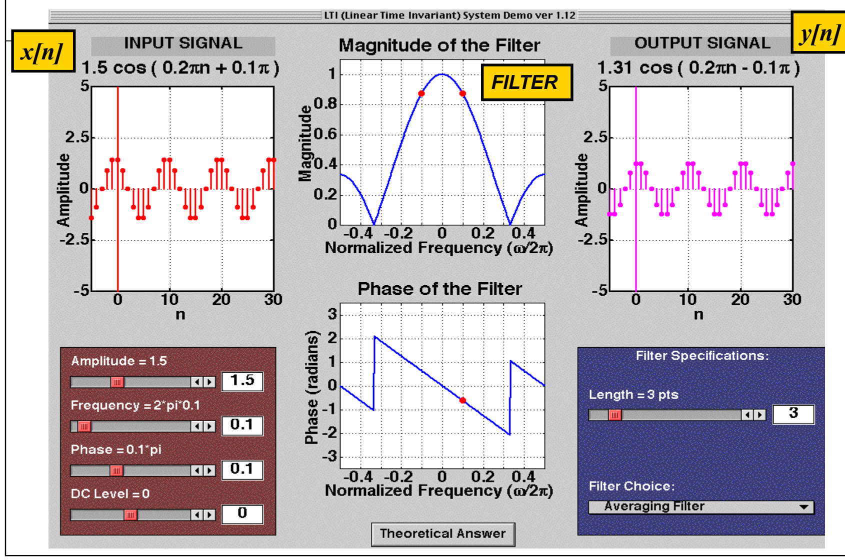
Eigenfunctions for LTI FIR Systems



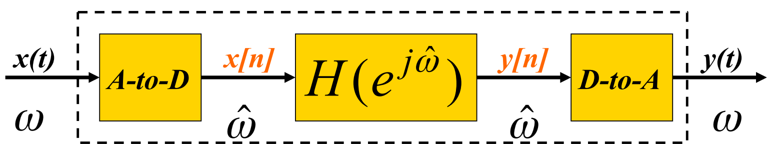
$$x[n] = e^{j\hat{\omega}n} \qquad y[n] = H(e^{j\hat{\omega}})e^{j\hat{\omega}n}$$

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} h[k]e^{-j\hat{\omega}k}$$

LTI Demo with Sinusoids



DIGITAL "FILTERING"



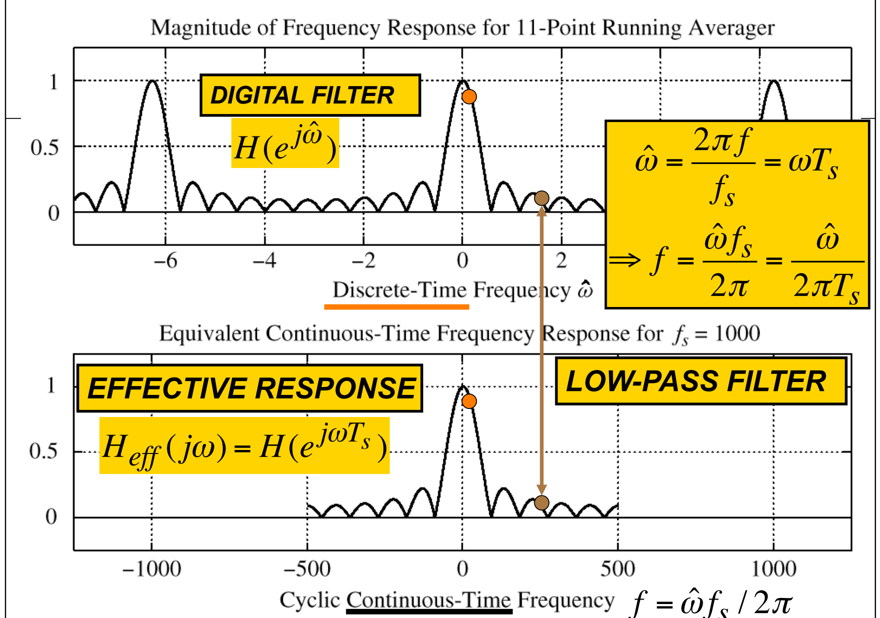
$H_{eff}(j\omega)$ "Effective" freq. resp.

If $x(t) = e^{j\omega_0 t}$ and $2\pi f_s > 2\omega_0$, then $x[n] = e^{j\hat{\omega}_0 n}$,

and $y[n] = H(e^{j\hat{\omega}_0})e^{j\hat{\omega}_0 n}$, where $\hat{\omega}_0 = \omega T_s$.

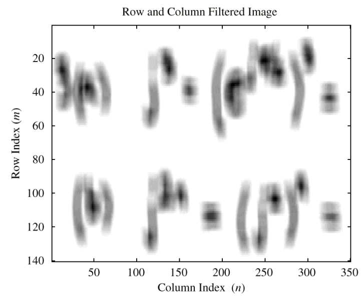
Therefore $y(t) = H_{eff}(j\omega_0)e^{j\omega_0 t}$, where

$$H_{eff}(j\omega) = H(e^{j\omega T_s})$$

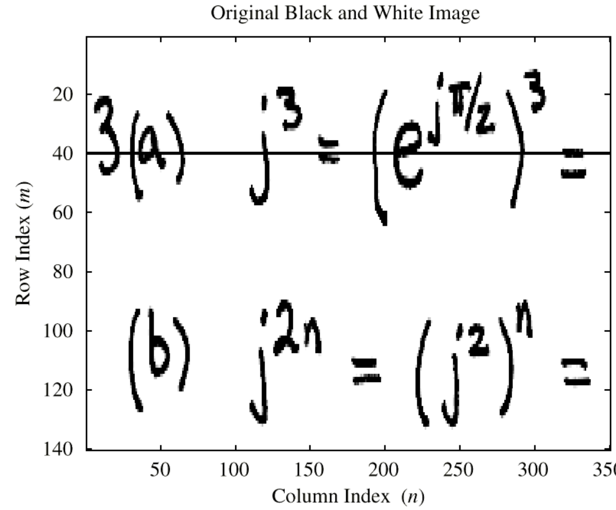


FILTER TYPES

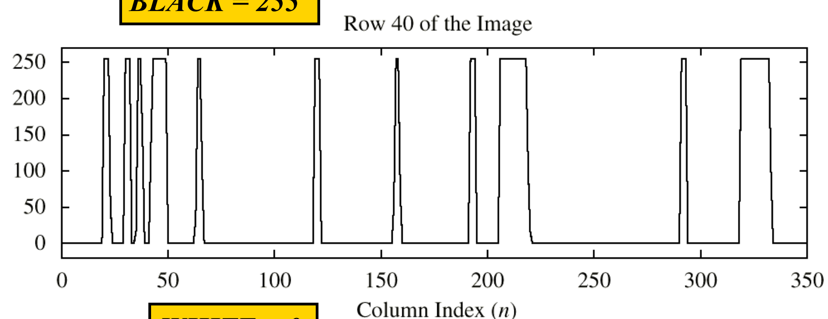
- LOW-PASS FILTER (LPF)
  - BLURRING
  - ATTENUATES HIGH FREQUENCIES
- HIGH-PASS FILTER (HPF)
  - SHARPENING for IMAGES
  - BOOSTS THE HIGHS
  - REMOVES DC
- BAND-PASS FILTER
  (BPF)



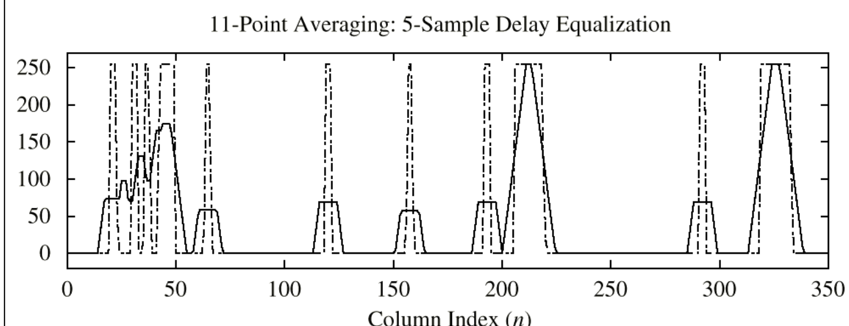
B & W IMAGE



ROW of B&W IMAGE



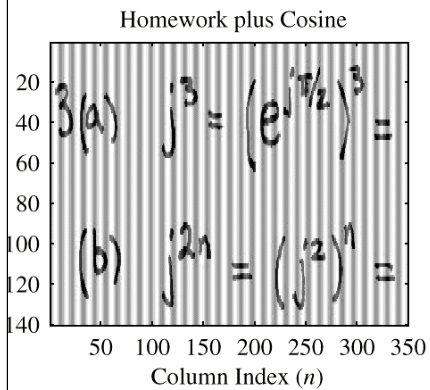
FILTERED ROW of IMAGE

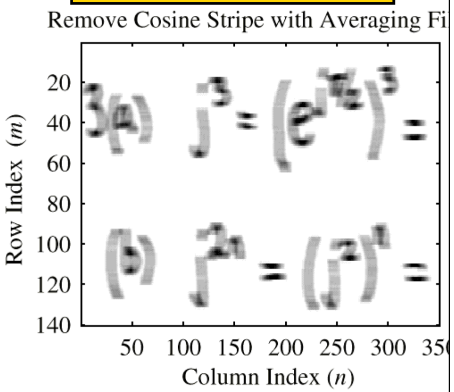


ADJUSTED DELAY by 5 samples

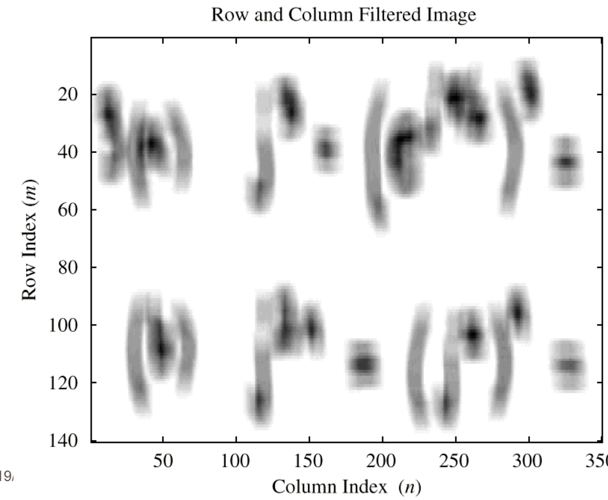
B&W IMAGE with COSINE

FILTERED: 11-pt AVG





FILTERED B&W IMAGE



LPF: BLUR

Simple Nulling Filter - I

- How could we get rid of a sinusoidal component?
- Make the frequency response zero at the frequency of the sinusoid.

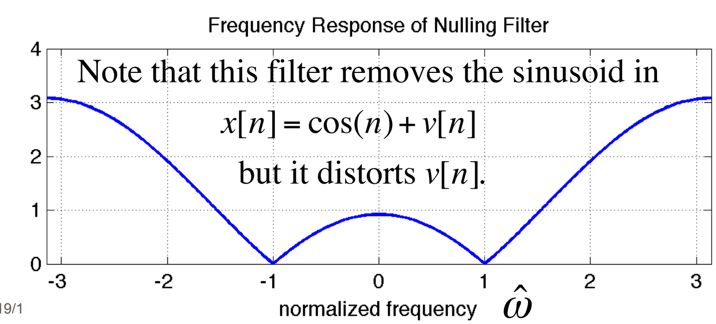
$$x[n] = \cos(\hat{\omega}_0 n) = 0.5e^{j\hat{\omega}_0 n} + 0.5e^{-j\hat{\omega}_0 n}$$

$$H(e^{j\hat{\omega}}) = (1 - e^{j\hat{\omega}_0} e^{-j\hat{\omega}})(1 - e^{-j\hat{\omega}_0} e^{-j\hat{\omega}})$$

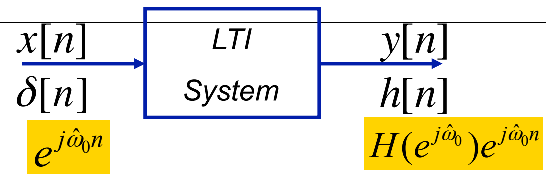
$$= 1 - 2\cos(\hat{\omega}_0) e^{-j\hat{\omega}} + e^{-j\hat{\omega}2}$$

Simple Nulling Filter – II

- ww = (-200:200)*pi/200;
  H = freqz([1, -2*cos(1), 1],1,ww);
  plot(ww,abs(H))



The Frequency Response



$$\boxed{H(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\hat{\omega}n}}$$

$$\boxed{\text{Periodic}: H(e^{j(\hat{\omega}+2\pi)}) = H(e^{j\hat{\omega}})}$$

$$\boxed{y[n] = A \cdot \left|H(e^{j\hat{\omega}_0})\right| \cos(\hat{\omega}_0 n + \varphi + \angle H(e^{j\hat{\omega}_0}))}$$

Discrete-Time Fourier Transform

- **Definition of the DTFT:**

Discrete-time Fourier Transform

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n}$$

Inverse Discrete-time Fourier Transform

$$x[n] = \frac{1}{2\pi}\int_0^{2\pi} X(e^{j\hat{\omega}})e^{j\hat{\omega}n}\,d\hat{\omega}$$

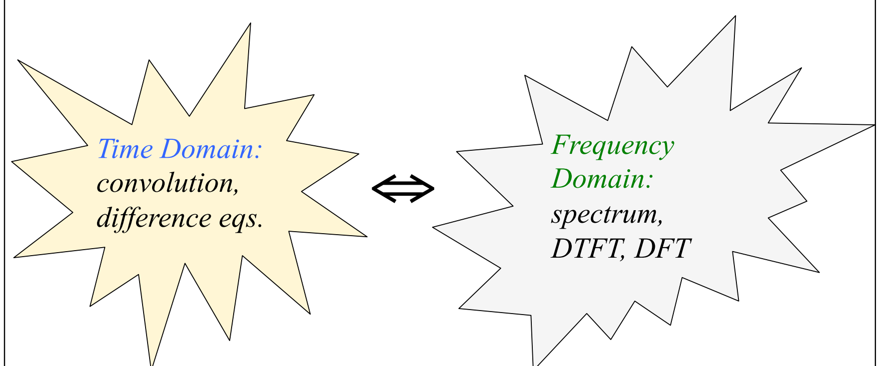
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\hat{\omega}})e^{j\hat{\omega}n}\,d\hat{\omega}$$

$$\boxed{\text{Periodic}: X(e^{j(\hat{\omega}+2\pi)}) = X(e^{j\hat{\omega}})}$$

What is a Transform?

- Change a problem from one domain to another to make it easier to solve
- Has to be invertible
  - Transform into new domain
  - Get back out (inverse must be unique)

Two Domains



Periodicity of DTFT

Can show that for integer m: $X(e^{j\hat{\omega}}) = X(e^{j(\hat{\omega}+2m\pi)})$

$$X(e^{j(\hat{\omega}+2\pi)}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\hat{\omega}+2m\pi)n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n}\underbrace{e^{-j2\pi mn}}_{1} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n}$$

Existence of DTFT

- **Discrete-time Fourier transform (DTFT) exists – provided that the sequence is absolutely-summable**

$$\left|X(e^{j\hat{\omega}})\right| = \left|\sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n}\right|$$

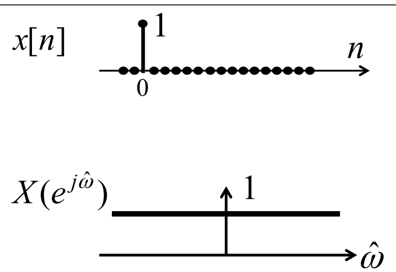
$$\le \sum_{n=-\infty}^{\infty}\left|x[n]e^{-j\hat{\omega}n}\right| = \sum_{n=-\infty}^{\infty}\left|x[n]\right| < \infty$$

- **DTFT applies to discrete time sequences, regardless of length (as long as it is <u>absolute summable</u>);**

DTFT of a Single Sample

$$x[n] = \begin{Bmatrix} 1, & n=0 \\ 0, & \text{elsewhere} \end{Bmatrix} = \delta[n]$$

Unit Impulse sequence



$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\hat{\omega}n}$$

$$= \sum_{n=0}^{0} e^{-j\hat{\omega}n} = 1$$

$$x[n] = \delta[n] \overset{\mathcal{F}}{\Leftrightarrow} X(e^{j\hat{\omega}}) = 1$$

Othogonality makes it work.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\hat{\omega}})e^{j\hat{\omega}n}\,d\hat{\omega} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{k=-\infty}^{\infty} x[k]e^{-j\hat{\omega}k}\right)e^{j\hat{\omega}n}\,d\hat{\omega}$$

$$= \sum_{k=-\infty}^{\infty} x[k]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\hat{\omega}(n-k)}\,d\hat{\omega}\right)$$

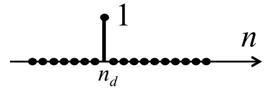
$$= \sum_{k=-\infty}^{\infty} x[k]\frac{\sin\pi(n-k)}{\pi(n-k)}$$

$$= x[n]$$

Delayed Unit Impulse

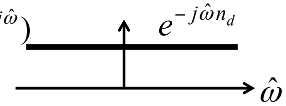
$$x_d[n] = \delta[n-n_d] = \begin{cases} 1, & n = n_d \\ 0, & \text{elsewhere} \end{cases}$$

$x[n]$



$$X_d(e^{j\hat{\omega}}) = \sum_{n=n_d}^{n_d} e^{-j\hat{\omega} n} = e^{-j\hat{\omega} n_d}$$

$X(e^{j\hat{\omega}})$



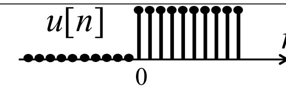
Generalizes to the <u>delay property</u>

$$x_d[n] = x[n-n_d] \Leftrightarrow$$

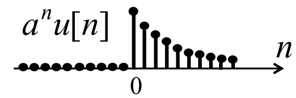
$$X_d(e^{j\hat{\omega}}) = X(e^{j\hat{\omega}})e^{-j\hat{\omega} n_d}$$

DTFT of Right-Sided Exponential

Unit Step Function: $u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$



$$x[n] = a^n u[n], \quad |a| < 1$$



$$X(e^{j\hat{\omega}}) = \sum_{n=0}^{\infty} a^n e^{-j\hat{\omega} n}$$

$$= \sum_{n=0}^{\infty} (ae^{-j\hat{\omega}})^n = \frac{1}{1-ae^{-j\hat{\omega}}} \quad \text{if } |a| < 1$$

Plotting: Magnitude and Angle Form

$$x[n] = a^n u[n] \Leftrightarrow X(e^{j\hat{\omega}}) = \frac{1}{1-ae^{-j\hat{\omega}}}$$

$$X(e^{j\hat{\omega}}) = |X(e^{j\hat{\omega}})| e^{j\angle X(e^{j\hat{\omega}})}$$

$$|X(e^{j\hat{\omega}})|^2 = X(e^{j\hat{\omega}})X^*(e^{j\hat{\omega}}) = \frac{1}{1-ae^{-j\hat{\omega}}} \cdot \frac{1}{1-ae^{j\hat{\omega}}}$$

$$= \frac{1}{1+a^2-2a\cos(\hat{\omega})}$$

$$\angle X(e^{j\hat{\omega}}) = \arctan\left(\frac{-a\sin(\hat{\omega})}{1-a\cos(\hat{\omega})}\right)$$

Magnitude and Angle Plots

EVEN Function

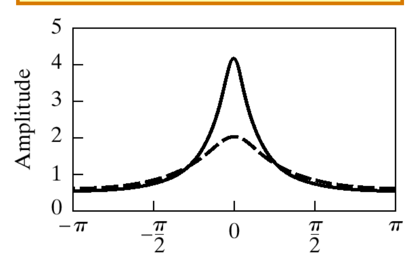
$$|X(e^{j\hat{\omega}})| = \frac{1}{(1+a^2-2a\cos(\hat{\omega}))^{1/2}}$$

$$|X(e^{-j\hat{\omega}})| = |X(e^{j\hat{\omega}})|$$

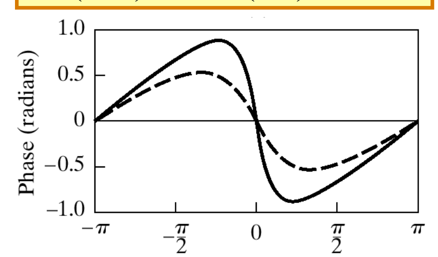
ODD Function

$$\angle X(e^{j\hat{\omega}}) = \arctan\left(\frac{-a\sin(\hat{\omega})}{1-a\cos(\hat{\omega})}\right)$$

$$\angle X(e^{-j\hat{\omega}}) = -\angle X(e^{j\hat{\omega}})$$



Radian Frequency ($\hat{\omega}$)



Radian Frequency ($\hat{\omega}$)

Can you always evaluate the inverse DTFT integral?

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\hat\omega})e^{j\hat\omega n}\,d\hat\omega$$

$$X(e^{j\hat\omega}) = \frac{1}{1+0.3e^{-j\hat\omega}} \quad \Rightarrow x[n] = ?$$

$$x[n] = \int_{-\pi}^{\pi} \frac{1}{1+0.3e^{-j\hat\omega}}\, e^{j\hat\omega n}\,d\hat\omega \quad ??$$

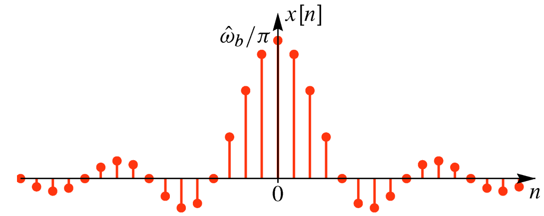
$$\boxed{x[n] = a^n u[n] \Leftrightarrow X(e^{j\hat\omega}) = \frac{1}{1-ae^{-j\hat\omega}}}$$

SINC Function

A "sinc" function or sequence

$$x[n] = \frac{\sin(0.25\pi\, n)}{\pi\, n}, \quad -\infty < n < \infty$$

$$X(e^{j\hat\omega}) = \sum_{n=-\infty}^{\infty} \frac{\sin(0.25\pi n)}{\pi n} e^{-j\hat\omega n} \quad ??$$



SINC Function

A "sinc" function or sequence

$$x[n] = \frac{\sin(0.2\pi\, n)}{\pi\, n}, \quad -\infty < n < \infty$$

Consider an ideal band-limited

$$X(e^{j\hat\omega}) = \begin{cases} 1, & |\hat\omega| \le 0.2\pi \\ 0, & 0.2\pi < |\hat\omega| \le \pi \end{cases}$$

Discrete-time Fourier Transform Pair

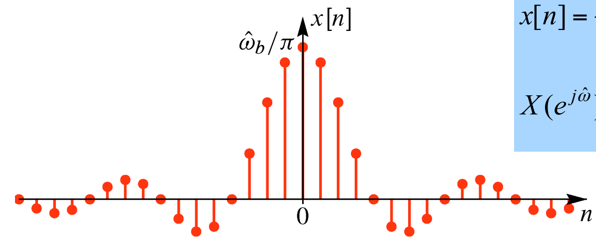
$$x[n] = \frac{1}{2\pi}\int_{-0.2\pi}^{0.2\pi} e^{j\hat\omega n}\,d\hat\omega = \left.\frac{e^{j\hat\omega n}}{2\pi\, jn}\right|_{-0.2\pi}^{0.2\pi}$$

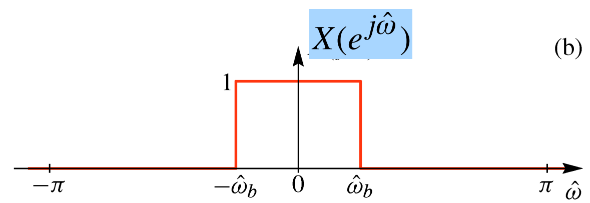
$$= \frac{e^{j0.2\pi n} - e^{-j0.2\pi n}}{2\pi\, jn} = \frac{\sin(0.2\pi\, n)}{\pi\, n}$$

$$x[n] = \frac{\sin(\hat\omega_b n)}{\pi\, n} \overset{DTFT}{\Leftrightarrow} X(e^{j\hat\omega}) = \begin{cases} 1, & |\hat\omega| \le \hat\omega_b \\ 0, & \hat\omega_b < |\hat\omega| \le \pi \end{cases}$$

SINC Function – Rectangle DTFT pair

$$x[n] = \frac{\sin(\hat\omega_b n)}{\pi\, n} \overset{DTFT}{\Leftrightarrow} X(e^{j\hat\omega}) = \begin{cases} 1, & |\hat\omega| \le \hat\omega_b \\ 0, & \hat\omega_b < |\hat\omega| \le \pi \end{cases}$$

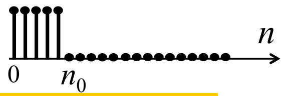




DTFT of Rectangular Pulse

A "rectangular" sequence of length L

$$x[n] = \begin{cases} 1, & 0 \le n \le L-1 \\ 0, & \text{elsewhere} \end{cases}$$



$$X(e^{j\hat{\omega}}) = \sum_{n=0}^{L-1} e^{-j\hat{\omega}n} = \frac{1 - e^{-jL\hat{\omega}}}{1 - e^{-j\hat{\omega}}}$$

$$= \frac{e^{-j(L-1)\hat{\omega}/2}\left(\sin \frac{L\hat{\omega}}{2}\right)}{\left(\sin \frac{\hat{\omega}}{2}\right)}$$

Discrete-time Fourier Transform Pair

Dirichlet Function: $D_L(e^{j\hat{\omega}})$

Summary of DTFT Pairs

$$x[n] = \delta[n - n_d] \Leftrightarrow X(e^{j\hat{\omega}}) = e^{-j\hat{\omega}n_d}$$

$$x[n] = a^n u[n] \Leftrightarrow X(e^{j\hat{\omega}}) = \frac{1}{1 - ae^{-j\hat{\omega}}}$$

$$x[n] = \frac{\sin(\hat{\omega}_c n)}{\pi n} \Leftrightarrow X(e^{j\hat{\omega}}) = \begin{cases} 1 & |\hat{\omega}| \le \hat{\omega}_c \\ 0 & \hat{\omega}_c < |\hat{\omega}| < \pi \end{cases}$$

$$x[n] = \begin{cases} 1 & 0 \le n < L \\ 0 & \text{otherwise} \end{cases} \Leftrightarrow X(e^{j\hat{\omega}}) = \frac{\sin(\frac{1}{2}L\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})} e^{-j\hat{\omega}(L-1)/2}$$

Using the DTFT

- The DTFT provides a ***frequency-domain*** representation that is invaluable for thinking about signals and solving DSP problems.
- To use it effectively you must
  - know **PAIRS**: the Fourier transforms of certain important signals
  - know ***properties*** and certain key ***theorems***
  - be able to combine time-domain and frequency domain methods appropriately

Summary

Discrete-time Fourier Transform (DTFT)

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n}$$

Inverse Discrete-time Fourier Transform

$$x[n] = \frac{1}{2\pi}\int_0^{2\pi} X(e^{j\hat{\omega}})e^{j\hat{\omega}n}\,d\hat{\omega}$$

Frequency Response and the DTFT

- The frequency response is the DTFT of the impulse response

$$H(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\hat{\omega}n}$$

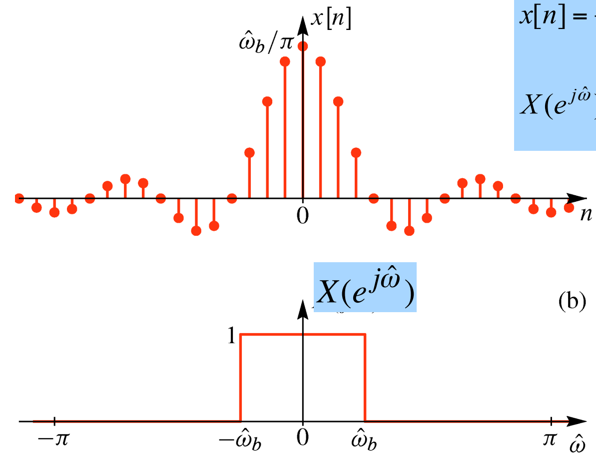
- Therefore the impulse response is the IDFT of the frequency response

$$h[n] = \frac{1}{2\pi}\int_{0}^{2\pi} H(e^{j\hat{\omega}})e^{j\hat{\omega}n}\,d\hat{\omega}$$

SINC Function – Rectangle DTFT pair

$$x[n] = \frac{\sin(\hat{\omega}_b n)}{\pi n} \overset{DTFT}{\Leftrightarrow}$$

$$X(e^{j\hat{\omega}}) = \begin{cases} 1, & |\hat{\omega}| \le \hat{\omega}_b \\ 0, & \hat{\omega}_b < |\hat{\omega}| \le \pi \end{cases}$$



Ideal Lowpass Filter

$$h_{lp}[n] = \frac{\sin(\hat{\omega}_c n)}{\pi n} \overset{DTFT}{\Leftrightarrow}$$

$$H_{lp}(e^{j\hat{\omega}}) = \begin{cases} 1, & |\hat{\omega}| \le \hat{\omega}_c \\ 0, & \hat{\omega}_c < |\hat{\omega}| \le \pi \end{cases}$$