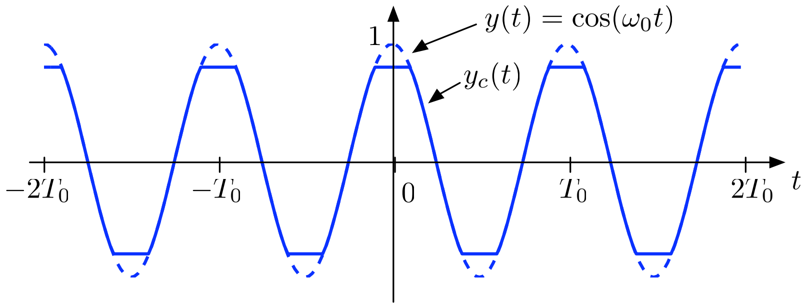
EE102A Winter 2012-13 Pauly
Signal Processing and Linear Systems I

Problem Set #4 Solutions

Due: Wednesday, February 6 at 5 PM.

1. An amplifier has an input of $x(t) = \cos(\omega_0 t)$. Unfortunately, the output is limited by the rail voltage of the amplifier, and the result is the clipped waveform $y_c(t)$ shown below:



We will use the Fourier series to characterize the effect of this clipping.

a) Is the amplifier a linear system? Is it time-invariant?

Solution:

The amplifier clips at the same voltage, for any input. Hence if we scale the input, the cosine will be clipped at a different level, and won't be a scaled version of the same clipped waveform. Hence, this is a *non-linear* system. However, we can still use frequency domain methods to analyze it!

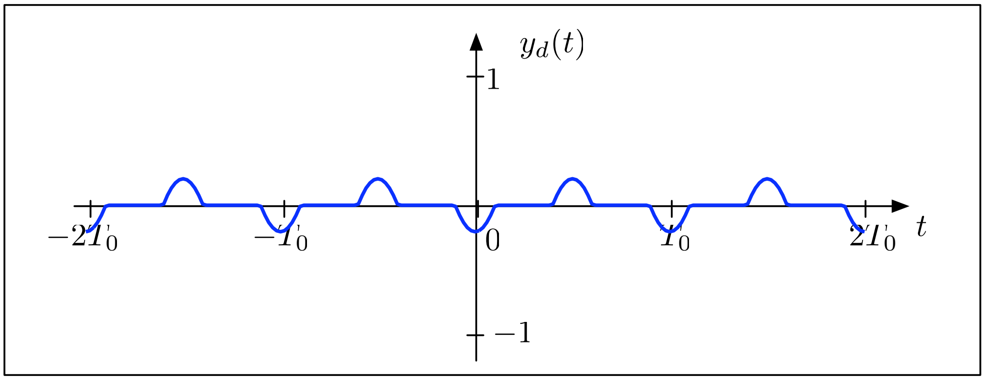
The clipping depends on the signal itself. If we delay the signal, it will still be clipped at the same level, and will produce the a delayed version of the same clipped signal. Hence, this system is *time invariant*.

b) We can think of the clipped signal as the ideal signal $y(t) = \cos(\omega_0 t)$ plus a distortion term $y_d(t)$, so that

$$y_c(t) = y(t) + y_d(t).$$

Sketch the error signal $y_d(t)$.

Solution:



c) What is the symmetry of the Fourier series coefficients of the error term $y_d(t)$? Are they real, imaginary, or complex?

Solution:

The signal is *real and even* so this means that the Fourier series coefficients are *real and even*.

d) Which of the Fourier series coefficients of $y_d(t)$ are non-zero? For example, is D_0 non-zero, or D_1 non-zero? You don't need to calculate the coefficients.

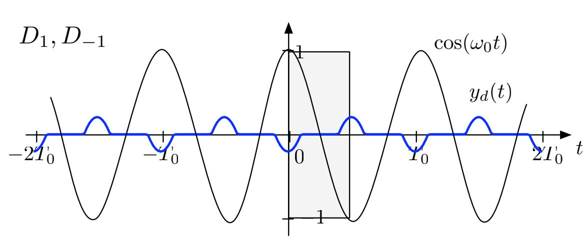
Hint: When considering the Fourier series coefficients integral over the whole period $-T_0/2$ to $T_0/2$, look for symmetries over *half* the period, such as $-T_0/2$ to 0, and 0 to $T_0/2$.

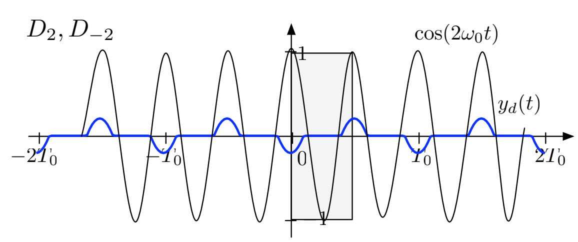
Solution:

First, since the Fourier series coefficients are real and even, the Fourier series involves only cosine terms, since cosine is real and even

$$y_d = \sum_{n=-\infty}^{\infty} D_n \cos(n\omega_0 t)$$

From the plot of part (b) we can see that $D_0 = 0$, since that is just the average value. For the higher order terms, the Fourier series coefficients will be $y_d(t)$ multiplied by $\cos(n\omega_0 t)$, integrated over one period. This is illustrated below for $n = \pm 1$ and $n = \pm 2$.





The shaded region is half of one period, from 0 to $T_0/2$. For D_1, D_{-1}, the integral over the shaded region is an odd function (centered on $T_0/4$) multiplied by an odd function $y_d(t)$, which is even. The integral will be non-zero, as will all of the n odd terms.

For D_2, D_{-2}, the integral over the half period is of an even function times an odd function, which is odd. The integral will then be zero. This will be true for all of the n even terms.

Hence the non-zero coefficients are n odd.

2. $f(t)$ is a periodic signal with a period T_0, and has a Fourier series with coefficients $\{D_n\}$.

   (a) What are the Fourier series coefficients for the delayed signal $f(t-\tau)$?

   Solution:

$$f(t-\tau) = \sum_{n=-\infty}^{\infty} D_{n,\tau} e^{jn\omega_0 t}$$

where

$$D_{n,\tau} = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} f(t-\tau) e^{-j\omega_0 nt} dt$$

Let $t' = t - \tau$,

$$\begin{aligned} D_{n,\tau} &= \frac{1}{T_0} \int_{t_0-\tau}^{t_0-\tau+T_0} f(t') e^{-j\omega_0 n(t'+\tau)} dt' \\ &= e^{-jn\omega_0\tau} \left(\frac{1}{T_0}\right) \int_{t_0-\tau}^{t_0-\tau+T_0} f(t') e^{-j\omega_0 nt'} dt' \\ &= e^{-jn\omega_0\tau} D_n \end{aligned}$$

where we have used the fact that $f(t)$ is periodic in the last step. We get the same Fourier series no matter where we choose the one period to integrate over. The Fourier series is then

$$f(t-\tau) = \sum_{n=-\infty}^{\infty} \left(D_n e^{-jn\omega_0\tau}\right) e^{jn\omega_0 t}$$

   (b) What are the Fourier series coefficients for the time scaled signal $f(at)$. Note that the period of $f(at)$ is T_0/a.

   Solution:

   We wish to find a Fourier series for $f(at)$ which will have a period T_0/a, and a fundamental frequency $a\omega_0$. The Fourier series is

$$f(at) = \sum_{n=-\infty}^{\infty} D_{n,a} e^{jan\omega_0 t}$$

where

$$D_{n,a} = \frac{1}{T_0/a} \int_{t_0}^{t_0+T_0/a} f(at) e^{-ja\omega_0 nt} dt$$

If we let $\tau = at$, then

$$\begin{aligned} D_{n,a} &= \frac{a}{T_0} \int_{at_0}^{at_0+T_0} f(\tau) e^{-jn\omega_0\tau} \frac{d\tau}{a} \\ &= \frac{1}{T_0} \int_{at_0}^{at_0+T_0} f(\tau) e^{-jn\omega_0\tau} d\tau \\ &= D_n. \end{aligned}$$

The Fourier series coefficients are the same as in the unscaled case. The Fourier series is then

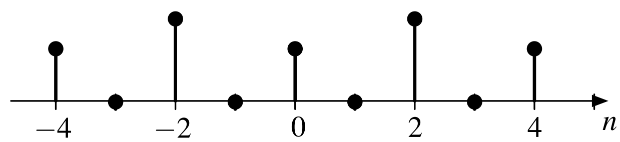
$$f(at) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0(at)}.$$

which is the orignal Fourier series, evaluated at at.

(c) You find that all of the odd Fourier series coefficients are zero (i.e. $D_1, D_{-1}, D_3, D_{-3}, \cdots$ are all zero). What can you conclude about $f(t)$?

Solution

If we sketch what the spectrum looks like, we get



The frequencies are all multiples of $2\omega_0$. This is a Fourier series that has a fundamental frequency of $2\omega_0$. The period of a signal with this fundmental frequency is

$$T_0' = \frac{2\pi}{2\omega_0} = \frac{1}{2}\left(\frac{2\pi}{\omega_0}\right) = \frac{1}{2}T_0.$$

Hence $f(t)$ has a period $T_0/2$. What we have computed is the Fourier series over an interval that includes two cycles of $f(t)$.

Another acceptable answer is that $f(t) = f(t \pm T_0/2)$, since this means the same thing.

(d) Two functions $f(t)$ and $\tilde{f}(t)$ have Fourier series coefficients D_n and $\tilde{D}_n$, respectively, and

$$\begin{aligned} \left|\tilde{D}_n\right| &= |D_n| \\ \angle\tilde{D}_n &= \angle D_n + n\frac{\pi}{2} \end{aligned}$$

Find a simple expression for $\tilde{f}(t)$ in terms of $f(t)$.

Solution:

We know from Problem 2a of HW4 that a delayed signal $f(t-\tau)$ has Fourier series coefficients $e^{-jn\omega_0\tau}D_n$, which looks a lot like what we have here with $-\omega_0\tau = \frac{\pi}{2}$, or

$$\tau = -\frac{\pi}{2\omega_0} = -\frac{\pi}{2(2\pi/T_0)} = -\frac{T_0}{4}$$

Then

$$\tilde{f}(t) = f(t-\tau) = f\left(t + \frac{T_0}{4}\right).$$

We can also work it out by substituting into the Fourier series directly, and seeing what happens. The signal $\tilde{f}(t)$ is

$$\tilde{f}(t) = \sum_{n=-\infty}^{\infty} \tilde{D}_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} e^{jn\pi/2} D_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} D_n e^{jn(\omega_0 t + \pi/2)} = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 (t + \frac{\pi}{2\omega_0})}.$$

This is then

$$\tilde{f}(t) = f\left(t + \frac{\pi}{2\omega_0}\right).$$

We can simplify this by using the fact that $\omega_0 = \frac{2\pi}{T_0}$, so

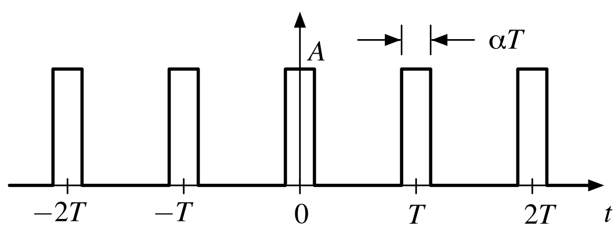
$$\tilde{f}(t) = f\left(t + \frac{\pi}{2(2\pi/T_0)}\right) = f\left(t + \frac{T_0}{4}\right).$$

A phase shift of $\pi/4$ per sample in the frequency domain produces a $T_0/4$ shift in the time domain. Here the phase increment is positive, and the shift in time is negative.

3. Switching amplifiers are a very efficient way to generate a time-varying output voltage from a fixed supply voltage. They are particularly useful in high-power applications.

   The basic idea is that an output voltage a is generated by rapidly switching between zero and the supply voltage A. The output is then lowpass filtered to remove the harmonics generated by the switching operation. For our purposes we can consider the lowpass filter as an integrator over many switching cycles, so the output voltage is the average value of the switching waveform.. Varying the switching rate varies the output voltage. In this problem we will only consider the case where the desired output voltage is constant.

   We can analyze this system with the Fourier series. If the output pulses are spaced by T, the waveform the amplifier generates immediately before the lowpass filter is



   The duty cycle of the switching amplifier is α, and the width of the pulses is αT. When $\alpha = 1$, the amplifier is constantly on and produces its maximum output A.

   (a) Reducing the duty cycle reduces the output voltage. After the lowpass filter, only the zero-frequency spectral component D_0 remains, and this will be the output voltage. Find the value of D_0 as a function of the duty cycle α. If the desired output voltage is a and the supply voltage is A, what should α be?
   Solution:

   The zero frequency spectral component is

$$D_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt = \frac{1}{T}\int_{-\alpha T/2}^{\alpha T/2} Adt = \frac{\alpha T A}{T} = \alpha A.$$

   If we want an output of a volts, we want

$$\alpha A = a$$

or

$$\alpha = a/A.$$

(b) The lowpass filter must suppress (average out) the harmonics generated by the switching waveform. The first harmonic is the most difficult to suppress since it tends to be large, and is closest in frequency. Find an expression for the amplitude of the first harmonic D_1 as a function of the duty cycle α.

Solution:

$$\begin{aligned}
D_1 &= \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-j\frac{2\pi t}{T}}\,dt \\
&= \frac{1}{T}\int_{-\alpha T/2}^{\alpha T/2} Ae^{-j\frac{2\pi t}{T}}\,dt \\
&= \frac{A}{T(-j2\pi/T)}\; e^{-j\frac{2\pi t}{T}}\Big|_{-\alpha T/2}^{\alpha T/2} \\
&= \frac{A}{T(-j2\pi/T)}\left(e^{-j\frac{2\pi(\alpha T/2)}{T}} - e^{-j\frac{2\pi(-\alpha T/2)}{T}}\right) \\
&= \frac{A}{\pi}\sin(\alpha\pi) \\
&= A\alpha\,\mathrm{sinc}(\alpha)
\end{aligned}$$

where either of the last two answers were acceptable.

(c) What duty cycle α results in the largest magnitude first harmonic?

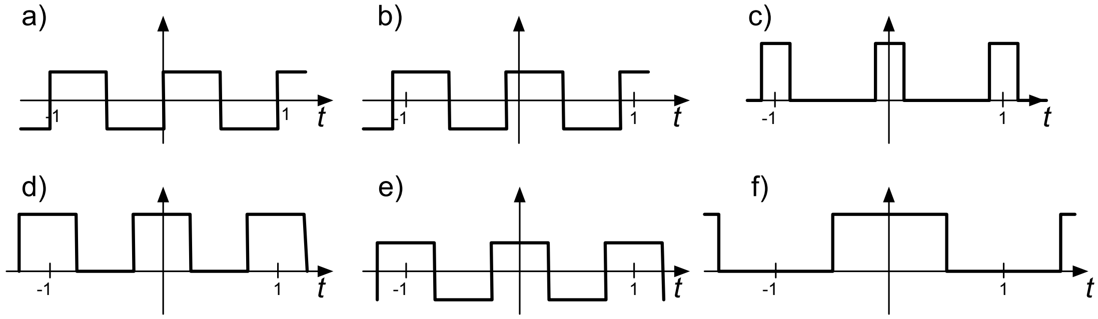
Solution:

From the previous part,

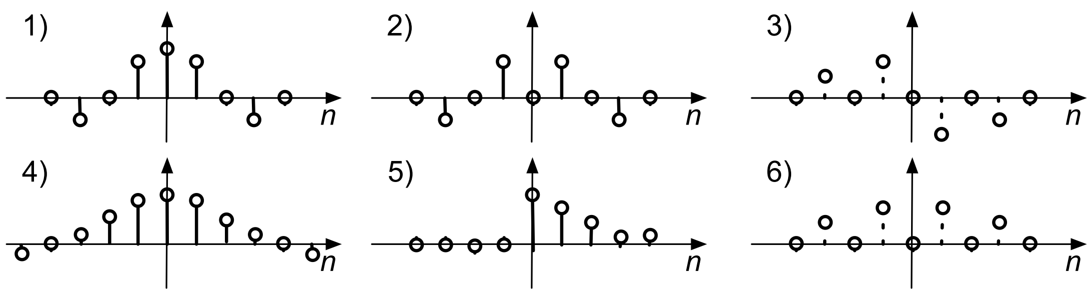
$$D_1 = \frac{A}{\pi}\sin(\alpha\pi)$$

This has a maximum at $\alpha\pi = \pi/2$, or $\alpha = 1/2$. Note that α must be between $\alpha = 0$ which is completely off to $\alpha = 1$ which is completely on.

4. Six different periodic signals (a-f) are plotted below. Assume they are all real.



Six different Fourier series coefficient spectra (1-6) are plotted below. Solid lines are real coefficients, and dashed lines are imaginary.



Each spectrum may correspond to one or more of the signals above, or maybe none at all. For each spectrum, decide which signals (if any) it corresponds to. In each case, provide a very brief description of your reasoning.

Spectrum 1: **Solution:**
This is real and even, so the signal is real and even, perhaps (c),(d), (e), or (f). It has a non-zero average value (D_0 is not zero), so (e) is ruled out. The rest are all square waves, which have

$$D_n = \frac{T}{\tau} \operatorname{sinc}\left(\frac{T}{\tau} n\right)$$

This depends only on the ratio of the square pulse length T to the period τ. The first zero is $n = 2$. The zeros of the sinc occur at the integers, so $(T/\tau)2 = 1$, or $2T = \tau$. The period is twice the length of the square pulse. This is both (d) and (f).

Spectrum 2: **Solution**
This is a real, even spectrum, so the signal is real and even. $D_0 = 0$, so it has zero average value. This is (e).

Spectrum 3: **Solution**
This spectrum odd, so it corresponds to an odd signal. The spectrum is Hermitian, so it is a real signal. This can only be (a).

Spectrum 4: **Solution**
This follows the same logic as (1), except now the first sinc zero is at $n = 4$, so $4T = \tau$, which is (c).

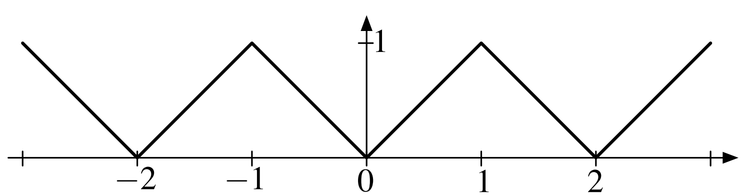
Spectrum 5: **Solution**
This spectrum is not Hermitian, so it does not correspond to a real signal. It is not one of the signals.

Spectrum 6: **Solution**
This is also not Hermitian, so it also is not one of the signals. In fact, it is anti-Hermitian (even imaginary part, odd real part), which corresponds to a purely imaginary signal.

5. For the lab, we are going to need the Fourier series of a triangle waveform, in addition to the sawtooth waveform described in class. When we found the Fourier series for the sawtooth waveform we used the integral

$$\int_0^1 te^{-j\omega t}dt = \begin{cases} \frac{1}{2} & \text{if } \omega = 0 \\ \frac{je^{-j\omega}}{\omega} + \frac{e^{-j\omega}-1}{\omega^2} & \text{if } \omega \neq 0 \end{cases}.$$

Use this result to show that the Fourier series coefficients for the triangle waveform



are

$$D_n = \begin{cases} \frac{1}{2} & \text{if } n = 0 \\ \frac{(-1)^n-1}{(n\pi)^2} & \text{if } n \neq 0 \end{cases}$$

Simplify this further if you can. Which terms are non-zero?

Solution:

$$D_n = \frac{1}{2}\int_{-1}^{1} f(t)e^{-jn\omega_0 t}dt$$

where $f(t) = |t|$, so

$$D_n = \frac{1}{2}\int_0^1 (te^{-jn\omega_0 t} + te^{jn\omega_0 t})dt$$

If $n = 0$, this is

$$D_n = \frac{1}{2}\int_0^1 2tdt = \frac{1}{2}t^2\Big|_0^1 = \frac{1}{2}.$$

If $n \neq 0$, and using $\omega_0 = 2\pi/2 = \pi$, and the result from the sawtooth waveform given above,

$$
\begin{aligned}
D_n &= \frac{1}{2}\int_0^1 (te^{-jn\omega_0 t} + te^{jn\omega_0 t})dt \\
&= \frac{1}{2}\left(\int_0^1 (te^{-jn\pi t} + te^{jn\pi t})dt\right) \\
&= \frac{1}{2}\left(\frac{je^{-jn\pi}}{n\pi} + \frac{e^{-jn\pi}-1}{(n\pi)^2} + \frac{je^{jn\pi}}{-n\pi} + \frac{e^{jn\pi}-1}{(-n\pi)^2}\right) \\
&= \frac{1}{2}\left(\frac{j(-1)^n}{n\pi} + \frac{(-1)^n-1}{(n\pi)^2} - \frac{j(-1)^n}{n\pi} + \frac{(-1)^n-1}{(n\pi)^2}\right) \\
&= \frac{(-1)^n-1}{(n\pi)^2}
\end{aligned}
$$

where we have used the fact that $e^{-jn\pi} = e^{jn\pi} = (-1)^n$. This can be further simplified by noting that $(-1)^n - 1 = 0$ if n is even, and -2 if n is odd, so

$$
D_n = \begin{cases} \frac{1}{2} & n = 0 \\ -\frac{2}{(n\pi)^2} & \text{n odd} \\ 0 & \text{n even; } n \neq 0 \end{cases}
$$

Laboratory 4

In this lab we will use matlab to compute the Fourier series for several signals, and compare the errors that results from the approximation of the signals by truncated Fourier series. The first signal will be the sawtooth waveform we discussed in class. This has has a discontinuity that causes Gibbs oscillations on either side of the discontinuity. The second is the triangle signal that you found the Fourier series for in Problem 1b. This has the same values as the sawtooth waveform for $t = [0, 1)$. However, since it is continuous, it is much better behaved.

Task 1: Write an m-file that evalutates a Fourier series Write an m-file that takes a set of Fourier series coefficients, a fundamental frequency, and a vector of output times, and computes the truncated Fourier series evaluated at these times. The declaration and help for the m-file might be

```
function fn = myfs(Dn,omega0,t)
%
%  fn = myfs(Dn,omega0,t)
%
%  Evaluates the truncated Fourier Series at times t
%
%   Dn     -- vector of Fourier series coefficients
%               assumed to run from -N:N, where length(Dn) is 2N+1
%   omega0 -- fundamental frequency
%   t      --  vector of times for evaluation
%
%   fn     --  truncated Fourier series evaluated at t
%
```

The output of the m-file should be

$$f_N(t) = \sum_{n=-N}^{N} D_n e^{j\omega_0 nt}$$

The length of the vector `Dn` should be $2N + 1$. You will need to calculate N from the length of `Dn`.

Solution:

```
function fn = myfs(Dn,omega0,t)
% fn = myfs(Dn,omega0,t)
% Evaluates the truncated Fourier Series at times t
% Dn -- vector of Fourier series coefficients
% assumed to run from -N:N, where length(Dn) is 2N+1
```

```
% omega0 -- fundamental frequency
% t -- vector of times for evaluation
% fn -- truncated Fourier series evaluated at t

N = (length(Dn)-1)/2;
fn = zeros(size(t));

for n = -N:N
    D_n = Dn(n+N+1);
    fn = fn + D_n*exp(j*omega0*n*t);
end
```

Task 2: Evaluate the Fourier series of the sawtooth waveform Verify the output of your routine by checking the Fourier series coefficients for the sawtooth waveform described in the class notes. Use the matlab `subplot` command to put multiple plots on a page. In particular the command

```
>> subplot(311)
```

indicates that you are generating a 3x1 array of plots (the first two coefficients) and that the current output should go to the first of these plots (the third coefficient). The other three plots can be addressed by changing the last index. For example, to plot the third of the three plots,

```
>> subplot(313)
```

Calculate the Fourier series coefficients of the sawtooth waveform for $N = 8$ using the expression from the lecture notes, and store them in a vector `Dn`. Plot the coefficients using a `stem` plot

```
>> subplot(311)
>> n = [-8:8]
>> stem(n,real(Dn),'-')
>> hold
>> stem(n,imag(Dn),'--')
```

In this case `Dn` is complex, so you need to plot each. The `hold` command simply puts the next plot on the same axes. The third argument to the stem plot specifies the line type, just as in `plot`. Then define a vector of the times you want to evaluate, and the fundamental frequency

```
>> t = [-2:0.01:2]
>> omega0 = 2*pi;
```

Then plot the ideal function, and the truncated Fourier series approximation

```
>> ft = mod(t,1);
>> fn = myfs(Dn,omega0,t);
>> subplot(312)
>> plot(t,ft,t,fn);
```

In the third subplot, plot the approximation error,

```
>> subplot(313)
>> plot(t,ft-fn);
```

Note the size of the error, for comparison with the next task. Make sure you label the axes of the plots as you go along.

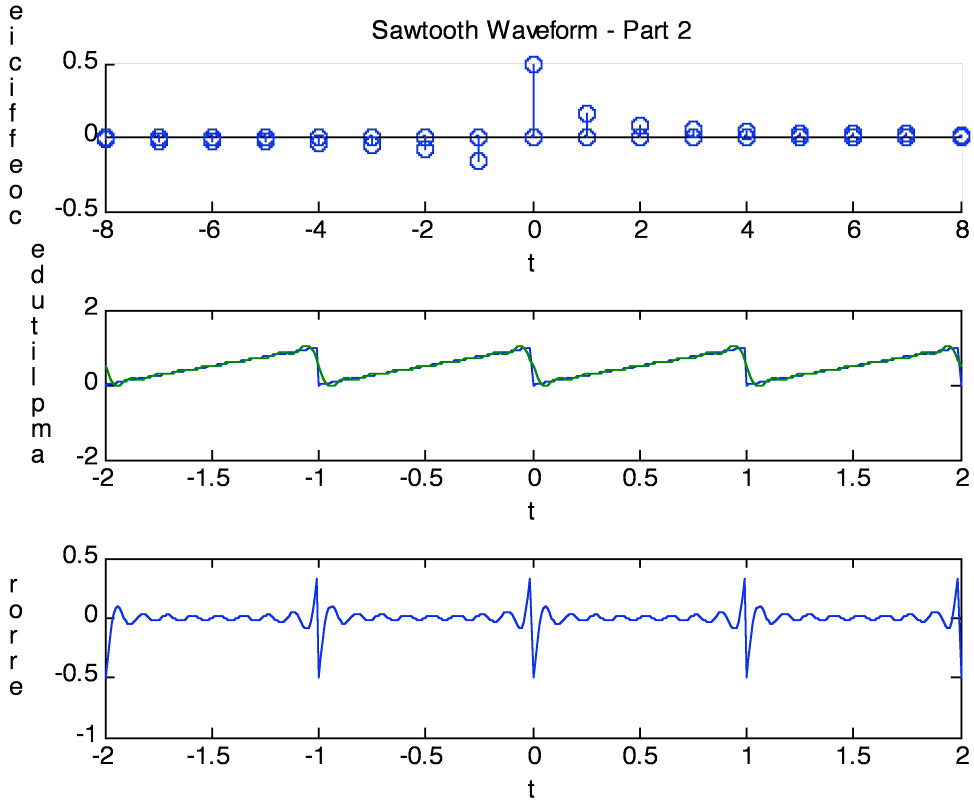
Solution:

```
% Creating sawtooth waveform coefficients Dn
N = 8;
for n = -N:N
    if n ~= 0
        Dn(n+N+1) = j/(2*pi*n);
    else
        Dn(n+N+1) = 1/2;
    end
end

% plotting coefficients
subplot(311);
n = [-8:8];
stem(n,real(Dn),'-');
hold
stem(n,imag(Dn),'--');
xlabel('t'); ylabel('coefficients'); title('Sawtooth Waveform - Part 2');

% plotting approximation and function
t = [-2:0.01:2];
omega0 = 2*pi;
ft = mod(t,1);
fn = myfs(Dn,omega0,t);
subplot(312)
plot(t,ft,t,fn);
xlabel('t'); ylabel('amplitude');

% plotting error
subplot(313);
plot(t,ft-fn);
xlabel('t'); ylabel('error');
```



Task 3: Evaluate the Fourier series of the triangle waveform Repeat the steps of Task 2 for the case of the triangular signal from this homework. Compute the integral square error for Task 2 and this task by squaring the error, and summing. For example, for the Task 2 signals,

```
>> e2 = sum((ft-fn).^2)*0.01;
```

where the 0.01 is the time sample width. What is the ratio of the integral square error for the two signals?

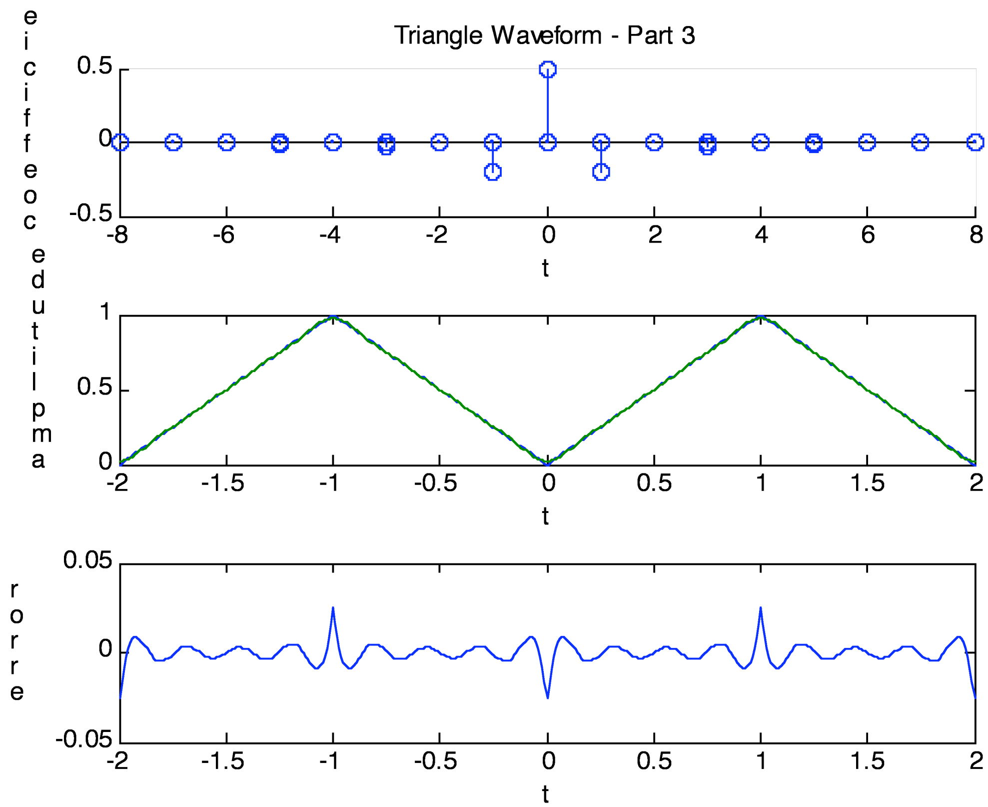
Solution:

```
% Creating triangle waveform coefficients Dn
N = 8;
for n = -N:N
    if n ~= 0
        Dn(n+N+1) = ((-1)^n - 1)/(n*pi)^2;
    else
        Dn(n+N+1) = 1/2;
    end
end
```

```
% plotting coefficients
subplot(311);
n = [-8:8];
stem(n,real(Dn),'-');
hold
stem(n,imag(Dn),'--');
xlabel('t'); ylabel('coefficients');
title('Triangle Waveform - Part 3');
% plotting approximation and function
t = [-2:0.01:2];
omega0 = pi; % this has changed because the new waveform has different period
ft2 = [(0:0.01:1) (0.99:-0.01:0.01) (0:0.01:1) (0.99:-0.01:0)];
fn2 = myfs(Dn,omega0,t);
subplot(312)
plot(t,ft2,t,fn2);
xlabel('t'); ylabel('amplitude');

% plotting error
subplot(313);
plot(t,ft2-fn2);
xlabel('t'); ylabel('error');

% Calculating integral square error
e2 = sum((ft-fn).^2)*0.01;
e22 = sum((ft2-fn2).^2)*0.01;
rat = e2/e22
```



Ratio of Integral Square error = 241.9 (the error of the sawtooth waveform was much greater than the error of the triangle waveform).