EE 382V: Social Computing **Fall 2018**

Homework 1: September 25

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1.1 Question1

Consider a bipartite graph with a set of boys and girls [and the number of boys is equal to the number of girls]. Suppose that every girl likes at least $d \geq 1$ boys and every boy likes at most d girls. Show that there exists a perfect matching between boys and girls.

1.1.1 Answer1

According to Halls [Marriage] theorem, if there no constricted set, then a perfect matching exists.

Let the set A represent the set of girls, and set B represent the set of boys.

Assume that there is a subset $S \subseteq A$, and its set of neighbors in the set of boys $N(S)$, such that it forms a constricted set where $|S| > |N(S)|$.

Let d_1 be the minimum degree of any girl, and d_2 be the maximum degree of any boy. Then, $d_1 \geq d$ and $d_2 \leq d$, so $d_1 \geq d_2$.

Let $D(S)$ be the sum of the degree of nodes within the subset S. Then $d_1|S| \leq D(S)$, since every girl node of S has a degree of at least d_1. Likewise, $d_2|N(S)| \geq D(S)$, since every boy node in $N(S)$ has a degree of at most $d2$ edges.

So,

$d_1|S| \leq D(S) \leq d2|N(S)|$

$d_1|S| \leq d2|N(S)|$

$|S| \leq \frac{d_2}{d_1}|N(S)|$

Since $d_1 \leq d_2$, $|S| \leq |N(S)|$, which contradicts S and $N(S)$ being a constricted set.

Thus, by Halls [Marriage] Theorem, a perfect matching exists.

1.2 Question2

Show that König–Egervry Theorem implies Halls Theorem.

1.2.1 Answer2

Halls [Marriage] theorem states that if there is no constricted set, then a perfect matching exists.

The König–Egervry theorem states that in any bipartite graph, the number of edges in a maximum matching is equal to the number of vertices in a minimum vertex cover.

Proof: No constricted set implies a perfect matching exists (by contradiction).

Let there be bipartite graph $G = (A, B, E)$ that consists of two partitioned sets A and B of vertices and a set E or edges between the two. Let M represent the set of maximum matching edges and V represent the minimum vertex cover.

Assume that there is no constricted set and no perfect matching. Then the maximum matching size $m < |A|$. By the König–Egervry theorem, the minimum vertex cover also has size $m < |A|$.

Let $X = A \cap V$ and $Y = B \cap V$. Since the set V is a vertex cover on A, B, then $|X| + |Y| = |V| = m < |A|$.

Rearranging this, $|Y| < |A| - |X| = |A - X|$. Also, since V is a vertex cover, then there are no edges between $A - X$ and $B - Y$.

So $N(A - X) \leq |Y| < |A - X|$, which is a constricted set and contradicts our assumption. Therefore a perfect matching must exist.

Proof: A perfect matching exists implies no constricted set.

If a perfect matching exists, then any set $S \subseteq A$ is matched to $|S|$ vertices in B.

So, $\forall S \subseteq A$, $|N(S)| = |S|$, so no constricted set exists.

1.3 Question3

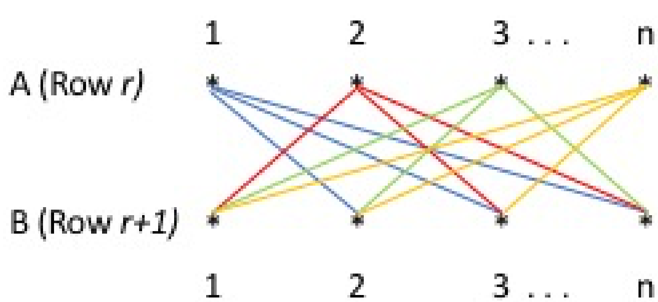
An $r \times n$ Sudoku is an $r \times n$ matrix with entries $1, \ldots, n$ such that each number appears at most once in each row and column. Show that any $r \times n$ Sudoku can be extended to an $(r+1) \times n$ Sudoku whenever $r < n$.

1.3.1 Answer3

Proof by induction.

Base case ($r = 1$). Show that a $1 \times n$ Sudoku can be extended to $2 \times n$ for $n > 1$. One way to extend is to copy row 1 of any $1 \times n$ Sudoku and insert it into row 2, but shift every element in the row by 1, so that the element at column i in row 1 is at column $(i + 1)\%n$.

Induction case. Assume any $k \times n$ Sudoku can be extended to $(k + 1) \times n$ for $k < r$. We'll show a $r \times n$ Sudoku can be extended to $(r + 1) \times n$ for $r < n$.



We have a bipartite graph $G = (A, B, E)$ where $A = 1, 2, 3, \ldots n$, $B = 1, 2, 3, \ldots n$, and edges $e_{ij} \in E$ that connect $i \in A$ to $j \in B$ where the i^{th} column of the Sudoku does not have the number j. If there is a perfect matching on G, we can extend the Sudoku by a row, as long as the number of resultant rows is less than or equal to n.

To do this, take the matching (i, j) of the perfect matching and place j on row $r + 1$ at column i.

Proof that a perfect matching exists:

First, note that in a Sudoku with r rows, column i has r distinct numbers in it. Therefore, column i has $(n - r)$ edges to the $(n - r)$ numbers not in column i. Then every element $j \in B$ appears exactly once in each row and at most once in any of the n columns, so there are $(n - r)$ column where i can be placed.

Thus, every element $j \in B$ is adjacent to $(n - r)$ edges. Using the answer to problem 1, we can prove G has a perfect matching by letting $d1 = d2 = (n - r)$. Thus a perfect matching exists and the Sudoku can be extended.