EE 382V: Social Computing **Fall 2018**

Homework 2: October 21

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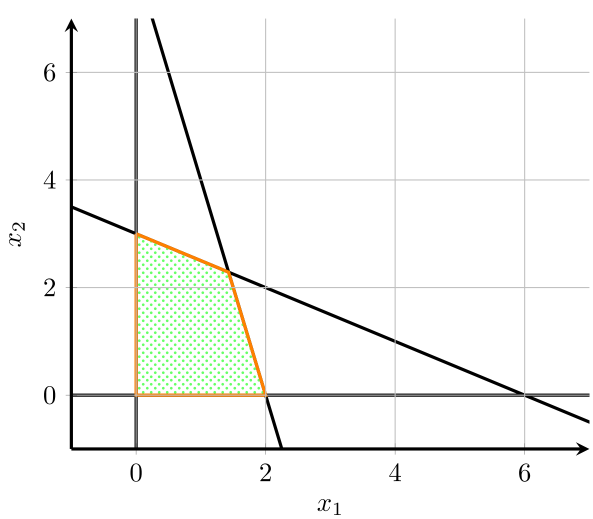
1.1 Question 1

Consider the following linear program.

max $2x_1 + 6x_2$
subject to
$4x_1 + x_2 \leq 8$,
$x_1 + 2x_2 \leq 6$,
$x_1, x_2 \geq 0$,

(a) Draw the feasible solution space on x_1,x_2 plane.
(b) Give all the vertices of the feasible solution space.
(c) What is the optimal value of the objective function?
(d) Give the dual of the above primal problem.
(e) Draw the feasible solution space for the dual version.
(f) What is the optimal solution of the dual version of the problem?

1.1.1 Answer 1a



1.1.2 Answer 1b

Feasible region corner vertices:
$(0, 0)$
$(0, 3)$
$(10/7, 16/7)$
$(2, 0)$

1.1.3 Answer 1c

By the simplex algorithm the optimal value is at one of the vertices. Evaluating the objective function at each of the vertices, we find the the optimal value of the Primal objective function is 18 at $(x_1, x_2) = (0, 3)$

1.1.4 Answer 1d

Dual Problem:

max $2x_1 + 6x_2$
subject to
$4x_1 + x_2 2 \leq 8$
$4_1 + 2x_2 2 \leq 6$
$x1, x2 \geq 0$
is equivalent to

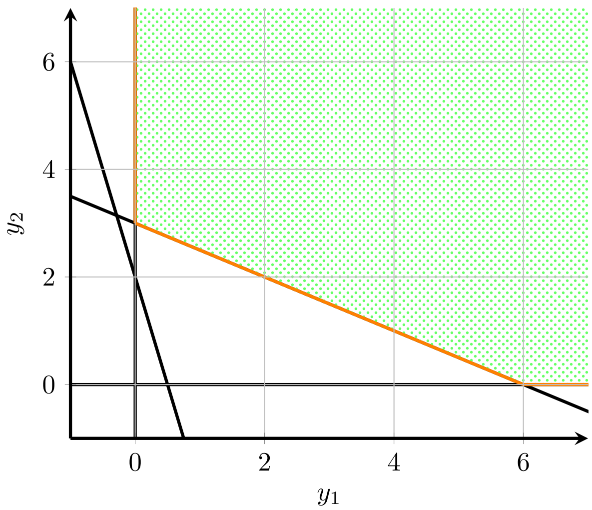
min $-2x_1 - 6x_2$
subject to
$4x_1 + x_2 2 \leq 8$
$4_1 + 2x_2 2 \leq 6$
$x1, x2 \geq 0$

By the appendix in the duality notes, the dual is given by

max $8y'_1 + 6y'_2$
subject to
$4y'_1 + y'_2 \leq -2,$
$y'_1 + 2y'_2 \leq -6,$
$y'_1, y'_2 \geq 0,$
then by letting $y_1 = -y'_1, y_2 = -y'_2$ we arrive at

min $8y_1 + 6y_2$
subject to
$4y_1 + y_2 \geq 2,$
$y_1 + 2y_2 \geq 6,$
$y_1, y_2 \geq 0,$

1.1.5 Answer 1e



1.1.6 Answer 1f

Again by the simplex method we only need to check the value at the vertices of the feasible space. The optimum solution for the Dual objective function is $(y_1, y_2) = (0, 3)$, which gives an optimum value of 18 (same as for the primal).

1.2 Question 2

Show that the man-oriented Gale-Shapley algorithm gives the woman-pessimal stable matching, i.e., each woman is paired with her worst valid partner of all stable matchings.

1.2.1 Answer 2

Proof by contradiction:

Let $M = \{\dots,(m,w),\dots\}$ be the set of male optimal marriages from the output of the male-oriented Gale-Shapley algorithm.

Suppose that there exists a stable matching that is not woman pessimal, represented by $M' = \{\dots,(m',w),\dots,(m,w'),\dots$ such that m is not the pessimal choice for w. In this scenario , w prefers m over m'. However, if the stable marriage is also male optimal, then m prefers w over w'.

The above case is a contradiction because this creates a blocking pair, which proves that such a marriage M' is not stable. Therefore, a male optimal pairing must be woman pessimal.

1.3 Question 3

Suppose that you are given an instance of a stable marriage problem, i.e., the ordered lists of men preferences and the ordered list of women preferences.

(a) You are given a particular couple (m, w) as an additional input. Give an algorithm to determine if there exists a stable marriage with m assigned to w.
(b) What certificate can you give to your friend to show that there is no stable marriage with (m, w) pair?
(c) What certificate can you give to your friend to show that there is a stable marriage with (m, w) pair?

1.3.1 Answer 3a

Modify the Gale-Shapely stable marriage algorithm with a constraint that the $(m*, w*)$ couple is already assigned (married), and check for potential blocking pairs with already assigned $(m*, w*)$ before each new couple engagement. This will be accomplished by having the woman reject any man that would create a blocking pair with $(m*, w*)$. If there is at least one free man left that has no else to propose to, then no stable marriage set exists with the already assigned $(m*, w*)$ couple.

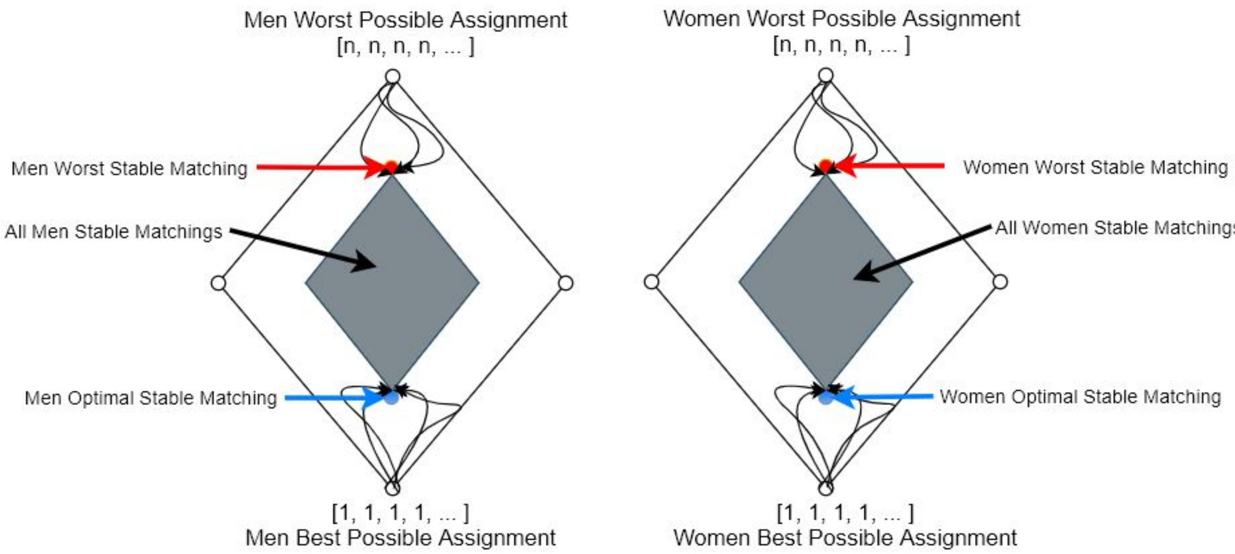
Inputs:
1. $mPref$: data structure containing preferences for each man
2. $wPref$: data structure containing preferences for each woman
3. $m*$: The man that has already been coupled with $w*$
4. $w*$: The woman that has already been couple with $m*$
5. $mrank$: where $mrank[i][j]$ indicates the rank of woman j for man i; higher preference correlates to lower rank
6. $wrank$: where $wrank[i][j]$ indicates the rank of man j for woman i; higher preference correlates to lower rank

Algorithm:
1. freelist = list of men who are free (initially has all men except $m*$)
2. while (freelist is not empty):
    3. choose some man m from freelist
    4. if m has already attempted a proposal to every woman except w*
        5. Print no stable marriage exists with given $(m*, w*)$ assignment
        6. EXIT
    7. m proposes to his most preferred woman w that he has not already proposed to, who is not $w*$
    8. if w is free:
        9. if $(mrank[m][w] > mrank[m][w*])$ and $(wrank[w*][m*] > wrank[w*][m])$: //blocking pair
            10. add m back to freelist // reject m; w still free
        11. else: // (m, w) is will not create a blocking pair
            12. w accepts proposal (assign m to w)
    13. else: // w is already engaged to m'
        14. if $wrank[w][m] > wrank[w][m']$: // w prefers m'
            15. add m back to freelist // reject m; w still engaged to m'
        16. else:
            17. if $(mrank[m][w] > mrank[m][w*])$ and $(wrank[w*][m*] > wrank[w*][m])$: // blocking
                18. add m back to freelist // reject m; w still engaged to m'
            19. else: // (m, w) is will not create a blocking pair
                20. add m' back to the freelist // reject m'

21. w gets engaged to m
22. Print stable marriage exists given $(m*, w*)$ assignment

1.3.2 Answer 3b

The certificate that can be shown to prove that there is no stable marriage with the (m, w) pair is to show the list of all possible stable marriages and show that the (m, w) is not included in any of them. As illustrated in the hasse diagram from the last lecture's scribe notes, the Gale-Shapely algorithm only produces the optimal/pessimal stable marriage sets at the top and bottom points in the hasse diagram region of all stable matchings. However, all possible stable marriage are required to show that the (m, w) is not a part of any of them, and therefore proves that no stable marriage set exists with that pair.



1.3.3 Answer 3c

The certificate that can be shown to prove that there is a stable marriage with the (m, w) pair is to show one instance of any stable marriage set that includes the (m, w) pair. Such a stable marriage set could be produced using the algorithm produced in answer 3a above. A Hasse diagram, as described in answer 3b above, could be used as a visual representation of all stable marriage sets. If at least one of them includes the (m, w) pair, it could be used as the certificate to show that a stable marriage with that pair exists.