Zumobot case study

Control Theory Module

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Homework assignment

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Chapter 1

Introduction

One of the most used study cases in Control Theory is the Inverted Pendulum analysis, as it presents an unstable open-loop characteristic but is also possible to stabilize it on a closed-loop configuration.

Here we present the analysis of the Zumobot 32U4, a small robot available in the market. The Zumo 32U4 robot is a complete, versatile robot controlled by an Arduino-compatible ATmega32U4 microcontroller. The Zumo 32U4 robot can be programmed from a computer using any operating system that supports the Arduino environment.

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Chapter 2

System Analysis

An inverted pendulum is basically a consistent mass on the ground (usually wheeled), connected through a frictionless joint to a pendulum, so that the pendulum can freely rotate around the joint and fall down by its own weight. The goal in this system is to provide an horizontal force to the mass on the ground with a direction contrary to the inclination of the pendulum with respect to the vertical axis, so that the pendulum holds in its highest position.

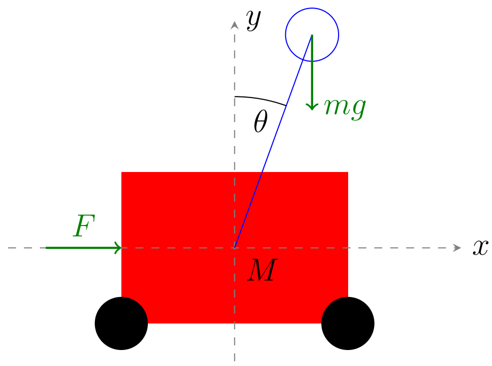

Figure 2.1: Physical representation of the inverted pendulum

The case study in this work is based on a commercial Zumobot 32U4 (See figure 2.2) robot from Pololu manufacturer. It consist on a rigid case that holds 2 brushless DC motors, an Arduino-compatible ATmega32U4 microcontroller and a set of batteries to power up the system. The motors are connected to a belt that rotates between to sets of wheels, sufficiently high to make the Zumobot able to stay in a vertical position without any contact between the case and the ground. The stucture is well suited for an inverted pendulum configuration, considering its physical characteristics and its simple programmability.

The goal position of the Zumobot is a vertical position with respect to the vertical axis. The Zumobot needs to be previously adjusted (frontal sensors holder dismounted), as the frontal part will face the ground, and the posterior part will be

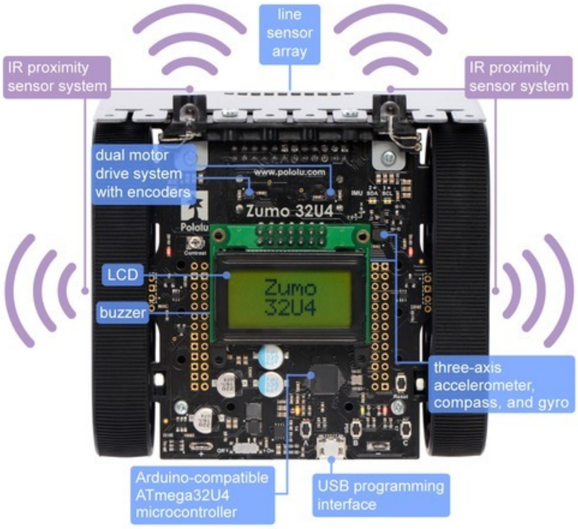

Figure 2.2: Pololu Zumobot

held up.

Figure 2.3: Stabilized Zumobot position

The device includes several helpful sensors and peripherals:

- AVR ATmega 32U4 microcontroller, with 16MHz crystal oscillator.
- Two micrometal gearmotors, driven by on-board TI DRV8838 motor divers.
- 3-axis accelerometer + 3-axis gyroscope

To power the motors, the microcontroller generates a PWM digital signal and a direction bit for each wheel (left and right). The Arduino environment provides a library to configure and control the PWM value with low effort. Along with the corresponding firmware, the whole actuator part of the system is complete.

The sensor part of the system uses the accelerometer and gyroscope to provide a value of the angle in the y axis, which is the axis that shows the angle θ defined in figure 2.1. The firmware in the Arduino controller gets the value from the gyroscope, and performs an adjustment of the value based on the accelerometer reading. It should be noted that the manufacturer of the robot states in the technical documentation[Pol] that the accelerometer and gyroscope readings are likely to be influenced by external noise from the DC motors and the batteries, and the values obtained should only be considered for rough estimation.

The control system is then defined as shown in the figure 2.4.

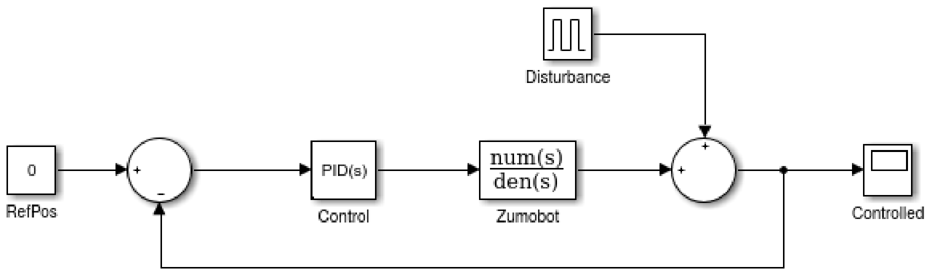

Figure 2.4: Control diagram of the inverted pendulum

The dynamics of the system can be divided into the following elements:

- DC motor behavior to convert a DC voltage to a force
- Robot behavior defined by the physical characteristics

With this clarification, the control diagram of the system becomes:

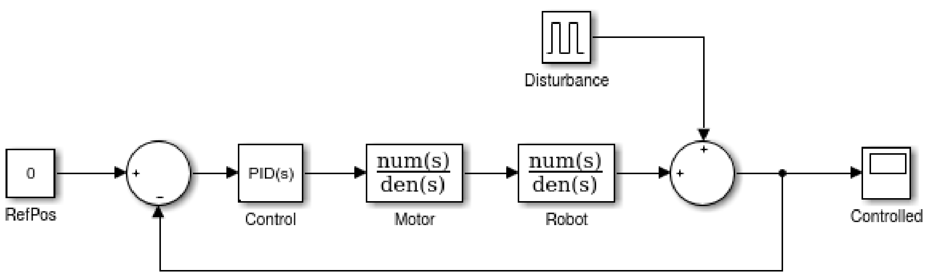

Figure 2.5: Control diagram of the inverted pendulum - analysis elements

To design the controller for the system, the dynamics of the motor and robot need to be defined. This is done in the next chapter by two different approaches.

Chapter 3

Mathematical development

Through the numerous literature it is possible to find several approaches to obtain a model for the Inverted Pendulum elements. For this work the focus is mostly on the dynamic behavior of the robot, as it is the one that influences the most the response of the system. To completely model the dynamic behavior of the robot, we need to consider the equations that govern the movement as a rigid body.

3.1 First approach: Sum of forces

The first approach considered here is described in [Sul03]. The methodology uses traditional dynamic physics to obtain the equations of motion. The final equation is then linearized and transformed into complex frequency domain.

3.1.1 Dynamic behavior

The equations of motion are obtained from the sum of forces in the cart for the horizontal direction (see figure 3.1).

$$F - b \cdot \dot{x} - N = M \cdot \ddot{x} \tag{3.1}$$

Now considering the pendulum itself, the force applied in the horizontal direction due to the momentum of the pendulum is determined as:

$$\tau = r \cdot F = I \cdot \ddot{\theta} \tag{3.2}$$

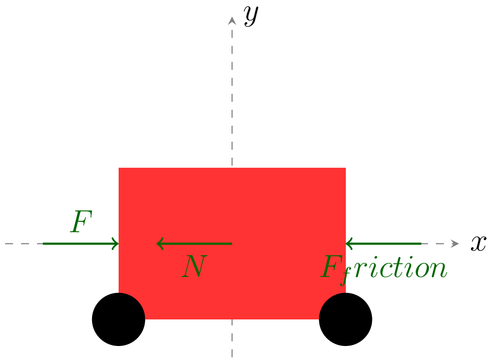

Figure 3.1: Cart forces

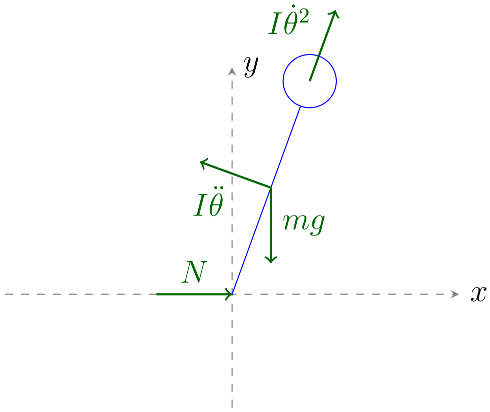

Figure 3.2: Pendulum forces

Given the fact that the moment of inertia of a pendulum of mass m is defined as $I = m \cdot L^2$, the previous equation can be rewritten as:

$$F = \frac{I \cdot \ddot{\theta}}{r} = \frac{m \cdot l^2 \cdot \ddot{\theta}}{l} = m \cdot l \cdot \ddot{\theta} \tag{3.3}$$

Obtaining the component of the force defined in 3.3 in the horizontal direction:

$$F = m \cdot l \cdot \ddot{\theta} \cdot \cos\theta \tag{3.4}$$

Now, the component of the centripetal force acting on the pendulum is similar to the one in 3.3, but the horizontal component of this force is:

$$F = m \cdot l \cdot \dot{\theta}^2 \cdot \sin\theta \tag{3.5}$$

Summing the defined forces present in the horizontal direction of the pendulum in 3.4 and 3.5, we obtain the following expression:

$$N = m \cdot \ddot{x} + m \cdot l \cdot \ddot{\theta} \cdot \cos\theta - m \cdot l \cdot \dot{\theta}^2 \cdot \sin\theta \tag{3.6}$$

Now we can substitute 3.6 into 3.1, we obtain the first equation of motion:

$$F = (M + m)\ddot{x} + b \cdot \dot{x} + m \cdot l \cdot \ddot{\theta} \cdot \cos\theta - m \cdot l \cdot \dot{\theta}^2 \cdot \sin\theta \tag{3.7}$$

To get the second equation of motion, we sum the forces perpendicular to the pendulum. The vertical components of this forces are considered here to get:

$$P \cdot \sin\theta + N \cdot \cos\theta - m \cdot g \cdot \sin\theta = m \cdot l \cdot \ddot{\theta} + m \cdot \ddot{x} \cdot \cos\theta \tag{3.8}$$

To get rid of the P and N terms, sum the moments around the center of gravity of the pendulum:

$$-P \cdot l \cdot \sin\theta - N \cdot l \cdot \cos\theta = I \cdot \ddot{\theta} \tag{3.9}$$

Summing up the equations 3.8 and 3.9, we obtain the second equation of movement:

$$(I + m \cdot l^2)\ddot{\theta} + m \cdot g \cdot l \cdot \sin\theta = -m \cdot l \cdot \ddot{x} \cdot \cos\theta \tag{3.10}$$

The obtained equations are non-linear, so they are linearized around the operating point, defined as the top vertical position or πrad from the stable equilibrium position. We need also to define a small angle deviation from the top vertical position, so that $\theta = \pi + \phi$.

Under this circunstances, we can deduce that $\cos\theta \approx -1$, $\sin\theta \approx -\phi$, and $\dot{\theta}^2 \approx 0$. Applying this relations into our movement equations, we obtain:

$$F = (M + m)\ddot{x} + b \cdot \dot{x} - m \cdot l \cdot \ddot{\phi} \tag{3.11}$$

$$(I + m \cdot l^2)\ddot{\phi} - m \cdot g \cdot l \cdot \phi = m \cdot l \cdot \ddot{x} \tag{3.12}$$

To obtain the transfer function of the linearized system of equations analytically, we perform the Laplace transform of the system equations:

$$F(s) = (M + m)s^2 \cdot X(s) + b \cdot s \cdot X(s) - m \cdot l \cdot s^2 \cdot \Phi(s) \tag{3.13}$$

$$(I + m \cdot l^2)s^2 \cdot \Phi(s) - m \cdot g \cdot l \cdot \Phi(s) = m \cdot l \cdot s^2 \cdot X(s) \tag{3.14}$$

To unify the equations, we solve 3.13 for $X(s)$ and then replace it into 3.14, obtaining:

$$\frac{\Phi(s)}{F(s)} = \frac{m \cdot l \cdot s}{q \cdot s^3 + b(l + m \cdot l^2)s^2 - m \cdot g \cdot l(M+m)s - b \cdot m \cdot g \cdot l} \tag{3.15}$$

Where:

$$q = (M+m)(l + m \cdot l^2) - (m \cdot l)^2 \tag{3.16}$$

Assuming a coefficient of friction as zero, we can represent the equation as:

$$\frac{\Phi(s)}{F(s)} = \frac{K_p}{\frac{s^2}{A_p^2} - 1} \tag{3.17}$$

With:

$$K_p = \frac{1}{(M+m)g}; A_p = \pm\sqrt{\frac{(M+m)m \cdot g \cdot l}{(M+m)(l+m \cdot l^2) - (m \cdot l)^2}} \tag{3.18}$$

3.1.2 Actuator system

The actuation mechanism consist on a DC motor that drives a belt system around two wheels. The overall transfer function of the actuation mechanism will depend upon the motor and the belt system.

The torque to be delivered by the motor is:

$$T_L = (M+m)r^2\dot{\omega} \tag{3.19}$$

The relation between Torque and Force can be expressed as:

$$T_L \propto r^2; F \propto r \tag{3.20}$$

The motor dynamics can be represented with the well known transfer function [Sul03] [Zac]:

$$\omega(s) = K_m \frac{V(s)}{\tau s + 1} \tag{3.21}$$

Where τ is the time constant and depends on the load, and K_m is the steady-state gain of the motor.

$$\frac{T(s)}{E(s)} = K_m \frac{(M+m)r \cdot s}{\tau s + 1} \tag{3.22}$$

3.1.3 Simulation

The obtained model is analyzed with computational tools (MATLAB), to validate its behavior and calculate an appropiate controller. As shown in figure 3.3, the system has a pole in the right part of the complex plane, fitting in the definition of an unstable system.

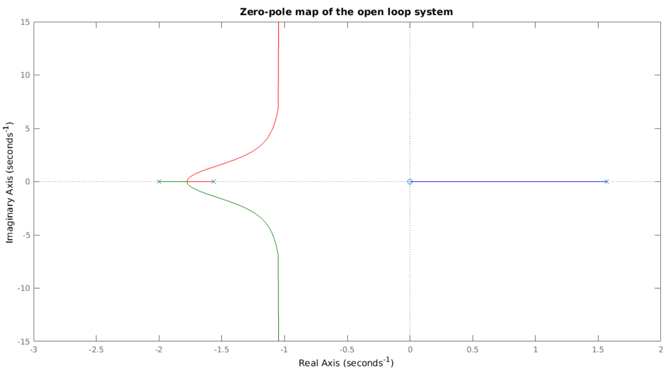

Figure 3.3: Zero-pole diagram for the open-loop system

The open-loop response shown in figure 3.4 confirms the unstable behavior.

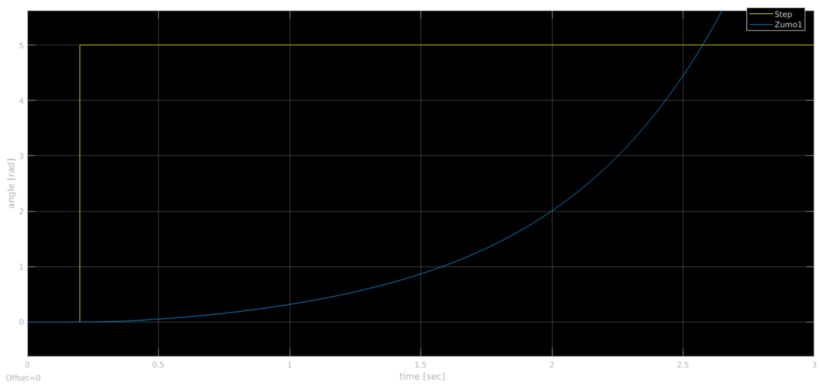

Figure 3.4: Open-loop step response of the system

3.2 Second approach - Euler-Lagrange

To represent the dynamics of the system as in [Jer12] and [Lun02], the initial definition is the natural form of the Lagrangian in classical mechanics:

$$\mathcal{L} = E_k - E_p \tag{3.23}$$

Where $E_k = \frac{1}{2}mv^2$ and $E_p = mgh$. Equation 3.23 can then be rewriten as:

$$\mathcal{L} = \frac{1}{2}mv^2 - mgh \tag{3.24}$$

The Euler-Lagrange equation states that:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{\partial \mathcal{L}}{\partial \theta} \tag{3.25}$$

The pendulum is a stiff bar of lenght L which is supported at one end by a frictionless pin.

TODO: Insert image

From the figure we can state that:

$$\begin{aligned} x &= L\sin\theta \quad & \dot{x} &= L\cos\theta\dot{\theta} \\ y &= L\cos\theta \quad & \dot{y} &= -L\sin\theta\dot{\theta} \end{aligned} \tag{3.26}$$

Velocity is a vector representing the change in the position in the coordinates x and y. Hence:

$$v^2 = \dot{x}^2 + \dot{y}^2 \tag{3.27}$$

With the coordinates found in 3.26, we can substitute in 3.27 to obtain:

$$\begin{aligned} v^2 &= L^2\cos\theta^2\dot{\theta}^2 + L^2\sin\theta^2\dot{\theta}^2 \\ v^2 &= L^2\dot{\theta}^2 \end{aligned} \tag{3.28}$$

Substituting 3.26 and 3.28 into 3.24, we obtain:

$$\mathcal{L} = \frac{1}{2}mL^2\dot{\theta}^2 - mgL\cos\theta \tag{3.29}$$

To perform the Euler-Lagrange equation presented in 3.25, we need to compute the partial derivatives. First we compute the left part:

$$\frac{\partial \mathcal{L}}{\partial \theta} = mgL\sin\theta \tag{3.30}$$

Now we compute the inner part of the right element:

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mL^2\dot{\theta} \tag{3.31}$$

And now we can compute the outer derivative of 3.31:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = mL^2\ddot{\theta} \tag{3.32}$$

Now that we have all the terms, we can write the equation:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) &= \frac{\partial \mathcal{L}}{\partial \theta} \\ mL^2\ddot{\theta} &= mgL\sin\theta \\ \ddot{\theta} &= \frac{g}{l}\sin\theta \end{aligned} \tag{3.33}$$

TODO: Insert development of second part of angular acceleration

The total accelerations present on the pendulum can then be stated as:

$$\ddot{\theta} = \ddot{\theta}_g + \ddot{\theta}_x = \left(\frac{g}{l}\right)\sin\theta - \left(\frac{\dot{x}}{l}\right)\cos\theta \tag{3.34}$$

The obtained equation is non-linear, so we apply an approximation based on the fact that the operation point implies an angle $\theta \approx 0$. It means that $\sin\theta \approx \theta$ and $\cos\theta \approx 1$. Applying this on 3.34, we obtain:

$$l\ddot{\theta} - g\theta = -\ddot{x} \tag{3.35}$$

To obtain the transfer function, we perform the Laplace transform on 3.35:

$$ls^2\Theta(s) - g\Theta(s) = -s^2X(s) \tag{3.36}$$

Now we solve for the variables Θ and X:

$$ls^2\Theta(s) - g\Theta(s) = -s^2X(s) \tag{3.37}$$

TODO: Complete

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