Zumobot case study

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Contents

Introduction

One of the most used study cases in Control Theory is the Inverted Pendulum analysis, as it presents an unstable open-loop characteristic but is also possible to stabilize it on a closed-loop configuration.

Here we present the analysis of the Zumobot 32U4, a small robot available in the market. The Zumo 32U4 robot is a complete, versatile robot controlled by an Arduino-compatible ATmega32U4 microcontroller. The Zumo 32U4 robot can be programmed from a computer using any operating system that supports the Arduino environment.

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1 System analysis

An inverted pendulum can be represented as a rigid body pendulum connected to a cart moving on a horizontal axis. Controlling the stability of the Zumobot can be compared to balancing the inverted pendulum on the moving cart.

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2 Mathematical development

Through the numerous literature is possible to find several approaches to obtain a model for the Inverted Pendulum.

To fully model the dynamic behavior of the zumo robot, we need to consider the equations that govern the movement as rigid body.

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2.1 First approach - Sum of forces

The first approach considered here is described in [1].

2.1.1 Dynamic behavior

The equations of motion are obtained from the sum of forces in the cart for the horizontal direction.

$$F - b \cdot \dot{x} - N = M \cdot \ddot{x} \tag{1}$$

Now considering the pendulum itself, the force applied in the horizontal direction due to the momentum of the pendulum is determined as:

$$\tau = r \cdot F = I \cdot \ddot{\theta} \tag{2}$$

Given the fact that the moment of inertia of a pendulum of mass m is defined as $I = m \cdot L^2$, the previous equation can be rewritten as:

$$F = \frac{I \cdot \ddot{\theta}}{r} = \frac{m \cdot l^2 \cdot \ddot{\theta}}{l} = m \cdot l \cdot \ddot{\theta} \tag{3}$$

Obtaining the component of the force defined in 3 in the horizontal direction:

$$F = m \cdot l \cdot \ddot{\theta} \cdot \cos\theta \tag{4}$$

Now, the component of the centripetal force acting on the pendulum is similar to the one in 3, but the horizontal component of this force is:

$$F = m \cdot l \cdot \dot{\theta}^2 \cdot \sin\theta \tag{5}$$

Summing the defined forces present in the horizontal direction of the pendulum in 4 and 5, we obtain the following expression:

$$N = m \cdot \ddot{x} + m \cdot l \cdot \ddot{\theta} \cdot \cos\theta - m \cdot l \cdot \dot{\theta}^2 \cdot \sin\theta \tag{6}$$

Now we can substitute 6 into 1, we obtain the first equation of motion:

$$F = (M+m)\ddot{x} + b \cdot \dot{x} + m \cdot l \cdot \ddot{\theta} \cdot \cos\theta - m \cdot l \cdot \dot{\theta}^2 \cdot \sin\theta \tag{7}$$

To get the second equation of motion, we sum the forces perpendicular to the pendulum. The vertical components of this forces are considered here to get:

$$P \cdot \sin\theta + N \cdot \cos\theta - m \cdot g \cdot \sin\theta = m \cdot l \cdot \ddot{\theta} + m \cdot \ddot{x} \cdot \cos\theta \tag{8}$$

To get rid of the P and N terms, sum the moments around the center of gravity of the pendulum:

$$-P \cdot l \cdot \sin\theta - N \cdot l \cdot \cos\theta = I \cdot \ddot{\theta} \tag{9}$$

Summing up the equations 8 and 9, we obtain the second equation of movement:

$$(I + m \cdot l^2)\ddot{\theta} + m \cdot g \cdot l \cdot \sin\theta = -m \cdot l \cdot \ddot{x} \cdot \cos\theta \tag{10}$$

The obtained equations are non-linear, so they are linearized around the operating point, defined as the top vertical position or πrad from the stable equilibrium position. We need also to define a small angle deviation from the top vertical position, so that $\theta = \pi + \phi$.

Under this circunstances, we can deduce that $\cos\theta \approx -1$, $\sin\theta \approx -\phi$, and $\dot{\theta}^2 \approx 0$. Applying this relations into our movement equations, we obtain:

$$F = (M+m)\ddot{x} + b \cdot \dot{x} - m \cdot l \cdot \ddot{\phi} \tag{11}$$

$$(I + m \cdot l^2)\ddot{\phi} - m \cdot g \cdot l \cdot \phi = m \cdot l \cdot \ddot{x} \tag{12}$$

To obtain the transfer function of the linearized system of equations analytically, we perform the Laplace transform of the system equations:

$$F(s) = (M+m)s^2 \cdot X(s) + b \cdot s \cdot X(s) - m \cdot l \cdot s^2 \cdot \Phi(s) \tag{13}$$

$$(I + m \cdot l^2)s^2 \cdot \Phi(s) - m \cdot g \cdot l \cdot \Phi(s) = m \cdot l \cdot s^2 \cdot X(s) \tag{14}$$

To unify the equations, we solve 13 for $X(s)$ and then replace it into 14, obtaining:

$$\frac{\Phi(s)}{F(s)} = \frac{m \cdot l \cdot s}{q \cdot s^3 + b(l + m \cdot l^2)s^2 - m \cdot g \cdot l(M+m)s - b \cdot m \cdot g \cdot l} \tag{15}$$

Where:

$$q = (M+m)(l + m \cdot l^2) - (m \cdot l)^2 \tag{16}$$

Assuming a coefficient of friction as zero, we can represent the equation as:

$$\frac{\Phi(s)}{F(s)} = \frac{K_p}{\frac{s^2}{A_p^2} - 1} \tag{17}$$

With:

$$K_p = \frac{1}{(M+m)g}; A_p = \pm\sqrt{\frac{(M+m)m \cdot g \cdot l}{(M+m)(l + m \cdot l^2) - (m \cdot l)^2}} \tag{18}$$

2.1.2 Actuator system

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2.2 Second approach - Euler-Lagrange

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3 System model

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4 Regulator

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5 Implementation

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List of Figures

List of Tables

References

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