Zumobot case study

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29.6.2017

Contents

Introduction

One of the most used study cases in Control Theory is the Inverted Pendulum analysis, as it presents an unstable open-loop characteristic but is also possible to stabilize it on a closed-loop configuration.

Here we present the analysis of the Zumobot 32U4, a small robot available in the market. The Zumo 32U4 robot is a complete, versatile robot controlled by an Arduino-compatible ATmega32U4 microcontroller. The Zumo 32U4 robot can be programmed from a computer using any operating system that supports the Arduino environment.

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1 System analysis

An inverted pendulum can be represented as a rigid body pendulum connected to a cart moving on a horizontal axis. Controlling the stability of the Zumobot can be compared to balancing the inverted pendulum on the moving cart.

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2 Mathematical development

Through the numerous literature is possible to find several approaches to obtain a model for the Inverted Pendulum.

To fully model the dynamic behavior of the zumo robot, we need to consider the equations that govern the movement as rigid body.

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2.1 First approach - Sum of forces

The first approach considered here is described in [1].

2.1.1 Dynamic behavior

The equations of motion are obtained from the sum of forces in the cart for the horizontal direction.

$$F - b \cdot \dot{x} - N = M \cdot \ddot{x} \tag{1}$$

Now considering the pendulum itself, the force applied in the horizontal direction due to the momentum of the pendulum is determined as:

$$\tau = r \cdot F = I \cdot \ddot{\theta} \tag{2}$$

Given the fact that the moment of inertia of a pendulum of mass m is defined as $I = m \cdot L^2$, the previous equation can be rewritten as:

$$F = \frac{I \cdot \ddot{\theta}}{r} = \frac{m \cdot l^2 \cdot \ddot{\theta}}{l} = m \cdot l \cdot \ddot{\theta} \tag{3}$$

Obtaining the component of the force defined in 3 in the horizontal direction:

$$F = m \cdot l \cdot \ddot{\theta} \cdot \cos\theta \tag{4}$$

Now, the component of the centripetal force acting on the pendulum is similar to the one in 3, but the horizontal component of this force is:

$$F = m \cdot l \cdot \dot{\theta}^2 \cdot \sin\theta \tag{5}$$

Summing the defined forces present in the horizontal direction of the pendulum in 4 and 5, we obtain the following expression:

$$N = m \cdot \ddot{x} + m \cdot l \cdot \ddot{\theta} \cdot \cos\theta - m \cdot l \cdot \dot{\theta}^2 \cdot \sin\theta \tag{6}$$

Now we can substitute 6 into 1, we obtain the first equation of motion:

$$F = (M + m)\ddot{x} + b \cdot \dot{x} + m \cdot l \cdot \ddot{\theta} \cdot \cos\theta - m \cdot l \cdot \dot{\theta}^2 \cdot \sin\theta \tag{7}$$

To get the second equation of motion, we sum the forces perpendicular to the pendulum. The vertical components of this forces are considered here to get:

$$P \cdot \sin\theta + N \cdot \cos\theta - m \cdot g \cdot \sin\theta = m \cdot l \cdot \ddot{\theta} + m \cdot \ddot{x} \cdot \cos\theta \tag{8}$$

To get rid of the P and N terms, sum the moments around the center of gravity of the pendulum:

$$-P \cdot l \cdot \sin\theta - N \cdot l \cdot \cos\theta = I \cdot \ddot{\theta} \tag{9}$$

Summing up the equations 8 and 9, we obtain the second equation of movement:

$$(I + m \cdot l^2)\ddot{\theta} + m \cdot g \cdot l \cdot \sin\theta = -m \cdot l \cdot \ddot{x} \cdot \cos\theta \tag{10}$$

The obtained equations are non-linear, so they are linearized around the operating point, defined as the top vertical position or πrad from the stable equilibrium position. We need also to define a small angle deviation from the top vertical position, so that $\theta = \pi + \phi$.

Under this circunstances, we can deduce that $\cos\theta \approx -1$, $\sin\theta \approx -\phi$, and $\dot{\theta}^2 \approx 0$. Applying this relations into our movement equations, we obtain:

$$F = (M + m)\ddot{x} + b \cdot \dot{x} - m \cdot l \cdot \ddot{\phi} \tag{11}$$

$$(I + m \cdot l^2)\ddot{\phi} - m \cdot g \cdot l \cdot \phi = m \cdot l \cdot \ddot{x} \tag{12}$$

To obtain the transfer function of the linearized system of equations analytically, we perform the Laplace transform of the system equations:

$$F(s) = (M+m)s^2 \cdot X(s) + b \cdot s \cdot X(s) - m \cdot l \cdot s^2 \cdot \Phi(s) \tag{13}$$

$$(I + m \cdot l^2)s^2 \cdot \Phi(s) - m \cdot g \cdot l \cdot \Phi(s) = m \cdot l \cdot s^2 \cdot X(s) \tag{14}$$

To unify the equations, we solve 13 for $X(s)$ and then replace it into 14, obtaining:

$$\frac{\Phi(s)}{F(s)} = \frac{m \cdot l \cdot s}{q \cdot s^3 + b(l + m \cdot l^2)s^2 - m \cdot g \cdot l(M+m)s - b \cdot m \cdot g \cdot l} \tag{15}$$

Where:

$$q = (M+m)(l + m \cdot l^2) - (m \cdot l)^2 \tag{16}$$

Assuming a coefficient of friction as zero, we can represent the equation as:

$$\frac{\Phi(s)}{F(s)} = \frac{K_p}{\frac{s^2}{A_p^2} - 1} \tag{17}$$

With:

$$K_p = \frac{1}{(M+m)g}; A_p = \pm\sqrt{\frac{(M+m)m \cdot g \cdot l}{(M+m)(l + m \cdot l^2) - (m \cdot l)^2}} \tag{18}$$

2.1.2 Actuator system

The actuation mechanism consist on a DC motor that drives a belt sytem around two wheels. The overall transfer function of the actuation mechanism will depend upon the motor and the belt system.

The torque to be delivered by the motor is:

$$T_L = (M+m)r^2\dot{\omega} \tag{19}$$

The relation between Torque and Force can be expressed as:

$$T_L \propto r^2; F \propto r \tag{20}$$

The motor dynamics can be represented with the well known tansfer function:

$$\omega(s) = K_m \frac{V(s)}{\tau s + 1} \tag{21}$$

Where τ is the time constant and depends on the load, and K_m is the steady-state gain of the motor.

$$\frac{F(s)}{E(s)} = K_m \frac{(M+m)r \cdot s}{\tau s + 1} \tag{22}$$

2.2 Second approach - Euler-Lagrange

To represent the dynamics of the system, the initial definition is the natural form of the Lagrangian in classical mechanics:

$$\mathcal{L} = E_k - E_p \tag{23}$$

Where $E_k = \frac{1}{2}mv^2$ and $E_p = mgh$. Equation 23 can then be rewriten as:

$$\mathcal{L} = \frac{1}{2}mv^2 - mgh \tag{24}$$

The Euler-Lagrange equation states that:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{\partial \mathcal{L}}{\partial \theta} \tag{25}$$

The pendulum is a stiff bar of lenght L which is supported at one end by a frictionless pin.

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From the figure we can state that:

$$\begin{aligned} x &= L\sin\theta\dot{x} = L\cos\theta\dot{\theta} \\ y &= L\cos\theta\dot{y} = -L\sin\theta\dot{\theta} \end{aligned} \tag{26}$$

Velocity is a vector representing the change in the position in the coordinates x and y. Hence:

$$v^2 = \dot{x}^2 + \dot{y}^2 \tag{27}$$

With the coordinates found in 27, we can substitute in 27 to obtain:

$$v^2 = L^2\cos\theta^2\dot{\theta}^2 + L^2\sin\theta^2\dot{\theta}^2 \tag{28}$$

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3 System model

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4 Regulator

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5 Implementation

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