Doctoral Programme in Economics

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Problem set 5

Microeconomics III

Viviens Martín, Javier

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1. Mechanism design.

1.1. First Best

The lender maximizes their utility subject to participation constraint for both types. This can be written in the following problem:

$$\begin{aligned} \max_{\{x_i, t_i\}_{i=h,l}} \quad & t_i - r x_i \\ \text{s.t.} \quad & \underline{\theta}\psi(\underline{x}) \geq \underline{t} \\ & \bar{\theta}\psi(\bar{x}) \geq \bar{t} \end{aligned} \tag{1}$$

By monotonicity of the utility function of the lender, we know the constraints will be binding. Hence, we can substitute t_i in the objective function and take the first order condition wrt x_i:

$$\theta_i \psi(x_i) - r = 0 \tag{2}$$

So we have that:

$$(\underline{x}, \underline{t}) = \left(\psi'^{-1}\left(\frac{r}{\underline{\theta}}\right), \underline{\theta}\psi\left(\psi'^{-1}\left(\frac{r}{\underline{\theta}}\right)\right)\right) \tag{3}$$

$$(\bar{x}, \bar{t}) = \left(\psi'^{-1}\left(\frac{r}{\bar{\theta}}\right), \bar{\theta}\psi\left(\psi'^{-1}\left(\frac{r}{\bar{\theta}}\right)\right)\right) \tag{4}$$

1.2.

Because Revelation Principle applys and thus we know that if some allocation (x, t) solves the problem, it has that it can be implemented by a direct and truthful mechanism, in which each type reveals their type and take the contract designed for them.

1.3.

$$\begin{aligned} \max_{\bar{x}, \bar{t}, \underline{x}, \underline{t}} \quad & p(\bar{t} - r\bar{x}) + (1-p)(\underline{t} - r\underline{x}) & \\ \text{s.t.} \quad & \underline{\theta}\psi(\underline{x}) - \underline{t} \geq 0 & \text{IR}(\underline{\theta}) \\ & \bar{\theta}\psi(\bar{x}) - \bar{t} \geq 0 & \text{IR}(\bar{\theta}) \\ & \underline{\theta}\psi(\underline{x}) - \underline{t} \geq \underline{\theta}\psi(\bar{x}) - \bar{t} & \text{IC}(\underline{\theta}) \\ & \bar{\theta}\psi(\bar{x}) - \bar{t} \geq \bar{\theta}\psi(\underline{x}) - \underline{t} & \text{IC}(\bar{\theta}) \end{aligned} \tag{5}$$

1.4.

Now we have that $\bar{t} = \bar{\theta}\psi(\bar{x}) - \bar{U}$ and $\underline{t} = \underline{\theta}\psi(\underline{x}) - \underline{U}$. Using this in the problem 5:

$$\begin{aligned}
\max_{\bar{x},\bar{U},\underline{x},\underline{U}} \quad & p(\bar{\theta}\psi(\bar{x}) - \bar{U} - r\bar{x}) + (1-p)(\underline{\theta}\psi(\underline{x}) - \underline{U} - r\underline{x}) & \\
\text{s.t.} \quad & \underline{\theta}\psi(\underline{x}) - [\underline{\theta}\psi(\underline{x}) - \underline{U}] \geq 0 & \text{IR}(\underline{\theta}) \\
& \bar{\theta}\psi(\bar{x}) - [\bar{\theta}\psi(\bar{x}) - \bar{U}] \geq 0 & \text{IR}(\bar{\theta}) \\
& \underline{\theta}\psi(\underline{x}) - [\underline{\theta}\psi(\underline{x}) - \underline{U}] \geq \underline{\theta}\psi(\bar{x}) - [\bar{\theta}\psi(\bar{x}) - \bar{U}] & \text{IC}(\underline{\theta}) \\
& \bar{\theta}\psi(\bar{x}) - [\bar{\theta}\psi(\bar{x}) - \bar{U}] \geq \bar{\theta}\psi(\underline{x}) - [\underline{\theta}\psi(\underline{x}) - \underline{U}] & \text{IC}(\bar{\theta})
\end{aligned} \tag{6}$$

Which I can rewrite as:

$$\begin{aligned}
\max_{\bar{x},\bar{U},\underline{x},\underline{U}} \quad & p(\bar{\theta}\psi(\bar{x}) - r\bar{x}) + (1-p)(\underline{\theta}\psi(\underline{x}) - r\underline{x}) - [p\bar{U} + (1-p)\underline{U}] & \\
\text{s.t.} \quad & \underline{U} \geq 0 & \text{IR}(\underline{\theta}) \\
& \bar{U} \geq 0 & \text{IR}(\bar{\theta}) \\
& \underline{U} \geq [\underline{\theta} - \bar{\theta}]\psi(\bar{x}) + \bar{U} & \text{IC}(\underline{\theta}) \\
& \bar{U} \geq [\bar{\theta} - \underline{\theta}]\psi(\underline{x}) + \underline{U} & \text{IC}(\bar{\theta})
\end{aligned} \tag{7}$$

The objective function of the lender is a positive function of the expected productivity of the borrowers, a negative function of the cost, r, and a negative function of the utility of the borrowers. In the first best case, the lender is able to distinguish across types and therefore it can maximize its utility by setting borrowers utility to zero. Now that types is private information, the lender has to pay some information rent and therefore some utility will be positive and hence the lender will be worse-off.

1.5.

First, notice that $\text{IC}(\bar{\theta})$ implies $\text{IR}(\bar{\theta})$, so that we know this constraint will always hold and will never bind. So we can get rid of it. Then, combining both ICs, we have that:

$$-[\bar{\theta} - \underline{\theta}]\psi(\underline{x}) \geq [\underline{\theta} - \bar{\theta}]\psi(\bar{x}) \tag{8}$$

$$[\bar{\theta} - \underline{\theta}]\psi(\underline{x}) \leq [\bar{\theta} - \underline{\theta}]\psi(\bar{x}) \tag{9}$$

This implies that:

$$\psi(\bar{x}) \geq \psi(\underline{x}) \implies \bar{x} \geq \underline{x} \tag{10}$$

Since $\psi(\cdot)$ is increasing in x. Then, realize that the IR($\underline{\theta}$) has to bind. If would not be optimal otherwise, since they could always decrease $\underline{U}$ and $\bar{U}$ by ρ units (with $\rho \in (0, \underline{U}]$) without making any other constraint being violated and increase profits. Hence, it has to be that $\underline{U} = 0$.

With $\underline{U} = 0$, $\bar{x} \geq \underline{x}$ and IR($\bar{\theta}$) redundant, we can write the constraints remaining as:

$$\bar{U} \in \left[(\bar{\theta} - \underline{\theta})\psi(\underline{x}), (\bar{\theta} - \underline{\theta})\psi(\bar{x})\right] \tag{11}$$

From all these possible values of $\bar{U}$, at equilibrium will be the minimum possible. The derivative of the objetive function wrt $\bar{U}$ is $-p < 0$, so the lower $\bar{U}$ the higher the objective function will be. Hence, we have that $\bar{U} = (\bar{\theta} - \underline{\theta})\psi(\underline{x})$. We can substitute this in the objective function:

$$\max_{\bar{x}, \underline{x}} \quad p(\bar{\theta}\psi(\bar{x}) - r\bar{x}) + (1-p)(\underline{\theta}\psi(\underline{x}) - r\underline{x}) - p(\bar{\theta} - \underline{\theta})\psi(\underline{x}) \tag{12}$$

THe FOCs are:

$$\bar{\theta}\psi'(\bar{x}) - r = 0 \tag{13}$$

$$(1-p)(\underline{\theta}\psi'(\underline{x}) - r) - p(\bar{\theta} - \underline{\theta})\psi'(\underline{x}) = 0 \tag{14}$$

From 13:

$$\bar{x} = \psi'^{-1}\left(\frac{r}{\bar{\theta}}\right) \tag{15}$$

The first as in the first best. From 14 we have that:

$$(1-p)r = \psi'(\underline{x})((1-p)\underline{\theta} - p(\bar{\theta} - \underline{\theta})) \tag{16}$$

$$\frac{(1-p)r}{\underline{\theta} - p\bar{\theta}} = \psi'(\underline{x}) \tag{17}$$

$$\underline{x} = \psi'^{-1}\left(\frac{(1-p)r}{\underline{\theta} - p\bar{\theta}}\right) \tag{18}$$

Which is less than the first best.

2. All Pay Auction with Incomplete Information

2.1.

Any agent $i = 1, 2$ solves:

$$\max_{b_i} \quad v_i P(b_i > b_j) - b_i \tag{19}$$

b_j is a function of v_j, so we can rewrite 19 as:

$$\begin{aligned} \max_{b_i} \quad & v_i P(b_i > f(v_j)) - b_i \\ \max_{b_i} \quad & v_i P(f^{-1}(b_i) > v_j) - b_i \end{aligned} \tag{20}$$

Since $v_i \sim Uniform(0,1)$, we know that:$P(v_j \leq x) = x$, so we can rewrite as:

$$\max_{b_i} \quad v_i f^{-1}(b_i) - b_i \tag{21}$$

The FOC is:

$$v_i \frac{\partial f^{-1}(b_i)}{\partial b_i} - 1 = 0 \tag{22}$$

$$\frac{\partial f^{-1}(b_i)}{\partial b_i} = \frac{1}{v_i} \tag{23}$$

Now make the guess that $f(\cdot)$ is increasing and quadratic. For instance, take $f(x) = x^2 \implies f^{-1}(x) = \sqrt{x} \implies f^{-1\prime}(x) = \frac{1}{2\sqrt{x}}$. Using this result in 23:

$$\frac{1}{2\sqrt{b_i}} = \frac{1}{v_i} \implies b_i = \frac{v_i^2}{4} \neq v_i^2 \tag{24}$$

So the guess is wrong. Now make the guess $f(x) = \frac{x^2}{2} \implies f^{-1}(x) = \sqrt{2x} \implies f^{-1\prime}(x) = \frac{1}{\sqrt{2x}}$. Using this in 23 again:

$$\frac{1}{\sqrt{2b_i}} = \frac{1}{v_i} \implies b_i = \frac{v_i^2}{2} \tag{25}$$

So the guess is right. So we have can characterize the symmetric equilibrium in which $b_i = \frac{v_i^2}{2}$ $\forall i \in \{1,2\}$

2.2.

Nothing it's said about the number of players, so I assume there are N of them. The generic problem is:

$$\max_{b_i} \quad v_i P(b_i \geq b_j \forall j \neq i) - b_i \tag{26}$$

Taking $b_j = f(v_j)$ we can write:

$$\max_{b_i} \quad v_i P(b_i \geq f(v_j) \forall j \neq i) - b_i \tag{27}$$

$$\max_{b_i} \quad v_i P(f^{-1}(b_i) \geq v_j \forall j \neq i) - b_i \tag{28}$$

Now, let $v_i \sim G$, so that $P(v_i \leq x) = G(x)$. Then, and assuming v_i are independent:

$$\max_{b_i} \quad v_i G(f^{-1}(b_i))^{N-1} - b_i \tag{29}$$

The FOC of the problem is:

$$v_i(N-1)G(f^{-1}(b_i))^{N-2} g(f^{-1}(b_i)) f^{-1'}(b_i) - 1 = 0 \tag{30}$$

Since we are considering the symmetric equilibrium, it has to be that $b_i = f(v_i)$. Using this in 30:

$$v_i(N-1)G(v_i)^{N-2} g(v_i) f^{-1'}(f(v_i)) = 1 \tag{31}$$

Using the fact that $f^{-1'}(f(v_i)) = \frac{1}{f'(v_i)}$,

$$f'(v_i) = v_i(N-1)G(v_i)^{N-2} g(v_i) \tag{32}$$

Hence,

$$f(v_i) = \int_0^y x(N-1)G(x)^{N-2} g(x) dx \tag{33}$$

is the symmetric equilibrium with N bidders and a continuously differentiable distribution of v, $G(v)$, assuming realizations of v are independent.

3. Again Mechanism Design

First, let me introduce some notation:

$$\Theta = \left\{ \underbrace{[\theta_1^{'}, \theta_2^{'}]}_{\theta_\alpha}, \underbrace{[\theta_1^{'}, \theta_2^{''}]}_{\theta_\beta}, \underbrace{[\theta_1^{''}, \theta_2^{'}]}_{\theta_\gamma}, \underbrace{[\theta_1^{''}, \theta_2^{''}]}_{\theta_\delta} \right\} \tag{34}$$

3.1.

The social choice function is such that:

$$f(\theta_\alpha) = b \qquad f(\theta_\beta) = f(\theta_\gamma) = f(\theta_\delta) = a \tag{35}$$

First, notice that:

$$\forall \theta \in \Theta \quad \forall i \in I: \qquad a \succsim x \quad \forall x \in X \implies \nexists x \in X : x \succ a \tag{36}$$

So $f(\cdot)$ is going to be ex post efficient for $\theta_\gamma, \theta_\beta$ and θ_δ. Now consider θ_α:

$$\forall i \in I: \quad b \succsim x \quad \forall x \in X \implies \nexists x \in X : x \succ b \tag{37}$$

So we have that $f(\theta)$ is ex-post efficient.

3.2.

Let's examine case by case the conditions so that Proposition 23.C.2 holds:

- Agent 1:

  We need:

$$u_1(\underbrace{f(\theta_\alpha)}_{b},\theta_1^{'}) \geq u_1(\underbrace{f(\theta_\gamma}_{a},\theta_1^{'}) \tag{38}$$
$$b \sim a \qquad \checkmark$$

$$u_1(\underbrace{f(\theta_\beta)}_{a},\theta_1^{'}) \geq u_1(\underbrace{f(\theta_\delta)}_{a},\theta_1^{'}) \tag{39}$$
$$a \sim a \qquad \checkmark$$

$$u_1(\underbrace{f(\theta_\gamma)}_{a},\theta_1^{''}) \geq u_1(\underbrace{f(\theta_\alpha)}_{b},\theta_1^{''}) \tag{40}$$
$$a \succ b \qquad \checkmark$$

$$u_1(\underbrace{f(\theta_\delta)}_{a},\theta_1^{''}) \geq u_1(\underbrace{f(\theta_\beta)}_{a},\theta_1^{''}) \tag{41}$$
$$a \sim a \qquad \checkmark$$

- Agent 2:

$$u_2(\underbrace{f(\theta_\alpha)}_{b},\theta_2^{'}) \geq u_2(\underbrace{f(\theta_\beta}_{a},\theta_2^{'}) \tag{42}$$
$$b \sim a \qquad \checkmark$$

$$u_2(\underbrace{f(\theta_\gamma)}_{a},\theta_2^{'}) \geq u_2(\underbrace{f(\theta_\delta)}_{a},\theta_2^{'}) \tag{43}$$
$$a \sim a \qquad \checkmark$$

$$u_2(\underbrace{f(\theta_\beta)}_{a},\theta_2^{''}) \geq u_2(\underbrace{f(\theta_\alpha)}_{b},\theta_2^{''}) \tag{44}$$
$$a \succ b \qquad \checkmark$$

$$u_2(\underbrace{f(\theta_\delta)}_{a},\theta_2^{''}) \geq u_2(\underbrace{f(\theta_\gamma)}_{a},\theta_2^{''}) \tag{45}$$
$$a \sim a \qquad \checkmark$$

Thus $f(\cdot)$ is truthfully implementable.

3.3.

No, it's not their unique weakly dominant strategy. Notice that $\hat{\theta}_i = \theta_i^{''}$ is also a dominant strategy for any $i = 1, 2$. This implies that when they are $\theta^{''}$ they will tell the truth but when they are $\theta^{'}$ they will lie and pretend to be $\theta^{''}$. This is so because $f(\theta_i^{''}, \theta_{-i}) = a \; \forall \theta_{-i}$ and $a \gtrsim x \quad \forall x \in X, \forall \theta \in \Theta$. This is, they know that if they say they are $\theta_i^{''}$, then a will be implemented independently of what others do, and a is weakly preferred to any other outcome in all the states of the world and for all agents. So there is no other strategy that does better than $\hat{\theta}_i = \theta_i^{''}$.
Finally, if $\hat{\theta}_i = \theta_i^{''}$ is implemented, then the outcome a will always be implemented, independently of the state of the world. Hence, when $\Theta = \theta_\alpha$, a and not b is implemented, which means that $f(\cdot)$ is not implemented.