Doctoral Programme in Economics

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Problem set 4

Microeconomics III

Viviens Martín, Javier

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1. Monopolistic Screening with Many Types.

1.1.

Since there is complete information and we have a monopolistic seller, we know that Individual Rationality will be binding for all types:

$$\theta_i q_i - p_i = 0 \implies p_i = \theta_i q_i \tag{1}$$

Princial solves:

$$\max_{q_i} q_i \theta_i - \frac{q_i^2}{2} \tag{2}$$

FOC:

$$\theta_i - q_i = 0 \impliedby \theta_i = q_i \tag{3}$$

SOC:

$$-1 < 0 \tag{4}$$

The solution is a contract for each type:

$$(p_i, q_i) = (\theta_i^2, \theta_i) \quad \forall i \tag{5}$$

1.2.

The revelation principal states that if some allocation can be implemented through some (indirect) mechanism, then it can be implemented through a direct and truthful mechanism. A direct mechanism can be achieved via screening through menus. Therefore, we can directly look at the principals problem:

$$\begin{aligned}
\max_{(p_i,q_i)_{i=1}^n} \quad & \sum_{i=1}^{N} \pi_i (p_i - c(q_i)) \\
\text{s.t.} \quad & \theta_i q_i - p_i \geq 0 \qquad \forall i \qquad \text{IR, N constraints} \\
& \theta_i q_i - p_i \geq \theta_i q_j - p_j \forall j \neq i \qquad \forall i \qquad \text{IC, Nx(N-1) constraints}
\end{aligned} \tag{6}$$

There are a total of $N + N \times (N-1) = N^2$ constraints.

1.3.

Take any two consumers i and j such that $\theta_i > \theta_j$. I need to show that $q_i \geq q_j$. From the IC of consumer i:

$$\theta_i q_i - p_i \geq \theta_i q_j - p_j \implies \theta_i(q_i - q_j) \geq p_i - p_j \tag{7}$$

From the IC of consumer j:

$$\theta_j q_j - p_j \geq \theta_j q_i - p_i \implies \theta_j(q_i - q_j) \leq p_i - p_j \tag{8}$$

Combining both:

$$\theta_i(q_i - q_j) \geq \theta_j(q_i - q_j) \tag{9}$$

Since $\theta_i Z_j$, it has to be that

$$(q_i - q_j) \geq 0 \implies q_i \geq q_j \tag{10}$$

So we have that:

$$\theta_i > \theta_j \implies q_i \geq q_j \quad \blacksquare \tag{11}$$

q* is increasing in θ.

1.4.

IR for θ_1 binds:

$$\theta_1 q_1 - p_1 = 0 \tag{12}$$

Downward adjacency IC for consumer 2 binds:

$$\theta_2 q_2 - p_2 = \theta_2 q_1 - p_1 \tag{13}$$

Combining 12 and 13:

$$\theta_2 q_2 - p_2 = \theta_2 q_1 - p_1 > \theta_1 q_1 - p_1 = 0 \tag{14}$$

So we have that participation constrain holds for individual 2:

$$\theta_2 q_2 - p_2 > 0 \tag{15}$$

Now consider consumer 3. Downward adjacency IC for consumer 3 binds:

$$\theta_3 q_3 - p_3 = \theta_3 q_2 - p_2 > \theta_2 q_2 - p_2 > 0 \tag{16}$$

So the participation constraint for consumer 3 holds:

$$\theta_3 q_3 - p_3 > 0 \tag{17}$$

By continuing iterating this process, we have that all participation constraints holds.
Now, take any consumer i. Downward adjacency IC for consumer i binds:

$$\theta_i q_i - p_i = \theta_i q_{i-1} - p_{i-1} \tag{18}$$

Similarly, downward adjacency IC for consumer $i-1$ also binds:

$$\theta_{i-1} q_{i-1} - p_{i-1} = \theta_{i-1} q_{i-2} - p_{i-2} \implies p_{i-1} - p_{i-2} = \theta_{i-1}(q_{i-1} - q_{i-2}) \tag{19}$$

$$p_{i-1} - p_{i-2} = \theta_{i-1}(q_{i-1} - q_{i-2}) < \theta_i(q_{i-1} - q_{i-2}) \tag{20}$$

$$\theta_i q_{i-2} - p_{i-2} < \theta_i q_{i-1} - p_{i-1} = \theta_i q_i - p_i \tag{21}$$

So we have that:

$$\theta_i q_i - p_i > \theta_i q_{i-2} - p_{i-2} \tag{22}$$

This is, all the IC constraint that make consumer not wanting to pretender they are a lower type hold. The only constraints remaining are the IC that make consumers not wanting to be a higher type.
Consider consumer i. Their upwards adjacency IC is:

$$\theta_i q_i - p_i \geq \theta_i q_{i+1} - p_{i+1} \tag{23}$$

But we know that $i+1$'s downwards adjacency IC binds:

$$\theta_{i+1} q_{i+1} - p_{i+1} = \theta_{i+1} q_i - p_i \implies p_{i+1} - p_i = \theta_{i+1}(q_{i+1} - q_i) > \theta_i(q_{i+1} - q_i) \tag{24}$$

So we have that:

$$p_{i+1} - p_i > \theta_i(q_{i+1} - q_i) \implies \theta_i q_i - p_i > \theta_i q_{i+1} - p_{i+1} \tag{25}$$

This is, the upwards adjacency IC always holds. This implies that all the upwards CI hold, in the same way that downward adjacency IC holding implied downward ICs holding.

1.5.

First, consider the participation constraint for consumer 1. Assume it doesn't bind:

$$\theta_1 q_1 > p_1 \tag{26}$$

Notice that from the N constraints that are binding, p_1 only appears in:

$$\theta_2 q_2 - p_2 \geq \theta_2 q_1 - p_1 \tag{27}$$

Then, the principal could increase p_1, increase their profits without inducing any distorion, since a higher p_1 does not violate 27.

Now, consider one of the downward adjacency IC. Let's assume it does not bind for some i:

$$\theta_i q_i - p_i > \theta_i q_{i-1} - p_{i-1} \tag{28}$$

As before, notice that pi only appears in another equation:

$$\theta_{i+1} q_{i+1} - p_{i+1} = \theta_{i+1} q_i - p_i \tag{29}$$

Then, the principal could increase p_i without creating any distorsion or violating any constraint. So it has to be that they are binding.

1.6.

From the IR of consumer 1 we have that:

$$p_1 = \theta_1 q_1 \tag{30}$$

From IC of consumer 2 with consumer 1, that also binds, we have that:

$$\theta_2 q_2 - p_2 = \theta_2 q_1 - p_1 = \theta_2 q_1 - \theta_1 q_1 \implies p_2 = \theta_2 q_{2+q_1(\theta_1-\theta_2)}(31)$$

We continue iterating and we can get an expression for any p_i as a function of θ and q:

$$p_i = \theta_i q_i + \sum_{k=1}^{i-1} q_k(\theta_k - \theta_{k+1}) \tag{32}$$

We can plug this in the objective function:

$$\max_{(q_i)_{i=1}^n} \quad \sum_{i=1}^{N} \pi_i \left(\theta_i q_i + \sum_{k=1}^{i-1} q_k(\theta_k - \theta_{k+1}) - c(q_i) \right) \tag{33}$$

The FOC wrt q_i is:

$$\pi_i\theta_i - \pi_i c'(q_i) + \sum_{k=i+1}^{N} \pi_k q_i(\theta_i - \theta_{i+1}) = 0 \tag{34}$$

Denote

$$\frac{\sum_{k=i+1}^{N} \pi_k}{\pi_i} = \tilde{\pi}_i \tag{35}$$

Then we can write:

$$\theta_i - q_i + q_i(\theta_i - \theta_{i+1})\tilde{\pi}_i = 0 \tag{36}$$

$$q_i^* = \frac{\theta_i}{1 - (\theta_i - \theta_{i+1})\frac{\sum_{k=i+1}^{N} \pi_k}{\pi_i}} \tag{37}$$

Equations 32 and 37 are the solution of the principal's problem. Notice that the highest type is receiving the same quality as in the first best but at a lower price.

2. Independent Private Value Auctions

Denote $u(b_i)$ the utility of bidding b_i.

2.1.

I will divide all strategies into two subgroups:

- $b_i > v_i$:

  Consider three scenarios:

  1. Second highest bid is strictly larger than b_i:

  $$u(b_i) = u(v_i) = 0 \tag{38}$$

  2. Second highest bid is smaller than v_i:

  $$u(b_i) = u(v_i) = v_i - b \leq 0 \tag{39}$$

  3. Second highest bid $\in (v_i, b_i]$:

  $$u(bi) = v_i - b < 0 = u(v_i) \tag{40}$$

  So v_i weakly dominates all the strategies that bid higher than v_i.

- $b_i < v_i$: Consider again three scenarios:

  1. second highest bid is strictly lower than b_i:

  $$u(b_i) = u(v_i) = v_i - b > 0 \tag{41}$$

  2. second highest bid is larger than v_i:

  $$u(b_i) = u(v_i) = 0 \tag{42}$$

  3. second highest bid $\in [b_i, v_i)$:

  $$u(b_i) = 0 < v_i - b = u(v_i) \tag{43}$$

  So bidding v_i weakly dominates bidding less than v_i.

2.2.

We are looking for a function $b(v_i)$ such that $b'(v_i)$ exists and is strictly greater than zero for all v_i. The generic bidder maximizes:

$$(v_i - b_i)P(b_i \geq b(v_1), ..., b_i \geq b(v_N)) \tag{44}$$

Notice that

$$P(b_i \geq b(v_j)) = P(v_j \leq b^{-1}(b_i)) = F\left(b^{-1}(b_i)\right) \tag{45}$$

Since types are drawn independently from the same distribution, we can write 45 as:

$$(v_i - b_i)[F\left(b^{-1}(b_i)\right)]^{n-1} \tag{46}$$

The FOC wrt b_i is:

$$-[F\left(b^{-1}(b_i)\right)]^{n-1} + (v_i - b_i)(N-1)[F\left(b^{-1}(b_i)\right)]^{n-2} f\left(b^{-1}(b_i)\right) \frac{1}{b'(b^{-1}(b_i))} = 0 \tag{47}$$

Since equilibrium has to be symmetric, it has to be that $b_i = b(v_i)$. Then, we can rewrite as:

$$b'(v_i)F(v_i)^{n-1} = (v_i - b(v_i))(N-1)F(v_i)^{n-2}f(v_i) \tag{48}$$

$$v_i(N-1)F(v_i)^{n-2}f(v_i) = \underbrace{b'(v_i)F(v_i)^{n-1} + b(v_i)(N-1)F(v_i)^{n-2}f(v_i)}_{\frac{\partial F(v_i)^{n-1}b(v_i)}{\partial v_i}} \tag{49}$$

$$v_i(N-1)F(v_i)^{n-2}f(v_i) = \frac{\partial F(v_i)^{n-1}b(v_i)}{\partial v_i} \tag{50}$$

$$\int_0^{v_i} x(n-1)F(x)^{n-2}f(x)dx = F(v_i)^{n-1}b(v_i) \tag{51}$$

$$[xF(x)^{n-1}]_0^{v_i} - \int_0^{v_i} F(x)^{n-1}dx = F(v_i)^{n-1}b(v_i) \tag{52}$$

$$b(v_i) = v_i - \frac{\int_0^{v_i} F(x)^{n-1}dx}{F(v_i)^{n-1}} \tag{53}$$

This function is clearly symmetric, differentiable and increasing in v. The intuition of this result is that people are bidding their value minus an information rent. This information rent depends on their type and it is increasing in v. The higher your type is, the less likely it is that someone has higher type than you, the higher your information rent is and the less you'll bid respectively to your type.

2.3.

Let's start with the second price auction. Everyone's bide their own valuation. The probability of individual i bidding less than y is given by $F(y)$. Then, the probability of y being the maximum bid is the probability of y being the maximum value which is equal to:

$$F^N(y) \tag{54}$$

The probability of y being the second highest bid is equal to the probability of all bids being lower than y plus the probability that only one of the bids out of N is larger (there are N possible combinations of this happening):

$$F^N(y) + N(1-F(y))F^{N-1}(y) \tag{55}$$

Taking the derivative of this expression we can get the density of the second highest value/bid:

$$F^{N-1}(y)Nf(y) + N(N-1)F^{N-2}(y)f(y) - N(N-1)F^{N-1}(y)f(y) - NF^{N-1}(y)f(y) = \tag{56}$$

$$= N(N-1)f(y)(F^{N-2}(y) - F^{N-1}(y)) \tag{57}$$

Then, to know the expected profit of the auctioneer we only need to know the expected second highest bid which is equal to the expected second highest value which is equal to:

$$\int N(N-1)f(y)(F^{N-2}(y) - F^{N-1}(y))ydy \tag{58}$$

In the case of the first price auction, we are interested in the highest value of $b(v_i)$ given in equation 53. Insteead of using 53, which is nicer for interpretation, let me use one of the intermediate steps, equation 51. Then, we have that:

$$b(v_i) = \frac{1}{F^{n-1}(y)} \int_0^{v_i} x(n-1)F(x)^{n-2} f(x)dx \tag{59}$$

The expectation of the bid is then:

$$E[b(v_i)] = \int \frac{1}{F^{n-1}(y)} \int_0^{v_i} \left(x(n-1)F(x)^{n-2} f(x)dx \right) f(y)dy \tag{60}$$

But we are not interested in the expectation of $b(v_i)$ but its expectation of it when v_i is the largest draw in the sample. From equation 54 we can recover the density of the larges draw: $NF^{N-1}(y)f(y)$. Using this density on the previous equation:

$$E[b(v_i^{max})] = \int \frac{1}{F^{n-1}(y)} \int_0^{v_i} \left(x(n-1)F(x)^{n-2} f(x)dx \right) NF^{N-1} f(y)dy \tag{61}$$

$$N(N-1) \int \int_0^{v_i} \left(x(n-1)F(x)^{n-2} f(x)dx \right) f(y)dy \tag{62}$$

$$N(N-1) \int y f(y) F^{N-2}(y)(1 - F(y))dy \tag{63}$$

which is exactly the same as in the second price auction.

3. Procurement

3.1.

The revelation principle states that any allocation that is implementable can also be implemented through a direct and truthful mechanism. This implies that we can solve the problem using directly a direct and truthful mechanism. A direct mechanism in this context implies that the choice set of any agent is the choice set of their type. Thruthful mechanismm implies that any agent has no incentive to pretend to be a different type. All in all, we can write the problem of the principal as:

$$\begin{aligned} \max_{(q_i,t_i)_{i=h,l}} \quad & \pi(q_h - t_h) + (1-\pi)(q_l - t_l) \\ \text{s.t.} \quad & t_i - c_i(q_i) \geq 0 \quad \forall i \\ & t_l - c_l(q_l) \geq t_h - c_l(q_h) \\ & t_h - c_h(q_h) \geq t_l - c_h(q_l) \end{aligned} \tag{64}$$

3.2.

To compute the Pareto Efficient production quantities, assume that the principal and the contractor have the same weight in the social welfare function. Then q_i maximizes:

$$\pi_i(q_i - t_i + t_i - c_i(q_i)) \tag{65}$$

Consequently, the optimal production quantity for type i is given by:

$$q_i : \quad c_i^{'}(q_i) = 1 \tag{66}$$

Notice that if $c_i^{'}(q_i) < 1 \ \forall q_i$ the optimal quantity would be $q_i = \infty$ and if $c_i^{'}(q_i) > 1 \ \forall q_i$ the optimal quantity would be $q_i = 0$.

3.3.

Using the fact that $c_H(q) > c_L(q) \ \forall q$, we have that:

$$t_l - c_l(q_l) \geq t_h - c_l(q_h) > t_h - c_h(q_h) \geq 0 \tag{67}$$

So we have that

$$t_l - c_l(q_l) > 0 \tag{68}$$

The low type IR contraint is redundant and will never bind.

3.4.

We can get rid of the IR of the low type. This implies the the IR of high type is binding. Assume not. Then the principal could pay a lower t and increase profit. So it has to be that IR for high type binds.

Given that IR_h binds and IR_L does not, it has to be that IC of low type binds. Assume not. Then, we would have:

$$\begin{aligned} & t_l - c_l(q_l) > 0 \\ & t_l - c_l(q_l) > t_h - c_l(q_h) > t_h - c_h(q_h) = 0 \\ & t_h - c_h(q_h) \geq t_l - c_h(q_l) \end{aligned} \tag{69}$$

Notice that we could decrease t_l up to the point the the IC binds. The principal would increase profits without violating any of the other constraints where t_l also appears. Then we have that the IC for the low type binds. Then we would have:

$$t_l - c_l(q_l) = t_h - c_l(q_h) \implies t_l = t_h - c_l(q_h) + c_l(q_l) \tag{70}$$

Using this in the IC of the hight:

$$t_h - c_h(q_h) \geq t_h - c_l(q_h) + c_l(q_l) - c_h(q_l) \tag{71}$$

$$c_h(q_l) - c_h(q_h) \geq c_l(q_l) - c_l(q_h) \tag{72}$$

$$c_h(q_l) - c_l(q_l) \geq c_h(q_h) - c_l(q_h) \tag{73}$$

since $c_h(q) > c_l(q)$ and $c_h'(q) > c_l'(q)$ $\forall q$, it has to be that $q_l \geq q_h$. Then, we can write the IC of the high type as:

$$t_l - t_h \leq c_h(q_l) - c_h(q_h) \tag{74}$$

From the IC of low type binding we know that:

$$t_l - t_h = c_l(q_l) - c_l(q_h) \leq c_h(q_l) - c_h(q_h) \tag{75}$$

So, given that $q_l \geq q_h$, the the IC of the high type is redundant. Therefore the problem of the principal becomes just:

$$\max_{q_h, q_l} \quad \pi(q_h - c_h(q_h)) + (1-\pi)(q_l - c_h(q_h) - c_l(q_h) + c_l(q_l)) \tag{76}$$

The FOC wrt to q_l is:

$$c_l'(q_l) = 1 \tag{77}$$

so the low type will produce the efficient quantity. The FOC wrt q_h is:

$$\pi(1 - c_h'(q_h)) - (1-\pi)(c_h'(q_h) + c_l'(q_h)) = 0 \tag{78}$$

$$c_h'(q_h) = 1 - \underbrace{\frac{1-\pi}{\pi}}_{>0} \underbrace{(c_h'(q_h) + c_l'(q_h))}_{>0} < 0 \tag{79}$$

With $c(\cdot)$ convex, this implies that the high type will produce a lower quantity than the efficiency one. This is the usual result we have seen in class: the principal prefers to distort the market in order to being able to identify the types. This distortion makes high type producing less than efficient and low type being better off than in an equilibrium where types are observed, since low type is producing the same but a higher price.