Doctoral Programme in Economics

2022-2023

*Problem set 1*

Latest version

# Econometrics III

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## 1. Exercise 1

$$y_t = 1 + 1.3y_{t-1} - 0.4y_{t-2} + u_t \qquad u_t \sim \mathcal{N}(0, 1) \tag{1}$$

### 1.1. Is $y_t$ stationary?

Let me write the companion form of this VAR(2):

$$\begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \underbrace{\begin{bmatrix} 1.3 & -0.4 \\ 1 & 0 \end{bmatrix}}_{F} \begin{bmatrix} y_{t-1} \\ y_{t-2} \end{bmatrix} + \begin{bmatrix} u_t \\ 0 \end{bmatrix} \tag{2}$$

We know that a VAR is stable if and only if the eigenvalues of the matrix $F$ are smaller than one in modulus, and that stability of a VAR implies stationarity. The eigenvalues of $F$ are the roots of the following equation:

$$\lambda^2 - 1.3\lambda + 0.4 \tag{3}$$

and are equal to 0.8 and 0.5, so I can conclude that $y_t$ is stationary.

### 1.2. Determine $E[y_t]$

By stationarity, we know that $E[y_t] = \mu$, which does not depend on $t$. Therefore I can write:

$$\begin{aligned} E[y_t] &= \mu \\ E[1 + 1.3y_{t-1} - 0.4y_{t-2} + u_t &= \mu \\ 1 + 1.3E[y_{t-1}] - 0.4E[y_{t-2}] + E[u_t] &= \mu \\ 1 + 1.3\mu - 0.4\mu &= \mu \\ \mu = \frac{1}{1 - 1.3 + 0.4} &= 10 \end{aligned} \tag{4}$$

### 1.3. Determine $Var(y_t)$

Let me generalize notation a bit and then I'll solve for the variance:

$$y_t = c + \pi_1 y_{t-1} + \phi_2 y_{t-2} + u_t \tag{5}$$

From the previous section, first notice that we can write:

$$c = \mu(1 - \phi_1 - \phi_2) \tag{6}$$

Now, some algebra:

$$\begin{aligned} y_t &= \mu(1 - \phi_1 - \phi_2) + \phi_1 y_{t-1} + \phi_2 y_{t-2} + u_t \\ y_t &= \mu + \phi_1(y_{t-1} - \mu) + \phi_2(y_{t-2} - \mu) + u_t \\ y_t - \mu &= \phi_1(y_{t-1} - \mu) + \phi_2(y_{t-2} - \mu) + u_t \end{aligned} \tag{7}$$

The variance of $y_t$ is:

$$\begin{aligned} Var(y_t) = \gamma_0 = E[(y_t - \mu)(y_t - \mu)] = \\ E[(y_t - \mu)(\phi_1(y_{t-1} - \mu) + \phi_2(y_{t-2} - \mu) + u_t)] = \\ \phi_1 \underbrace{E[(y_t - \mu)(y_{t-1} - \mu)]}_{\gamma_1} + \phi_2 \underbrace{E[(y_t - \mu)(y_{t-1} - \mu)]}_{\gamma_2} + \underbrace{E[(y_t - \mu)u_t]}_{\sigma_u^2} \end{aligned} \tag{8}$$

So we have that

$$\gamma_0 = \phi_1 \gamma_1 + \phi_2 \gamma_2 + \sigma_u^2 \tag{9}$$

Similarly:

$$\begin{aligned} \gamma_1 = E[(y_t - \mu)(y_{t-1} - \mu)] = \\ = E[(\phi_1(y_{t-1} - \mu) + \phi_2(y_{t-2} - \mu) + u_t)(y_{t-1} - \mu)] = \\ \phi_1 \gamma_0 + \phi_2 \gamma_1 \end{aligned} \tag{10}$$

$$\gamma_1 = \phi_1 \gamma_0 + \phi_2 \gamma_1 \tag{11}$$

$$\begin{aligned} \gamma_2 = E[(y_t - \mu)(y_{t-2} - \mu)] = \\ E[(\phi_1(y_{t-1} - \mu) + \phi_2(y_{t-2} - \mu) + u_t)(y_{t-2} - \mu)] = \\ = \phi_1 \gamma_1 + \phi_2 \gamma_0 \end{aligned} \tag{12}$$

$$\gamma_2 = \phi_1 \gamma_1 + \phi_2 \gamma_0 \tag{13}$$

Equations (9), (11) and (13) constitute a system of 3 equations and 3 unkowns, so we can solve. In this specific case, this system looks like:

$$\begin{cases} \gamma_0 = 1.3\gamma_1 - 0.4\gamma_2 + 1 \\ \gamma_1 = 1.3\gamma_0 - 0.4\gamma_1 \\ \gamma_2 = 1.3\gamma_1 - 0.4\gamma_0 \end{cases} \tag{14}$$

The solution to this system is:

$$\gamma_0 = \frac{700}{81} \qquad \gamma_1 = \frac{650}{81} \qquad \gamma_2 = \frac{565}{81} \tag{15}$$

Hence:

$$Var(y_t) = \frac{700}{81} \approx 8.64 \tag{16}$$

### 1.4. Determine $\rho_1$ and $\rho_2$.

We know that $\rho_i = \frac{\gamma_i}{\gamma_0}$, so we can directly compute this with the information of the previous section:

$$\rho_1 = \frac{650}{700} = \frac{13}{14} \qquad \rho_2 = \frac{565}{700} = \frac{113}{140} \tag{17}$$

### 1.5. Simulate the IRF in the computer and report the graph.

I don't understand exactly what does it mean "simulate" the IRF. In this case, that we know $\phi_1$, $\phi_2$ and $\sigma_u^2$, we know that the effect of $u_t$ on $y_{t+h}$ is given by $F^h_{(1,1)}$. The impulse response function is shown in figure 1:

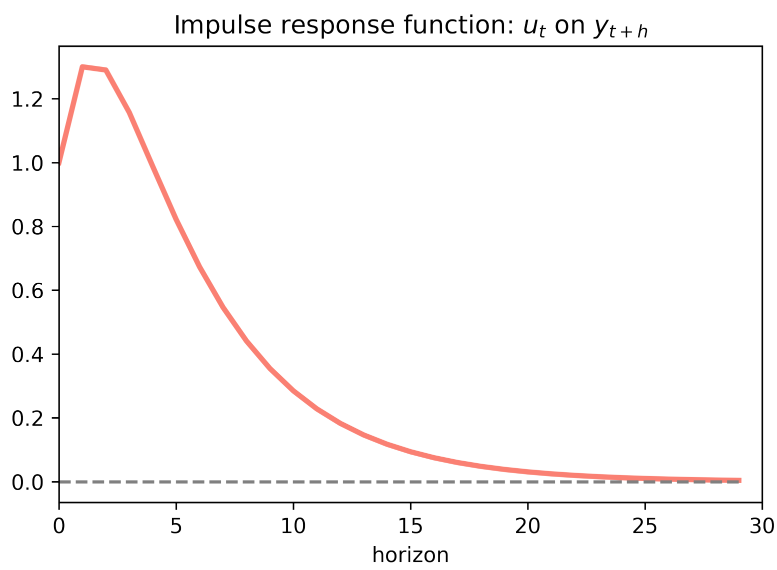


Figure 1

**1.6. Simulate one series of this process for T=500 and plot the sample autocorrelation and the sample partial autocorrelation. How does it compare with the theoretical ones? Are they equal? Why?**

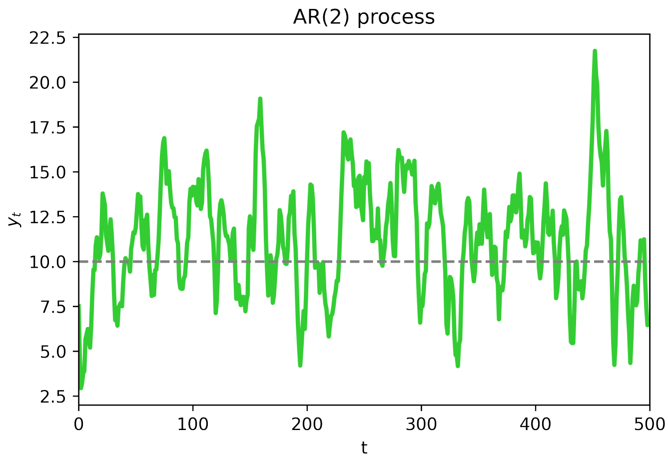


Figure 2

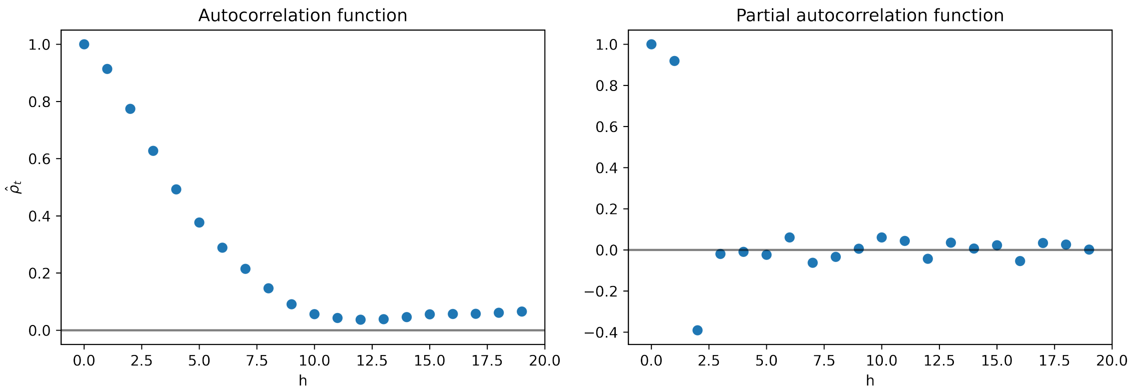


Figure 4: Autocorrelation and partial autocorrelation functions.

Sample autocorrelations are not equal to the theoretical values. They are similar though. This is so because we are a random sample of this AR(2) process, and hence we are not estimating the autocorrelation of the AR(2) process but from a random sample of this process, which are two different things. Therefore, each time we draw a random sample we will find different values for the sample moments. However, it is true that these moments have a distribution around the theoretical values. Therefore, the larger the sample is the closer we should get to the theoretical values ($\hat{\rho} \to \rho$ as $T \to \infty$). Other way to check this is to take $N$ different random samples of size T. Then we should find that $\frac{1}{N}\sum_{n=1}^{N}\hat{\rho} \to \rho$ as $N \to \infty$.

## 2. Exercise 2

$$y_t = 1 + u_t - 1.3u_{t-1} + 0.4u_{t-2} \qquad u_t \sim \mathcal{N}(0,1) \tag{18}$$

### 2.1. Is $y_t$ stationary?

$$E[y_t] = E[1 + u_t - 1.3u_{t-1} + 0.4u_{t-2}] = 1 \tag{19}$$

$$Var(y_t) = Var(1 + u_t - 1.3u_{t-1} + 0.4u_{t-2}) = 1 + 1.69 + 0.16 = 2.85 \tag{20}$$

where I have used the fact that shocks are iid.

$$\gamma_j = E[(y_t - \mu)(y_{t-j} - \mu)] = E[(u_t - 1.3u_{t-1} + 0.4u_{t-2})(u_{t-j} - 1.3u_{t-1-j} + 0.4u_{t-2-j})] \tag{21}$$

$$\gamma_1 = -1.3E[u_{t-1}^2] - 0.52E[u_{t-2}^2] = -1.82 \tag{22}$$

$$\gamma_2 = 0.4E[u_{t-2}^2] = 0.4 \tag{23}$$

$$\gamma_j = 0 \quad \forall j > 2 \tag{24}$$

The mean and the autocovariances do not depend on t, and the variance does not explode (it is finite), therefore I can conclude that $y_t$ is a stationary process.

### 2.2. Is $y_t$ invertible?

Consider the following expression:

$$1 - 1.3z + 0.4z^2 \tag{25}$$

It has two roots: 1.25 and 2. Both roots are outside the unit circle, therefore, it ha to be that the process is invertible.

### 2.3. Determine $E[y_t]$

From previous section, $E[y_t] = 1$.

### 2.4. Determine $Var(y_t)$

From previous section, $Var(y_t) = 2.85$.

### 2.5. Determine $\rho_1$ and $\rho_2$

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{-1.82}{2.85} \approx -0.64 \tag{26}$$

$$\rho_2 = \frac{\gamma_2}{\gamma_0} = \frac{0.4}{2.85} \approx 0.14 \tag{27}$$

### 2.6. Plot Impulse Response Function

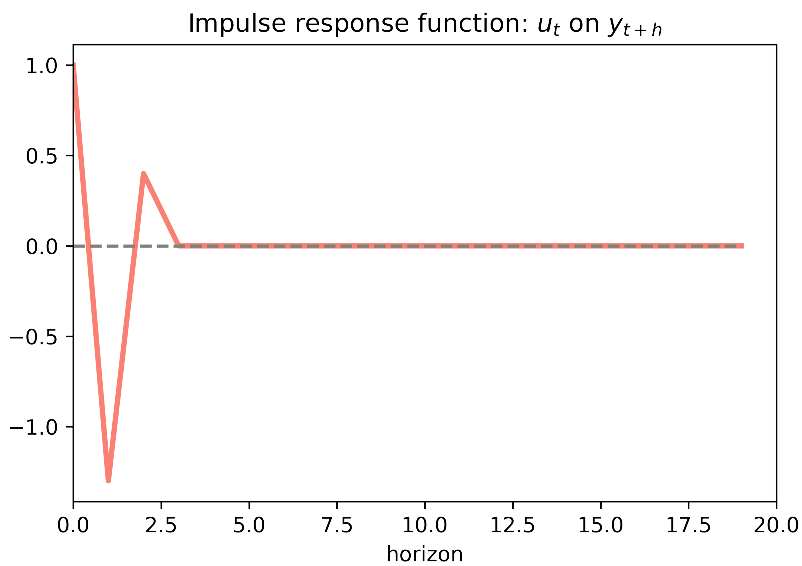


Figure 5

**2.7. Simulate one series of this process for T=500 and plot the sample autocorrelation and the sample partial autocorrelation. How does it compare with the theoretical ones? Are they equal? Why?**

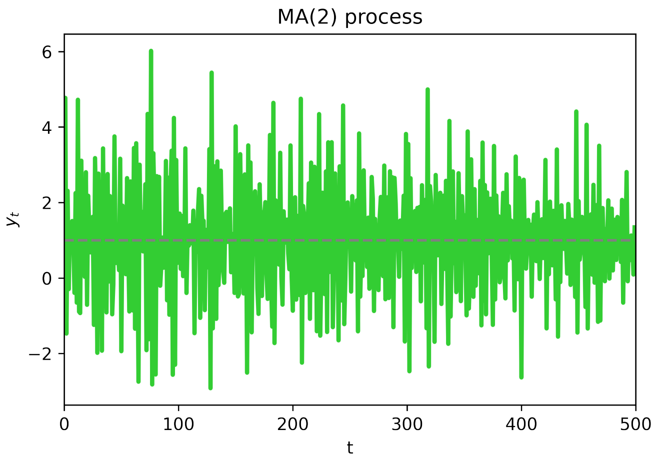


Figure 6

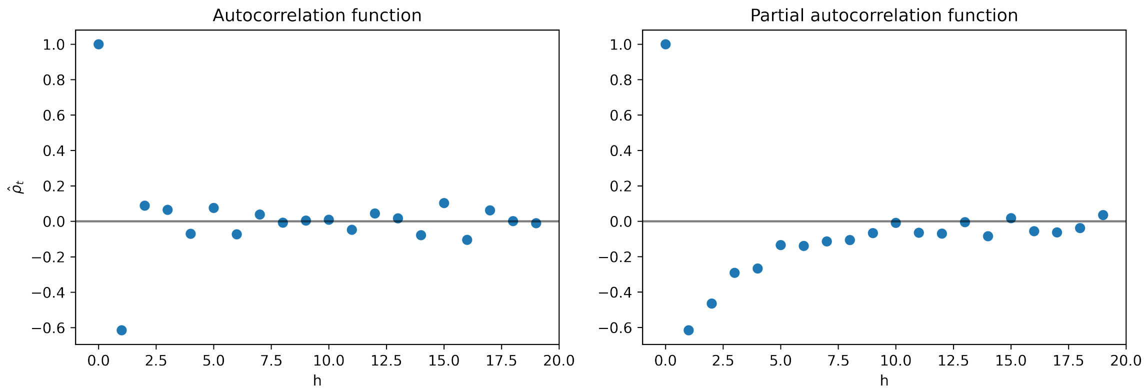


Figure 8: Autocorrelation and partial autocorrelation functions.

Same answer as in previous exercise

**3. Download from Fred the data on US unemployment rate from 1948 to 2020.**

I downloaded the monthly unemployment rate (in %) seasonally adjusted. The total number of periods is $T = 875$.

### 3.1. Plot the autocorrelation function and the partial autocorrelation for 24 lags.

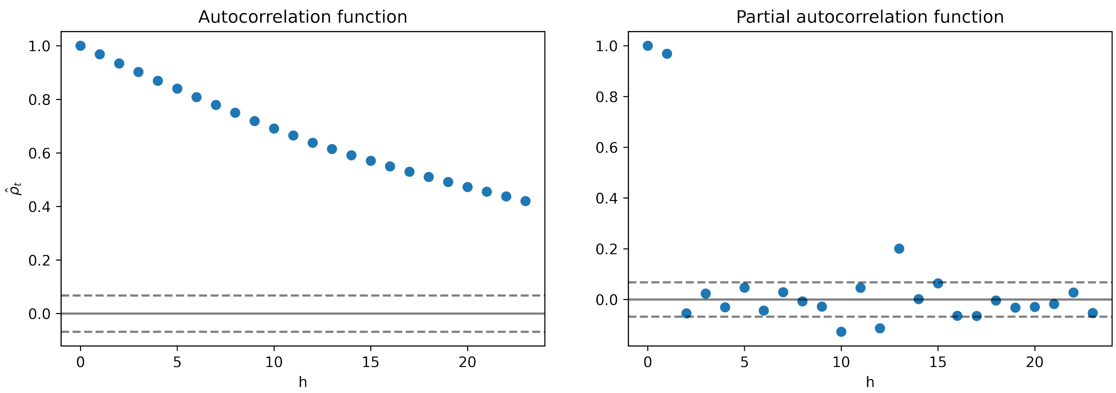


Figure 10: Autocorrelation and partial autocorrelation functions.

### 3.2. Do you think an AR(p) or an MA(q) could fit the data well? Suggest values for p and/or q.

I believe an AR process would fit the data better than a MA process. Just looking at the functions in figure 10 it is enough to argue so. Clearly these figures are more similar to the ones in exercise 1 than to the figures in exercise 2. In general, what we observe, a soft decay in the autocorrelation function with a sharp decay in the partial autocorrerlation function, is something typical to observe in AR processes (with MA processes being the other way around).

For the number of $p$, from the partial autocorrelation function in figure 10 we could also observe that $p = 1$ would be enough. If we look at the different information criteria, we have that both the Hannan-Quinn Information Criterion (HQIC) and the Bayesian Information Criterion (BIC) suggest $p = 1$ whereas the Akaike Information Criterion (AIC) suggests $p = 2$. This is in line with what I argued using the graph of the partial autocorrelation function.