Doctoral Programme in Economics

2022-2023

Problem set 1

Macroeconomics II

Viviens Martín, Javier

29^{th} *January* 2022

Exercise 1

The problem of the representative generation born at period t is:

$$\begin{aligned} \max_{m_t,k_t,c_{t+1}} \quad & u(c_{t+1}) \\ \text{s.t.} \quad & \frac{m_t}{p_t} + s_t = w_t \qquad (\lambda) \\ & c_{t+1} = \frac{m_t}{p_{t+1}} + R_{t+1}s_t \qquad (\mu) \end{aligned} \tag{1}$$

The First Order Conditions of this problem are:

$$[c_{t+1}]: \qquad u'(c_{t+1}) - \mu = 0 \tag{2}$$

$$[s_t]: \qquad -\lambda + R_{t+1}\mu = 0 \tag{3}$$

$$[m_t]: \qquad -\lambda\frac{1}{p_t} + \mu\frac{1}{p_{t+1}} = 0 \tag{4}$$

From equations (3) and (4) we can already see which is the condition for an interior solutions, this is, the condition for an equilibrium in which agents use both money and capital to save:

$$R_{t+1} = \frac{p_t}{p_{t+1}} \tag{5}$$

Equation (5) is the so called no arbitrage condition, which implies both money and capital have the same return and therefore people are indifferent between using one or another to save.

The firm solves:

$$\max_{K_t,L_t} F(K_t, L_t) - R_tK_t - w_tL_t \tag{6}$$

Labour is supplied inelastically, so at equilibrium it has to be that $L_t = 1$. We can rewrite the previous problem as:

$$\max_{K_t} f(k) - R_tk_t - w_t \tag{7}$$

where $f(k) = F(K, 1)$. The first order conditions yields:

$$R_t = f'(k_t) \tag{8}$$

and

$$w_t = f(k_t) - f'(k_t)k_t \tag{9}$$

The market clearing condition in the goods market is:

$$f(k_t) = c_t + s_t \qquad \forall t \tag{10}$$

and the market clearing condition in the money market:

$$M = m_t \quad \forall t \tag{11}$$

From the budget constraint of the household:

$$c_{t+1} = \frac{m_t}{p_{t+1}} + R_{t+1} s_t \tag{12}$$

using no arbitrage condition,

$$c_{t+1} = \frac{m_t}{p_{t+1}} + \frac{p_t}{p_{t+1}} s_t = \frac{p_t}{p_{t+1}} \left[\frac{m_t}{p_t} + s_t \right] \tag{13}$$

using the budget constraint when young:

$$c_{t+1} = \frac{p_t}{p_{t+1}} w_t \tag{14}$$

Then, we can use the market clearing condition to write:

$$f(k_{t+1}) = \frac{p_t}{p_{t+1}} w_t + s_{t+1} = f'(k_{t+1})(f(k_t) - f'(k_t)k_t) + k_{t+2} \tag{15}$$

$$k_{t+2} = f(k_{t+1}) - f'(k_{t+1})[f(k_t) - f'(k_t)k_t] \tag{16}$$

Equation 16 characterizes the dynamics of the capital in equilibrium.

Using the budget constraint when young and the market clearing condition in the money market we can write:

$$M = m_t = p_t(w_t - s_t) = p_t(f(k_t) - f'(k_t)k_t - k_{t+1}) \tag{17}$$

At the steady state becomes:

$$M = p_t(f(k) - f'(k)k - k) \tag{18}$$

$$p_t = \frac{M}{(f(k) - f'(k)k - k)} = p_{t+1} \tag{19}$$

Therefore, the conditions for which both money and capital are used in the steady state are:

- The steady state level of capital,k, is such that f'(k) = 1
- prices are constant in every period and equal to $\frac{M}{(f(k)-f'(k)k-k)}$.

Corollary: a change in the endowment of money for the initial old has no effect in the steady state real variables (capital and consumption). It only affects prices. This is, if we have to economies, A and B, such that $M^A = 2M^B$, we would have that $p^A = 2p^B$ and $k^A = k^B$.

Now, assume there is a lump-sum transfer of money. The first order conditions don't change, only the market clearing condition for the money market change in this context. Now we have:

$$M_t = p_t(f(k) - f'(k)k - k) \tag{20}$$

$$M_{t+1} = p_{t+1}(f(k) - f'(k)k - k) \tag{21}$$

$$\frac{p_t}{p_{t+1}} = \frac{1}{1+\sigma} < 1 \tag{22}$$

Now, the steady state capital level, $\tilde{k}$ has to be such that $f'(\tilde{k}) = \frac{1}{1+\sigma}$

Assume that f is increasing and concave.

$$f'(\tilde{k}) < f'(k) \implies \tilde{k} > k \tag{23}$$

Then, the level of capital in the steady state in which both money and capital are used increases. The intuition behind this result is the following: the increase in the money supply translates into an increase of prices: we would have inflation at the steady state, where each period prices grow at a constant rate $(1 + \sigma)$. This reduces the return of holding money. In order to have money holdings in equilibrium, we need that the return of capital being also lower, otherwise capital would dominate money and nobody would want to hold money. The way to decrease the return of savings is to increase savings.

To ilustrate better this situation, let's compare again two economies, A and B. In the economy A the supply of money is fixed, whereas in the economy B in increases a rate σ, which is transferred to the old each period in a lump sum. The real value of money in the economy B is lower than in the economy A. Hence, in the economy B people will decide to hold less money (in real terms) and more capital in comparison with economy A.

Exercise 2

As before, I will assume that are not explicitly specified in the statement. This time, I will assume that the production function is $f(n_t) = n_t$ as in the lectures. The problem of a representative household is:

$$\begin{aligned} \max_{c_{t+1}, n_t} \quad & u(c_{t+1} + h(c_{t+1}^g)) - g(n_t) \\ \text{s.t.} \quad & m_t = p_t n_t \\ & p_{t+1} c_{t+1} = m_t \end{aligned} \tag{24}$$

The solution of this problem is given by:

$$\frac{p_t}{p_{t+1}} u' \left(\frac{p_t}{p_{t+}} n_t + h\left(c_{t+1}^g\right) \right) = g'(n_t) \tag{25}$$

The government budget constraint is given by:

$$p_t c_t^g = \sigma M_{t-1} \qquad \forall t \tag{26}$$

The market clearing condition in the market for goods is:

$$c_{t+1} + c_{t+1}^g = n_{t+1} \quad \forall t \tag{27}$$

And the market clearing condition in the market for money is:

$$m_t = M_t = M_{t-1} + \sigma M_{t-1} \quad \forall t \tag{28}$$

From the money market clearing condition at period t and the budget constraint of the generation born at period t we get that:

$$m_t = p_t n_t = M_t = M_{t-1} + \sigma M_{t-1} \tag{29}$$

From market clearing condition in market of good at period t+1 and the budget constraint of the gobernment at period t+1 and the generation born at period t we get:

$$n_{t+1} = c_{t+1} + c_{t+1}^g = \frac{m_t}{p_{t+1}} + \frac{\sigma M_t}{p_{t+1}} = \frac{M_{t+1}}{p_{t+1}} \tag{30}$$

Thus,

$$\frac{p_t}{p_{t+1}} = \frac{M_t / n_t}{M_t(1+\sigma)/n_{t+1}} = \frac{n_{t+1}}{(1+\sigma) n_t} \tag{31}$$

We can use equation (31) in equation (25) to get:

$$\boxed{n_{t+1}u'\left(\frac{n_{t+1}}{(1+\sigma)}+h\left(\frac{\sigma}{(1+\sigma)}n_{t+1}\right)\right)=g'(n_t)n_t(1+\sigma)} \tag{32}$$

where I have used:

$$c^g_{t+1}=\frac{\sigma M_t}{p_{t+1}}=\frac{\sigma p_t n_t}{p_{t+1}}=\frac{\sigma n_{t+1}}{(1+\sigma)} \tag{33}$$

Equation (32) is the difference equation characterizing the equilibrium sequence $\{n_t\}_{t=1}^{\infty}$.

Now take $h(c^g_t)=c^g_t$. Equation (32) becomes:

$$n_{t+1}u'\left(\frac{n_{t+1}}{(1+\sigma)}+\frac{\sigma}{(1+\sigma)}n_{t+1}\right)=g'(n_t)n_t(1+\sigma) \tag{34}$$

$$n_{t+1}u'(n_{t+1})=g'(n_t)n_t(1+\sigma) \tag{35}$$

Which is the same as in the equilibrium with lump-sum transfer of money. Instead, what is doing te government here is transfering the new money created in a lump sum, but instead of giving the money, is buying goods with that money and giving the goods in lump-sum. As long as $h(c)=c$, this makes no difference.

Finally, consider the steady state version of 35:

$$u'(n)=g'(n)(1+\sigma) \tag{36}$$

$$\frac{dn}{d\sigma}=\frac{g'(n)}{u''(n)-g''(n)(1+\sigma)}<0 \tag{37}$$

The money creation decreases the employment level at the steady state. This is so because the money creation is reducing the return on labour by increasing prices. Therefore, the optimal level of σ would be zero.

Exercise 3

The problem of the representative household of the generation born at period t is:

$$\begin{aligned}\max_{c_{t+1},n_t}\quad & u(c_{t+1})-g(n_t)\\ \text{s.t.}\quad & b_t+g_t=p_t n_t\\ & p_{t+1}c_{t+1}=\rho^b b_t+\rho^g g_t\end{aligned} \tag{38}$$

In order to make both moneys being used at equilibrium, we need to impose no arbitrage condition, this is, $\rho^b = \rho^g = \rho$. You can see this from the first order conditions for an interior solution of the problem:

$$\begin{aligned} [b_t]: \quad & -\lambda_1 + \lambda_2 \rho^b = 0 \\ [g_t]: \quad & -\lambda_1 + \lambda_2 \rho^g = 0 \\ & \frac{\rho^g}{\rho^b} = 1 \implies \rho^g = \rho^b \end{aligned} \tag{39}$$

We can rewrite the problem as:

$$\begin{aligned} \max_{c_{t+1}, n_t} \quad & u(c_{t+1}) - g(n_t) \\ \text{s.t.} \quad & b_t + g_t = p_t n_t \\ & p_{t+1} c_{t+1} = \rho(b_t + g_t) \end{aligned} \tag{40}$$

The solution of this problem is given by:

$$\frac{p_t}{p_{t+1}} \rho u' \left(\frac{p_t}{p_{t+1}} \rho n_t \right) = g'(n_t) \tag{41}$$

Market for green money clears in every period:

$$\rho^g g_{t-1} = g_t \qquad \forall t \tag{42}$$

Where g_0 is the endowmnet for intial old. Similarly for the blue money:

$$\rho^b b_{t-1} = b_t \qquad \forall t \tag{43}$$

The market for goods clears at every period:

$$c_t = n_t \tag{44}$$

From the budget constraint of the household born at generation t:

$$c_{t+1} = \frac{b_t + g_t}{p_{t+1}} \rho \tag{45}$$

Imposing market clearing condition of money markets:

$$c_{t+1} = \frac{\rho(b_{t-1} + g_{t-1})}{p_{t+1}} \rho \tag{46}$$

And now using the market clearing condition in the goods market:

$$n_{t+1} = \frac{\rho(b_{t-1} + g_{t-1})}{p_{t+1}} \rho \tag{47}$$

The same for generation born at $t-1$:

$$n_t = \frac{(b_{t-1} + g_{t-1})}{p_t}\rho \tag{48}$$

Combining 47 and 48:

$$\frac{p_t}{p_{t+1}} = \frac{n_{t+1}}{\rho n_t} \tag{49}$$

So at equilibrium, 41 becomes:

$$n_{t+1}u'(n_{t+1}) = g'(n_t)n_t \tag{50}$$

Which is exactly the same difference equation as when there is only one currency.

The relative price of green money in terms of blue money is:

$$\frac{\rho^g}{\rho^b} = \frac{\rho}{\rho} = 1 \qquad \text{no arbitrage condition} \tag{51}$$

If only green money had value in equilibrium, it has to be that at equilibrium $b_t = 0$ which implies that $\rho^b < \rho^g$. From the market clearing in the blue money market:

$$\rho^b b_{t-1} = b_t = 0 \implies \rho^b = 0 \tag{52}$$

Then, consumption when old would be given by:

$$c_{t+1} = \frac{\rho^g g_t}{p_{t+1}} \implies c_{t+1} = \frac{\rho^g p_t n_t}{p_{t+1}} \tag{53}$$

$$\frac{p_t}{p_{t+1}}\rho^g u'\left(\frac{p_t}{p_{t+1}}\rho^g n_t\right) = g'(n_t \tag{54}$$

Using market clearing conditions:

$$\left.\begin{aligned} n_{t+1} &= c_{t+1} = \tfrac{\rho^g g_t}{p_{t+1}} = \tfrac{\rho^{g2} g_{t-1}}{p_{t+1}} \\ n_t &= c_t = \tfrac{\rho^g g_{t-1}}{p_t} \end{aligned}\right\} \implies \frac{p_t}{p_{t+1}} = \frac{n_{t+1}}{n_t \rho^g} \tag{55}$$

Thus equation 54 becomes:

$$n_{t+1}u'(n_{t+1}) = g'(n_t)n_t \tag{56}$$

The difference equation does not change.