Doctoral Programme in Economics

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Problem set 3

Microeconomics III

Viviens Martín, Javier

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1. Screening.

1.1.

For me it's not clear what the question is asking, since only says "optimal contract", but I don't know if the agency knows the costs or not. So I will solve for both.

1. Costs are observable for the agency.

   The agency solves:

$$\begin{aligned} \max_{p_i, q_i} \quad & \beta(q_i) - p_i \\ \text{s.t.} \quad & p_i \geq c_i q_i \end{aligned} \tag{1}$$

   We know that the constraint will be binding, so we can just solve:

$$\max_{q_i} \quad \beta(q_i) - c_i q_i \tag{2}$$

   So the optimal contract is $(p_i, q_i) = (c_i\beta'^{-1}(c_i), \beta'^{-1}(c_i))$, which depends on the company's type.

2. Agency doesn't observe cost.

   The previous menu of contracts is no longer optimanl since the low cost company will take the high cost company contract. Hence, now the agency solves:

$$\begin{aligned} \max_{p_h, q_h, p_l, q_l} \quad & b[\beta(q_l) - p_l] + (1-b)[\beta(q_h) - p_h] \\ \text{s.t.} \quad & p_l \geq c_l q_l \\ & p_h \geq c_h q_h \\ & p_l - c_l q_l \geq p_h - c_l q_h \\ & p_h - c_h q_h \geq p_l - c_h q_l \end{aligned} \tag{3}$$

   We know that $p_l - c_l q_l \underbrace{\geq}_{3^{rd}\text{constraint}} p_h - c_l q_h > p_h - c_h q_h \underbrace{\geq}_{2^{nd}\text{constraint}} 0$. So we have that the first constraint will never bind. This implies that the third constraint will always bind. Suppose not. Then, the firm could find a lower p_l and increase profits. This is not problematic with the fourth constraint, nor with the first since it is not binding.

   Now, notice that q_l has to be grater than q_h. Adding the third and fourth constraints:

$$\begin{aligned} p_l - c_l q_l + p_h - c_h q_h \geq p_h - c_l q_h + p_l - c_h q_l \\ c_h(q_l - q_h) \geq c_l(q_l - q_h) \end{aligned} \tag{4}$$

Recall that $c_h > c_l$. Then it has to be that $(q_l - q_h) \geq 0 \implies q_l \geq q_h$.

Now, notice the last two results imply that the fourth constraint is not binding. The third constraint is binding:

$$p_l - c_l q_l = p_h - c_l q_h \implies p_l - p_h = c_l(q_l - q_h) \tag{5}$$

We can rewrite the fourth constraint as:

$$p_l - p_h \leq c_h(q_l - q_h) \tag{6}$$

But we know that:

$$p_l - p_h = c_l(q_l - q_h) < c_h(q_l - q_h) \tag{7}$$

So it has to be that the constraint is not binding. Finally, the second constraint has to be binding in equilibrium. Supoose not. Then, we could find a lower p_h that increases profits that is not problematic with the other two constraint where p_h appears (third because it appears in the RHS and fourht because it's not binding).

All in all, we have constraints two and three binding and one and four not binding. We can rewrite the problem as:

$$\max_{q_h, q_l} \quad b[\beta(q_l) - c_h q_h - c_l(q_l - q_h)] + (1-b)[\beta((q_h) - c_h q_h] \tag{8}$$

The first order condition wrt q_l is:

$$b[\beta'(q_l) - c_l] = 0 \implies q_l = \beta'^{-1}(c_l) \tag{9}$$

Which is exactly the same quantity as in the first best case. The FOC wrt q_h is:

$$b[c_l - c_h] + (1-b)[\beta'(q_h) - c_h] = 0 \implies q_h = \beta'^{-1}\left(c_h + \frac{b(c_h - c_l)}{1-b}\right) \tag{10}$$

So the optimal contracts would be:

$$\boxed{\begin{aligned} (p_h, q_h) &= \left(c_h \beta'^{-1}\left(c_h + \frac{b(c_h - c_l)}{1-b}\right), \beta'^{-1}\left(c_h + \frac{b(c_h - c_l)}{1-b}\right)\right) \\ (p_l, q_l) &= \left(c_h \beta'^{-1}\left(c_h + \frac{b(c_h - c_l)}{1-b}\right) + c_l\left(\beta'^{-1}(c_l) - \beta'^{-1}\left(c_h + \frac{b(c_h - c_l)}{1-b}\right)\right), \beta'^{-1}(c_l)\right) \end{aligned}} \tag{11}$$

1.2. Compare the first best and the second best.

The quantity supplied by the low cost firm does not change from one equilibrium to another. On the other hand, the high type will provide less quantity in the first best equilibrium. Since $\beta(\cdot)$ is concave,

$$c_h < (c_h + \frac{b(c_h - c_l)}{1-b} \implies \beta'^{-1}(c_h) \geq \beta'^{-1}\left(c_h + \frac{b(c_h - c_l)}{1-b}\right) \tag{12}$$

In terms of equilibrium prices, high cost companies will also be paid lest (exactly by the same arguments as before), whereas low cost companies will be paid more in the second best. To see this, observe that we can write the price in the second best as:

$$p_l = c_l q_l + p_h - c_l q_h = \underbrace{c_l q_l}_{p_l \text{in the first best, since quantities coincide}} + \underbrace{c_h q_h - c_l q_h}_{>0} \tag{13}$$

All in all, we have that the low firm, which is the "best" in this scenario, produces the same quantity for a higher price. So they gain. This is the information rent we saw in class. High cost companies produce less for a lower price such that they make zero profits anyway.

1.3.

The first best does not change, since c is observed, the prices and quantities of a company with c_i is just $(c_i\beta'^{-1}(c_i), \beta'^{-1}(c_i))$.

In the second best, the agency solves:

$$\begin{aligned} \max_{q(c),p(c)} \quad & E[\beta(q(c)) - p(c)] = \int [\beta(q(c)) - p(c)] f(c) dc \\ \text{s.t.} \quad & p(c) \geq q(c)c \quad \forall c \\ & p(c) - q(c)c \geq p(c') - q(c')c \quad \forall (c, c') \end{aligned} \tag{14}$$

Denote $V(c, c') = p(c') - q(c')c$. The necessary FOC for the IC is:

$$\left.\frac{\partial V(c,c')}{\partial c'}\right|_{(c,c)} = 0 \tag{15}$$

$$p'(c) - q'(c)c = 0 \tag{16}$$

The necessary SOC is:

$$\left.\frac{\partial^2 V(c,c')}{\partial^2 c'}\right|_{(c,c)} \leq 0 \tag{17}$$

$$p''(c) - q''(c)c \leq 0 \tag{18}$$

The total derivative of $V(c, c')$ is:

$$\frac{dV(c,c)}{dc} = \underbrace{p'(c) - q'(c)c}_{=0} - q(c) \leq 0 \tag{19}$$

So we have that, at equilibrium, the profit of the company has to decrease with the type, c. This implies that the company with highest cost. $\bar{c}$, will make zero profit at equilibrium. This is an equivalent condition as c^h having the IR binding in the previous section. Then, we have that:

$$\frac{dV(c,c)}{dc} = -q(c) \implies V(c,c) = \int_c^{\bar{c}} q(s)ds \tag{20}$$

Recall that $V(c, c') = p(c') - q(c')c$. Then, we can writ:

$$p(c) = \int_c^{\bar{c}} q(s)ds + q(c)c \tag{21}$$

Substituting the expression for $p(c)$ in the objective function of the agency:

$$E[\beta(q(c)) - p(c)] = E[\beta(q(c)) - \int_c^{\bar{c}} q(s)ds - q(c)c] \tag{22}$$

So the agency maximizes:

$$\begin{aligned}
&\max_{(q(c))} \int_{\underline{c}}^{\bar{c}} \left(\beta(q(c)) - \underbrace{\int_c^{\bar{c}} q(s)ds}_{(c-\bar{c})q(c)} - q(c)c \right) f(c)dc \\
&\max_{(q(c))} \frac{1}{\bar{c} - \underline{c}} \int_{\underline{c}}^{\bar{c}} (\beta(q(c)) - (c - \bar{c})q(c) - q(c)c)\, dc \\
&\max_{(q(c))} \frac{1}{\bar{c} - \underline{c}} \int_{\underline{c}}^{\bar{c}} (\beta(q(c)) - (2c - \bar{c})q(c))\, dc
\end{aligned} \tag{23}$$

This equation already contains all the incentive compatibility constraints and the individual rationality constraints. Therefore, maximizing wrt q(c) will solve the problem. Hence, we would have that the maximum is at:

$$\beta^{'}(q(c)) = 2c - \bar{c} \tag{24}$$

So the solution to the second best is given by:

$$q(c) = \beta^{'-1}(2c - \bar{c}) \tag{25}$$

It was most difficult to get this result than the one in the previous section, but the conclusions are exactly the same: the most efficient type will produce exactly the same in the first best and in the second best, but it will be able to get some information rent from the agency. On the other hand, the company less efficient (with the highest cost), will make zero profit and will produce less than in the first best. All the firms whose cost lies between $(\underline{c}, \bar{c})$ will be in a in-between situation. The lower cost, the close will get that company to the first best and the more information rent it will gent. On the other hand, the higher cost a firm has, the less it will produce in equilibrium and the less profits it will make.

2. Moral Hazard with 3 types

2.1.

As usual with this set up, the IR will be binding. This implies that wage will be:

$$w(e^h) = \frac{25}{9} \qquad w(e^m) = \frac{64}{25} \qquad w(e^l) = \frac{16}{9} \tag{26}$$

The utilities of the principal will be given by:

$$\begin{cases} \frac{20}{3} - \frac{25}{9} = 35/9 & \text{if} \quad e = e^h \\ \frac{10}{2} - \frac{64}{25} = 61/25 & \text{if} \quad e = e^m \\ \frac{10}{3} - \frac{16}{9} = 14/9 & \text{if} \quad e = e^l \end{cases} \tag{27}$$

So the optimal contract is a wage of 25/9 in exchange of an effort of e^h.

2.2.

2.2.1

The participation constraint is:

$$\frac{1}{3}\sqrt{w_h} + \frac{2}{3}\sqrt{w_l} \geq \frac{4}{3} \tag{28}$$

The IC constraints are:

$$\frac{1}{3}\sqrt{w_h} + \frac{2}{3}\sqrt{w_l} - \frac{4}{3} \geq \frac{1}{2}\sqrt{w_h} + \frac{1}{2}\sqrt{w_l} - \frac{8}{5} \tag{29}$$

$$\frac{1}{3}\sqrt{w_h} + \frac{2}{3}\sqrt{w_l} - \frac{4}{3} \geq \frac{2}{3}\sqrt{w_h} + \frac{1}{3}\sqrt{w_l} - \frac{5}{3} \tag{30}$$

Notice that for any contract such that the principal pays the same in both states ($w_h = w_b$), we have that the IC are redundant. Therefore we would have the same wage as in the first best, the IR binding and the IC not binding.

2.2.2

IR:

$$\frac{1}{2}\sqrt{w_h} + \frac{1}{2}\sqrt{w_l} - \frac{8}{5} \geq 0 \tag{31}$$

IC:

$$\frac{1}{2}\sqrt{w_h} + \frac{1}{2}\sqrt{w_l} - \frac{8}{5} \geq \frac{1}{3}\sqrt{w_h} + \frac{2}{3}\sqrt{w_l} - \frac{4}{3} \tag{32}$$

$$\frac{1}{2}\sqrt{w_h} + \frac{1}{2}\sqrt{w_l} - \frac{8}{5} \geq \frac{2}{3}\sqrt{w_h} + \frac{1}{3}\sqrt{w_l} - \frac{5}{3} \tag{33}$$

Notice that an effort equal to e^m can never be induced. If we sum the two IC constraints, we get:

$$\sqrt{w_h} + \sqrt{w_l} - \frac{16}{5} \geq \sqrt{w_h} + \sqrt{w_l} - \frac{9}{3} \tag{34}$$

$$3 \geq 3.2 \tag{35}$$

which is not true. This means that the two IC constraints are not implementable at the same time. If we make agent not wanting to deviate to an effor e_l, then it has to be that it wants to deviate to an effort e_h and viceversa.

2.2.3

IR:

$$\frac{2}{3}\sqrt{w_h} + \frac{1}{3}\sqrt{w_l} - \frac{5}{3} \geq 0 \tag{36}$$

IC:

$$\frac{2}{3}\sqrt{w_h} + \frac{1}{3}\sqrt{w_l} - \frac{5}{3} \geq \frac{1}{2}\sqrt{w_h} + \frac{1}{2}\sqrt{w_l} - \frac{8}{5} \tag{37}$$

$$\frac{2}{3}\sqrt{w_h} + \frac{1}{3}\sqrt{w_l} - \frac{5}{3} \geq \frac{1}{3}\sqrt{w_h} + \frac{2}{3}\sqrt{w_l} - \frac{4}{3} \tag{38}$$

Following the hint, let's assume that IC_l binds whereas IC_m does not. Then, from 38 binding:

$$\sqrt{w_h} = \sqrt{w_l} + 1 \tag{39}$$

Using this result in 36:

$$\sqrt{w_l} \geq 1 \tag{40}$$

Let's assume further it binds. It would imply that $w_l = 1$ and $w_h = 4$. With these wages, the three constraints would be: IR

$$\frac{2}{3}2 + \frac{1}{3} - \frac{5}{3} = 0 \tag{41}$$

It is binding, so cool. The IC would be:

$$0 \geq \frac{2}{3} + \frac{2}{3} - \frac{4}{3} = 0 \tag{42}$$

It is binding as well, so my guesses are verified. The only thing missing is check that the last constraint is not violated:

$$0 \geq 1 + \frac{1}{2} - \frac{8}{5} = -\frac{1}{10} \tag{43}$$

So the wages that induce e^h maximizing profit of the principal are $w_h = 4$ and $w_l = 1$

Finally, to find the optimal contract, I have to compare the outcome of the principal when they induce e_l and when they induce e_h (e_m I don't consider since it's not implemebtable).

$$\pi(e_h) = \frac{20}{3} - \frac{8}{3} - \frac{1}{3} = \frac{11}{3} \tag{44}$$

$$\pi(e_l) = \frac{10}{3} - \frac{16}{9} = \frac{14}{9} \tag{45}$$

Clearly, $\pi(e_h) > \pi(e_l)$, so the optimal contract would be $w_h = 4$, $w_l = 1$ and agents chooses $e = e_h$.

3. Moral Hazard with 3 types again

3.1.

The problem of the principal is:

$$\begin{aligned} \max_{w,e} \quad & x(e) - w \\ \text{s.t.} \quad & \sqrt{w} - e^2 \geq \bar{u} \end{aligned} \tag{46}$$

We know that the budget constraint will be binding. Hence, we have that

$$w = (114 + e^2) \tag{47}$$

So the equilibrium wage will be 16900 if the principal asks for an effort of 4 and 22500 if they ask for an effort of 6. Therefore, there are two possible contracts: $(w, e) = (16900, 4)$ or $(w, e) = (22500, 6)$.

$$E[u((w, e) = (22500, 6))] = 27500 \tag{48}$$

$$E[u((w, e) = (16900, 4))] = 23100 \tag{49}$$

Clearly, the first contract yields a higher expected payoff, therefore the first best contract would be $(w, e) = (22500, 6)$.

3.2.

Now the contract offered by the principal cannot include the effort level, e, nor the wage can be a function of it, since it's not observable. However, it can offer a wage as a function of the outcome observed. The principal maximizes the expected utility as before:

$$\max_{w(x)} \quad (x^g - w^g)\pi(x^g \mid e) - (x^b - w^b)\pi(x^b \mid e) \tag{50}$$

Where x^g denotes the good state where 60,000 is realized and x^b the state where 30,000 is realized. The wages w^g, w^b denote the wages paid in each of the sates when realized. Given w^g, w^b, the agent solves:

$$\max_e \sqrt{w^g}\pi(x^g \mid e) + \sqrt{w^b}\pi(x^b \mid e) - e^2 \tag{51}$$

We can solve this backwards. Firs, notice that

$$\pi(x^g \mid e = 6) = 2/3 \tag{52}$$

and

$$\pi(x^g \mid e = 4) = 1/3 \tag{53}$$

So we can rewrite 51 as:

$$\max\{\sqrt{w^g}2/3 + \sqrt{w^b}1/3 - 6^2, \sqrt{w^g}1/3 + \sqrt{w^b}2/3 - 4^2\} \tag{54}$$

The agent will choose effort equal to six if:

$$\begin{gathered}\sqrt{w^g}2/3 + \sqrt{w^b}1/3 - 6^2 \geq \sqrt{w^g}1/3 + \sqrt{w^b}2/3 - 4^2 \\ \sqrt{w^g} - 60 \geq \sqrt{w^b}\end{gathered} \tag{55}$$

Combining this with the fact that the agent has to be better off than the reservation utility, this is:

$$\sqrt{w^g}2/3 + \sqrt{w^b}1/3 - 6^2 \geq 114 \tag{56}$$

$$\sqrt{w^g}2 + \sqrt{w^b} = 450 \implies \sqrt{w^b} = 450 - 2\sqrt{w^g} \tag{57}$$

Combining **??** and 55 :

$$\sqrt{w^g} - 60 \geq 450 - 2\sqrt{w^g} \tag{58}$$

$$3\sqrt{w^g} \geq 510 \implies w^g \geq 28900 \implies w^b = 12100 \tag{59}$$

The wage contract that offers a wage equal to 28900 if x = 60k and 12100 if x = 30k is the contract that induces e=6 with the minimum cost.

I can proceed exactly the same way to compute the wage contract that induces e=4. This time we have:

$$\sqrt{w^g} - 60 \leq \sqrt{w^b} \tag{60}$$

and

$$\sqrt{w^g}1/3 + \sqrt{w^b}2/3 - 4^2 \geq 114 \tag{61}$$

$$\sqrt{w^g} + \sqrt{w^b}2 \geq 390 \tag{62}$$

$$\sqrt{w^g} = (390 - 2\sqrt{w^b}) \tag{63}$$

$$\sqrt{w^b} \geq 110 \implies w^b = 12100 \implies w^g = 28900 \tag{64}$$

The expected wage that the principal would pay is 17700, which is greater than the wage in the first best. Therefore, the principal would be better of without differentiating and paying the first best 16900 independently of the outcome realized. Since wage does not depend on effort, the agent clearly would choose the lowest effort possible.

Hence, **he wage contract that offers a wage equal to 16900, independently of the x obserseved, is the contract that induces e=4 with the minimum cost.**

We can use these results in the objective function of the principal to examine which is the contract that maximizes the principal welfare. Inducing effort equal to 4, principal gets:

$$B = \frac{2}{3}30,000 + \frac{1}{3}60,000 - 16900 = 23100 \tag{65}$$

Inducing effort equal to 6 at the minimum cost, the principal gets:

$$B = \frac{1}{3}(30,000 - 12100) + \frac{2}{3}(60,000 - 28900) = 26,700 \tag{66}$$

So clearly, the principal prefers to induce e=6. Hence, the second best is:

$$w(x) = \begin{cases} 12100 & \text{if} \quad x = 30,000 \\ 28900 & \text{if} \quad x = 60,000 \end{cases} \tag{67}$$

and the agent would make an effort of $e = 6$.

3.3.

Already answered.