Doctoral Programme in Economics

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Problem set 2

Latest version

Macroeconomics II

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Question 1

Before I start the exercise, I'll assume $f(n) = n$, there is a constant total population of measure 1 in the two islands such that α people live in Island 1 and $(1 - \alpha)$ people live in Island 2, $\alpha \in (0, 1)$.

The problem of a generic household born in Island 1 is:

$$\begin{aligned} \max_{c^1_{t+1}, n^1_t} \quad & u(c^1_{t+1}) - g(n^1_t) \\ \text{s.t.} \quad & m^1_t = p_t n^1_t \\ & p_{t+1} c^1_{t+1} = m^1_t + \tau_{t+1} \end{aligned} \tag{1}$$

We can rewrite as:

$$\max_{n^1_t} \quad u\left(\frac{p_t}{p_{t+1}} n^1_t + \tau_{t+1}\right) - g(n^1_t) \tag{2}$$

and the first order condition is given by:

$$\frac{p_t}{p_{t+1}} u'\left(\frac{p_t}{p_{t+1}} n^1_t + \tau_{t+1}\right) = g'(n^1_t) \tag{3}$$

Where I have assumed that the lump sum transfer is the same for any household in the island 1.

The problem of a generic household born in Island 2 is:

$$\begin{aligned} \max_{c^2_{t+1}, n^2_t} \quad & u(c^2_{t+1}) - g(n^2_t) \\ \text{s.t.} \quad & m^2_t = p_t n^2_t \\ & p_{t+1} c^2_{t+1} = m^2_t \end{aligned} \tag{4}$$

We can rewrite as:

$$\max_{n^2_t} \quad u\left(\frac{p_t}{p_{t+1}} n^2_t\right) - g(n^2_t) \tag{5}$$

and the first order condition is given by:

$$\frac{p_t}{p_{t+1}} u'\left(\frac{p_t}{p_{t+1}} n^2_t\right) = g'(n^2_t) \tag{6}$$

At this point we have already an interesting result:

$$\frac{u'(c^i_{t+1})}{g'(n^i_t)} = \frac{p_{t+1}}{p_t} \quad \forall i, \forall t \tag{7}$$

This is, the ratio of marginal utility of consumption over teh marginal disutility of labour is equal in the two islands in every period. Therefore, we know that:

$$\begin{aligned} c_{t+1}^1 = c_{t+1}^2 &\implies n_t^1 = n_t^2 \\ c_{t+1}^1 > c_{t+1}^2 &\implies n_t^1 < n_t^2 \end{aligned} \tag{8}$$

The market clearing conditions are:

- Goods market:

$$\alpha c_t^1 + (1-\alpha)c_t^2 = y_t = \alpha n_t^1 + (1-\alpha)n_t^2 \quad \forall t \tag{9}$$

- Money market

$$\alpha m_t^1 + (1-\alpha)m_t^2 = M_t = (1+`\sigma)M_{t-1} \quad \forall t \tag{10}$$

Combining 9, 10 and the budget constraints:

$$M_t = \alpha m_t^1 + (1-\alpha)m_t^2 = p_t(\alpha n_t^1 + (1-\alpha)n_t^2) = p_t y_t \tag{11}$$

Therefore, at equilibrium:

$$\frac{p_t}{p_{t+1}} = \frac{y_{t+1}}{y_t(1+\sigma)} \tag{12}$$

Also, we have that:

$$y_{t+1} = \alpha c_{t+1}^1 + (1-\alpha)c_{t+1}^2 = \alpha\frac{m_t^1}{p_{t+1}} + \alpha\frac{\tau_{t+1}}{p_{t+1}} + (1-\alpha)\frac{m_t^2}{p_{t+1}} = \frac{M_t}{p_{t+1}} + \alpha\tau_{t+1}\frac{1}{p_{t+1}} \tag{13}$$

$$y_{t+1} = \alpha n_{t+1}^1 + (1-\alpha)n_{t+1}^2 = [\alpha m_{t+1}^1 + (1-\alpha)m_{t+1}^2]\frac{1}{p_{t+1}} = \frac{M_{t+1}}{p_{t+1}} \tag{14}$$

Combining 13 and 14:

$$M_{t+1} = M_t + \alpha\tau_{t+1} \implies \tau_{t+1} = \frac{\sigma M_t}{\alpha} \tag{15}$$

This is: the lump sum transfer that people receive in Island 1 is just the total amount of new money created, σM_t divided by the total amount of people that live in the island, α.

Now, consider an equilibrium such that $n_t^1 = n_t^2 = n_t$. Then, we would have that:

$$c_{t+1}^2 = \frac{p_t}{p_{t+1}}n_t c_{t+1}^1 = \frac{p_t}{p_{t+1}}n_t + \frac{\sigma M_t}{p_{t+1}} = c_{t+1}^2 + \frac{\sigma M_t}{p_{t+1}} \tag{16}$$

From the equilibrium condition 8, we know that if $n_t^1 = n_t^2 = n_t$, it has to be that $c_{t+1}^1 = c_{t+1}^2$ which impliies that $\sigma M_t / p_{t+1} = 0 \implies p_{t+1} = \infty$. We have found the first type of equilibrium, call it autarky:

$$\boxed{n_t^1 = n_t^2 = 0, \quad c_{t+1}^1 = c_{t+1}^2 = 0 \quad p_t/p_{t+1} = 0} \tag{17}$$

Now, consider an equilibrium such that $n_t^1 > n_t^2 \implies m_t^1 > m_t^2$. The budget constraint of the people in island 1 is:

$$c_{t+1}^1 = \frac{m_t^1}{p_{t+1}} + \frac{\sigma M_t}{\alpha p_{t+1}} > \frac{m_t^2}{p_{t+1}} + \frac{\sigma M_t}{\alpha p_{t+1}} = c_{t+1}^2 \tag{18}$$

So we have that

$$c_{t+1}^1 > c_{t+1}^2 + \frac{\sigma M_t}{\alpha p_{t+1}} \geq c_{t+1}^2 \implies c_{t+1}^1 > c_{t+1}^2 \tag{19}$$

We have that $n_t^1 > n_t^2$ and $c_{+1}^1 > c_{t+1}^2$, which is a violation of the equilibrium condition 8. Therefore, there cannot exists an equilibrium such that $n_t^1 > n_t^2$.

Finnally, consider $n_t^1 < n_t^2$. In this case, we have that:

$$c_{t+1}^1 = \frac{p_t n_t^1}{p_{t+1}} + \frac{\sigma M_t}{\alpha p_{t+1}} < \frac{p_t n_t^2}{p_{t+1}} + \frac{\sigma M_t}{\alpha p_{t+1}} = c_{t+1}^2 + \frac{\sigma M_t}{\alpha p_{t+1}} \tag{20}$$

Notice that we could have,

$$c_{t+1}^2 < c_{t+1}^1 < c_{t+1}^2 + \frac{\sigma M_t}{\alpha p_{t+1}} \tag{21}$$

So a competitive equilibrium is a sequence $\left(\{n_t^i\}_{t=1}^{\infty}\right)_{i=1,2}$ such that:

$$u'\left(\frac{n_t^2(n_{t+1}^1 + n_{t+1}^2)}{(1+\sigma)(n_t^1 + n_t^2)}\right)(n_{t+1}^1 + n_{t+1}^2) = g'(n_t^2)(n_t^1 + n_t^2)(1+\sigma) \tag{22}$$

and

$$u'\left(\frac{(n_{t+1}^1 + n_{t+1}^2)}{(1+\sigma)(n_t^1 + n_t^2)}\left[n_t^1 + \frac{\sigma}{\alpha}(n_t^1 + n_t^2)\right]\right)(n_{t+1}^1 + n_{t+1}^2) = g'(n_t^1)(n_t^1 + n_t^2)(1+\sigma) \tag{23}$$

$$n_t^1 < n_t^2 \quad \forall t \tag{24}$$

$$c_{t+1}^1 > c_{t+1}^2 \quad \forall t \implies \frac{n_t^1}{n_t^2} > \frac{1 - \sigma/\alpha}{1 + \sigma/\alpha} \tag{25}$$

At the steady state, we have that $n_t^1 = n^1$ and $n_t^2 = n^2$. Equations 22 and 23 becomes:

$$u'\left(\frac{n^2}{1+\sigma}\right) = g'(n^2)(1+\sigma) \tag{26}$$

and

$$u'\left(\frac{n^1 + \frac{\sigma}{\alpha}(n^1 + n^2)}{(1+\sigma)}\right) = g'(n^1)(1+\sigma) \tag{27}$$

From equation 26 we have that:

$$\frac{dn^2}{d\sigma} = \frac{u''(\cdot)\frac{n^2}{(1+\sigma)^2} + g'(n^2}{u''(\cdot)\frac{1}{1+\sigma} - g''(n^2)(1+\sigma)} \tag{28}$$

The denominator is clearly negative. Therefore, the labour supply at the steady state in the island 2 will increase if $u''(\cdot)\frac{n^2}{(1+\sigma)^2} + g'(n^2) < 0$ and decrease otherwise. From equation 27:

$$\frac{dn^1}{d\sigma} = \frac{u''(\cdot)\frac{\alpha n^1 - n^1 - n^2}{\alpha(1+\sigma)^2} - g'(n^1)}{u''(\cdot)\frac{\alpha+\sigma}{\alpha(1+\sigma)} - g''(n^1)(1+\sigma)} \tag{29}$$

As before, the denominator is negative and the derivative will have the opposite sign as the numerator.

Honestly, I don't know the sign of these derivatives, but my intuition tells me that both are negative. As the money growth increases, in both islands there will be less labour supplied, and in the limit they will be in the autarky equilibrium, where basically the return of labour is zero and nobody works.

Question 2

Again, I assume $f(n) = n$. I also assume there is no money growth: $m_t = M \ \forall t$. The problem of a generic household born at time t is:

$$\begin{aligned} \max_{c_{t+1}, n_t} \quad & E\left[u(c_{t+1}) - \epsilon_t g(n_t)\right] \\ \text{s.t.} \quad & m_t = p_t n_t \\ & p_{t+1} c_{t+1} = m_t \end{aligned} \tag{30}$$

The first order condition of this problem is:

$$E\left[\frac{p_t}{p_{t+1}} u'\left(\frac{p_t}{p_{t+1}} n_t\right)\right] = \epsilon_t g'(n_t) \tag{31}$$

The market clearing conditions are:

- Market for goods:

$$c_t = n_t \tag{32}$$

- Market for money:

$$m_t = M \tag{33}$$

Combining both with the budget constraint of the households, we get:

$$p_t = \frac{M}{n_t} \tag{34}$$

And consequently,

$$\frac{p_t}{p_{t+1}} = \frac{n_{t+1}}{n_t} \tag{35}$$

Using 35 in 31:

$$E\left[n_{t+1}u'(n_{t+1})\right] = \epsilon_t g'(n_t)n_t \quad \equiv n_t = \xi(\epsilon_t) \tag{36}$$

Note that the left hand side is constant, since the shocks are i.i.d. (each period, people have the same expectation of the labour that people in the future will supply). Therefore, the labour supply at period t only depends on the realization of the shock at period t.

$$\frac{dn_t}{d\epsilon_t} = -\frac{g'(n_t)n_t}{\epsilon_t(g'(n_t) + g''(n_t)n_t)} < 0 \tag{37}$$

where I have assumed that $\epsilon_t > 0$, this is, it cannot be that a shock makes you want to work. Hence, the shock has a negative effect on labour supply: the bigger the shock, the less labour people will supply in equilibrium. This is so because the expected utility of consumption is independent of the shock (people know that in the future they will consume exactly what next generation produces, which is not affected by the shock).

Regardin prices, from equation 35 we can observe that the schock affects positively prices. Since it decreases labour supply (n goes down, M/n goes up) it increases price at equilibrium. We know that markets have to clear in equilibrium. Therefore, a shock that induces lower labour supply translates into higher price in order to make supply ($p_t n_t$) equal to demand ($m_t = M$).

Question 2: Second trial

In the previous attempt I didn't solve for the SREE, so I will redo it but this time I'll try to follow more closely Cooper's notes.

Agents solve:

$$\max_n E\left[\frac{p_t}{p_{t+1}}n - \epsilon g(n)\right] \tag{38}$$

The FOC is:

$$E\left[\frac{p_t}{p_{t+1}}u'\left(\frac{p_t}{p_{t+1}}n\right)\right] = \epsilon g'(n) \tag{39}$$

The market clearing condition assuming no money growth is:

$$p_t n_t = M \tag{40}$$

therefore,

$$\frac{p_t}{p_{t+1}} = \frac{n(\epsilon')}{n(\epsilon)} \tag{41}$$

Equation 38 becomes

$$E[n(\epsilon')u'(n(\epsilon'))] = \epsilon g'(n(\epsilon))n(\epsilon) \tag{42}$$

Since ϵ is iid, we have that the LHS is constant. This implies that the RHS has to be constant as well:

$$\epsilon g'(n(\epsilon))n(\epsilon) = k \quad \forall \epsilon \tag{43}$$

Consequently, it has to be that if ϵ goes up, $g'(n(\epsilon))n(\epsilon)$ goes down. By the assumption of g being increasing and strictly convex, we have that n has to go down. Therefore, we have that $n(\epsilon)$ is decreasing in ϵ.

Hence, in equilibrium, employment will response negatively (in the opposite direction) to variations in ϵ whereas the prices will respond positively (in the same direction).

The intuition behind this result is that, when there is a positive shock in ϵ, people dislike more working and therefore they will supply less labour. At equilibrium it has to be that the supply is equal to the demand. In this case, the demand is fixed, M, therefore, a lower labour supply n has to be compensated with a higher price p in order the make demand equal supply.

Another way to see this is that ϵ is basically increasing the marginal disutility of labour. Therefore, in order to make equation 38 hold in equilibrium it has to be that the real return of labour increases. ϵ has no effect on labour supply tomorrow $n_t = n(\epsilon)$, and therefore it has to be that it increases the real wages by increasing today's prices, $p(\epsilon)$.

Question 3

Business as usual, we have the following first order condition:

$$E\left[\frac{p_t}{p_{t+1}}u'\left(\frac{p_t}{p_{t+1}}n_t\right)\right] = g'(n_t) \tag{44}$$

For a fraction $1-\lambda$ of agents, equation 44 is:

$$u'(n_t) = g'(n_t) \implies 1 = n_t\theta \implies n_t = \theta^{-1} \tag{45}$$

This is, a fraction $1-\lambda$ of agents will supply $n_t = \theta^{-1}$ every period.

Market clearing condition implies:

$$M = p_t n_t = p_t\left(n_t\lambda + (1-\lambda)\theta^{-1}\right) \implies p_t = M\left[n_t\lambda + (1-)\theta^{-1}\right]^{-1} \tag{46}$$

Consequently,

$$\frac{p_t}{p_{t+1}} = \frac{n_{t+1}\lambda + (1-\lambda)\theta^{-1}}{n_t\lambda + (1-\lambda)\theta^{-1}} \tag{47}$$

Using this in equation 44:

$$\boxed{n_{t+1}\lambda + (1-\lambda)\theta^{-1} = \theta n_t(n_t\lambda + (1-\lambda)\theta^{-1})} \tag{48}$$

Equation 48 is the difference equation characterizing this economy.

From 48, we can observe there is only one steady state where $n = \theta^{-1}$, this is, both types supply the same amount of labour, and prices are in fact constant.

There is only one steady state: the autarky steady state does no longer exists because $1-\lambda$ fraction of the population is supplying labour every period. Hence, money has to have some value, otherwise markets wouldn't clear.

Let me rewrite the difference equation as:

$$n_{t+1}\lambda + (1-\lambda)\theta^{-1} = \theta\lambda n_t^2 + (1-\lambda)n_t \tag{49}$$

To study local stability we need to check whether the following derivative is between 0 and 1.

$$\frac{dn_{t+1}}{dn_t} = \theta n_t + \frac{1-\lambda}{\lambda} \tag{50}$$

Equation 50 at the steady state becomes:

$$1 + \frac{1-\lambda}{\lambda} > 1 \tag{51}$$

So the steady state equilibrium is not locally stable. Too see why it is not locally stable, and to confirm that there exists only one steady state, we could just look at equation 48 graphically in figure 1:

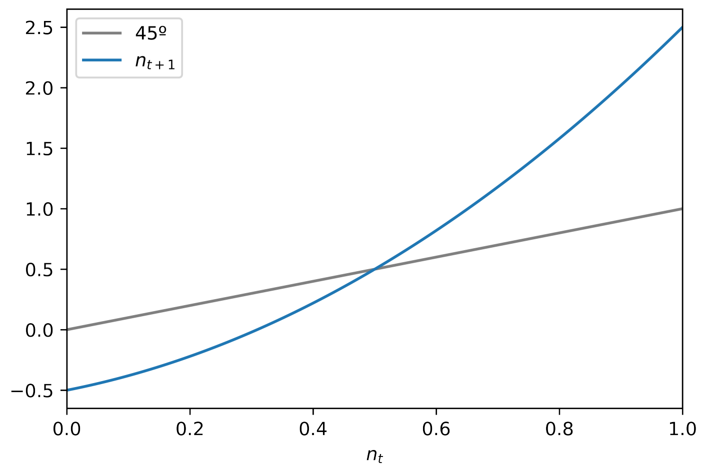

Figure 1: $\lambda = 0.5$, $\theta = 2$.

Notice that the steady state does not depend on λ. So λ plays no role on the steady state. In terms of dynamics, we have that:

$$\frac{d\frac{dn_{t+1}}{dn_t}}{d\lambda} < 0 \tag{52}$$

this is, the higher λ, the less the labour supply tomorrow will react to a change in the labour supplt today. In any case, we have that $\frac{dn_{t+1}}{dn_t} > 1 \ \forall \lambda$, so the stability of the steady state does not depend on λ. You can also observe graphically these facts about the effect of λ in figure 2.

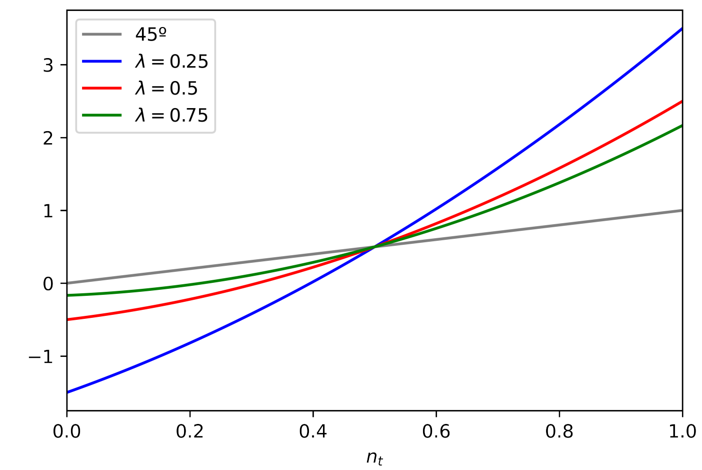

Figure 2: $\theta = 2$.