From equation (1) to equation (5)

$$I'(\mu') = \int_{-1}^{1} \left[\frac{1}{2} + \frac{3}{16}(3\mu'^2-1)(\mu^2-\frac{1}{3})\right] I(\mu)\, d\mu$$

let $I(\mu) = \sum_{i=0}^{\infty} I_i P_i(\mu)$

$$I'(\mu') = \int_{-1}^{1} d\mu \left[\frac{1}{2}P_0(\mu) + \frac{1}{8}(3\mu'^2-1)P_2(\mu)\right] \sum_{i=0}^{\infty} I_i P_i(\mu)$$

Since Legendre polynomials are orthonormal: $\int_{-1}^{1} P_n(x) P_m(x)\, dx = \frac{2}{2n+1}\delta_{mn}$,

$$I'(\mu') = \int_{-1}^{1} d\mu\, \frac{1}{2}P_0(\mu)\cdot I_0 P_0(\mu) + \int_{-1}^{1} d\mu\, \frac{1}{8}(3\mu'^2-1)P_2(\mu)\, I_2 P_2(\mu)$$

$$= \frac{I_0}{2}\cdot 2 + \frac{I_2}{8}(3\mu'^2-1)\cdot\frac{2}{5}$$

$$= I_0 + \frac{3}{20} I_2 (\mu'^2 - \frac{1}{3})$$

* I think in the paper, they are using a different normalization for $P_n(\mu)$, since they let the 2nd term be $I_2(\mu^2-\frac{1}{3}) = \frac{2}{3} I_2 P_2(\mu)$

$$I'(\mu') = I(\mu')(1-\tau) + \tau\left[I_0 + \frac{3}{20} I_2(\mu'^2-\frac{1}{3})\right]$$

$$= \tau[I_0'(\mu') - I(\mu')] + I(\mu')$$ where $I_0'(\mu') = I_0 + \frac{3}{20} I_2(\mu'^2 - \frac{1}{3})$