Question 1

1. $O = \{ low, medium, high \}$

$S = \{ Healthy, Unhealthy \}$

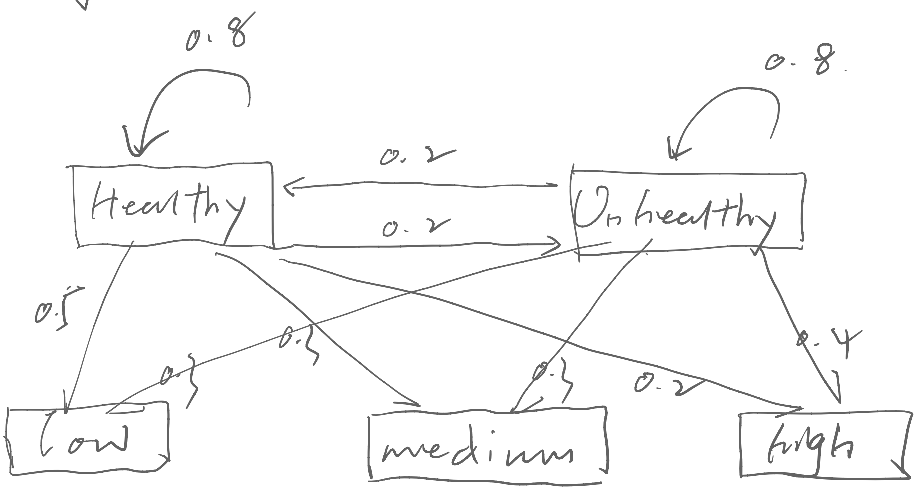
$\lambda = [\pi, a, b]$

> Initial state distribution $\pi = [0.5 \quad 0.5]$

> transition matrix $a = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$

> Emission matrix $b = \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}$

> Diagram



2. $t=1$. $\alpha_H = 0.5(0.5\times0.8 + 0.5\times0.2) = 0.25$

$\alpha_L = 0.3(0.5\times0.2 + 0.5\times0.8) = 0.15$

$t=2$ $\alpha_H = 0.5(0.25\times0.8 + 0.15\times0.2) = 0.115$

$\alpha_L = 0.3(0.5\times0.2 + 0.5\times0.8) = 0.051$

$$P(t_2 = \text{Healthy} \mid t_1 = \text{low}, t_2 = \text{low}) = \frac{\alpha_{H,t=2}}{\alpha_{H,t=2} + \alpha_{L,t=2}} = \frac{0.115}{0.115+0.051}$$

$= 0.69$

3. $\delta_0(H) = 0.5$

$\delta_1(H) = \max \begin{cases} 0.5\times0.8\times0.3 = 0.2 \\ 0.5\times0.2\times0.3 = 0.05 \end{cases}$

$\delta_2(H) = \max \begin{cases} 0.5\times0.2\times0.3 = 0.12 \\ 0.5\times0.8\times3 = 0.03 \end{cases}$

$\delta_3(H) = \max \begin{cases} 0.2\times0.8\times0.5 = 0.08 \\ 0.12\times0.2\times0.5 = 0.012 \end{cases}$

Question 2

1. Likelihood:

$$P(X \mid \mu_k) = \prod_{i=1}^{m} \prod_{k=1}^{k} \left(P(z^i = k, \pi_k) P(x^i \mid z=k; \mu) \right)$$

$$= \prod_{i=1}^{m} \prod_{k=1}^{k} \left(\pi_k \text{Bernoulis}(x^i, \mu_k) \right)$$

Then, do log transformation.

$$P = \sum_{i=1}^{m} \sum_{k=1}^{k} I(z^i = k) [\pi_k \text{Bernouli}(x^i, \mu_k)]$$

$$E = \sum_{i=1}^{m} \sum_{k=1}^{k} \{ r_k^i \log \pi_k + x^i \log \mu_k + (1-x^i) \log(1-\mu_k) \}$$

$$\frac{\partial E}{\partial \mu_k} = 0.$$

$$\sum_{i=1}^{m} r_k \left(\frac{x_j^i}{\mu_k} - \frac{1-x_j^i}{\mu_k} \right) = 0$$

$$\Rightarrow \quad \mu_{kj} = \frac{\sum_{i=1}^{m} r_k^i x_j^i}{\sum_{i=1}^{m} r_k^i}$$

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2. Similar to part-1

$$\frac{\partial E}{\partial \mu_{kj}} = \frac{(\sum r_k^i x_j^i) + \alpha - 1}{\mu_{kj}} - \frac{\sum r_k^i (1-x_j^i) + \beta - 1}{1-\mu_{kj}} = 0$$

$$\Rightarrow \mu_{kj} = \frac{\sum_{i=1}^{n} r_k^i x_j^i + \alpha + 1}{(\sum_{i=1}^{m} r_k) + \alpha + \beta - 2.}$$

Question 3:

$$f(u) = \arg\min_{v \in V} \| x - v \|^2$$

$$= (\arg\min_a \| x - a v \|^2) * u$$

$$= \arg\min_a (x^T x + 2 a u^T x + a^2 \underbrace{(u^T u)}_{=1}) * u.$$

$$\frac{\partial (x^T x + 2 a u^T x + a^2)}{\partial a} = 0$$

$$\Rightarrow \quad a = \frac{-2 u^T x}{-2} = u^T x$$

$$\therefore f(u) = u^T x\, u$$

$$\Rightarrow \arg\min_{u : u^T u = 1} \sum_{i=1}^{m} \| x^i - f_u(x) \|^2$$

$$= \arg\min \sum_{i=1}^{m} \| x^i - u^T x u \|^2$$

$$= \arg\min \sum (x^{iT} x - 2 (u^T x)^2 + u^T u (u^T x)^2)$$

$$= \arg\min_u \sum_{i=1}^{m} -(u^T x^i)^2$$

$$= \arg\min_{u: u^T u = 1} u^T (\sum_{i=1}^{m} x^i x^{iT}) u.$$