COMP 540 Assignment #1

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0 Background refresher(30 points)

- Plot the categorical distribution.

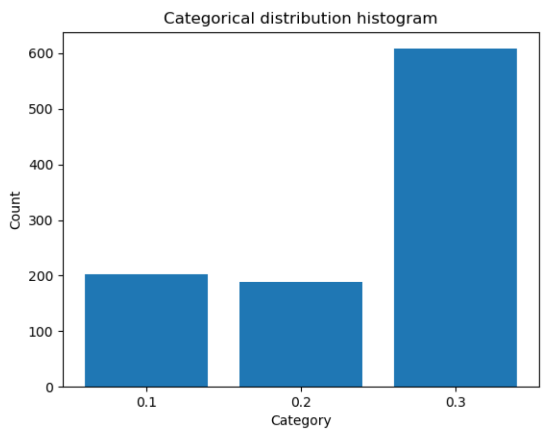

Figure 1: Categorical distribution

- Plot the Univariate normal distribution with mean of and standard deviation of 1.

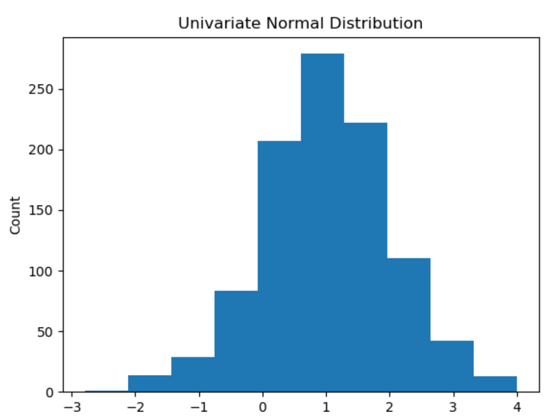

Figure 2: Univariate Normal Distribution

- Produce a scatter plot of the samples for a 2-D Gaussian.

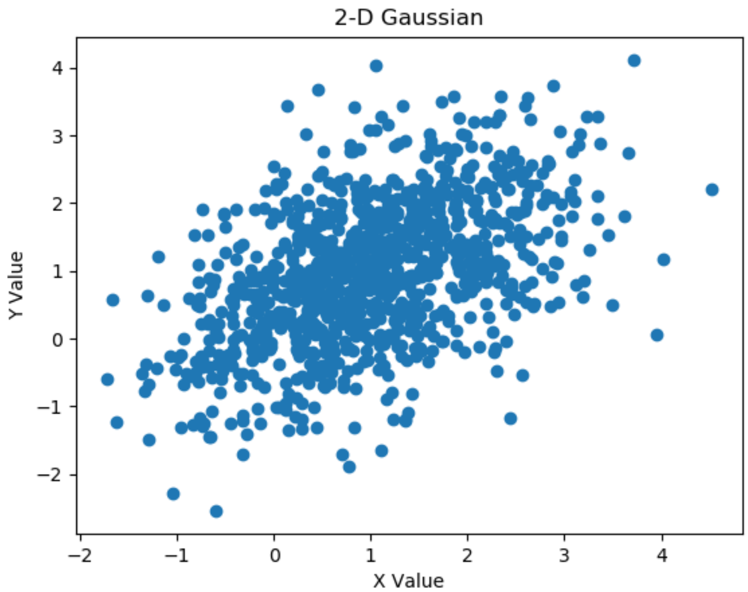

Figure 3: Univariate Normal Distribution

- Test mixture sampling code Code can be seen in sampler.py. Mixture Gaussian plot is shown below

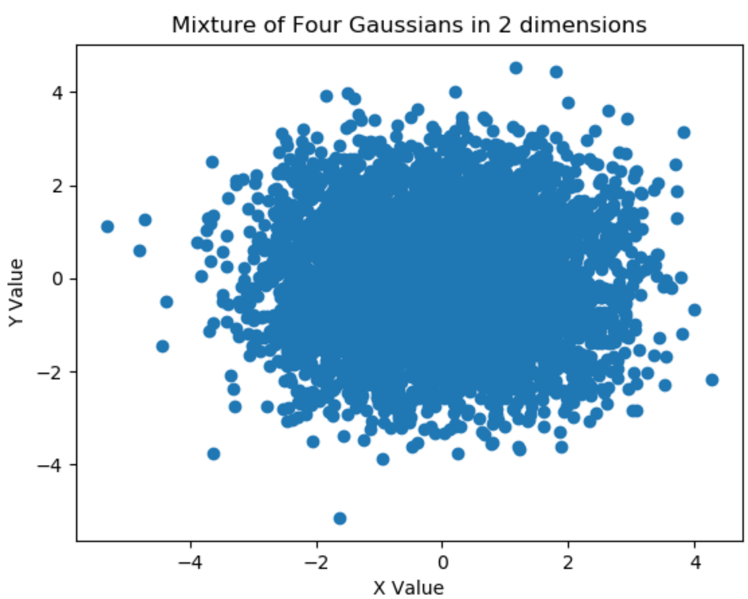

Figure 4: Univariate Normal Distribution

- Prove that the sum of two independent Poisson random variables is also a Poisson random variable. Suppose $X \sim \mathcal{P}(\lambda)$ and $Y \sim \mathcal{P}(\mu)$. Now Prove that $X + Y \sim \mathcal{P}(\lambda + \mu)$.

$$
\begin{aligned}
P(X+Y=k) &= \sum_{i=0}^{k} P(X+Y=k, X=i) \\
&= \sum_{i=0}^{k} P(Y=k-i, X=i) \\
&= \sum_{i=0}^{k} P(Y=k-i)P(X=i) \\
&= \sum_{i=0}^{k} e^{-\mu}\frac{\mu^{k-i}}{(k-i)!}e^{-\lambda}\frac{\lambda^i}{i!} \\
&= e^{-(\mu+\lambda)}\frac{1}{k!}\sum_{i=0}^{k}\frac{k!}{i!(k-i)!}\mu^{k-i}\lambda^i \\
&= e^{-(\mu+\lambda)}\frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}\mu^{k-i}\lambda^i \\
&= \frac{(\mu+\lambda)^k}{k!}\cdot e^{-(\mu+\lambda)}
\end{aligned}
$$

So $X + Y \sim \mathcal{P}(\lambda + \mu)$.

- Consider the vectors $\boldsymbol{u} = \begin{bmatrix}1 & 2\end{bmatrix}^{\mathrm{T}}$ and $\boldsymbol{v} = \begin{bmatrix}2 & 3\end{bmatrix}^{\mathrm{T}}$. Define the matrix $\boldsymbol{M} = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$. Compute the eigenvalues and eigenvectors of $\boldsymbol{M}$.

$$
\begin{aligned}
\boldsymbol{M} &= \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} \\
&= \begin{bmatrix}1 & 2\end{bmatrix}^{\mathrm{T}} \cdot \begin{bmatrix}2 & 3\end{bmatrix} \\
&= \begin{bmatrix}2 & 3 \\ 4 & 6\end{bmatrix}
\end{aligned}
$$

$$
\left|\lambda \boldsymbol{I} - \boldsymbol{M}\right| = \begin{vmatrix}\lambda - 2 & -3 \\ -4 & \lambda - 6\end{vmatrix} = 0
$$

Then we can solve it as

$$
\begin{aligned}
(\lambda-2)(\lambda-6) - 12 &= 0 \\
\lambda^2 - 8\lambda &= 0 \\
\lambda(\lambda - 8) &= 0 \\
\lambda_1 = 0, \lambda_2 &= 8
\end{aligned}
$$

Let $\lambda = 0$:

$$
\begin{aligned}
(\lambda \boldsymbol{I} - \boldsymbol{M}) \cdot \begin{bmatrix} x_1 & x_2 \end{bmatrix}^{\mathrm{T}} &= 0 \\
\begin{bmatrix} -2 & -3 \\ -4 & -6 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
-2x_1 - 3x_2 &= 0
\end{aligned}
$$

Let $x_1 = 3$, Then $x_2 = 2$. So eigenvector is $\begin{bmatrix} 3 & 2 \end{bmatrix}^{\mathrm{T}}$.
Let $\lambda = 8$:

$$
\begin{aligned}
(\lambda \boldsymbol{I} - \boldsymbol{M}) \cdot \begin{bmatrix} x_1 & x_2 \end{bmatrix}^{\mathrm{T}} &= 0 \\
\begin{bmatrix} 6 & -3 \\ -4 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
-6x_1 - 3x_2 &= 0 \\
-4x_1 + 2x_2 &= 0
\end{aligned}
$$

Let $x_1 = 1$, Then $x_2 = 2$. So eigenvector is $\begin{bmatrix} 1 & 2 \end{bmatrix}^{\mathrm{T}}$.
Thus, when eigenvalue $\lambda = 0$, eigenvector is $\begin{bmatrix} 3 & 2 \end{bmatrix}^{\mathrm{T}}$, when eigenvalue $\lambda = 8$, eigenvector is $\begin{bmatrix} 1 & 2 \end{bmatrix}^{\mathrm{T}}$.

- Provide one example for each of the following cases.
  As for $(A+B)^2 \neq A^2 + 2AB + B^2$. Suppose $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Since $A \cdot B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $B \cdot A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.
  So $(A+B)^2 = A^2 + AB + BA + B^2 \neq A^2 + 2AB + B^2$.
  As for $AB = 0, A \neq 0, B \neq 0$, Suppose $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

- Show that $\boldsymbol{A}$ is orthogonal.
  Given $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u} = 1$ and $\boldsymbol{A} = \boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$.

$$
\begin{aligned}
\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} &= (\boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})^{\mathrm{T}}(\boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}) \\
&= (\boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})(\boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}) \\
&= \boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} + 4\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} \\
&= \boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} + 4\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} \\
&= \boldsymbol{I}
\end{aligned}
$$

  So $\boldsymbol{A}$ is orthogonal.

1 Locally weighted linear regression(20 points)