COMP 540 Assignment #1

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0 Background refresher(30 points)

- Plot the categorical distribution.

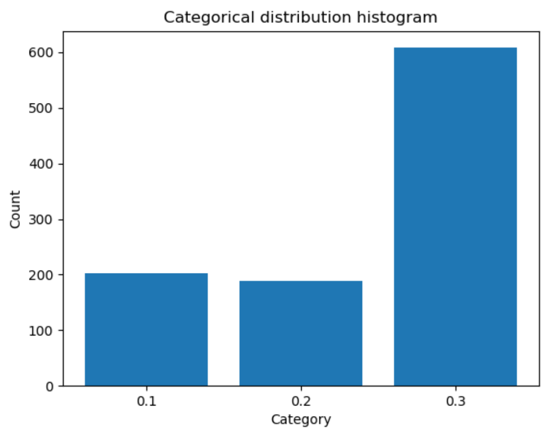


Figure 1: Categorical distribution

- Plot the Univariate normal distribution with mean of and standard deviation of 1.

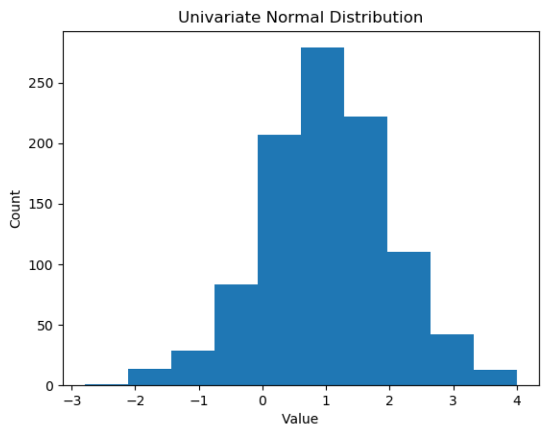


Figure 2: Univariate Normal Distribution

- Produce a scatter plot of the samples for a 2-D Gaussian.

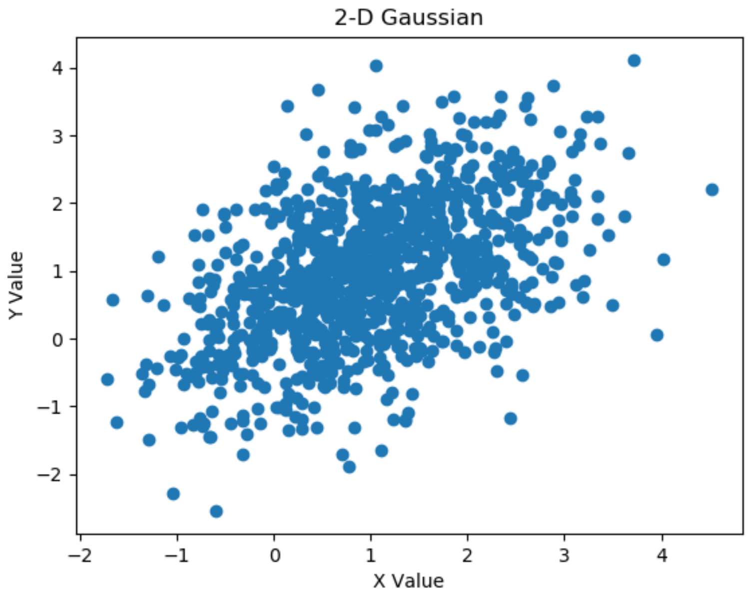


Figure 3: Univariate Normal Distribution

- Test mixture sampling code Code can be seen in sampler.py. Mixture Gaussian plot is shown below

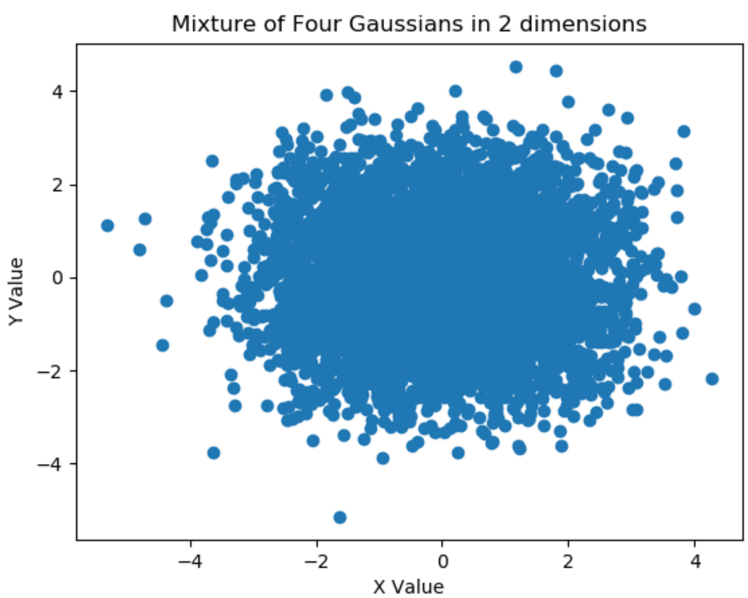


Figure 4: Univariate Normal Distribution

- Prove that the sum of two independent Poisson random variables is also a Poisson random variable. Suppose $X \sim \mathcal{P}(\lambda)$ and $Y \sim \mathcal{P}(\mu)$. Now Prove that $X + Y \sim \mathcal{P}(\lambda + \mu)$.

$$
\begin{aligned}
P(X+Y=k) &= \sum_{i=0}^{k} P(X+Y=k, X=i) \\
&= \sum_{i=0}^{k} P(Y=k-i, X=i) \\
&= \sum_{i=0}^{k} P(Y=k-i)P(X=i) \\
&= \sum_{i=0}^{k} e^{-\mu} \frac{\mu^{k-i}}{(k-i)!} e^{-\lambda} \frac{\lambda^i}{i!} \\
&= e^{-(\mu+\lambda)} \frac{1}{k!} \sum_{i=0}^{k} \frac{k!}{i!(k-i)!} \mu^{k-i} \lambda^i \\
&= e^{-(\mu+\lambda)} \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} \mu^{k-i} \lambda^i \\
&= \frac{(\mu+\lambda)^k}{k!} \cdot e^{-(\mu+\lambda)}
\end{aligned}
$$

So $X + Y \sim \mathcal{P}(\lambda + \mu)$.

- Find α, μ_1 and σ_1.
  We have X_0 and X_1 be continuous random variables. If

$$
\begin{aligned}
p(X_0 = x_0) &= \alpha_0 e^{-\frac{(x_0-\mu_0)^2}{2\sigma_0^2}} \\
P(X_1 = x_1 | X_0 = x_0) &= \alpha_1 e^{-\frac{(x_1-x_0)^2}{2\sigma^2}}
\end{aligned}
$$

$$
\begin{aligned}
p(X_1 = x_1) &= \int P(X_1 = x_1 | X_0 = x_0) \cdot p(X_0 = x_0) dx_0 \\
&= \alpha_0 \alpha_1 \int e^{-\frac{\sigma^2(x_0-\mu_0)^2 + \sigma_0^2(x_1-x_0)^2}{2\sigma_0^2\sigma^2}} dx_0 \\
&= \alpha_0 \alpha_1 \int e^{-\frac{(\sigma^2+\sigma_0^2)x_0^2 - 2(\sigma^2\mu_0 + \sigma_0^2 x_1)x_0 + \sigma^2\mu_0^2 + \sigma_0^2 x_1^2}{2\sigma_0^2\sigma^2}} dx_0 \\
&= \alpha_0 \alpha_1 \int e^{-\frac{1}{2\sigma_0^2\sigma^2}[(\sqrt{\sigma^2+\sigma_0^2}x_0 - \frac{-\sigma^2\mu_0+\sigma_0^2 x_1}{\sqrt{\sigma^2+\sigma_0^2}})^2 + \sigma^2\mu_0^2 + \sigma_0^2 x_1^2 - \frac{\sigma^2\mu_0^2+\sigma_0^2 x_1^2}{\sigma^2+\sigma_0^2}]} dx_0
\end{aligned}
$$

Since

$$
\int \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1
$$

$$
\begin{aligned}
p(X_1 = x_1) &= \frac{\alpha\alpha_0\sqrt{2\pi}\sigma\sigma_0}{\sqrt{\sigma^2+\sigma_0^2}} e^{-\frac{1}{2\sigma_0^2\sigma^2}(\sigma^2\mu_0^2+\sigma_0^2x_1^2-\frac{\sigma^2\mu_0^2+\sigma_0^2x_1^2}{\sigma^2+\sigma_0^2})} \\
&= \alpha e^{-\frac{(x_1-\mu_0)^2}{2(\sigma^2+\sigma_0^2)}}
\end{aligned}
$$

Thus we can solve that:

$$
\begin{aligned}
\alpha &= \frac{\alpha\alpha_0\sqrt{2\pi}\sigma\sigma_0}{\sqrt{\sigma^2+\sigma_0^2}} \\
\mu_1 &= \mu_0 \\
\sigma_1 &= \sqrt{\sigma^2+\sigma_0^2}
\end{aligned}
$$

- Consider the vectors $\boldsymbol{u} = \begin{bmatrix}1 & 2\end{bmatrix}^{\mathrm{T}}$ and $\boldsymbol{v} = \begin{bmatrix}2 & 3\end{bmatrix}^{\mathrm{T}}$. Define the matrix $\boldsymbol{M} = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$. Compute the eigenvalues and eigenvectors of $\boldsymbol{M}$.

$$
\begin{aligned}
\boldsymbol{M} &= \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} \\
&= \begin{bmatrix}1 & 2\end{bmatrix}^{\mathrm{T}} \cdot \begin{bmatrix}2 & 3\end{bmatrix} \\
&= \begin{bmatrix}2 & 3 \\ 4 & 6\end{bmatrix}
\end{aligned}
$$

$$
\left|\lambda\boldsymbol{I} - \boldsymbol{M}\right| = \begin{vmatrix}\lambda - 2 & -3 \\ -4 & \lambda - 6\end{vmatrix} = 0
$$

Then we can solve it as

$$
\begin{aligned}
(\lambda-2)(\lambda-6) - 12 &= 0 \\
\lambda^2 - 8\lambda &= 0 \\
\lambda(\lambda-8) &= 0 \\
\lambda_1 = 0, \lambda_2 &= 8
\end{aligned}
$$

Let $\lambda = 0$:

$$
\begin{aligned}
(\lambda\boldsymbol{I} - \boldsymbol{M}) \cdot \begin{bmatrix}x_1 & x_2\end{bmatrix}^{\mathrm{T}} &= 0 \\
\begin{bmatrix}-2 & -3 \\ -4 & -6\end{bmatrix} \cdot \begin{bmatrix}x_1 \\ x_2\end{bmatrix} &= \begin{bmatrix}0 \\ 0\end{bmatrix} \\
-2x_1 - 3x_2 &= 0
\end{aligned}
$$

Let $x_1 = 3$, Then $x_2 = -2$. So eigenvector is $\begin{bmatrix}3 & -2\end{bmatrix}^{\mathrm{T}}$.
Let $\lambda = 8$:

$$
\begin{aligned}
(\lambda \boldsymbol{I}-\boldsymbol{M}) \cdot \begin{bmatrix} x_1 & x_2 \end{bmatrix}^{\mathrm{T}} &= 0 \\
\begin{bmatrix} 6 & -3 \\ -4 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
6x_1 - 3x_2 &= 0 \\
-4x_1 + 2x_2 &= 0
\end{aligned}
$$

Let $x_1 = 1$, Then $x_2 = 2$. So eigenvector is $\begin{bmatrix} 1 & 2 \end{bmatrix}^{\mathrm{T}}$.
Thus, when eigenvalue $\lambda = 0$, eigenvector is $\begin{bmatrix} 3 & -2 \end{bmatrix}^{\mathrm{T}}$, when eigenvalue $\lambda = 8$, eigenvector is $\begin{bmatrix} 1 & 2 \end{bmatrix}^{\mathrm{T}}$.

- Provide one example for each of the following cases.
  As for $(A+B)^2 \neq A^2 + 2AB + B^2$. Suppose $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Since $A \cdot B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $B \cdot A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.
  So $(A+B)^2 = A^2 + AB + BA + B^2 \neq A^2 + 2AB + B^2$.
  As for $AB = 0, A \neq 0, B \neq 0$, Suppose $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

- Show that $\boldsymbol{A}$ is orthogonal.
  Given $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u} = 1$ and $\boldsymbol{A} = \boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$.

$$
\begin{aligned}
\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} &= (\boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})^{\mathrm{T}}(\boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}) \\
&= (\boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})(\boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}) \\
&= \boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} + 4\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} \\
&= \boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} + 4\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} \\
&= \boldsymbol{I}
\end{aligned}
$$

  So $\boldsymbol{A}$ is orthogonal.

- Prove the following assertions.
  As for $f(x) = x^3$ for $x \geq 0$.

$$
\begin{aligned}
f(x) &= x^3 \\
f''(x) &= 6x
\end{aligned}
$$

  Since $x \geq 0$, $f''(x) = 6x \geq 0$. Thus $f(x) = x^3$ is convex for $x \geq 0$.
  As for $f(x_1, x_2) = max(x_1, x_2)$. Let $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{y} = (y_1, y_2)$ and $\lambda \in [0, 1]$.

$$
\begin{aligned}
f(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}) &= f(\lambda x_1 + (1-\lambda)y_1, \lambda x_2 + (1-\lambda)y_2) \\
&= max(\lambda x_1 + (1-\lambda)y_1, \lambda x_2 + (1-\lambda)y_2) \\
&\leq max(\lambda x_1, \lambda x_2) + max((1-\lambda)y_1, (1-\lambda)y_2) \\
&= \lambda max(x_1, x_2) + (1-\lambda) max(y_1, y_2) \\
&= \lambda f(\boldsymbol{x}) + (1-\lambda) f(\boldsymbol{y})
\end{aligned}
$$

  So $f(x_1, x_2) = max(x_1, x_2)$ is convex on R^2.
  As for function $f+g$. If univariate functions f and g are convex on S, then $f''(x) \geq 0$ and $g''(x) \geq 0$.

So $(f+g)^{''}(x) = f^{''}(x) + g^{''}(x) \geq 0$. Thus if univariate functions f and g are convex on S, then $f+g$ is convex on S.
As for fg. If univariate functions f and g are convex and non-nigegative on S.

$$\begin{aligned}(fg)^{''}(x) &= (f^{'}g + fg^{'})^{'}(x) \\ &= (f^{''}g + f^{'}g^{'} + f^{'}g^{'} + fg^{''})(x) \\ &= (f^{''}g + 2f^{'}g^{'} + fg^{''})(x)\end{aligned}$$

Since f and g have their minimum within S at the same point. Before the minimum point, both f and g are decreasing. After the minimum point, both f and g are increasing. So $f^{'}g^{'} \geq 0$, $f^{''} \geq 0$, $g^{''} \geq 0$, $f \geq 0$ and $g \geq 0$. Thus $(fg)^{''}(x) = (f^{''}g + 2f^{'}g^{'} + fg^{''})(x) \geq 0$. Then fg is convex on S.

- Find the highest entropy of categorical distribution.
  The entropy of a categorical distribution on K values is defined as

$$H(p) = -\sum_{i=1}^{K} p_i log(p_i)$$

The probability and constrains are defined below:

$$P(X = x_i) = p_i \quad for \quad i = 1, 2, ..., K$$

$$s.p. \begin{cases} & \sum_{i=1}^{K} p_i = 1 \\ p_i \geq 0 \quad for \quad i = 1, 2, ..., K \end{cases}$$

Constrain function is

$$\varphi(p_i) = \sum_{i=1}^{K} p_i - 1 = 0$$

Define Lagrange multipliers:

$$\begin{aligned}\mathcal{L} &= H(p) + \lambda\varphi(p_i) \\ &= -\sum_{i=1}^{K} p_i log(p_i) + \lambda(\sum_{i=1}^{K} p_i - 1)\end{aligned}$$

To find the highest entropy, we should find the point where derivative is 0.

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial p_i} &= -log(p_i) - 1 + \lambda = 0 \\ p^* &= e^{\lambda - 1}\end{aligned}$$

Thus when all $p_i, i = 1, 2, ..., K$ are equally equal to $e^{\lambda-1}$, the categorical distribution has the highest entropy.

1 Locally weighted linear regression(20 points)

- Find an appropriate diagonal matrix W.

$$J(\theta) = (X\theta - y)^T W (X\theta - y)$$

Let W be

$$W = \begin{bmatrix} \frac{1}{2}w^{(1)} & & & 0 \\ & \frac{1}{2}w^{(2)} & & \\ & 0 & \ddots & \\ & & & \frac{1}{2}w^{(i)} \end{bmatrix}$$

X is the $m \times d$ input matrix and y is a $m \times 1$ vector.

$$X = \begin{bmatrix} x_1^{(1)} & x_2^{(1)} & x_3^{(1)} & \cdots & x_d^{(1)} \\ x_1^{(2)} & x_2^{(2)} & x_3^{(2)} & \cdots & x_d^{(2)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ x_1^{(m)} & x_2^{(m)} & x_3^{(m)} & \cdots & x_d^{(m)} \end{bmatrix}$$

$$y = \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(m)} \end{bmatrix}$$

$$\theta = \begin{bmatrix} \theta_{(1)} \\ \theta_{(2)} \\ \vdots \\ \theta_{(d)} \end{bmatrix}$$

$$X\theta - y = \begin{bmatrix} (\theta_1 x_1^{(1)} + \theta_2 x_2^{(1)} + \cdots + \theta_d x_d^{(1)}) - y^{(1)} \\ (\theta_1 x_1^{(2)} + \theta_2 x_2^{(2)} + \cdots + \theta_d x_d^{(2)}) - y^{(2)} \\ \vdots \\ (\theta_1 x_1^{(m)} + \theta_2 x_2^{(m)} + \cdots + \theta_d x_d^{(m)}) - y^{(m)} \end{bmatrix}$$

$$W(X\theta - y) = \begin{bmatrix} \frac{1}{2}w^{(1)} \times ((\theta_1 x_1^{(1)} + \theta_2 x_2^{(1)} + \cdots + \theta_d x_d^{(1)}) - y^{(1)}) \\ \frac{1}{2}w^{(2)} \times ((\theta_1 x_1^{(2)} + \theta_2 x_2^{(2)} + \cdots + \theta_d x_d^{(2)}) - y^{(2)}) \\ \vdots \\ \frac{1}{2}w^{(m)} \times ((\theta_1 x_1^{(m)} + \theta_2 x_2^{(m)} + \cdots + \theta_d x_d^{(m)}) - y^{(m)}) \end{bmatrix}$$

$$\begin{aligned} (X\theta - y)^T W (X\theta - y) = & \frac{1}{2}w^{(1)} \times ((\theta_1 x_1^{(1)} + \theta_2 x_2^{(1)} + \cdots + \theta_d x_d^{(1)}) - y^{(1)})^2 \\ & + \frac{1}{2}w^{(2)} \times ((\theta_1 x_1^{(2)} + \theta_2 x_2^{(2)} + \cdots + \theta_d x_d^{(2)}) - y^{(2)})^2 + \cdots \\ & + \frac{1}{2}w^{(m)} \times ((\theta_1 x_1^{(m)} + \theta_2 x_2^{(m)} + \cdots + \theta_d x_d^{(m)}) - y^{(m)})^2 \end{aligned}$$

So $J(\theta) = (X\theta - y)^T W (X\theta - y)$ can be written in the from $J(\theta) = (X\theta - y)^T W (X\theta - y)$ when choosing W as above.

-

2 Properties of the linear regression estimator(10 points)

3 Implementing linear regression and regularized linear regression(90 points)

3.1 Implementing linear regression(45 points)

- A1: Computing the cost function $J(\theta)$
  See the implementation for the *loss* function in the file *linear_regressor.py*.

- A2: Implementing gradient descent
  See the implementation for the *train* function in the file *linear_regressor.py*.

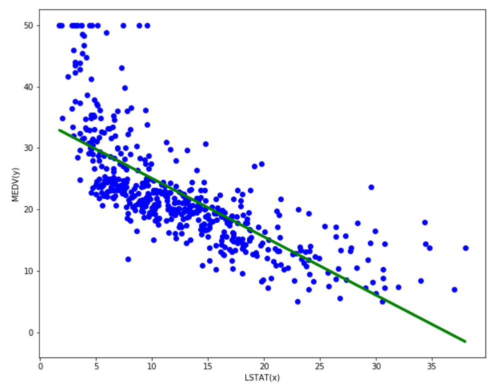


Figure 5: Fitting a linear model

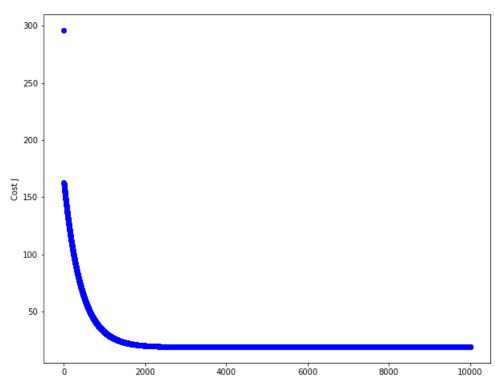


Figure 6: Convergence of gradient descent

- A3: Predicting on unseen data
  For lower status percentage = 5, we predict a median home value of 298034.49
  For lower status percentage = 50, we predict a median home value of -129482.13

- B1: Feature normalization

- B2: Loss function and gradient descent

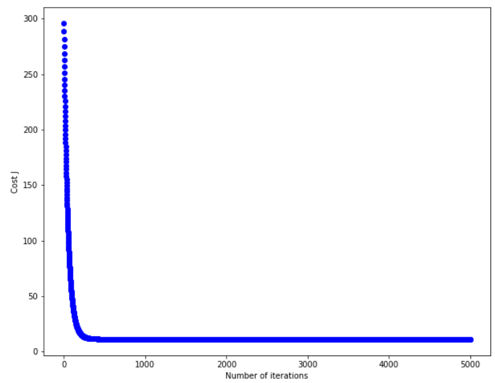
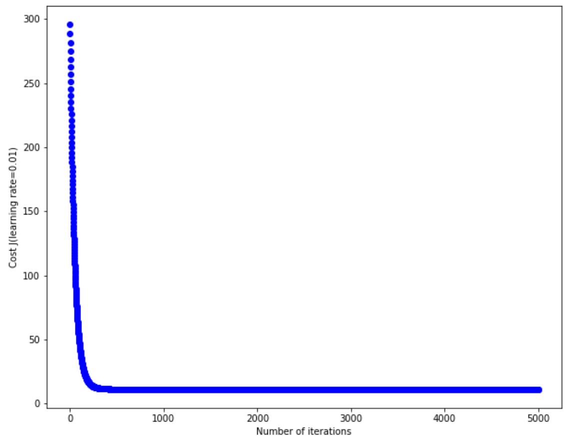


Figure 7: Convergence of gradient descent with multiple variables

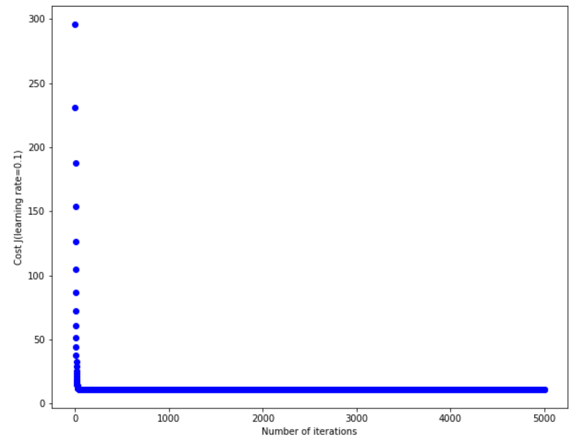
- B3: Making predictions on unseen data
  For average home in Boston suburbs, we predict a median home value of 230467.11

- B4: Normal equations
  Theta computed by direct solution is: $[3.64594884e+01-1.08011358e-014.64204584e-022.05586264e-022.68673382e+00-1.77666112e+013.80986521e+006.92224640e-04-1.47556685e+003.06049479e-01-1.23345939e-02-9.52747232e-019.31168327e-03-5.24758378e-01]$
  For average home in Boston suburbs, we predict a median home value of 230406.54.

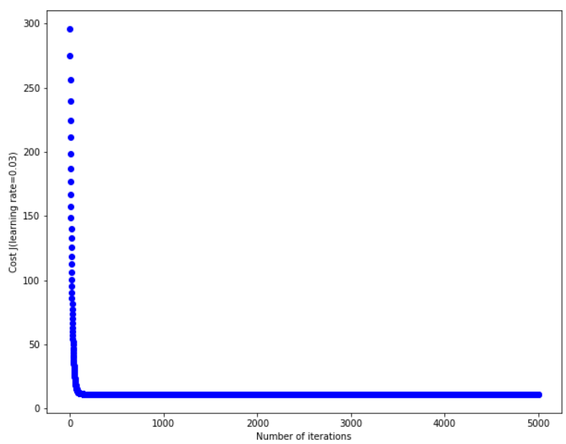
- B5: Exploring convergence of gradient descent



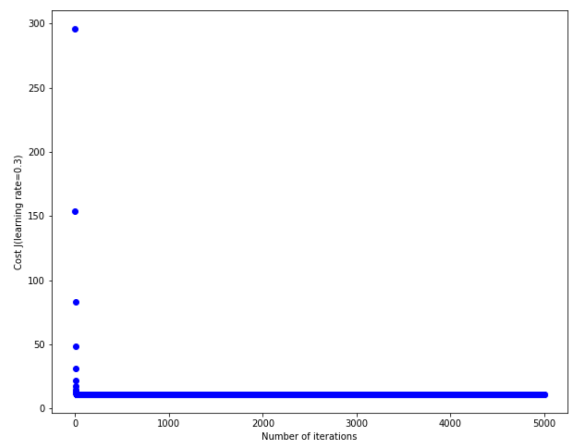
((a)) learning rate=0.01



((b)) learning rate=0.03



((c)) learning rate=0.1

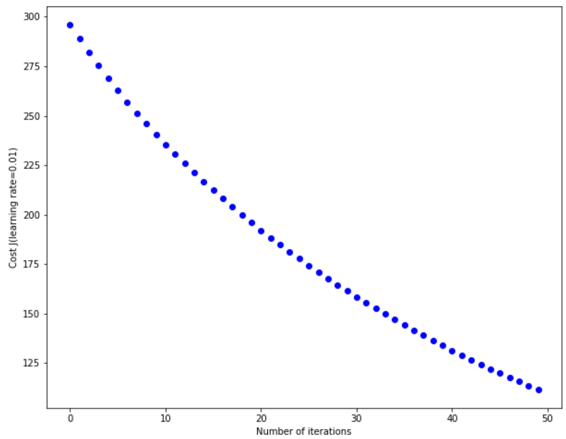


((d)) learning rate=0.3

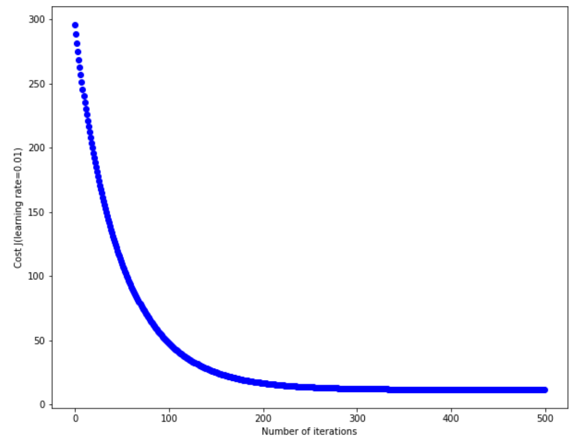
Figure 8: Convergence of gradient descent with multiple variables according to different learning rates

According to the above figures, when learning $rate = 0.01$ is a good choice.

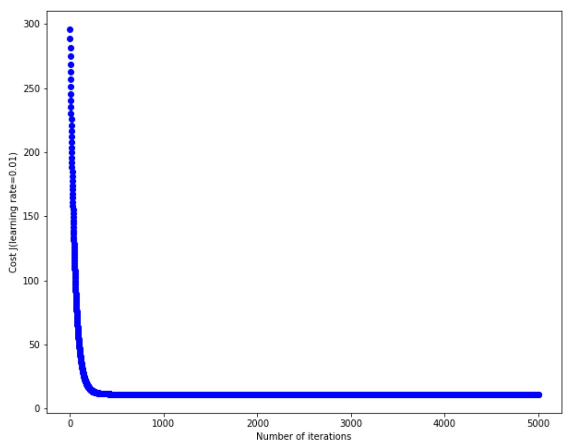
When learning rate = 0.01, chosing different iteration times.



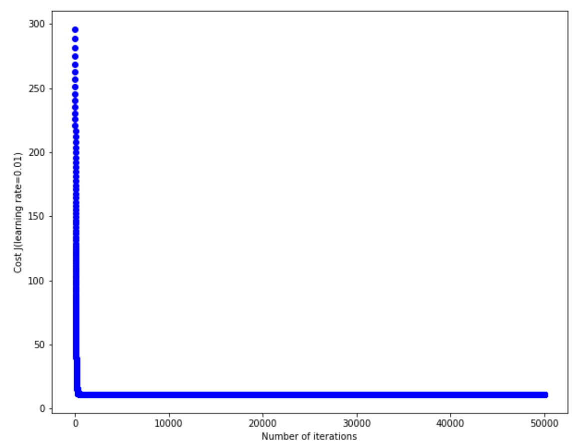
((a)) iteration time = 50



((b)) iteration time = 500



((c)) iteration time = 5000



((d)) iteration time = 50000

Figure 9: Convergence of gradient descent with multiple variables according to different iteration times

According to the above figures, iteration time is 500 is a good choice.