COMP 540 Assignment #1

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0 Background refresher(30 points)

- Plot the categorical distribution.

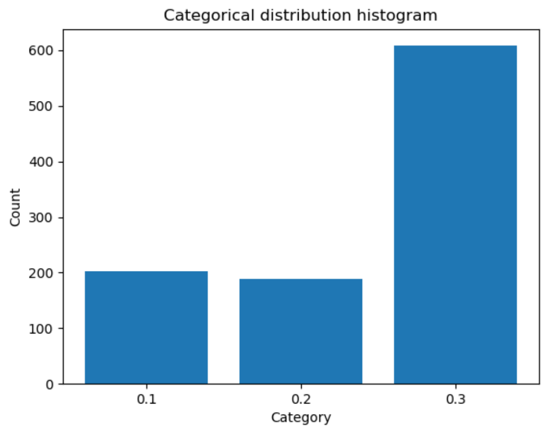


Figure 1: Categorical distribution

- Plot the Univariate normal distribution with mean of and standard deviation of 1.

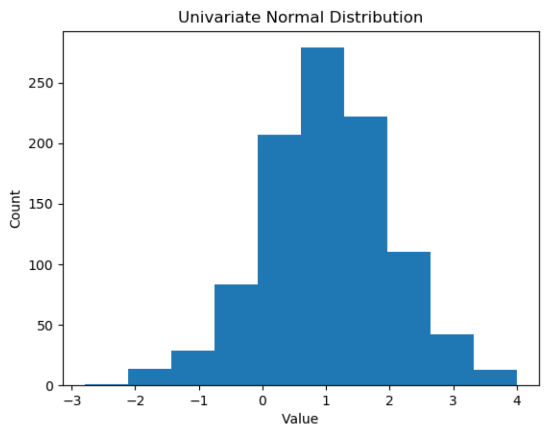


Figure 2: Univariate Normal Distribution

- Produce a scatter plot of the samples for a 2-D Gaussian.

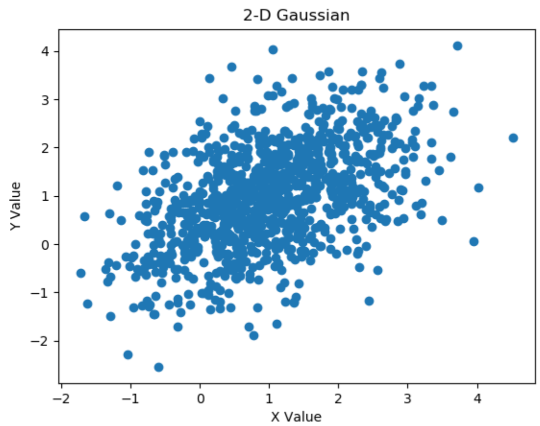


Figure 3: Univariate Normal Distribution

- Test mixture sampling code Code can be seen in sampler.py. Mixture Gaussian plot is shown below

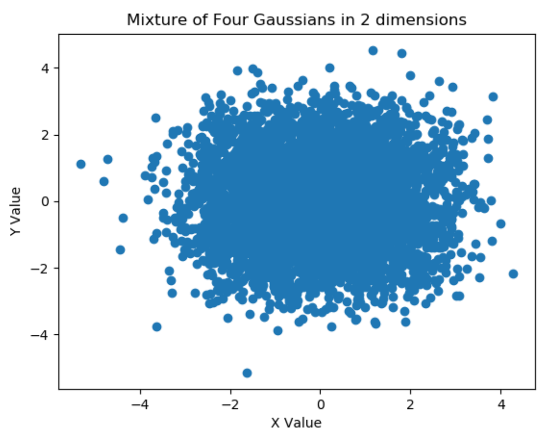


Figure 4: Univariate Normal Distribution

- Prove that the sum of two independent Poisson random variables is also a Poisson random variable. Suppose $X \sim \mathcal{P}(\lambda)$ and $Y \sim \mathcal{P}(\mu)$. Now Prove that $X + Y \sim \mathcal{P}(\lambda + \mu)$.

$$
\begin{aligned}
P(X+Y=k) &= \sum_{i=0}^{k} P(X+Y=k, X=i) \\
&= \sum_{i=0}^{k} P(Y=k-i, X=i) \\
&= \sum_{i=0}^{k} P(Y=k-i)P(X=i) \\
&= \sum_{i=0}^{k} e^{-\mu} \frac{\mu^{k-i}}{(k-i)!} e^{-\lambda} \frac{\lambda^i}{i!} \\
&= e^{-(\mu+\lambda)} \frac{1}{k!} \sum_{i=0}^{k} \frac{k!}{i!(k-i)!} \mu^{k-i} \lambda^i \\
&= e^{-(\mu+\lambda)} \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} \mu^{k-i} \lambda^i \\
&= \frac{(\mu+\lambda)^k}{k!} \cdot e^{-(\mu+\lambda)}
\end{aligned}
$$

So $X+Y \sim \mathcal{P}(\lambda+\mu)$.

- Find α, μ_1 and σ_1.
  We have X_0 and X_1 be continuous random variables. If

$$
\begin{aligned}
p(X_0 = x_0) &= \alpha_0 e^{-\frac{(x_0-\mu_0)^2}{2\sigma_0^2}} \\
P(X_1 = x_1 | X_0 = x_0) &= \alpha_1 e^{-\frac{(x_1-x_0)^2}{2\sigma^2}}
\end{aligned}
$$

$$
\begin{aligned}
p(X_1 = x_1) &= \int P(X_1 = x_1 | X_0 = x_0) \cdot p(X_0 = x_0) dx_0 \\
&= \alpha_0 \alpha_1 \int e^{-\frac{\sigma^2 (x_0-\mu_0)^2 + \sigma_0^2 (x_1-x_0)^2}{2\sigma_0^2\sigma^2}} dx_0 \\
&= \alpha_0 \alpha_1 \int e^{-\frac{(\sigma^2+\sigma_0^2)x_0^2 - 2(\sigma^2\mu_0 + \sigma_0^2 x_1)x_0 + \sigma^2\mu_0^2 + \sigma_0^2 x_1^2}{2\sigma_0^2\sigma^2}} dx_0 \\
&= \alpha_0 \alpha_1 \int e^{-\frac{1}{2\sigma_0^2\sigma^2}[(\sqrt{\sigma^2+\sigma_0^2}x_0 - \frac{-\sigma^2\mu_0+\sigma_0^2 x_1}{\sqrt{\sigma^2+\sigma_0^2}})^2 + \sigma^2\mu_0^2 + \sigma_0^2 x_1^2 - \frac{\sigma^2\mu_0^2+\sigma_0^2 x_1^2}{\sigma^2+\sigma_0^2}]} dx_0
\end{aligned}
$$

Since

$$
\int \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1
$$

$$
\begin{aligned}
p(X_1 = x_1) &= \frac{\alpha\alpha_0\sqrt{2\pi}\sigma\sigma_0}{\sqrt{\sigma^2+\sigma_0^2}} e^{-\frac{1}{2\sigma_0^2\sigma^2}(\sigma^2\mu_0^2 + \sigma_0^2 x_1^2 - \frac{\sigma^2\mu_0^2+\sigma_0^2 x_1^2}{\sigma^2+\sigma_0^2})} \\
&= \alpha e^{-\frac{(x_1-\mu_0)^2}{2(\sigma^2+\sigma_0^2)}}
\end{aligned}
$$

Thus we can solve that:

$$
\begin{aligned}
\alpha &= \frac{\alpha\alpha_0\sqrt{2\pi}\sigma\sigma_0}{\sqrt{\sigma^2+\sigma_0^2}} \\
\mu_1 &= \mu_0 \\
\sigma_1 &= \sqrt{\sigma^2+\sigma_0^2}
\end{aligned}
$$

- Show that if $P(A|B,C) > P(A|B)$ then $P(A|B,C^C) < P(A|B)$

$$
P(A|B,C) = \frac{P(A\bigcap B\bigcap C)}{P(B\bigcap C)}
$$

$$
\begin{aligned}
\frac{P(A\bigcap B\bigcap C)}{P(B\bigcap C)} &> \frac{P(A\bigcap B)}{P(B)} \\
\frac{P(C|A\bigcap B)}{P(C|B)} &> 1 \\
P(C|A\bigcap B) &> P(C|B) \\
1-P(C|A\bigcap B) &< 1-P(C|B) \\
P(C^C|A\bigcap B) &< P(C^C|B) \\
\frac{P(C^C|A\bigcap B)}{P(C^C|B)} &< 1
\end{aligned}
$$

$$
\begin{aligned}
P(A|B,C^C) &= \frac{P(A\bigcap B\bigcap C^C)}{P(B\bigcap C^C)} \\
&= \frac{P(C^C|A\bigcap B)P(A\bigcap B)}{P(C^C|B)P(B)} \\
&= \frac{P(C^C|A\bigcap B)}{P(C^C|B)P(B)}P(A|B) \\
&< P(A|B)
\end{aligned}
$$

Thus, if $P(A|B,C) > P(A|B)$ then $P(A|B,C^C) < P(A|B)$.

- Consider the vectors $\boldsymbol{u} = \begin{bmatrix}1 & 2\end{bmatrix}^{\mathrm{T}}$ and $\boldsymbol{v} = \begin{bmatrix}2 & 3\end{bmatrix}^{\mathrm{T}}$. Define the matrix $\boldsymbol{M} = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$. Compute the eigenvalues and eigenvectors of $\boldsymbol{M}$.

$$
\begin{aligned}
\boldsymbol{M} &= \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} \\
&= \begin{bmatrix}1 & 2\end{bmatrix}^{\mathrm{T}} \cdot \begin{bmatrix}2 & 3\end{bmatrix} \\
&= \begin{bmatrix}2 & 3 \\ 4 & 6\end{bmatrix}
\end{aligned}
$$

$$
\left|\lambda\boldsymbol{I}-\boldsymbol{M}\right| = \begin{vmatrix}\lambda-2 & -3 \\ -4 & \lambda-6\end{vmatrix} = 0
$$

Then we can solve it as

$$
\begin{aligned}
(\lambda-2)(\lambda-6)-12&=0\\
\lambda^2-8\lambda&=0\\
\lambda(\lambda-8)&=0\\
\lambda_1=0,\lambda_2&=8
\end{aligned}
$$

Let $\lambda = 0$:

$$
\begin{aligned}
(\lambda\boldsymbol{I}-\boldsymbol{M})\cdot\begin{bmatrix}x_1 & x_2\end{bmatrix}^{\mathrm{T}}&=0\\
\begin{bmatrix}-2 & -3\\ -4 & -6\end{bmatrix}\cdot\begin{bmatrix}x_1\\ x_2\end{bmatrix}&=\begin{bmatrix}0\\ 0\end{bmatrix}\\
-2x_1-3x_2&=0
\end{aligned}
$$

Let $x_1 = 3$, Then $x_2 = -2$. So eigenvector is $\begin{bmatrix}3 & -2\end{bmatrix}^{\mathrm{T}}$.
Let $\lambda = 8$:

$$
\begin{aligned}
(\lambda\boldsymbol{I}-\boldsymbol{M})\cdot\begin{bmatrix}x_1 & x_2\end{bmatrix}^{\mathrm{T}}&=0\\
\begin{bmatrix}6 & -3\\ -4 & 2\end{bmatrix}\cdot\begin{bmatrix}x_1\\ x_2\end{bmatrix}&=\begin{bmatrix}0\\ 0\end{bmatrix}\\
6x_1-3x_2&=0\\
-4x_1+2x_2&=0
\end{aligned}
$$

Let $x_1 = 1$, Then $x_2 = 2$. So eigenvector is $\begin{bmatrix}1 & 2\end{bmatrix}^{\mathrm{T}}$.
Thus, when eigenvalue $\lambda = 0$, eigenvector is $\begin{bmatrix}3 & -2\end{bmatrix}^{\mathrm{T}}$, when eigenvalue $\lambda = 8$, eigenvector is $\begin{bmatrix}1 & 2\end{bmatrix}^{\mathrm{T}}$.

- Show that if A is positive semi-definite, then all eigenvalues of A are non-negative.
  Definition of an eigenvalue and eigenvector is:

  $$A\boldsymbol{v} = \lambda\boldsymbol{v}$$

  If A is positive semi-definite, $\boldsymbol{x}^T A\boldsymbol{x} \geq 0$ for all $\boldsymbol{x}$.

  $$\boldsymbol{v}^T A\boldsymbol{v} = \boldsymbol{v}^T\boldsymbol{v}\lambda$$

  Since $\boldsymbol{v}^T\boldsymbol{v}$ is necessarily a positive number, in order for $\boldsymbol{v}^T A\boldsymbol{v}$ to be greater than or equal to 0, λ must be $\lambda \geq 0$.

- Provide one example for each of the following cases.
  As for $(A+B)^2 \neq A^2 + 2AB + B^2$. Suppose $A = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$ and $B = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$. Since $A \cdot B = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ and $B \cdot A = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$.
  So $(A+B)^2 = A^2 + AB + BA + B^2 \neq A^2 + 2AB + B^2$.
  As for $AB = 0, A \neq 0, B \neq 0$, Suppose $A = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$ and $B = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$.

- Show that $\boldsymbol{A}$ is orthogonal.
  Given $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u} = 1$ and $\boldsymbol{A} = \boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$.

$$\begin{aligned}
\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} &= (\boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})^{\mathrm{T}}(\boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}) \\
&= (\boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})(\boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}) \\
&= \boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} + 4\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} \\
&= \boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} + 4\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} \\
&= \boldsymbol{I}
\end{aligned}$$

  So $\boldsymbol{A}$ is orthogonal.

- Prove the following assertions.
  –As for $f(x) = x^3$ for $x \geq 0$.

$$\begin{aligned}
f(x) &= x^3 \\
f^{''}(x) &= 6x
\end{aligned}$$

  Since $x \geq 0$, $f^{''}(x) = 6x \geq 0$. Thus $f(x) = x^3$ is convex for $x \geq 0$.
  –As for $f(x_1, x_2) = max(x_1, x_2)$. Let $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{y} = (y_1, y_2)$ and $\lambda \in [0, 1]$.

$$\begin{aligned}
f(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}) &= f(\lambda x_1 + (1-\lambda)y_1, \lambda x_2 + (1-\lambda)y_2) \\
&= max(\lambda x_1 + (1-\lambda)y_1, \lambda x_2 + (1-\lambda)y_2) \\
&\leq max(\lambda x_1, \lambda x_2) + max((1-\lambda)y_1, (1-\lambda)y_2) \\
&= \lambda max(x_1, x_2) + (1-\lambda)max(y_1, y_2) \\
&= \lambda f(\boldsymbol{x}) + (1-\lambda)f(\boldsymbol{y})
\end{aligned}$$

  So $f(x_1, x_2) = max(x_1, x_2)$ is convex on R^2.
  –As for function $f + g$. If univariate functions f and g are convex on S, then $f^{''}(x) \geq 0$ and $g^{''}(x) \geq 0$. So $(f+g)^{''}(x) = f^{''}(x) + g^{''}(x) \geq 0$. Thus if univariate functions f and g are convex on S, then $f + g$ is convex on S.
  –As for fg. If univariate functions f and g are convex and non-negegative on S.

$$\begin{aligned}
(fg)^{''}(x) &= (f^{'}g + fg^{'})^{'}(x) \\
&= (f^{''}g + f^{'}g^{'} + f^{'}g^{'} + fg^{''})(x) \\
&= (f^{''}g + 2f^{'}g^{'} + fg^{''})(x)
\end{aligned}$$

  Since f and g have their minimum within S at the same point. Before the minimum point, both f and g are decreasing. After the minimum point, both f and g are increasing. So $f^{'}g^{'} \geq 0$, $f^{''} \geq 0$, $g^{''} \geq 0$, $f \geq 0$ and $g \geq 0$. Thus $(fg)^{''}(x) = (f^{''}g + 2f^{'}g^{'} + fg^{''})(x) \geq 0$. Then fg is convex on S.

- Find the highest entropy of categorical distribution.
  The entropy of a categorical distribution on K values is defined as

$$H(p) = -\sum_{i=1}^{K} p_i log(p_i)$$

  The probability and constrains are defined below:

$$P(X = x_i) = p_i \quad for \quad i = 1, 2, ..., K$$

$$s.p. \begin{cases} \sum_{i=1}^{K} p_i = 1 \\ p_i \geq 0 \quad for \quad i = 1, 2, ..., K \end{cases}$$

Constrain function is

$$\varphi(p_i) = \sum_{i=1}^{K} p_i - 1 = 0$$

Define Lagrange multipliers:

$$\begin{aligned} \mathcal{L} &= H(p) + \lambda\varphi(p_i) \\ &= -\sum_{i=1}^{K} p_i log(p_i) + \lambda(\sum_{i=1}^{K} p_i - 1) \end{aligned}$$

To find the highest entropy, we should find the point where derivative is 0.

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial p_i} &= -log(p_i) - 1 + \lambda = 0 \\ p^* &= e^{\lambda - 1} \end{aligned}$$

Thus when all $p_i, i = 1, 2, ..., K$ are equally equal to $e^{\lambda-1}$, the categorical distribution has the highest entropy.

1 Locally weighted linear regression(20 points)

- Find an appropriate diagonal matrix W.

$$J(\theta) = (X\theta - y)^T W (X\theta - y)$$

Let W be

$$W = \begin{bmatrix} \frac{1}{2}w^{(1)} & & & \\ & \frac{1}{2}w^{(2)} & & 0 \\ & 0 & \ddots & \\ & & & \frac{1}{2}w^{(i)} \end{bmatrix}$$

X is the $m \times d$ input matrix and y is a $m \times 1$ vector.

$$X = \begin{bmatrix} x_1^{(1)} & x_2^{(1)} & x_3^{(1)} & \cdots & x_d^{(1)} \\ x_1^{(2)} & x_2^{(2)} & x_3^{(2)} & \cdots & x_d^{(2)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ x_1^{(m)} & x_2^{(m)} & x_3^{(m)} & \cdots & x_d^{(m)} \end{bmatrix}$$

$$y = \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(m)} \end{bmatrix}$$

$$\theta = \begin{bmatrix} \theta_{(1)} \\ \theta_{(2)} \\ \vdots \\ \theta_{(d)} \end{bmatrix}$$

$$X\theta - y = \begin{bmatrix} (\theta_1 x_1^{(1)} + \theta_2 x_2^{(1)} + \cdots + \theta_d x_d^{(1)}) - y^{(1)} \\ (\theta_1 x_1^{(2)} + \theta_2 x_2^{(2)} + \cdots + \theta_d x_d^{(2)}) - y^{(2)} \\ \vdots \\ (\theta_1 x_1^{(m)} + \theta_2 x_2^{(m)} + \cdots + \theta_d x_d^{(m)}) - y^{(m)} \end{bmatrix}$$

$$W(X\theta - y) = \begin{bmatrix} \frac{1}{2} w^{(1)} \times ((\theta_1 x_1^{(1)} + \theta_2 x_2^{(1)} + \cdots + \theta_d x_d^{(1)}) - y^{(1)}) \\ \frac{1}{2} w^{(2)} \times ((\theta_1 x_1^{(2)} + \theta_2 x_2^{(2)} + \cdots + \theta_d x_d^{(2)}) - y^{(2)}) \\ \vdots \\ \frac{1}{2} w^{(m)} \times ((\theta_1 x_1^{(m)} + \theta_2 x_2^{(m)} + \cdots + \theta_d x_d^{(m)}) - y^{(m)}) \end{bmatrix}$$

$$\begin{aligned} (X\theta - y)^T W (X\theta - y) = \frac{1}{2} w^{(1)} \times ((\theta_1 x_1^{(1)} + \theta_2 x_2^{(1)} + \cdots + \theta_d x_d^{(1)}) - y^{(1)})^2 \\ + \frac{1}{2} w^{(2)} \times ((\theta_1 x_1^{(2)} + \theta_2 x_2^{(2)} + \cdots + \theta_d x_d^{(2)}) - y^{(2)})^2 + \cdots \\ + \frac{1}{2} w^{(m)} \times ((\theta_1 x_1^{(m)} + \theta_2 x_2^{(m)} + \cdots + \theta_d x_d^{(m)}) - y^{(m)})^2 \end{aligned}$$

So $J(\theta) = (X\theta - y)^T W (X\theta - y)$ can be written in the from $J(\theta) = (X\theta - y)^T W (X\theta - y)$ when choosing W as above.

- Solve for θ

$$\begin{aligned} J(\theta) &= (X\theta - y)^T W (X\theta - y) \\ &= ((X\theta)^T - y^T) W (X\theta - y) \\ &= ((X\theta)^T W - y^T W)(X\theta - y) \\ &= \theta^T X^T W X \theta - 2\theta^T X^T W y + y^T W y \end{aligned}$$

$$\begin{aligned} \nabla_\theta J(\theta) = 2X^T W X \theta - 2X^T W y = 0 \\ \theta = (X^T W X)^{-1} X^T W y \\ = (X^T X)^{-1} X^T y \end{aligned}$$

Thus $\theta = (X^T X)^{-1} X^T y$.

- Write down an algorithm for calculating θ by batch gradient descent for locally weighted linear regression.
  We know that $\theta = (X^T X)^{-1} X^T y$.
  Algorithm is

Algorithm 1: Calculating θ by batch gradient descent

Input: $\boldsymbol{x}$
Output: θ
for $i < 10000$ **do**
  $\theta_j \leftarrow \theta_j - \alpha \sum_i w^{(i)} (\sum_k (\theta_k x_x^{(i)}) - y^{(i)}), j = 0$
  $\theta_j \leftarrow \theta_j - \alpha \sum_i w^{(i)} (\sum_k (\theta_k x_x^{(i)}) - y^{(i)}) x_j^{(i)}, j \neq 0$

Locally weighted linear regression is a non-parametric method.

2 Properties of the linear regression estimator(10 points)

- Show that $E[\theta] = \theta^*$

$$\begin{aligned}\theta &= (X^TX)^{-1}X^Ty \\ &= (X^TX)^{-1}X^T(X\theta^* + \epsilon) \\ &= (X^TX)^{-1}X^TX\theta^* + (X^TX)^{-1}X^T\epsilon\end{aligned}$$

$$\begin{aligned}E(\theta) &= E((X^TX)^{-1}X^TX\theta^* + (X^TX)^{-1}X^T\epsilon) \\ &= E((X^TX)^{-1}X^TX\theta^*) + E(X^TX)^{-1}X^T\epsilon) \\ &= \theta^* + (X^TX)^{-1}X^TE(\epsilon) \\ &= \theta^*\end{aligned}$$

- Show the variance of the least squares estimator is $Var(\theta) = (X^TX)^{-1}\sigma^2$. Suppose $\boldsymbol{b}$ is an estimator of θ

$$\begin{aligned}Var(\theta) &= E[(\boldsymbol{b} - \theta)(\boldsymbol{b} - \theta)^T|X] \\ &= E[(X^TX)^{-1}X^T\epsilon\epsilon^TX(X^TX)^{-1}|X] \\ &= (X^TX)^{-1}X^TE[\epsilon\epsilon^T|X]X(X^TX)^{-1} \\ &= (X^TX)^{-1}\sigma^2\end{aligned}$$

3 Implementing linear regression and regularized linear regression(90 points)

3.1 Implementing linear regression(45 points)

- A1: Computing the cost function $J(\theta)$
  See the implementation for the *loss* function in the file *linear_regressor.py*.

- A2: Implementing gradient descent
  See the implementation for the *train* function in the file *linear_regressor.py*.

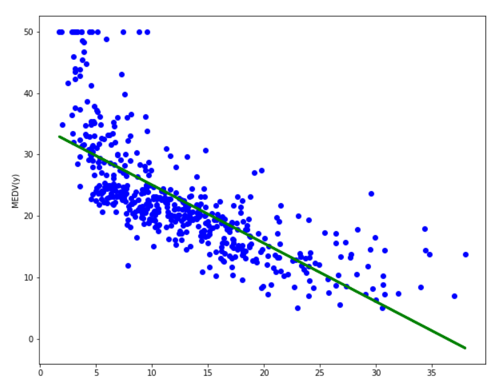


Figure 5: Fitting a linear model

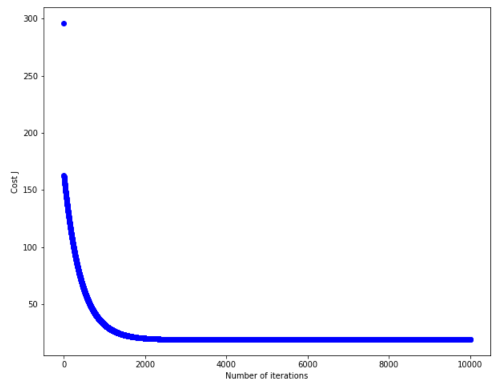


Figure 6: Convergence of gradient descent

- A3: Predicting on unseen data
  For lower status percentage = 5, we predict a median home value of 298034.49
  For lower status percentage = 50, we predict a median home value of -129482.13

- B1: Feature normalization
  See the implementation for the $feature_normalize$ function in the file *utils.py*.

- B2: Loss function and gradient descent
  See the implementation for the *train* and *loss* function in the file *linear_regressor_multi.py*.

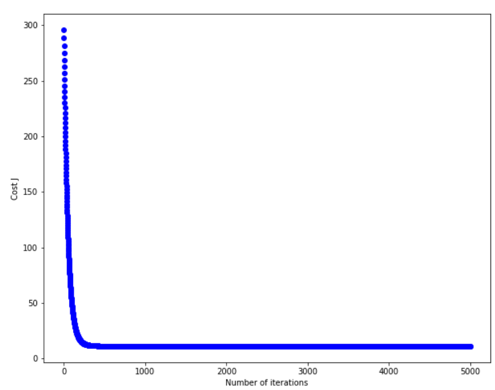
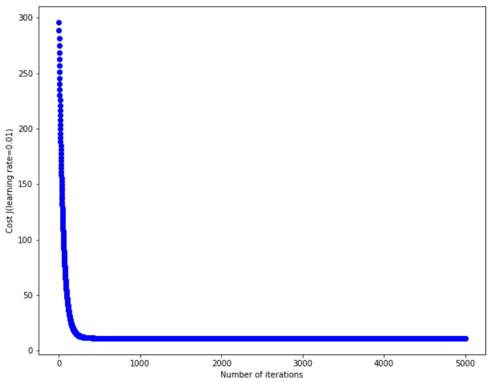


Figure 7: Convergence of gradient descent with multiple variables

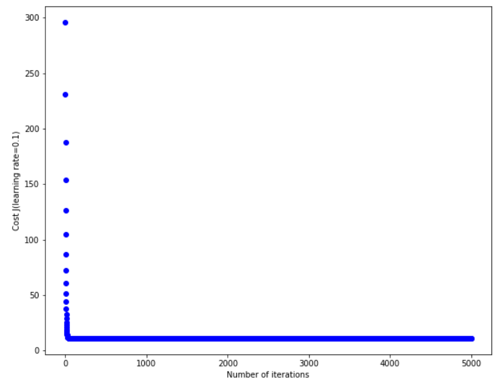
- B3: Making predictions on unseen data
  For average home in Boston suburbs, we predict a median home value of 230467.11

- B4: Normal equations
  See the implementation for the $normal_eqn$ function in the file *linear_regressor_multi.py*.
  Theta computed by direct solution is: $[3.64594884e + 01 - 1.08011358e - 014.64204584e - 022.05586264e - 022.68673382e + 00 - 1.77666112e + 013.80986521e + 006.92224640e - 04 - 1.47556685e + 003.06049479e - 01 - 1.23345939e - 02 - 9.52747232e - 019.31168327e - 03 - 5.24758378e - 01]$
  For average home in Boston suburbs, we predict a median home value of 230406.54.

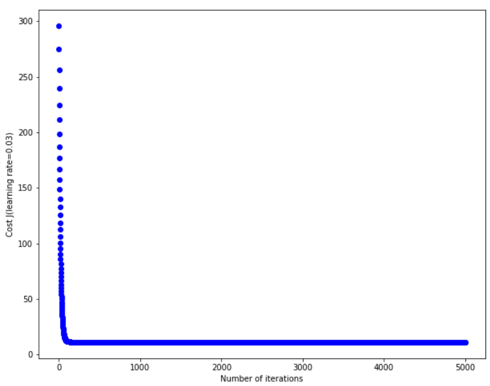
- B5: Exploring convergence of gradient descent



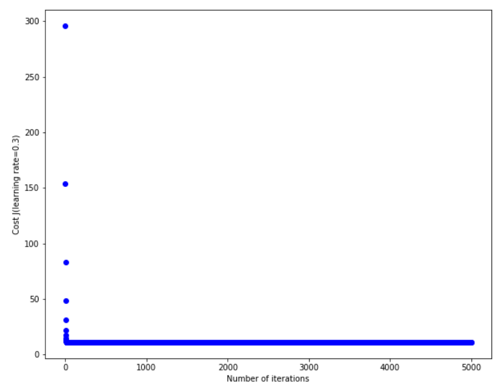
((a)) learning rate=0.01



((b)) learning rate=0.03



((c)) learning rate=0.1

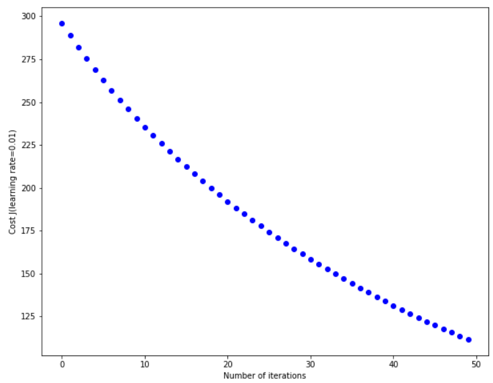


((d)) learning rate=0.3

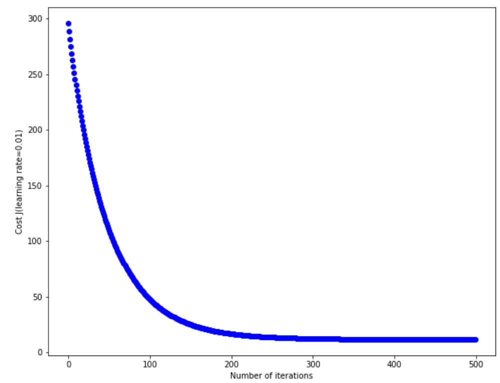
Figure 8: Convergence of gradient descent with multiple variables according to different learning rates

According to the above figures, when learning $rate = 0.01$ is a good choice.

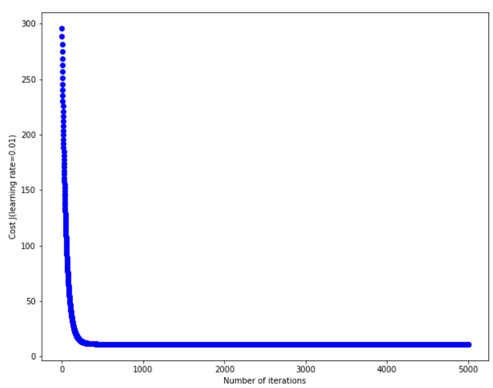
When learning rate = 0.01, chosing different iteration times.



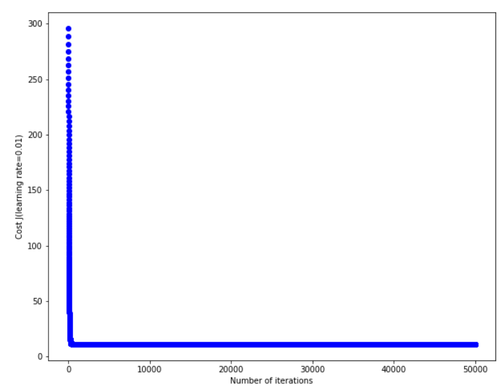
((a)) iteration time = 50



((b)) iteration time = 500



((c)) iteration time = 5000



((d)) iteration time = 50000

Figure 9: Convergence of gradient descent with multiple variables according to different iteration times

According to the above figures, iteration time is 500 is a good choice.

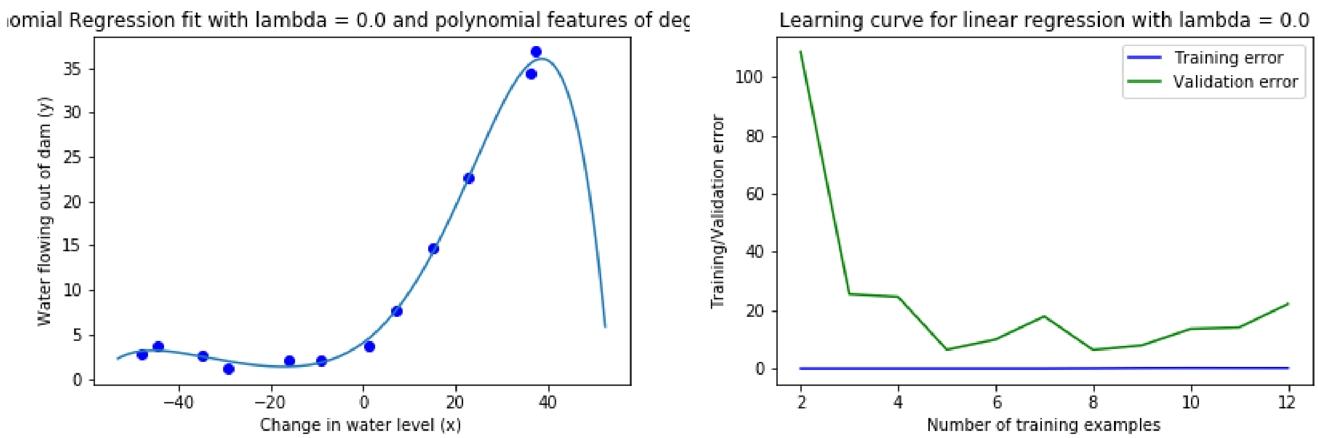
3.2 Implementing regularized linear regression(45 points)

- A1: Regularized linear regression cost function
  See the implementation for the *loss* function in the class *Reg_LinearRegression_SquaredLoss* in the file *reg_linear_regressor_multi.py*.

- A2: Gradient of the Regularized linear regression cost function
  See the implementation for the $grad_loss$ function in the class *Reg_LinearRegression_SquaredLoss* in the file *reg_linear_regressor_multi.py*.

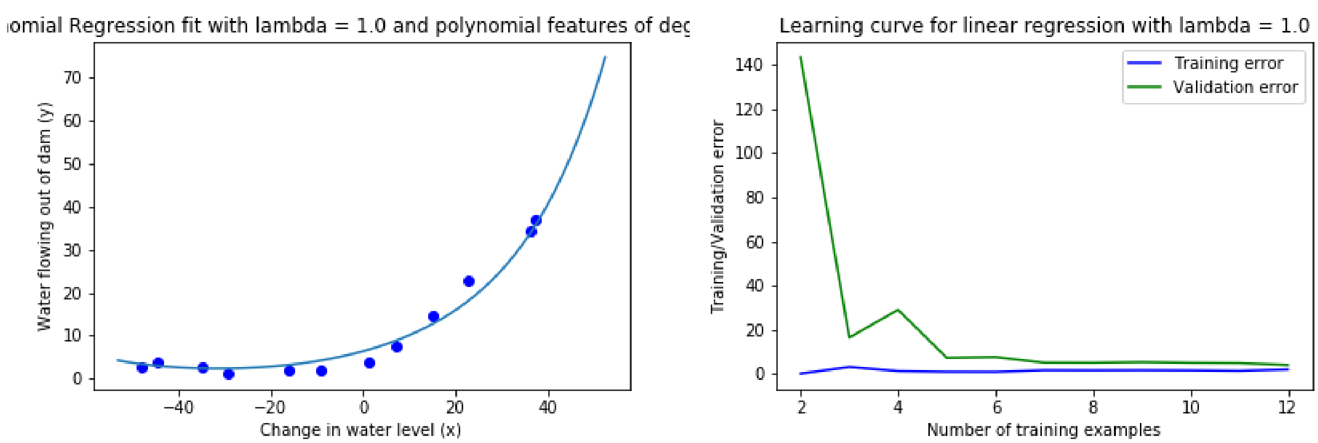
- A3: Learning curves

- A4: Adjusting the regularization parameter



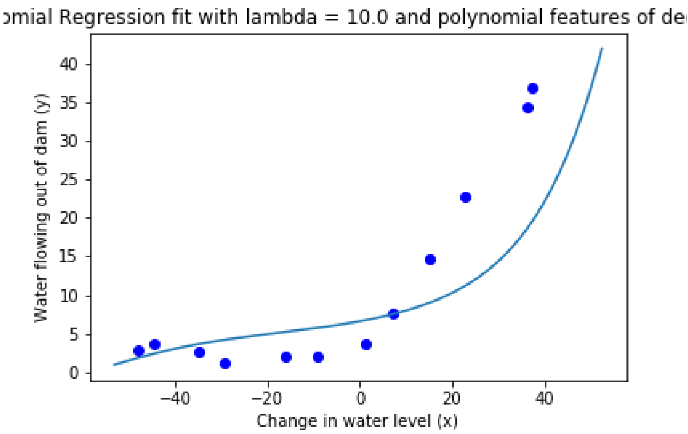
((a)) Polynomial Regression fit ((b)) Learning curve for linear regression

Figure 10: $\lambda = 0.0$

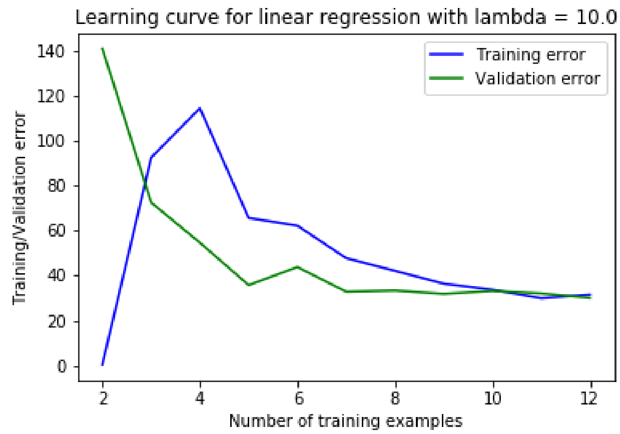


((a)) Polynomial Regression fit ((b)) Learning curve for linear regression

Figure 11: $\lambda = 1.0$

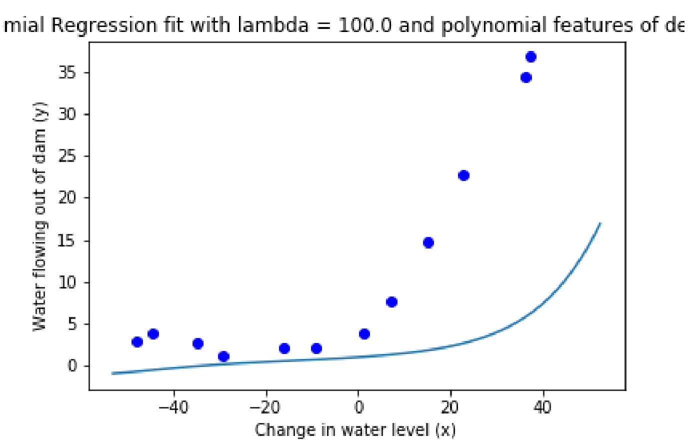


((a)) Polynomial Regression fit

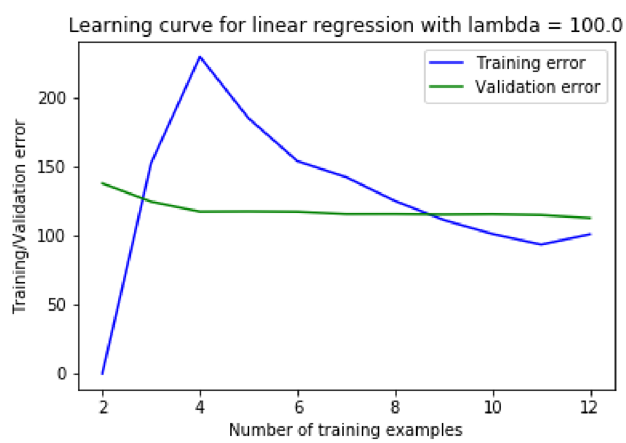


((b)) Learning curve for linear regression

Figure 12: $\lambda = 10.0$



((a)) Polynomial Regression fit



((b)) Learning curve for linear regression

Figure 13: $\lambda = 100.0$

- A5: Selecting λ using a validation set According to the figure above, the best λ is 3.

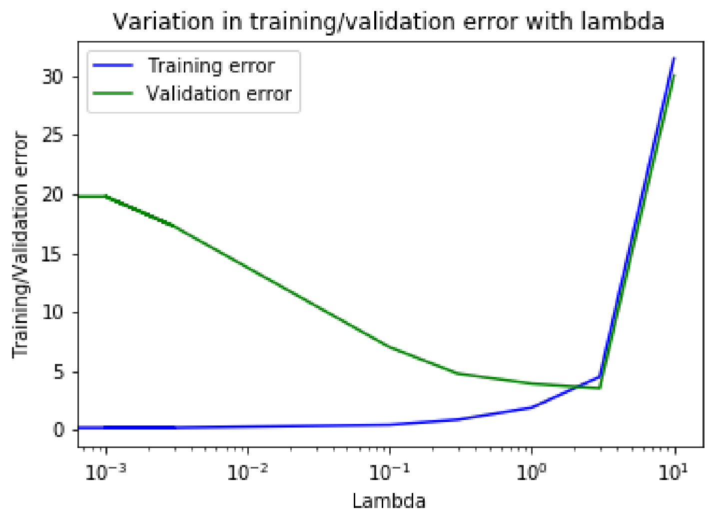


Figure 14: Fitting a linear model

- A6: Computing test set error
  Test error is 8.936718220214782.

- A7: Plotting learning curves with randomly selected examples

- A8: Comparing ridge regression and lasso regression models
  Lasso regression converges faster than Ridge regression.

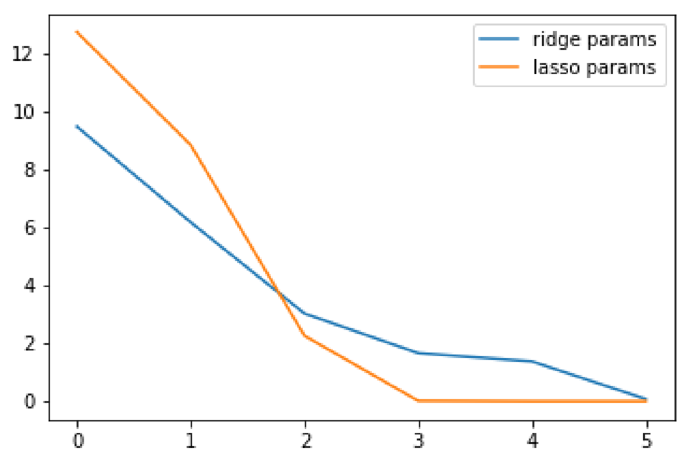


Figure 15: Fitting a linear model

4 Extra Credits

- The lowest achievable error on the test set with $\lambda = 0$
  Test error is 12.930601133114878 when $\lambda = 0$.

- Select the best value for λ and report the test set error with the best λ.
  The best $\lambda = 30.0$. Test error at $\lambda = 30.0$ is 13.34301316716897.

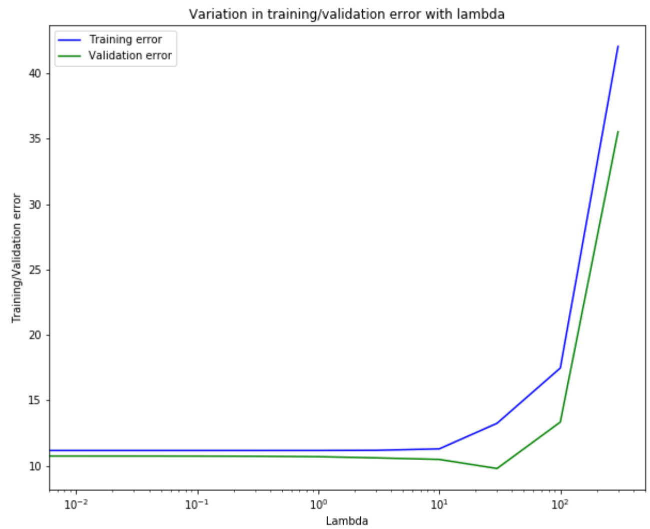


Figure 16: Fitting a linear model

- What is the test set error with quadratic features with the best λ chosen with the validation set?
  The best $\lambda = 30.0$. Test error at $\lambda = 30.0$ is 4.742818184669156.

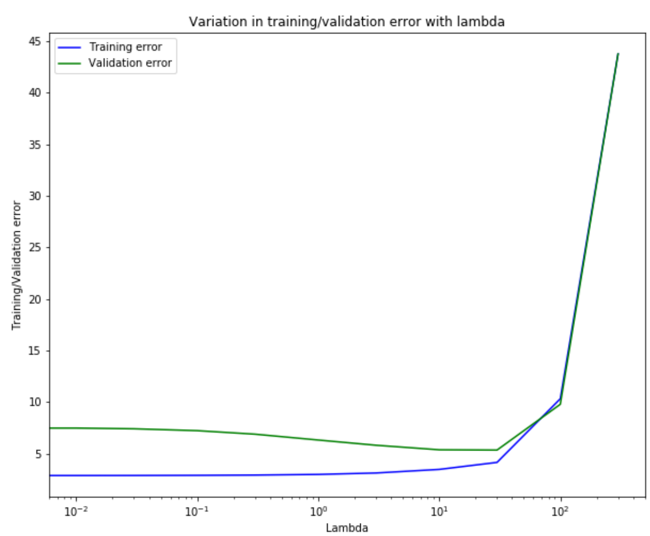


Figure 17: Fitting a linear model

- What is the test set error with cubic features with the best λ chosen with the validation set?
  The best $\lambda = 300.0$. Test error at $\lambda = 30.0$ is 6.558244031190886.

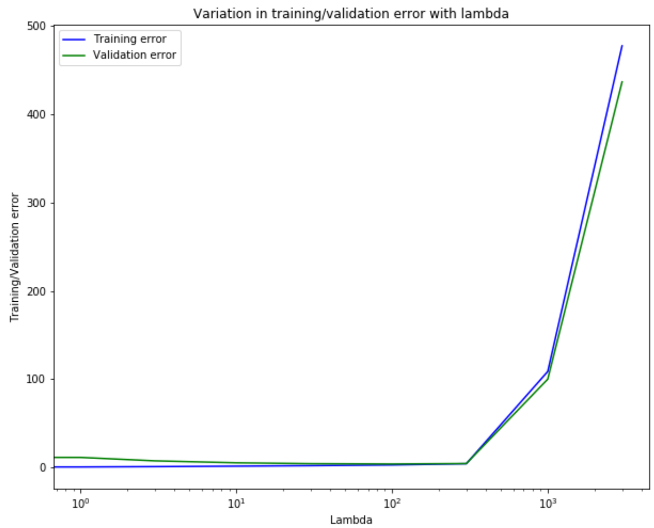


Figure 18: Fitting a linear model