COMP 540 Assignment #1

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0 Background refresher(30 points)

- Plot the categorical distribution.

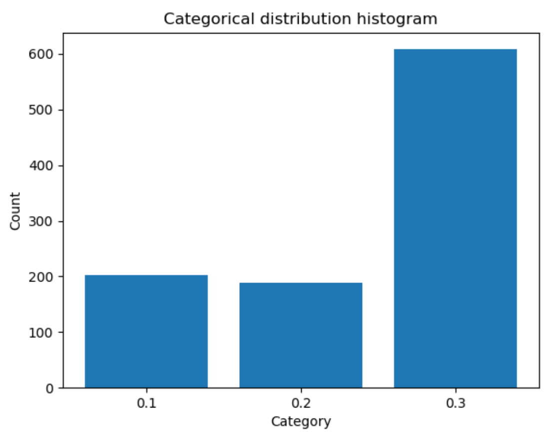

Figure 1: Categorical distribution

- Plot the Univariate normal distribution with mean of and standard deviation of 1.

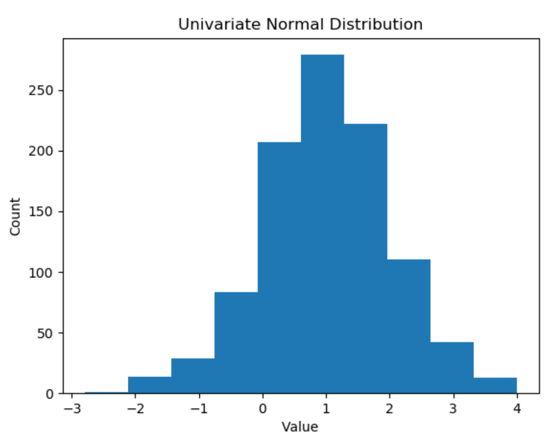

Figure 2: Univariate Normal Distribution

- Produce a scatter plot of the samples for a 2-D Gaussian.

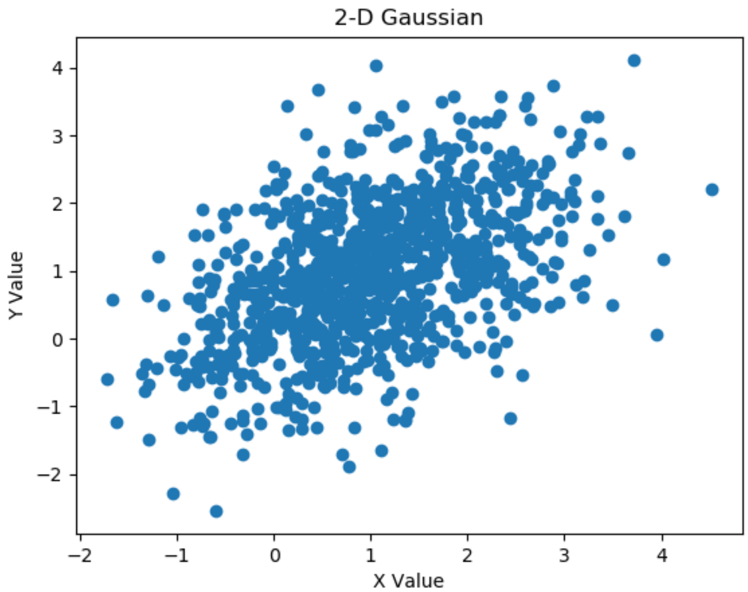

Figure 3: Univariate Normal Distribution

- Test mixture sampling code Code can be seen in sampler.py. Mixture Gaussian plot is shown below

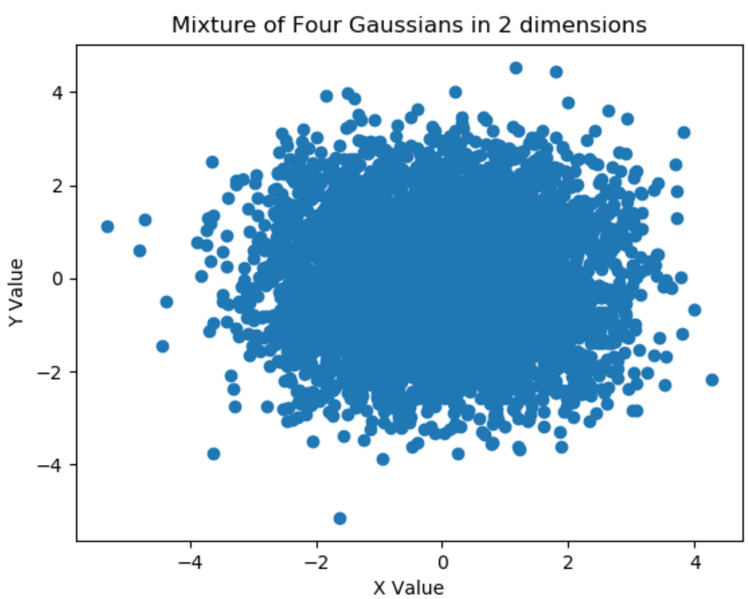

Figure 4: Univariate Normal Distribution

- Prove that the sum of two independent Poisson random variables is also a Poisson random variable. Suppose $X \sim \mathcal{P}(\lambda)$ and $Y \sim \mathcal{P}(\mu)$. Now Prove that $X + Y \sim \mathcal{P}(\lambda + \mu)$.

$$\begin{aligned}
P(X+Y=k) &= \sum_{i=0}^{k} P(X+Y=k, X=i) \\
&= \sum_{i=0}^{k} P(Y=k-i, X=i) \\
&= \sum_{i=0}^{k} P(Y=k-i)P(X=i) \\
&= \sum_{i=0}^{k} e^{-\mu}\frac{\mu^{k-i}}{(k-i)!}e^{-\lambda}\frac{\lambda^{i}}{i!} \\
&= e^{-(\mu+\lambda)}\frac{1}{k!}\sum_{i=0}^{k}\frac{k!}{i!(k-i)!}\mu^{k-i}\lambda^{i} \\
&= e^{-(\mu+\lambda)}\frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}\mu^{k-i}\lambda^{i} \\
&= \frac{(\mu+\lambda)^k}{k!}\cdot e^{-(\mu+\lambda)}
\end{aligned}$$

So $X + Y \sim \mathcal{P}(\lambda + \mu)$.

- Find α, μ_1 and σ_1.
We have X_0 and X_1 be continuous random variables. If

$$\begin{aligned}
p(X_0 = x_0) &= \alpha_0 e^{-\frac{(x_0-\mu_0)^2}{2\sigma_0^2}} \\
P(X_1 = x_1 | X_0 = x_0) &= \alpha_1 e^{-\frac{(x_1-x_0)^2}{2\sigma^2}}
\end{aligned}$$

$$\begin{aligned}
p(X_1 = x_1) &= \int P(X_1 = x_1|X_0 = x_0)\cdot p(X_0 = x_0)dx_0 \\
&= \alpha_0\alpha_1 \int e^{-\frac{\sigma^2(x_0-\mu_0)^2+\sigma_0^2(x_1-x_0)^2}{2\sigma_0^2\sigma^2}} dx_0 \\
&= \alpha_0\alpha_1 \int e^{-\frac{(\sigma^2+\sigma_0^2)x_0^2-2(\sigma^2\mu_0+\sigma_0^2x_1)x_0+\sigma^2\mu_0^2+\sigma_0^2x_1^2}{2\sigma_0^2\sigma^2}} dx_0 \\
&= \alpha_0\alpha_1 \int e^{-\frac{1}{2\sigma_0^2\sigma^2}[(\sqrt{\sigma^2+\sigma_0^2}x_0-\frac{-\sigma^2\mu_0+\sigma_0^2x_1}{\sqrt{\sigma^2+\sigma_0^2}})^2+\sigma^2\mu_0^2+\sigma_0^2x_1^2-\frac{\sigma^2\mu_0^2+\sigma_0^2x_1^2}{\sigma^2+\sigma_0^2}]} dx_0
\end{aligned}$$

Since

$$\int \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx = 1$$

$$
\begin{aligned}
p(X_1 = x_1) &= \frac{\alpha\alpha_0\sqrt{2\pi}\sigma\sigma_0}{\sqrt{\sigma^2+\sigma_0^2}} e^{-\frac{1}{2\sigma_0^2\sigma^2}(\sigma^2\mu_0^2+\sigma_0^2x_1^2-\frac{\sigma^2\mu_0^2+\sigma_0^2x_1^2}{\sigma^2+\sigma_0^2})} \\
&= \alpha e^{-\frac{(x_1-\mu_0)^2}{2(\sigma^2+\sigma_0^2)}}
\end{aligned}
$$

Thus we can solve that:

$$
\begin{aligned}
\alpha &= \frac{\alpha\alpha_0\sqrt{2\pi}\sigma\sigma_0}{\sqrt{\sigma^2+\sigma_0^2}} \\
\mu_1 &= \mu_0 \\
\sigma_1 &= \sqrt{\sigma^2+\sigma_0^2}
\end{aligned}
$$

- Consider the vectors $\boldsymbol{u} = \begin{bmatrix}1 & 2\end{bmatrix}^{\mathrm{T}}$ and $\boldsymbol{v} = \begin{bmatrix}2 & 3\end{bmatrix}^{\mathrm{T}}$. Define the matrix $\boldsymbol{M} = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$. Compute the eigenvalues and eigenvectors of $\boldsymbol{M}$.

$$
\begin{aligned}
\boldsymbol{M} &= \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} \\
&= \begin{bmatrix}1 & 2\end{bmatrix}^{\mathrm{T}} \cdot \begin{bmatrix}2 & 3\end{bmatrix} \\
&= \begin{bmatrix}2 & 3 \\ 4 & 6\end{bmatrix}
\end{aligned}
$$

$$
\left|\lambda\boldsymbol{I} - \boldsymbol{M}\right| = \begin{vmatrix}\lambda-2 & -3 \\ -4 & \lambda-6\end{vmatrix} = 0
$$

Then we can solve it as

$$
\begin{aligned}
(\lambda-2)(\lambda-6)-12 &= 0 \\
\lambda^2 - 8\lambda &= 0 \\
\lambda(\lambda-8) &= 0 \\
\lambda_1 = 0, \lambda_2 &= 8
\end{aligned}
$$

Let $\lambda = 0$:

$$
\begin{aligned}
(\lambda\boldsymbol{I} - \boldsymbol{M}) \cdot \begin{bmatrix}x_1 & x_2\end{bmatrix}^{\mathrm{T}} &= 0 \\
\begin{bmatrix}-2 & -3 \\ -4 & -6\end{bmatrix} \cdot \begin{bmatrix}x_1 \\ x_2\end{bmatrix} &= \begin{bmatrix}0 \\ 0\end{bmatrix} \\
-2x_1 - 3x_2 &= 0
\end{aligned}
$$

Let $x_1 = 3$, Then $x_2 = -2$. So eigenvector is $\begin{bmatrix}3 & -2\end{bmatrix}^{\mathrm{T}}$.
Let $\lambda = 8$:

$$
\begin{aligned}
(\lambda \boldsymbol{I} - \boldsymbol{M}) \cdot \begin{bmatrix} x_1 & x_2 \end{bmatrix}^{\mathrm{T}} &= 0 \\
\begin{bmatrix} 6 & -3 \\ -4 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
6x_1 - 3x_2 &= 0 \\
-4x_1 + 2x_2 &= 0
\end{aligned}
$$

Let $x_1 = 1$, Then $x_2 = 2$. So eigenvector is $\begin{bmatrix} 1 & 2 \end{bmatrix}^{\mathrm{T}}$.
Thus, when eigenvalue $\lambda = 0$, eigenvector is $\begin{bmatrix} 3 & -2 \end{bmatrix}^{\mathrm{T}}$, when eigenvalue $\lambda = 8$, eigenvector is $\begin{bmatrix} 1 & 2 \end{bmatrix}^{\mathrm{T}}$.

- Provide one example for each of the following cases.
  As for $(A+B)^2 \neq A^2 + 2AB + B^2$. Suppose $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Since $A \cdot B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $B \cdot A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.
  So $(A+B)^2 = A^2 + AB + BA + B^2 \neq A^2 + 2AB + B^2$.
  As for $AB = 0, A \neq 0, B \neq 0$, Suppose $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

- Show that $\boldsymbol{A}$ is orthogonal.
  Given $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u} = 1$ and $\boldsymbol{A} = \boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$.

$$
\begin{aligned}
\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} &= (\boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})^{\mathrm{T}}(\boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}) \\
&= (\boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})(\boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}) \\
&= \boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} + 4\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} \\
&= \boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} + 4\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} \\
&= \boldsymbol{I}
\end{aligned}
$$

  So $\boldsymbol{A}$ is orthogonal.

- Prove the following assertions.
  As for $f(x) = x^3$ for $x \geq 0$.

$$
\begin{aligned}
f(x) &= x^3 \\
f^{''}(x) &= 6x
\end{aligned}
$$

  Since $x \geq 0$, $f^{''}(x) = 6x \geq 0$. Thus $f(x) = x^3$ is convex for $x \geq 0$.
  As for $f(x_1, x_2) = max(x_1, x_2)$. Let $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{y} = (y_1, y_2)$ and $\lambda \in [0, 1]$.

$$
\begin{aligned}
f(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) &= f(\lambda x_1 + (1-\lambda)y_1, \lambda x_2 + (1-\lambda)y_2) \\
&= max(\lambda x_1 + (1-\lambda)y_1, \lambda x_2 + (1-\lambda)y_2) \\
&\leq max(\lambda x_1, \lambda x_2) + max((1-\lambda)y_1, (1-\lambda)y_2) \\
&= \lambda max(x_1, x_2) + (1-\lambda)max(y_1, y_2) \\
&= \lambda f(\boldsymbol{x}) + (1-\lambda)f(\boldsymbol{y})
\end{aligned}
$$

  So $f(x_1, x_2) = max(x_1, x_2)$ is convex on R^2.
  As for function $f + g$. If univariate functions f and g are convex on S, then $f^{''}(x) \geq 0$ and $g^{''}(x) \geq 0$.

So $(f+g)^{''}(x) = f^{''}(x) + g^{''}(x) \geq 0$. Thus if univariate functions f and g are convex on S, then $f+g$ is convex on S.
As for fg. If univariate functions f and g are convex and non-nigegative on S.

$$\begin{aligned}(fg)^{''}(x) &= (f^{'}g + fg^{'})^{'}(x) \\ &= (f^{''}g + f^{'}g^{'} + f^{'}g^{'} + fg^{''})(x) \\ &= (f^{''}g + 2f^{'}g^{'} + fg^{''})(x)\end{aligned}$$

Since f and g have their minimum within S at the same point. Before the minimum point, both f and g are decreasing. After the minimum point, both f and g are increasing. So $f^{'}g^{'} \geq 0$, $f^{''} \geq 0$, $g^{''} \geq 0$, $f \geq 0$ and $g \geq 0$. Thus $(fg)^{''}(x) = (f^{''}g + 2f^{'}g^{'} + fg^{''})(x) \geq 0$. Then fg is convex on S.

1 Locally weighted linear regression(20 points)