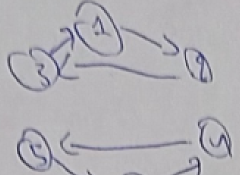
Types of questions: - What are the specific laws for particular kinds of system.
- What are the rules for the allowable laws?

Dynamical laws law of motion

1 system but 2 cycles



(Trapped on one cycle is called having a conserved quantity. (a quantity of a system that doesn't change with time)

For a system to be allowable:
It needs to be reversible (deterministic into both the past and future). Just flip the arrows and check for any conflicts

$$\boxed{a = b^2 + c^2 - 2bc\cos\theta}$$

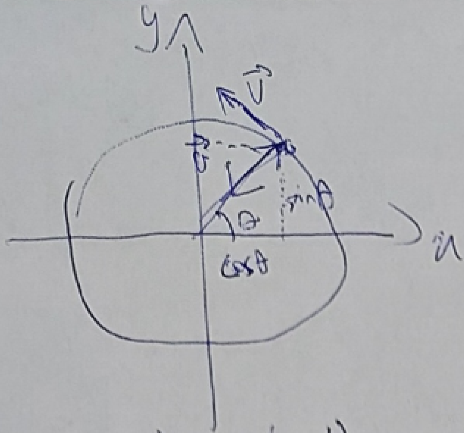
$$\vec{A}\cdot\vec{B} = |A|\,|B|\cos\theta = A_x B_x + A_y B_y + A_z B_z$$

$$\vec{r}_n(t) = x(t),\ y(t),\ z(t) \qquad v_i = \frac{dx_i}{dt} = \dot{x}_i \qquad a_i = \dot{v}_i = \ddot{x}_i$$

If $\theta = \omega t \Rightarrow \frac{2\pi}{\omega} = \text{period}$ $\qquad \omega = \text{angular frequency}$

In case of centrifugal force (circular motion): if $x = 1$
$\Rightarrow v = \omega$
$a = \omega^2$

Lecture One



unit circle ($r=1$)

Example: angle increases linearly (as a function of time)

$\Rightarrow \theta = \omega t$

$$\left.\begin{aligned} x(t) &= \cos \omega t \\ y(t) &= \sin \omega t \end{aligned}\right\} \Rightarrow \left.\begin{aligned} v_x &= -\omega \sin \omega t \\ v_y &= \omega \cos \omega t \end{aligned}\right\} \Rightarrow \begin{aligned} a_x &= -\omega^2 \cos \omega t \\ a_y &= -\omega^2 \sin \omega t \end{aligned}$$

To prove that $\vec{v}$ is perpendicular to $\vec{r}$, we need to show that $\vec{r} \cdot \vec{v} = 0$:

$$\vec{r} \cdot \vec{v} = r_x v_x + r_y v_y = -\omega \sin \omega t \cos \omega t + \omega \sin \omega t \cos \omega t = 0$$

$\Rightarrow$ it is perpendicular

$a_x = -\omega^2 \cos \omega t = -\omega^2 \cdot x(t)$ $\Rightarrow$ acceleration is ~~in the same direction~~ parallel to the position

$a_y = -\omega^2 \sin \omega t = -\omega^2 y(t)$ but is in the opposite direction (coefficient is -ve)

Lecture 2:

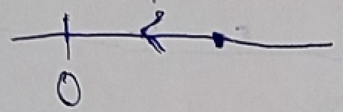
Aristotle's Law

$$\vec{F} = m\vec{v}$$

$$F(t) = mv = m\dot{x} = m\,\frac{x(t+\Delta) - x(t)}{\Delta}$$

$$\Rightarrow x(t+\Delta) = \frac{\Delta}{m}F(t) + x(t) \quad (1)$$

Spring pulling object (Harmonic oscillator):



so $\boxed{F = -kx}$ plug in (1)

Discrete form

$$-k\frac{\Delta}{m}x(t) + x(t) = x(t+\Delta)$$

$$x(t)\Big[\underbrace{1 - k\frac{\Delta}{m}}_{\text{slightly } < 1}\Big] = x(t+\Delta)$$

so goes back

Continuous Form

$$m\frac{dx}{dt} = -kx$$

$$\frac{dx}{dt} = -\frac{k}{m}x$$

$$\frac{dx}{x} = -\frac{k}{m}dt$$

$$\int \frac{1}{x}\,dx = \int -\frac{k}{m}\,dt$$

$$\ln(x) = -\frac{k}{m}t + c$$

$$e \qquad e$$

$$\Rightarrow x = Ae^{-k/m\,t}$$

When $t=0$, $x = A$ so $A = x(0)$

$$\Rightarrow \boxed{x = x(0)Ae^{-k/m\,t}}$$

You can predict the future, but not the past. If you displace the object a tiny infinitesimal amount, too small to be detectable, and run it backwards, where does it go to? (All trajectories come in and go to the origin) so you can't tell where it came from. But if you have an infinite accuracy you could.

So Aristotle's law, apart from being experimentally incorrect, is also irreversible.

"The laws of physics is that they exist in frames of reference in which Newton's laws are true."

Newton's 2nd Law:

$F = ma = m\dot{v} = m\ddot{x}$; Is it predictive? i.e. if you know where it is at one instant of time and you know its mass and the force on it, do you know where it is the next instant of time?

$$F = m\ddot{x} \quad \ddot{x} = a = \dot{v} = \frac{d}{dt}\left[\frac{x(t+\Delta) - x(t)}{\Delta}\right] = \frac{\frac{x(t+\Delta) - x(t)}{\Delta} - \frac{x(t) - x(t-\Delta)}{\Delta}}{\Delta}$$

$$\frac{F}{m} = \frac{x(t+\Delta) - 2x(t) + x(t-\Delta)}{\Delta^2}$$

$$\boxed{\Delta^2 \frac{F}{m} + 2x(t) - x(t-\Delta) = x(t+\Delta)}$$

Now we have a formula that tells us where we'll be next, but we not only have to know our position now, but our position at a previous instance in time.

Initial conditions to have: $x(t)$, $\dot{x}(t)$

where you are now $\rightarrow$ $\leftarrow$ Tells you where you was a moment ago

If we know both, we can predict the future with these Newton's equations.

Momentum: $\vec{p} = m\vec{v}$ $\therefore$ $p = m\dot{x}$

$$\dot{p} = \frac{d}{dt} m\dot{x} = m\frac{d\dot{x}}{x} = m\ddot{x} = F \Rightarrow \boxed{F = \dot{p}} \quad \boxed{p = m\dot{x}}$$

2 equations to form: The phase space which is predictive using 2 first-order equation (instead of one in Aristotle's law)

* Newton's law is reversible in the sense that if you record a movie of an accelerating object, and you run it backwards, acceleration stays the same, but if you run a movie backward of a velocity, the velocity changes sign.

* $F = -kx$

$m\ddot{x} = -kx$ (set $m = 1$ and $k = 1$ in order to not carry constants around)

$\ddot{x} = -x$ (Which function's 2nd derivative is same as itself but with a (-) sign? cos() or sin())

$\ddot{x} = -x$

So $x = c\cos(t - t_0)$

$p = \dot{x} = -c\sin(t - t_0)$ (m is 1)

(oscillates and then comes to rest,

x is max and v is min, and when it gets to the origin, x is min and v is max. Exchanges $\dot{x}$ and x)

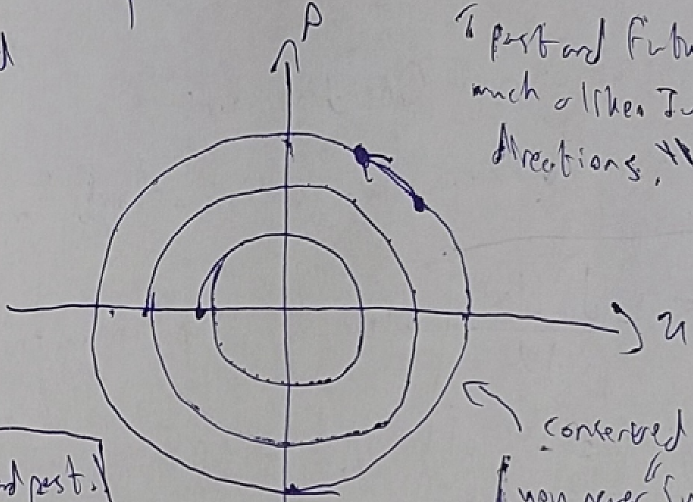
then, $\boxed{x^2 + p^2 = c^2}$

c is radius and is determined via initial conditions.

$\dot{x} = -x$ (Aristotle's version)

$x = x(0)\,e^{-t}$

(comes to rest)



past and future are very much alike. Just different directions.

conserved quantity (you never "jump" from one cycle to the other)

$\boxed{\text{It is predictable to both future and past.}}$

Since energy → $c^2 = \frac{\dot{x}^2}{2} + \frac{x^2}{2}$

Harmonic Oscillator

To know if it is reversible, replace t with $-t$.

$F(x) = \frac{d^2x}{dt^2}$: $t \to -t$ but $t^2 \to t^2$ and so does dt^2.

In Aristotle's case: $F(x) = \frac{dx}{dt}$, $t \to -t$ and $dt \to -dt$. The equation changes

Newton's laws:

1st law: In the absence of force, the object moves at a constant velocity. (redundant, special case of 2nd law)

2nd law: $\vec{F} = m\frac{d^2\vec{r}}{dt^2} = m\frac{d\vec{v}}{dt}$

3rd law: $\vec{F}_{ij} = -\vec{F}_{ji}$ along line of centers

exerted by j on i

For i particles:

$$m_i \frac{d^2 \vec{r}_i}{dt^2} = \sum_{j \neq i} \vec{F}_{ij} = m_i \frac{d\vec{v}_i}{dt} = \frac{d\vec{p}_i}{dt}$$

$$\frac{d}{dt}\vec{P}_{total} = \sum_i \sum_j \vec{F}_{ij}$$

: Each pair of particle is counted twice, one is $\vec{F}_{ij}$ and the other $\vec{F}_{ji}$. So the result is from Newton's third law that is equal to zero

so $\boxed{\frac{d}{dt}\vec{P}_{total} = 0 \quad \text{so total momentum is conserved}}$

Conservation of Energy in One dimensions

In one dimensions, you can always think of any function as being the derivative of some other function.

So $\boxed{F(x) = -\frac{dV(x)}{dx} \qquad \int F(x)\,dx = -V(x)}$ where V is potential energy

; Now, let's Define E where

$$\underbrace{\frac{1}{2}m\dot{x}^2}_{T} + \underbrace{V(x)}_{V} = E$$

$$\frac{1}{2}(m)(2)(\ddot{x})(\dot{x}) + \frac{dV(x)}{dx}\cdot\dot{x} = \frac{dE}{dt}$$

$$\dot{x}\left[m\ddot{x} + \frac{dV(x)}{dx}\right] = \frac{dE}{dt}$$

$$\dot{x}\left[m\ddot{x} - \left(-\frac{dV(x)}{dx}\right)\right] = \frac{dE}{dt}$$

$$\dot{x}(F - F) = \frac{dE}{dt}$$

$$\frac{dE}{dt} = 0$$

Conservation of Energy in i dimensions:

$$F_i(x) = -\frac{\partial V(x)}{\partial x_i}$$

Question: Is there always a V whose partial derivatives give you F_i? The answer is no, I would then seem that there is no such thing as potential energy which has the property that its derivative is equal to the force, its a Law of Physics. (not something that can be ~~derived~~ obtained by experiments)

$$\therefore \sum_i \frac{1}{2} m \dot{x}_i^2 + V(x) = E$$

$$\sum_i \frac{1}{\cancel{2}} m \cancel{(2)} \dot{x}_i \ddot{x}_i + \frac{\partial V}{\partial x_i} \frac{dx}{dt} = \frac{dE}{dt}$$

$$\sum_i \dot{x}_i \left[m\ddot{x} + \frac{\partial V}{\partial x_i} \right] = \frac{dE}{dt}$$

$$\frac{dE}{dt} = 0$$

Conservation of Energy for more than one particle in several dimensions:

Let x_i be this whole column: $x_1, y_1, z_1, x_2, y_2, z_2, \ldots$

$F_i(x) = m_i \ddot{x}_i$

allowing for possibility of different masses

(Then do same as last point)

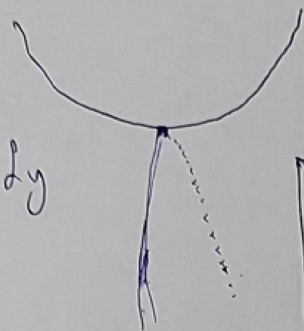
Lecture 3:

What does "stationary" mean? It means $\frac{dV(x)}{dx} = 0$ (at local minima, maxima, and saddle points)

or when $\frac{dV}{dx}\,\delta x = \delta V$ (δ is pronounced as delta and means small change)

Saddle points:

$$\delta V = \frac{\partial V}{\partial x}\delta x + \frac{\partial V}{\partial y}\delta y$$



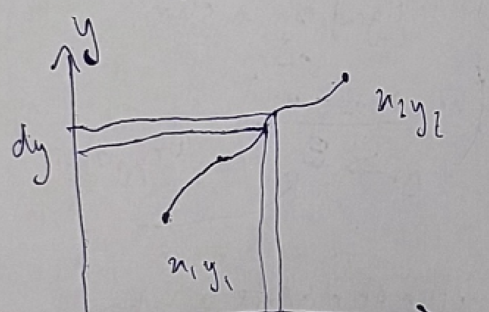
One partial derivative will increase and the other decrease (if 2 directions)

General law of Nature for equilibrium:

$$\delta V = 0$$

"Calculus of variations" is the problem of finding functions which minimize (stationaryze) some quantity (make its derivative = 0).

Example: Pick two points and ask what curve minimizes the distance between them:



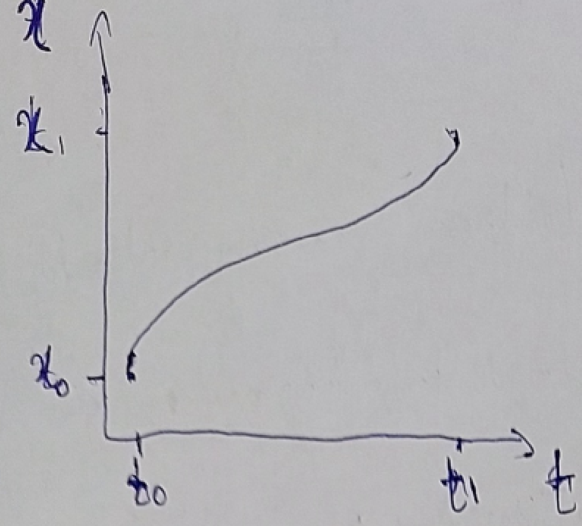
Length of segment $ds = \sqrt{dx^2 + dy^2}$

$$= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$\Rightarrow S_{12} = \underbrace{\int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx}_{S}$$

"Find a curve that has the property that the integral along it (S) is minimal ~~segment~~ (condition is $\delta \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = 0$)" To solve this, we need to reduce it to a differential equation.

* minimize a function: make its derivative $= 0$



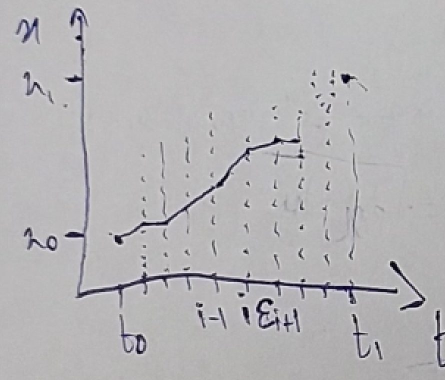
Given: you have the positions at 2 different times and you're asked to fill in the trajectory in between.

When you have questions like this in physics, you have to minimize something. In this case, it is the Action.

$$A = \int_{t_0}^{t_1} dt\, \mathcal{L}(x\ \dot{x})$$

To minimize the Action: $\delta \int_{t_0}^{t_1} dt \mathcal{L} = 0$

We have to reduce it to a differential equation. To do so, we'll ~~take~~ replace the problem with an approximate problem. We divide the time axis into little increments. Replace the trajectory by a sequence of integrals:



We will ~~replace~~ shrink ε to 0. This way we'll change this piecewise function to a curve.

We have to replace the integral by a sum:

$$\int dt \mathcal{L} \to \varepsilon \sum \mathcal{L}\left(x_i, \frac{\overbrace{x_{i+1} - x_i}^{\Delta x_i}}{\underbrace{\varepsilon}_{\Delta t \to \varepsilon}}\right)$$

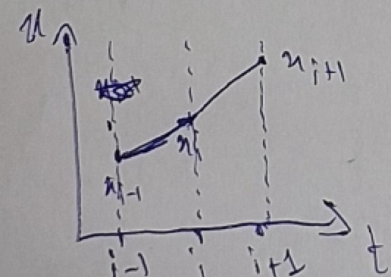
approximation to the velocity between i and $i+1$

Now we've replaced the continuous function by a function of discrete variables.

* i stands for the time the particle was at, not dimension or particle number.

Now, let's differentiate the expression with respect to x_i: we are keeping everything fixed but varying x_i. We are finding the change in the action if we vary x_i a little bit.

(only the neighboring intervals are affected)



$$\frac{\partial}{\partial x_i}\left[\varepsilon \mathcal{L}\left(x_i, \frac{x_{i+1} - x_i}{\varepsilon}\right) + \varepsilon \mathcal{L}\left(x_{i-1}, \frac{x_i - x_{i-1}}{\varepsilon}\right)\right]$$

$$= \varepsilon\left[\frac{\partial \mathcal{L}(x_i \cdot v_i)}{\partial x_i} + \frac{dv_i}{dx_i}\frac{\partial \mathcal{L}}{\partial v_i} - \frac{dv_i}{dx_i}\frac{\partial \mathcal{L}}{\partial v_{i+1}}\right]$$

$$= \varepsilon\left[\frac{\partial \mathcal{L}(x_i\ v_i)}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial \mathcal{L}}{\partial v_i} - \frac{1}{\varepsilon}\frac{\partial \mathcal{L}}{\partial v_{i+1}}\right]$$

$$= \varepsilon\left[\frac{\partial \mathcal{L}(x_i\ v_i)}{\partial x_i} + \frac{\frac{\partial \mathcal{L}}{\partial v_i} - \frac{\partial \mathcal{L}}{\partial v_{i+1}}}{\varepsilon}\right]$$ ε is small $\Rightarrow$ it is a derivative

$$= \varepsilon\left[\frac{\partial \mathcal{L}(x_i\ v_i)}{\partial x_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v_i}\right]$$

$$\Rightarrow \delta A = -\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v_i} + \frac{\partial \mathcal{L}}{\partial x} = 0$$ (to make action stationary)

~~The general ...~~ Euler-Lagrange equation: $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial x}$

* We can find an $\mathcal{L}(x, \dot{x})$ such that when you write ~~down~~ an Euler-Lagrange equation, it is Newton's equation.

For $\mathcal{L}(x, \dot{x}) = T - V(x) = \frac{1}{2}m\dot{x}^2 - V(x)$:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{d}{dt}[m\dot{x}] = m\ddot{x} \quad \Big| \quad \frac{\partial \mathcal{L}}{\partial x} = -\frac{dV(x)}{dx} = F(x)$$

$\Rightarrow F = m\ddot{x}$ (Newton's 2nd law derived)

For many particles and directions of space: $\mathcal{L}(x_i\ \dot{x}_i)$ is a function of all the coordinates and velocities ~~...~~ directions:

At each interval, there's a seperate equation for each coordinate:

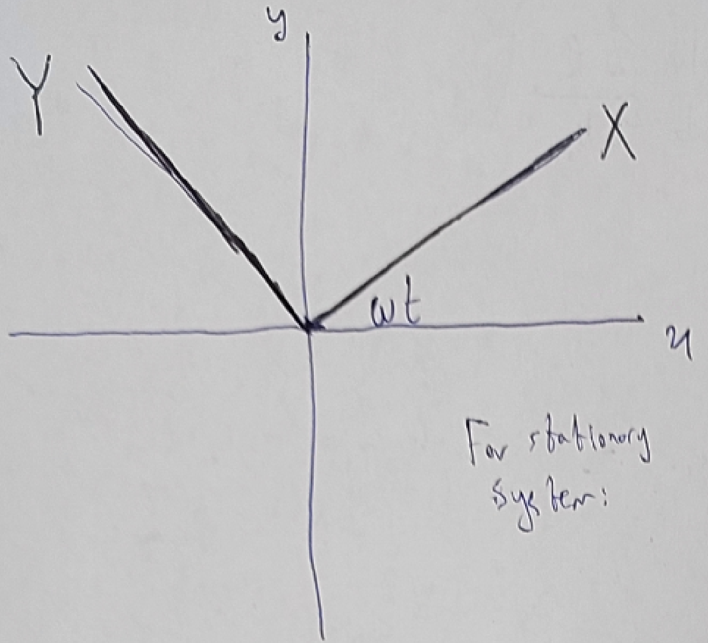
$$\Rightarrow \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} = \frac{\partial \mathcal{L}}{\partial x_i}$$

This time $T = \sum \frac{m}{2}\dot{x}$ and $V = V(x_i)$

⋮ Same as before

$$m\ddot{x}_i = -\frac{\partial V}{\partial x_i} = F_i(x)$$

* The Action Principle is true in any coordinate system, so the Lagrangian is an extremely convenient tool for changing coordinates



$$\boxed{\begin{aligned} x &= X\cos\omega t + Y\sin\omega t \\ y &= -X\sin\omega t + Y\cos\omega t \end{aligned}}$$

For stationary system: $\mathcal{L} = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right)$ (No $V(r)$ (simplest case))

* We have to rewrite the action of the Lagrangian in terms of X + Y.

$$\dot{x} = \dot{X}\cos\omega t - \omega X\sin\omega t + \dot{Y}\sin\omega t + \omega Y\cos\omega t$$
$$\dot{y} = -\dot{X}\sin\omega t - \omega X\cos\omega t + \dot{Y}\cos\omega t - \omega Y\sin\omega t$$

$$\vdots$$

$$\therefore \mathcal{L} = \frac{m}{2}\left(\dot{x}^2+\dot{y}^2\right) = \frac{m}{2}\left(\dot{X}^2+\dot{Y}^2\right) \underbrace{+\frac{m\omega^2}{2}\left(X^2+Y^2\right)}_{\text{centrifugal force}} \underbrace{+\frac{m\omega}{2}\left(\dot{X}Y-\dot{Y}X\right)}_{\text{coriolis force}}$$

(leads to a velocity-dependent force)

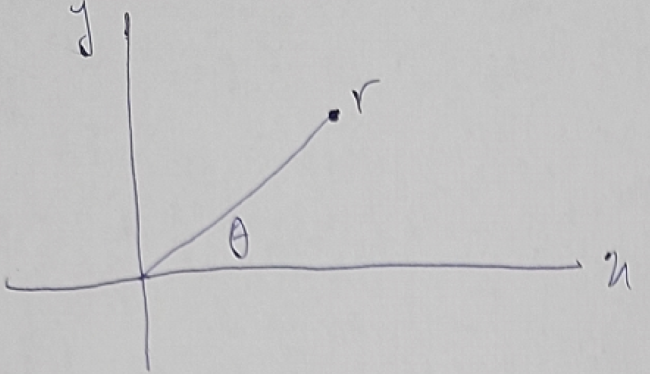
If $\omega = 0$ (system is not rotating), we get that things move in straight lines in the XY system.

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\ddot{X} + \frac{m\omega}{2}\dot{Y} = \underbrace{\omega^2 m X - \frac{m\omega}{2}\dot{Y}}_{\frac{\partial \mathcal{L}}{\partial x}}$$

$$\Rightarrow m a_X = \underbrace{\omega^2 m X}_{\text{centrifugal}} - \underbrace{m\omega\dot{Y}}_{\text{Depends on velocity}}$$

$$m a_Y = \omega^2 m Y + m\omega\dot{X}$$

Let's consider a system of ordinary polar coordinates (not moving).



$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$\dot{x} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$$
$$\dot{y} = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta}$$

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = m\ddot{r} = \frac{\partial \mathcal{L}}{\partial r} = \underbrace{mr\dot{\theta}^2}_{\text{centrifugal force}} - \frac{dV}{dr}$$ } Euler-Lagrange equations for r

Euler-Lagrange equations for θ:

$$\frac{\partial \mathcal{L}}{\partial \theta} = 0$$ ($\mathcal{L}$ doesn't depend on θ)

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{d}{dt}\left[mr^2\dot{\theta}\right] = 0$$

← Angular momentum (conserved quantity)

So Angular momentum is conserved. It followed from assuming that V did not depend on θ.
If it did, we would've gotten another term on the right hand side $\left(-\frac{\partial V}{\partial \theta}\right)$

* General rule: If there is a Lagrangian that doesn't depend on a coordinate, there will always be a conservation law.
* Coordinates that don't appear in the Lagrangian are called cyclic coordinates

* Lagrangians package problems in a simple and minimal kind of way. One function of all the coordinates and velocities completely determine
* The Euler-Lagrange equations are a very efficient tool for changing coordinates (invariant of the coordinates).

Lecture 4:

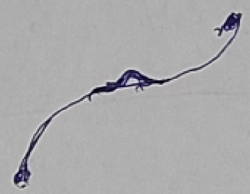
- You will learn about symmetry and conservation laws, which are at the heart of classical mechanics
- The principle of least Action, the Euler-Lagrange Equations, and the connections between symmetry and conservation laws, are 1/2 of the heart of ~~a~~ the subject. The other half is the Hamiltonian formulation, and they are very closely related.

* Mathematical Manipulations that we do over and over and over again. Often, we calculate the variation ~~of~~ something when we change things by a small parameter. Imagine a small parameter that we want to shift things by. (We shift things by an amount proportional to the small parameter).

* For any $F(x,y)$: $\delta F(x,y) = \frac{\partial F}{\partial x}\delta x + \frac{\partial F}{\partial y}\delta y$ (where δx and $\delta y = k\delta$ / are proportional to the small parameter δ)

"First order change in F"

* Suppose there's an integral: $\delta \int^{2} dt\, \mathcal{L}(x, \dot{x})$

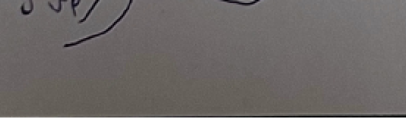


- We are studying the problem of making this integral stationary;
- The Euler-Lagrange equations come from supposing the above curve is the trajectory, the solution. Now let's vary it a bit (move it up and down) and require that the first order change in the Action is 0.
- The Action can change for 2 reasons: - The Lagrangian at x changes when you shift the curve,
  - The velocity changes

$$\Rightarrow \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} - \frac{\partial \mathcal{L}}{\partial x_i} = 0$$

coming from change of velocity

you take $\frac{d}{dt}$ because the change in the curve consist of 2 parts:



* Fun Fact: If you want to explicitly write the Lagrangian of the Standard model of particle physics, it would take a whole page

* In the context of the general/abstract notation of classical mechanics, we don't call things x; we call them q.

$$\underbrace{\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot q_i}}_{\text{"momentum conjugate to } q\text{"}} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

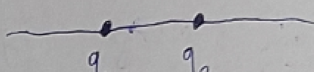
$\frac{\partial \mathcal{L}}{\partial \dot x} = \frac{\partial}{\partial \dot x}\left[\frac{m\dot x^2}{2} - V(x)\right] = m\dot x$, which is momentum, which is a more general concept. It is the variation of the derivative of the Lagrangian with respect to the time derivative of a coordinate. It's full name is:

"The canonical momentum conjugate to the coordinate q_i"

* Some random guy just woke up and decided that p's and x's aren't nice and decided to replace them with q's

$$\Rightarrow \frac{d}{dt}p_i = \frac{\partial \mathcal{L}}{\partial q_i} = \dot p_i$$

* Consider $\mathcal{L} = \frac{\dot q_1^2 + \dot q_2^2}{2} - V(q_1 - q_2)$ ← potential energy depending on displacement



$\dot p_1 = -V'(q_1 - q_2)$ $\quad$ ($-V'(q_1-q_2) = \frac{\partial \mathcal{L}}{\partial q_1}$; F' means the derivative of F with respect to it's argument)

$\dot p_2 = V'(q_1 - q_2)$ $\quad$ (When q_2 varies, $(q_1 - q_2)$ varies in the opposite direction as to when q_1 varies. An additive/positive increase in q_1 corresponds to an additive increase in $(q_1 - q_2)$, but if we have a positive additive increase to q_2, then $(q_1 - q_2)$ decreases)

$$\dot p_1 + \dot p_2 = 0$$

$$\frac{d}{dt}(p_1 + p_2) = 0$$ ← conservation law

* Consider the case where $V(aq_1 + bq_2) \Rightarrow \begin{cases} (\dot p_1 = -aV'(aq_1 + bq_2)) \times b \\ (\dot p_2 = -bV'(aq_1 + bq_2)) \times a \end{cases}$

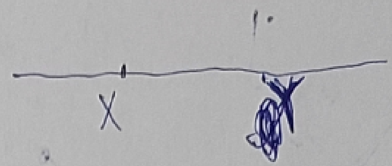
$$\begin{cases} b\dot p_1 = -abV'(aq_1 + bq_2) \\ a\dot p_2 = -abV'(aq_1 + bq_2) \end{cases} \quad -$$

$$b\dot p_1 - a\dot p_2 = 0$$

$$\frac{d}{dt}(bp_1 - ap_2) = 0$$ ← conservation law

* What could a and b be?

- Suppose you have 2 particles of masses m and M



$$\mathcal{L} = \frac{m\dot{x}^2}{2} + \frac{M\dot{y}^2}{2} - V(x-y)$$

position not ~~coord~~ direction

Let $q_1 = \sqrt{m}\,x$, $q_2 = \sqrt{M}\,y$ $\Rightarrow \mathcal{L} = \frac{\dot{q}_1^2}{2} + \frac{\dot{q}_2^2}{2} - V\left(\frac{q_1}{\sqrt{m}} - \frac{q_2}{\sqrt{M}}\right)$

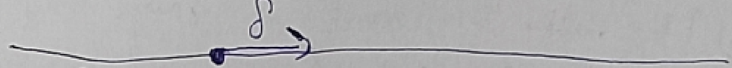
$\left(a = \frac{1}{\sqrt{m}} ; b = \frac{1}{\sqrt{M}}\right)$

The kinetic terms come up simple this way, at the cost of additional constants in V.

* The simplest symmetry; $\mathcal{L} = \frac{\dot{q}^2}{2}$ (One free particle of mass = 1)

- What is meant by symmetry?
  A coordinate change in q that does not affect the Lagrangian -
- We have 2 ways to change the coordinate of an object in a system;
  - Passive coordinate transformation; moving the center of the coordinates.
  - Active coordinate transformation; moving the object.
- Lets think of a change in q (δq) and set it equal to a number δ

$$\delta q = \delta$$



- $q' = q + \delta q = q + \delta$ (q' is coordinate after transformation)

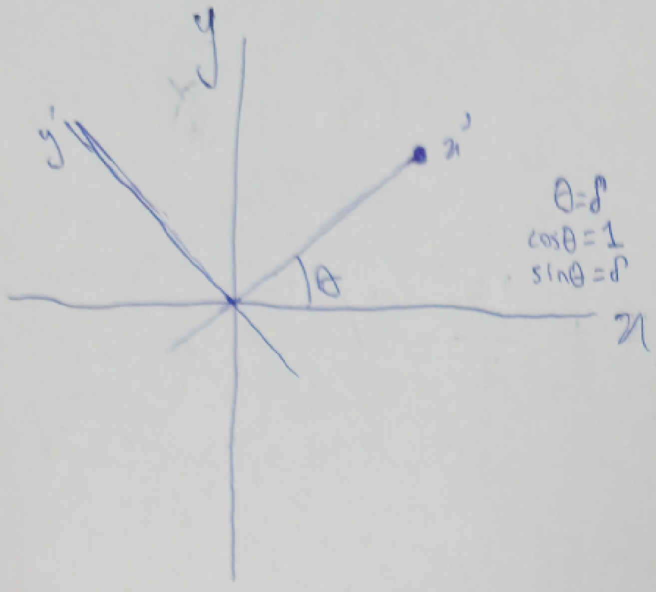
$\Rightarrow \dot{q}' = \frac{d}{dt}q + \cancel{\frac{d}{dt}\delta}^{\,0} = \dot{q}$ (does not change)

$\Rightarrow \delta\mathcal{L} = 0$

This is a symmetry. Symmetry ~~means~~ is when the change in the Lagrangian is 0 after a coordinate change. The Lagrangian has the same form if we change coordinates.

- We also have a conservation laws $\dot{p} = 0$ (the canonical momentum conjugate to q is conserved)

*We are done with Translational Symmetry. Let's do Rotational symmetry



$\theta = \delta$
$\cos\theta = 1$
$\sin\theta = \delta$

$$\mathcal{L} = \frac{m}{2}\left[\dot{x}^2 + \dot{y}^2\right] - V(x^2+y^2)$$

$$x' = x\cos\theta + y\sin\theta$$
$$y' = -x\sin\theta + y\cos\theta$$

$$\Rightarrow x' = x + y\delta$$
$$y' = y - x\delta$$

* where δ is an infinitesimally small quantity

$$\Rightarrow \delta x = y\delta \Rightarrow \delta\dot{x} = \dot{y}\delta$$
$$\delta y = -x\delta \qquad \delta\dot{y} = -\dot{x}\delta$$

Change in x^2+y^2:

$$\delta(x^2+y^2) = 2x\delta x + 2y\delta y$$
$$= 2xy\delta - 2yx\delta$$
$$= 0$$

$\Rightarrow$ This is saying that the distance from the Origin doesn't change when we rotate the coordinates.

$$\Rightarrow \delta V = 0$$

Do the same for $\delta(\dot{x}^2+\dot{y}^2)$. We find out it is equal to 0 $\Rightarrow$ We have a symmetry.

* General Definition

$\delta q_i = f_i(q)\delta$; where f is a function that changes the coordinates. It might depend on the position too

$$\delta\dot{q}_i = \frac{d}{dt}\delta q_i$$

$$\delta\mathcal{L}(q_i\,\dot{q}_i) = \sum_i \underline{\frac{\partial\mathcal{L}}{\partial q_i}}\delta q_i + \underline{\frac{\partial\mathcal{L}}{\partial\dot{q}_i}}\delta\dot{q}_i$$
$$= \sum_i \dot{p}_i\delta q_i + p_i\delta\dot{q}_i$$
$$= \frac{d}{dt}\sum_i p_i\delta q_i$$
$$= \frac{d}{dt}\sum_i p_i f(q)\delta$$

If we have a symmetry:

$$\delta L(q_i \, \dot{q}_i) = 0$$

$$\frac{d}{dt} \sum_i p_i f(q) \delta = 0$$

$$\delta \cdot \frac{d}{dt} \sum_i p_i f(q) = 0$$

$$\frac{d}{dt} \Big[\underbrace{\sum_i p_i f(q)}_{Q\text{: conserved quantity}} \Big] = 0$$

Then if we have a symmetry of type: $\delta q_i = f_i(q) \delta$,

$$Q = \sum_i p_i f_i(q)$$

- If we go back to previous example where $\delta q_1 = \delta$, $\delta q_2 = \delta$

$$Q = \sum_i p_i \overbrace{f_i(q)}^{1} = p_1 + p_2 \checkmark$$

* $f_i(q)$ is the coefficient of δ

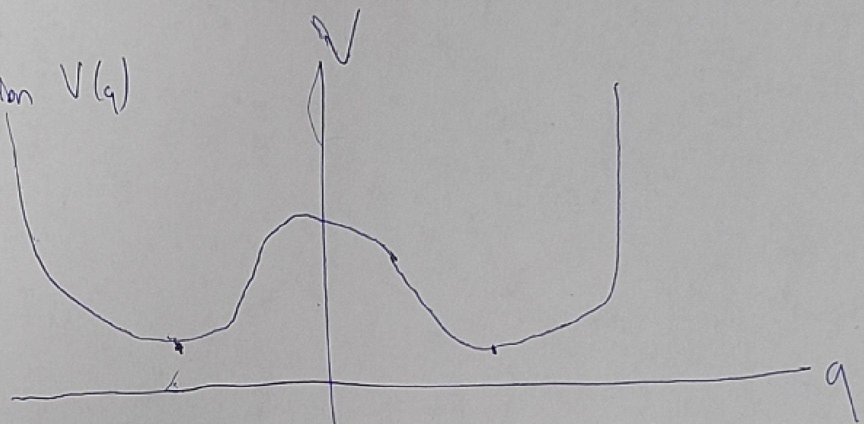
- If $\delta q_1 = b p_1$; $\delta q_2 = -a p_2$

$\Rightarrow Q = b p_1 - a p_2 \checkmark$ which is exactly the quantity we found earlier.

- If $\delta x = y \delta$; $\delta y = -x \delta$ (rotated coordinates example)

$\Rightarrow Q = \sum_i p_i f_i = p_x y - p_y x =$ angular momentum

* Consider a potential energy function $V(q)$



$V(q) = V(-q)$

We are reflecting in the vertical axis. There is no notion of building up this reflection by a lot of little transformations.

Thus, this type of symmetry in classical mechanics does not have a conservation law (There is one in QM)

This type of symmetry would be called "discrete symmetry".

The other ones that can be built up from little incremental steps are called "continuous symmetries."

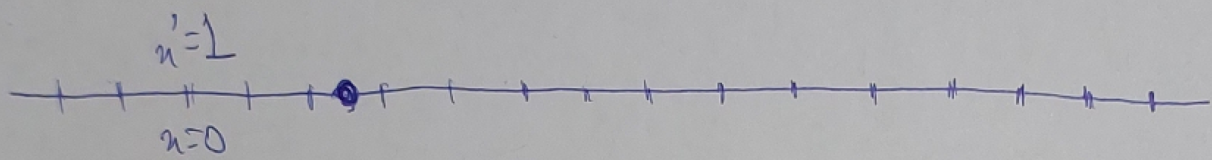
Question from me:

Can't the reflection be thought about as the sum of little incremental rotations with respect to the origin on the q-z plane, where z is the 3rd dimension?

Lecture 5:

- Make symmetries clearer.
- Difference between Active and Passive Transformations
- Energy conservation
- Hamiltonian and Examples
- Non-standard Lagrangian

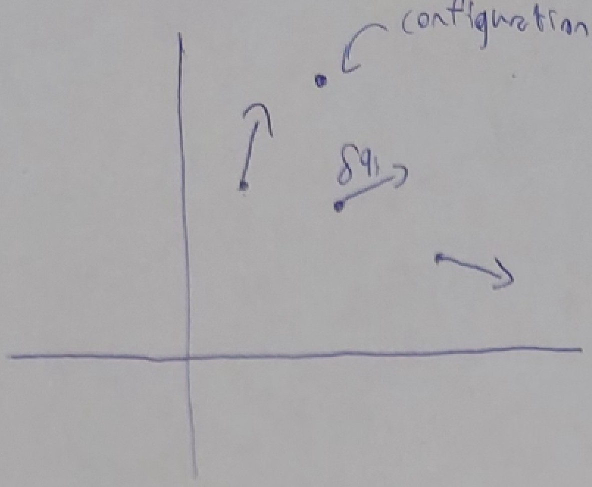
Passive Vs. Active Transformations:



"Perform the transformation $x \longrightarrow x+1$"

- Imagine the line is coordinatized (every point is labelled)
- It could mean 2 things: - "Relabel the point so that every point labelled x becomes $x+1$
  (If there was a potential energy, it wouldn't change because the particle hasn't moved
  - Take the particle from where it is, and move it. In this case, the potential energy changes.

It is easier to think about Active transformations.



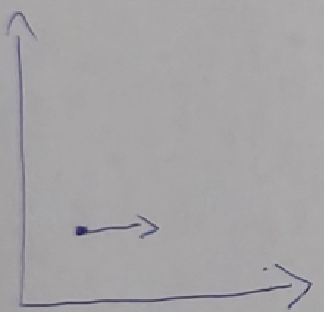
$$\delta q_i = f_i(q)\,\delta$$

q-space

The shift may depend on where you are. We also assume that f_i does not depend on time. if $x \longrightarrow x + \delta$

$\dot{x} \longrightarrow \dot{x}$ (velocity does not change)

If the Lagrangian at all points doesn't change when you make a transformation, then that is a symmetry.



If the potential energy doesn't change when you shift to x only, then it depends on y. $(V(y))$

If it doesn't change when you shift y, then it might depend on x. $(V(x))$

If it doesn't change for any transformation in x and y, then it is a constant $(V = k;\ \frac{\partial V}{\partial x} = 0)$

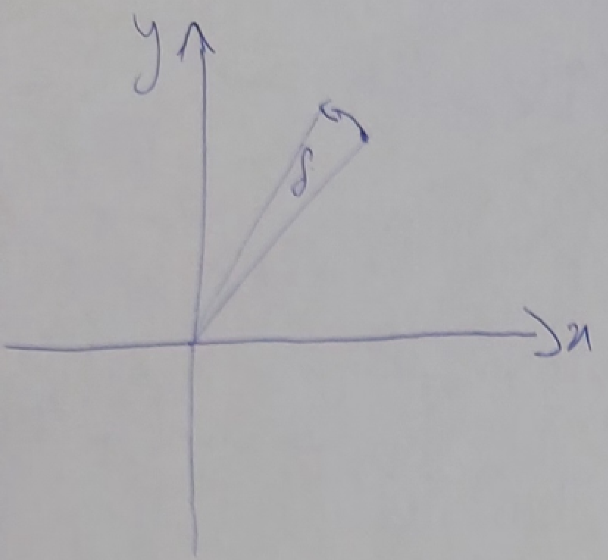
The conserved quantity is

$$Q = \sum_i p_i\, f_i(q)$$

Now we see why we do not restrict f_i to be very small but multiply it by δ.

Rotation:

Rule: $\delta \dot{q} = \frac{d}{dt} \delta q$



$$\delta x = -y \delta$$
$$\delta y = x \delta$$

If this is a rigid rotation, it shouldn't change the length of the vector.

$$r^2 = x^2 + y^2$$

$$\delta r^2 = 2x\delta x + 2y \delta y = -2xy\delta + 2yx\delta = 0$$, so it doesn't change.

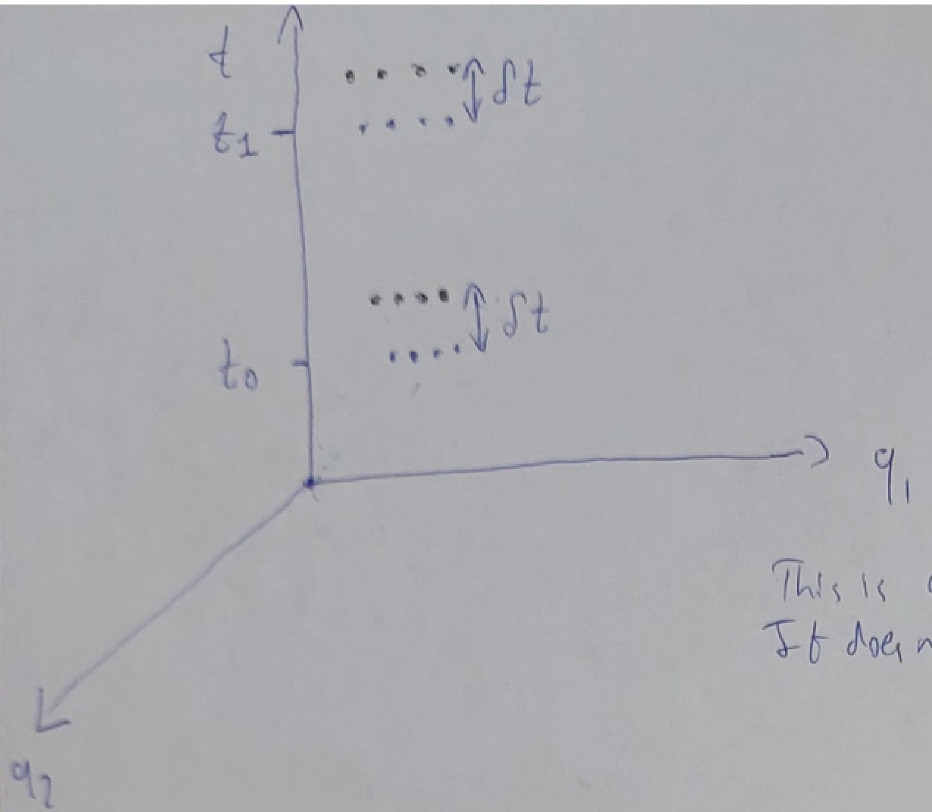
Assuming the Lagrangian doesn't change (V depends on the radial distance, and T doesn't change):

$$Q = -y p_x + x p_y = [\vec{r} \times \vec{p}]_z \quad \text{(cross product)}$$

Angular momentum.

* Is energy conservation associated with a symmetry?

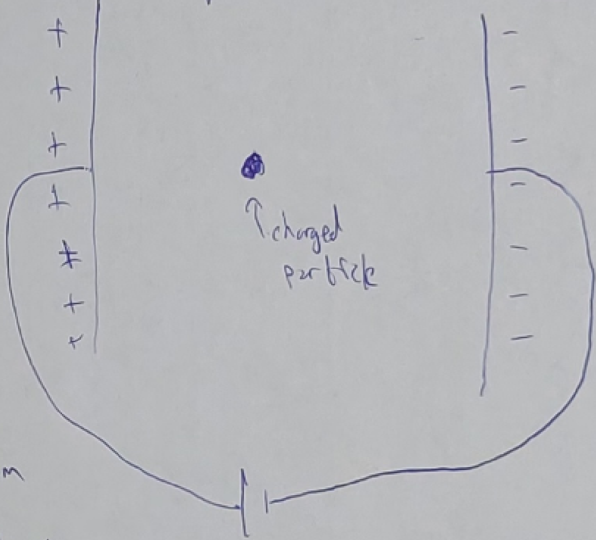
Yes, but not of the types we've seen till now. It is not a shift of the q's



There is an experiment and we time-translated it. We started later and finished later.

This is a time translation invariant experiment. It does not depend on time.

An example of a time translation variant experiment: Capacitor



There will be a Lagrangian and a potential energy of the particle that is proportional to the distance. The particle will accelerate. If the charges on the plates were constant in time, the experiment would be time-translation invariant. But if we connect the plates to a generator, starting from no charge on them and slowly ramp up the charge (in a way that we have full control over), the experiment would stop being time-translation invariant.

Another experiment is a spring put into an electromagnetic field. It will affect the atoms and interatomic spacing, therefore affecting the spring constant.

Time-translation invariance implies energy conservation.
If there is no time-translation invariance, then energy is being transferred between the system being studied and some external system.

Time translation invariance and Lagrangians:

Generalizing: By looking at the Lagrangian, how do we know if it has time-translation invariance?

If the Lagrangian is in the form:

$L(q, \dot{q})$: no explicit dependence on time in the parameters. Therefore, it has time-translation invariance. (Even if q and $\dot{q}$ implicitly depend on time)

$L(q, \dot{q}, t)$: no time-translation invariance

Ask yourself the question: "If you know q and $\dot{q}$, do you know the Lagrangian, or do you also need to know the time?"

$$\boxed{\frac{\partial L}{\partial \dot{q}} = p} \quad \text{by definition}$$

and since $\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0$ (Euler-Lagrange Equation)

$$\boxed{\frac{\partial L}{\partial q} = \frac{d}{dt} p = \dot{p}}$$

$$\frac{dL(q_i, \dot{q}_i)}{dt} = \sum_i \left[\frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i\right]$$

$$= \sum_i \left[\dot{p}_i \dot{q}_i + p_i \ddot{q}_i\right]$$

$$= \frac{d}{dt}\left[\sum_i p_i \dot{q}_i\right]$$

$$\Rightarrow \frac{dL}{dt} = \frac{d}{dt}\sum_i p_i \dot{q}_i$$

$$\frac{d}{dt}\left[L - \sum_i p_i \dot{q}_i\right] = 0$$

conserved quantity

$$L - \sum p_i \dot{q}_i = -H$$

$$\boxed{H = \sum p_i \dot{q}_i - L}$$

Take an example:

$$L = \frac{m\dot{x}^2}{2} - V(x)$$

$$p = m\dot{x}$$

$$\Rightarrow H = m\dot{x}\dot{x} - \left[\frac{m}{2}\dot{x}^2 - V(x)\right]$$

$$\boxed{H = \frac{m\dot{x}^2}{2} + V(x) = E}$$

If there is time dependence:

$$\frac{dL}{dt}(q_i, \dot{q}_i; t) = \sum_i \frac{\partial L}{\partial q_i} q_i + \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i + \frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{dL}{dt} - \frac{d}{dt}\sum p_i \dot{q} = \frac{\partial L}{\partial t} \leftarrow \text{differentiating the parameters}$$

accounts for changes in q and $\dot{q}$ (the true derivative)

$$-\frac{dH}{dt} = \frac{\partial L}{\partial t}$$

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

In the Lagrangian: $L = \frac{m}{2}\dot{x}^2 - \frac{k(t)}{2}x^2$: $\frac{\partial L}{\partial t} = -\frac{\dot{k}}{2}x^2$

$$\Rightarrow \frac{dH}{dt} = \frac{\dot{k}(t)}{2}x^2$$

For any value of x other than $x = 0$, the time derivative of the Energy is non-zero

* When the Lagrangian is written of the form $T - V$, then the Hamiltonian is $T + V$.

When it is written in an unfamiliar form, the Hamiltonian tells you what the rule is for the definition of energy. If you treat the whole system, energy would be conserved.

*Summary:

$L(q, \dot{q}, t)$

$p = \frac{\partial L}{\partial \dot{q}}$

$\dot{p} = \frac{\partial L}{\partial q}$

If we have a symmetry of type: $\delta q_i = f_i(q)\,\varepsilon$

$$\Rightarrow Q = \sum_i p_i f_i(q) \; ; \; \dot{Q} = 0$$

Generally:

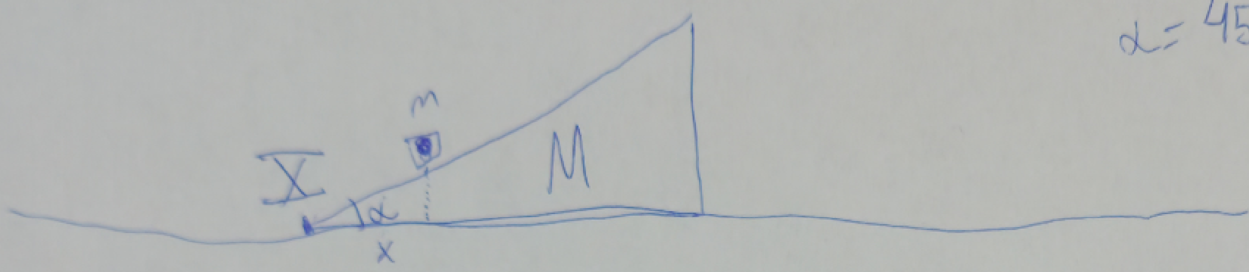
$$\frac{dL}{dt} = \sum_i \frac{d}{dt} p\dot{q} + \frac{\partial L}{\partial t}$$

where $H = \sum p\dot{q} - L$

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

Lecture 6

- Example of a wedge
- Example of a double pendulum
- Hamiltonian and Harmonic oscillator in Phase space.
- Hamilton Equations



$\alpha = 45°$

m is constrained to move along the hypotenuse of M. M is sliding on a frictionless surface. Find the motion of this system (motion equations for both)

Step 1: Assign a set of sufficient coordinates

X is the position of the wedge.

x is the position of the particle ~~with respect~~ relative to the corner

Step 2: Write down the Lagrangian $(T-V)$:

$V_{m_x} = \dot{X} + \dot{x}$ (full velocity which is the rate at which x changes + velocity of the wedge)

$\tan\alpha = \frac{y}{x} \Rightarrow y = x\tan 45 = x$

$V_{m_y} = \dot{y} = \dot{x}$

$$T = \frac{M}{2}\dot{X}^2 + \frac{m}{2}(\dot{X}+\dot{x})^2 + \frac{m}{2}\dot{y}^2 = \frac{M}{2}\dot{X}^2 + \frac{m}{2}(\dot{X}+\dot{x})^2 + \frac{m}{2}\dot{x}^2$$

$V = GPE = mgy = mgx$ (the wedge is not moving vertically so it's $V=0$)

$$\mathcal{L} = \frac{M}{2}\dot{X}^2 + \frac{m}{2}(\dot{X}+\dot{x})^2 + \frac{m}{2}\dot{x}^2 - mgx$$

$$P_X \neq M\dot{X}$$

$$P_X = \frac{\partial L}{\partial \dot{X}} = M\dot{X} + m(\dot{X}+\dot{x})$$

$$P_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + m(\dot{X}+\dot{x})$$

There is a translation symmetry (if we move the whole system)

$$\delta X = \varepsilon \quad ; \quad \delta x = 0$$

$$Q = \sum_i p_i f_i(q) = 1 P_X + 0 P_x = P_X$$

$$\dot{P}_X = \frac{\partial L}{\partial X}$$; X does not appear in the Lagrangian. It is a cyclic variable.

$$\dot{P}_X = 0$$ (conserved quantity Q)

There is no symmetry with respect to vertical motion. The wedge isn't allowed to move up and down, and when the particle moves up and down, V changes, and so does the Lagrangian. Therefore no symmetry.

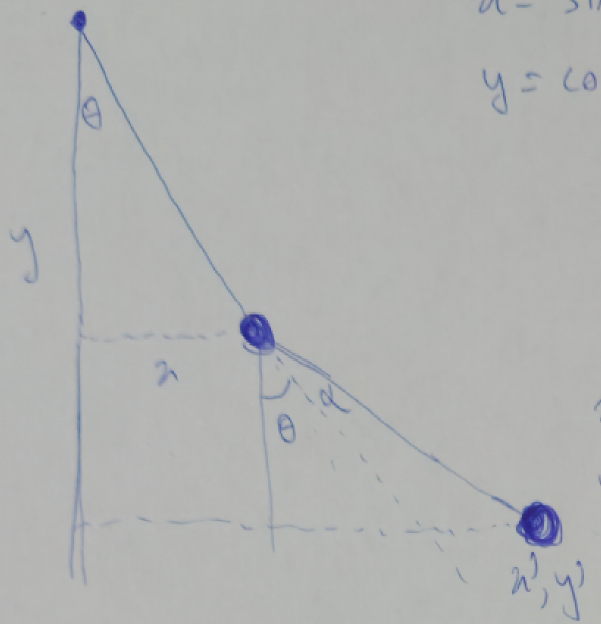
$$\dot{P}_x = \frac{\partial L}{\partial x} = -mg$$

Double Pendulum:

Take length of strings and masses of bobs $= 1$

$x = \sin\theta \qquad \dot{x} = \dot{\theta}\cos\theta$

$y = \cos\theta \qquad \dot{y} = -\dot{\theta}\sin\theta$



$x' = x + \sin(\theta+\alpha) = \sin\theta + \sin(\theta+\alpha)$

$y' = y + \cos(\theta+\alpha) = \cos\theta + \cos(\theta+\alpha)$

$\dot{x}' = \dot{\theta}\cos\theta + (\dot{\theta}+\dot{\alpha})\cos(\theta+\alpha)$

$\dot{y}' = -\dot{\theta}\sin\theta - (\dot{\theta}+\dot{\alpha})\sin(\theta+\alpha)$

$T = \dot{\theta}^2 + \frac{(\dot{\theta}+\dot{\alpha})^2}{2} + \dot{\theta}(\dot{\theta}-\dot{\alpha})\cos\alpha$, assuming no gravitational field

θ does not appear, therefore it is a cyclic coordinate

$\Rightarrow \dot{P}_\theta = 0$

$V = -mg\cos\theta - mg(\cos(\alpha+\theta)+\cos\theta)$

$= -2g\cos\theta - g\cos(\theta+\alpha)$

- In Hamilton's equations of motion, the focus is not on the q and $\dot{q}$'s (as in the Lagrangian), but on the q's and p's (momenta conjugate to q's).
- The number of things you need to know to get a mechanical system started is twice the number of degrees of freedoms (twice the number of q's, p's, or $\dot{q}$'s).
- In general, for good mechanical systems:

$\dot{q} = q(q\,p)$; $\dot{q}$'s can be solved for in terms of q's and p's

- The Hamiltonian is thought of as a function of the p's and the q's

$$H = \sum p\dot{q} - L$$

$$H(p\,q) = H$$

- Let's take an example: the Harmonic Oscillator:

$$L = \underbrace{\frac{1}{2\omega}\dot{q}^2}_{T} - \underbrace{\frac{\omega}{2}q^2}_{V} \quad \text{where } \omega \text{ is a parameter}$$

$$p = \frac{\partial L}{\partial \dot{q}} = \frac{\dot{q}}{\omega} \Rightarrow \dot{q} = \omega p$$

$$H(p\,q) = p \cdot \omega p - \frac{1}{2\omega}(\omega p)^2 + \frac{\omega}{2}q^2$$

$$= \omega p^2 - \frac{p^2\omega}{2} + \frac{\omega}{2}q^2$$

$$= \frac{\omega}{2}(p^2 + q^2)$$

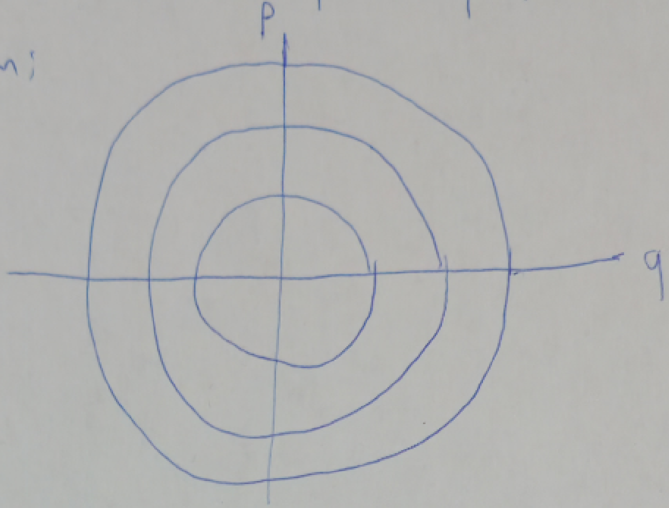
* When you are doing the Hamiltonian version of classical mechanics, you get rid of the $\dot{q}$'s by solving for them.
* Sometimes, you can't solve equations, because they don't have solutions.
* You can write Lagrangians for which there will not be solutions. They are bad

systems. They are systems ~~that~~ which cannot come from Quantum Mechanics.

* Unless it is possible to solve for the velocities in terms of the momenta, it would not be considered as a legitimate Lagrangian.

- The Hamiltonian is the energy ~~and~~ and it is proportional to q^2+p^2.
- Let's draw the phase space (space of q's and p's)

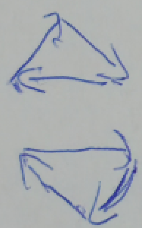
For one degree of freedom;



The surfaces/trajectories are circles of different radii. Different radii means different values of p^2+q^2 and therefore different values of energy.

Whatever value of q^2+p^2 ~~is~~ when you begin, it must stay that (it must stay in ~~the~~ a circle), because we know this value is proportional to the energy and energy better be conserved.

- Remember the example from Lecture 1 (of conservation laws):



For the die.

If you are on a cycle, you'll stay in it. If the laws of physics divide the space up into separate cycles, then there is a conservation law.

- We will derive Hamilton's equations:

We use a standard trick of calculus:

$$\delta F(q p) = \frac{\partial F}{\partial q}\delta q + \frac{\partial F}{\partial p}\delta p$$

If you have $\delta F(qp)$ given in the form: $A\,\delta q + B\delta p$.

$\Rightarrow$ You know that $A = \frac{\partial F}{\partial q}$ and $B = \frac{\partial F}{\partial p}$

* We know that $H = \sum \dot{q} p - L$

$$\therefore\ \delta H = \sum \dot{q}\delta p + p\delta\dot{q} - \frac{\partial L}{\partial \dot{q}}\delta\dot{q} - \frac{\partial L}{\partial q}\delta q$$

$$= \sum \dot{q}\delta p + \cancel{p\delta\dot{q}} - \cancel{p\delta\dot{q}} - \dot{p}\,\delta q$$

$$= \sum \dot{q}\delta p - \dot{p}\,\delta q$$

$$\Rightarrow \boxed{\begin{aligned}\frac{\partial H}{\partial p_i} &= \dot{q}_i \\ \frac{\partial H}{\partial q_i} &= -\dot{p}_i\end{aligned}}$$

Two first order equations

(Lagrangians are 2nd order, and there are half as many of them)

* For the harmonic oscillator: $H = \frac{\omega}{2}(p^2 + q^2)$

$$\left.\begin{aligned}(1)\quad \frac{\partial H}{\partial p} &= \omega p = \dot{q} \\ (2)\quad \frac{\partial H}{\partial q} &= \omega q = -\dot{p}\end{aligned}\right\}\ \text{equations of motion}$$

The familiar equations of motion are second degree:

$\Rightarrow \ddot{q} = \omega\dot{p}$ (Differentiate (1) W.R.T time)

$\ddot{q} = -\omega^2 q$ (substitute $\dot{p} = -\omega q$ from (2))

Note that: $q \neq x$

it is an abstract coordinate. If you plot p against x (not q), you will get ellipses, not circles

Lecture 7:

- History of Classical Mechanics and what Lagrange was looking for
- Hamilton derivation of the conservation of energy
- Flow in phase space
- Divergence of the phase space flow
- Liouville's Theorem and examples
- Poisson Brackets

* Consider a system described by: $H(q_i, p_i)$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$\frac{dH(q_i,p_i)}{dt} = \sum \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial q_i}\dot{q}_i$$

$$= \sum -\frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q_i} + \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i}$$

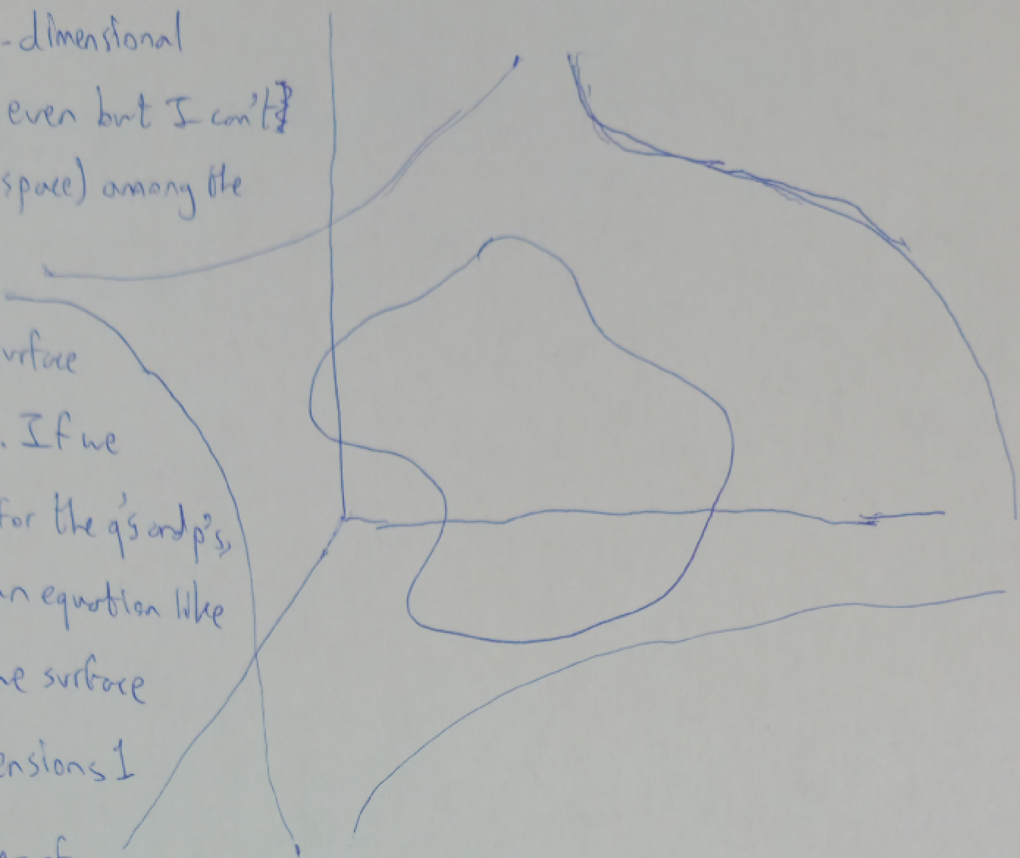
$$= 0$$; thus energy is conserved

If the number of q's is n, then the total number of dimensions in the space is $2n$.

$H(q_i p_i) = E$; where E is a number that just happens to be the total energy of the system.

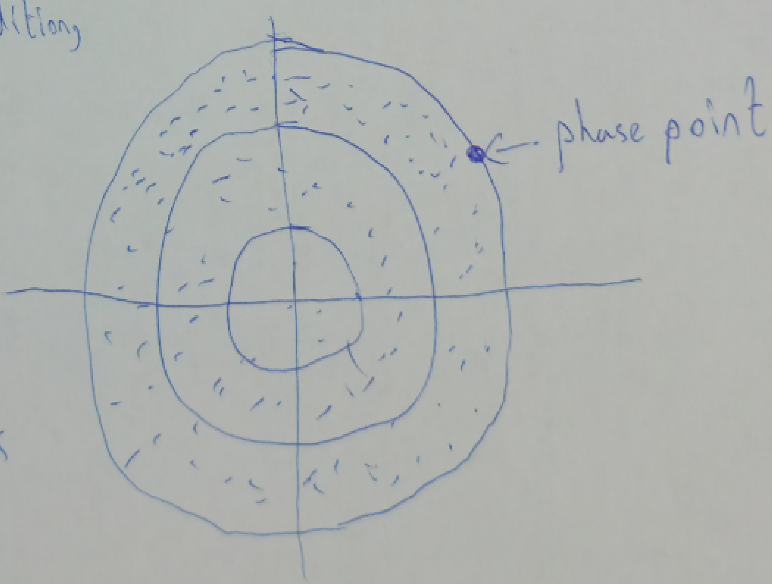
The solutions of this equation are just some contours (surfaces) in the phase space. of constant value (initial E)

One equation in an n-dimensional space (it has to be even but I can't draw a 4-dimensional space) among the collection of unknowns /variables gives you a surface of 1 less dimension. If we have 2N dimensions for the q's and p's, and we write down an equation like that, it defines some surface in the space (of dimensions 1 less than the total number of dimensions. That is what writing one equation does. "Each value of E corresponds to a surface in the phase space of one lower dimension"



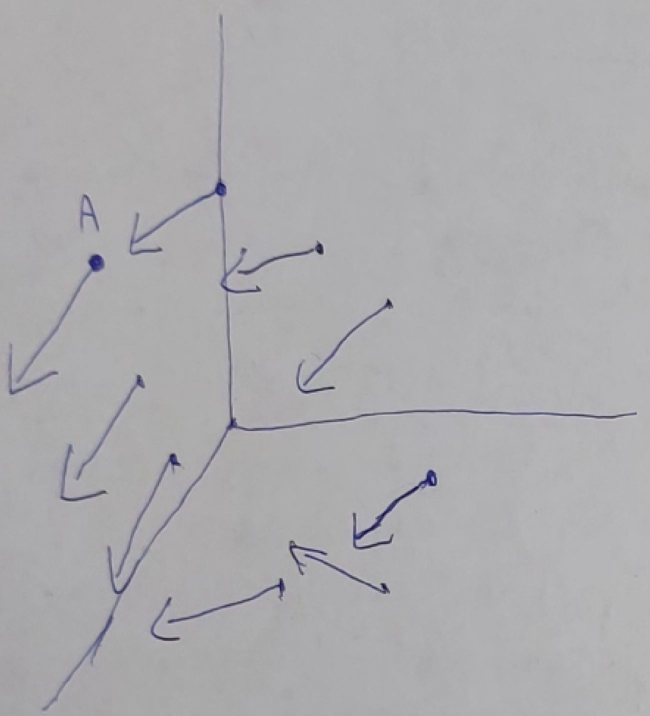
The way the system moves through phase space stays on surfaces of constant H. Once the Energy is fixed, the system will continue to travel on the same surface. An example is the harmonic oscillator: $H = \frac{\omega}{2}(p^2+q^2)$

- Instead of taking a given initial condition let's imagine all possible starting points and indicate them by points on the plane.
- The sprinkling of points can be thought about as a fluid, an imaginary fluid composed of a lot of dust (fluid dust) where all particles move in a way which is dictated by the two equations.



* More generally, for an arbitrary phase space (always even dimensional), imagine that the phase space is filled up with dust. The dust corresponds to every ~~single~~ possible starting point, then start the clock and the dust moves through the phase space and defines a flow of an imaginary fluid. We are interested in a property of the fluid: Having a divergence or convergence of the flow.

- Flow vectors:

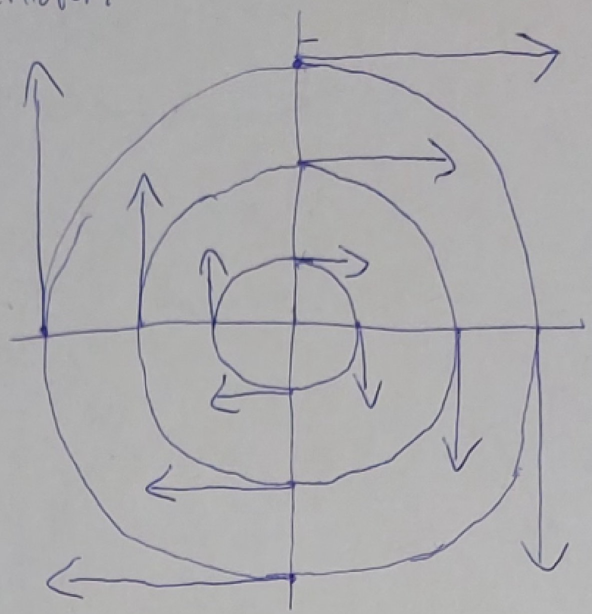


Points move. Their motion only depends on where they are in the phase space. The components of position / location in the phasespace are q and p. The components of "velocity" (not the honest velocity of mechanics $\dot{q}$, but the velocity in phase space) are $\dot{q}$ and $\dot{p}$. We can make a vector out of them:

$$\begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{p}_i \\ \vdots \end{pmatrix}$$

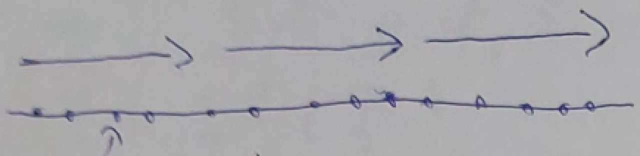
can be thought of as a vector in the phasespace showing the direction of flow. Flow is the $\dot{p}$'s and $\dot{q}$'s and they are given by known functions of "position" in the phase space (not q only), if we know the Hamiltonian. So it means that whenever a point in the flow gets to point A, it always has the same velocity, which is determined by the value of H, and its derivatives at point A. This means we can take the space and fill it up with a vector field (a vector at every position)

For the system describing the Harmonic oscillator:



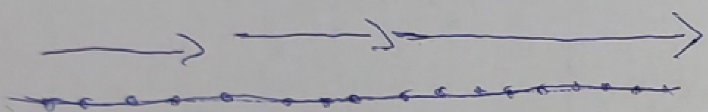
The question to ask about the fluid is whether it is compressible.

We will start with a one-dimensional example:



points in the Fluid

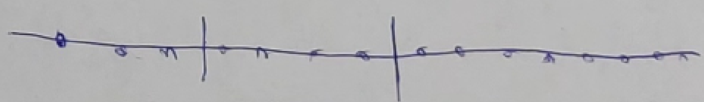
The fluid moves and what we want to know is whether the points get squeezed together or spread apart (whether the fluid is compressible). If a fluid is strictly compressible, it has a constant density (you cannot change the distance between neighboring particles.



If velocity varies along the axis, it's clear that in the region the "jump" in velocity took place, the points will have to be spreading apart

* The only way to have an incompressible fluid in one dimension is for the velocity to be absolutely constant and independent of where you are. (It can depend on time but we're assuming explicitly that H doesn't depend on time)

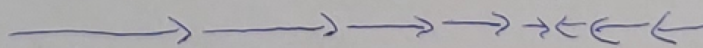
$$\frac{d\vec{v}}{dx} = 0$$



* The conditions for an incompressible fluid can be thought of in 2 different ways.
  - Require ~~that~~ the same amount that enters ~~passes~~ the region from the left goes out from the right
  - Stick to the region and follow it as it moves. It's volume should not change.

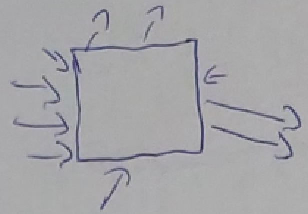
If $\frac{dv}{dx} > 0$:

If $\frac{dv}{dx} < 0$:



If $\frac{dv}{dx} = 0$:

In case of 2 dimensions:



(flowing in = going out)

$V_x(x y)$
$V_y(x\ y)$

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0$$ (similar cases for more dimensions)

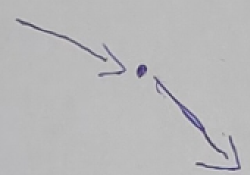
- 3 dimensions: $$\underbrace{\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}} = 0$$

This quantity has a name: The divergence of the vector field: $\vec{\nabla} \cdot \vec{V}$

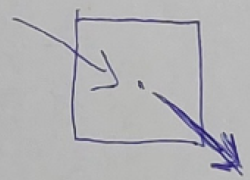
* Let's return to Lecture 1:

The arrows are very similar to the idea of flow in phase space. They tell you where to go next:

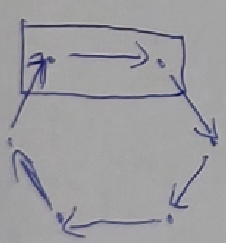
- What is true of a good law is that at every point, what flows in must flow out:
If you are at a point, you'll know where you'll be next, and where you were before.



- Sounds a bit familiar?



Example of good laws



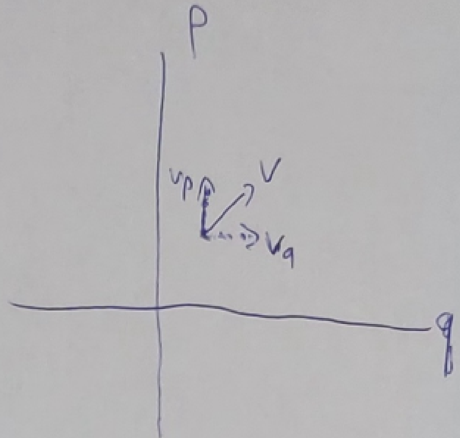
If you take 2 points, and follow them, they stay 2 points. The volume is conserved if you follow the system.
Or If we focus on a particular region/node, the number of arrows in is the same as the number of arrows out

- The question is: Is the flow of the fluid in phase space, dictated by Hamilton's equations, is ~~compressible~~ such that the fluid is incompressible?

Liouville's Theorem: Says that the flow in phase space is incompressible, and it is the closest analog that we have of the idea that as many points come in as go out.

$$V_p = \dot{p} = -\frac{\partial H}{\partial q}$$

$$V_q = \dot{q} = \frac{\partial H}{\partial p}$$



The divergence is calculated as:

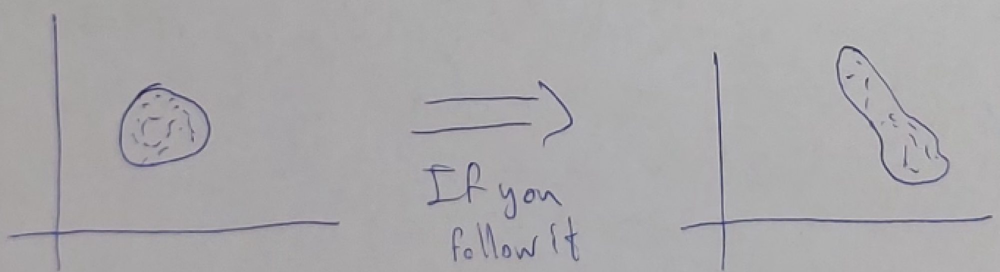
$$\vec{\nabla} \cdot \vec{V} = \frac{\partial V_p}{\partial p} + \frac{\partial V_q}{\partial q}$$

$$= -\frac{\partial}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial}{\partial q}\frac{\partial H}{\partial p}$$

$$= -\frac{\partial}{\partial q}\frac{\partial H}{\partial p} + \frac{\partial}{\partial q}\frac{\partial H}{\partial p} \quad \text{(property of partial derivatives)}$$

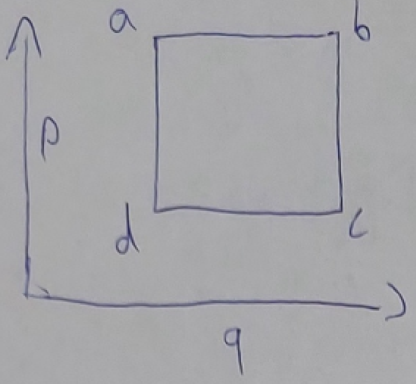
$$= 0$$

This is Liouville's theorem and it is a consequence of Hamilton's equations.



Same volume

So the particles won't coalesce and lose memory



a, b, c, d are defined by some function F.

$\frac{\partial F}{\partial q}$ at the bottom is $F(c)-F(d)$

$\frac{\partial F}{\partial q}$ at the top is $F(b)-F(a)$

so $\frac{\partial}{\partial p}\frac{\partial F}{\partial q} = F(b)-F(a)-F(c)+F(d)$

(Vertical difference of 2 horizontal differences)

$\left.\begin{array}{l}\frac{\partial F}{\partial p} \text{ at the left is } F(a)-F(d) \\ \frac{\partial F}{\partial p} \text{ at the right is } F(b)-F(c)\end{array}\right\} \Rightarrow \frac{\partial}{\partial q}\frac{\partial F}{\partial p} = F(b)-F(c)-F(a)+F(d)$

$\therefore \frac{\partial}{\partial p}\frac{\partial F}{\partial q} = \frac{\partial}{\partial q}\frac{\partial F}{\partial p}$ so the order of differentiation doesn't matter if p and q are independent

* Take a box of gas and expand it (almost like the universe is expanding)! The "fluid" (positions of molecules in real space) are spreading out from each other. The fluid in ordinary space is not incompressible. They are streching away from each other. What this means is that in momentum space, the points must be getting closer together. Think about it: what happens to the momenta of particles in a gas if you expand the gas? It goes down. If the gas compresses, the momenta increase

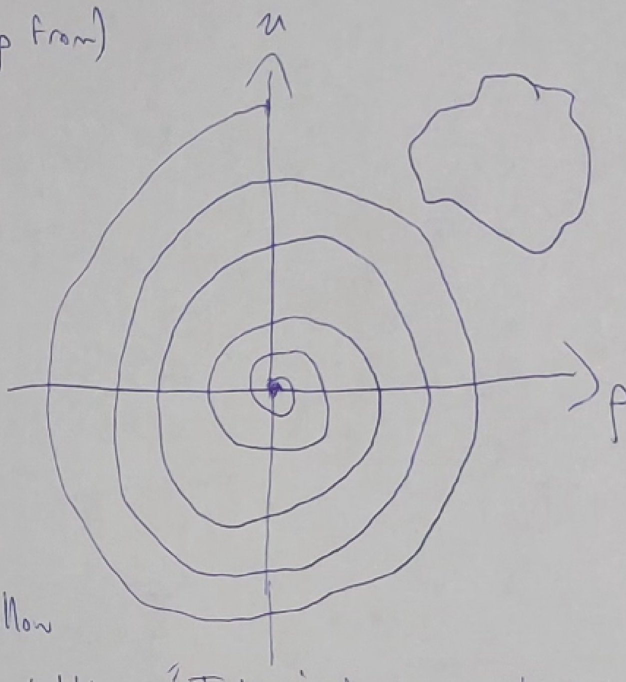
Example that doesn't follow Liouville's theorem:

We've already proved it, so how can we provide an example that doesn't follow it?
The theorem is a consequence of Hamilton's equations, so if we provide a system that doesn't have a proper Hamiltonian / Lagrangian, we can say it doesn't follow Liouville's theorem.
The system is a damped harmonic oscillator (harmonic oscillator in a fluid):

$$m\ddot{x} = -kx - \underbrace{c\dot{x}}_{\text{velocity dependent viscosity term.}}$$

* Viscosity cannot be derived from a Hamiltonian.

Let $p = m\dot{x}$ (There is no Lagrangian to derive p from)



The viscous drag force sucks energy, therefore the radius of the orbit in phase space gets smaller with time.

(I did x vs p because I drew the curve counter-clockwise)

If you start with a patch of phase space and follow every point and they all get shrunken down to a little volume very close to the center, so this system doesn't follow Liouville's theorem.

"It gets harder and harder to distinguish the points on phase space."

$$V_x = \dot{x} = \frac{p}{m} \quad (\text{since } p = m\dot{x}) \Rightarrow m\ddot{x} = -kx - \frac{cp}{m}$$

$$V_p = \dot{p} = -kx - \frac{cp}{m} \quad \Bigg| \quad \frac{\partial V_x}{\partial x} = 0 \text{ but } \frac{\partial V_p}{\partial p} = -\frac{c}{m} \neq 0$$

$\Rightarrow$ The divergence is not 0, it is negative, so the flow vectors are pointing inwards (converging).
This velocity divergence is pushing everything into the origin.

- Work is energy that is exchanged with a system that you're not accounting for

Poisson Brackets:

$$\frac{d}{dt} F(qp) = \sum_i \frac{\partial F}{\partial q_i} \dot{q} + \frac{\partial F}{\partial p_i} \dot{p} = \sum_i \frac{\partial F}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q_i}$$

Given any two functions F and G:

$$\sum_i \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} = \{F, G\}$$

It follows that: $\dot{F}(qp) = \{F, H\}$

Lecture 8:

- Rehearsing Poisson Brackets
- Poisson Brackets Algebra
- Poisson Brackets Algebra Example
- Angular Momentum
- Symmetry and Conservation Laws
- Poisson Brackets of components of Angular Momentum
- Gyroscope

* Poisson Brackets:

$$\sum_i \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} = \{F, G\}$$

* Set of Rules for Poisson Brackets:

1.) They are anti-symmetric:

$$\{A, B\} = -\{B, A\}$$

2.) $\{A+B, C\} = \{A, C\} + \{B, C\}$

3.) $\{\lambda A, B\} = \lambda \cdot \{A, B\}$

4.) $\{AB, C\} = \{A, C\} B + A\{B, C\}$

Special Cases:

$$\{q_i, q_j\} = \{p_i, p_j\} = 0 \quad \text{For all } i \neq j \text{ and } i = j$$

Any function of p, with a function of p is zero

$$\{q_i, p_j\} = \begin{matrix} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{matrix} = \delta_{ij}$$

$$\dot{F} = \{F, H\}$$

*Let's use these rules to calculate the equations of motion of a Harmonic Oscillator:

$$H(p,q) = \frac{p^2}{2m} + \frac{k}{2} q^2$$

$$\dot{q} = \{q, H\} = \left\{q, \frac{p^2}{2m}\right\}$$

"Poissinating" q with q results in 0 so we don't include it"

$$= \frac{1}{2m} \{q, p^2\} = \frac{1}{2m} \{q, p \cdot p\}$$

$$= \frac{1}{2m} \cdot \left(\{q, p\} p + p \{q, p\} \right) = \frac{\cancel{2} p}{\cancel{2} m} \cdot \cancel{\{q, p\}}^{\,1} = \frac{p}{m}$$

$$\dot{p} = \left\{p, \frac{k}{2} q^2\right\} = \frac{k}{2} \{p, q^2\} = \frac{k}{2} \cdot 2q \cdot \{p, q\} = k \times q \times -1 = -kq$$

Note for futures: $i\hbar \{A, C\} \sim [A, C]$ ← commutator

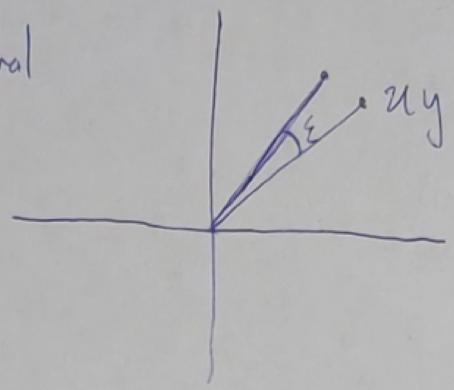
↑ Planck's constant

*Angular Momentum:

If there's a symmetry that does not change the Lagrangian / Hamiltonian of form $\delta q_i = f_i(q)\varepsilon$, then there is conserved quantity Q

where $Q = \sum_i p_i \delta q_i$

If we rotate the system with a small infinitesimal angle ε:



$$\delta x = -y\varepsilon$$

$$\delta y = x\varepsilon \qquad \delta z = 0$$

then $L = Q = -p_x y + p_y x = xp_y - yp_x$

This is the angular momentum for a particle moving in 2 dimensions.

Angular momentum is a vector in 3d. $xp_y - yp_x$ is the z-component of angular momentum. Then

$$L_z = xp_y - yp_x$$

$$L_x = yp_z - zp_y \qquad \text{So } \vec{L} = \vec{r} \times \vec{p}$$

$$L_y = zp_x - xp_z$$

If you are rotating about some arbitrary axis, just take the axis and dot product L with that direction

$\{x, L_z\} = \{x, xp_y - yp_x\} = -y\{x, p_x\} = -y$

x doesn't have a poisson bracket with x, $\underbrace{\text{nor with } p_y}$, nor with y, only with p_x

(only q's and p's with the same index give poisson brackets (Special case #2))

$\{y, L_z\} = \{y, xp_y - yp_x\} = \{y, xp_y\} = x\{y, p_y\} = x$

$\{z, L_z\} = 0$

Comparing the results with the δ's, we find out that taking the Poisson Bracket with respect to L, gives the small change in a variable when you rotate a bit.

so $\delta p_x = -p_y$

$\delta p_y = p_x$

$\delta p_z = 0$

$\{x, L_z\} = -y \qquad \{p_x, L_z\} = \{p_x, x\} p_y = -p_y$

$\{y, L_z\} = x \qquad \{p_y, L_z\} = \{p_y, -y\} p_x = p_x$

$\{z, L_z\} = 0 \qquad \{p_z, L_z\} = 0$

We found a special case of something very general:

The poisson bracket of something with an angular momentum is just the change in that something when you make a rotation.

Remember: Taking a poisson bracket of something with H, gives you the small change in that something when you change time a little bit.

- Let's try translations:

What is the quantity that is conserved by virtue of translation-invariance?

The momentum. Let's check if the poisson bracket of something with p, gives the small change in it.

$\delta q = \varepsilon, \; f = 1 \qquad \delta F = \frac{dF}{dq} \cdot \varepsilon$

Does $\{F(q), p\}$ give $\frac{dF}{dq}$ $\qquad \{F(q), p\} \overset{?}{=} \frac{dF}{dq}$

Take $F(q) = q^n$: $\{q^n, p\} = q\{q^{n-1}, p\} + q^{n-1}\{q\,p\}$

$$= q\underbrace{(n-1)q^{n-2}} + q^{n-1} = n\,q^{n-1}$$

Induction hypothesis: $\{q^{n-1}, p\} = (n-1)q^{n-2}$

$\{q, p\} = 1 \Rightarrow$ it works for the base case

$\Rightarrow$ By mathematical Induction, for any function $F = q^n$:

$$\{F, p\} = \frac{dF}{dq}$$

"When you poissinate with a momentum, you get the small change in that something when you translate q"

A symmetry operation is constructed by taking the Poisson Bracket with the quantity that is expected to be conserved.

For every symmetry, there is a conservation law. We take the conserved quantity and we take its poisson brackets with things, and that gives us the small change / the transformation

*We've talked about symmetries and how they generate things. L_z is the generator of rotations about the z-axis. They generate infinitesimal rotations about the z-axis by Poisson Bracketing. The generator of translations is momentum p. The

Generator of time translations is the Hamiltonian. To compute how something changes under a small shift of whatever you're doing, you take the Poisson Bracket.

- Let's give a name to all these conserved generators: G

$$\Rightarrow \quad \frac{dG}{dt} = \{G, H\} = 0$$

(taking time derivative is the same as taking the Poisson Bracket with H)

"G doesn't change as the system evolves"

If $\{G, H\} = 0 \Rightarrow \{H, G\} = 0$

$\Rightarrow$ H doesn't change when you do a transformation. This shows a symmetry

$\{G, H\}$ (conservation laws)

$\{H, G\}$ (symmetry)

so if you know what is conserved, you can figure out the symmetry.

If you know that G is conserved, and you want to know the transformation, just Poisson Bracket every thing, and it will tell you how it transforms.

* So all this was not just a little game of defining something so that Mr. Poisson could avoid writing down a complicated expression. The thing has real power.
* Poisson Brackets of components of Angular Momentum:

$$\{L_x, L_z\} = \{yp_z - zp_y, xp_y - yp_x\} = p_z x - zp_x = -L_y$$

so $\{L_z, L_x\} = L_y$

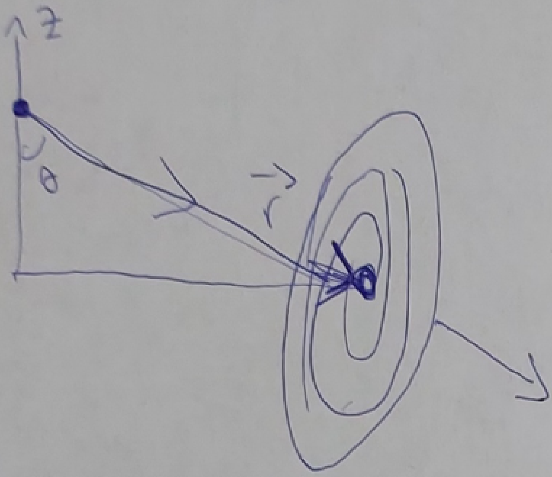
$\{L_x, L_y\} = L_z$

$\{L_y, L_z\} = L_x$

- For a gyroscope, the energy is: $H = \frac{1}{2I}\left(L_x^2 + L_y^2 + L_z^2\right)$ (If there is no external force)

$$\dot{L_x} = \{L_x, H\} = \{L_x, \tfrac{1}{2I}(L_x^2+L_y^2+L_z^2)\} = \frac{1}{2I}\{L_x, L_y^2 + L_z^2\} \quad (\{L_x, L_x\} = 0)$$

$$= \frac{1}{2I}\left(2L_y \cdot \{L_x, L_y\} + \{L_x, L_z^2\}\right) = \frac{1}{2I}\left(2L_y \cdot L_z - 2L_y L_z\right) = 0$$



- Including the torque applied by ~~...~~ its weight will add a potential energy, which is $\propto L_z$

$$V = cL_z$$

$$\{L_x, cL_z\} = -cL_y$$
$$\{L_y, cL_z\} = cL_x$$
$$\{L_z, cL_z\} = 0 \quad \text{so } \theta \text{ doesn't change}$$

The gyroscope is rotating in the ~~xz~~ xy-plane

Lecture 9:

- Electric & Magnetic Forces
- Fields
- Vector Calculus
- The del operator
- Kronecker Delta, Levi Civita Symbol
- Basic Theorems
- Gauge Transformation
- The Force Law

Mathematical Interlude:

- Field: a physical quantity which depends on space and time. (temperature at any position ...)
  This is a scalar field. A vector field is something like the wind velocity.
  The gravitational field is neither a vector nor a scalar. It is a tensor.
  Fields vary from place to place.
- The effect of a field is local. It influences things wherever the field happens to be.
- We can do various things to a field to create other fields like differentiating them.
- The derivative of any function is another function, so the derivative of any field is another field.

We introduce a fake vector: del: $\vec{\nabla}$

It is and isn't a vector.

$$\vec{\nabla} \longrightarrow \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

It never stands by itself. It always acts on something.

$$\vec{\nabla}\cdot S(x,y,z) = \left(\frac{\partial S}{\partial x}, \frac{\partial S}{\partial y}, \frac{\partial S}{\partial z}\right)$$ where S is a scalar.

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$; it is called divergence

Cross Product:

$$(\vec{V}\times\vec{A})_x = V_y A_z - V_z A_y$$
$$(\vec{V}\times\vec{A})_y = V_z A_x - V_x A_z$$
$$(\vec{V}\times\vec{A})_z = V_x A_y - V_y A_x$$

Kronecker Symbol:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$$

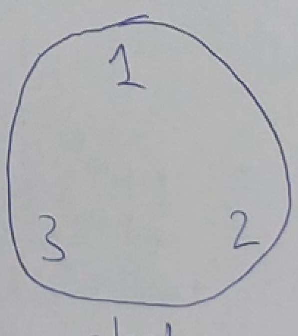
Levi-Civita Symbol:

$\epsilon_{ijk} = 0$ if any ~~pair~~ two are equal or if all three are equal

$\epsilon_{123} = 1$ $\quad \epsilon_{132} = -1$

$\epsilon_{231} = 1$ $\quad \epsilon_{321} = -1$

$\epsilon_{312} = 1$ $\quad \epsilon_{213} = -1$



clock

1 if permutation is clockwise on the clock.

-1 if permutation is counter-clockwise.

Then $(\vec{\nabla} \times \vec{A})_i = \epsilon_{ijk} V_j A_k$

$$V_i A_j - V_j A_i = \varepsilon_{ijk} (\vec{V} \times \vec{A})_k$$

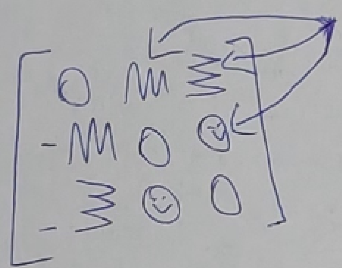
$(\vec{\nabla} \times \vec{A})$ returns the curl of the vector field, which is also a vector field

- What type of information is contained in the curl?

Divergence measures the tendency of the field to spread out away from a point, roughly speaking.

The curl measures the tendency of the field to circulate around some point.

Matrix Representation:



You have 3 independent quantities.
The other 3 are negative because it is antisymmetric

* $\nabla \cdot (\vec{\nabla} \times \vec{A}) = 0$ "The divergence of any curl is always zero".
* If and only if a field is a curl, then its divergence is zero
* If the divergence of a field is zero, then that field must be a curl.
* $\nabla \times (\nabla S) = 0$

The curl of any gradient of a scalar is zero.

- The Magnetic Field is denoted by $\vec{B}$

$$\vec{\nabla} \cdot \vec{B} = 0$$

The Magnetic field is divergence-free, therefore it is the curl of something; The Vector Potential.

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

B is the real thing. A is not unique. It is just a vector field whose curl is the magnetic field

$$\text{let } \vec{A}' = \vec{A} + \nabla S \leftarrow \text{This is a gauge transformation}$$

$$\vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \vec{A} + \vec{\nabla} \times \nabla S = \vec{\nabla} \times \vec{A} \quad \text{(curl of any gradient is zero)}$$

The Physical forces on charged Particles are gauge-invariant. The Magnetic field is gauge invariant.

$$* \quad \vec{F} = e\left(-\nabla V(x)\right) \qquad U = e \cdot V \quad \text{where } U \text{ is the potential energy}$$

gradient of the scalar potential

$$= e\left(\vec{E}(x) + \frac{\vec{v}}{c} \times \vec{B}\right) \quad \text{(Electric Field)}$$

We've seen examples of velocity-dependent forces like viscous forces, but this is very different. A viscous force acts along the direction of the velocity. This force is perpendicular to both the velocity and the magnetic field.

It is called the Lorentz Force.

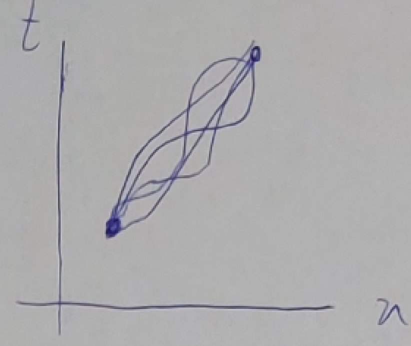
It is similar to the coriolis force.

$$F_{coriolis} = -2m\left(\vec{v} \times \vec{\Omega}\right)$$

For a particle in a Magnetic Field

$$\mathcal{A} = \int \mathcal{L}(x, \dot{x})\, dt$$

$$= \int \left[\frac{m}{2}\dot{x}^2 + \frac{e}{c} A\cdot\dot{x}\right] dt$$



But A is not unique.

It is subject to Gauge Transformations.

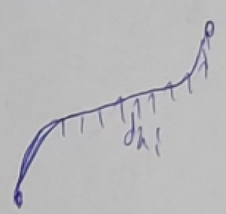
What will happen if we change A from one gauge to another. Is the Action going to change, and will it change the equations of motion.

$$\int A_i \frac{dx_i}{dt}\, dt = \int A_i\, dx_i$$

Applying a gauge transformation:

$$\int \left(A_i + \frac{\partial S}{\partial x_i}\right)\frac{dx_i}{dt}\, dt$$

The change in the action is: $\int \frac{\partial S}{\partial x_i} dx_i$



You take the difference of S between 2 close ~~points~~ points and sum them up.

They all cancel out except the endpoints

so $$\int_{t_i}^{t_f} \frac{\partial S}{\partial x_i}\, dx_i = S(x_f) - S(x_i)$$

Using the action principle, the endpoints are fixed, so the contribution to the action doesn't depend on the trajectory at all (just the endpoints).

Then the equations of motion and the choice of correct trajectory will not change when you do a gauge transformation.

So the Action is no gauge-invariant in itself, but the equations of motion that follow from it are.

Lecture 10:

$B_z = b$ $\quad A_x = 0$ $\quad \dfrac{\partial A_y}{\partial x} - \dfrac{\partial A_x}{\partial y} = b = B_z$ for both cases

$B_x = 0$ $\quad A_y = bx$

$B_y = 0$ $\quad A_z = 0$

<u>OR</u> $A_x = -by$

$A_y = 0$

$A_z = 0$

Let $S = cxy \Rightarrow \nabla_x S = cy$ & $\nabla_y S = cx$

So $A_x = 0 \longrightarrow cy$

$A_y = bx \longrightarrow bx + cx$

$A_z = 0 \longrightarrow 0$

Choose c as $-b$ $\Rightarrow A_x = -by$

$A_y = 0$

$A_z = 0$

So they are the same (the transformations)

The physical phenomena that occur should be independent of the gauge.

* for a particle in a Magnetic Field:

$$\mathcal{L} = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{e}{c}(A_x \dot{x} + A_y \dot{y} + A_z \dot{z})$$

$$P_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} + \frac{e}{c} A_x(x)$$

$$\dot{P}_x = m\ddot{x} + \frac{e}{c}\left(\frac{\partial A}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z}\right)$$

$$\dot{p}_x = \frac{\partial L}{\partial x}$$

$$m\ddot{x} + \frac{e}{c}\left(\frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z}\right) = \frac{e}{c}\left[\frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_y}{\partial x}\dot{y} + \frac{\partial A_z}{\partial x}\dot{z}\right]$$

$$ma_x = \frac{e}{c}\left[\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\dot{y} + \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\dot{z}\right]$$

$$ma_x = \frac{e}{c}\left(B_z\dot{y} - B_y\dot{z}\right) = \frac{e}{c}\left(v \times B\right)_x$$

Hamiltonian form:

$$P_x = m\dot{x} + \frac{e}{c}A_x$$
$$P_y = m\dot{y} + \frac{e}{c}A_y$$
$$P_z = m\dot{z} + \frac{e}{c}A_z$$

Rearranging: $\dot{x} = \dfrac{P_x - \frac{e}{c}A_x}{m}$

$\vdots$

$$H = \sum_i P_i\dot{x}_i - L(x, \dot{x})$$
$$= P_x\dot{x} + P_y\dot{y} + P_z\dot{z} - \frac{m}{2}\dot{x}^2 - \frac{m}{2}\dot{y}^2 - \frac{m}{2}\dot{z}^2 - \frac{e}{c}A_x\dot{x} - \frac{e}{c}A_y\dot{y} - \frac{e}{c}A_z\dot{z}$$

Plug $\dot{x} = \dfrac{P_x - \frac{e}{c}A_x}{m}$ and the other components to get:

$$H = \frac{\left(P_x - \frac{e}{c}A_x\right)^2}{2m} + \frac{\left(P_y - \frac{e}{c}A_y\right)^2}{2m} + \frac{\left(P_z - \frac{e}{c}A_z\right)^2}{2m}$$

* The ordinary old fashioned kinetic energy is conserved because the force is perpendicular to the velocity.
* Symmetry is just the shift of cyclic coordinates (do not appear in Lagrangian)

$$\left.\begin{array}{l} A_z = 0 \\ A_x = 0 \\ A_y = bx \end{array}\right\}$$ Plug into Lagrangian:

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{e}{c}(bx\dot{y})$$

$p_x = m\dot{x}$

$p_y = m\dot{y} + \frac{e}{c}bx$

$p_z = m\dot{z}$

by $p = \frac{\partial L}{\partial \dot{x}}$

Cyclic coordinates: y and z [they do not appear in the Lagrangian]

Then p_z is conserved; $\dot{p}_z = 0$, so $a_z = 0$

$$\dot{p}_y = \frac{d}{dt}\left(m\dot{y} + \frac{e}{c}bx\right) = 0$$

$$ma_y + \frac{e}{c}bv_x = 0 \Rightarrow ma_y = -\frac{e}{c}bv_x$$

For a_x, we have two ways to get it: Work out it's equation of motion

OR change gauges: $A_z = 0$, $A_x = -by$, $A_y = 0$

$$\Rightarrow L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{e}{c}(b\dot{x}y)$$

$\Rightarrow p_x$ changes: $p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} - \frac{e}{c}by$

$\Rightarrow \dot{p}_x = 0 \Rightarrow ma_x = \frac{e}{c}bv_y$

* $ma_x = \frac{e}{c}(v \times B)_x$

$ma_y = \frac{e}{c}(v \times B)_y$

$m\vec{a} = \frac{e}{c}(\vec{v} \times \vec{B})$ Acceleration is perpendicular to both B and v. Velocity is constant, $\vec{B}$ is constant $\Rightarrow a$ is constant.

What is the solution of a particle moving so that it's acceleration is always perpendicular to its velocity and is constant?

A circle.

Then $\frac{mv^2}{r} = \frac{e}{c} v B$

then $r = \frac{cmv}{eB}$

* Legitimate Lagrangians are those that lead to a sensible Hamiltonian.