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National University of Computer and Emerging Sciences

Islamabad Campus

Multivariable Calculus (MT1008) (CS)

Date: May 22, 2024

Final Exam

Total Time (Hrs): 3

Total Marks: 200

Total Questions: 12

Course Instructor(s):

Dr. Hamda Khan, Dr. Imran Shahzad, Mr. Arif Hussain

Roll No ____________ Course Section ____________ Student Signature ____________

Instructions:

1. Attempt questions in the given order.
2. Show full steps for scoring full credit.
3. Solve the questions using techniques learnt in this course. Using a method other than the required method will result in deduction or zero marks.
4. Start each question on a new page.

Do not write below this line.

Attempt all the questions.

Q1: In an ancient city, the royal palace grounds occupied the center disk of radius 1 kilometer. Common people live in a ring-shaped region with inner radius 1 kilometer and outer radius 3 kilometers that was approximately uniformly filled with residences. What was the average distance of a common residence in that city from the city center? **[20 marks]**

Q2: A town is the shape of a rectangle, with vertices $(0,0), (8,0), (8,4)$ and $(0,4)$. The population density is modeled by the function $(x, y) = xy^2$. They want to place their city hall so that it is centered relative to their population, not necessarily geographically. Determine the "population center" of this town. Also sketch the shape of the town. (Assume the units are miles, and the density is "hundreds of people per square mile"). **[20 marks]**

Q3: A tube in the shape of a right circular cylinder of radius 4m & height 0.5m, emits heat from its curved surface only. The heat emission rate, in Wm^{-2}, is given by $\frac{1}{2}e^{-2z}\sin^2\theta$, where θ and z are standard cylindrical polar coordinates, whose origin is at the center of one of the flat faces of the cylinder. Given that the cylinder is contained in the part of space for which $z \geq 0$, determine the total heat emission rate from the tube. **[10 marks]**

Q4: Let E be the solid region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the inverted cone (or upside-down cone) $z = 4 - \sqrt{x^2 + y^2}$, and let S be the surface/boundary of E with outward pointing normal vectors. Sketch the solid and find the flux through S of the field: **[10 marks]**

$F(x, y, z) = e^{3z}i + (xy + xz)j + \cos(xy)\,\boldsymbol{k}$

Q5: To reduce shipping distances between the manufacturing facilities and a major consumer, a well-known computer brand intends to start production of a new controlling chip for their microprocessors at their two plants. The cost of producing x_1 chips at Thailand is $C_1 = 0.002x_1^2 + 4x_1 + 500$ and cost of producing x_2 chips at Malaysia is $C_2 = 0.005x_2^2 + 4x_2 + 275$. The computer manufacturer buys them for RM150 per chip. Find the quantity that should be produced at each location to maximize the profit. Also find the profit. **[20 marks]**

Q6: Gradient descent method can be applied to various machine learning algorithms. It provides a general framework for optimizing models by iteratively refining their parameters based on the cost function. To show its effectiveness, use steepest descent method to find minimum of

$$f(x_1, x_2) = x_1^2 - x_1x_2 + x_2^2$$

such that the error does not exceed by 0.05 for the function. Start with initial approximation of $(x_1, x_2) = \left(1, \frac{1}{2}\right)$. **[20 marks]**

Q7a: Find the limit: **[10 marks]**

$$\lim_{(x,y)\to(0,0)} \frac{1 - cos(x^2 + y^2)}{x^2 + y^2}$$

Q7b: Let $f(x, y) = \begin{cases} \frac{\sin(x^2+y^2)}{x^2+y^2} & (x, y) \neq (0,0) \\ 1 & (x, y) = (0,0). \end{cases}$

Check the continuity of $f(x, y)$ at $(0,0)$. **[10 marks]**

Q8a: After finishing your final exam you head to the theater to catch the midnight screening of the new Star Wars movie. As you wait to be seated you decide to pass the time by determining the number of people in line. Your math adrenaline is still pumping from the exam, you decide to do this by using a line integral. In particular, the wall against which people are lined up is given by the curve C with $(x(t), y(t)) = (6t^2, 4\sqrt{3}\,t^3)$ for $0 \leq t \leq 1$ with distances measured in meters. Also, the number of people in the line at a given (x, y) is $\frac{5}{2}x + \frac{13}{2}$ people per meter. Determine the number of people in line, i.e., find $\int_C \left(\frac{5}{2}x + \frac{13}{2}\right) ds$. Note: Your answer should be a whole number. **[10 marks]**

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Q8b: A fluid velocity field is $\mathbf{F} = x\mathbf{i} + z\mathbf{j} + y\mathbf{k}$. Find the flow along the helix:
$r(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}, 0 \le t \le \frac{\pi}{2}$. **[10 marks]**

Q9: A highway passes by the small town of Las Cienegas. From Las Cienegas, the highway is 7 miles to north and 5 miles to the east. Assume the highway is straight as it passes through the region. The town wants to build an access road to the highway. Using Lagrange Multiplier find the shortest possible distance from Las Cienegas to the highway? **[20 marks]**

Q10a: Integrate f over the line segment C joining the origin to the point (1, 1, 1). Where:
$f(x, y, z) = x - 3y^2 + z$ **[08 marks]**

Q10b: Consider another path, formed from two line-segments C_1 from origin to (1, 1, 0) and C_2 from (1, 1, 0) to (1, 1, 1). Integrate $f(x, y, z) = x - 3y^2 + z$ along the second path. **[06 marks]**

Q10c: Does the line integral along C is equal to the line integral along $C_1 U C_2$? If not, does it mean line integral is not independent of path? State the reason. **[06 marks]**

Q11: What is the difference between a vector and a vector field? Find work done by the force field $F = (e^x \cos y + yz)\mathbf{i} + (xz - e^x \sin y)\mathbf{j} + (xy + z)\mathbf{k}$ in moving an object, along a smooth curve, joining the points (1, 0, 5) to (2, π/2, 7). **[10 marks]**

Q12: Sketch the region for the integration and then evaluate the double integral $\int_0^{2\sqrt{ln3}} \int_{\frac{y}{2}}^{\sqrt{ln3}} e^{x^2}\, dxdy$. Hint: Reverse the order of integration. **[10 marks]**

End of Question Paper