Strassen's Matrix Multiplication

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[partially based on slides by Prof. Welch]

Matrix Multiplication

Consider two n x n matrices A and B

Recall that the matrix product $C = AB$ of two n x n matrices is defined as the n x n matrix that has the coefficient

$$c_{kl} = \Sigma_m a_{km} b_{ml}$$

in row k and column l, where the sum ranges over the integers from 1 to n; the scalar product of the k^{th} row of a with the l^{th} column of B.

The straightforward algorithm uses $O(n^3)$ scalar operations.

Can we do better?

Idea: Use Divide and Conquer

The divide and conquer paradigm is important general technique for designing algorithms. In general, it follows the steps:

- divide the problem into subproblems
- recursively solve the subproblems
- combine solutions to subproblems to get solution to original problem

Divide-and-Conquer

Let write the product A B = C as follows:

$$\begin{array}{|c|c|}\hline A_0 & A_1 \\ \hline A_2 & A_3 \\ \hline \end{array} \times \begin{array}{|c|c|}\hline B_0 & B_1 \\ \hline B_2 & B_3 \\ \hline \end{array} = \begin{array}{|c|c|}\hline A_0\times B_0+A_1\times B_2 & A_0\times B_1+A_1\times B_3 \\ \hline A_2\times B_0+A_3\times B_2 & A_2\times B_1+A_3\times B_3 \\ \hline \end{array}$$

- Divide matrices A and B into four submatrices each
- We have 8 smaller matrix multiplications and 4 additions. Is it faster?

Divide-and-Conquer

Let us investigate this recursive version of the matrix multiplication.

Since we divide A, B and C into 4 submatrices each, we can compute the resulting matrix C by

- 8 matrix multiplications on the submatrices of A and B,
- plus $\Theta(n^2)$ scalar operations

Divide-and-Conquer

- Running time of recursive version of straightfoward algorithm is

  $T(n) = 8T(n/2) + \Theta(n^2)$ and $T(2) = \Theta(1)$

  where $T(n)$ is running time on an $n \times n$ matrix

- Master theorem gives us:

  $T(n) = \Theta(n^3)$

- Can we do fewer recursive calls (fewer multiplications of the $n/2 \times n/2$ submatrices)?

Strassen's Matrix Multiplication

$$A \times B = C.$$

$$\begin{array}{|c|c|}\hline A_{11} & A_{12} \\ \hline A_{21} & A_{22} \\ \hline \end{array} \times \begin{array}{|c|c|}\hline B_{11} & B_{12} \\ \hline B_{21} & B_{22} \\ \hline \end{array} = \begin{array}{|c|c|}\hline C_{11} & C_{12} \\ \hline C_{21} & C_{22} \\ \hline \end{array}$$

$P_1 = (A_{11}+ A_{22})(B_{11}+B_{22})$

$P_2 = (A_{21} + A_{22}) * B_{11}$

$P_3 = A_{11} * (B_{12} - B_{22})$

$P_4 = A_{22} * (B_{21} - B_{11})$

$P_5 = (A_{11} + A_{12}) * B_{22}$

$P_6 = (A_{21} - A_{11}) * (B_{11} + B_{12})$

$P_7 = (A_{12} - A_{22}) * (B_{21} + B_{22})$

$C_{11} = P_1 + P_4 - P_5 + P_7$

$C_{12} = P_3 + P_5$

$C_{21} = P_2 + P_4$

$C_{22} = P_1 + P_3 - P_2 + P_6$

Strassen's Matrix Multiplication

- Strassen found a way to get all the required information with only 7 matrix multiplications, instead of 8.

- Recurrence for new algorithm is
  - $T(n) = 7T(n/2) + \Theta(n^2)$

Solving the Recurrence Relation

Applying the Master Theorem to

$T(n) = a\, T(n/b) + f(n)$

with $a=7$, $b=2$, and $f(n)=\Theta(n^2)$.

Since $f(n) = O(n^{\log_b(a)-\varepsilon}) = O(n^{\log_2(7)-\varepsilon})$,

case a) applies and we get

$T(n)= \Theta(n^{\log_b(a)}) = \Theta(n^{\log_2(7)}) = O(n^{2.81})$.

Discussion of Strassen's Algorithm

- Not always practical
  - constant factor is larger than for naïve method
  - specially designed methods are better on sparse matrices
  - issues of numerical (in)stability
  - recursion uses lots of space
- Not the fastest known method
  - Fastest known is $O(n^{2.3727})$ [Winograd-Coppersmith algorithm improved by V. Williams]
  - Best known lower bound is $\Omega(n^2)$