Q.1 A square matrix, whose all elements below the main diagonal are zero is called an upper triangular matrix.

So according to the definition,

$$\begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet \\ \phi & & \bullet & \bullet \\ & & & \bullet \end{bmatrix} \leftarrow \text{upper triangular matrix.}$$

Now, In the question,

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

using Gaussian elimination

$R_1 \leftarrow R_1 - R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$R_2 \leftarrow 2R_1 - R_2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$R_3 \leftarrow R_2 - R_3$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$$ ~~main diagonal~~

→ ~~upper triangular matrix A~~.

Further,

$R_2 \leftarrow R_2 - R_3$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

→ main diagonal

→ upper triangular matrix A.

Q2

Wednesday, February 21, 2024 21:09

Q.2: A square matrix whose all elements above the main diagonal are zero is called a "Lower triangular matrix".

for example,

$$\begin{bmatrix} \bullet & & & \phi \\ \bullet & \bullet & & \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix}$$

Now, Based on Question Q2.

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

using Gaussian elimination,

$$R_1 \leftarrow R_1 - R_3$$

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$R_1 \leftarrow R_1 - R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$ → ~~main diagonal~~

~~Lower triangular matrix~~.

Furthermore,

$$R_2 \leftarrow R_2 - R_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$ → main diagonal

→ Lower triangular matrix.

Point to Note here,

→ Q2 which is Lower triangular matrix, and it is transpose of upper triangular matrix (Q1).

$$U^T = L$$

$$L^T = U$$

Q3

Thursday, February 22, 2024 13:09

Q.3 We have

$$B = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix}$$

Given, the matrix B is of rank 2.

$$\text{rank}(B) = 2$$

It means, one of the Row having all zero's element.

let us bring c to Row 3 using Gausian elemenation

$$R_3 \leftarrow R_3 - R_2$$

$$B = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & -c & d-2 & 0 \end{bmatrix}$$

to be Rank 2, Row 3 elements should be 0.

therefore $-c = 0 \Rightarrow c = 0$

$$d - 2 = 0 \Rightarrow d = 2$$

So, $\boxed{c = 0 \text{ and } d = 2.}$ will provide $\text{rank}(B) = 2$.

Q4. Let us define, what is eigenvalue and eigenvectors are,
Let C be an $n \times n$ matrix.

1. An eigenvalue of C is a scalar λ such that the equation $Cv = \lambda v$ has a nontrivial solution.
2. An eigenvector of C is a nonzero vector 'v' in $\mathbb{R}^n$ such that $Cv = \lambda v$, for some scalar λ.

If $Cv = \lambda v$ for $v \neq 0$, we say that λ is the "eigenvalue" for v, and that v is an "eigenvector" for λ.

In short:

> Let C be an $n \times n$ matrix and suppose,
> $$\det(\lambda I - C) = 0 \text{ for some } \lambda \in \mathbb{C}.$$
> Then λ is an eigenvalue of C and thus there exists a nonzero vector $v \in \mathbb{C}^n$ such that $Cv = \lambda v$.

So, In Given Question,

1. First, we have to find λ (eigenvalues) of C by solving the equation $\det(\lambda I - C) = 0$.
2. then for each λ, we have to find the basic eigenvectors $V \neq 0$ by finding the basic solutions to $(\lambda I - C)V = 0$.

eigenvalues:

$$C = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

According to the equation
$$\det(\lambda I - C) = 0$$

where, $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ← Identity matrix.

$$\det\left(\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}\right) = 0$$

$$\Rightarrow \det\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}\right) = 0$$

$$\Rightarrow \det\left(\begin{bmatrix} \lambda & 0 & -2 \\ 0 & \lambda-2 & 0 \\ -2 & 0 & \lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow \lambda\left(\lambda(\lambda-2) - 0\right) - 0\left(0\cdot\lambda - 0\cdot -2\right) + (-2)\left(0 - (-2)(\lambda-2)\right) = 0$$

$$\Rightarrow \lambda^2(\lambda-2) - 2\left(+2(\lambda-2)\right) = 0$$

$$\Rightarrow \lambda^2(\lambda-2) - 4(\lambda-2) = 0$$

$$\Rightarrow (\lambda-2)(\lambda^2-4) = 0$$

this can be written as,

$$\lambda - 2 = 0 \Rightarrow \boxed{\lambda = 2}$$

$$\lambda^2 - 4 = 0 \Rightarrow \lambda^2 - 2^2 = 0 \Rightarrow \boxed{\lambda = 2} \text{ and } \boxed{\lambda = -2}.$$

Hence, $\boxed{\text{eigenvalue} = \pm 2}$.

Now, we need to find the eigenvectors for $\lambda = 2$. First we find the eigenvectors for $\lambda = 2$, we wish to find all vectors $V \neq 0$. Such that $AV = 2V$, These are the solutions to $(2I - C)V = 0$.

So, $(2I - C)V = 0$

$$\left(2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}\right)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So, $(2I - C)v = 0$

$$\left(2\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \begin{bmatrix}0&0&2\\0&2&0\\2&0&0\end{bmatrix}\right)\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\left(\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix} - \begin{bmatrix}0&0&2\\0&2&0\\2&0&0\end{bmatrix}\right)\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}2&0&-2\\0&0&0\\-2&0&2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

The augmented matrix for this s/m and corresponding RREF are given by,

$$\left[\begin{array}{ccc|c}2&0&-2&0\\0&0&0&0\\-2&0&2&0\end{array}\right] \Rightarrow \dots \rightarrow \left[\begin{array}{ccc|c}1&0&-1&0\\0&0&0&0\\0&0&0&0\end{array}\right]$$

The solution is any vector of the form

$$\begin{bmatrix}t\\0\\t\end{bmatrix} = t\begin{bmatrix}1\\0\\1\end{bmatrix}$$

This gives the basic eigenvector for $\lambda = 2$ as

$$\begin{bmatrix}1\\0\\1\end{bmatrix}$$ → eigenvector for $\lambda = 2$.

Now, we have show $Cv = \lambda v$ for $\lambda = 2$ and associated eigen vectors v.

$$\Rightarrow \begin{bmatrix}0&0&2\\0&2&0\\2&0&0\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}0.1+0.0+2.1\\0.1+2.0+0.1\\2.1+0.0+0.1\end{bmatrix} \Rightarrow \begin{bmatrix}0+0+2\\0+0+0\\2+0+0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}2\\0\\2\end{bmatrix} \Rightarrow \boxed{2\begin{bmatrix}1\\0\\1\end{bmatrix} \Rightarrow \lambda v}$$ proof

⇒ Now consider $\boxed{\lambda = -2}$

eigen vector

$$(-2I - C)v = \vec{0}$$

$$\Rightarrow \left(-2\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \begin{bmatrix}0&0&2\\0&2&0\\2&0&0\end{bmatrix}\right)\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \left(\begin{bmatrix}-2&0&0\\0&-2&0\\0&0&-2\end{bmatrix} - \begin{bmatrix}0&0&2\\0&2&0\\2&0&0\end{bmatrix}\right)\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}-2&0&-2\\ &-4&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 0 & -2 \\ 0 & -4 & 0 \\ -2 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

equation,

$-2x - 2z = 0 \Rightarrow x + z = 0 \Rightarrow x = -z$

$-4y = 0 \Rightarrow y = 0$

$-2x - 2z = 0 \Rightarrow x + z = 0 \Rightarrow x = -z$

eigenvector when eigenvalue $\lambda = -2$

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \leftarrow \text{eigenvector}$$

Now, we have to show $CV = \lambda V$ for $\lambda = -2$ and $v = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\Rightarrow \boxed{-2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}$$

L.H.S = R.H.S.

overall, eigenvalues are $= +2, -2$

and eigenvectors are $= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Q5

Wednesday, February 21, 2024 18:50

Q.5

$$A = \begin{bmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{bmatrix}_{4\times 5}$$

The space spanned by the rows of A is called the row space of A, which is V here. denoted by RS(A). It is a subspace of R^5

Row Reduced form of A.

① $R_1 \leftarrow R_2 - R_3$

$$\begin{bmatrix} 1 & 7 & 0 & 3 & -1 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{bmatrix}$$

② $R_2 \leftarrow R_2 - R_1$

$$\begin{bmatrix} 1 & 7 & 0 & 3 & -1 \\ 0 & 0 & -1 & -5 & 0 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{bmatrix}$$

③ $R_4 \leftarrow R_4 - 3R_3$

$$\begin{bmatrix} 1 & 7 & 0 & 3 & -1 \\ 0 & 0 & -1 & -5 & 0 \\ 2 & 14 & 0 & 6 & 1 \\ 0 & 0 & -1 & -5 & -3 \end{bmatrix}$$

④ $R_3 \leftarrow R_3 - 2R_1$

$$\begin{bmatrix} 1 & 7 & 0 & 3 & -1 \\ 0 & 0 & -1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & -1 & -5 & 3 \end{bmatrix}$$

⑤ $R_4 \leftarrow R_4 - R_2$

$$\begin{bmatrix} 1 & 7 & 0 & 3 & -1 \\ 0 & 0 & -1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix}$$

⑥ $R_4 \leftarrow R_4 + R_3$

$$\begin{bmatrix} 1 & 7 & 0 & 3 & -1 \\ 0 & 0 & -1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

⑦ $R_3 \leftarrow \frac{1}{3} R_3$

$$\begin{bmatrix} 1 & 7 & 0 & 3 & -1 \\ 0 & 0 & -1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

⑧ $R_1 \leftarrow R_1 + R_3$

$$\begin{bmatrix} 1 & 7 & 0 & 3 & 0 \\ 0 & 0 & -1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

⑨ $R_2 \leftarrow -R_2$

$$\begin{bmatrix} 1 & 7 & 0 & 3 & 0 \\ 0 & 0 & +1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

⇒ A "basis" for RS(A) consists of the non-zero rows in the reduced matrix.

⇒ $\boxed{\{(1,7,0,3,0), (0,0,1,5,0), (0,0,0,0,1)\}}$

PART-2 · Tell which vectors $(x_1, x_2, x_3, x_4, x_5)$ are elements of V.

For that let's expressed the vectors $(x_1, x_2, x_3, x_4, x_5)$ in terms of basis vector

$v_1 = 1 \times x_1 + 7x_2 + 0 \times x_3 + 3 \times x_4 + 0 \times x_5$

$\boxed{v_1 = x_1 + 7x_2 + 3x_4}$

$v_2 = 0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 5x_4 + 0 \cdot x_5$

$\boxed{v_2 = x_3 + x_4}$

$v_2 = 0 \cdots$

$\boxed{v_2 = x_3 + x_4}$