Course: CS972 Linear Algebra
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To accomplish the assignment, I have used/referenced the following resources:

- "NO BULLSHIT guide to Linear Algebra" by Ivan Savov
- "Linear Algebra" by David Cherney
- Linear Algebra course on Khan Academy
- Several YouTube videos on Linear Algebra to brush up on basics
- Time to time consultation with ChatGPT to understand certain terminologies with examples
- Assistance from a few eMasters batchmates who helped me understand questions and approaches.

Overall, I am still in the process of adjusting to Linear Algebra, but I feel that this assignment was a good hands-on opportunity to learn and apply linear algebra concepts. Hopefully, in the upcoming assignments, I will be able to provide more concise and accurate answers. Thank you!

08/02/2024.

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(P-1)

SUBJECT: LA - Assignment -1

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Q1. PART: 1

Linear Functions map any linear combination of Inputs to the same linear combination of outputs.

and A Function f is linear if it satisfies the equation

$$\boxed{f(\alpha x+\beta y) = \alpha f(x) + \beta f(y)} \quad \text{--- (1)}$$

For any two I/p x & y. and for all constant α and β.

Now, we need to define a linear function

$f: \mathbb{R}^n \to \mathbb{R}$ and

$$f(x_1, x_2 \ldots x_n) = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n \quad \text{--- (2)}$$

Next, eq (2) can be rewritten In eq (1) Form as consider R.H.S of eq (1).

$$\Rightarrow \alpha(a_1 x_1 + a_2 x_2 + \ldots + a_n x_n) + \beta(a_1 y_1 + a_2 y_2 + \ldots + a_n y_n)$$

$$\Rightarrow \alpha a_1 x_1 + \beta a_1 y_1 + \alpha a_2 x_2 + \beta a_2 y_2 + \ldots + \alpha a_n x_n + \beta a_n y_n$$

$$\Rightarrow a_1\underbrace{(\alpha x_1 + \beta y_1)}_{z_1} + a_2\underbrace{(\alpha x_2 + \beta y_2)}_{z_2} + \ldots + a_n\underbrace{(\alpha x_n + \beta y_n)}_{z_n}$$

$$\Rightarrow a_1 z_1 + a_2 z_2 + a_3 z_n + \ldots + a_n z_n$$

which can be written as

$$f(z_1, z_2, \ldots z_n)$$

Hence, L.H.S = R.H.S

Proof for linearity

(P-2)

Now, we have to prove S is precisely the null space of f.

For f to be null space,

$f(x) = 0$

def (1) { lets define "null space" → Null space of a function f(x) are the set of Inputs for which the Function's outputs is zero.

In short,

$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$

$\hookrightarrow (a_1, a_2, \ldots, a_n) \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0$

this can be written as $Ax = 0$ — | where $A = (a_1, a_2, \ldots a_n)$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

So according to the definition (def (1)) the s/m $Ax = 0$ has solution x and the solution is called "null space" of $Ax = 0$.

therefore the Solution $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in (1)^n$

is nothing but the set of S.

therefore, S is precisely the null space.

Q.1 PART 2.

Now, we need to show that the dim(S) is $\underline{n-1}$, if not all a_i's are 0.

To Prove this we can use the 'rank-nullity theorem".

So, according to 'Ranknullity theorem" the dim of the domain of a linear function is equal to the sum of the Rank of the function, and the dimension of it's null space.

above theorem can be written as,

$$\text{rank}(f) + \underbrace{\text{nullity}(f)}_{\dim(\mathcal{N}(f))} = \text{Dim}(\mathbb{C}^n) \quad \text{—— (1)}$$

$$\text{rank}(f) + \text{nullity}(f) = n. \quad \text{—— (2)}$$

In above case, the domain of f is $\mathbb{C}^n$. So dimension is n (as stated in above eq).

the Rank of f is the dimension of it's Image, which is a subspace of $\mathbb{C}$. and provided that not all a_i's are 0, f is not the "zero function", therefore it's Image has dimension '1'.

therefore the above eq (2) will be

$$\text{rank}(f) + \dim(\mathcal{N}(f)) = n$$

$$1 + \dim(S) = n$$

$$\boxed{\dim(S) = n-1.}$$ → Based on Rank-nullity theorem.

Q.2.

let us define a linear Function $f: \mathbb{C}^n \to \mathbb{C}^m$ Such that the null space of f is S as follow.

$$f(x_1, x_2, \ldots, x_n) = (a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + \cdots + a_{1,n}x_n,\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n,\ \ldots,\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n) \quad \text{--- eq ①}$$

For, the null space of f then $f(x) = 0$. that means each component is zero,

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = 0$$
$$\vdots$$
$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n = 0$$

thus, this s/m Solution space is S. therefore null space of f is S.

⇒ The matrix representation of f is like a "family photo" of all the coefficients from eq ①, lined up row by row!

conti---

$$F = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & a_{m,3} & \cdots & a_{m,n} \end{bmatrix}_{m \times n}$$

Row colour

Q.3: Now, let's perform Gaussian elimination on the columns of F to obtain a matrix F'.

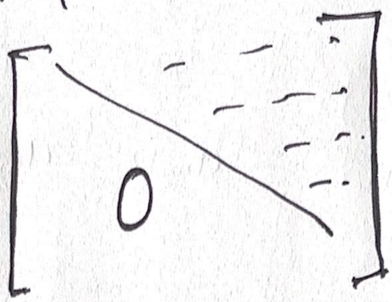
Gaussian elimination algorithm proceeds in two phases:

- a Forward phase in which we move Left to Right.
- a backward phase in which we move Righ to Left.

1. Forward phase (Left to Right)
   - obtain a pivot in the leftmost column.
   - Substract the row with the pivot from all rows below it to obtain zeros in the entire column.
   - look for a leading One in the next column and Repeat.
2. Backward phase (Right to Left)
   - Find the rightmost pivot and use it to eliminate all numbers above the pivot in it's column.
   - Move one column to the left and repeat.

So, aftere applying Gaussian elimination method as stated in previous page on the column of F then we get F'.

It means we get a uppere triangle matrix,

$F' =$ 

We know that both solution are same.

therefore, $Fx = 0$ & $F'x = 0$ has same solution.

$Fx = 0$ has solution space is S. Hence, $F'x = 0$ has same solution space S.

In short, Since Gaussian elimination does not change the null space of a matrix. Hence, the null space of F' is also S.

It's worth to mention here, the null space of F' will contain vectores $(x_1, x_2, \ldots, x_n)$ that satisfy the reduced row echelon form of the s/m of linear equations, which is equivalent to the original s/m of linear equations. therefore, the null space of F' is also S.

Q.4

We can use F' to find a basis for the vector space S.

as stated in Gaussian elimination during Q.3

we can first find the pivot columns of F' (the columns with a leading 1 in the ~~row each~~ REF).

The indices of these pivot columns correspond to the dependent variables in the s/m of linear equations.

The remaining columns (non-pivot columns) (like free variables ☺).

For each free variables, we give it a moment to shine by setting it to "1" and dimming the lights (set to 0) on the others (sorry for using these words ☺), This way, we choreograph a basis for S. ~~ensuring every vector knows it's moves~~.

So, let us keep going to find basis for vector space S.

To find basis for S from F', identify the pivot columns (columns without leading 1's), and we will use the corresponding columns of the original matrix (before Gaussian elimination) as a basis for S.

Continue--

To understand this,

let's consider F' as a 3×4 matrix.

as choosing $m \times n$ would bit difficult to explain.

So, In this case, F' would look like this

$$F' = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In this example, columns 1 and 2 are the pivot columns in F'. so, we select column 1 and 2 from the original matrix F given.

$$F = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \end{bmatrix}$$

the selected columns are:

$$\begin{bmatrix} a_{11} \\ a_{22} \\ a_{31} \end{bmatrix}$$

and $\begin{bmatrix} a_{1,2} \\ a_{2,2} \\ a_{3,2} \end{bmatrix}$

These coloums form a basis for the vector space S. representing independent solutions to the s/m of linear equations. ~~this can be extended to~~

It means,

↳ Any vector can be expressed as a linear combination of the basis vectors.

↳ The vectors that form the basis are linearly Independent.