Chapter 10

Conics and Polar Coordinates

Section 10.1

The Parabola

The Parabola

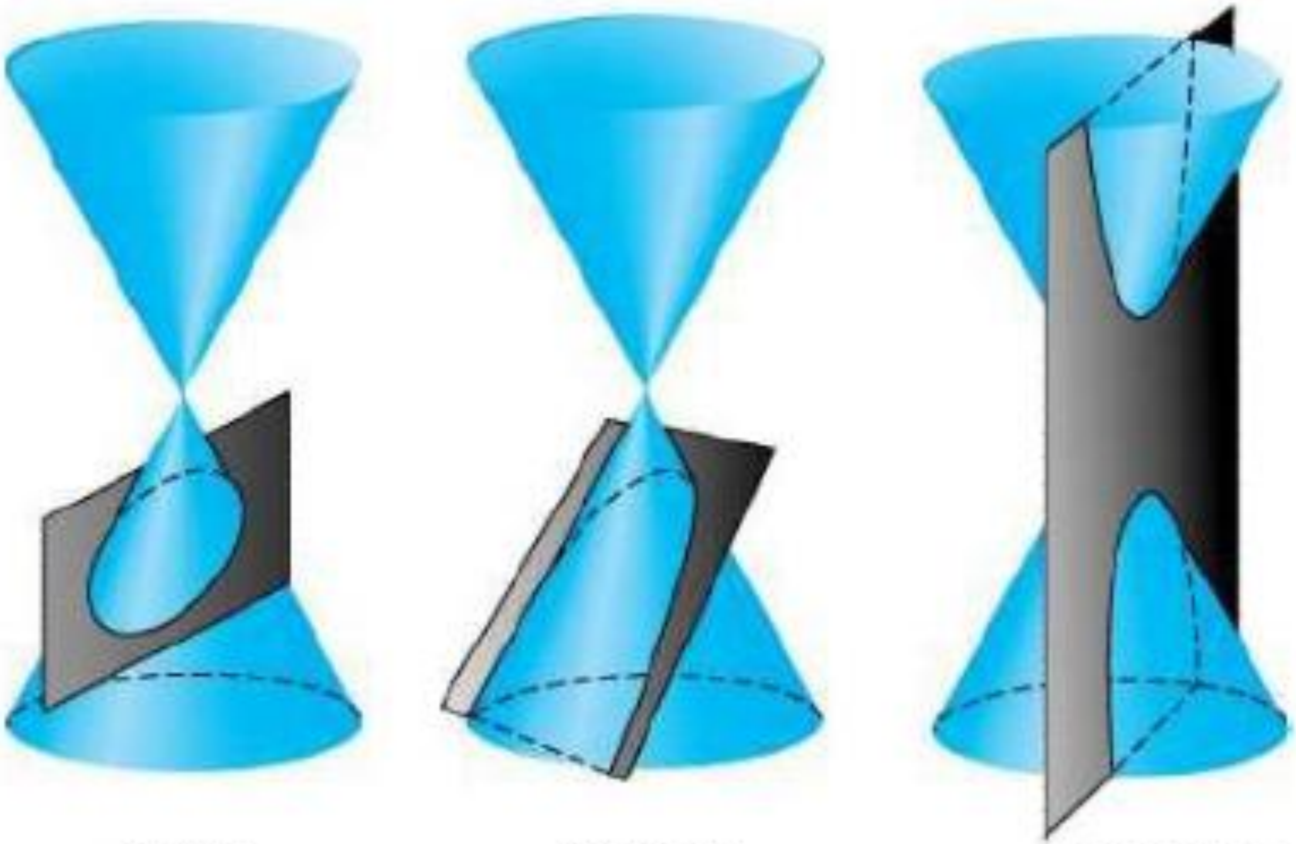


Figure 1

The Parabola

In the plane let ℓ be a fixed line (the **directrix**) and F be a fixed point (the **focus**) not on the line, as in Figure 2. The set of points P for which the ratio of the distance $|PF|$ from the focus to the distance $|PL|$ from the line is a positive constant e (the **eccentricity**), that is, the set of points P that satisfy

$$\boxed{|PF| = e|PL|}$$

is called a **conic.** If $0 < e < 1$, the conic is an **ellipse**; if $e = 1$, it is a **parabola**; if $e > 1$, it is a **hyperbola.**

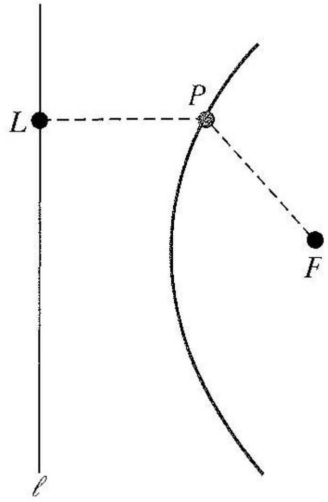


Figure 2

The Parabola

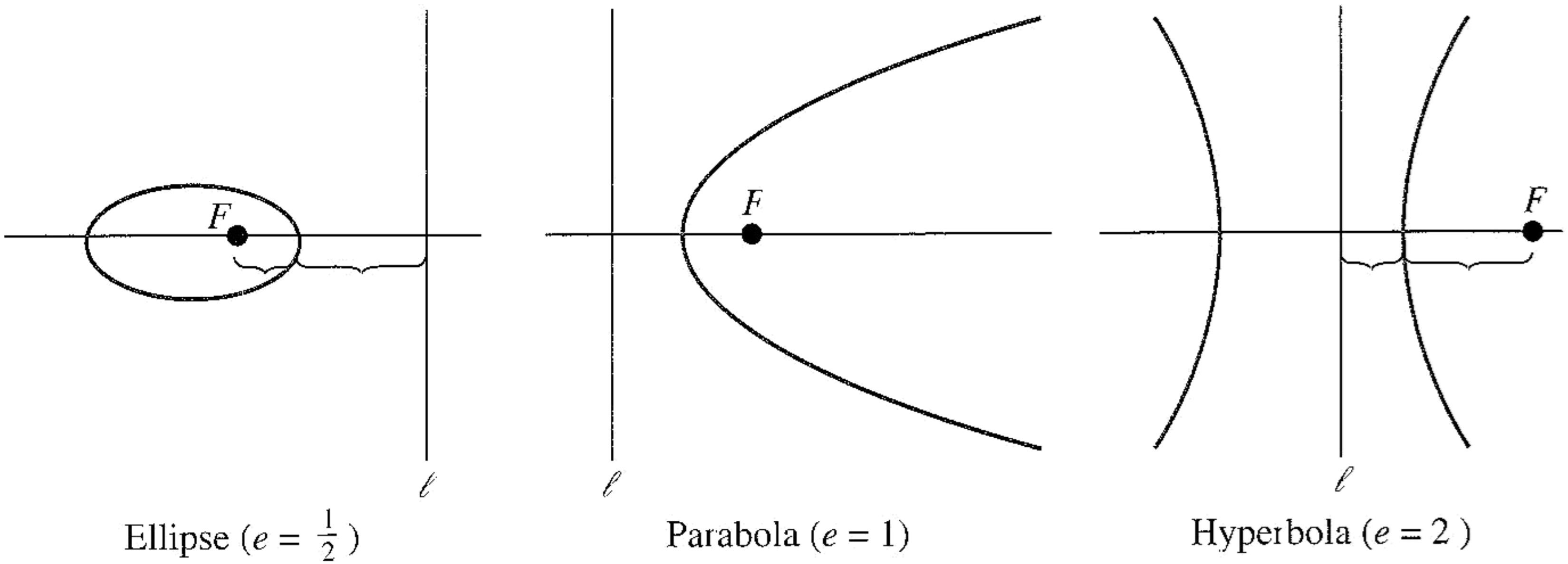


Figure 3

The Parabola ($e = 1$)

Since a parabola is symmetric with respect to its axis, it is natural to place one of the coordinate axes, for instance, the x-axis, along this axis. Let the focus F be to the right of the origin, say at (p,0), and the directrix to the left with equation $x = -p$. Then the vertex is at the origin. All this show in Figure 4.

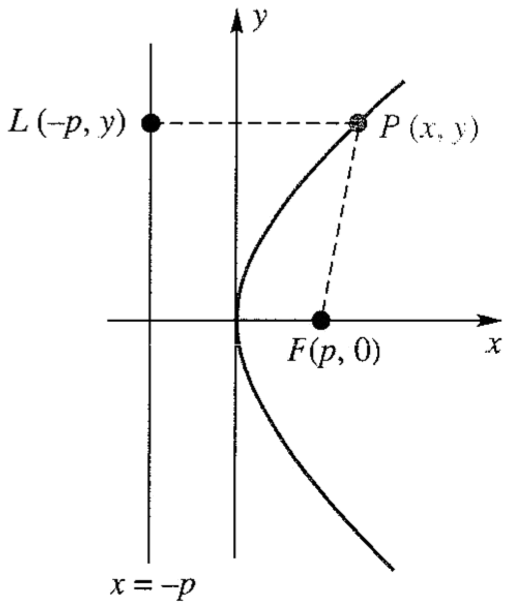


Figure 4

The Parabola ($e = 1$)

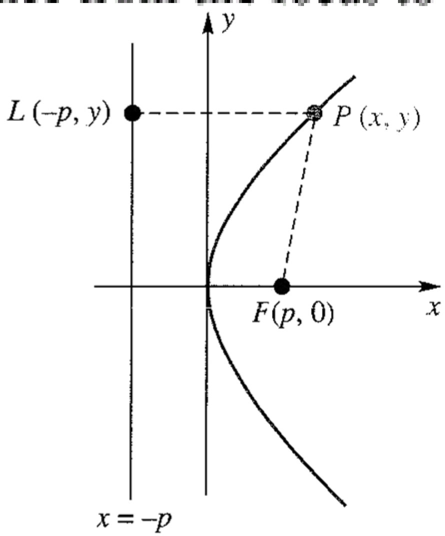
From the condition $|PF| = |PL|$ and the distance formula, we get

$$\sqrt{(x-p)^2 + (y-0)^2} = \sqrt{(x+p)^2 + (y-y)^2}$$

After squaring both sides and simplifying, we obtain

$$\boxed{y^2 = 4px}$$

This is called the **standard equation** of a horizontal parabola (horizontal axis) opening to the right. Note that $p > 0$ and that p is the distance from the focus to the vertex.



EXAMPLE 1 Find the focus and directrix of the parabola with equation $y^2 = 12x$.

SOLUTION Since $y^2 = 4(3)x$, we see that $p = 3$. The focus is at $(3, 0)$; the directrix is the line $x = -3$. ■

There are three variants of the standard equation. If we interchange the roles of x and y, we obtain the equation $x^2 = 4py$. It is the equation of a vertical parabola with focus at $(0, p)$ and directrix $y = -p$. Finally, introducing a minus sign on one side of the equation causes the parabola to open in the opposite direction. All four cases are shown in Figure 5.

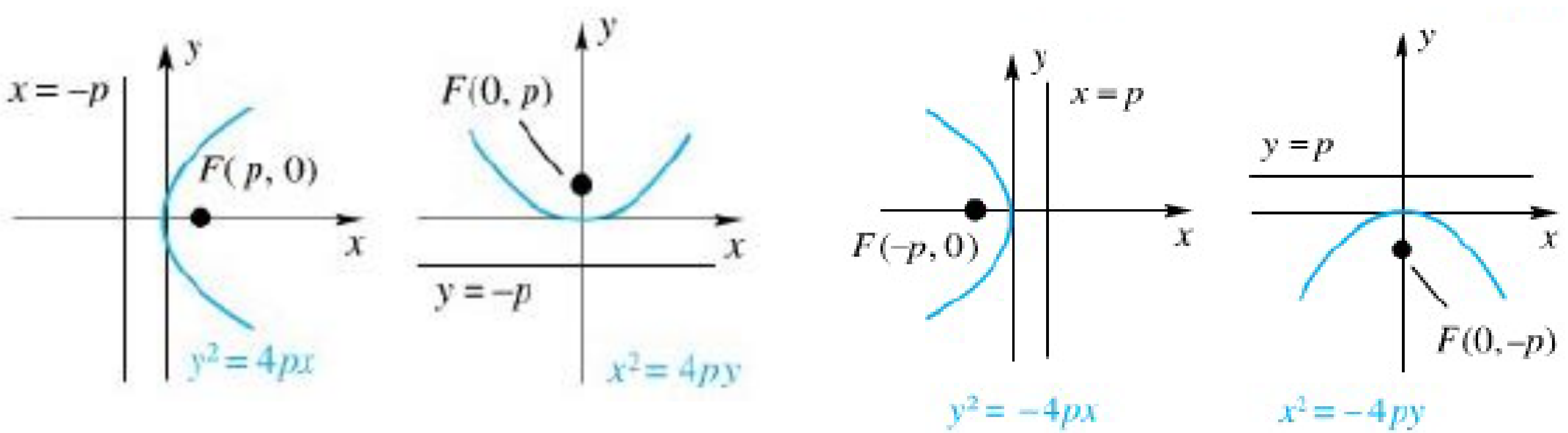


Figure 5

EXAMPLE 2 Determine the focus and directrix of the parabola $x^2 = -y$ and sketch the graph.

SOLUTION We write $x^2 = -4\left(\frac{1}{4}\right)y$, from which we conclude that $p = \frac{1}{4}$. The form of the equation tells us that the parabola is vertical and opens down. The focus is at $\left(0, -\frac{1}{4}\right)$; the directrix is the line $y = \frac{1}{4}$. The graph is shown in Figure 6. ■

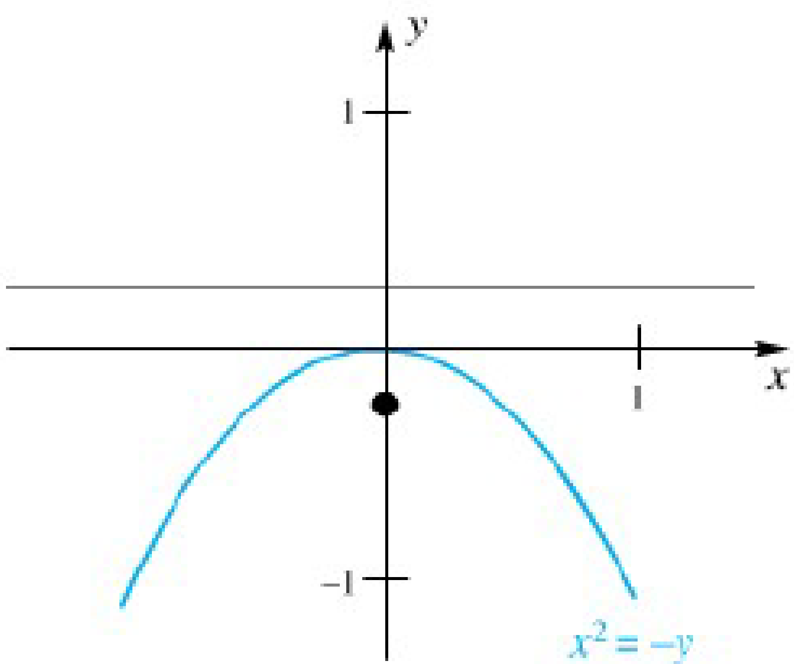


Figure 6

Section 10.2

Ellipses and Hyperbolas

Standard Equation of the Ellipse

Standard Equation of the Ellipse For the ellipse, $0 < e < 1$, and so $(1 - e^2)$ is positive. To simplify notation, let $b = a\sqrt{1 - e^2}$. Then the equation derived above takes the form

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

which is called the **standard equation of an ellipse.** Since $c = ae$, the numbers a, b, and c satisfy the Pythagorean relationship $a^2 = b^2 + c^2$. In Figure 5, the shaded right triangle captures the condition $a^2 = b^2 + c^2$. Thus, the number $2a$ is the **major diameter,** whereas $2b$ is the **minor diameter.**

Standard Equation of the Ellipse

The Ellipse $(0 < e < 1)$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$b^2 + c^2 = a^2$$

$$e = \frac{c}{a}$$

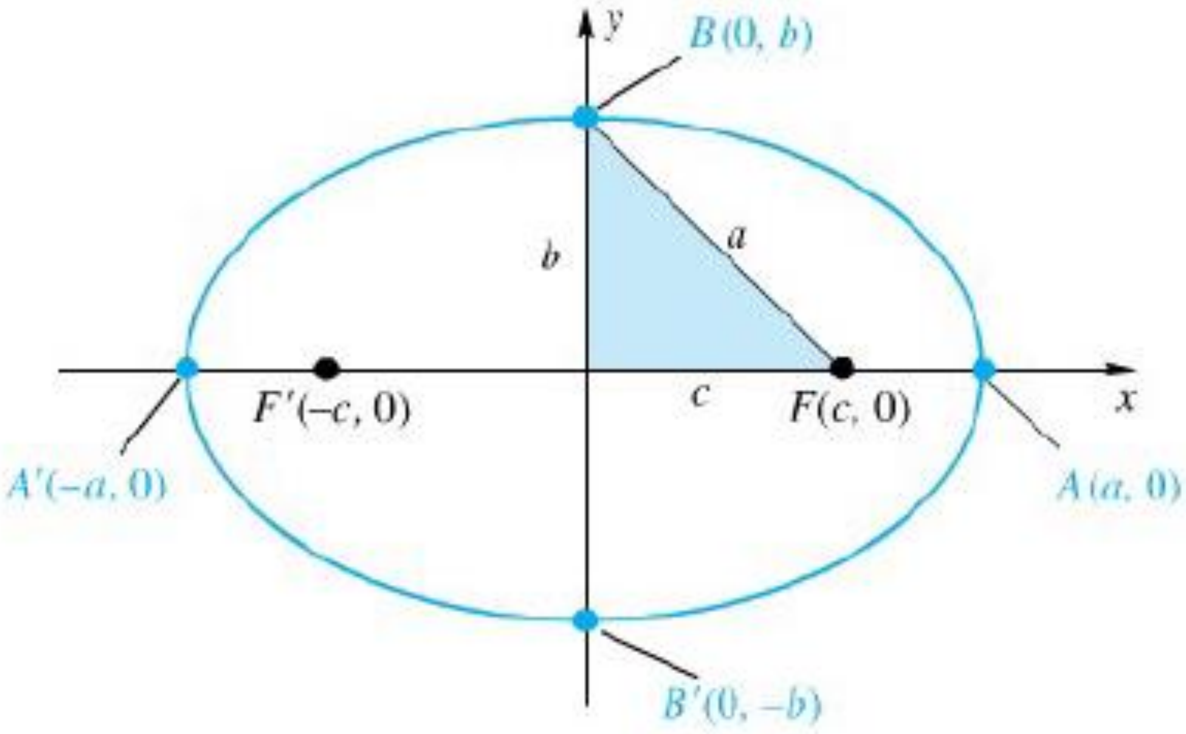


Figure 5

EXAMPLE 1 Sketch the graph of

$$\frac{x^2}{36} + \frac{y^2}{4} = 1$$

and determine its foci and eccentricity.

SOLUTION Since $a = 6$ and $b = 2$, we calculate

$$c = \sqrt{a^2 - b^2} = \sqrt{36 - 4} = 4\sqrt{2} \approx 5.66$$

The foci are at $(\pm c, 0) = \left(\pm 4\sqrt{2}, 0\right)$, and $e = c/a \approx 0.94$. The graph is sketched in Figure 7. ■

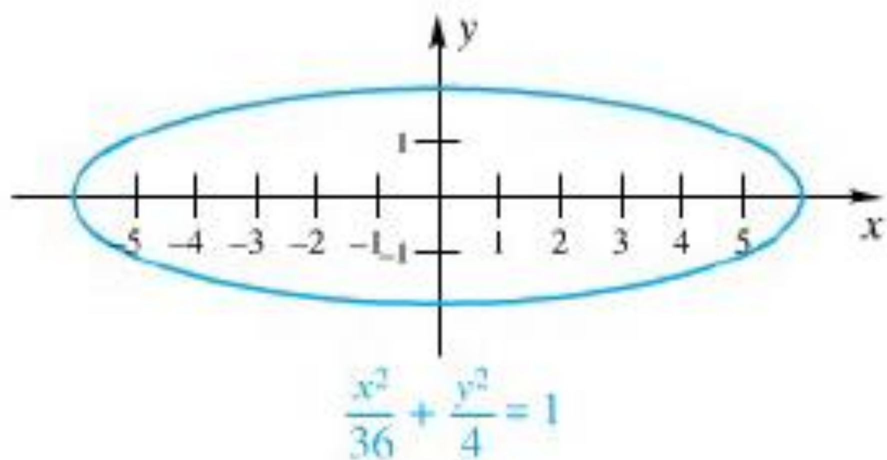


Figure 7

EXAMPLE 2 Sketch the graph of

$$\frac{x^2}{16} + \frac{y^2}{25} = 1$$

and determine its foci and eccentricity.

SOLUTION The larger square is now under y^2, which tells us that the major axis is vertical. Noting that $a = 5$ and $b = 4$, we conclude that $c = \sqrt{25 - 16} = 3$. Thus, the foci are $(0, \pm 3)$, and $e = c/a = \frac{3}{5} = 0.6$ (Figure 8). ■

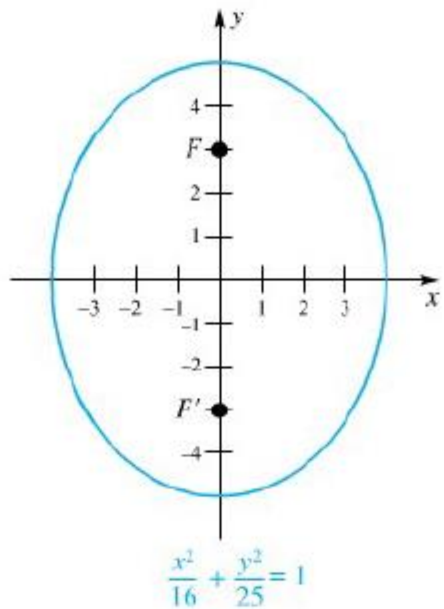


Figure 8

Standard Equation of the Hyperbola

Standard Equation of the Hyperbola For the hyperbola, $e > 1$ and so $e^2 - 1$ is positive. If we let $b = a\sqrt{e^2 - 1}$, then the equation $x^2/a^2 + y^2/(1 - e^2)a^2 = 1$, which was derived earlier, takes the form

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$$

This is called the **standard equation of a hyperbola.** Since $c = ae$, we now obtain $c^2 = a^2 + b^2$. (Note how this differs from the corresponding relationship for an ellipse.)

To interpret b, observe that if we solve for y in terms of x we get

$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$$

Standard Equation of the Hyperbola

For large x, $\sqrt{x^2 - a^2}$ behaves like x (i.e., $\left(\sqrt{x^2 - a^2} - x\right) \to 0$ as $x \to \infty$; see Problem 70) and hence y behaves like

$$y = \frac{b}{a}x \quad \text{or} \quad y = -\frac{b}{a}x$$

More precisely, the graph of the given hyperbola has these two lines as asymptotes.

The important facts for the hyperbola are summarized in Figure 9. As with the ellipse, there is an important right triangle (shaded in the diagram) that has legs a and b. This **fundamental triangle** determines the rectangle centered at the origin having sides of length $2a$ and $2b$. The extended diagonals of this rectangle are the asymptotes mentioned above.

Standard Equation of the Hyperbola

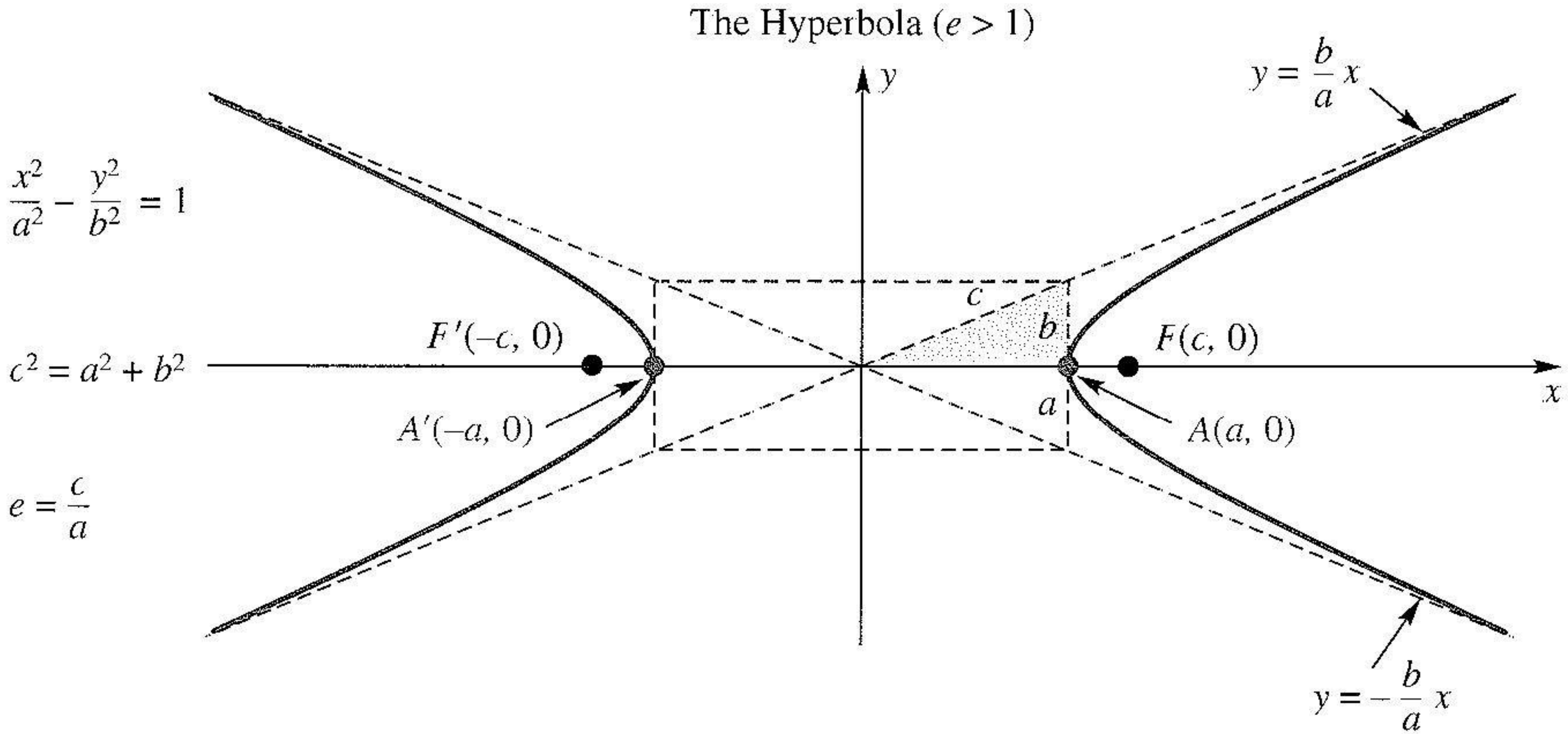


Figure 9

EXAMPLE 3 Sketch the graph of

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

showing the asymptotes. What are the equations of the asymptotes? What are the foci?

SOLUTION We begin by determining the fundamental triangle; it has horizontal leg 3 and vertical leg 4. After drawing it, we can indicate the asymptotes and sketch the graph (Figure 10). The asymptotes are $y = \frac{4}{3}x$ and $y = -\frac{4}{3}x$. Since $c = \sqrt{a^2 + b^2} = \sqrt{9 + 16} = 5$, the foci are at $(\pm 5, 0)$. ■

$y = -\frac{b}{a}x$ $y = \frac{b}{a}x$

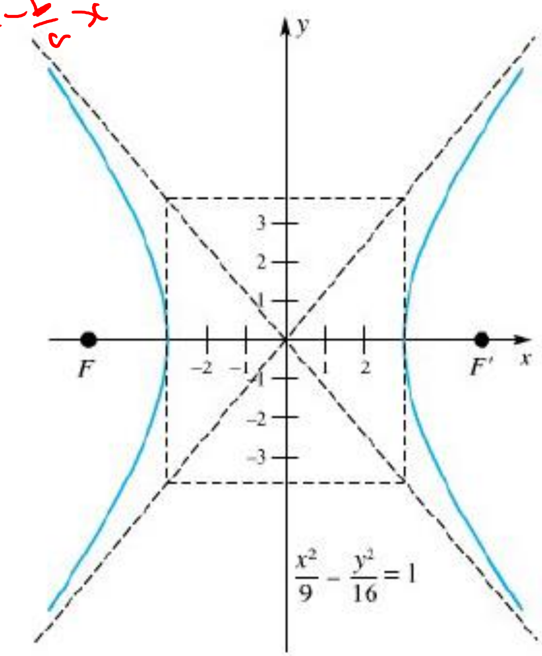


Figure 10

EXAMPLE 4 Determine the foci of

$$-\frac{x^2}{4}+\frac{y^2}{9}=1$$

and sketch its graph.

SOLUTION We note immediately that this is a vertical hyperbola, which is determined by the fact that the plus sign is associated with the y^2 term. Thus, $a=3$, $b=2$, and $c=\sqrt{9+4}=\sqrt{13}\approx 3.61$. The foci are at $(0,\pm\sqrt{13})$ (Figure 11). ■

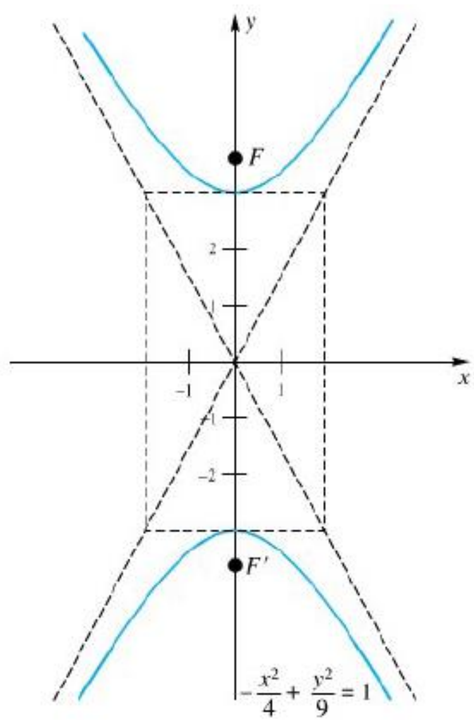


Figure 11

Section 10.3

Translation and Rotation of Axes

Translations

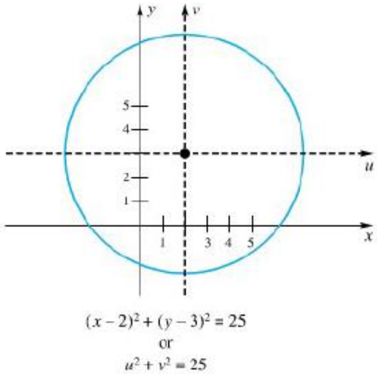


Figure 1

The case of a circle is instructive. The circle of radius 5 centered at (2, 3) has equation

$$(x-2)^2+(y-3)^2=25$$

or, in equivalent expanded form,

$$x^2+y^2-4x-6y=12$$

The same circle with its center at the origin of the uv-coordinate system (Figure 1) has the simple equation

$$u^2+v^2=25$$

The introduction of new axes does not change the shape or size of a curve, but it may greatly simplify its equation. It is this *translation* of axes and the corresponding change of variables in an equation that we wish to investigate.

Translations

Translations If new axes are chosen in the plane, every point will have two sets of coordinates, the old ones, (x, y), relative to the old axes and the new ones, (u, v), relative to the new axes. The original coordinates are said to undergo a **transformation.** If the new axes are parallel, respectively, to the original axes and have the same directions and scales, then the transformation is called a **translation of axes.**

From Figure 2, it is easy to see how the new coordinates (u, v) relate to the old ones (x, y). Let (h, k) be the old coordinates of the new origin. Then

$$\boxed{u = x - h, \qquad v = y - k}$$

or, equivalently,

$$\boxed{x = u + h, \qquad y = v + k}$$

EXAMPLE 1 Find the new coordinates of $P(-6, 5)$ after a translation of axes to a new origin at $(2, -4)$.

SOLUTION Since $h = 2$ and $k = -4$, it follows that

$$u = x - h = -6 - 2 = -8 \qquad v = y - k = 5 - (-4) = 9$$

The new coordinates are $(-8, 9)$. ■

EXAMPLE 2 Given the equation $4x^2 + y^2 + 40x - 2y + 97 = 0$, find the equation of its graph after a translation with new origin $(-5, 1)$.

SOLUTION In the equation, we replace x by $u + h = u - 5$ and y by $v + k = v + 1$. We obtain

$$4(u - 5)^2 + (v + 1)^2 + 40(u - 5) - 2(v + 1) + 97 = 0$$

or

$$4u^2 - 40u + 100 + v^2 + 2v + 1 + 40u - 200 - 2v - 2 + 97 = 0$$

This simplifies to

$$4u^2 + v^2 = 4$$

or

$$u^2 + \frac{v^2}{4} = 1$$

which we recognize as the equation of an ellipse.

Completing the Square

Completing the Square Given a complicated second-degree equation, how do we know what translation will simplify the equation and bring it to a recognizable form? We can complete the square to eliminate the first-degree terms of any expression of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0, \qquad A \neq 0, \quad C \neq 0$$

EXAMPLE 3 Make a translation that will eliminate the first-degree terms of

$$4x^2 + 9y^2 + 8x - 90y + 193 = 0$$

and use this information to sketch the graph of the given equation.

SOLUTION Recall that to complete the square of $x^2 + ax$ we must add $a^2/4$ (the square of half the coefficient of x). Using this, we rewrite the given equation by adding the same numbers to both sides.

$$\begin{aligned}
4(x^2 + 2x \qquad) + 9(y^2 - 10y \qquad) &= -193 \\
4(x^2 + 2x + 1) + 9(y^2 - 10y + 25) &= -193 + 4 + 225 \\
4(x + 1)^2 + 9(y - 5)^2 &= 36 \\
\frac{(x + 1)^2}{9} + \frac{(y - 5)^2}{4} &= 1
\end{aligned}$$

The translation $u = x + 1$ and $v = y - 5$ transforms this to

$$\frac{u^2}{9} + \frac{v^2}{4} = 1$$

which is the standard form of a horizontal ellipse. The graph is shown in Figure 3. ■

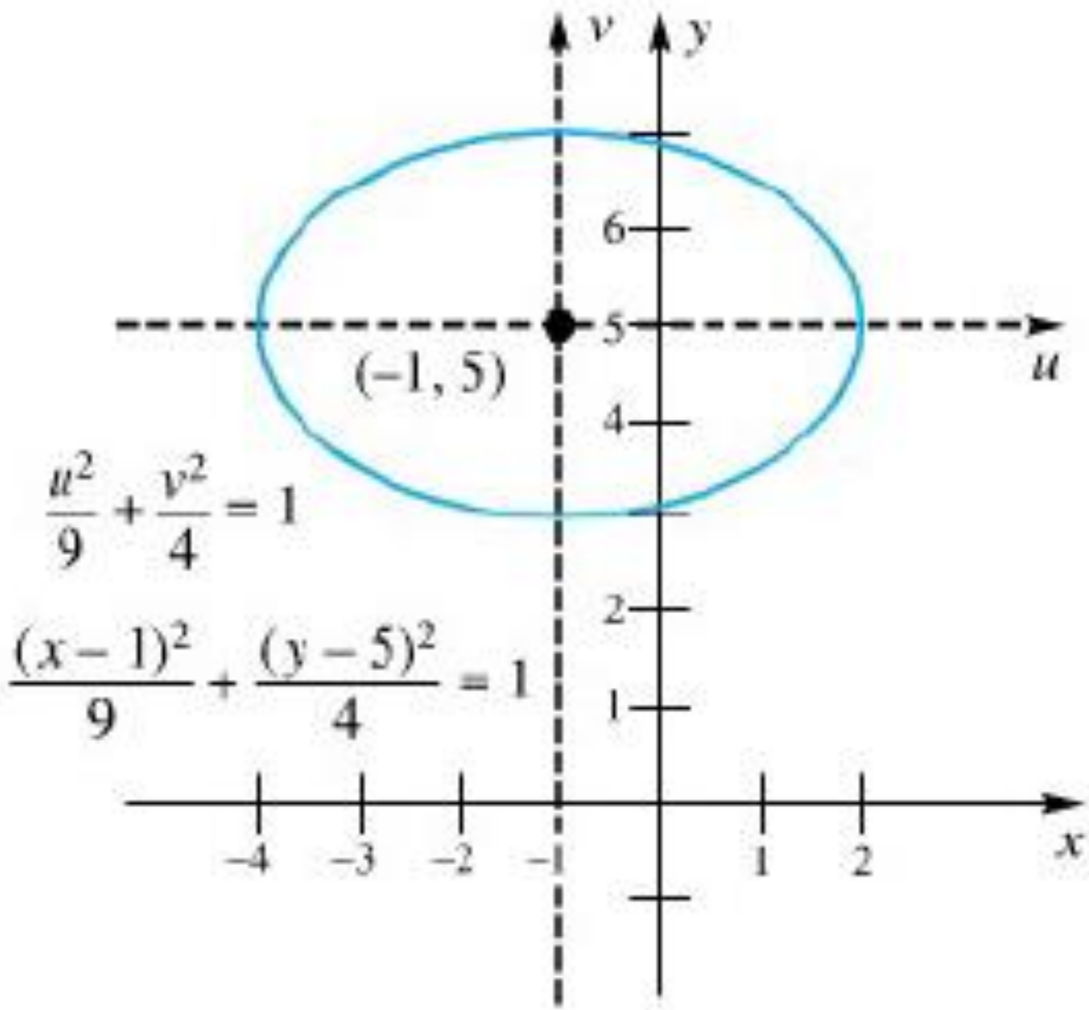


Figure 3

General Second-Degree Equation

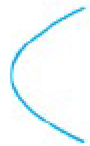
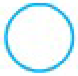
General Second-Degree Equations Now we ask an important question. Is the graph of an equation of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

always a conic? The answer is no, unless we admit certain limiting forms. The following table indicates the possibilities with a sample equation for each.

Thus, the graphs of the general quadratic equation above fall into three general categories, but yield nine different possibilities, including limiting forms.

General Second-Degree Equation

Conics		Limiting Forms	
1. $(AC = 0)$ Parabola: $y^2 = 4x$		Parallel lines: $y^2 = 4$	
		Single line: $y^2 = 0$	
		Empty set: $y^2 = -1$	
2. $(AC > 0)$ Ellipse: $\frac{x^2}{9} + \frac{y^2}{4} = 1$		Circle: $x^2 + y^2 = 4$	
		Point: $2x^2 + y^2 = 0$	
		Empty set: $2x^2 + y^2 = -1$	
3. $(AC < 0)$ Hyperbola: $\frac{x^2}{9} - \frac{y^2}{4} = 1$		Intersecting lines: $x^2 - y^2 = 0$	

EXAMPLE 5 Use a translation to simplify

$$4x^2 - y^2 - 8x - 6y - 5 = 0$$

and sketch its graph.

SOLUTION We rewrite the equation as follows:

$$\begin{aligned} 4(x^2 - 2x \qquad) - (y^2 + 6y \qquad) &= 5 \\ 4(x^2 - 2x + 1) - (y^2 + 6y + 9) &= 5 + 4 - 9 \\ 4(x - 1)^2 - (y + 3)^2 &= 0 \end{aligned}$$

Let $u = x - 1$ and $v = y + 3$, which results in

$$4u^2 - v^2 = 0$$

or

$$(2u - v)(2u + v) = 0$$

This is the equation of two intersecting lines (Figure 5).

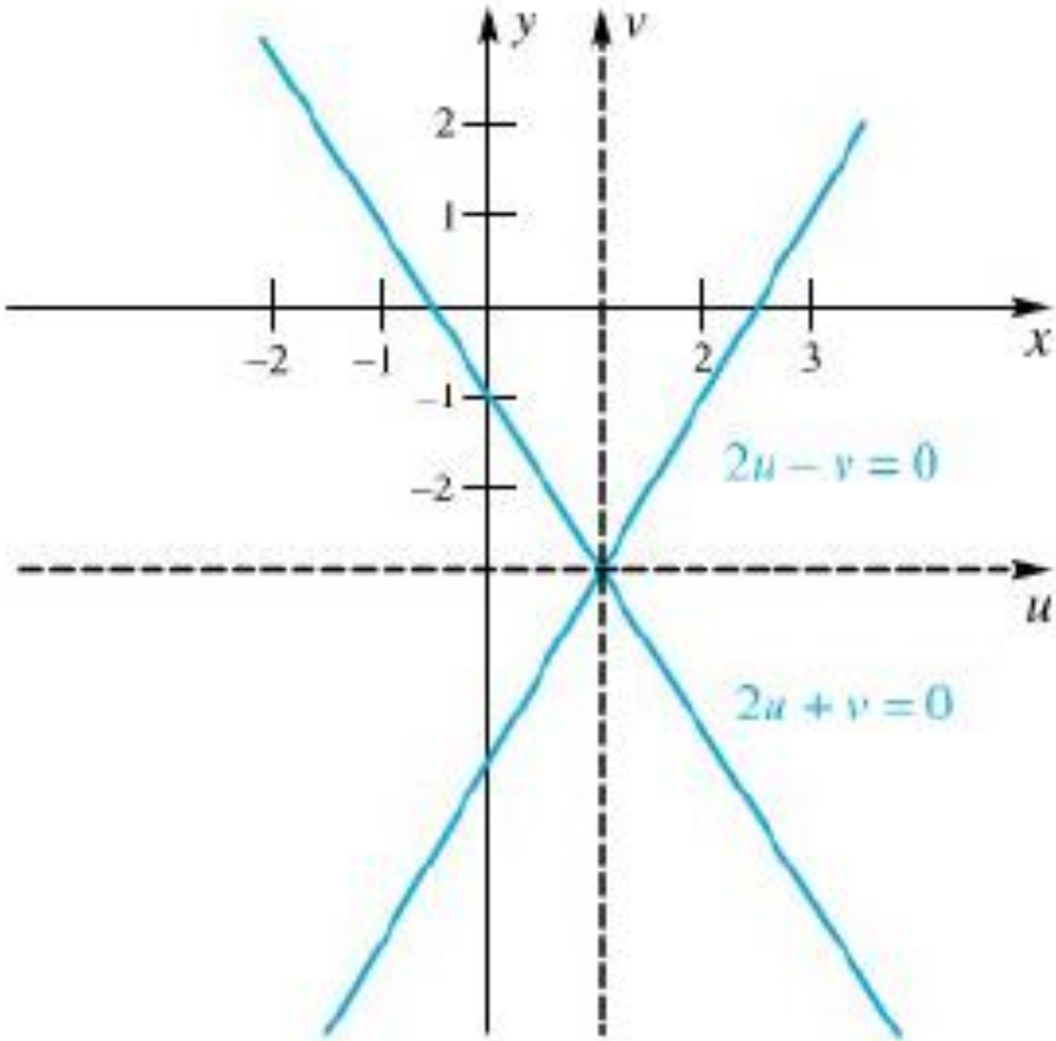


Figure 5

Rotations

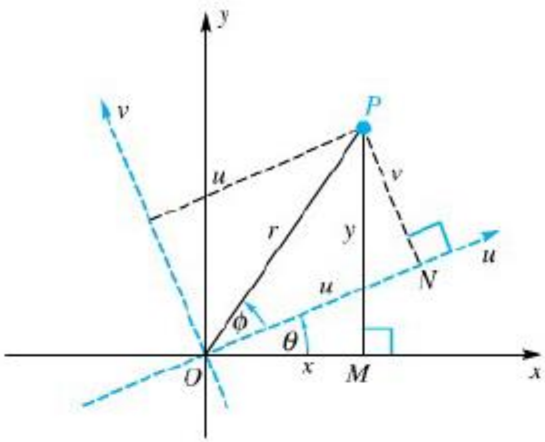


Figure 6

Rotations Introduce a new pair of coordinate axes, the u- and v-axes, with the same origin as the x- and y-axes but rotated through an angle θ, as shown in Figure 6. A point P then has two sets of coordinates: (x, y) and (u, v). How are they related?

Let r denote the length of OP, and let ϕ denote the angle from the positive u-axis to OP. Then x, y, u, and v have the geometric interpretations shown in the diagram.

Looking at the right triangle OPM, we see that

$$\cos(\phi + \theta) = \frac{x}{r}$$

so

$$\begin{aligned} x = r\cos(\phi + \theta) &= r(\cos\phi\cos\theta - \sin\phi\sin\theta) \\ &= (r\cos\phi)\cos\theta - (r\sin\phi)\sin\theta \end{aligned}$$

Consideration of triangle OPN shows that $u = r\cos\phi$ and $v = r\sin\phi$. Thus,

$$\boxed{x = u\cos\theta - v\sin\theta}$$

Similar reasoning leads to

$$\boxed{y = u\sin\theta + v\cos\theta}$$

These formulas determine a transformation called a **rotation of axes.**

EXAMPLE 7 Find the new equation that results from $xy = 1$ after a rotation of axes through $\theta = \pi/4$. Sketch the graph.

SOLUTION The required substitutions are

$$x = u\cos\frac{\pi}{4} - v\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}(u - v)$$

$$y = u\sin\frac{\pi}{4} + v\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}(u + v)$$

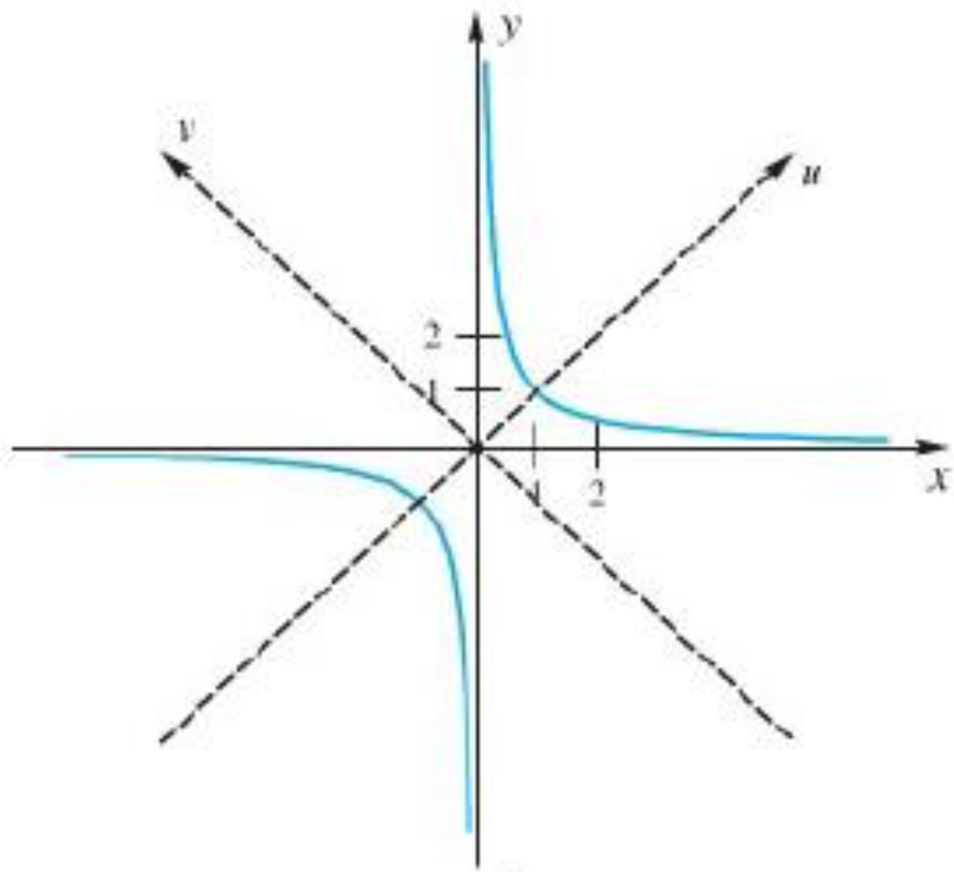
The equation $xy = 1$ takes the form

$$\frac{\sqrt{2}}{2}(u - v)\frac{\sqrt{2}}{2}(u + v) = 1$$

which simplifies to

$$\frac{u^2}{2} - \frac{v^2}{2} = 1$$

This we recognize as the equation of a hyperbola with $a = b = \sqrt{2}$. Note that the cross-product term has disappeared as a result of the rotation. The choice of the angle $\theta = \pi/4$ was just right to make this happen. The graph is shown in Figure 7. ■



$$xy = 1$$
or
$$\frac{u^2}{2} - \frac{v^2}{2} = 1$$

Figure 7

Determining the Angle θ

Determining the Angle θ How do we know what rotation to make in order to eliminate the cross-product term? Consider the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

If we make the substitutions

$$x = u\cos\theta - v\sin\theta$$

$$y = u\sin\theta + v\cos\theta$$

this equation takes the form

$$au^2 + buv + cv^2 + du + ev + f = 0$$

where a, b, c, d, e, and f are numbers that depend on θ. We could find expressions for all of them, but we really care only about b. When we do the necessary algebra, we find that

$$\begin{aligned} b &= B(\cos^2\theta - \sin^2\theta) - 2(A - C)\sin\theta\cos\theta \\ &= B\cos 2\theta - (A - C)\sin 2\theta \end{aligned}$$

To make $b = 0$, we require that

$$B\cos 2\theta = (A - C)\sin 2\theta$$

or

$$\boxed{\cot 2\theta = \frac{A - C}{B}}$$

EXAMPLE 8 Make a rotation of axes to eliminate the cross-product term in

$$4x^2 + 2\sqrt{3}xy + 2y^2 + 10\sqrt{3}x + 10y = 5$$

Then sketch the graph.

SOLUTION

$$\cot 2\theta = \frac{A - C}{B} = \frac{4 - 2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$$

which means that $2\theta = \pi/3$ and $\theta = \pi/6$. The appropriate substitutions are

$$x = u\frac{\sqrt{3}}{2} - v\frac{1}{2} = \frac{\sqrt{3}u - v}{2}$$

$$y = u\frac{1}{2} + v\frac{\sqrt{3}}{2} = \frac{u + \sqrt{3}v}{2}$$

Our equation transforms first to

$$4\frac{\left(\sqrt{3}u - v\right)^2}{4} + 2\sqrt{3}\frac{\left(\sqrt{3}u - v\right)\left(u + \sqrt{3}v\right)}{4}$$

$$+ 2\frac{\left(u + \sqrt{3}v\right)^2}{4} + 10\sqrt{3}\frac{\sqrt{3}u - v}{2} + 10\frac{u + \sqrt{3}v}{2} = 5$$

and, after simplifying, to

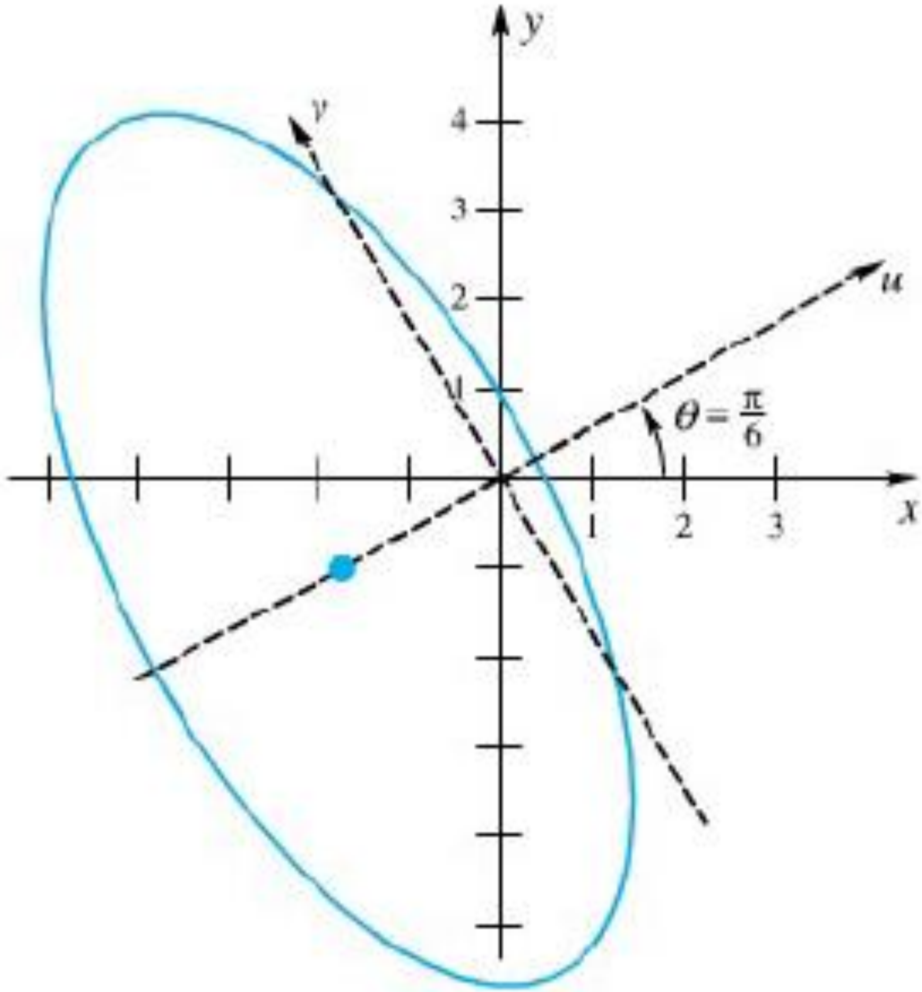
$$5u^2 + v^2 + 20u = 5$$

To put this equation in recognizable form, we complete the square.

$$5(u^2 + 4u + 4) + v^2 = 5 + 20$$

$$\frac{(u + 2)^2}{5} + \frac{v^2}{25} = 1$$

We identify the last equation as that of a vertical ellipse with center at $u = -2$ and $v = 0$ and with $a = 5$ and $b = \sqrt{5}$. This allows us to draw the graph shown in Figure 8. If we wanted to carry the simplifying process further, we would make the translation $r = u + 2, s = v$, which results in the standard equation $r^2/5 + s^2/25 = 1$. ■



$$4x^2 + 2\sqrt{3}xy + 2y^2 + 10\sqrt{3}x + 10y = 5$$

$$\text{or } \frac{(u+2)^2}{5} + \frac{v^2}{25} = 1$$

Figure 8