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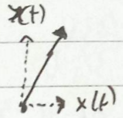
A) parametric Representation of Curves in the plane

1.) Plane curve



$$x = f(t), \quad y = g(t)$$

ex: eliminating param

$x = t^2 + 2t, \quad y = t - 3, \quad -2 \le t \le 3$

$t = y + 3$

$\Rightarrow x = (y+3)^2 + 2(y+3) = y^2 + 8y + 15$

$x + 1 = (y+4)^2$

2.) Differentiation & Integration

$x = f(t), \ y = g(t) \Rightarrow \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$

$\int \frac{dy}{dx}$ ✓ ✗

ex: $\frac{dy}{dx}$ at $t = \frac{\pi}{6}$ if $x = 5\cos t, \ y = 4\sin t$

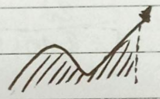
$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{4\cos t}{-5\sin t} = -\dfrac{4}{5}\cot t = \dfrac{-4\sqrt{3}}{5}$

ex: $\int_1^3 y\,dx$ where $x = 2t - 1 \quad y = t^2 + 2$

$dx = 2dt$ when $x = 1, t = 1$ and $x = 3, t = 2$

$\Rightarrow \int_1^2 (t^2+2)\cdot 2\,dt = \frac{26}{3}$

3.) Area under a Parametric Curve

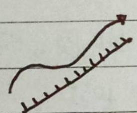


$$A = \int_a^b y(t)\cdot x'(t)\,dt$$

ex: $x(t) = t - \sin t, \ y(t) = 1 - \cos t \quad 0 \le t \le 2\pi$

$A = \int_a^b y(t)\cdot x'(t)\,dt = \int_0^{2\pi} (1-\cos t)(1-\cos t)\,dt = \int_0^{2\pi} \left(\frac{3}{2} - 2\cos t + \frac{\cos 2t}{2}\right) dt = 3\pi$

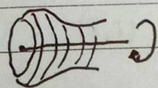
4.) Arc Length



$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

ex: $x(t) = 3\cos t, \ y(t) = 3\sin t$

$s = \int_0^{\pi} \sqrt{(-3\sin t)^2 + (3\cos t)^2} = \int_0^{\pi} 3\,dt = 3\pi$



5.) Surface area

around x-axis: $S = 2\pi \int_a^b y(t)\sqrt{x'(t)^2 + y'(t)^2}\,dt$ around y-axis: $S = 2\pi \int_a^b x(t)\sqrt{x'(t)^2 + y'(t)^2}\,dt$

JOYKO® 36 Lines, 6 mm

B] Polar Coordinate

*Polar to cartesian

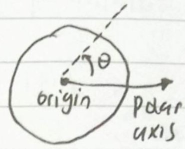
$x = r\cos\theta$

$y = r\sin\theta$

*Cartesian to polar

$r^2 = x^2 + y^2$

$\tan\theta = y/x$



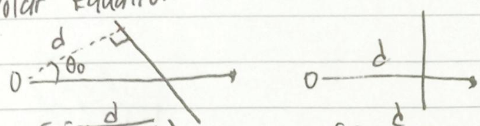
ex: Polar of $(-3, \sqrt{3})$

$r^2 = (-3)^2 + (\sqrt{3})^2 = 12$

$\tan\theta = \frac{\sqrt{3}}{-3} \Rightarrow (r, \theta) = (2\sqrt{3}, \frac{5\pi}{6}) \vee (-2\sqrt{3}, -\frac{\pi}{6})$

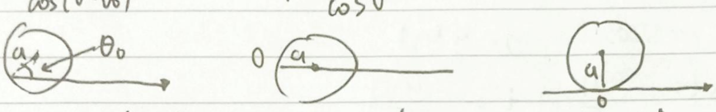
•] Polar Equation

Line:



$r = \frac{d}{\cos(\theta - \theta_0)}$ $\qquad$ $r = \frac{d}{\cos\theta}$

Circle:



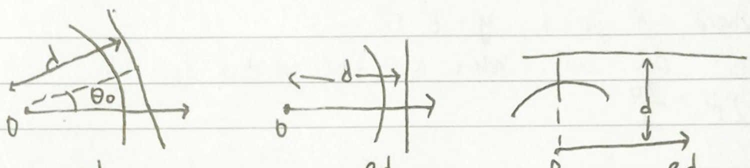
$r = 2a\cos(\theta - \theta_0)$ $\qquad$ $r = 2a\cos\theta$ $\qquad$ $r = 2a\sin\theta$

2a = jari" lingkaran

Ellipse 0<e<1

parabola e=1

hype e>1



$r = \frac{ed}{1 + e\cos(\theta - \theta_0)}$ $\qquad$ $r = \frac{ed}{1 + e\cos\theta}$ $\qquad$ $r = \frac{ed}{1 + e\sin\theta}$

ex: reverse to cartesian $r = \frac{2}{1 - \cos\theta}$

$r - r\cos\theta = 2$

$r - x = 2$

$r^2 = x^2 + 4x + 4$

$x^2 + y^2 = x^2 + 4x + 4 \Rightarrow y^2 = 4(x+1)$ parabola vertex at $(-1, 0)$

C] Graphs of Polar Equation

1.) Symmetric test

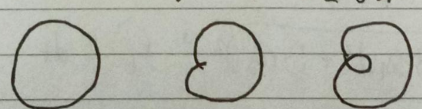
(r, θ) to $(r, -\theta) \rightarrow$ x-axis

(r, θ) to $(-r, -\theta) \rightarrow$ y-axis

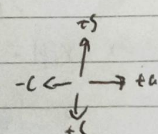
(r, θ) to $(-r, \theta) \rightarrow$ origin

2.) Cardiods and Limaçons

$r = a \pm b\cos\theta$ $\qquad$ $r = a \pm b\sin\theta$



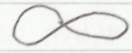
a > b $\qquad$ a = b $\qquad$ a < b



when $\frac{a}{2} < b < a$ start to looks cardiodoid

3.) Lemniscates

$r^2 = \pm a \cos 2\theta$ $\quad$ $r^2 = \pm a \sin 2\theta$



4.) Roses

$r = a \cos n\theta$ $\quad$ $r = a \sin n\theta$ $\quad \begin{cases} \text{has } n \text{ leaves if } n \text{ is odd} \\ \text{has } 2n \text{ leaves if } n \text{ is even} \end{cases}$

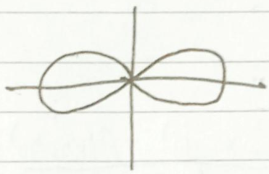
5.) Spirals

$r = a\theta$

ex: $r^2 = 8 \cos 2\theta$

since: $\cos(-2\theta) = \cos 2\theta$ origin sim.

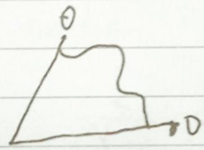
θ	r
0	± 2.8
$\pi/12$	± 2.6
$\pi/6$	± 2
$\pi/4$	0



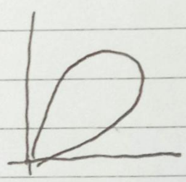
D] Calculus in polar coordinate

1.) Area of sector

$$A = \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta$$



ex: Area of one leaf of $r = 4 \sin 2\theta$



$$A = \frac{1}{2} \int_0^{\pi/2} 16 \sin^2 2\theta \, d\theta = 8 \int_0^{\pi/2} \frac{1 - \cos 4\theta}{2} \, d\theta = 4 \int_0^{\pi/2} d\theta - \int_0^{\pi/2} \cos 4\theta . 4 \, d\theta$$

$$= 2\pi$$

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A] Derivatives of Function of 2 or More Variables

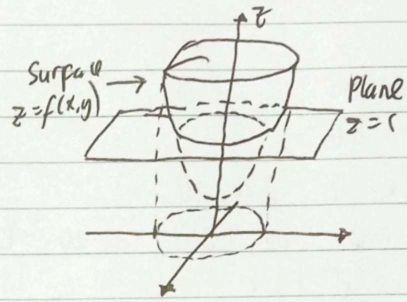
* Real valued function of 2 var, is a function f that assigned to each ordered pair (x,y) in some set D

ex: (1) $f(x,y) = x^2 + 3y^2$

(2) $g(x,y) = 2x\sqrt{y}$

* Level curves

Projection of the curves on the xy-plane is called level curve.



B] Partial

1.) Partial Derivative

$$f_x(x_0,y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

ex: f_x and f_y of $f(x,y) = x^2 + 3y^3$

$f_x(x,y) = 2xy + 0$ → treat y as constant

$f_y(x,y) = x^2 + 9y^2$ → treat x as constant

Symbols → $\frac{\partial}{\partial x}$ or $\frac{\partial}{\partial y}$

2) Higher Partial Derivatives

$f_{xx} = \frac{\partial^2 f}{\partial x^2}$ $f_{yy} = \frac{\partial^2 f}{\partial y^2}$

$f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$ $f_{yx} = \frac{\partial^2 f}{\partial x \partial y}$

ex: $f(x,y) = xe^y - \sin\left(\frac{x}{y}\right) + x^3y^2$

$f_x(x,y) = e^y - \frac{1}{y}\cos\left(\frac{x}{y}\right) + 3x^2y^2$

$f_y(x,y) = xe^y + \frac{x}{y^2}\cos\left(\frac{x}{y}\right) + 2x^3y$

$f_{xx}(x,y) = \frac{1}{y^2}\sin\left(\frac{x}{y}\right) + 6xy^2$

$f_{yy}(x,y) = xe^y + \frac{x^2}{y^4}\sin\left(\frac{x}{y}\right) - \frac{2x}{y^3}\cos\left(\frac{x}{y}\right) + 2x^3$

$f_{xy}(x,y) = e^y - \frac{x}{y^3}\sin\left(\frac{x}{y}\right) + \frac{1}{y^2}\cos\left(\frac{x}{y}\right) + 6x^2y$

$= f_{yx}(x,y)$

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3) More Than Two Variables

$$f_x(x,y,z) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y, z) - f(x,y,z)}{\Delta x}$$

ex: $f = xy + 2yz + 2zx$

$f_x = y + 3z$
$f_y = x + 2z$
$f_z = 2y + 3x$

c) Limit & Continuity

* Limits

if $f(x,y)$ is a polynomial, then

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

if $f(x,y) = p(x,y)/q(x,y)$, where p & q are polynomial, then

$$\lim_{(x,y)\to(a,b)} f(x,y) = \frac{p(a,b)}{q(a,b)}$$

provided $q(a,b) \neq 0$, else the limit does not exist

ex: $\lim_{(x,y)\to(0,0)} \dfrac{x^2+y^2+1}{x^2-y^2} \begin{matrix} \} \; 1 \\ \} \; 0 \end{matrix}$ DNE

ex: $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ has no lim at origin

$$\left.\begin{array}{l} * \lim\limits_{(x,0)\to(0,0)} f(x,0) = \lim\limits_{(x,0)\to(0,0)} \dfrac{x^2-0}{x^2+0} = +1 \\ * \lim\limits_{(0,y)\to(0,0)} \dfrac{0-y^2}{0+y^2} = -1 \end{array}\right\} \text{DNE at } (0,0)$$

* Continuity at a point

if $g(x,y)$ continuous at (a,b) and $f(x)$ continuous at $g(a,b)$, then $f \circ g$ defined $(f \circ g)(x,y) = f(g(x,y))$ is continuous at (a,b)

$$\text{continuous} \left\{ \begin{array}{l} \text{① } f \text{ has value at } (a,b) \\ \text{② } f \text{ has limit at } (a,b) \\ \text{③ } f(a,b) = \lim\limits_{(x,y)\to(a,b)} f(x,y) \end{array} \right.$$

* Mixed partial

if continuous $\to f_{xy} = f_{yx}$

D] Differentiability

f is locally linear if:

$$f(a+h) = f(a) + hm + h\,\varepsilon(h)$$

where $\lim_{h\to 0} \varepsilon(h) = 0$

Thus, f is differentiable at p if:

$$\lim_{\varepsilon(h)\to 0}\ \lim_{h\to 0} f(p+h) = f(p) + \nabla f(p)\cdot h + \varepsilon(h)\cdot h$$

Theorem A: If $F(x,y)$ has continuous partial derivatives $f_x(x,y)$ and $f_y(x,y)$ on a disk D whose interior contains (a,b) then $f(x,y)$ is differentiable at (a,b)

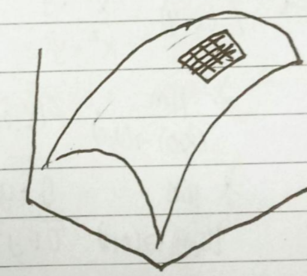
$$\boxed{\nabla f(x,y) = \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \end{pmatrix}}$$

* Equation of Tangent Plane at (a,b)

$$\boxed{z = f(a,b) + \nabla f(a,b)\cdot\langle x-a,\, y-b\rangle}$$

* property of ∇

1.) $\nabla[f(p)+g(p)] = \nabla f(p) + \nabla g(p)$
2.) $\nabla[\alpha f(p)] = \alpha \nabla f(p)$
3.) $\nabla[f(p)\cdot g(p)] = f(p)\nabla g(p) + g(p)\nabla f(p)$



* Continuity vs. Differentiability

If f is differentiable at p, then f is continuous at p

E] Directional Derivatives and Gradient

For any unit vector u, let:

$$D_u f(p) = \lim_{h\to 0} \frac{f(p+hu) - f(p)}{h}$$

Thus

$$\boxed{D_u f(x,y) = u\cdot\nabla f(x,y) = D_u f(x,y) = u_1 f_x(x,y) + u_2 f_y(x,y)}$$

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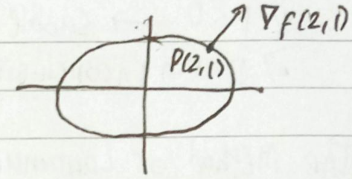
* Maximum Rate of Change

$$D_u f(p) = \|\nabla f(p)\| \cos\theta \quad \begin{cases} \text{max when } \theta = 0 \\ \text{min when } \theta = \pi \end{cases}$$

* Level Curves and Gradient

The gradient of f at a point P is perpendicular to the level curve of f that goes through P



F] Chain Rule

1.) First Version

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

2.) Second Version

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

3.) Implicit function

$$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y}$$

G] Tangent Planes and Approximation

Tangent Plane:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Approximation

$$dz = f_x(x, y)\,\Delta x + f_y(x, y)\,\Delta y$$

H] Maxima and Minima

i.) $f(p_0)$ is a global maximum value of f on S if $f(p_0) \geq f(p)$ for all p in S

ii) ——— " ——— minimum -"- ——— $\leq$ -"- ———

ii) either global maximum or minimum

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* Second partial test

$D = D(x_0, y_0) = f_{xx}(x_0, y_0) \cdot f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0)$

- $D > 0$ $\begin{cases} f_{xx} < 0 \rightarrow \text{local maximum} \\ f_{xx} > 0 \rightarrow \text{local minimum} \end{cases}$
- $D < 0 \rightarrow$ saddle point
- $D = 0 \rightarrow$ inconclusive

I| The Method of Lagrange Multipliers

To maximize/minimize $f(p)$ subject to constraint $g(p) = 0$

$$\boxed{\nabla f(p) = \lambda \nabla g(p) \text{ and } g(p) = 0}$$

Each such point P is a critical point for the constrained extremum problem

examples:

- (differentiability and tangent plane) of $f(x,y) = xe^y + x^2y$ at $(2,0)$

$\frac{\partial f}{\partial x} = e^y + 2xy$ and $\frac{\partial f}{\partial y} = xe^y + x^2$

$\nabla f(2,0) = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$

$z = f(2,0) + \nabla f(2,0) \cdot \langle x-2, y \rangle = 2 + \langle 1, 6 \rangle \cdot \langle x-2, y \rangle$
$= 2 + x - 2 + 6y = x + 6y$ //

- (directional derivative) $f(x,y) = 4x^2 - xy + 3y^2$ at $(2,-1)$ in direction of vector $\vec{a} = 4i + 3j$

$D_u f(2,-1) = \langle \frac{4}{5}, \frac{3}{5} \rangle \cdot \langle 17, -8 \rangle = \frac{4}{5} \cdot 17 + \frac{3}{5} \cdot (-8) = \frac{44}{5}$

- (Chain rule) $z = x^3y$; $x = 2t$; $y = t^2$. Find dz/dt

$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} = 3x^2y \cdot 2 + x^3 \cdot 2t = 40t^4$

- (extrema) of $f(x,y) = 3x^3 + y^2 - 9x + 4y$

Since $F_x = 9x^2 - 9$ and $F_y = 2y + 4$, by solving it $F_x = F_y = 0 \Rightarrow (1,-2)$ and $(-1,-2)$
Now $F_{xx} = 18x$; $F_{yy} = 2$, $F_{xy} = 0$ thus $D = 18 \cdot 2 = 36 > 0$ por $(1,-2)$
Thus $F_{xx} = 18 > 0 \rightarrow$ local minimum
Then for $(-1,-2)$ $D = -36 < 0 \rightarrow$ saddle point

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i) (Method of Lagrange Multipliers)

Find minimum of $f(x,y,z) = 3x + 2y + z + 5$ subject to the constraint $g(x,y,z) = 9x^2 + 4y^2 - z = 0$ (4)

$\nabla f = \lambda \nabla g$

$$\begin{array}{c} (1) \\ (2) \\ (3) \end{array} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 18x \\ 8y \\ -1 \end{pmatrix}$$

- From (3), $\lambda = -1$, we get $x = -\frac{1}{6}$, $y = -\frac{1}{4}$, $z = \frac{1}{2}$

so $f\left(-\frac{1}{6}, -\frac{1}{4}, \frac{1}{2}\right) = \frac{9}{2}$ //