Section 10.4

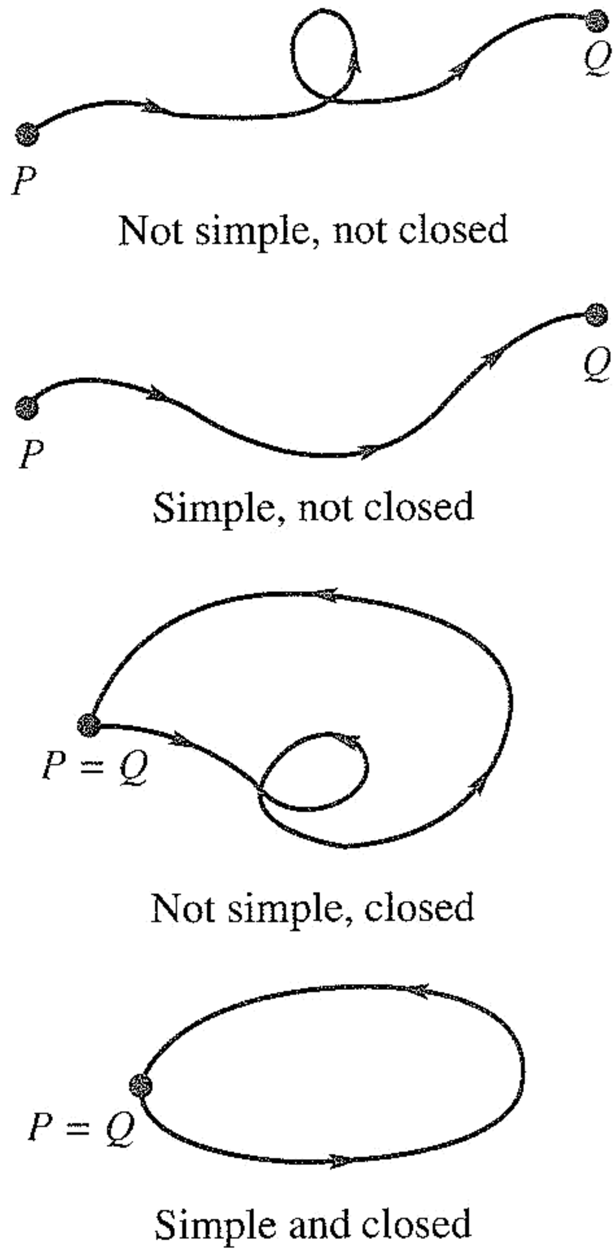
Parametric Representation of Curves in the Plane

A Plane Curve

A **plane curve** is determined by a pair of parametric equations

$$x = f(t), \qquad y = g(t), \qquad t \text{ in } I$$

with f and g continuous on the interval I. Usually I is a closed interval $[a, b]$. Think of t, called the **parameter,** as measuring time. As t advances from a to b, the point (x, y) traces out the curve in the xy-plane. When I is the closed interval $[a, b]$, the points $P = (x(a), y(a))$ and $Q = (x(b), y(b))$ are called the **initial** and **final end points.** If the curve has end points that coincide, then we say that the curve is **closed.** If distinct values of t yield distinct points in the plane (except possibly for $t = a$ and $t = b$), we say the curve is a **simple** curve (Figure 1). The pair of relationships $x = f(t)$, $y = g(t)$, together with the interval I is called the **parametrization** of the curve.



P
Q
Not simple, not closed
P
Q
Simple, not closed
P = Q
Not simple, closed
P = Q
Simple and closed

Figure 1

Eliminating the Parameter To recognize a curve given by parametric equations, it may be desirable to eliminate the parameter. Sometimes this can be accomplished by solving one equation for t and substituting in the other (Example 1). Often we can make use of a familiar identity, as in Example 2.

EXAMPLE 1 Eliminate the parameter in

$$x = t^2 + 2t, \qquad y = t - 3, \qquad -2 \le t \le 3$$

Then identify the corresponding curve and sketch its graph.

SOLUTION From the second equation, $t = y + 3$. Substituting this expression for t in the first equation gives

$$x = (y + 3)^2 + 2(y + 3) = y^2 + 8y + 15$$

or

$$x + 1 = (y + 4)^2$$

This we recognize as a parabola with vertex at $(-1, -4)$ and opening to the right.

In graphing the given equation, we must be careful to display only that part of the parabola corresponding to $-2 \le t \le 3$. A table of values and the graph are shown in Figure 2. The arrowhead indicates the curve's *orientation*, that is, the direction of increasing t. ■

t	x	y
–2	0	–5
–1	–1	–4
0	0	–3
1	3	–2
2	8	–1
3	15	0

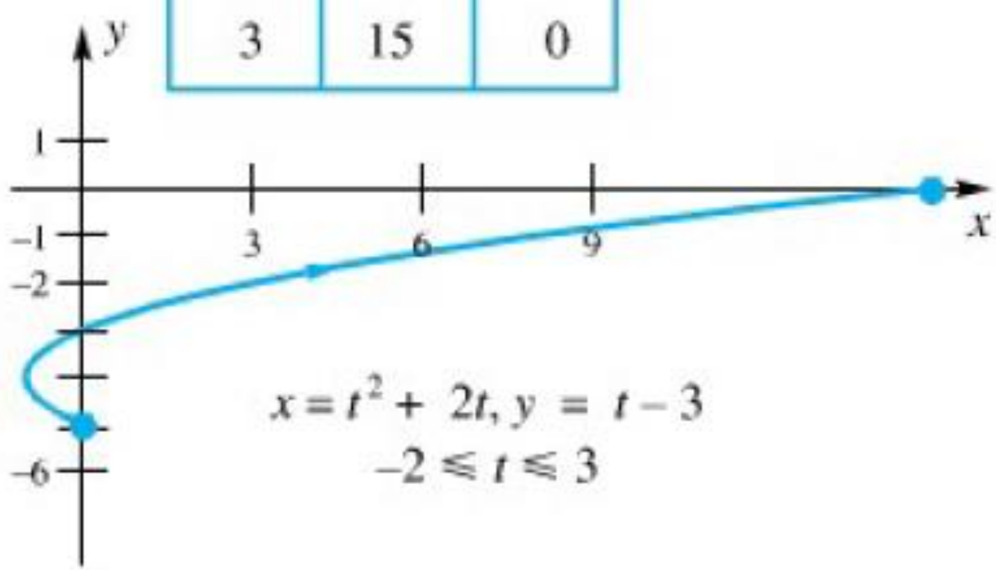


Figure 2

EXAMPLE 2 Show that

$$x = a \cos t, \qquad y = b \sin t, \qquad 0 \le t \le 2\pi$$

represents the ellipse shown in Figure 3.

SOLUTION We solve the equations for $\cos t$ and $\sin t$, then square, and add.

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \cos^2 t + \sin^2 t = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

A quick check of a few values for t convinces us that we do get the complete ellipse. In particular, $t = 0$ and $t = 2\pi$ give the same point, namely, $(a, 0)$.

If $a = b$, we get the circle $x^2 + y^2 = a^2$. ■

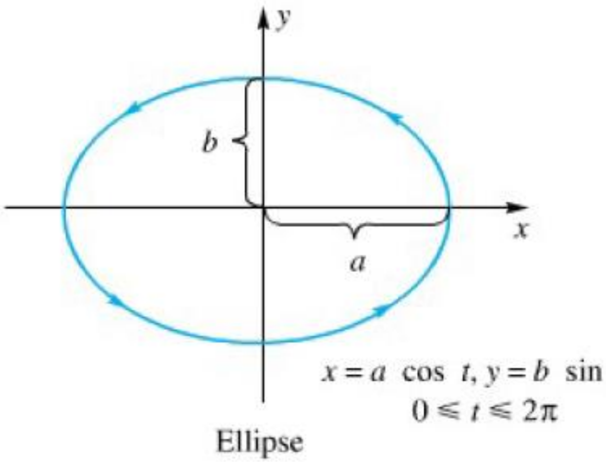


Figure 3

Calculus for Curves Defined Parametrically

Theorem A

Let f and g be continuously differentiable with $f'(t) \neq 0$ on $\alpha < t < \beta$. Then the parametric equations

$$x = f(t), \qquad y = g(t)$$

define y as a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

EXAMPLE 6 Find the first two derivatives dy/dx and d^2y/dx^2 for the function determined by

$$x = 5\cos t, \qquad y = 4\sin t, \qquad 0 < t < 3$$

and evaluate them at $t = \pi/6$ (see Example 2).

SOLUTION Let y' denote $\dfrac{dy}{dx}$. Then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4\cos t}{-5\sin t} = -\frac{4}{5}\cot t$$

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt} = \frac{\frac{4}{5}\csc^2 t}{-5\sin t} = -\frac{4}{25}\csc^3 t$$

At $t = \pi/6$,

$$\frac{dy}{dx} = \frac{-4\sqrt{3}}{5}, \qquad \frac{d^2y}{dx^2} = \frac{-4}{25}(8) = -\frac{32}{25}$$

The first value is the slope of the tangent line to the ellipse $x^2/25 + y^2/16 = 1$ at the point $\left(5\sqrt{3}/2, 2\right)$. You can check that this is so by implicit differentiation.

EXAMPLE 7 Evaluate (a) $\int_1^3 y\,dx$ and (b) $\int_1^3 xy^2\,dx$, where $x = 2t - 1$ and $y = t^2 + 2$.

SOLUTION From $x = 2t - 1$, we have $dx = 2\,dt$. When $x = 1, t = 1$ and when $x = 3, t = 2$.

(a) $$\int_1^3 y\,dx = \int_1^2 (t^2 + 2)2\,dt = 2\left[\frac{t^3}{3} + 2t\right]_1^2 = \frac{26}{3}$$

(b) $$\int_1^3 xy^2\,dx = \int_1^2 (2t - 1)(t^2 + 2)^2 2\,dt$$

$$= 2\int_1^2 (2t^5 - t^4 + 8t^3 - 4t^2 + 8t - 4)\,dt = 86\tfrac{14}{15}$$ ■

Exercise

1. For the following parametric equations, sketch the graphs, determine whether the graph close or not and simple or not, and find the cartesian equation of the curve by removing the parameter.
   a) $x = 2\sqrt{t-2},\ y = 3\sqrt{4-t};\ 2 \le t \le 4$
   b) $x = -2\sin r,\ y = -3\cos r;\ 0 \le r \le 4\pi$

2. Determine $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ from the following parametric equations.
   a) $x = 6s^2,\ y = -2s^3;\ s \ne 0$
   b) $x = 1 - \cos t,\ y = 1 + \sin t;\ t \ne n\pi$

3. Evaluate the following integrals.
   a) $\int_0^1 (x^2 - 4y)\,dx$, where $x = t + 1,\ y = t^3 + 4$.
   b) $\int_1^{\sqrt{3}} xy\,dy$, where $x = \sec t,\ y = \tan t$.

Area under a Parametric Curve

Consider the non-self-intersecting plane curve defined by the parametric equations

$$x = x(t), \quad y = y(t), \quad \text{for } a \le t \le b$$

and assume that $x(t)$ is differentiable. The area under this curve is given by

$$A = \int_a^b y(t)x'(t)\,dt. \tag{10.2.7}$$

Example 10.2.4: Finding the Area under a Parametric Curve

Find the area under the curve of the cycloid defined by the equations

$$x(t)=t-\sin t,\quad y(t)=1-\cos t,\quad \text{for } 0\le t\le 2\pi. \tag{10.2.8}$$

Solution

Using Equation 10.2.7, we have

$$\begin{aligned}
A &= \int_a^b y(t)x'(t)\,dt \\
&= \int_0^{2\pi}(1-\cos t)(1-\cos t)\,dt \\
&= \int_0^{2\pi}(1-2\cos t+\cos^2 t)\,dt \\
&= \int_0^{2\pi}\left(1-2\cos t+\frac{1+\cos(2t)}{2}\right)dt \\
&= \int_0^{2\pi}\left(\frac{3}{2}-2\cos t+\frac{\cos(2t)}{2}\right)dt \\
&= \frac{3t}{2}-2\sin t+\frac{\sin(2t)}{4}\Big|_0^{2\pi} \\
&= 3\pi
\end{aligned}$$

Arc Length of a Parametric Curve

Consider the plane curve defined by the parametric equations

$$x = x(t), \quad y = y(t), \quad \text{for } t_1 \le t \le t_2$$

and assume that $x(t)$ and $y(t)$ are differentiable functions of t. Then the arc length of this curve is given by

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \tag{10.2.10}$$

Example 10.2.5: Finding the Arc Length of a Parametric Curve

Find the arc length of the semicircle defined by the equations

$$x(t) = 3\cos t, \quad y(t) = 3\sin t, \quad \text{for } 0 \leq t \leq \pi.$$

Solution

The values $t = 0$ to $t = \pi$ trace out the blue curve in Figure 10.2.8. To determine its length, use Equation 10.2.10:

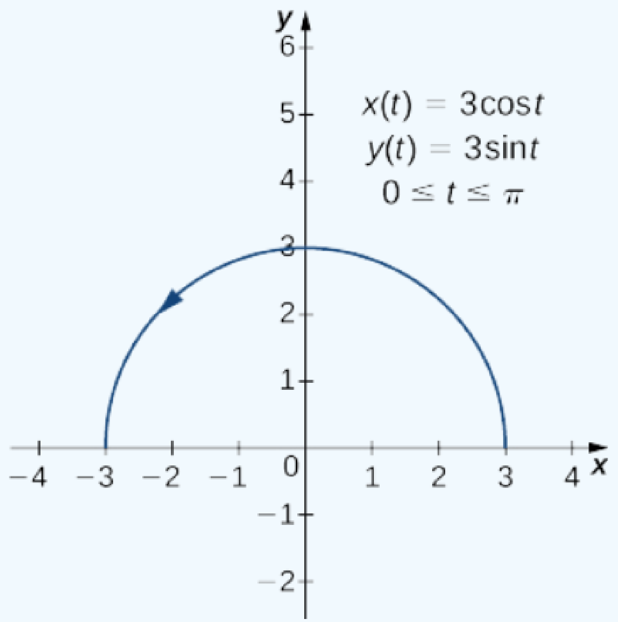


Figure 10.2.8: The arc length of the semicircle is equal to its radius times π.

$$\begin{aligned} s &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^{\pi} \sqrt{(-3\sin t)^2 + (3\cos t)^2}\, dt \\ &= \int_0^{\pi} \sqrt{9\sin^2 t + 9\cos^2 t}\, dt \\ &= \int_0^{\pi} \sqrt{9(\sin^2 t + \cos^2 t)}\, dt \\ &= \int_0^{\pi} 3\, dt = 3t\Big|_0^{\pi} \\ &= 3\pi \text{ units.} \end{aligned}$$

Surface Area

- The surface area of a volume of revolution revolved around the x-axis is given by

$$S = 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

- If the curve is revolved around the y-axis, then the formula is

$$S = 2\pi \int_a^b x(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

Example 10.2.6: Finding Surface Area

Find the surface area of a sphere of radius r centered at the origin.

Solution

We start with the curve defined by the equations

$$x(t) = r\cos t, \quad y(t) = r\sin t, \quad \text{for } 0 \le t \le \pi.$$

This generates an upper semicircle of radius r centered at the origin as shown in the following graph.

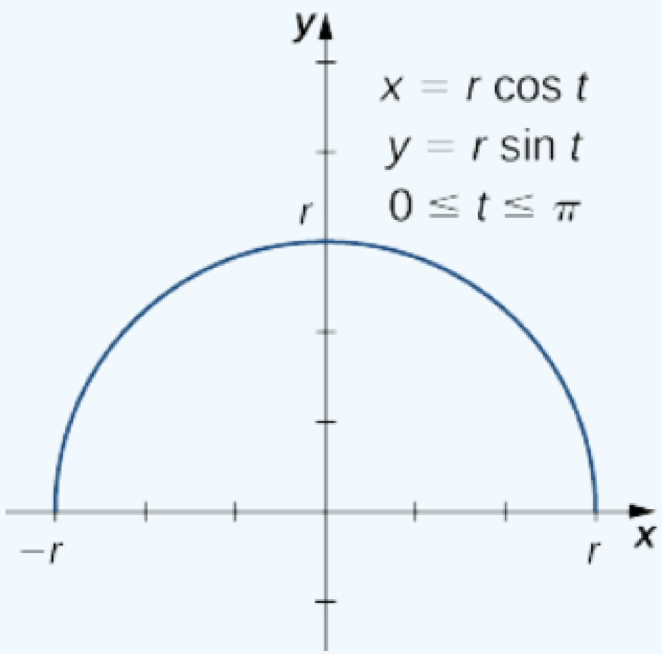


Figure 10.2.10: A semicircle generated by parametric equations.

When this curve is revolved around the x-axis, it generates a sphere of radius r. To calculate the surface area of the sphere, we use Equation 10.2.12:

$$\begin{aligned} S &= 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\,dt \\ &= 2\pi \int_0^\pi r\sin t\sqrt{(-r\sin t)^2 + (r\cos t)^2}\,dt \\ &= 2\pi \int_0^\pi r\sin t\sqrt{r^2\sin^2 t + r^2\cos^2 t}\,dt \\ &= 2\pi \int_0^\pi r\sin t\sqrt{r^2(\sin^2 t + \cos^2 t)}\,dt \\ &= 2\pi \int_0^\pi r^2\sin t\,dt \\ &= 2\pi r^2\left(-\cos t\Big|_0^\pi\right) \\ &= 2\pi r^2(-\cos\pi + \cos 0) \\ &= 4\pi r^2 \text{ units}^2. \end{aligned}$$

This is, in fact, the formula for the surface area of a sphere.

Exercise

1. Determine the area under the parametric curve given by $x = 6(\theta - \sin\theta), y = 6(\theta - \cos\theta)$, for $0 \le \theta \le 2\pi$.
2. Determine the length of the parametric curve given by $x = 3\sin t\,, y = 3\cos t\,$, for $0 \le \theta \le 2\pi$.
3. Determine the surface area of the solid obtained by rotating the following parametric curve about the $x-$axis.

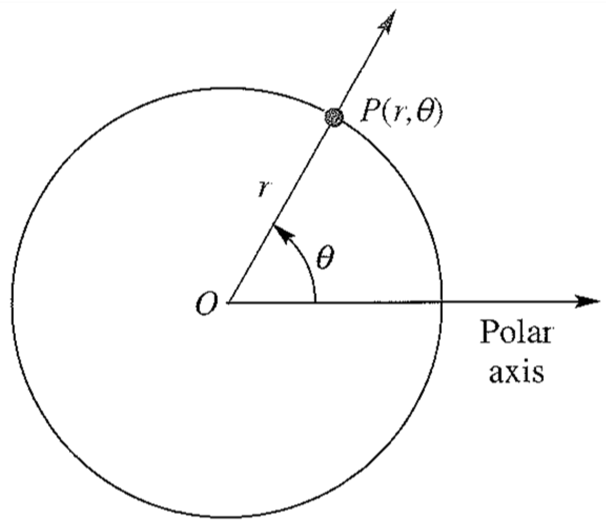
$$x = \cos^3\theta\,, y = \sin^3\theta\,, \text{for } 0 \le \theta \le \frac{\pi}{2}$$

Section 10.5

The Polar Coordinate System

Polar Coordinates

- We start with a fixed half-line, called the **polar axis**, emanating from a fixed point *O*, called the **pole** of **origin**.
- If r is the radius of the circle and θis one of the angles that the ray makes with the polar axis, the (r, θ) is a pair of **polar coordinates** for *P* (Figure 2).



Polar Coordinates

Figure 2

EXAMPLE 1 Graph the polar equation $r = 8 \sin \theta$.

SOLUTION We substitute multiples of $\pi/6$ for θ and calculate the corresponding r-values. See the table in Figure 5. Note that as θ increases from 0 to 2π the graph in Figure 5 is traced twice.

θ	r
0	0
$\pi/6$	4
$\pi/3$	6.93
$\pi/2$	8
$2\pi/3$	6.93
$5\pi/6$	4
π	0
$7\pi/6$	−4
$4\pi/3$	−6.93
$3\pi/2$	−8
$5\pi/3$	−6.93
$11\pi/6$	−4

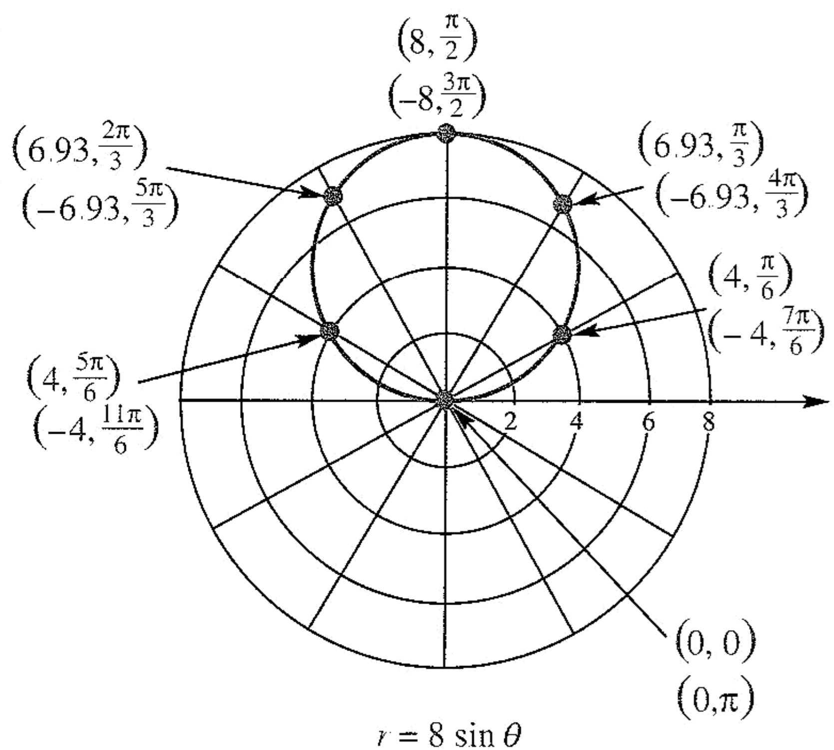


Figure 5

Relation to Cartesian Coordinates

Relation to Cartesian Coordinates We suppose that the polar axis coincides with the positive x-axis of the Cartesian system. Then the polar coordinates (r, θ) of a point P and the Cartesian coordinates (x, y) of the same point are related by the equations

Polar to Cartesian	**Cartesian to Polar**
$x = r\cos\theta$	$r^2 = x^2 + y^2$
$y = r\sin\theta$	$\tan\theta = y/x$

That this is true for a point P in the first quadrant is clear from Figure 7 and is easy to show for points in the other quadrants.

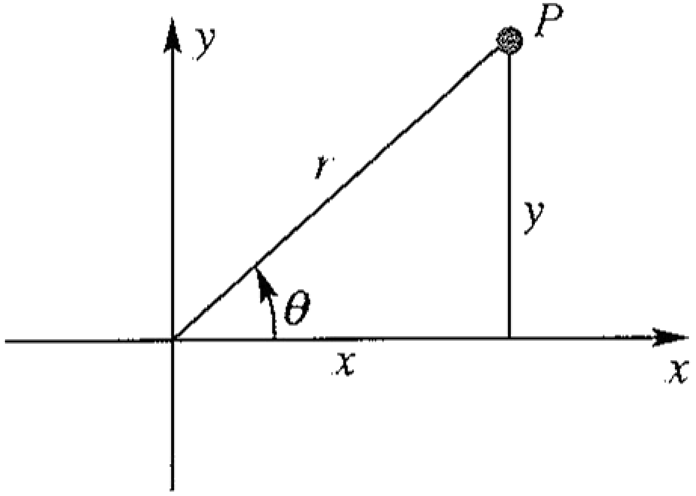


Figure 7

EXAMPLE 3 Find the Cartesian coordinates corresponding to $(4, \pi/6)$ and polar coordinates corresponding to $(-3, \sqrt{3})$.

SOLUTION If $(r, \theta) = (4, \pi/6)$, then

$$x = 4\cos\frac{\pi}{6} = 4\cdot\frac{\sqrt{3}}{2} = 2\sqrt{3}$$

$$y = 4\sin\frac{\pi}{6} = 4\cdot\frac{1}{2} = 2$$

If $(x, y) = (-3, \sqrt{3})$, then (see Figure 8)

$$r^2 = (-3)^2 + \left(\sqrt{3}\right)^2 = 12$$

$$\tan\theta = \frac{\sqrt{3}}{-3}$$

One value of (r, θ) is $(2\sqrt{3}, 5\pi/6)$. Another is $(-2\sqrt{3}, -\pi/6)$.

EXAMPLE 4 Show that the graph of $r = 8 \sin \theta$ (Example 1) is a circle and that the graph of $r = 2/(1 - \cos \theta)$ (Example 2) is a parabola by changing to Cartesian coordinates.

SOLUTION If we multiply $r = 8 \sin \theta$ by r, we get

$$r^2 = 8r \sin \theta$$

which, in Cartesian coordinates, is

$$x^2 + y^2 = 8y$$

and may be written successively as

$$x^2 + y^2 - 8y = 0$$

$$x^2 + y^2 - 8y + 16 = 16$$

$$x^2 + (y - 4)^2 = 16$$

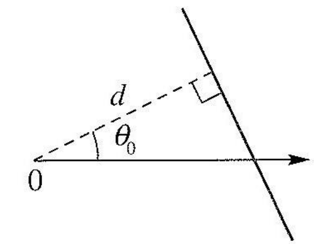
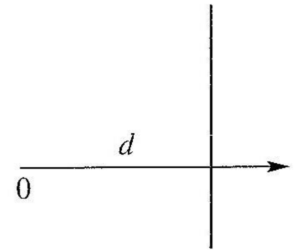
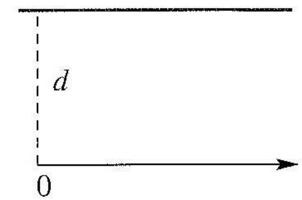
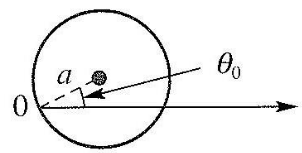
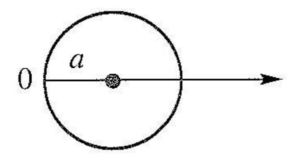
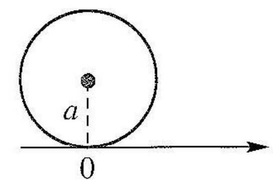
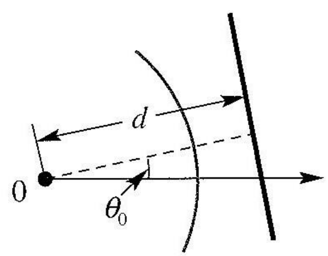
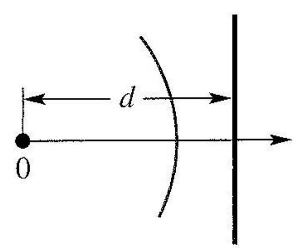
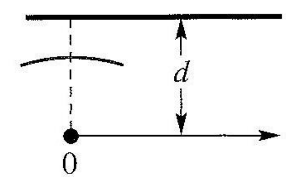
The latter is the equation of a circle of radius 4 centered at $(0, 4)$.

The second equation is handled by the following steps.

$$\begin{aligned} r &= \frac{2}{1 - \cos\theta} \\ r - r\cos\theta &= 2 \\ r - x &= 2 \\ r &= x + 2 \\ r^2 &= x^2 + 4x + 4 \\ x^2 + y^2 &= x^2 + 4x + 4 \\ y^2 &= 4(x + 1) \end{aligned}$$

We recognize the last equation as that of a parabola with vertex at $(-1, 0)$ and focus at the origin. ▒

Summary of Polar Equations

Type of Figure	General Case	$\theta_0 = 0$	$\theta_0 = \pi/2$
Line	 $r = \dfrac{d}{\cos(\theta - \theta_0)}$	 $r = \dfrac{d}{\cos\theta}$	 $r = \dfrac{d}{\sin\theta}$
Circle	 $r = 2a\cos(\theta - \theta_0)$	 $r = 2a\cos\theta$	 $r = 2a\sin\theta$
Ellipse $(0 < e < 1)$ **Parabola** $(e = 1)$ **Hyperbola** $(e > 1)$	 $r = \dfrac{ed}{1 + e\cos(\theta - \theta_0)}$	 $r = \dfrac{ed}{1 + e\cos\theta}$	 $r = \dfrac{ed}{1 + e\sin\theta}$

EXAMPLE 5 Find the equation of the horizontal ellipse with eccentricity $\frac{1}{2}$, focus at the pole, and vertical directrix 10 units to the right of the pole.

SOLUTION

$$r = \frac{\frac{1}{2}\cdot 10}{1 + \frac{1}{2}\cos\theta} = \frac{10}{2 + \cos\theta}$$ ■

EXAMPLE 6 Identify and sketch the graph of $r = \dfrac{7}{2 + 4\sin\theta}$.

SOLUTION The equation suggests a conic with vertical major axis. Putting it into the form shown in the polar equations chart gives

$$r = \frac{7}{2 + 4\sin\theta} = \frac{\frac{7}{2}}{1 + 2\sin\theta} = \frac{2\left(\frac{7}{4}\right)}{1 + 2\sin\theta}$$

which we recognize as the polar equation of a hyperbola with $e = 2$, focus at the pole, and horizontal directrix $\frac{7}{4}$ units above the polar axis (Figure 12). ■

Exercise

1. Find the Cartesian equations from the Polar equations below.
   a) $r - 5\cos\theta = 0$
   b) $r^2 - 6r\cos\theta - 4r\sin\theta + 9 = 0$
2. Name the curve for each polar equations below. If it is conic, determine its eccentricity.
   a) $r = \dfrac{3}{\sin\theta}$
   b) $r = \dfrac{6}{2+\sin\theta}$
   c) $r = \dfrac{4}{2+2\cos\left(\theta-\frac{\pi}{3}\right)}$
   d) $r = \dfrac{4}{\frac{1}{2}+\cos(\theta-\pi)}$

Section 10.6

Graphs of Polar Equations

The Symmetric Test

1. The graph of a polar equation is symmetric about the x-axis (the polar axis) if replacing (r, θ) by $(r, -\theta)$ (or by $(-r, \pi - \theta)$) produces an equivalent equation (Figure 1).
2. The graph of a polar equation is symmetric about the y-axis (the line $\theta = \pi/2$) if replacing (r, θ) by $(-r, -\theta)$ (or by $(r, \pi - \theta)$) produces an equivalent equation (Figure 2).
3. The graph of a polar equation is symmetric about the origin (pole) if replacing (r, θ) by $(-r, \theta)$ (or by $(r, \pi + \theta)$) produces an equivalent equation (Figure 3).

Because of the multiple representation of points in polar coordinates, symmetries may exist that are not identified by these three tests (see Problem 39).

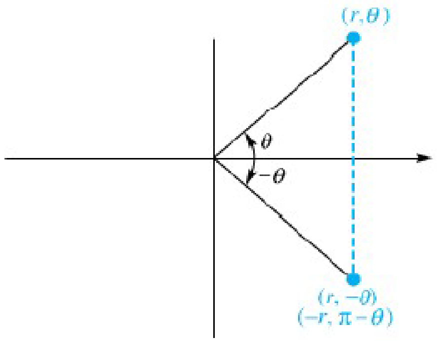


Figure 1

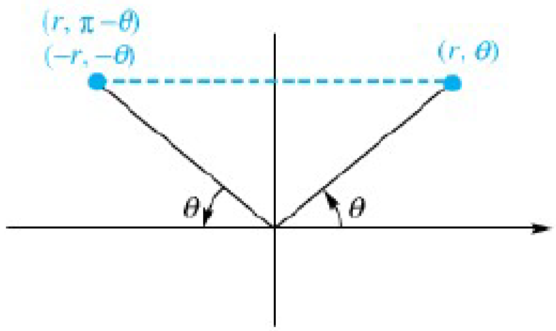


Figure 2

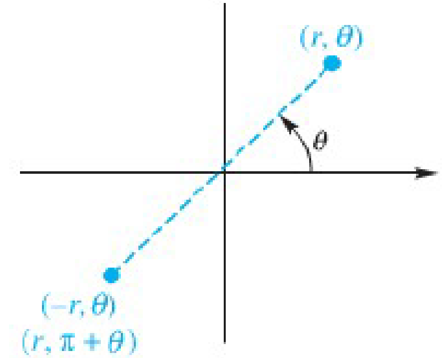


Figure 3

Cardioids and Limaçons

$$r = a \pm b\cos\theta \qquad r = a \pm b\sin\theta$$

with a and b positive. Their graphs are called **limaçons,** with the special cases in which $a = b$ referred to as **cardioids.** Typical graphs are shown in Figure 4.

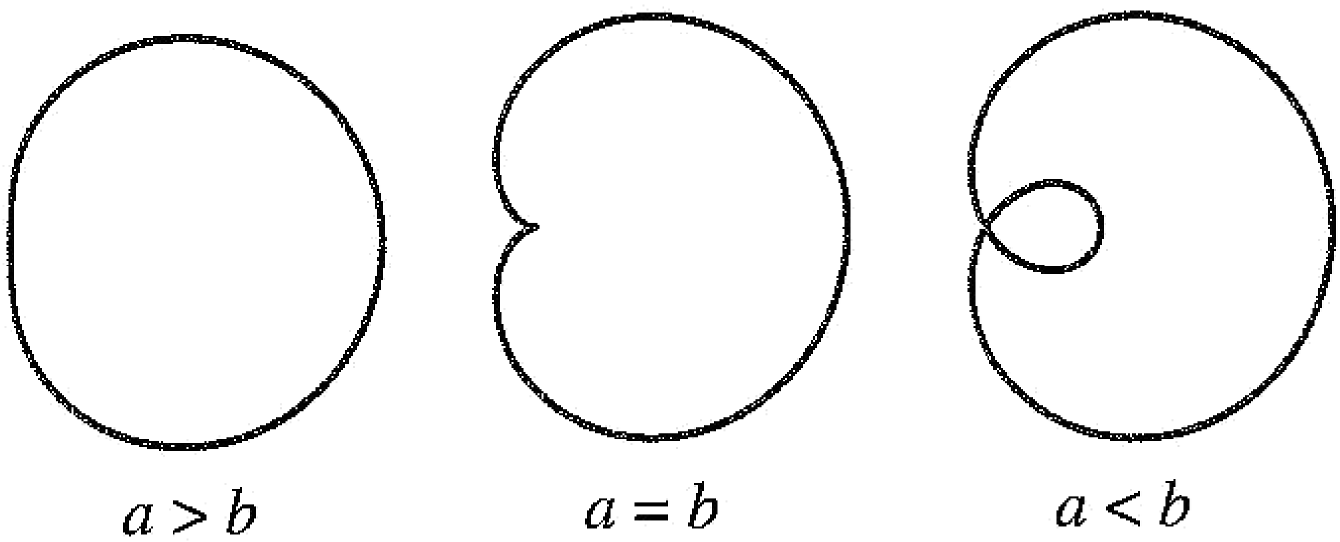


Figure 4

EXAMPLE 1 Analyze the equation $r = 2 + 4\cos\theta$ for symmetry and sketch its graph.

SOLUTION Since cosine is an even function ($\cos(-\theta) = \cos\theta$), the graph is symmetric with respect to the x-axis. The other symmetry tests fail. A table of values and the graph appear in Figure 5. ■

θ	r
0	6
$\pi/6$	5.5
$\pi/3$	4
$\pi/2$	2
$7\pi/12$	1.0
$2\pi/3$	0
$3\pi/4$	–0.8
$5\pi/6$	–1.5
π	–2

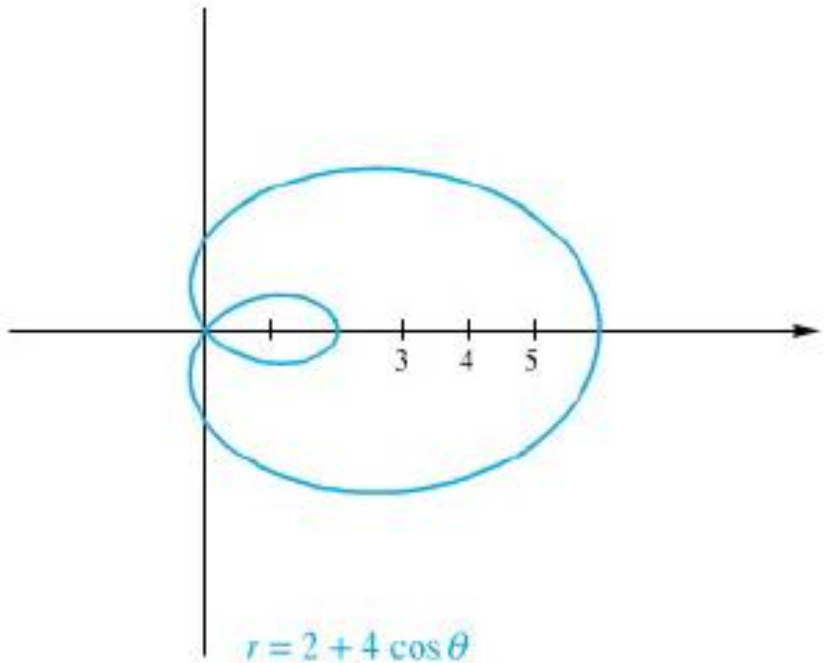


Figure 5

Lemniscates

Lemniscates The graphs of

$$r^2 = \pm a \cos 2\theta \qquad r^2 = \pm a \sin 2\theta$$

are figure-eight-shaped curves called **lemniscates**.

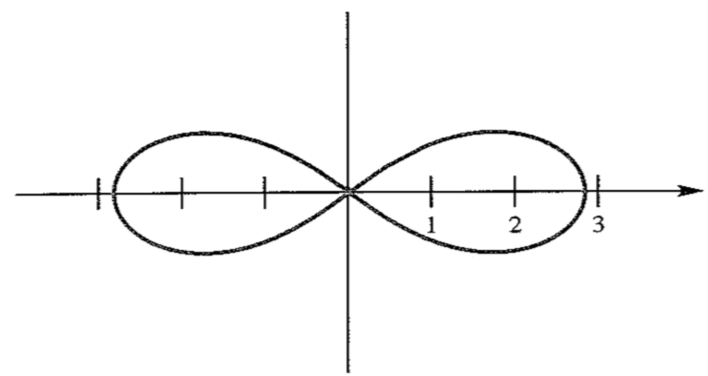
EXAMPLE 2 Analyze the equation $r^2 = 8\cos 2\theta$ for symmetry and sketch its graph.

SOLUTION Since $\cos(-2\theta) = \cos 2\theta$ and

$$\cos[2(\pi - \theta)] = \cos(2\pi - 2\theta) = \cos(-2\theta) = \cos 2\theta$$

the graph is symmetric with respect to both axes. Clearly, it is also symmetric with respect to the origin. A table of values and the graph are shown in Figure 6.

θ	r
0	± 2.8
$\pi/12$	± 2.6
$\pi/6$	± 2
$\pi/4$	0



$r^2 = 8\cos 2\theta$

Figure 6

Roses

Roses Polar equations of the form

$$r = a \cos n\theta \qquad r = a \sin n\theta$$

represent flower-shaped curves called **roses**. The rose has n leaves if n is odd and $2n$ leaves if n is even.

EXAMPLE 3 Analyze $r = 4 \sin 2\theta$ for symmetry and sketch its graph.

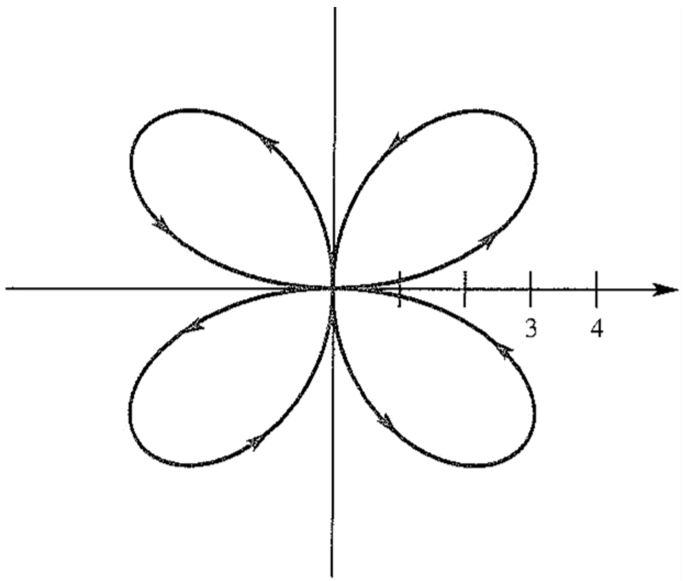
SOLUTION You can check that $r = 4 \sin 2\theta$ satisfies all three symmetry tests. For example, it meets Test 1 since

$$\sin(2(\pi - \theta)) = \sin(2\pi - 2\theta) = -\sin 2\theta$$

and so replacing (r, θ) by $(-r, \pi - \theta)$ produces an equivalent equation.

A rather extensive table of values for $0 \le \theta \le \pi/2$, a somewhat briefer one for $\pi/2 \le \theta \le 2\pi$, and the corresponding graph are shown in Figure 7. The arrows on the curve indicate the direction $P(r, \theta)$ moves as θ increases from 0 to 2π.

θ	r	θ	r
0	0	$2\pi/3$	–3.5
$\pi/12$	2	$5\pi/6$	–3.5
$\pi/8$	2.8	π	0
$\pi/6$	3.5	$7\pi/6$	3.5
$\pi/4$	4	$4\pi/3$	3.5
$\pi/3$	3.5	$3\pi/2$	0
$3\pi/8$	2.8	$5\pi/3$	–3.5
$5\pi/12$	2	$11\pi/6$	–3.5
$\pi/2$	0	2π	0



$r = 4 \sin 2\theta$

Figure 7

Spirals

Spirals The graph of $r = a\theta$ is called a **spiral of Archimedes;** the graph of $r = ae^{b\theta}$ is called a **logarithmic spiral.**

EXAMPLE 4 Sketch the graph of $r = \theta$ for $\theta \geq 0$.

SOLUTION We omit a table of values, but note that the graph crosses the polar axis at $(0, 0)$, $(2\pi, 2\pi)$, $(4\pi, 4\pi)$, ... and crosses its extension to the left at (π, π), $(3\pi, 3\pi)$, $(5\pi, 5\pi)$, ..., as in Figure 8.

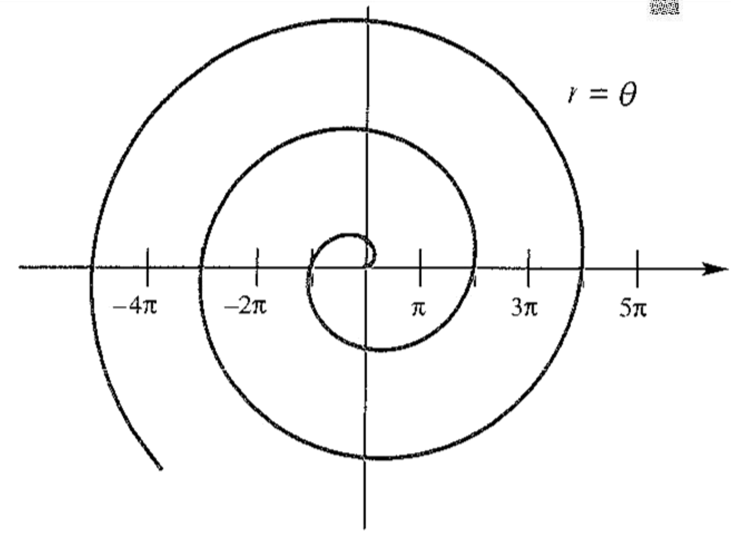


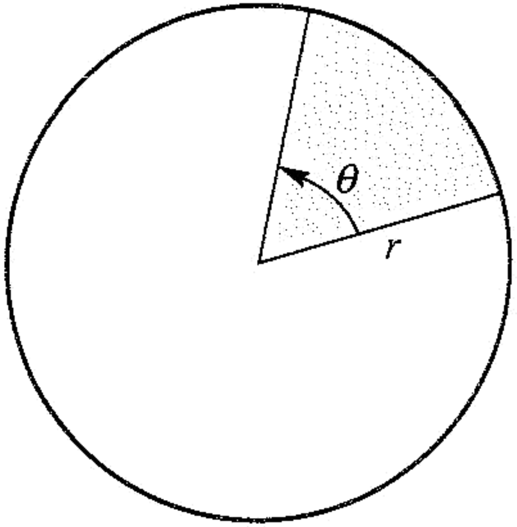
Figure 8

Section 10.7

Calculus in Polar coordinates

Area of a Sector

Area of a sector: $A = \dfrac{1}{2}\theta r^2$



$A = \frac{1}{2}\theta r^2$

Figure 1

Area Formula

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta$$

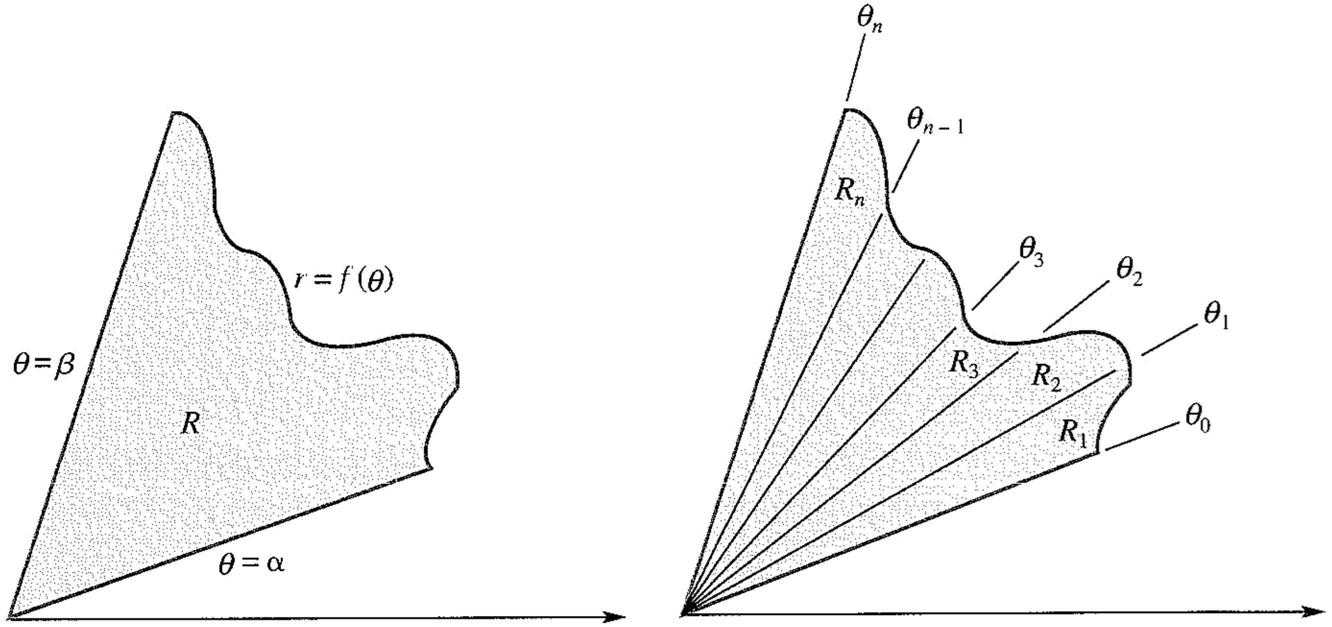


Figure 2

EXAMPLE 2 Find the area of one leaf of the four-leaved rose $r = 4 \sin 2\theta$.

SOLUTION The complete rose was sketched in Example 3 of the previous section. Here we show only the first-quadrant leaf (Figure 5). This leaf is 4 units long and averages about 1.5 units in width, giving 6 as an estimate for its area. The exact area A is given by

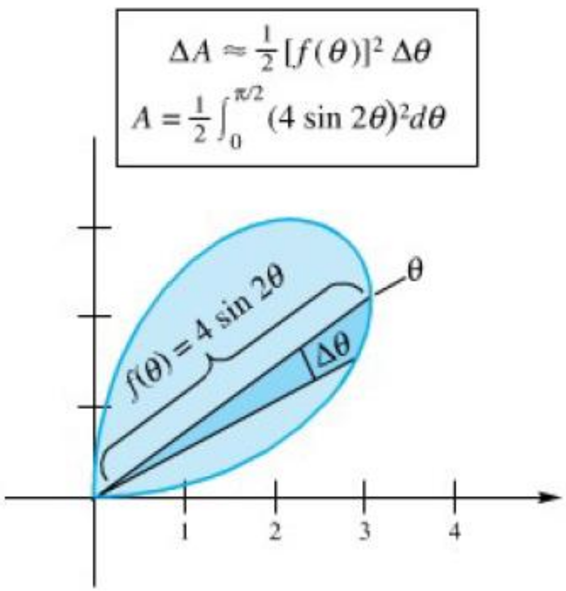


Figure 5

$$A = \frac{1}{2}\int_0^{\pi/2} 16 \sin^2 2\theta \, d\theta = 8 \int_0^{\pi/2} \frac{1 - \cos 4\theta}{2} \, d\theta$$

$$= 4 \int_0^{\pi/2} d\theta - \int_0^{\pi/2} \cos 4\theta \cdot 4 \, d\theta$$

$$= [4\theta]_0^{\pi/2} - [\sin 4\theta]_0^{\pi/2} = 2\pi$$

EXAMPLE 3 Find the area of the region outside the cardioid $r = 1 + \cos\theta$ and inside the circle $r = \sqrt{3}\sin\theta$.

SOLUTION The graphs of the two curves are sketched in Figure 6. We will need the θ-coordinates of the points of intersection. Let's try solving the two equations simultaneously.

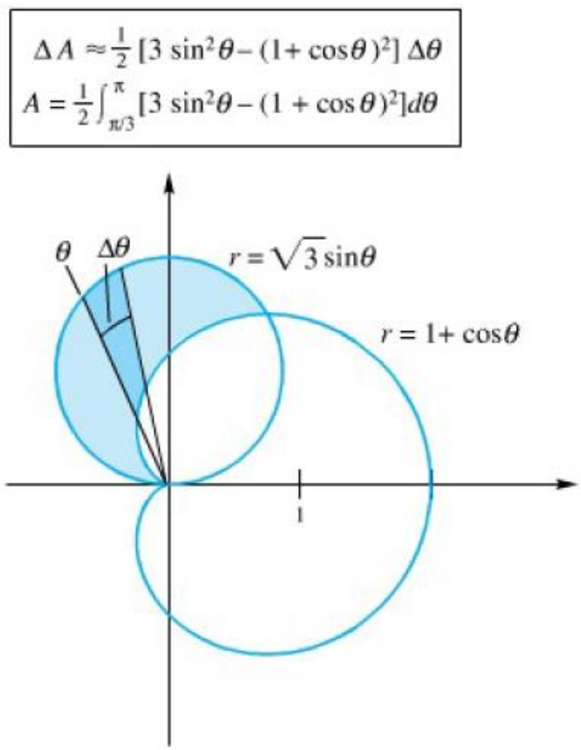


Figure 6

$$1 + \cos\theta = \sqrt{3}\sin\theta$$

$$1 + 2\cos\theta + \cos^2\theta = 3\sin^2\theta$$

$$1 + 2\cos\theta + \cos^2\theta = 3(1 - \cos^2\theta)$$

$$4\cos^2\theta + 2\cos\theta - 2 = 0$$

$$2\cos^2\theta + \cos\theta - 1 = 0$$

$$(2\cos\theta - 1)(\cos\theta + 1) = 0$$

$$\cos\theta = \frac{1}{2} \quad \text{or} \quad \cos\theta = -1$$

$$\theta = \frac{\pi}{3} \quad \text{or} \quad \theta = \pi$$

Now slice, approximate, and integrate.

$$A = \frac{1}{2}\int_{\pi/3}^{\pi}[3\sin^2\theta - (1+\cos\theta)^2]\,d\theta$$

$$= \frac{1}{2}\int_{\pi/3}^{\pi}[3\sin^2\theta - 1 - 2\cos\theta - \cos^2\theta]\,d\theta$$

$$= \frac{1}{2}\int_{\pi/3}^{\pi}\left[\frac{3}{2}(1-\cos 2\theta) - 1 - 2\cos\theta - \frac{1}{2}(1+\cos 2\theta)\right]d\theta$$

$$= \frac{1}{2}\int_{\pi/3}^{\pi}[-2\cos\theta - 2\cos 2\theta]\,d\theta$$

$$= \frac{1}{2}[-2\sin\theta - \sin 2\theta]_{\pi/3}^{\pi}$$

$$= \frac{1}{2}\left[2\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right] = \frac{3\sqrt{3}}{4} \approx 1.299$$

Exercise

1. Find the area of the region inside the limaçon $r = 2 + \cos\theta$.
2. Find the area of the region inside the lemniscate $r^2 = 9\sin 2\theta$
3. Find the area of the region inside circle $r = 3\sin\theta$ and outside cardioid $r = 1 + \sin\theta$.

End of Chapter 10