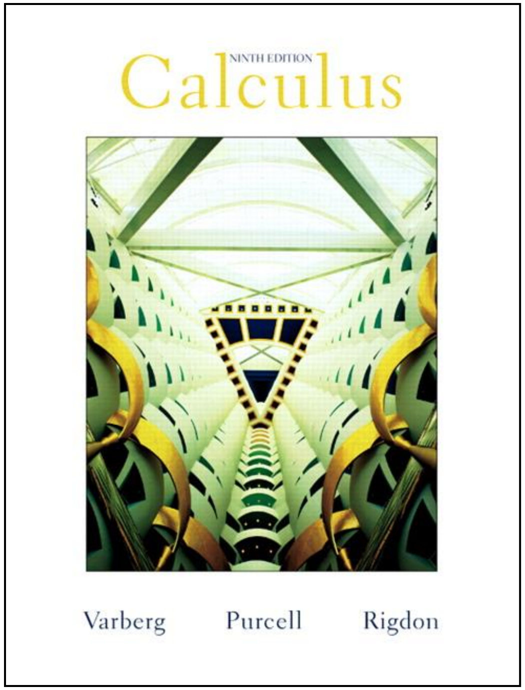
Varberg, Calculus 9e



Chapter 12

Derivatives for Functions of Two or More Variables

Section 12.1

Functions of Two or More Variables

Function of Two or More Variables

- **Real-valued function of two real variables**, the is a function *f* the assigns to each ordered pair (x,y) in some set *D* of the plane a (unique) real number $f(x,y)$. Examples are

$$f(x, y) = x^2 + 3y^2 \qquad (1)$$

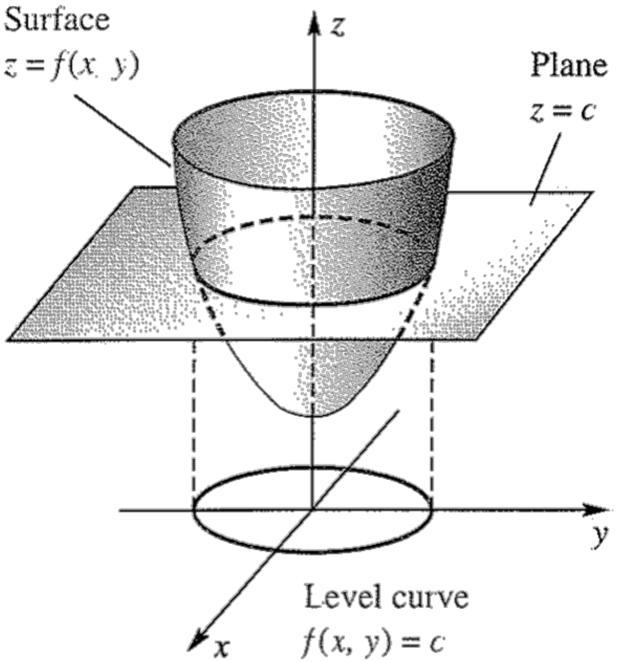
$$g(x, y) = 2x\sqrt{y} \qquad (2)$$

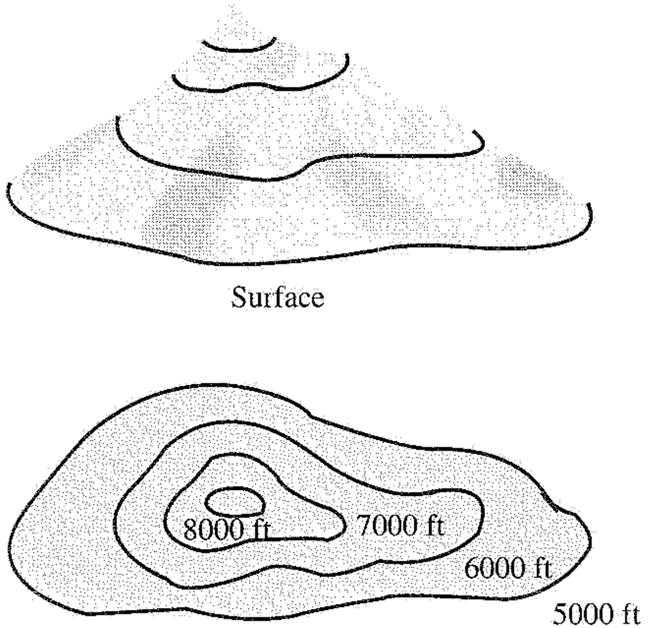
Function of Two or More Variables

- The set D is called the **domain** , the set of all points (x,y) in the plane for which the function rule makes sense and gives a real number value.
- The **range** of a function is its set of values. If $z = f(x,y)$, we call x an y the **independent variables** and z the **dependent variable**.

Level Curves

- The projection of the curve on the *xy*-plane is called a **level curve**, and a collection of such curves is a **contour plot** or a **contour map**.

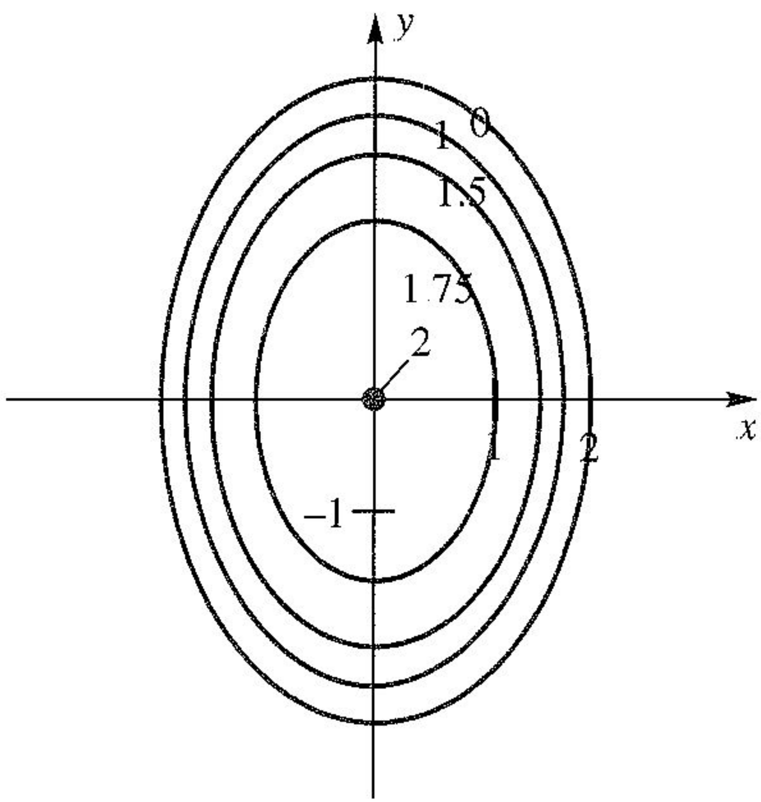




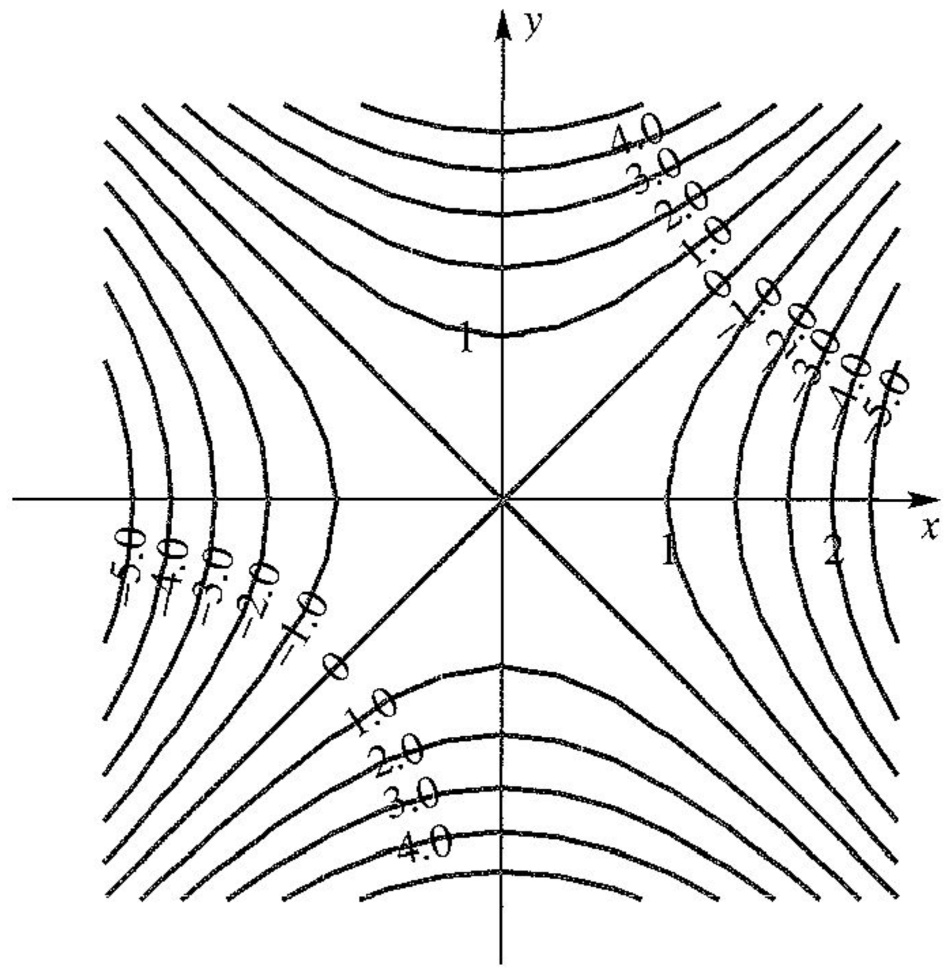
Surface

Contour map
with level curves

Contour Map $z = \frac{1}{3}\sqrt{36 - 9x^2 - 4y^2}$



Contour Map $z = y^2 - x^2$

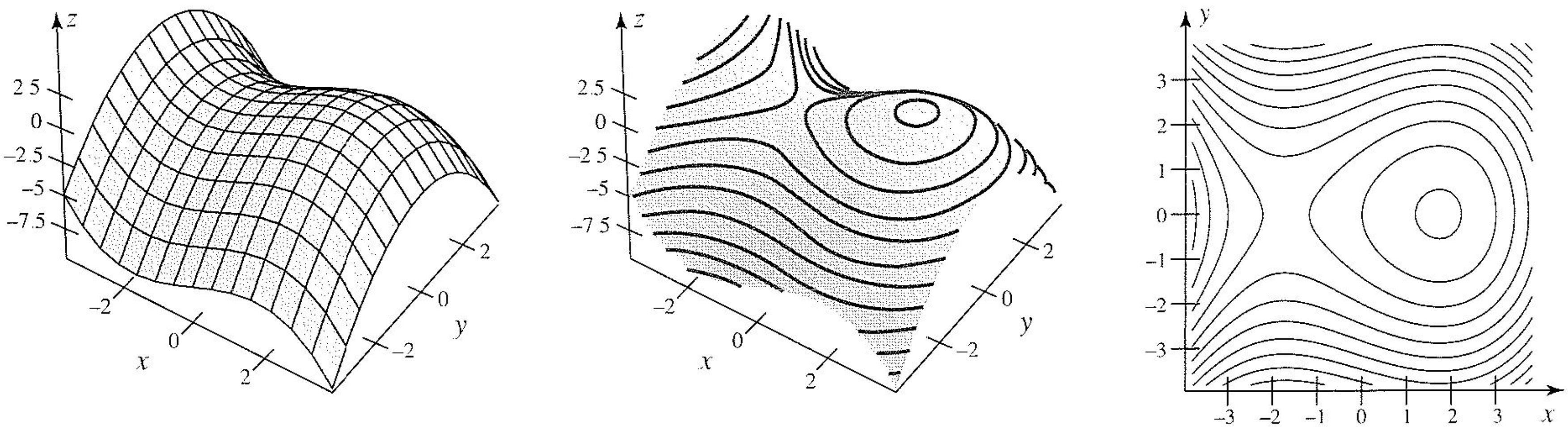


EXAMPLE 4 Draw contour maps for the surfaces corresponding to $z = \frac{1}{3}\sqrt{36 - 9x^2 - 4y^2}$ and $z = y^2 - x^2$ (see Examples 2 and 3, and Figures 4 and 5).

SOLUTION The level curves of $z = \frac{1}{3}\sqrt{36 - 9x^2 - 4y^2}$ corresponding to $z = 0, 1, 1.5, 1.75, 2$ are shown in Figure 12. They are ellipses. Similarly, in Figure 13, we show the level curves of $z = y^2 - x^2$ for $z = -5, -4, -3, \ldots, 2, 3, 4$. These curves are hyperbolas unless $z = 0$. The level curve for $z = 0$ is a pair of intersecting lines. ■

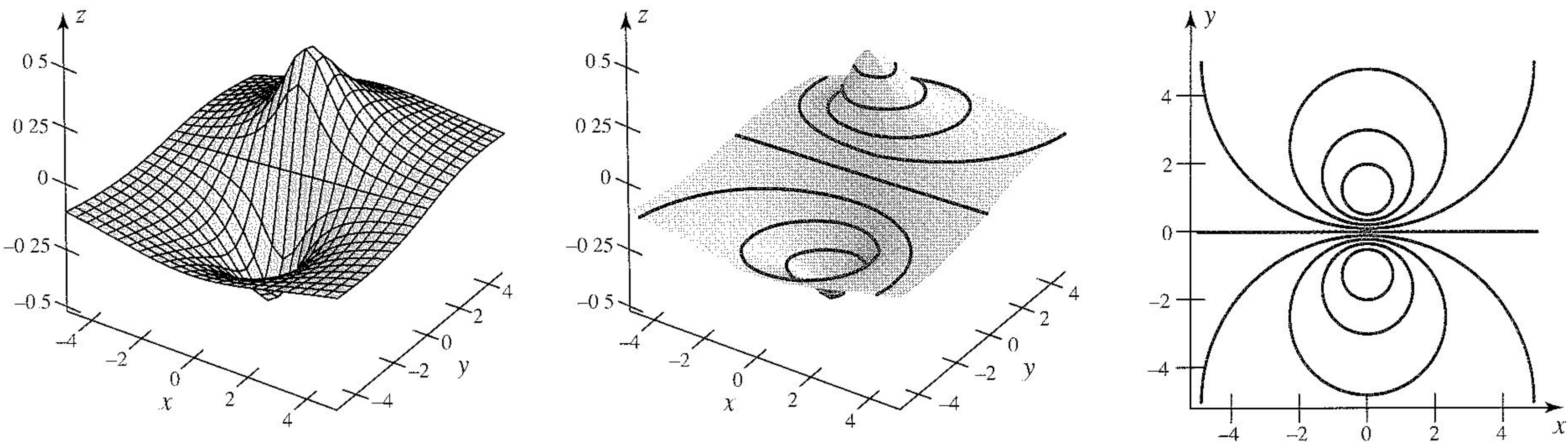
Computer Graphs and Level Curves

- In Figures 15 through 19, we have drawn five more surfaces



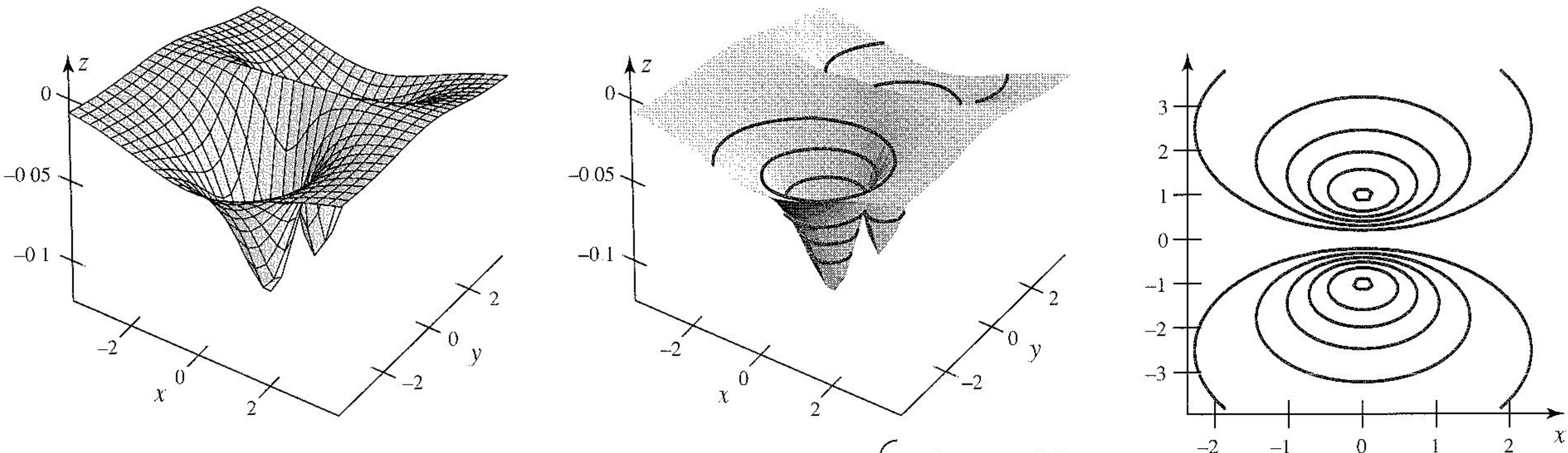
$$z = x - \left(\tfrac{1}{9}\right)x^3 - \left(\tfrac{1}{2}\right)y^2 \qquad \begin{cases} -3.8 \le x \le 3.8 \\ -3.8 \le y \le 3.8 \end{cases}$$

Figure 15



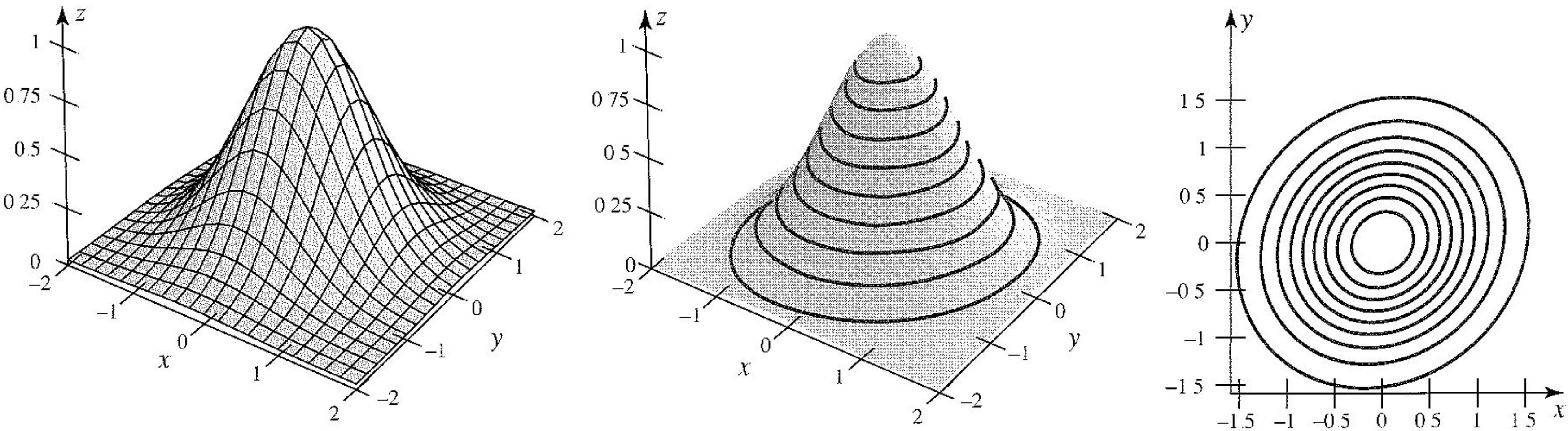
$$z = \frac{y}{(1 + x^2 + y^2)} \qquad \begin{cases} -5 \le x \le 5 \\ -5 \le y \le 5 \end{cases}$$

Figure 16



$$z = -1 + \cos\left(\frac{y}{1 + x^2 + y^2}\right) \quad \begin{cases} -3.8 \le x \le 3.8 \\ -3.8 \le y \le 3.8 \end{cases}$$

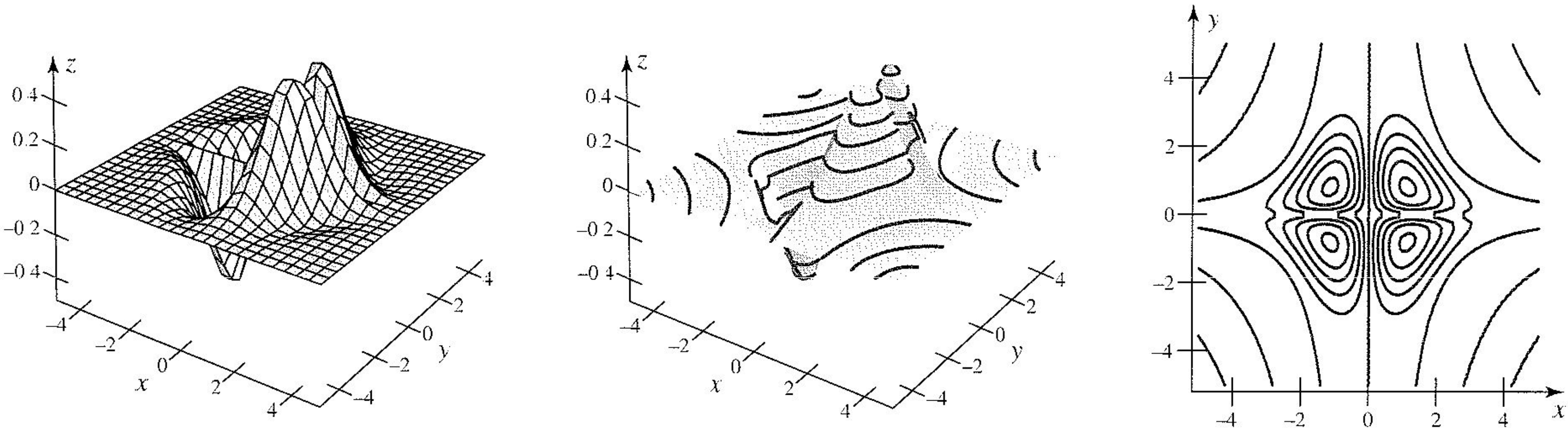
Figure 17



$$z = e^{-x^2 - y^2 + xy/4}$$

$$\begin{cases} -2 \le x \le 2 \\ -2 \le y \le 2 \end{cases}$$

Figure 18



$$z = e^{-(x^2+y^2)/4}\sin(x\sqrt{|y|}) \qquad \begin{cases} -5 \le x \le 5 \\ -5 \le y \le 5 \end{cases}$$

Figure 19

Section 12.2

Partial Derivatives

Partial Derivatives

- Suppose that f is a function of two variables x and y. If y is held constant, say $y = y_0$, then f(x,y_0) is a function of the single variable x. Its derivative at $x = x_0$ is called the **partial derivative of f with respect to x** at (x_0, y_0)

$$\boxed{f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}}$$

Similarly, the partial derivative of f with respect to y at (x_0, y_0) is denoted by $f_y(x_0, y_0)$ and is given by

$$\boxed{f_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}}$$

EXAMPLE 1 Find $f_x(1, 2)$ and $f_y(1, 2)$ if $f(x, y) = x^2y + 3y^3$.

SOLUTION To find $f_x(x, y)$, we treat y as a constant and differentiate with respect to x, obtaining

$$f_x(x, y) = 2xy + 0$$

Thus,

$$f_x(1, 2) = 2 \cdot 1 \cdot 2 = 4$$

Similarly, we treat x as a constant and differentiate with respect to y, obtaining

$$f_y(x, y) = x^2 + 9y^2$$

and so

$$f_y(1, 2) = 1^2 + 9 \cdot 2^2 = 37$$

Partial Derivatives

The symbol ∂ is special to mathematics and is called the partial derivative sign. The symbols $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial y}$ represent linear operators, much like the linear operators D_x and $\dfrac{d}{dx}$ that we encountered in Chapter 2.

EXAMPLE 2 If $z = x^2 \sin(xy^2)$, find $\partial z/\partial x$ and $\partial z/\partial y$.

SOLUTION

$$\begin{aligned}
\frac{\partial z}{\partial x} &= x^2 \frac{\partial}{\partial x}[\sin(xy^2)] + \sin(xy^2)\frac{\partial}{\partial x}(x^2) \\
&= x^2 \cos(xy^2)\frac{\partial}{\partial x}(xy^2) + \sin(xy^2)\cdot 2x \\
&= x^2 \cos(xy^2)\cdot y^2 + 2x\sin(xy^2) \\
&= x^2y^2\cos(xy^2) + 2x\sin(xy^2)
\end{aligned}$$

$$\frac{\partial z}{\partial y} = x^2\cos(xy^2)\cdot 2xy = 2x^3y\cos(xy^2)$$

Geometric an Physical Interpretations

Geometric and Physical Interpretations Consider the surface whose equation is $z = f(x, y)$. The plane $y = y_0$ intersects this surface in the plane curve QPR (Figure 1), and the value of $f_x(x_0, y_0)$ is the slope of the tangent line to this curve at $P(x_0, y_0, f(x_0, y_0))$. Similarly, the plane $x = x_0$ intersects the surface in the plane curve LPM (Figure 2), and $f_y(x_0, y_0)$ is the slope of the tangent line to this curve at P.

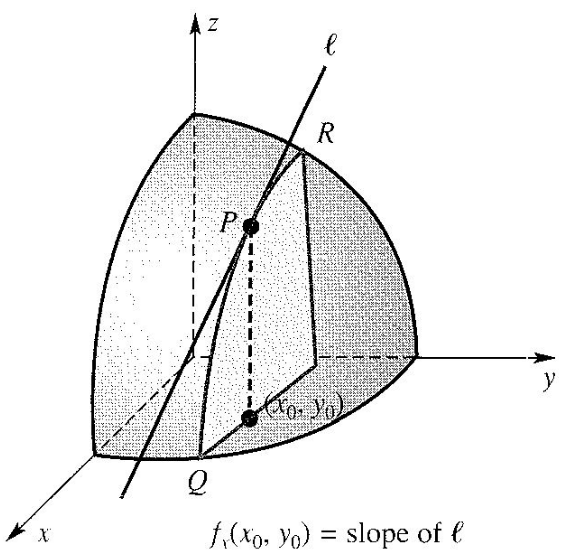


Figure 1

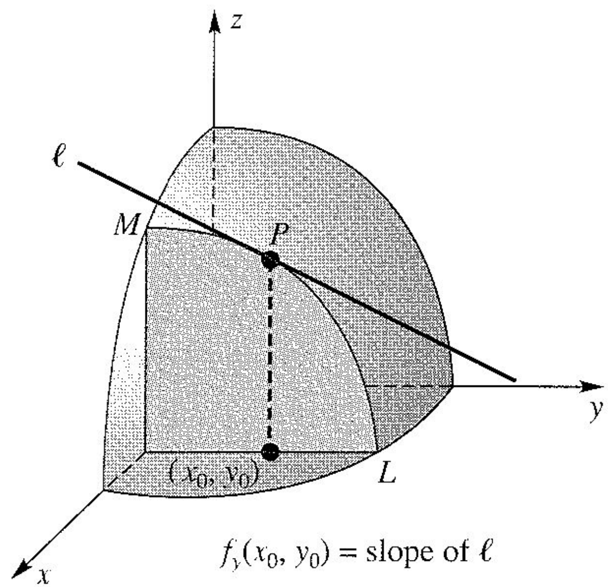


Figure 2

EXAMPLE 3 The surface $z = f(x, y) = \sqrt{9 - 2x^2 - y^2}$ and the plane $y = 1$ intersect in a curve as in Figure 1. Find parametric equations for the tangent line at $(\sqrt{2}, 1, 2)$.

SOLUTION

$$f_x(x, y) = \tfrac{1}{2}(9 - 2x^2 - y^2)^{-1/2}(-4x)$$

and so $f_x(\sqrt{2}, 1) = -\sqrt{2}$. This number is the slope of the tangent line to the curve at $(\sqrt{2}, 1, 2)$; that is, $-\sqrt{2}/1$ is the ratio of rise to run along the tangent line. It follows that this line has direction vector $\langle 1, 0, -\sqrt{2} \rangle$ and, since it goes through $(\sqrt{2}, 1, 2)$,

$$x = \sqrt{2} + t, \qquad y = 1, \qquad z = 2 - \sqrt{2}t$$

provide the required parametric equations.

Higher Partial Derivatives

Higher Partial Derivatives Since a partial derivative of a function of x and y is, in general, another function of these same two variables, it may be differentiated partially with respect to either x or y, resulting in four **second partial derivatives** of f.

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \qquad f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} \qquad f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y}$$

EXAMPLE 5 Find the four second partial derivatives of

$$f(x, y) = xe^y - \sin(x/y) + x^3y^2$$

SOLUTION

$$f_x(x, y) = e^y - \frac{1}{y}\cos\left(\frac{x}{y}\right) + 3x^2y^2$$

$$f_y(x, y) = xe^y + \frac{x}{y^2}\cos\left(\frac{x}{y}\right) + 2x^3y$$

$$f_{xx}(x, y) = \frac{1}{y^2}\sin\left(\frac{x}{y}\right) + 6xy^2$$

$$f_{yy}(x, y) = xe^y + \frac{x^2}{y^4}\sin\left(\frac{x}{y}\right) - \frac{2x}{y^3}\cos\left(\frac{x}{y}\right) + 2x^3$$

$$f_{xy}(x, y) = e^y - \frac{x}{y^3}\sin\left(\frac{x}{y}\right) + \frac{1}{y^2}\cos\left(\frac{x}{y}\right) + 6x^2y$$

$$f_{yx}(x, y) = e^y - \frac{x}{y^3}\sin\left(\frac{x}{y}\right) + \frac{1}{y^2}\cos\left(\frac{x}{y}\right) + 6x^2y$$

Higher Partial Derivatives

- Partial derivatives of the third and higher orders are defined analogously, and the notation for them is similar.

$$\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right] = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y\,\partial x}\right) = \frac{\partial^3 f}{\partial y^2\,\partial x} = f_{xyy}$$

More Than Two Variables

More Than Two Variables Let f be a function of three variables, x, y, and z. The **partial derivative of f with respect to x** at (x, y, z) is denoted by $f_x(x, y, z)$ or $\partial f(x, y, z)/\partial x$ and is defined by

$$f_x(x, y, z) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}$$

EXAMPLE 6 If $f(x, y, z) = xy + 2yz + 3zx$, find f_x, f_y, and f_z.

SOLUTION To get f_x, we think of y and z as constants and differentiate with respect to the variable x. Thus,

$$f_x(x, y, z) = y + 3z$$

To find f_y, we treat x and z as constants and differentiate with respect to y:

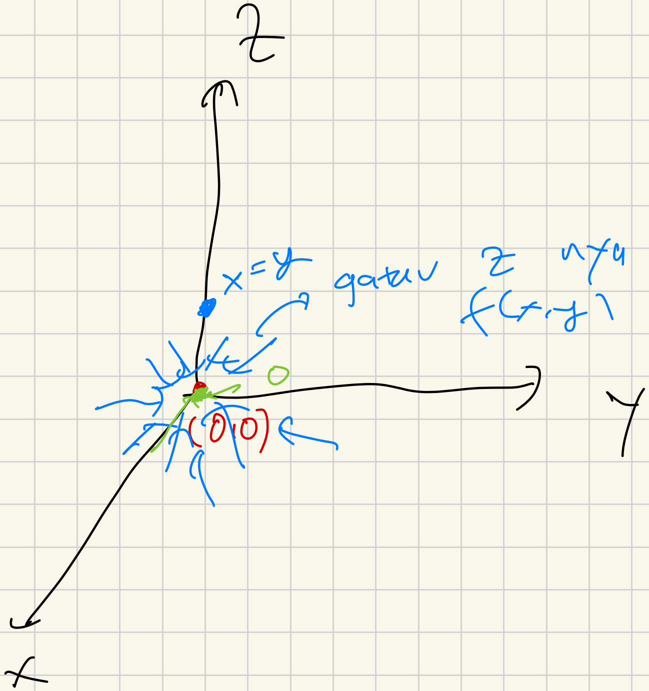
$$f_y(x, y, z) = x + 2z$$

Similarly,

$$f_z(x, y, z) = 2y + 3x$$

Section 12.3

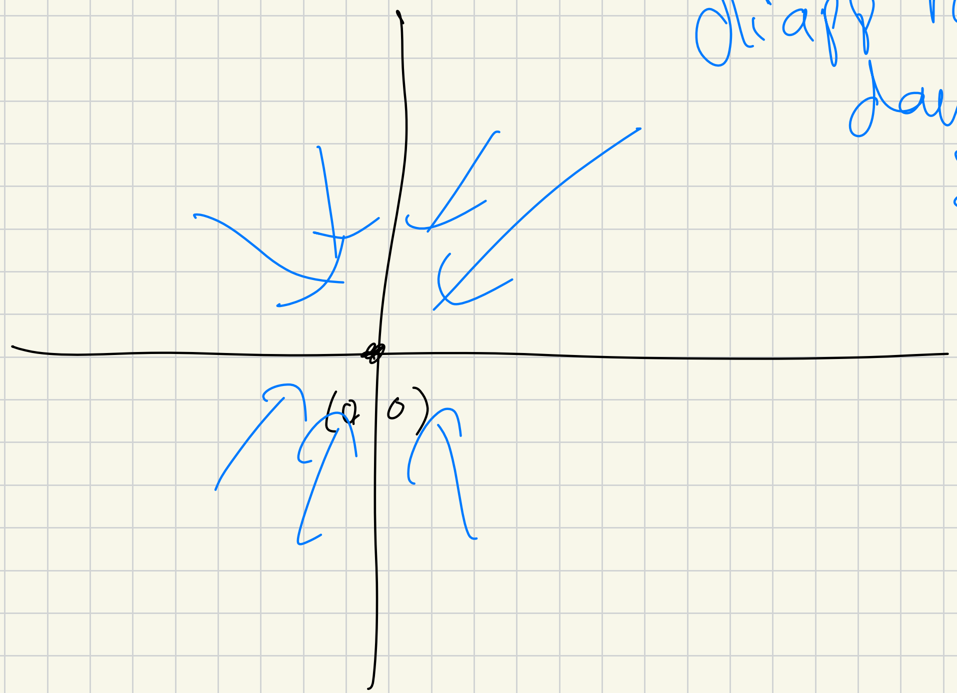
Limits and Continuity



$$\lim_{(x,y)\to(0,0)}$$

$$f(x,y) \to (0,0)$$

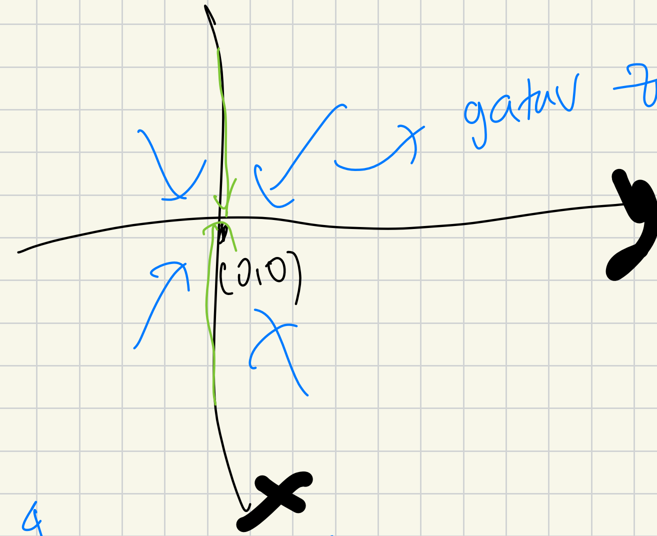
diapproach dari segala arah!



(1) Cara awam / brute force

(2) Polar

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{(x^2+y^2)^2}$$



gatau ∃ nya berapa

cek $x = y$

$$\lim_{y\to 0} \frac{(y^4)}{(2y^2)^2} = \lim_{y\to 0} \frac{y^4}{4y^4} = \frac{1}{4}$$

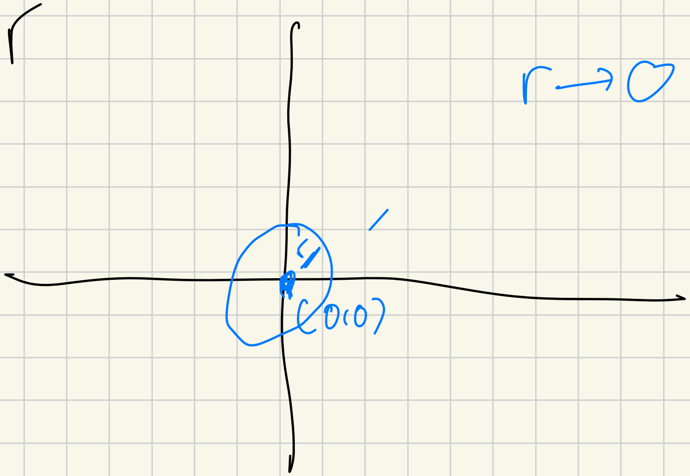
ga sama

$$\lim_{\substack{x=0 \\ y\to 0}} \frac{0}{(y^2)^2} = \frac{0}{y^4} = 0$$

Gak Kontinu

$(x, y \to a, b)$

2) Polar



$r \to 0$

$x = r\cos\theta$

$y = r\sin\theta$

$r^2 = x^2 + y^2$

$$\lim_{r\to 0} \frac{\cancel{r^2}\cos^2\theta\ \cancel{r^2}\sin^2\theta}{\cancel{(r^2)^2}} = \cos^2\theta \sin^2\theta$$

menghasilkan ≠ berbeda

di θ — berbeda.

$f(x,y)$

Gak Kontinu.

Jadi segala θ berbeda $\lim_{r\to 0} f(x,y)$ sama converted

Kontinu

$$f(x,y) = (1 - x^2 - y^2)^{-1/2}$$

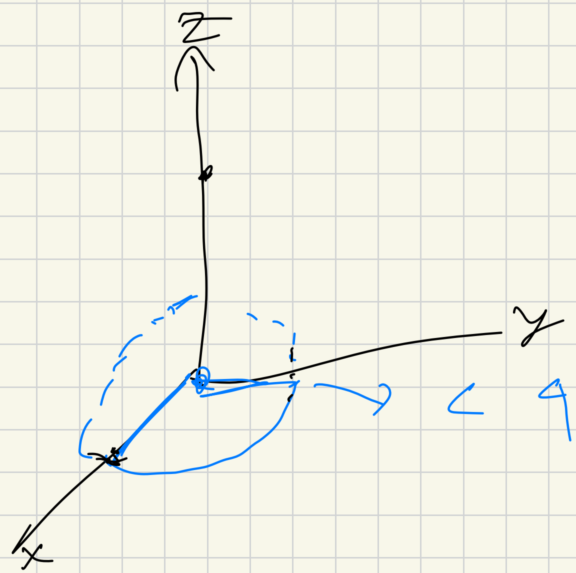
$$1$$

$$= \frac{}{\sqrt{1 - x^2 - y^2}} \leq 0$$

Bar kontinu set x, y values harus ketika

$$1 - x^2 - y^2 > 0$$

$$x^2 + y^2 < 1$$



Set x, y / Region x, y nya itu lingkaran dgn jari² < 4 pusat (0, 0)

Limits

Definition Limit of a Function of Two Variables

To say that $\lim_{(x,y)\to(a,b)} f(x, y) = L$ means that for each $\varepsilon > 0$ (no matter how small) there is a corresponding $\delta > 0$ such that $|f(x, y) - L| < \varepsilon$, provided that $0 < \|(x, y) - (a, b)\| < \delta$.

Note several aspects of this definition.

1. The path of approach to (a, b) is irrelevant. This means that if different paths of approach lead to different L-values then the limit does not exist.
2. The behavior of $f(x, y)$ at (a, b) is irrelevant; the function does not even have to be defined at (a, b). This follows from the restriction $0 < \|(x, y) - (a, b)\|$.

Limits

Theorem A

If $f(x, y)$ is a polynomial, then

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

and if $f(x, y) = p(x, y)/q(x, y)$, where p and q are polynomials, then

$$\lim_{(x,y)\to(a,b)} f(x, y) = \frac{p(a, b)}{q(a, b)}$$

provided $q(a, b) \neq 0$. Furthermore, if

$$\lim_{(x,y)\to(a,b)} p(x, y) = L \neq 0 \quad \text{and} \quad \lim_{(x,y)\to(a,b)} q(x, y) = 0$$

then

$$\lim_{(x,y)\to(a,b)} \frac{p(x, y)}{q(x, y)}$$

does not exist.

EXAMPLE 1 Evaluate the following limits if they exist:

(a) $\lim_{(x,y)\to(1,2)} (x^2y + 3y)$ and (b) $\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2 + 1}{x^2 - y^2}$

SOLUTION

(a) The function whose limit we seek is a polynomial, so by Theorem A

$$\lim_{(x,y)\to(1,2)} (x^2y + 3y) = 1^2 \cdot 2 + 3 \cdot 2 = 8$$

(b) The second function is a rational function, but the limit of the denominator is equal to 0, while the limit of the numerator is 1. Thus, by Theorem A, this limit does not exist.

EXAMPLE 2 Show that the function f defined by

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

has no limit at the origin (Figure 3).

has no limit at the origin (Figure 3).

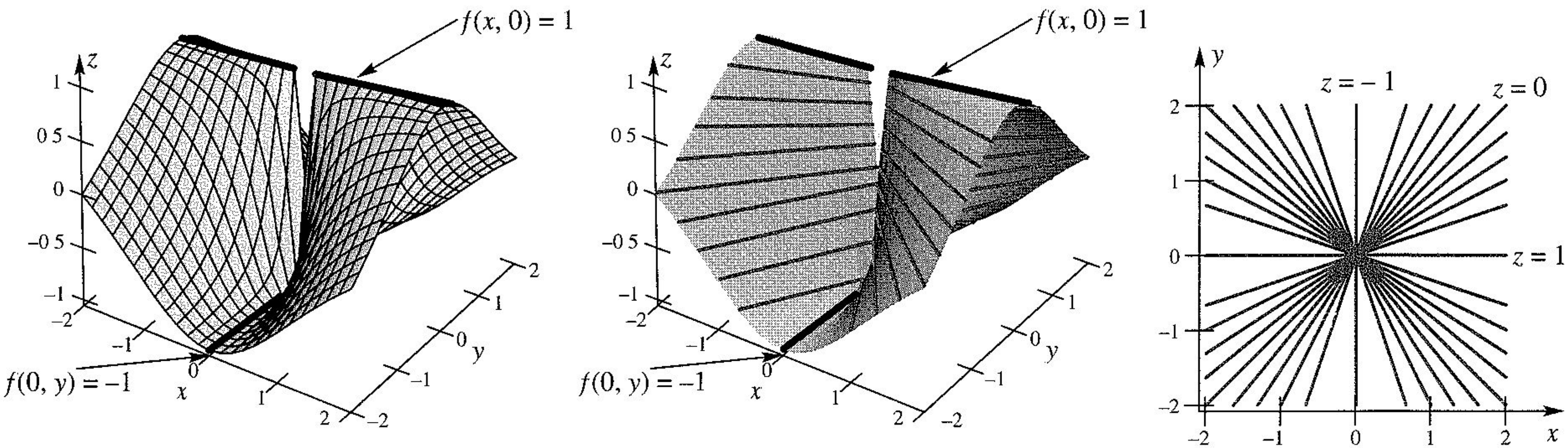


Figure 3

SOLUTION The function f is defined everywhere in the xy-plane except at the origin. At all points on the x-axis different from the origin, the value of f is

$$f(x, 0) = \frac{x^2 - 0}{x^2 + 0} = 1$$

Thus, the limit of $f(x, y)$ as (x, y) approaches $(0, 0)$ along the x-axis is

$$\lim_{(x,0)\to(0,0)} f(x, 0) = \lim_{(x,0)\to(0,0)} \frac{x^2 - 0}{x^2 + 0} = +1$$

Similarly, the limit of $f(x, y)$ as (x, y) approaches $(0, 0)$ along the y-axis is

$$\lim_{(0,y)\to(0,0)} f(0, y) = \lim_{(0,y)\to(0,0)} \frac{0 - y^2}{0 + y^2} = -1$$

Thus, we get different values depending on how $(x, y) \to (0, 0)$. In fact, there are points arbitrarily close to $(0, 0)$ at which the value of f is 1 and other points equally close at which the value of f is -1. Therefore, the limit cannot exist at $(0, 0)$. ▒

Continuity at a Point

Continuity at a Point To say that $f(x, y)$ is **continuous** at the point (a, b), we require the following: (1) f has a value at (a, b), (2) f has a limit at (a, b), and (3) the value of f at (a, b) is equal to the limit there. In summary, we require that

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

Theorem B **Composition of Functions**

If a function g of two variables is continuous at (a, b) and a function f of one variable is continuous at $g(a, b)$, then the composite function $f \circ g$, defined by $(f \circ g)(x, y) = f(g(x, y))$, is continuous at (a, b).

Theorem C **Equality of Mixed Partials**

If f_{xy} and f_{yx} are continuous on an open set S, then $f_{xy} = f_{yx}$ at each point of S.

Section 12.4

Differentiability

Differentiability

- A function f is **locally linear at** a if there is a constant m such that

$$f(a+h) = f(a) + hm + h\varepsilon(h)$$

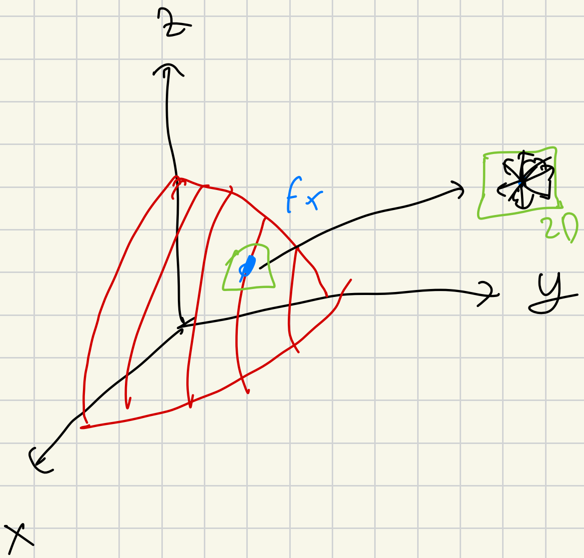
where $\varepsilon(h)$ is a function satisfying $\lim_{h \to 0} \varepsilon(h) = 0$

Definition Local Linearity for a Function of Two Variables

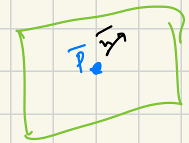
We say that f is **locally linear** at (a, b) if

$$f(a+h_1, b+h_2) = f(a,b) + h_1 f_x(a,b) + h_2 f_y(a,b) + h_1\varepsilon_1(h_1,h_2) + h_2\varepsilon_2(h_1,h_2)$$

where $\varepsilon_1(h_1, h_2) \to 0$ as $(h_1, h_2) \to 0$ and $\varepsilon_2(h_1, h_2) \to 0$ as $(h_1, h_2) \to 0$.



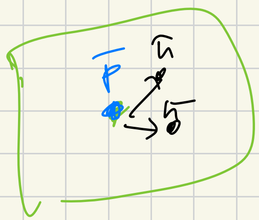
locally linear



$$f(\bar{p}+\bar{h}) = f(\bar{p}) + \nabla f \cdot \vec{h} +$$

$\bar{p} \to (x,y)$

$$\nabla f = \begin{pmatrix} f_x \\ f_y \end{pmatrix} \qquad \bar{h} = \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

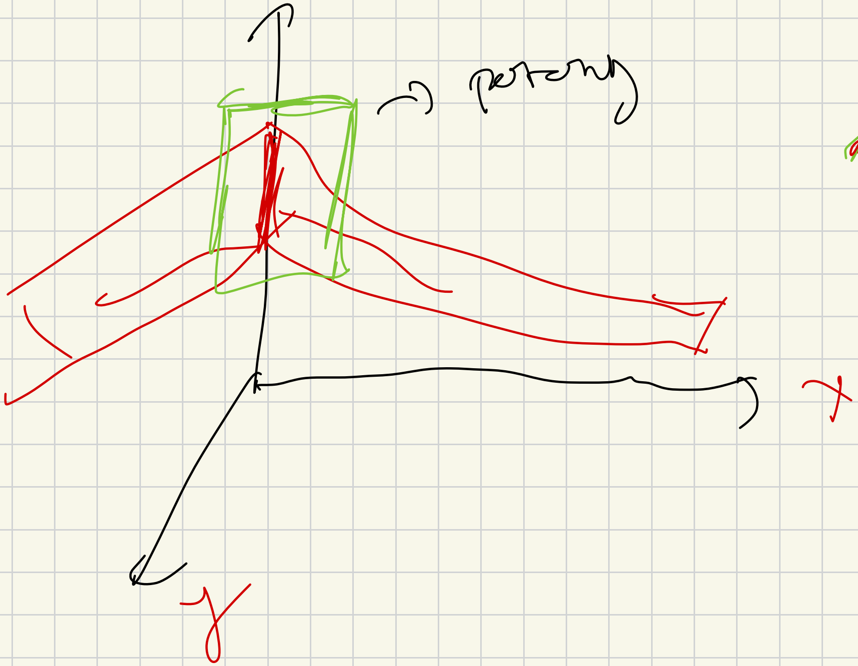


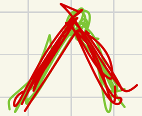
$$f(\bar{p}+\bar{h}) = f(\bar{p}) + f_x \cdot \Delta x + f_y \cdot \Delta y$$

Persamaan bidang singgung

$f(x,y)$ terdiferensialkan atau tidak

Jika $f(x,y)$ terdiferensialkan $\rightarrow$ f_x, f_y kontinu





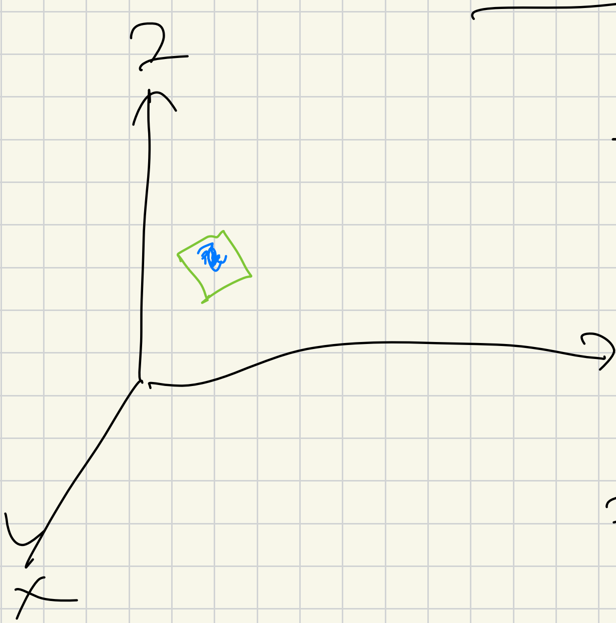
tdk terdiferensialkan

$$f(x,y) = \underline{x^2 y} + \underline{y^2 x} \quad , \quad \underline{\bar{p}\left(\overset{x}{2}, \overset{y}{2}\right)}$$

$f(x,y) \rightarrow$ Polinomial $\rightarrow$ Kontinu

maka <u>terdiferensialkan.</u>

$$\bar{h}\begin{pmatrix}\Delta x \\ \Delta y\end{pmatrix}$$



$$f(\bar{p}+\bar{h}) = f(\bar{p}) + \nabla_f \cdot \bar{h}$$

$$z = 16 + \begin{pmatrix}12 \\ 12\end{pmatrix} \cdot \begin{pmatrix}x-2 \\ y-2\end{pmatrix}$$

$$z = 16 + 12(x-2) + 12(y-2)$$

$$z = 16 + 12x - 24 + 12y - 24$$

Bidang Singgung $\Rightarrow$ $\boxed{z = 12x + 12y - 32}$

$$f(x,y,z) = 3x^2 - 2y^2 + xz^2, \quad p = (1,2,-1)$$

$$\nabla_f = \begin{pmatrix} 6x + z^2 \\ -4y \\ 2xz \end{pmatrix} = \begin{pmatrix} 7 \\ -8 \\ -2 \end{pmatrix}$$

$$f(\bar{p} + \bar{h}) = w \rightarrow \text{tangent hyperplane}$$

$$w = (3(1)^2 - 2(2)^2 + (1)(-1)^2) + \begin{pmatrix} 7 \\ -8 \\ -2 \end{pmatrix} \begin{pmatrix} x-1 \\ y-2 \\ z+1 \end{pmatrix}$$

$$w = -4 + (7x - 7 - 8y + 16 - 2z - 2)$$

$$w = 7x - 8y - 2z + 3$$

Differentiability

Definition Differentiability for a Function of Two or More Variables

The function f is **differentiable** at $\mathbf{p}$ if it is locally linear at $\mathbf{p}$. The function f is differentiable on an open set R if it is differentiable at every point in R.

The vector $(f_x(\mathbf{p}), f_y(\mathbf{p})) = f_x(\mathbf{p})\mathbf{i} + f_y(\mathbf{p})\mathbf{j}$ is denoted $\nabla f(\mathbf{p})$ and is called the **gradient** of f. Thus, f is differentiable at $\mathbf{p}$ if and only if

$$f(\mathbf{p} + \mathbf{h}) = f(\mathbf{p}) + \nabla f(\mathbf{p}) \cdot \mathbf{h} + \varepsilon(\mathbf{h}) \cdot \mathbf{h}$$

where $\varepsilon(\mathbf{h}) \to \mathbf{0}$ as $\mathbf{h} \to \mathbf{0}$. The operator ∇ is read "del" and is often called the **del operator.**

In the sense described above, *the gradient becomes the analog of the derivative*. We point out several aspects of our definitions.

1. The derivative $f'(x)$ is a number, whereas the gradient $\nabla f(\mathbf{p})$ is a vector.
2. The products $\nabla f(\mathbf{p}) \cdot \mathbf{h}$ and $\varepsilon(\mathbf{h}) \cdot \mathbf{h}$ are dot products.
3. The definitions of differentiability and gradient are easily extended to any number of dimensions.

Differentiability

Theorem A

If $f(x, y)$ has continuous partial derivatives $f_x(x, y)$ and $f_y(x, y)$ on a disk D whose interior contains (a, b), then $f(x, y)$ is differentiable at (a, b).

Differentiability

If the function f is differentiable at $\mathbf{p}_0$, then, when $\mathbf{h}$ has small magnitude

$$f(\mathbf{p}_0 + \mathbf{h}) \approx f(\mathbf{p}_0) + \nabla f(\mathbf{p}_0) \cdot \mathbf{h}$$

Letting $\mathbf{p} = \mathbf{p}_0 + \mathbf{h}$, we find that the function T defined by

$$T(\mathbf{p}) = f(\mathbf{p}_0) + \nabla f(\mathbf{p}_0) \cdot (\mathbf{p} - \mathbf{p}_0)$$

should be a good approximation to $f(\mathbf{p})$ if $\mathbf{p}$ is close to $\mathbf{p}_0$. The equation $z = T(\mathbf{p})$ defines a plane that approximates f near $\mathbf{p}_0$. Naturally, this plane is called the **tangent plane.** See Figure 4.

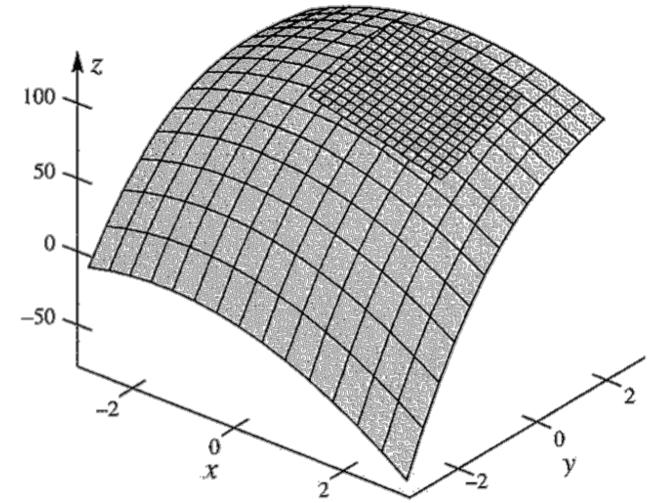


Figure 4

EXAMPLE 1 Show that $f(x, y) = xe^y + x^2y$ is differentiable everywhere and calculate its gradient. Then find the equation of the tangent plane at $(2, 0)$.

SOLUTION We note first that

$$\frac{\partial f}{\partial x} = e^y + 2xy \qquad \text{and} \qquad \frac{\partial f}{\partial y} = xe^y + x^2$$

Both of these functions are continuous everywhere and so, by Theorem A, f is differentiable everywhere. The gradient is

$$\nabla f(x, y) = (e^y + 2xy)\mathbf{i} + (xe^y + x^2)\mathbf{j} = \langle e^y + 2xy, xe^y + x^2 \rangle$$

Thus,

$$\nabla f(2, 0) = \mathbf{i} + 6\mathbf{j} = \langle 1, 6 \rangle$$

and the equation of the tangent plane is

$$\begin{aligned} z &= f(2, 0) + \nabla f(2, 0) \cdot \langle x - 2, y \rangle \\ &= 2 + \langle 1, 6 \rangle \cdot \langle x - 2, y \rangle \\ &= 2 + x - 2 + 6y = x + 6y \end{aligned}$$

Rules for Gradients

Theorem B **Properties of ∇**

The gradient operator ∇ satisfies

1. $\nabla[f(\mathbf{p}) + g(\mathbf{p})] = \nabla f(\mathbf{p}) + \nabla g(\mathbf{p})$
2. $\nabla[\alpha f(\mathbf{p})] = \alpha \nabla f(\mathbf{p})$
3. $\nabla[f(\mathbf{p})g(\mathbf{p})] = f(\mathbf{p})\,\nabla g(\mathbf{p}) + g(\mathbf{p})\nabla f(\mathbf{p})$

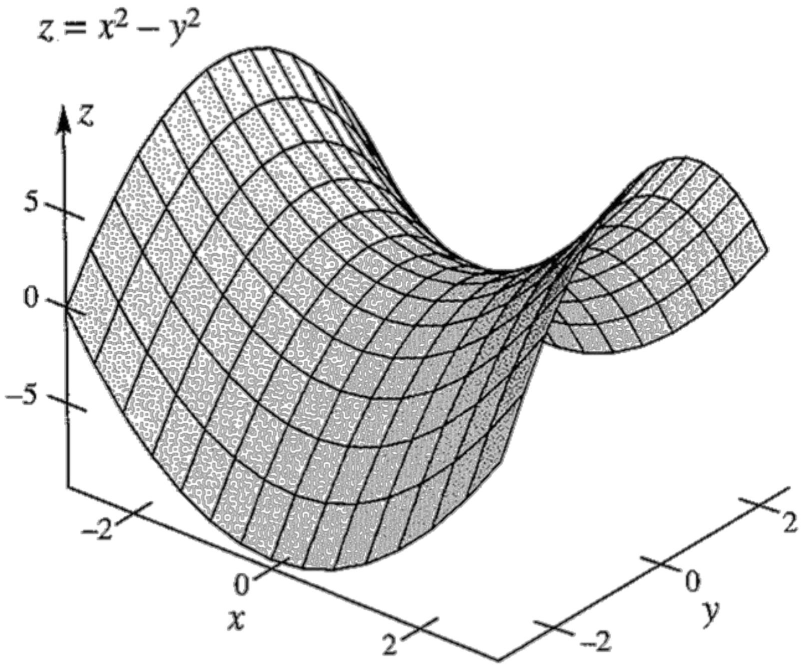
Continuity versus Differentiability

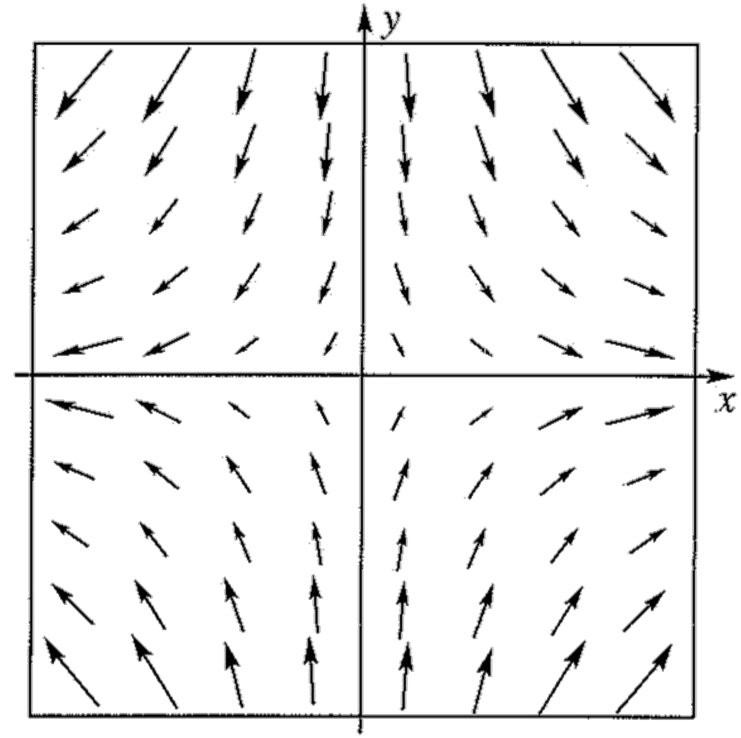
Theorem C

If f is differentiable at $\mathbf{p}$, then f is continuous at $\mathbf{p}$.

The Gradient Field

- The gradient ∇f associates with each point **p** in the domain of f a vector $\nabla f(p)$. The set of all these vectors is called the **gradient field** for f.





Section 12.5

Directional Derivatives and Gradients

Directional Derivative

Definition

For any unit vector $\mathbf{u}$, let

$$D_{\mathbf{u}} f(\mathbf{p}) = \lim_{h \to 0} \frac{f(\mathbf{p} + h\mathbf{u}) - f(\mathbf{p})}{h}$$

This limit, if it exists, is called the **directional derivative** of f at $\mathbf{p}$ in the direction $\mathbf{u}$.

Thus, $D_{\mathbf{i}} f(\mathbf{p}) = f_x(\mathbf{p})$ and $D_{\mathbf{j}} f(\mathbf{p}) = f_y(\mathbf{p})$. Since $\mathbf{p} = (x, y)$, we also use the notation $D_{\mathbf{u}} f(x, y)$. Figure 1 gives the geometric interpretation of $D_{\mathbf{u}} f(x_0, y_0)$. The vector $\mathbf{u}$ determines a line L in the xy-plane through (x_0, y_0). The plane through L perpendicular to the xy-plane intersects the surface $z = f(x, y)$ in a curve C. Its tangent at the point $(x_0, y_0, f(x_0, y_0))$ has slope $D_{\mathbf{u}} f(x_0, y_0)$. Another useful interpretation is that $D_{\mathbf{u}} f(x_0, y_0)$ measures the rate of change of f with respect to distance in the direction

$$f_y(\mathbf{p}) = \lim_{h \to 0} \frac{f(\mathbf{p} + h\mathbf{j}) - f(\mathbf{p})}{h}$$

Connection with the Gradient

Theorem A

Let f be differentiable at $\mathbf{p}$. Then f has a directional derivative at $\mathbf{p}$ in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ and

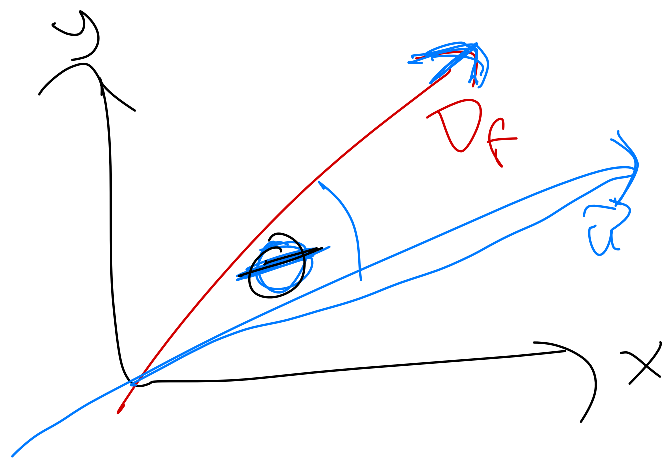
$$D_{\mathbf{u}} f(\mathbf{p}) = \underline{\mathbf{u}} \cdot \underline{\nabla f(\mathbf{p})} \quad = \|\nabla f(p)\| \cdot \|u\| \cdot \cos\theta$$

That is,

$$D_{\mathbf{u}} f(x, y) = u_1 f_x(x, y) + u_2 f_y(x, y)$$

$(i + j)$

$$\hat{u} = \frac{\langle \vec{u} \rangle}{\|\vec{u}\|}$$



$f(x,y) = x^2 y + y^2 x$ cari turunan berarah

di $(2,2)$ dengan arah $\vec{a}$ $\langle\langle 3, 4 \rangle\rangle$

$$\hat{u} = \frac{\langle u \rangle}{\|u\|} = \frac{\langle 3, 4 \rangle}{\sqrt{25}} = \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix}$$

$\|u\| \rightarrow \sqrt{\ldots^2 + \ldots^2}$

$$D_{\hat{u}} f(\bar{p}) = \nabla f(2,2) \cdot \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix}$$

$$= \begin{pmatrix} 12 \\ 12 \end{pmatrix} \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix} = \frac{36}{5} + \frac{48}{5}$$

$$= 84/5$$

$\theta = \frac{\pi}{6}$ $\quad \vec{u} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

Vektor Satuan

$\left(\quad \right)$

Vektor arah

$\langle \ldots \rangle \rightarrow i + j$

$$\frac{\langle \ldots \rangle}{\|(x^2 + y^2)\|} =$$

$$f(x, y) = x^3 - y^5 \qquad p = (2, -1)$$

$$\nabla_f = \begin{pmatrix} 3x^2 \\ -5y^4 \end{pmatrix} = \begin{pmatrix} 12 \\ -5 \end{pmatrix} \rightarrow \text{Direction}$$

rate of change, dapat dari panjang $|\nabla_f|$

$$\hat{\nabla f} = \frac{\langle 12, -5 \rangle}{\sqrt{144 + 25}} = \frac{(12, -5)}{13} \longrightarrow \text{Unit vector arah}$$

$$\|\langle 12, -5 \rangle\| = 13$$

turunan berarah thd sudut, hasil nya ∇f unit vektor

$$\| \quad \|$$

$$D_u f ($$

EXAMPLE 1 If $f(x, y) = 4x^2 - xy + 3y^2$, find the directional derivative of f at $(2, -1)$ in the direction of the vector $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j}$.

SOLUTION The unit vector $\mathbf{u}$ in the direction of $\mathbf{a}$ is $\left(\frac{4}{5}\right)\mathbf{i} + \left(\frac{3}{5}\right)\mathbf{j}$. Also, $f_x(x, y) = 8x - y$ and $f_y(x, y) = -x + 6y$; thus, $f_x(2, -1) = 17$ and $f_y(2, -1) = -8$. Consequently, by Theorem A,

$$D_{\mathbf{u}}f(2, -1) = \left\langle \tfrac{4}{5}, \tfrac{3}{5} \right\rangle \cdot \left\langle 17, -8 \right\rangle = \tfrac{4}{5}(17) + \tfrac{3}{5}(-8) = \tfrac{44}{5}$$

EXAMPLE 2 Find the directional derivative of the function $f(x, y, z) = xy \sin z$ at the point $(1, 2, \pi/2)$ in the direction of the vector $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

SOLUTION The unit vector $\mathbf{u}$ in the direction of $\mathbf{a}$ is $\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$. Also, $f_x(x, y, z) = y \sin z$, $f_y(x, y, z) = x \sin z$, and $f_z(x, y, z) = xy \cos z$, and so $f_x(1, 2, \pi/2) = 2$, $f_y(1, 2, \pi/2) = 1$, and $f_z(1, 2, \pi/2) = 0$. We conclude that

$$D_{\mathbf{u}}f\left(1, 2, \frac{\pi}{2}\right) = \frac{1}{3}(2) + \frac{2}{3}(1) + \frac{2}{3}(0) = \frac{4}{3}$$

Maximum Rate of Change

Maximum Rate of Change For a given function f at a given point $\mathbf{p}$, it is natural to ask in what direction the function is changing most rapidly, that is, in what direction is $D_{\mathbf{u}}f(\mathbf{p})$ the largest? From the geometric formula for the dot product (Section 11.3), we may write

$$D_{\mathbf{u}}f(\mathbf{p}) = \mathbf{u}\cdot\nabla f(\mathbf{p}) = \cancel{\|\mathbf{u}\|}\,\|\nabla f(\mathbf{p})\|\cos\theta = \|\nabla f(\mathbf{p})\|\cos\theta$$

where θ is the angle between $\mathbf{u}$ and $\nabla f(\mathbf{p})$. Thus, $D_{\mathbf{u}}f(\mathbf{p})$ is maximized when $\theta = 0$ and minimized when $\theta = \pi$. We summarize as follows.

$\nabla f(\mathbf{p})$

Theorem B

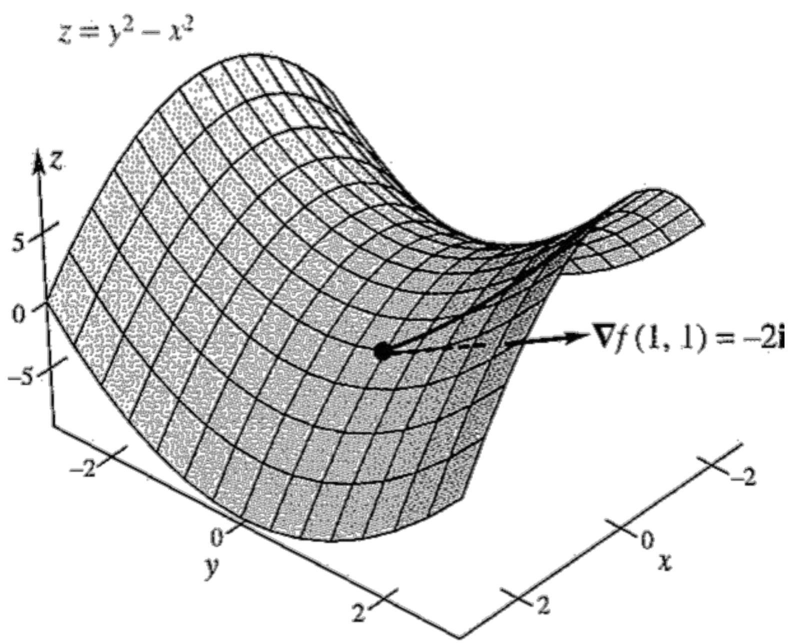
A function increases most rapidly at $\mathbf{p}$ in the direction of the gradient (with rate $\|\nabla f(\mathbf{p})\|$) and decreases most rapidly in the opposite direction (with rate $-\|\nabla f(\mathbf{p})\|$).

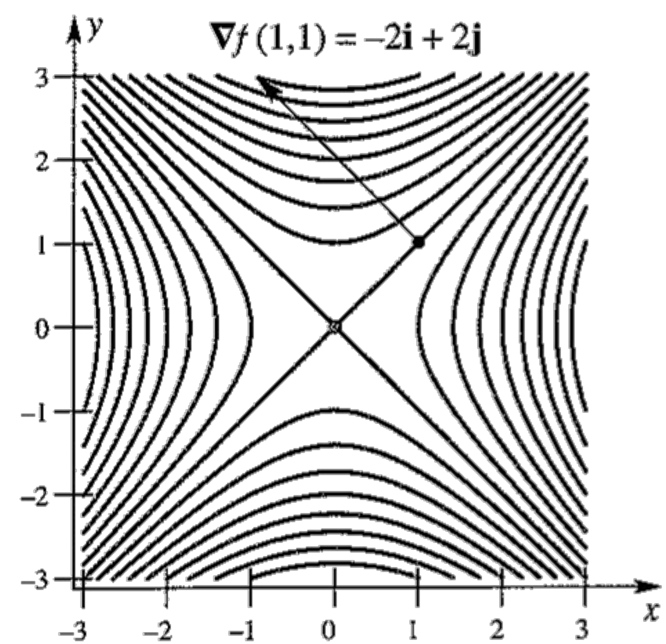
EXAMPLE 3 Suppose that a bug is located on the hyperbolic paraboloid $z = y^2 - x^2$ at the point $(1, 1, 0)$, as in Figure 2. In what direction should it move for the steepest climb and what is the slope as it starts out?

SOLUTION Let $f(x, y) = y^2 - x^2$. Since $f_x(x, y) = -2x$ and $f_y(x, y) = 2y$,

$$\nabla f(1, 1) = f_x(1, 1)\mathbf{i} + f_y(1, 1)\mathbf{j} = -2\mathbf{i} + 2\mathbf{j}$$

Thus, the bug should move from $(1, 1, 0)$ in the direction $-2\mathbf{i} + 2\mathbf{j}$, where the slope will be $\|-2\mathbf{i} + 2\mathbf{j}\| = \sqrt{8} = 2\sqrt{2}$.





Level Curves and Gradients

Theorem C

The gradient of f at a point P is perpendicular to the level curve of f that goes through P.

EXAMPLE 4 For the paraboloid $z = x^2/4 + y^2$, find the equation of its level curve that passes through the point $P(2, 1)$ and sketch it. Find the gradient vector of the paraboloid at P, and draw the gradient with its initial point at P.

SOLUTION The level curve of the paraboloid that corresponds to the plane $z = k$ has the equation $x^2/4 + y^2 = k$. To find the value of k belonging to the level curve through P, we substitute $(2, 1)$ for (x, y) and obtain $k = 2$. Thus, the equation of the level curve that goes through P is that of the ellipse

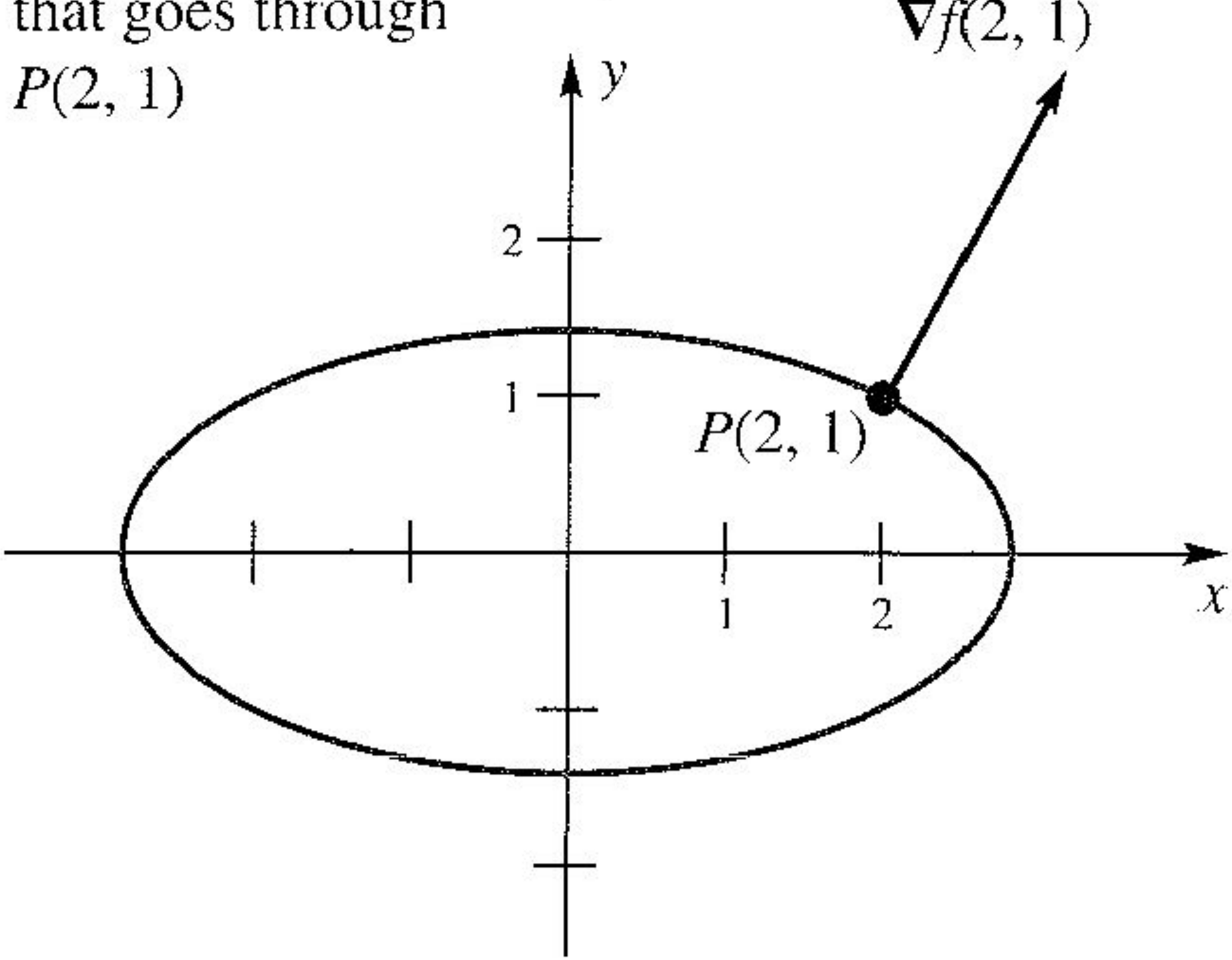
$$\frac{x^2}{8} + \frac{y^2}{2} = 1$$

Next let $f(x, y) = x^2/4 + y^2$. Since $f_x(x, y) = x/2$ and $f_y(x, y) = 2y$, the gradient of the paraboloid at $P(2, 1)$ is

$$\nabla f(2, 1) = f_x(2, 1)\mathbf{i} + f_y(2, 1)\mathbf{j} = \mathbf{i} + 2\mathbf{j}$$

The level curve and the gradient at P are shown in Figure 4.

The level curve of $z = \frac{x^2}{4} + y^2$ that goes through $P(2, 1)$



Section 12.6

The Chain Rule

f_x

f_y

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$f(\bar{p}+\bar{h}) = f(\bar{p}) + f_x \cdot \Delta x + f_y \cdot \Delta y$$

$$\underbrace{f(\bar{p}+\bar{h}) - f(\bar{p})}_{z} = f_x \cdot \Delta x + f_y \cdot \Delta y$$

$$\frac{z}{\Delta t} = \frac{f_x \, \Delta x}{\Delta t} + \frac{f_y \, \Delta y}{\Delta t}$$

$$\frac{dz}{dt} = \boxed{\frac{dz}{dx} \cdot \frac{dx}{dt} + \frac{dz}{dy} \cdot \frac{dy}{dt}}$$

First Version

First Version If $z = f(x, y)$, where x and y are functions of t, then it makes sense to ask for dz/dt, and there ought to be a formula for it.

Theorem A Chain Rule

Let $x = x(t)$ and $y = y(t)$ be differentiable at t, and let $z = f(x, y)$ be differentiable at $(x(t), y(t))$. Then $z = f(x(t), y(t))$ is differentiable at t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

EXAMPLE 1 Suppose that $z = x^3y$, where $x = 2t$ and $y = t^2$. Find dz/dt.

SOLUTION

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= (3x^2y)(2) + (x^3)(2t) \\ &= 6(2t)^2(t^2) + 2(2t)^3(t) \\ &= 40t^4\end{aligned}$$

Second Version

Second Version Suppose next that $z = f(x, y)$, where $x = x(s, t)$ and $y = y(s, t)$. Then it makes sense to ask for $\partial z/\partial s$ and $\partial z/\partial t$.

$x = \ldots s + \ldots t$
$y = \ldots s + \ldots t$

Theorem B **Chain Rule**

Let $x = x(s, t)$ and $y = y(s, t)$ have first partial derivatives at (s, t), and let $z = f(x, y)$ be differentiable at $(x(s, t), y(s, t))$. Then $z = f(x(s, t), y(s, t))$ has first partial derivatives given by

1. $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s}$;

2. $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t}$.

EXAMPLE 4 If $z = 3x^2 - y^2$, where $x = 2s + 7t$ and $y = 5st$, find $\partial z/\partial t$ and express it in terms of s and t.

SOLUTION

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$= (6x)(7) + (-2y)(5s)$$

$$= 42(2s + 7t) - 10st(5s)$$

$$= 84s + 294t - 50s^2t$$

Of course, if we substitute the expressions for x and y into the formula for z and then take the partial derivative with respect to t, we get the same answer:

$$\frac{\partial z}{\partial t} = \frac{\partial}{\partial t}[3(2s + 7t)^2 - (5st)^2]$$

$$= \frac{\partial}{\partial t}[12s^2 + 84st + 147t^2 - 25s^2t^2]$$

$$= 84s + 294t - 50s^2t$$

Implicit Function

Implicit Functions Suppose that $F(x, y) = 0$ defines y implicitly as a function of x, for example, $y = g(x)$, but that the function g is difficult or impossible to determine. We can still find dy/dx. One method for doing this, implicit differentiation, was discussed in Section 2.7. Here is another method.

Let's differentiate both sides of $F(x, y) = 0$ with respect to x using the Chain Rule. We obtain

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

Solving for dy/dx yields the formula

$$\boxed{\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y}}$$

EXAMPLE 6 Find dy/dx if $x^3 + x^2y - 10y^4 = 0$ using

(a) the Chain Rule, and (b) implicit differentiation.

SOLUTION

(a) Let $F(x, y) = x^3 + x^2y - 10y^4$. Then

$$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y} = -\frac{3x^2 + 2xy}{x^2 - 40y^3}$$

(b) Differentiate both sides with respect to x to obtain

$$3x^2 + x^2\frac{dy}{dx} + 2xy - 40y^3\frac{dy}{dx} = 0$$

Solving for dy/dx gives the same result as we obtained with the Chain Rule.

If z is an implicit function of x and y defined by the equation $F(x, y, z) = 0$, then differentiation of both sides with respect to x, holding y fixed, yields

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

If we solve for $\partial z/\partial x$ and note that $\partial y/\partial x = 0$, we get the first of the formulas below. A similar calculation holding x fixed and differentiating with respect to y produces the second formula.

$$\boxed{\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z}, \qquad \frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}}$$

EXAMPLE 7 If $F(x, y, z) = x^3e^{y+z} - y\sin(x - z) = 0$ defines z implicitly as a function of x and y, find $\partial z/\partial x$.

SOLUTION

$$\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z} = -\frac{3x^2e^{y+z} - y\cos(x - z)}{x^3e^{y+z} + y\cos(x - z)}$$

Section 12.7

Tangent Planes and Approximations

Tangent Planes

$z = f(x, y)$

$f(x, y) - z = 0$

$F(x, y, z) = 0$

Definition

Let $F(x, y, z) = k$ determine a surface, and suppose that F is differentiable at a point $P(x_0, y_0, z_0)$ of this surface, with $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$. Then the plane through P perpendicular to $\nabla F(x_0, y_0, z_0)$ is called the **tangent plane** to the surface at P.

Theorem A **Tangent Planes**

For the surface $F(x, y, z) = k$, the equation of the tangent plane at (x_0, y_0, z_0) is $\nabla F(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$; that is,

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

In particular, for the surface $z = f(x, y)$, the equation of the tangent plane at $(x_0, y_0, f(x_0, y_0))$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$z = z_0 + f_x(x - x_0) + f_y(y - y_0) + \frac{1}{2}(f_{xx}(x - x_0)^2 + f_{xy}(x - x_0)(y - y_0)$

dz $(z - z_0)$

EXAMPLE 1 Find the equation of the tangent plane (Figure 3) to $z = x^2 + y^2$ at the point $(1, 1, 2)$.

SOLUTION Let $f(x, y) = x^2 + y^2$, and note that $\nabla f(x, y) = 2x\mathbf{i} + 2y\mathbf{j}$. Thus, $\nabla f(1, 1) = 2\mathbf{i} + 2\mathbf{j}$, and from Theorem A, the required equation is

$$z - 2 = 2(x - 1) + 2(y - 1)$$

or

$$2x + 2y - z = 2$$

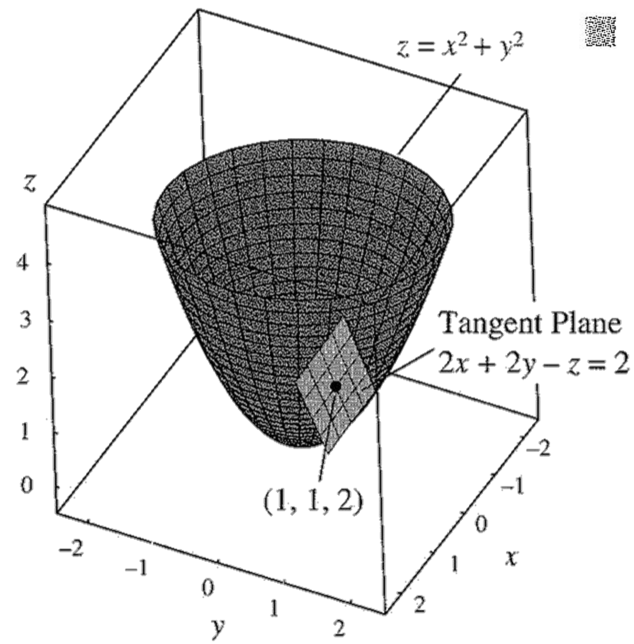


Figure 3

Differentials and Approximations

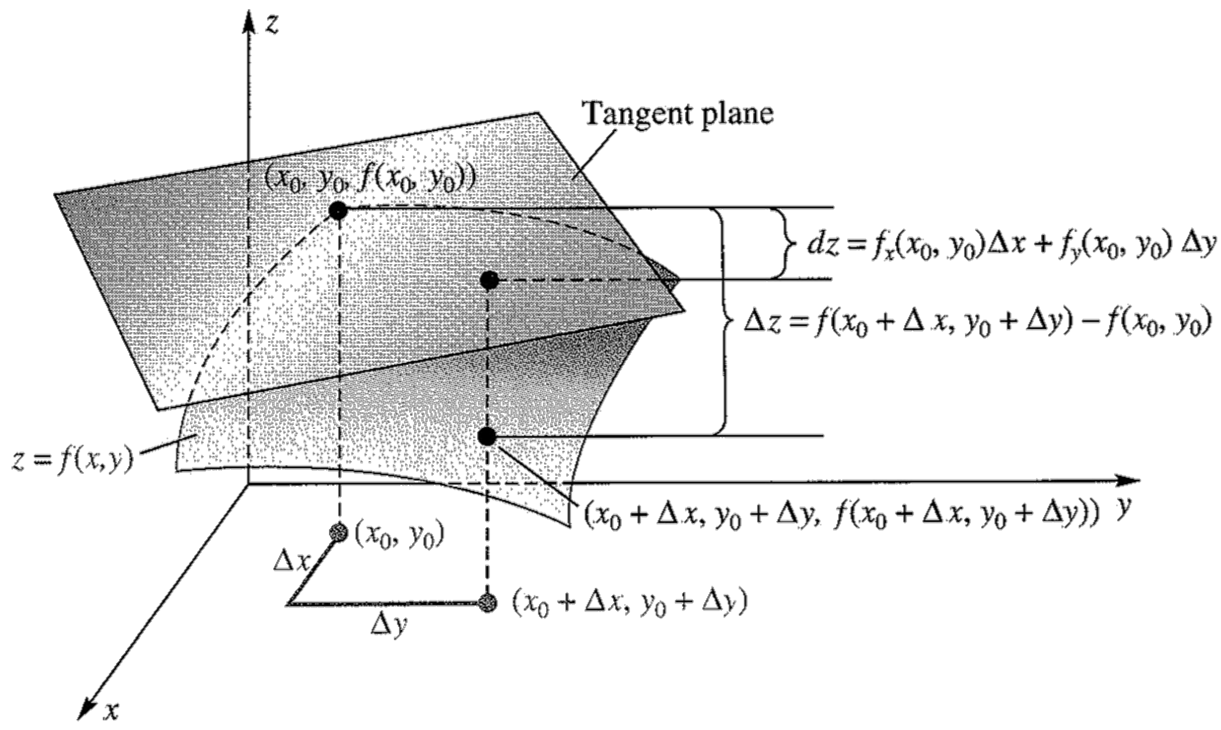
Definition

Let $z = f(x, y)$, where f is a differentiable function, and let dx and dy (called the differentials of x and y) be variables. The **differential of the dependent variable,** dz, also called the total **differential of f** and written $df(x, y)$, is defined by

$$dz = df(x, y) = f_x(x, y)\,dx + f_y(x, y)\,dy = \nabla f \cdot \langle dx, dy \rangle$$

Differentials and Approximations

The significance of dz arises from the fact that if $dx = \Delta x$ and $dy = \Delta y$ represent small changes in x and y, respectively, then dz will be a good approximation to Δz, the corresponding change in z. This is illustrated in Figure 5 and, while dz does not appear to be a very good approximation to Δz, you can see that it will get better and better as Δx and Δy get smaller.



EXAMPLE 3 Let $z = f(x, y) = 2x^3 + xy - y^3$. Compute Δz and dz as (x, y) changes from $(2, 1)$ to $(2.03, 0.98)$.

SOLUTION

$$\begin{aligned}
\Delta z &= f(2.03, 0.98) - f(2, 1) \\
&= 2(2.03)^3 + (2.03)(0.98) - (0.98)^3 - [2(2)^3 + 2(1) - 1^3] \\
&= 0.779062 \\
dz &= f_x(x, y)\,\Delta x + f_y(x, y)\,\Delta y \\
&= (6x^2 + y)\,\Delta x + (x - 3y^2)\,\Delta y
\end{aligned}$$

At $(2, 1)$ with $\Delta x = 0.03$ and $\Delta y = -0.02$,

$$dz = (25)(0.03) + (-1)(-0.02) = 0.77$$

Section 12.8

Maxima and Minima

Maxima and Minima

Definition

Let f be a function with domain S, and let $\mathbf{p}_0$ be a point in S.

(i) $f(\mathbf{p}_0)$ is a **global maximum value** of f on S if $f(\mathbf{p}_0) \geq f(\mathbf{p})$ for all $\mathbf{p}$ in S.

(ii) $f(\mathbf{p}_0)$ is a **global minimum value** of f on S if $f(\mathbf{p}_0) \leq f(\mathbf{p})$ for all $\mathbf{p}$ in S.

(iii) $f(\mathbf{p}_0)$ is a **global extreme value** of f on S if $f(\mathbf{p}_0)$ is either a global maximum value or a global minimum value.

We obtain definitions for **local maximum value** and **local minimum value** if in (i) and (ii) we require only that the inequalities hold on $N \cap S$, where N is some neighborhood of $\mathbf{p}_0$. $f(\mathbf{p}_0)$ is a **local extreme value** of f on S if $f(\mathbf{p}_0)$ is either a local maximum value or a local minimum value.

Theorem A **Max–Min Existence Theorem**

If f is continuous on a closed bounded set S, then f attains both a (global) maximum value and a (global) minimum value there.

Where Do Extreme Values Occurs?

Theorem B **Critical Point Theorem**

Let f be defined on a set S containing $\mathbf{p}_0$. If $f(\mathbf{p}_0)$ is an extreme value, then $\mathbf{p}_0$ must be a critical point; that is, either $\mathbf{p}_0$ is

1. a boundary point of S; or
2. a stationary point of f; or
3. a singular point of f.

EXAMPLE 1 Find the local maximum or minimum values of $f(x, y) = x^2 - 2x + y^2/4$.

SOLUTION The given function is differentiable throughout its domain, the xy-plane. Thus, the only possible critical points are the stationary points obtained by setting $f_x(x, y)$ and $f_y(x, y)$ equal to zero. But $f_x(x, y) = 2x - 2$ and $f_y(x, y) = y/2$ are zero only when $x = 1$ and $y = 0$. It remains to decide whether $(1, 0)$ gives a maximum or a minimum or neither. We will develop a simple tool for this soon, but for now we must use a little ingenuity. Note that $f(1, 0) = -1$ and

$$f(x, y) = x^2 - 2x + \frac{y^2}{4} = x^2 - 2x + 1 + \frac{y^2}{4} - 1$$

$$= (x - 1)^2 + \frac{y^2}{4} - 1 \geq -1$$

Thus, $f(1, 0)$ is actually a global minimum for f. There are no local maximum values. ■

Sufficient Conditions of Extrema

Theorem C Second Partials Test

Suppose that $f(x, y)$ has continuous second partial derivatives in a neighborhood of (x_0, y_0) and that $\nabla f(x_0, y_0) = \mathbf{0}$. Let

$$D = D(x_0, y_0) = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0)$$

Then

1. if $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then $f(x_0, y_0)$ is a local maximum value;
2. if $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then $f(x_0, y_0)$ is a local minimum value;
3. if $D < 0$, then $f(x_0, y_0)$ is not an extreme value ((x_0, y_0) is a saddle point);
4. if $D = 0$, then the test is inconclusive.

EXAMPLE 3 Find the extrema, if any, of the function F defined by $F(x, y) = 3x^3 + y^2 - 9x + 4y$.

SOLUTION Since $F_x(x, y) = 9x^2 - 9$ and $F_y(x, y) = 2y + 4$, the critical points, obtained by solving the simultaneous equations $F_x(x, y) = F_y(x, y) = 0$, are $(1, -2)$ and $(-1, -2)$.

Now $F_{xx}(x, y) = 18x$, $F_{yy}(x, y) = 2$, and $F_{xy} = 0$. Thus, at the critical point $(1, -2)$,

$$D = F_{xx}(1, -2) \cdot F_{yy}(1, -2) - F_{xy}^2(1, -2) = 18(2) - 0 = 36 > 0$$

Furthermore, $F_{xx}(1, -2) = 18 > 0$ and so, by Theorem C(2), $F(1, -2) = -10$ is a local minimum value of F.

In testing the given function at the other critical point, $(-1, -2)$, we find that $F_{xx}(-1, -2) = -18$, $F_{yy}(-1, -2) = 2$, and $F_{xy}(-1, -2) = 0$, which makes $D = -36 < 0$. Thus, by Theorem C(3), $(-1, -2)$ is a saddle point and $F(-1, -2)$ is not an extremum. ■

EXAMPLE 4 Find the minimum distance between the origin and the surface $z^2 = x^2y + 4$.

SOLUTION Let $P(x, y, z)$ be any point on the surface. The square of the distance between the origin and P is $d^2 = x^2 + y^2 + z^2$. We seek the coordinates of P that make d^2 (and hence d) a minimum.

Since P is on the surface, its coordinates satisfy the equation of the surface. Substituting $z^2 = x^2y + 4$ in $d^2 = x^2 + y^2 + z^2$, we obtain d^2 as a function of two variables x and y.

$$d^2 = f(x, y) = x^2 + y^2 + x^2y + 4$$

To find the critical points, we set $f_x(x, y) = 0$ and $f_y(x, y) = 0$, obtaining

$$2x + 2xy = 0 \quad \text{and} \quad 2y + x^2 = 0$$

By eliminating y between these equations, we get

$$2x - x^3 = 0$$

Thus, $x = 0$ or $x = \pm\sqrt{2}$. Substituting these values in the second of the equations, we obtain $y = 0$ and $y = -1$. Therefore, the critical points are $(0, 0)$, $\left(\sqrt{2}, -1\right)$, and $\left(-\sqrt{2}, -1\right)$. (There are no boundary points.)

To test each of these, we need $f_{xx}(x, y) = 2 + 2y$, $f_{yy}(x, y) = 2$, $f_{xy}(x, y) = 2x$, and

$$D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = 4 + 4y - 4x^2$$

Since $D\left(\pm\sqrt{2}, -1\right) = -8 < 0$, neither $\left(\sqrt{2}, -1\right)$ nor $\left(-\sqrt{2}, -1\right)$ yields an extremum. However, $D(0, 0) = 4 > 0$ and $f_{xx}(0, 0) = 2 > 0$; so $(0, 0)$ yields the minimum distance. Substituting $x = 0$ and $y = 0$ in the expression for d^2, we find $d^2 = 4$.

The minimum distance between the origin and the given surface is 2. ▩

Section 12.9

The Method of Lagrange Multipliers

Geometric Interpretation of the Method

Theorem A **Lagrange's Method**

To maximize or minimize $f(\mathbf{p})$ subject to the constraint $g(\mathbf{p}) = 0$, solve the system of equations

$$\nabla f(\mathbf{p}) = \lambda \nabla g(\mathbf{p}) \quad \text{and} \quad g(\mathbf{p}) = 0$$

for $\mathbf{p}$ and λ. Each such point $\mathbf{p}$ is a critical point for the constrained extremum problem, and the corresponding λ is called a Lagrange multiplier.

EXAMPLE 1 What is the greatest area that a rectangle can have if the length of its diagonal is 2?

SOLUTION Place the rectangle in the first quadrant with two of its sides along the coordinate axes; then the vertex opposite the origin has coordinates (x, y), with x and y positive (Figure 3). The length of its diagonal is $\sqrt{x^2 + y^2} = 2$, and its area is xy.

Thus, we may formulate the problem to be that of maximizing $f(x, y) = xy$ subject to the constraint $g(x, y) = x^2 + y^2 - 4 = 0$. The corresponding gradients are

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = y\,\mathbf{i} + x\,\mathbf{j}$$

$$\nabla g(x, y) = g_x(x, y)\mathbf{i} + g_y(x, y)\mathbf{j} = 2x\,\mathbf{i} + 2y\,\mathbf{j}$$

$$\nabla g(x, y) = g_x(x, y)\mathbf{i} + g_y(x, y)\mathbf{j} = 2x\,\mathbf{i} + 2y\,\mathbf{j}$$

Lagrange's equations thus become

(1) $$y = \lambda(2x)$$

(2) $$x = \lambda(2y)$$

(3) $$x^2 + y^2 = 4$$

which we must solve simultaneously. If we multiply the first equation by y and the second by x, we get $y^2 = 2\lambda xy$ and $x^2 = 2\lambda xy$, from which

(4) $$y^2 = x^2$$

From (3) and (4), we find that $x = \sqrt{2}$ and $y = \sqrt{2}$; and by substituting these values in (1), we obtain $\lambda = \frac{1}{2}$. Thus, the solution to equations (1) through (3), keeping x and y positive, is $x = \sqrt{2}$, $y = \sqrt{2}$, and $\lambda = \frac{1}{2}$.

We conclude that the rectangle of greatest area with diagonal 2 is the square having sides of length $\sqrt{2}$. Its area is 2. A geometric interpretation of this problem is shown in Figure 4. ∎

EXAMPLE 3 Find the minimum of $f(x, y, z) = 3x + 2y + z + 5$ subject to the constraint $g(x, y, z) = 9x^2 + 4y^2 - z = 0$.

SOLUTION The gradients of f and g are $\nabla f(x, y, z) = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\nabla g(x, y, z) = 18x\mathbf{i} + 8y\mathbf{j} - \mathbf{k}$. To find the critical points, we solve the equations

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \quad \text{and} \quad g(x, y, z) = 0$$

for (x, y, z, λ), in which λ is a Lagrange multiplier. This is equivalent, in the present problem, to solving the following system of four simultaneous equations in the four variables x, y, z, and λ.

(1) $$3 = 18x\lambda$$

(2) $$2 = 8y\lambda$$

(3) $$1 = -\lambda$$

(4) $$9x^2 + 4y^2 - z = 0$$

From (3), $\lambda = -1$. Substituting this result in equations (1) and (2), we get $x = -\frac{1}{6}$ and $y = -\frac{1}{4}$. By putting these values for x and y in equation (4), we obtain $z = \frac{1}{2}$. Thus, the solution of the foregoing system of four simultaneous equations is $\left(-\frac{1}{6}, -\frac{1}{4}, \frac{1}{2}, -1\right)$, and the only critical point is $\left(-\frac{1}{6}, -\frac{1}{4}, \frac{1}{2}\right)$. Therefore, the minimum of $f(x, y, z)$, subject to the constraint $g(x, y, z) = 0$, is $f\left(-\frac{1}{6}, -\frac{1}{4}, \frac{1}{2}\right) = \frac{9}{2}$. (How do we know that this value is a minimum rather than a maximum?) ■

End of Chapter 12

End of Chapter 12