Chapter 7

Techniques of Integration

Section 7.1

Basic Integration Rules

Standard Forms

Standard Integral Forms

Constants, Powers

1. $\int k\,du = ku + C$

2. $\int u^r\,du = \begin{cases} \dfrac{u^{r+1}}{r+1} + C & r \neq -1 \\ \ln|u| + C & r = -1 \end{cases}$

Exponentials

3. $\int e^u\,du = e^u + C$

4. $\int a^u\,du = \dfrac{a^u}{\ln a} + C,\ a \neq 1,\ a > 0$

Trigonometric Functions

5. $\int \sin u\,du = -\cos u + C$

6. $\int \cos u\,du = \sin u + C$

7. $\int \sec^2 u\,du = \tan u + C$

8. $\int \csc^2 u\,du = -\cot u + C$

9. $\int \sec u \tan u\,du = \sec u + C$

10. $\int \csc u \cot u\,du = -\csc u + C$

11. $\int \tan u\,du = -\ln|\cos u| + C$

12. $\int \cot u\,du = \ln|\sin u| + C$

Algebraic Functions

13. $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\dfrac{u}{a}\right) + C$

14. $\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{u}{a}\right) + C$

15. $\int \dfrac{du}{u\sqrt{u^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\left(\dfrac{|u|}{a}\right) + C = \dfrac{1}{a}\cos^{-1}\left(\dfrac{a}{|u|}\right) + C$

16. $\int \sinh u\,du = \cosh u + C$

17. $\int \cosh u\,du = \sinh u + C$

Substitution in Indefinite Integrals

Theorem A **Substitution in Indefinite Integrals**

Let g be a differentiable function and suppose that F is an antiderivative of f. Then, if $u = g(x)$,

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du = F(u) + C = F(g(x)) + C$$

EXAMPLE 2 Find $\int \frac{3}{\sqrt{5 - 9x^2}} dx$.

SOLUTION Think of $\int \frac{du}{\sqrt{a^2 - u^2}}$. Let $u = 3x$, so $du = 3\,dx$. Then

$$\int \frac{3}{\sqrt{5 - 9x^2}}\,dx = \int \frac{1}{\sqrt{5 - u^2}}\,du = \sin^{-1}\left(\frac{u}{\sqrt{5}}\right) + C$$

$$= \sin^{-1}\left(\frac{3x}{\sqrt{5}}\right) + C$$

EXAMPLE 6 Find $\int \frac{a^{\tan t}}{\cos^2 t}\, dt$.

SOLUTION Mentally, substitute $u = \tan t$.

$$\int \frac{a^{\tan t}}{\cos^2 t}\, dt = \int a^{\tan t}(\sec^2 t\, dt) = \frac{a^{\tan t}}{\ln a} + C$$

EXAMPLE 7 Evaluate $\displaystyle\int_2^5 t\sqrt{t^2-4}\,dt$.

SOLUTION Let $u = t^2 - 4$, so $du = 2t\,dt$; note that when $t = 2, u = 0$, and when $t = 5, u = 21$. Thus,

$$\begin{aligned}\int_2^5 t\sqrt{t^2-4}\,dt &= \frac{1}{2}\int_2^5 (t^2-4)^{1/2}(2t\,dt) \\ &= \frac{1}{2}\int_0^{21} u^{1/2}\,du \\ &= \left[\frac{1}{3}u^{3/2}\right]_0^{21} = \frac{1}{3}(21)^{3/2} \approx 32.08\end{aligned}$$

Section 7.2

Integration by Parts

Integration by Parts

Integration by Parts: Indefinite Integrals

$$\int u\,dv = uv - \int v\,du$$

Integration by Parts: Definite Integrals

$$\int_a^b u\,dv = \left[uv\right]_a^b - \int_a^b v\,du$$

EXAMPLE 1 Find $\int x \cos x \, dx$.

SOLUTION We wish to write $x \cos x \, dx$ as $u \, dv$. One possibility is to let $u = x$ and $dv = \cos x \, dx$. Then $du = dx$ and $v = \int \cos x \, dx = \sin x$ (we can omit the arbitrary constant at this stage). Here is a summary of this double substitution in a convenient format.

$$u = x \qquad\qquad dv = \cos x \, dx$$

$$du = dx \qquad\qquad v = \sin x$$

The formula for integration by parts gives

$$\int \underbrace{x}_{u} \underbrace{\cos x \, dx}_{dv} = \underbrace{x}_{u} \underbrace{\sin x}_{v} - \int \underbrace{\sin x}_{v} \underbrace{dx}_{du}$$
$$= x \sin x + \cos x + C$$

We were successful on our first try. Another substitution would be

$$u = \cos x \qquad dv = x\,dx$$

$$du = -\sin x\,dx \qquad v = \frac{x^2}{2}$$

This time the formula for integration by parts gives

$$\int \underbrace{(\cos x)}_{u}\,\underbrace{x\,dx}_{dv} = \underbrace{(\cos x)}_{u}\,\underbrace{\frac{x^2}{2}}_{v} - \int \underbrace{\frac{x^2}{2}}_{v}\,\underbrace{(-\sin x\,dx)}_{du}$$

which is correct but not helpful. The new integral on the right-hand side is more complicated than the original one. Thus, we see the importance of a wise choice for u and dv. ■

EXAMPLE 2 Find $\int_1^2 \ln x \, dx$.

SOLUTION We make the following substitutions:

$$u = \ln x \qquad dv = dx$$

$$du = \left(\frac{1}{x}\right) dx \qquad v = x$$

Then

$$\begin{aligned} \int_1^2 \ln x \, dx &= [x \ln x]_1^2 - \int_1^2 x \frac{1}{x} dx \\ &= 2 \ln 2 - \int_1^2 dx \\ &= 2 \ln 2 - 1 \approx 0.386 \end{aligned}$$

Reduction Formulas

Reduction Formulas A formula of the form

$$\int f^n(x)g(x)\,dx = h(x) + \int f^k(x)\,g(x)\,dx$$

where $k < n$, is called a **reduction formula** (the exponent on f is reduced). Such formulas can often be obtained via integration by parts.

EXAMPLE 7 Derive a reduction formula for $\int \sin^n x \, dx$.

SOLUTION Let $u = \sin^{n-1} x$ and $dv = \sin x \, dx$. Then

$$du = (n - 1) \sin^{n-2} x \cos x \, dx \quad \text{and} \quad v = -\cos x$$

from which

$$\int \sin^n x \, dx = -\sin^{n-1} x \cos x + (n - 1) \int \sin^{n-2} x \cos^2 x \, dx$$

If we replace $\cos^2 x$ by $1 - \sin^2 x$ in the last integral, we obtain

$$\int \sin^n x \, dx = -\sin^{n-1} x \cos x + (n - 1) \int \sin^{n-2} x \, dx - (n - 1) \int \sin^n x \, dx$$

After combining the first and last integrals above and solving for $\int \sin^n x \, dx$, we get the reduction formula (valid for $n \geq 2$)

$$\int \sin^n x \, dx = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n - 1}{n} \int \sin^{n-2} x \, dx$$

Section 7.3

Some Trigonometric Integrals

Trigonometric Forms

1. $\int \sin^n x\,dx$ and $\int \cos^n x\,dx$
2. $\int \sin^m x \cos^n x\,dx$
3. $\int \sin mx \cos nx\,dx$, $\int \sin mx \sin nx\,dx$, $\int \cos mx \cos nx\,dx$
4. $\int \tan^n x\,dx$, $\int \cot^n x\,dx$
5. $\int \tan^m x \sec^n x\,dx$, $\int \cot^m x \csc^n x\,dx$

Type 1 $\left(\int \sin^n x\,dx,\ \int \cos^n x\,dx\right)$

EXAMPLE 1 **(n Odd)** Find $\int \sin^5 x\,dx$.

SOLUTION

$$\begin{aligned}\int \sin^5 x\,dx &= \int \sin^4 x \sin x\,dx \\ &= \int (1-\cos^2 x)^2 \sin x\,dx \\ &= \int (1 - 2\cos^2 x + \cos^4 x)\sin x\,dx \\ &= -\int (1 - 2\cos^2 x + \cos^4 x)(-\sin x\,dx) \\ &= -\cos x + \tfrac{2}{3}\cos^3 x - \tfrac{1}{5}\cos^5 x + C\end{aligned}$$

EXAMPLE 2 **(*n* Even)** Find $\int \sin^2 x\, dx$ and $\int \cos^4 x\, dx$.

SOLUTION Here we make use of half-angle identities.

$$\int \sin^2 x\, dx = \int \frac{1 - \cos 2x}{2}\, dx$$

$$= \frac{1}{2}\int dx - \frac{1}{4}\int (\cos 2x)(2\, dx)$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

$$\int \cos^4 x\, dx = \int \left(\frac{1 + \cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)\, dx$$

$$= \frac{1}{4}\int dx + \frac{1}{4}\int (\cos 2x)(2)\, dx + \frac{1}{8}\int (1 + \cos 4x)\, dx$$

$$= \frac{3}{8}\int dx + \frac{1}{4}\int \cos 2x(2\, dx) + \frac{1}{32}\int \cos 4x(4\, dx)$$

$$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

Type 2 $\left(\int \sin^m x \cos^n x \, dx\right)$

EXAMPLE 3 **(m or n Odd)** Find $\int \sin^3 x \cos^{-4} x \, dx$.

SOLUTION

$$\begin{aligned}
\int \sin^3 x \cos^{-4} x \, dx &= \int (1 - \cos^2 x)(\cos^{-4} x)(\sin x) \, dx \\
&= -\int (\cos^{-4} x - \cos^{-2} x)(-\sin x \, dx) \\
&= -\left[\frac{(\cos x)^{-3}}{-3} - \frac{(\cos x)^{-1}}{-1}\right] + C \\
&= \frac{1}{3}\sec^3 x - \sec x + C
\end{aligned}$$

EXAMPLE 4 **(Both m and n Even)** Find $\int \sin^2 x \cos^4 x \, dx$.

SOLUTION

$$\int \sin^2 x \cos^4 x \, dx$$

$$= \int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{8} \int (1 + \cos 2x - \cos^2 2x - \cos^3 2x) \, dx$$

$$= \frac{1}{8} \int \left[1 + \cos 2x - \frac{1}{2}(1 + \cos 4x) - (1 - \sin^2 2x) \cos 2x\right] dx$$

$$= \frac{1}{8} \int \left[\frac{1}{2} - \frac{1}{2}\cos 4x + \sin^2 2x \cos 2x\right] dx$$

$$= \frac{1}{8}\left[\int \frac{1}{2} dx - \frac{1}{8} \int \cos 4x(4 \, dx) + \frac{1}{2} \int \sin^2 2x(2 \cos 2x \, dx)\right]$$

$$= \frac{1}{8}\left[\frac{1}{2} x - \frac{1}{8} \sin 4x + \frac{1}{6} \sin^3 2x\right] + C$$

Type 3 $\left(\int \sin mx \cos nx \, dx, \int \sin mx \sin nx \, dx, \int \cos mx \cos nx \, dx\right)$

1. $\sin mx \cos nx = \frac{1}{2}[\sin(m+n)x + \sin(m-n)x]$

2. $\sin mx \sin nx = -\frac{1}{2}[\cos(m+n)x - \cos(m-n)x]$

3. $\cos mx \cos nx = \frac{1}{2}[\cos(m+n)x + \cos(m-n)x]$

EXAMPLE 5 Find $\int \sin 2x \cos 3x \, dx$.

SOLUTION Apply product identity 1.

$$\begin{aligned}
\int \sin 2x \cos 3x \, dx &= \frac{1}{2} \int [\sin 5x + \sin(-x)] \, dx \\
&= \frac{1}{10} \int \sin 5x (5 \, dx) - \frac{1}{2} \int \sin x \, dx \\
&= -\frac{1}{10} \cos 5x + \frac{1}{2} \cos x + C
\end{aligned}$$

Type 4 $\left(\int \tan^n x\,dx,\ \int \cot^n x\,dx\right)$

EXAMPLE 8 Find $\int \cot^4 x\,dx$.

SOLUTION

$$\begin{aligned}\int \cot^4 x\,dx &= \int \cot^2 x\,(\csc^2 x - 1)\,dx \\ &= \int \cot^2 x \csc^2 x\,dx - \int \cot^2 x\,dx \\ &= -\int \cot^2 x\,(-\csc^2 x\,dx) - \int (\csc^2 x - 1)\,dx \\ &= -\tfrac{1}{3}\cot^3 x + \cot x + x + C\end{aligned}$$

Type 5 $\left(\int \tan^m x \sec^n x\, dx,\ \int \cot^m x \csc^n x\, dx\right)$

EXAMPLE 10 **(n Even, m Any Number)** Find $\int \tan^{-3/2} x \sec^4 x\, dx$.

SOLUTION

$$\begin{aligned}\int \tan^{-3/2} x \sec^4 x\, dx &= \int (\tan^{-3/2} x)(1 + \tan^2 x) \sec^2 x\, dx \\ &= \int (\tan^{-3/2} x) \sec^2 x\, dx + \int (\tan^{1/2} x) \sec^2 x\, dx \\ &= -2 \tan^{-1/2} x + \tfrac{2}{3} \tan^{3/2} x + C\end{aligned}$$

EXAMPLE 11 **(m Odd, n Any Number)** Find $\int \tan^3 x \sec^{-1/2} x\, dx$.

SOLUTION

$$\int \tan^3 x \sec^{-1/2} x\, dx = \int (\tan^2 x)(\sec^{-3/2} x)(\sec x \tan x)\, dx$$

$$= \int (\sec^2 x - 1) \sec^{-3/2} x\, (\sec x \tan x\, dx)$$

$$= \int \sec^{1/2} x\, (\sec x \tan x\, dx) - \int \sec^{-3/2} x\, (\sec x \tan x\, dx)$$

$$= \tfrac{2}{3}\sec^{3/2} x + 2 \sec^{-1/2} x + C$$

Section 7.4

Rationalizing Substitutions

Integrands Involving

Integrands Involving $\sqrt[n]{ax+b}$ If $\sqrt[n]{ax+b}$ appears in an integral, the substitution $u = \sqrt[n]{ax+b}$ will eliminate the radical.

EXAMPLE 1 Find $\displaystyle\int \frac{dx}{x-\sqrt{x}}$.

SOLUTION Let $u = \sqrt{x}$, so $u^2 = x$ and $2u\,du = dx$. Then

$$\int \frac{dx}{x-\sqrt{x}} = \int \frac{2u}{u^2-u}\,du = 2\int \frac{1}{u-1}\,du$$

$$= 2\ln|u-1| + C = 2\ln|\sqrt{x}-1| + C$$

EXAMPLE 2 Find $\int x\sqrt[3]{x-4}\,dx$.

SOLUTION Let $u = \sqrt[3]{x-4}$, so $u^3 = x - 4$ and $3u^2\,du = dx$. Then

$$\int x\sqrt[3]{x-4}\,dx = \int (u^3+4)u\cdot(3u^2\,du) = 3\int (u^6+4u^3)\,du$$

$$= 3\left[\frac{u^7}{7} + u^4\right] + C = \frac{3}{7}(x-4)^{7/3} + 3(x-4)^{4/3} + C$$

Integrands Involving

Integrands Involving $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$ and $\sqrt{x^2 - a^2}$ To rationalize these three expressions, we may assume that a is positive and make the following trigonometric substitutions.

	Radical	**Substitution**	**Restriction on t**
1.	$\sqrt{a^2 - x^2}$	$x = a \sin t$	$-\pi/2 \le t \le \pi/2$
2.	$\sqrt{a^2 + x^2}$	$x = a \tan t$	$-\pi/2 < t < \pi/2$
3.	$\sqrt{x^2 - a^2}$	$x = a \sec t$	$0 \le t \le \pi, t \ne \pi/2$

EXAMPLE 4 Find $\int \sqrt{a^2 - x^2}\, dx$.

SOLUTION We make the substitution

$$x = a \sin t, \qquad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$$

Then $dx = a \cos t\, dt$ and $\sqrt{a^2 - x^2} = a \cos t$. Thus,

$$\begin{aligned}
\int \sqrt{a^2 - x^2}\, dx &= \int a \cos t \cdot a \cos t\, dt = a^2 \int \cos^2 t\, dt \\
&= \frac{a^2}{2} \int (1 + \cos 2t)\, dt \\
&= \frac{a^2}{2}\left(t + \frac{1}{2}\sin 2t\right) + C \\
&= \frac{a^2}{2}(t + \sin t \cos t) + C
\end{aligned}$$

Now, $x = a \sin t$ is equivalent to $x/a = \sin t$ and, since t was restricted to make the sine function invertible,

$$t = \sin^{-1}\left(\frac{x}{a}\right)$$

Using the right triangle in Figure 1 (as we did in Section 6.8), we see that

$$\cos t = \cos\left[\sin^{-1}\left(\frac{x}{a}\right)\right] = \sqrt{1 - \frac{x^2}{a^2}} = \frac{1}{a}\sqrt{a^2 - x^2}$$

Thus,

$$\int \sqrt{a^2 - x^2}\, dx = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2 - x^2} + C$$

EXAMPLE 5 Find $\int \frac{dx}{\sqrt{9 + x^2}}$.

SOLUTION Let $x = 3 \tan t, -\pi/2 < t < \pi/2$. Then $dx = 3 \sec^2 t \, dt$ and $\sqrt{9 + x^2} = 3 \sec t$.

$$\int \frac{dx}{\sqrt{9 + x^2}} = \int \frac{3 \sec^2 t}{3 \sec t} \, dt = \int \sec t \, dt$$
$$= \ln|\sec t + \tan t| + C$$

The last step, the integration of sec t, was handled in Problem 56 of Section 7.1. Now $\tan t = x/3$, which suggests the triangle in Figure 3, from which we conclude that $\sec t = \sqrt{9 + x^2}/3$. Thus,

$$\int \frac{dx}{\sqrt{9 + x^2}} = \ln\left|\frac{\sqrt{9 + x^2} + x}{3}\right| + C$$
$$= \ln\left|\sqrt{9 + x^2} + x\right| - \ln 3 + C$$
$$= \ln\left|\sqrt{9 + x^2} + x\right| + K$$

EXAMPLE 6 Calculate $\int_2^4 \frac{\sqrt{x^2-4}}{x}\,dx$.

SOLUTION Let $x = 2\sec t$, where $0 \le t < \pi/2$. Note that the restriction of t to this interval is acceptable, since x is in the interval $2 \le x \le 4$ (see Figure 4). This is important because it allows us to remove the absolute value sign that normally appears when we simplify $\sqrt{x^2-a^2}$. In our case,

$$\sqrt{x^2-4} = \sqrt{4\sec^2 t - 4} = \sqrt{4\tan^2 t} = 2|\tan t| = 2\tan t$$

We now use the theorem on substitution in a definite integral (which requires changing the limits of integration) to write

$$\begin{aligned}
\int_2^4 \frac{\sqrt{x^2-4}}{x}\,dx &= \int_0^{\pi/3} \frac{2\tan t}{2\sec t}\, 2\sec t \tan t\, dt \\
&= \int_0^{\pi/3} 2\tan^2 t\, dt = 2\int_0^{\pi/3} (\sec^2 t - 1)\, dt \\
&= 2\Big[\tan t - t\Big]_0^{\pi/3} = 2\sqrt{3} - \frac{2\pi}{3} \approx 1.37
\end{aligned}$$

Section 7.5

Integration of Rational Functions Using Partial Fractions

Partial Fraction Decomposition (Linear Factors)

EXAMPLE 3 **Distinct Linear Factors** Decompose $(3x - 1)/(x^2 - x - 6)$ and then find its indefinite integral.

SOLUTION Since the denominator factors as $(x + 2)(x - 3)$, it seems reasonable to hope for a decomposition of the following form:

$$(1) \qquad \frac{3x - 1}{(x + 2)(x - 3)} = \frac{A}{x + 2} + \frac{B}{x - 3}$$

Our job is, of course, to determine A and B so that (1) is an identity, a task that we find easier after we have multiplied both sides by $(x + 2)(x - 3)$. We obtain

$$(2) \qquad 3x - 1 = A(x - 3) + B(x + 2)$$

or, equivalently,

$$(3) \qquad 3x - 1 = (A + B)x + (-3A + 2B)$$

However, (3) is an identity if and only if coefficients of like powers of x on both sides are equal; that is,

$$A + B = 3$$

$$-3A + 2B = -1$$

By solving this pair of equations for A and B, we obtain $A = \frac{7}{5}, B = \frac{8}{5}$. Consequently,

$$\frac{3x-1}{x^2-x-6} = \frac{3x-1}{(x+2)(x-3)} = \frac{\frac{7}{5}}{x+2} + \frac{\frac{8}{5}}{x-3}$$

and

$$\begin{aligned}\int \frac{3x-1}{x^2-x-6}\,dx &= \frac{7}{5}\int \frac{1}{x+2}\,dx + \frac{8}{5}\int \frac{1}{x-3}\,dx \\ &= \frac{7}{5}\ln|x+2| + \frac{8}{5}\ln|x-3| + C\end{aligned}$$

EXAMPLE 4 **Distinct Linear Factors** Find $\int \frac{5x+3}{x^3-2x^2-3x}\,dx$.

SOLUTION Since the denominator factors as $x(x+1)(x-3)$, we write

$$\frac{5x+3}{x(x+1)(x-3)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-3}$$

and seek to determine A, B, and C. Clearing the fractions gives

$$5x+3 = A(x+1)(x-3) + Bx(x-3) + Cx(x+1)$$

Substitution of the values $x=0$, $x=-1$, and $x=3$ results in

$$3 = A(-3)$$

$$-2 = B(4)$$

$$18 = C(12)$$

or $A=-1$, $B=-\frac{1}{2}$, $C=\frac{3}{2}$. Thus,

$$\int \frac{5x+3}{x^3-2x^2-3x}\,dx = -\int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{1}{x+1}\,dx + \frac{3}{2}\int \frac{1}{x-3}\,dx$$

$$= -\ln|x| - \frac{1}{2}\ln|x+1| + \frac{3}{2}\ln|x-3| + C$$

EXAMPLE 5 **Repeated Linear Factors** Find $\int \frac{x}{(x-3)^2}\,dx$.

SOLUTION Now the decomposition takes the form

$$\frac{x}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2}$$

with A and B to be determined. After clearing the fractions, we get

$$x = A(x-3) + B$$

If we now substitute the convenient value $x = 3$ and any other value, such as $x = 0$, we obtain $B = 3$ and $A = 1$. Thus,

$$\int \frac{x}{(x-3)^2}\,dx = \int \frac{1}{x-3}\,dx + 3\int \frac{1}{(x-3)^2}\,dx$$

$$= \ln|x-3| - \frac{3}{x-3} + C$$

EXAMPLE 6 **Some Distinct, Some Repeated Linear Factors** Find

$$\int \frac{3x^2 - 8x + 13}{(x+3)(x-1)^2}\,dx$$

SOLUTION We decompose the integrand in the following way:

$$\frac{3x^2 - 8x + 13}{(x+3)(x-1)^2} = \frac{A}{x+3} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

Clearing the fractions changes this to

$$3x^2 - 8x + 13 = A(x-1)^2 + B(x+3)(x-1) + C(x+3)$$

Substitution of $x = 1$, $x = -3$, and $x = 0$ yields $C = 2$, $A = 4$, and $B = -1$. Thus,

$$\int \frac{3x^2 - 8x + 13}{(x+3)(x-1)^2}\,dx = 4\int \frac{dx}{x+3} - \int \frac{dx}{x-1} + 2\int \frac{dx}{(x-1)^2}$$

$$= 4\ln|x+3| - \ln|x-1| - \frac{2}{x-1} + C$$

EXAMPLE 8 **A Repeated Quadratic Factor** Find $\int \frac{6x^2 - 15x + 22}{(x + 3)(x^2 + 2)^2} dx$.

SOLUTION Here the appropriate decomposition is

$$\frac{6x^2 - 15x + 22}{(x + 3)(x^2 + 2)^2} = \frac{A}{x + 3} + \frac{Bx + C}{x^2 + 2} + \frac{Dx + E}{(x^2 + 2)^2}$$

After considerable work, we discover that $A = 1$, $B = -1$, $C = 3$, $D = -5$, and $E = 0$. Thus,

$$\begin{aligned}
&\int \frac{6x^2 - 15x + 22}{(x + 3)(x^2 + 2)^2}\, dx \\
&\quad = \int \frac{dx}{x + 3} - \int \frac{x - 3}{x^2 + 2}\, dx - 5\int \frac{x}{(x^2 + 2)^2}\, dx \\
&\quad = \int \frac{dx}{x + 3} - \frac{1}{2}\int \frac{2x}{x^2 + 2}\, dx + 3\int \frac{dx}{x^2 + 2} - \frac{5}{2}\int \frac{2x\, dx}{(x^2 + 2)^2} \\
&\quad = \ln|x + 3| - \frac{1}{2}\ln(x^2 + 2) + \frac{3}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \frac{5}{2(x^2 + 2)} + C
\end{aligned}$$

Partial fraction Decomposition (Quadratic Factors)

Summary To decompose a rational function $f(x) = p(x)/q(x)$ into partial fractions, proceed as follows:

Step 1: If $f(x)$ is improper, that is, if $p(x)$ is of degree at least that of $q(x)$, divide $p(x)$ by $q(x)$, obtaining

$$f(x) = \text{a polynomial} + \frac{N(x)}{D(x)}$$

Step 2: Factor $D(x)$ into a product of linear and irreducible quadratic factors with real coefficients. By a theorem of algebra, this is always (theoretically) possible.

Step 3: For each factor of the form $(ax + b)^k$, expect the decomposition to have the terms

$$\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$$

Partial fraction Decomposition (Quadratic Factors)

Step 4: For each factor of the form $(ax^2 + bx + c)^m$, expect the decomposition to have the terms

$$\frac{B_1 x + C_1}{ax^2 + bx + c} + \frac{B_2 x + C_2}{(ax^2 + bx + c)^2} + \cdots + \frac{B_m x + C_m}{(ax^2 + bx + c)^m}$$

Step 5: Set $N(x)/D(x)$ equal to the sum of all the terms found in Steps 3 and 4. The number of constants to be determined should equal the degree of the denominator, $D(x)$.

Step 6: Multiply both sides of the equation found in Step 5 by $D(x)$ and solve for the unknown constants. This can be done by either of two methods: (1) Equate coefficients of like-degree terms or (2) assign convenient values to the variable x.

End of Chapter 7