No.

Date

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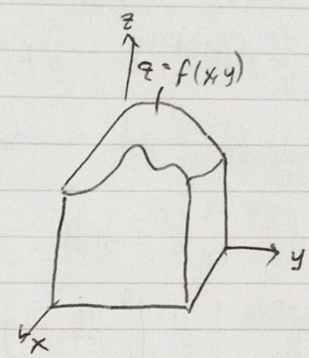
Multiple Integral

A. Double Integral over Rectangles



$$\iint_R f(x,y)\,dA = \lim_{\|P\|\to 0}\sum_{k=1}^{n} f(\bar{x}_k,\bar{y}_k)\,\Delta A_k$$

→ Have the same property with single integral

B. Iterated Integral

for $R = \{(x,y) : a \le x \le b,\ c \le y \le d\}$

1.) $V = \iint_R f(x,y)\,dA = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$

ex: $\int_0^3 \int_1^2 (2x+3y)\,dx\,dy$

inner integral: $\int_1^2 (2x+3y)\,dx = [x^2+3yx]_1^2 = 4+6y-(1+3y) = 3+3y$

$\Rightarrow \int_0^3 3+3y\,dy = \left[3y+\frac{3}{2}y^2\right]_0^3 = 9+\frac{27}{2} = \frac{45}{2}$

C. Double Integral over non rectangle

$$\iint_S f(x,y)\,dA = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx \quad \text{or} \quad \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\,dy$$

ex: $\int_3^5 \int_{-x}^{x^2} 4x+10y\,dy\,dx = \int_3^5 [4xy+5y^2]_{-x}^{x^2}\,dx$

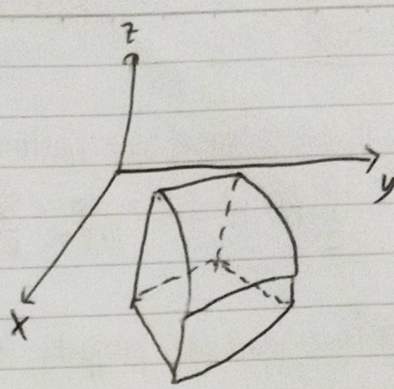
$= \int_3^5 [(4x^3+5x^4)-(-4x^2+5x^2)]\,dx$

$= \int_3^5 5x^4+4x^3-x^2\,dx = \left[x^5+x^4-\frac{x^3}{3}\right]_3^5$

$= \frac{10.180}{3} = 3393\frac{1}{3}$

D] Double Integral in Polar Coordinates

$V = \iint_R f(x,y)\, dA = \iint_R f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$



ex: $\iint_R e^{x^2+y^2}\, dA$ since $x^2+y^2=r^2$, $R = 1 \le r \le 3$, $0 \le \theta \le \frac{\pi}{4}$

$= \int_0^{\pi/4} \int_1^3 e^{r^2}\, r\, dr\, d\theta$

$= \int_0^{\pi/4} \frac{1}{2} e^{r^2} \Big|_1^3 \, d\theta = \int_0^{\pi/4} \frac{1}{2}(e^9 - e)\, d\theta = \frac{\pi}{8}(e^9 - e) = 3181$

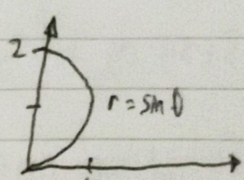
ex: Find the volume of solid under the surface $z = x^2+y^2$, above xy-plane, inside cylinder $x^2+y^2 = 2y$

$\hookrightarrow r = 2\sin\theta$

$V = 2 \iint_S (x^2+y^2)\, dA = 2 \int_0^{\pi/2} \int_0^{2\sin\theta} r^2 r\, dr\, d\theta$

$= 2 \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2\sin\theta} d\theta = 8 \int_0^{\pi/2} \sin^4\theta\, d\theta$

$= 8 \cdot \frac{3}{8} \cdot \frac{\pi}{2} = \frac{3\pi}{2}$



*probability integral

$I = \int_0^{\infty} e^{-x^2}\, dx$

$V_b = \int_{-b}^{b} \int_{-b}^{b} e^{-x^2-y^2}\, dy\, dx = 4\left[\int_0^b e^{-x^2}\, dx\right]^2 = 4I^2$

$V = \lim_{a\to\infty} \int_0^{2\pi} \int_0^a e^{-r^2}\, r\, dr\, d\theta = \lim_{a\to\infty} \int_0^{2\pi} \left[-\frac{1}{2} e^{-r^2}\right]_0^a d\theta$

$= \lim_{a\to\infty} \frac{1}{2} \int_0^{2\pi} [1 - e^{-a^2}]\, d\theta = \lim_{a\to\infty} \pi [1 - e^{-a^2}] = \pi$

$\therefore 4I^2 = \pi \iff I = \frac{1}{2}\sqrt{\pi}$

E] Applications of Double Integrals

* Actual mass m

$$m = \iint_S \delta(x,y)\, dA$$

ex: lamina with density $\delta(x,y) = xy$ is bounded by the x-axis, the line $x = 8$, and curve $y = x^{2/3}$. Find total mass

$$m = \int_0^8 \int_0^{x^{2/3}} xy \, dy\, dx = \int_0^8 \left[\frac{xy^2}{2}\right]_0^{x^{2/3}} dx = \frac{1}{2}\int_0^8 x^{7/3}$$

$$= \frac{1}{2}\left[\frac{3}{10} x^{10/3}\right]_0^8 = \frac{768}{5} = 153.6$$

* Center of Mass

$$\bar{x} = \frac{M_y}{m} = \frac{\iint_S x\,\delta(x,y)\, dA}{\iint_S \delta(x,y)\, dA} \qquad \bar{y} = \frac{M_x}{m} = \frac{\iint_S y\,\delta(x,y)\, dA}{\iint_S \delta(x,y)\, dA}$$

ex: center mass of lamina ex 1

$$M_y = \iint_S x\,\delta(x,y)\, dA = \int_0^8 \int_0^{x^{2/3}} x^2 y \, dy\, dx = \frac{1}{2}\int_0^8 x^{10/3} dx = \frac{12288}{13}$$

$$M_x = \iint_S y\,\delta(x,y)\, dA = \int_0^8 \int_0^{x^{2/3}} xy^2 \, dy\, dx = \frac{1}{3}\int_0^8 x^3 dx = \frac{1024}{3}$$

$$\therefore \bar{x} = \frac{M_y}{m} = \frac{80}{13} \approx 6.15 \qquad \bar{y} = \frac{M_x}{m} = \frac{20}{9} \approx 2.22$$

E] Surfarce Area

$$A(t) = \iint_S \sqrt{f_x^2 + f_y^2 + 1}\, dA$$

ex: $\int_0^{2\pi} \int_0^3 \sqrt{4r^2+1}\; r\, dr\, d\theta = \int_0^{2\pi} \frac{1}{8}\left[\frac{2}{3}(4r^2+1)^{3/2}\right]_0^3 d\theta$

$$= \int_0^{2\pi} \frac{1}{12}(37^{3/2} - 1)\, d\theta = \frac{\pi}{6}(37^{3/2} - 1) \approx 117.32$$

6) Triple Integral

$$\iiint_B f(x,y,z)\,dV.$$

ex: evaluate $\iiint_B x^2yz\,dV$ wher $B = \{(x,y,z) : 1 \le x \le 2,\ 0 \le y \le 1,\ 0 \le z \le 2\}$

$$= \int_0^2 \int_0^1 \int_1^2 x^2yz\,dx\,dy\,dz = \int_0^2 \int_0^1 \left[\frac{1}{3}x^3yz\right]_1^2 dy\,dz = \int_0^2 \int_0^1 \frac{7}{3}yz\,dy\,dz$$

$$= \frac{7}{3}\int_0^2 \left[\frac{1}{2}y^2z\right]_0^1 dz = \frac{7}{3}\int_0^2 \frac{1}{2}z\,dz = \frac{7}{6}\left[\frac{z^2}{2}\right]_0^2 = \frac{7}{3}$$

* General Region

$$\int_{a_1}^{a_2} \int_{\phi_1(x)}^{\phi_2(x)} \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

ex: Evaluate $f(x,y,z) = 2xyz$ over the solid region S in the first octant that's bounded by the parbolic cylinder $z = 2 - \frac{1}{2}x^2$ and the planes $z = 0$, $y = x$ and $y = 0$

$$\int_0^2 \int_0^x \int_0^{2-x^2/2} 2xyz\,dz\,dy\,dx = \int_0^2 \int_0^x \left[xyz^2\right]_0^{2-x^2/2} dy\,dx$$

$$= \int_0^2 \int_0^x \left(4xy - 2x^3y + \frac{1}{4}x^5y\right) dy\,dx = \int_0^2 \left(2x^3 - x^5 + \frac{1}{8}x^7\right) dx = \frac{4}{3}$$

* Cylindrical Coordinates

$$\int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} \int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta$$

ex: first octant bounded by $z = 4 - x^2 - y^2$ and $x^2 + y^2 = 2x$

$$V = \int_0^{\pi/2} \int_0^{2\cos\theta} \int_0^{4-r^2} r\,dz\,dr\,d\theta = \int_0^{\pi/2} \int_0^{2\cos\theta} r(4-r^2)\,dr\,d\theta = \int_0^{\pi/2} \left[2r^2 - \frac{1}{4}r^4\right]_0^{2\cos\theta} d\theta$$

$$= \int_0^{\pi/2} \left(8\cos^2\theta - 4\cos^4\theta\right) d\theta = 8 \cdot \frac{1}{2} \cdot \frac{\pi}{2} - 4\,\frac{3}{8} \cdot \frac{\pi}{2} = \frac{5\pi}{4}$$

* Chain Rule

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

for $z = f(x(s,t), y(s,t))$

$$\left.\begin{aligned}\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}\end{aligned}\right\} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial x}$$

$$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y}$$

* Tangent Plane

$$z - z_0 = f_x(a,b)(x - x_0) + f_y(a,b)(y - b)$$

* Differential & Approx

$$dz = \nabla f \cdot \langle dx, dy \rangle$$

* Maxima/Minima

abs a + abs y − c ≥ c → global minima

* Second Partial Test

$$D = D(x_0, y_0) = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0)$$

$D > 0$ $\begin{cases} f_{xx} < 0 & \text{local max val} \\ f_{xx} > 0 & \text{local min val} \end{cases}$

$D < 0$: not extreme val

$D = 0$: inconclusive

* Lagrange Mult.

$$\nabla f(p) = \lambda \nabla g(p) \qquad g(p) = 0$$

ex: $f(x,y) = x^2 + y^2$ $\quad xy = 1$ $\quad (1,1,2)$

$$\begin{pmatrix} 2x \\ 2y \end{pmatrix} = \lambda \begin{pmatrix} y \\ x \end{pmatrix}$$

- $2x = \lambda y$
- $2y = \lambda x$ $\qquad x = \frac{1}{y}$
- $xy = 1$ $\qquad 2 = \lambda y^2$

$-2y^2 = \frac{2}{y^2} = \lambda$

$y^4 = \pm 1$

$y = 1$: $x = 1$, $\lambda = 2$ $\qquad$ $y = -1$: $x = -1$, $\lambda = 2$

ex: e^{xy} $\quad x^3 + y^3 = 16$

$$\begin{pmatrix} y e^{xy} \\ x e^{xy} \end{pmatrix} = \lambda \begin{pmatrix} 3x^2 \\ 3y^2 \end{pmatrix}$$

MULTIPLE INTEGRAL

- Cari integral dalam, baru luar
- CONSTANT JADI cX GA ILANG
- URUTAN MATTER BUAT BATESAN

ex: Vol $z = x^2 + y^2$ above x-y plane inside $x^2 + y^2 = 2y$

- $z = r^2$
- $x^2 + y^2 = 2y$

$r = 2\sin\theta$

$$V = 2\int_0^{\pi/2}\int_0^{2\sin\theta} r^3\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^{\sin\theta} d\theta = 8\int_0^{\pi/2} \sin^4\theta\,d\theta$$

$$= 8\left(\frac{3}{8}\,\frac{\pi}{2}\right) = \frac{3\pi}{2}$$

* Application of Double Integral

$m = \iint_S \delta(x,y)\,dA$ dengan density $\delta(x,y)$

ex: $\delta(x,y) = xy$ bounded by x-axis $x = 8$ and $y = x^{2/3}$

$$m = \iint_S xy\,dA = \int_0^8\int_0^{x^{2/3}} xy\,dy\,dx$$

$$= \int_0^8 \left[\frac{xy^2}{2}\right]_0^{x^{2/3}} dx = \frac{1}{2}\int_0^8 x^{7/3}\,dx$$

$$= \frac{1}{2}\,\frac{3}{10}\,x^{10/3}\Big|_0^8 = \frac{768}{5}$$

* Center of Mass

$$\bar{x} = \frac{M_y}{m} = \frac{\iint x\,\delta(x,y)\,dA}{\iint \delta(x,y)\,dA} \qquad \bar{y} = \frac{M_x}{m} = \frac{\iint y\,\delta(x,y)\,dA}{\iint \delta(x,y)\,dA}$$

* Surface Area

$$A(G) = \iint_S \sqrt{f_x^2 + f_y^2 + 1}\,dA$$

ex: $z = x^2+y^2$ below $z=9$

inside circle $x^2+y^2=9$ $\quad r=3$

$$A(t)) = \int_0^{2\pi}\int_0^3 \sqrt{4r^2+1}\; r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{8}\left[\frac{2}{3}(4r^2+1)^{3/2}\right]_0^3 d\theta$$

$$= \int_0^{2\pi} \frac{1}{12}(37^{3/2}-1)\,d\theta = \frac{\pi}{6}(37^{3/2}-1) \approx 117.32$$

* Triple Integral

ex:

$$\iiint_B x^2yz\,dV \quad \text{where } B: x: 1-2,\; y: 0-1,\; z: 0-2$$

$$\int_0^2\int_0^1\int_1^2 x^2yz\,dx\,dy\,dz$$

$$= \int_0^2\int_0^1 \left[\frac{1}{3}x^3yz\right]_1^2 dy\,dz = \int_0^2\int_0^1 \frac{7}{3}yz\,dy\,dz$$

$$= \frac{7}{3}\int_0^2 \left[\frac{1}{2}y^2z\right]_0^1 dz = \frac{7}{3}\int_0^2 \frac{1}{2}z\,dz = \frac{7}{6}\left[\frac{z^2}{2}\right]_0^2 = \frac{7}{3}$$

* General Region

$f(x,y,z) = 2xyz$ in first octant bounded by $z = 2-\frac{1}{2}x^2$ and plane $z=0$ $y=x$ and $y=0$

$$\int_0^2\int_0^x\int_0^{2-\frac{x^2}{2}} 2\,xyz\,dz\,dy\,dx$$

$$= \int_0^2\int_0^x [xyz^2]_0^{2-x^2/2}\,dy\,dx$$

$$= \int_0^2\int_0^x 4xy - 2x^3y + \frac{x^5y}{4}\,dy\,dx$$

$$= \int_0^2 2x^3 - x^5 + \frac{1}{8}x^7\,dx = \frac{4}{3}$$

* Cylindrical Coordinates

$x = r\cos\theta \quad y = r\sin\theta \quad x^2+y^2 = r^2$

$$f(x,y,z) = f(r\cos\theta, r\sin\theta, z) = F(r,\theta,z)$$

ex: Volume in 1st octant bounded above by $z = 4-x^2-y^2$ laterally $x^2+y^2 = 2x$

$$V = \iiint_S 1\,dV = \int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^{4-r^2} r\,dz\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^{2\cos\theta} r(4-r^2)\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[2r^2 - \frac{1}{4}r^4\right]_0^{2\cos\theta} d\theta$$

$$= \int_0^{\pi/2} 8\cos^2\theta - 4\cos^4\theta = \frac{5\pi}{4}$$

* Spherical Coordinates

$x = \rho\sin\phi\cos\theta \quad y = \sin\phi\sin\theta \quad z = \rho\cos\phi$

$$\Delta V = \bar{\rho}^2\sin\bar{\phi}\,\Delta\rho\,\Delta\theta\,\Delta\phi$$

$$V = \iiint f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

* Jacobian

$$\iint f(x(u,v), y(u,v))\,|J(u,v)|\,du\,dv$$

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

ex: $\iint_R (x+y)\, e^{y-2x} \, dA$

batas: $\Rightarrow x+y=1$ $\quad y-2x=0$

$\Rightarrow x+y=5$ $\quad y-2x=2$

Change of coordinates

$u = x+y \qquad x = \frac{1}{3}(u-v)$

$v = y-2x \qquad y = \frac{1}{3}(v-2u)$

$$J = \begin{vmatrix} x_u & y_u \\ y_v & y_v \end{vmatrix} = |x_u y_v - x_v y_u| = \left|\frac{1}{9} - \left(-\frac{2}{9}\right)\right|$$

$$= \frac{1}{3}$$

$\Rightarrow x_u = \frac{1}{3} \qquad \Rightarrow x_v = -\frac{1}{3}$

$\Rightarrow y_v = \frac{1}{3} \qquad \Rightarrow y_u = \frac{2}{3}$

$$\int_0^2 \int_1^5 \frac{1}{3} u e^v \, du \, dv$$

$$= \frac{1}{3} \cdot \frac{1}{2} \int_0^2 u^2 e^v \Big|_{u=1}^{u=5} dv$$

$$= \frac{1}{\cancel{6}} \cdot \cancel{24}\, 4 \int_0^2 e^v \, dv = 4 \int_0^2 e^v \, dv$$

$$= 4 e^v \Big|_0^2 = 4(e^2 - 1) \quad //$$