# INTEGRAL TAK WAJAR / IMPROPER

## A] L'Hopital

$$\lim_{x\to u} \frac{f(x)}{g(x)} = \lim_{x\to u} \frac{f'(x)}{g'(x)}$$

ex: $\lim_{x\to 0} \frac{\tan 2x}{\ln(1+x)} = \lim_{x\to 0} \frac{2\sec^2 2x}{1/(1+x)} = \frac{2}{1} = 2$

## B] Cauchy's Mean Value Theorem

let f & g differentiable di (a,b) & continous di [a,b]. jika $g'(x) \neq 0$ di a-b
exist c di (a,b):

$$\boxed{\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}}$$

ex: $\lim_{x\to 1^+} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right) = \lim_{x\to 1^+} \frac{x\ln x - x + 1}{(x-1)\ln x} \overset{L}{=} \lim_{x\to 1^+} \frac{x\cdot\frac{1}{x} + \ln x - 1}{(x-1)(1/x) + \ln x} = \lim_{x\to 1^+} \frac{x\ln x}{x-1+\ln x\cdot x}$

$\overset{L}{=} \lim_{x\to 1^+} \frac{1+\ln x}{2+\ln x} = \frac{1}{2}$ //

## C] Indeterminate forms $0^0, \infty^0, 1^\infty$

tips: consider its logarithm!

ex: $\lim_{x\to 0^+} (x+1)^{\cot x}$ let $y = (x+1)^{\cot x}$

$\ln y = \cot x \cdot \ln(x+1) = \frac{\ln(x+1)}{\tan x}$

$\Rightarrow \lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(x+1)}{\tan x} \overset{L}{=} \lim_{x\to 0^+} \frac{\frac{1}{x+1}}{\sec^2 x} = 1 \iff e^{\ln y} = e^1 = e$ //

## D] Other: $\frac{0}{0}, \frac{\infty}{\infty}, 0\cdot\infty, \infty-\infty$

## E] One Finite Limit

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to -\infty} \int_a^b f(x)\,dx \quad \int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

$\Rightarrow$ converge jika mendekati satu value, else: divergent

ex: $\int_{-\infty}^{-1} xe^{-x^2} = -\frac{1}{2}\int_a^{-1} e^{-x^2}(-2x\,dx) = \left[-\frac{1}{2}e^{-x^2}\right]_a^{-1} = -\frac{1}{2}e^{-1} + \frac{1}{2}e^{-a^2} = -\frac{1}{2e}$ //

## F] Both Limits Infinite

If BOTH $\int_{-\infty}^0 f(x)\,dx$ and $\int_0^\infty f(x)\,dx$ converge, then $\int_{-\infty}^{\infty} f(x)\,dx$ converge

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^\infty f(x)\,dx$$

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6| Integrands that are Infinite at an end point

Suppose $\lim_{x \to b^-} |f(x)| = \infty$ then:

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$ ∴ if finite : converge, else : div

ex: $\int_0^2 \frac{dx}{\sqrt{4-x^2}} = \lim_{t \to 2^-} \int_0^t \frac{dx}{\sqrt{4-x^2}} = \lim_{t \to 2^-} \sin^{-1}\left(\frac{x}{2}\right)\Big|_0^t = \lim_{t \to 2^-} \left[\sin^{-1}\left(\frac{t}{2}\right) - \sin^{-1}\left(\frac{0}{2}\right)\right] = \frac{\pi}{2}$

7| Integrands that are infinite at interior point

$a < c < b$, suppose $\lim_{x \to c} |f(x)| = \infty$ then:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$ If BOTH finite : converge, else : div

ex: $\int_{-2}^1 \frac{1}{x^2}\,dx = \int_{-2}^1 \frac{1}{x^2}\,dx = \int_{-2}^0 \frac{1}{x^2}\,dx + \int_0^1 \frac{1}{x^2}\,dx$

$= \left(-\frac{1}{x}\right)\Big|_{-2}^0 + \left(-\frac{1}{x}\right)\Big|_0^1 = -\infty - \frac{1}{2} + \left(-\frac{1}{1}\right) + \infty$

∴ Divergent

## $\boxed{\text{9.1 INFINITE SEQUENCES}}$

Denote by $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}$

ex: explicit formula for the nth term as in $a_n = 3n - 2,\ n \geq 1$

recursion formula $a_1 = 1,\ a_n = a_{n-1} + 3,\ n \geq 2$

Convergence

if $\lim_{n \to \infty} a_n = L$ $\quad n \geq N \Rightarrow |a_n - L| < \varepsilon$

ex: $\lim_{n \to \infty} \frac{1}{n^p} = 0$ let an arbitrary $\varepsilon > 0$, choose $N > \sqrt[p]{1/\varepsilon}$. Then $n \geq N$ implies

$$|a_n - L| = \left|\frac{1}{n^p} - 0\right| = \frac{1}{n^p} \leq \frac{1}{N^p} < \frac{1}{\left(\sqrt[p]{1/\varepsilon}\right)^p} = \varepsilon$$

Menurunkan semua sifat limit //

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## Squeeze Theorem

ex: $\{a_n\}$ and $\{c_n\}$ converge to $L$, $a_n \le b_n \le c_n \to \{b_n\}$ converge

ex: $\lim\limits_{n\to\infty} \frac{\sin^3 n}{n} = 0 \Rightarrow \lim\limits_{n\to\infty}\left(\frac{-1}{n} \le \frac{\sin^3 n}{n} \le \frac{1}{n}\right) = 0$

## lim absolute

$\lim\limits_{n\to\infty} |a_n| = 0 \to \lim\limits_{n\to\infty} a_n = 0$

## Montonic Seq. Theorem

ex: $\frac{1}{2}, 1, \frac{9}{8}, \frac{25}{32}, \frac{9}{16}, \frac{49}{128}$ $\Rightarrow \frac{n^2}{2^n} > \frac{(n+1)^2}{2^{n+1}}$

$n^2 > \frac{(n+1)^2}{2}$

$2n^2 > n^2 + 2n + 1 \Rightarrow n^2 - 2n > 1$ true for $n \ge 3$

## Infinite Series

$\sum^{\infty} a_k$ converge if the sequence of partial sum $\{S_n\}$ converge to $S$

$S_n = \frac{a - ar^n}{1-r} = \frac{a}{1-r} - \frac{a}{1-r} r^n$ $\quad S_\infty = \frac{a}{1-r}, \ |r| < 1$

## $n^{th}$-term Test for Divergent

$\sum\limits_{n=1}^{\infty} a_n$ converge $\to \lim\limits_{n\to\infty} a_n = 0$

$\sum^{\infty} a_k$ diverge $\to \sum^{\infty} c a_k$ diverge

## Harmonic Series

$\sum^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \frac{1}{n} \ldots = \infty$

## Integral Test

$\sum\limits_{k=1}^{\infty} a_k$ conv $\iff \int_1^{\infty} f(x)\,dx$ conv

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## p-series Test

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots$$

a.) converge if $p > 1$

b.) diverge if $p \leq 1$

if $p \neq 1$ $\quad \int_1^t x^{-p}\,dx = \left[\frac{x^{1-p}}{1-p}\right]_1^t = \frac{t^{1-p}-1}{1-p}$

if $p = 1$ $\quad \int_1^t x^{-1}\,dx = \ln t$

## Ordinary Comparison Test

if $0 \leq a_n < b_n$

1.) if $\sum b_n$ converge, so does $\sum a_n$

2.) if $\sum a_n$ diverge, so does $\sum b_n$

ex: $\sum^{\infty} \frac{n}{5n^2-4} > \frac{n}{5n^2} = \frac{1}{5}\left(\frac{1}{n}\right) \rightarrow$ harmonic, so diverge

## Limit Comparison Test

if $a_n \geq 0$, $b_n > 0$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L \begin{cases} 0 < L < \infty, & \sum a_n \text{ and } \sum b_n \text{ conv./div together} \\ L = 0, & \text{and } \sum b_n \text{ converge, then } \sum a_n \text{ converge} \end{cases}$$

ex: $\sum_{n=1}^{\infty} \frac{3n-2}{n^3-2n^2+11}$ compare with $3/n^2$

$\Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(3n-2)/(n^3-2n+11)}{3/n^2} = \lim_{n\to\infty} \frac{3n^3-2n^2}{3n^2-6n^2+33} = 1$

$\therefore$ since $\sum 3/n^2$ converge, $a_n$ converge

## Ratio Test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho \begin{cases} \rho < 1, & \text{series converge} \\ \rho > 1, & \text{series diverge} \\ \rho = 1, & \text{inconclusive} \end{cases}$$

ex: $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ : $\rho = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!}\frac{n!}{2^n} = \lim_{n\to\infty} \frac{2}{n+1} = 0$ converge

Summary: 1. if $\lim_{n\to\infty} a_n \neq 0$, conclude diverge in a series
2. If $a_n$ involves $n!$, $r^n$, or $n^n$ ⇒ ratio test
3. if $a_n$ involves only constant powers of $n$, ⇒ Limit Comp test
4. else ⇒ Ordinary comp. test or Bounded sum test or Integral

## Alternating Series Test

$a_1 - a_2 + a_3 - a_4 + \ldots$

with $a_n > a_{n+1} > 0$ if $\lim_{n\to\infty} a_n = 0$ always converge

## Absolute Ratio Test

$$\lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} = \rho \begin{cases} \rho < 1, & \text{series converge absolutely} \\ \rho > 1, & \text{series diverge} \\ \rho = 1, & \text{inconclusive} \end{cases}$$

ex: $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^n}{n!}$

$$\Rightarrow \rho = \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} = \lim_{n\to\infty} \frac{3}{n+1} = 0$$

∴ conv. absolutely ⟶ converge.

## Conditional Converge

when $\sum u_n$ conv while $\sum |u_n|$ diverge

## The Convergence Set

$\sum a_n x^n$ with interval
- Single point $x = 0$
- an interval $(-R, R)$, + maybe endpoint
- the whole real line

ex: $\sum_{n=0}^{\infty} \frac{(x-1)^n}{(n+1)^2}$

$$\Rightarrow \rho = \lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{(n+2)^2} \cdot \frac{(n+1)^2}{(x-1)^n} \right|$$

$$= \lim_{n\to\infty} |x-1| \frac{(n+1)^2}{(n+2)^2} \to \text{L'hopital } \frac{1}{1}$$

$$= |x-1|$$

∴ converge $0 < x < 2$, also converge at both of end points

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# Taylor and McLaurin Series

Uniqueness theorem

$$c_n = \frac{f^{(n)}(a)}{n!}$$

## Taylor's Theorem

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad ; \lim_{n\to\infty} R_n(\cancel{x}) = 0$$

ex: compute $\int_0^{0.5} \sqrt{1+x^4}\, dx = 1 + \frac{1}{2}x - \frac{1}{8}x^8 + \frac{1}{16}x^{12} - \frac{5}{128}x^{16}$

$$= \left[x + \frac{x^5}{10} - \frac{x^9}{72} + \frac{x^{13}}{208} + \ldots\right] \approx 0.90\cancel{1}2$$

## Important McLaurin series

1. $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \qquad -1 < x < 1$
2. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \qquad -1 < x \leq 1$
3. $\tan^{-1}(x) = x - x^3/3 + x^5/5 - x^7/7 + \ldots \qquad -1 \leq x \leq 1$
4. $e^x = 1 + x + x^2/2! + x^3/3! + \ldots$
5. $\sin x = x - x^3/3! + x^5/5! - x^7/7! + \ldots$
6. $\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + \ldots$
7. $\sinh x = x + x^3/3! + x^5/5! + x^7/7! + \ldots$
8. $\cosh x = 1 + x^2/2! + x^4/4! + x^6/6! + \ldots$
9. $(1+x)^p = 1 + \binom{p}{1}x + \binom{p}{2}x^2 + \binom{p}{3}x^3 + \ldots \qquad -1 < x < 1$

## Taylor polynomial of order n

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

McLaurin polynomial

$$f(x) \approx p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n$$

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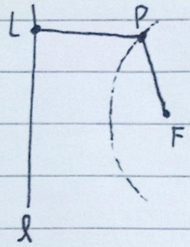
## CONIC AND POLAR COORDINATE

### PARABOLA

let $l$, (directrix), and $f$ (focus) and set of point $P$

the ratio is $e$ (eccentricity)

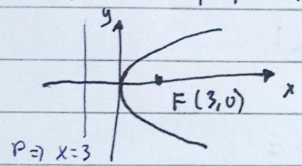
$$|PF| = e|PL| \quad \begin{cases} 0 < e < 1 & \text{ellips} \\ e = 1 & \text{parabola} \\ e > 1 & \text{hyperbola} \end{cases}$$




- standard equation of parabola $y^2 = 4px$ ; $\boxed{\text{Directrix}: x = -p}$

ex : $y^2 = 12x$

since $y^2 = 4(3)x$, $p = 3$. The focus is at $(3,0)$ ; directrix is the line $x = -3$

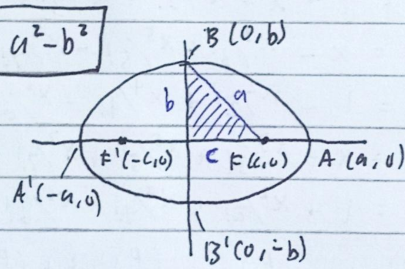



### ELLIPSE

$\boxed{a > b}$ $\boxed{b^2 + c^2 = a^2}$ $\boxed{c^2 = a^2 - b^2}$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

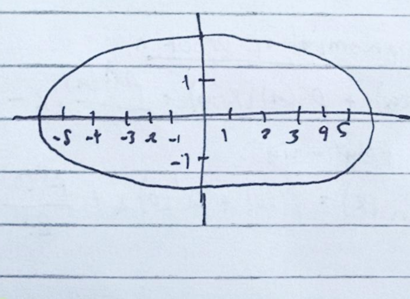
$\boxed{e = \frac{c}{a}}$




ex: $\frac{x^2}{36} + \frac{y^2}{4} = 1$

$c = \sqrt{a^2 - b^2} = \sqrt{36 - 4} = 4\sqrt{2}$

focus at $(\pm c, 0) = (\pm 4\sqrt{2}, 0)$




- Directrix & foci :

$\boxed{x = \pm \frac{a}{e} = \pm \frac{a^2}{c}}$ $\boxed{F(\pm c, 0)}$

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## HIPERBOLA

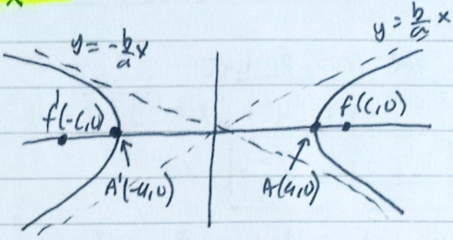
$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with asymptote $y=\pm\frac{b}{a}x$

→ foci & directrix

$f(\pm c,0)$ ; $x=\pm\frac{a}{e}$

$\boxed{c^2=a^2+b^2}$

$\boxed{e=\frac{c}{a}}$




ex: $\frac{x^2}{9}-\frac{y^2}{16}=1 \Rightarrow a=3$ ; $b=4$

→ asymptote: $y=\pm\frac{4}{3}x$ with $c=\sqrt{a^2+b^2}=\sqrt{9+16}=5$ ; $f(\pm 5,0)$

→ f location : - If $x^2$ positive → f on x axis, else on y axis

## TRANSLATONS

if $(u,v)$ new coordinate and $(h,k)$ new origin

$$\boxed{u=x-h, \quad v=y-k}$$

ex: $4x^2+y^2+40x-2y+97=0$ with new origin $(-5,1)$

replace: $4(u-5)^2+(v+1)^2+40(u-5)-2(v+1)+97=0 \Leftrightarrow u^2+\frac{v^2}{4}=1$

### Completing the Square

ex: $4x^2+9y^2+8x-90y+193=0$

$\Leftrightarrow 4(x+1)^2+9(y-5)^2=36 \Leftrightarrow \frac{(x+1)^2}{9}+\frac{(y-5)^2}{4}=1$

$u=x+1$ ; $v=y-5$

graph $\frac{u^2}{9}+\frac{v^2}{4}=1$ with origin $(-1,5)$

### General second Degree Equation

→ $y^2=4$ paralel lines

→ $y^2=0$ single line

→ $y^2=-1$ ; $2x^2+y^2=-1$ empty set

→ $x^2+y^2=4$ circle

→ $2x^2+y^2=0$ point

→ $x^2-y^2=0$ intersecting lines

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# ROTATIONS

$\boxed{x = u\cos\theta - v\sin\theta} \quad \boxed{y = u\sin\theta + v\cos\theta}$

eliminate cross product

$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

$$\boxed{\cot 2\theta = \frac{A-C}{B}}$$

ex: $4x^2 + 2\sqrt{3}\,xy + 2y^2 + 10\sqrt{3}x + 10y = 5$

$\cot 2\theta = \frac{4-2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$

$2\theta = \pi/3 \iff \theta = \pi/6$

$x = \sqrt{3}/2\, u - 1/2\, v$

$y = 1/2\, u + \sqrt{3}/2\, v$

$\iff 5u^2 + v^2 + 20u = 5$

$5(u^2 + 4u + 4) + v^2 = 25$

$\frac{(u+2)^2}{5} + \frac{v^2}{25} = 1$