Rumus Integral

$\int a\,dx = ax$

$\int \frac{a^x}{x} = \ln|x|$

$\int a^x dx = \frac{a^x}{\ln a}$

$\int \ln x = x\ln x - x$

$\int \sin x = -\cos x$

$\int \cos x = \sin x$

$\int \tan x = \ln|\sec x|$
$= -\ln|\cos x|$

$\int \cot x\,dx = \ln|\sin x|$

$\int \sec x\,dx = \ln|\sec x + \tan x|$

$\int \csc x\,dx = \ln|\csc x - \cot x|$

$\int \sec^2 x\,dx = \tan x$

$\int \sec x \tan x\,dx = \sec x$

$\int \csc^2 x\,dx = -\cot x$

$\int \tan^2 x\,dx = \tan x - x$

$\int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left|\frac{x+\sqrt{a^2+x^2}}{a}\right|$

$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x$

$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b|$

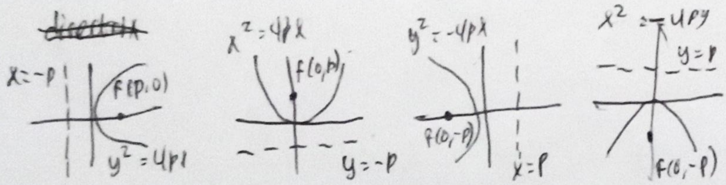
$\int \frac{x}{ax^2+b}\,dx = \frac{1}{2a}\ln|ax^2+b|$

$\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$

CONIC & POLAR

$e\begin{cases} 0<e<1 & \text{elips} \\ e=1 & \text{parabola} \\ e>1 & \text{hiperbola} \end{cases}$

* parabola
$y^2 = 4px$



* elips
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\boxed{a>b}$ $\boxed{b^2+c^2=a^2}$ $\boxed{a^2-b^2=c^2}$ $\boxed{e=\frac{c}{a}}$

Horizontal
Dir: $x = \pm\frac{a}{e} = \pm\frac{a^2}{c}$
$f(\pm c, 0)$
$Ver(\pm a, 0)$

Vertikal
Dir: $y = \pm\frac{a}{e} = \pm\frac{a^2}{c}$
$F(0, \pm c)$
$ver(0, \pm a)$

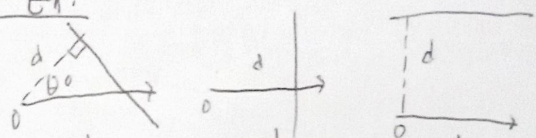
* Hiperbola
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $\boxed{a^2+b^2=c^2}$ $\boxed{\text{a selalu pos}}$ $\boxed{e=\frac{c}{a}}$

Horizontal)(2a
asymptote: $y = \pm\frac{b}{a}x$
$f(\pm c, 0)$
$Ver(\pm a, 0)$

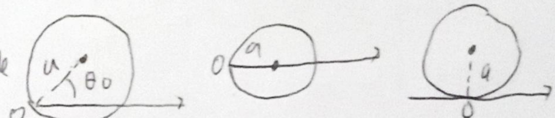
Vertikal
asymptote: $x = \pm\frac{b}{a}y$
$F(0, \pm c)$
$ver(0, \pm a)$

Relation
$x = r\cos\theta$ $r^2 = x^2 + y^2$
$y = r\sin\theta$ $\tan\theta = y/x$

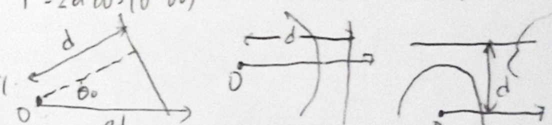
polar Eq.

line


$r = \frac{d}{\cos(\theta-\theta_0)}$ $r = \frac{d}{\cos\theta}$ $r = \frac{d}{\sin\theta}$

circle


$r = 2a\cos(\theta-\theta_0)$ $r = 2a\cos\theta$ $r = 2a\sin\theta$

conic


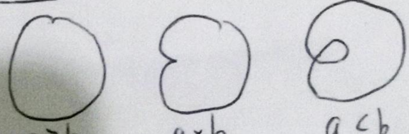
$r = \frac{ed}{1+e\cos(\theta-\theta_0)}$ $r = \frac{ed}{1+e\cos\theta}$ $r = \frac{ed}{1+e\sin\theta}$

↳ $r = \frac{ed}{1\pm e\cos\theta}$ dir: $x = \pm d$

$r = \frac{ed}{1\pm e\sin\theta}$ dir: $y = \pm d$

Symmetric test
- x axis → $(r, -\theta)$
- y axis → $(-r, -\theta)$ or $(r, \pi-\theta)$
- origin → $(-r, \theta)$

Cardioid



$a>b$ $a=b$ $a<b$

$r = a \pm b\cos\theta$
$r = a \pm b\sin\theta$

lemniscates

$r^2 = \pm a \cos 2\theta$

$r^2 = \pm a \sin 2\theta$

roses

$r = a \cos n\theta$ / $r = a \sin n\theta$ $\left\{ \begin{array}{l} n \text{ leaves if } n \text{ ganjil} \\ 2n \text{ leaves if } n \text{ genap} \end{array} \right.$

POLAR COORD

$$A = \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta$$

* Area $r = 2 + \cos\theta$ sym x-axis

$$A = 2 \cdot \frac{1}{2} \int_0^{\pi} 4 + 4\cos\theta + \frac{\cos 2\theta + 1}{2}$$

$$= \frac{9}{2}\theta + 4\sin\theta + \frac{1}{4}\sin 2\theta \Big|_0^{\pi} = \frac{9}{2}\pi$$

* Area $r^2 = 9 \sin 2\theta$

$r = \pm 3\sqrt{\sin 2\theta}$

0	0	sym x, y, pol
$\pi/4$	1	
$\pi/2$	0	
$3\pi/2$	-1 TTM	

$$4 \cdot \frac{1}{2} \int_0^{\pi/4} 9 \sin 2\theta = \cancel{2} \frac{9}{\cancel{2}} \cos 2\theta \Big|_0^{\pi/4} = 9.$$

* Area inside $r = 3\sin\theta$ outside $r = 1 + \sin\theta$

both y sym

$3\sin\theta = 1 + \sin\theta$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6} \vee \theta = \frac{5\pi}{6}$

$$2 \cdot \frac{1}{2} \int_{\pi/6}^{\pi/2} 9\sin^2\theta - 1^2 - 2\sin\theta - \sin^2\theta \, d\theta$$

$$= \int_{\pi/6}^{\pi/2} 4(1 - \cos 2\theta) - 1 - 2\sin\theta \, d\theta$$

$$= 3\theta - 2\sin 2\theta + 2\cos\theta \Big|_{\pi/6}^{\pi/2}$$

$$= \left(\frac{3}{2} - \frac{1}{2}\right)\pi - (0 - \cancel{\sqrt{3}}) + (0 - \cancel{\sqrt{3}}) = \pi$$

DERIVATIVE ≥2 VAR

* limit ~~& continuity~~

$$\lim_{(x,y) \to (a,b)} \frac{p}{q} \quad q \neq 0$$

* Continuity

$$\lim_{(x,y) \to (a,b)} f(x,y) = f(a,b)$$

(1) f has value at (a,b)

(2) f has limit at (a,b)

(3) $\lim = f$

* Differentiability

$f_x(a,b) \wedge f_y(a,b) \rightarrow$ differentiable at (a,b)

* Tangent plane

$$T(x,y) = f(x,y) + \nabla f(x,y) \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$$

ex: $xe^y + x^2 y$ at $(2,0)$

$f_x = e^y + 2xy \quad f_y = xe^y + x^2$

$$\nabla f(x,y) = \begin{pmatrix} e^y + 2xy \\ xe^y + x^2 \end{pmatrix} \quad \nabla f(2,0) = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$$

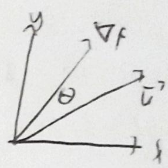
$$z = f(2,0) + \nabla f(2,0) \begin{pmatrix} x - 2 \\ y \end{pmatrix}$$

$$= 2 + \begin{pmatrix} 1 \\ 6 \end{pmatrix} \begin{pmatrix} x - 2 \\ y \end{pmatrix} = 2 + x - 2 + 6y = x + 6y$$

differentiable $\longrightarrow$ continuous

* Directional Derivatives

for $u = u_1 i + u_2 j$



$D_u f(p) = u \cdot \nabla f(p)$

$D_u f(x,y) = u_1 f_x(x,y) + u_2 f_y(x,y)$

$$\theta \rightarrow \vec{u} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

* dijadiin $\hat{u}$ dulu $\hat{u} = \frac{\langle u \rangle}{\|u\|} = \frac{\langle a, b \rangle}{\sqrt{a^2 + b^2}}$

ex: $4x^2 - xy + 3y^2$ at $(2,-1)$ in dir $a = 4i + 3j$

$f_x = 8x - y \quad f_y = -x + 6y \quad f_x(2,-1) = 17 \quad f_y(2,-1) = -8$

$$D_u f(2,-1) = \begin{pmatrix} 4/5 \\ 3/5 \end{pmatrix} \begin{pmatrix} 17 \\ -8 \end{pmatrix} = \frac{44}{5}$$

* Max Rate of Change

$$\nabla f(a,b) = f_x(a,b) i + f_y(a,b) j$$

slope: $\|\nabla f(a,b)\|$

ex: $z = y^2 - x^2$ at $(1,1,0)$

$f_x = -2x \quad f_y = 2y$

$\nabla f(1,1) = -2i + 2j \quad$ slope $= \sqrt{4+4} = 2\sqrt{2}$

* Chain Rule

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

for $z = f(x(s,t), y(s,t))$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{dx}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{dy}{\partial t}$$

$$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y}$$

* Tangent Plane

$$z - z_0 = f_x(a,b)(x - x_0) + f_y(a,b)(y - b)$$

* Differential & Approx

$$dz = \nabla f \cdot (dx, dy)$$

* Maxima/Minima

abs a + abs y − c ≥ c → global minima

* Second Partial Test

$$D = D(x_0, y_0) = f_{xx}(x_0,y_0)\, f_{yy}(x_0,y_0) - f_{xy}^2(x_0,y_0)$$

- $D > 0$
  - $f_{xx} < 0$ local max val
  - $f_{xx} > 0$ local min val
- $D < 0$: not extreme val
- $D = 0$: inconclusive

* Lagrange Mult.

$\nabla f(p) = \lambda \nabla g(p) \quad g(p) = 0$

ex: $f(x,y) = x^2 + y^2 \quad xy = 1 \quad (1,1,2)$

$$\begin{pmatrix}2x\\2y\end{pmatrix} = \lambda\begin{pmatrix}y\\x\end{pmatrix}$$

- $2x = \lambda y$
- $2y = \lambda x$
- $xy = 1$

$x = \frac{1}{y}$

$2 = \lambda y^2$

$2y^2 = \frac{2}{y^2} = \lambda$

$y^4 = \pm 1$

$y = 1$: $x = 1$, $\lambda = 2$ $\quad$ $y = -1$: $x = -1$, $\lambda = 2$

ex: $e^{xy} \quad x^3 + y^3 = 16$

$$\begin{pmatrix}ye^{xy}\\xe^{xy}\end{pmatrix} = \lambda\begin{pmatrix}3x^2\\3y^2\end{pmatrix}$$

MULTIPLE INTEGRAL

- Cari integral dalam, baru luar
- CONSTANT JADI ~~cX~~ GAILANG
- URUTAN MATTER BUAT BATESAN

ex: vol $z = x^2 + y^2$ above x-y plane inside $x^2 + y^2 = 2y$

- $z = r^2$
- $x^2 + y^2 = 2y$

$r = 2\sin\theta$

$$V = 2\int_0^{\pi/2}\int_0^{2\sin\theta} r^3\, dr\, d\theta$$

$$= 2\int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^{\sin\theta} d\theta = 8\int_0^{\pi/2}\sin^4\theta\, d\theta$$

$$= 8\left(\frac{3}{8}\,\frac{\pi}{2}\right) = \frac{3\pi}{2}$$

* Application of Double Integral

$m = \iint_S \delta(x,y)\, dA$ dengan density $\delta(x,y)$

ex: $\delta(x,y) = xy$ bounded by x-axis $x = 8$ and $y = x^{2/3}$

$$m = \iint_S xy\, dA = \int_0^8\int_0^{x^{2/3}} xy\, dy\, dx$$

$$= \int_0^8\left[\frac{xy^2}{2}\right]_0^{x^{2/3}} dx = \frac{1}{2}\int_0^8 x^{7/3}\, dx$$

$$= \frac{1}{2}\,\frac{3}{10}\, x^{10/3}\Big|_0^8 = \frac{768}{5}$$

* Center of Mass

$$\bar{x} = \frac{M_y}{m} = \frac{\iint x\,\delta(x,y)\, dA}{\iint \delta(x,y)\, dA} \qquad \bar{y} = \frac{M_x}{m} = \frac{\iint y\,\delta(x,y)\, dA}{\iint \delta(x,y)\, dA}$$

* Surface Area

$$A(G) = \iint_S \sqrt{f_x^2 + f_y^2 + 1}\, dA$$

ex: $z = x^2+y^2$ below $z=9$

inside circle $x^2+y^2=9$ $\quad r=3$

$$A(S) = \int_0^{2\pi}\int_0^3 \sqrt{4r^2+1}\; r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{8}\left[\frac{2}{3}(4r^2+1)^{3/2}\right]_0^3 d\theta$$

$$= \int_0^{2\pi} \frac{1}{12}(37^{3/2}-1)\,d\theta = \frac{\pi}{6}(37^{3/2}-1) \approx 117.32$$

* Triple Integral

ex:

$$\iiint_B x^2yz\,dV \quad \text{where } B: x: 1-2,\ y: 0-1,\ z: 0-2$$

$$\int_0^2\int_0^1\int_1^2 x^2yz\,dx\,dy\,dz$$

$$= \int_0^2\int_0^1 \left[\frac{1}{3}x^3yz\right]_1^2 dy\,dz = \int_0^2\int_0^1 \frac{7}{3}yz\,dy\,dz$$

$$= \frac{7}{3}\int_0^2\left[\frac{1}{2}y^2z\right]_0^1 dz = \frac{7}{3}\int_0^2 \frac{1}{2}z\,dz = \frac{7}{6}\left[\frac{z^2}{2}\right]_0^2 = \frac{7}{3}$$

* General Region

$f(x,y,z) = 2xyz$ in first octant bounded by $z = 2-\frac{1}{2}x^2$ and plane $z=0$ $y=x$ and $y=0$

$$\int_0^2\int_0^x\int_0^{2-\frac{x^2}{2}} 2\,xyz\;dz\,dy\,dx$$

$$= \int_0^2\int_0^x \left[xyz^2\right]_0^{2-x^2/2} dy\,dx$$

$$= \int_0^2\int_0^x 4xy - 2x^3y + \frac{x^5y}{4}\,dy\,dx$$

$$= \int_0^2 2x^3 - x^5 + \frac{1}{8}x^7\,dx = \frac{4}{3}$$

* Cylindrical Coordinates

$x = r\cos\theta \quad y = r\sin\theta \quad x^2+y^2 = r^2$

$f(x,y,z) = f(r\cos\theta, r\sin\theta, z) = F(r,\theta,z)$

ex. Volume in 1st octant bounded above by $z = 4-x^2-y^2$ laterally $x^2+y^2 = 2x$

$$V = \iiint_S 1\,dV = \int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^{4-r^2} r\,dz\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^{2\cos\theta} r(4-r^2)\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[2r^2 - \frac{1}{4}r^4\right]_0^{2\cos\theta} d\theta$$

$$= \int_0^{\pi/2} 8\cos^2\theta - 4\cos^4\theta = \frac{5\pi}{4}$$

* Spherical coordinates

$x = \rho\sin\phi\cos\theta \quad y = \sin\phi\sin\theta \quad z = \rho\cos\phi$

$$\Delta V = \bar{\rho}^2\sin\bar{\phi}\;\Delta\rho\,\Delta\theta\,\Delta\phi$$

$$V = \iiint f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi$$

* Jacobian

$$\iint f(x(u,v), y(u,v))\,|J(u,v)|\,du\,dv$$

$$J(u,v): \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

ex: $\iint_R (x+y)\, e^{y-2x}\, dA$

batas: $\cdot)\ x+y=1$ $\quad y-2x=0$

$\cdot)\ x+y=5$ $\quad y-2x=2$

change of coordinates

$u = x+y$ $\quad x = \frac{1}{3}(u-v)$

$v = y-2x$ $\quad y = \frac{1}{3}(v-2u)$

$J = \begin{vmatrix} x_u & y_u \\ y_v & y_v \end{vmatrix} = |x_u y_v - x_v y_u| = \left|\frac{1}{9} - \left(-\frac{2}{9}\right)\right|$

$= \frac{1}{3}$

$\cdot)\ x_u = \frac{1}{3}$ $\quad \cdot)\ x_v = -\frac{1}{3}$

$\cdot)\ y_v = \frac{1}{3}$ $\quad \cdot)\ y_u = \frac{2}{3}$

$\int_0^2 \int_1^5 \frac{1}{3} u e^v \, du \, dv$

$= \frac{1}{3} \cdot \frac{1}{2} \int_0^2 u^2 e^v \Big|_{u=1}^{u=5} dv$

$= \frac{1}{\cancel{6}} \cdot \cancel{24}^{4} \int_0^2 e^v \, dv = 4 \int_0^2 e^u \, dv$

$= 4 e^v \Big|_0^2 = 4(e^2 - 1)$ //

~ contoh" soal ~

$\circ)$ Lagrange

misal: $-\frac{2}{9} x = \lambda 2x$

$\frac{1}{2} y = \lambda 2y$

$\frac{1/2\, y}{2y} = \lambda = \frac{-2/9\, x}{2x}$

$\frac{1}{4} = \lambda = -\frac{1}{9}$

$\cdot)$ Tetrahedron $O(0,0,0)$ $A(2,0,0)$ $B(0,4,0)$ $C(0,0,4)$

$2x + y + z = 4$ $\quad 2x + y + 2 = 4$

$\to$ di $z = 2$ $\quad y = 2 - 2x$

$2 \le z \le 4 - 2x - y$ $\quad 2x + 2 + 0 = 4$

$0 \le y \le 2 - 2x$ $\quad x = 1$

$0 \le x \le 1$

$\int_0^1 \int_0^{2-2x} \int_2^{4-2x-y} 1 \, dz \, dy \, dx = \frac{2}{3}$

3D $\begin{cases} \text{scratch — triple } dz\,dy\,dx \\ \text{ada Z nya} \begin{cases} \text{polar triple } r\,dz\,dr\,d\theta \\ \text{double} \end{cases} \end{cases}$

2D $\begin{cases} \text{polar double } r\,dr\,d\theta \\ \text{single} \end{cases}$

$\cdot)$ rangkap 3

$\iiint 3\, dV$

didalam silinder $x^2 + y^2 = 4$ diatas $z = 0$

dibawah bid. $y + z = 3$

$r^2 = 4 \to r = 2$ $\quad z = 3 - y$

$= 3 - r\sin\theta$

$\int_0^{2\pi} \int_0^2 \int_0^{3 - r\sin\theta} 3r \, dz \, dr$

Jangan lupa Jacobian "r" utk integral polar !!!