Chapter 8

Indeterminate Forms and Improper Integrals

Section 8.1

Indeterminate Forms of Type 0/0

L'Hôpital's Rule

Theorem A **L'Hôpital's Rule for forms of type 0/0**

Suppose that $\lim_{x \to u} f(x) = \lim_{x \to u} g(x) = 0$. If $\lim_{x \to u}[f'(x)/g'(x)]$ exists in either the finite or infinite sense (i.e., if this limit is a finite number or $-\infty$ or $+\infty$), then

$$\lim_{x \to u} \frac{f(x)}{g(x)} = \lim_{x \to u} \frac{f'(x)}{g'(x)}$$

EXAMPLE 2 Find $\lim_{x\to 3} \frac{x^2 - 9}{x^2 - x - 6}$ and $\lim_{x\to 2^+} \frac{x^2 + 3x - 10}{x^2 - 4x + 4}$.

SOLUTION Both limits have the 0/0 form, so, by l'Hôpital's Rule,

$$\lim_{x\to 3} \frac{x^2 - 9}{x^2 - x - 6} = \lim_{x\to 3} \frac{2x}{2x - 1} = \frac{6}{5}$$

$$\lim_{x\to 2^+} \frac{x^2 + 3x - 10}{x^2 - 4x + 4} = \lim_{x\to 2^+} \frac{2x + 3}{2x - 4} = \infty$$

EXAMPLE 3 Find $\lim_{x\to 0} \frac{\tan 2x}{\ln(1 + x)}$.

SOLUTION Both numerator and denominator have limit 0. Hence,

$$\lim_{x\to 0} \frac{\tan 2x}{\ln(1 + x)} = \lim_{x\to 0} \frac{2 \sec^2 2x}{1/(1 + x)} = \frac{2}{1} = 2$$

Cauchy's Mean Value Theorem

Theorem B **Cauchy's Mean Value Theorem**

Let the functions f and g be differentiable on (a, b) and continuous on $[a, b]$. If $g'(x) \neq 0$ for all x in (a, b), then there exists a number c in (a, b) such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Note that this theorem reduces to the ordinary Mean Value Theorem for Derivatives (Theorem 3.6A) when $g(x) = x$.

Section 8.2

Other Indeterminate Forms

L'Hôpital's Rule for Form of Type ∞/ ∞

Theorem A **L'Hôpital's Rule for Forms of Type ∞/∞**

Suppose that $\lim_{x \to u} |f(x)| = \lim_{x \to u} |g(x)| = \infty$. If $\lim_{x \to u} [f'(x)/g'(x)]$ exists in either the finite or infinite sense, then

$$\lim_{x \to u} \frac{f(x)}{g(x)} = \lim_{x \to u} \frac{f'(x)}{g'(x)}$$

Here u may stand for any of the symbols a, a^-, a^+, $-\infty$, or ∞.

EXAMPLE 2 Show that, if a is any positive real number, $\lim_{x\to\infty} \frac{x^a}{e^x} = 0$.

SOLUTION Suppose as a special case that $a = 2.5$. Then three applications of l'Hôpital's Rule give

$$\lim_{x \to \infty} \frac{x^{2.5}}{e^x} \overset{\text{L}}{=} \lim_{x \to \infty} \frac{2.5x^{1.5}}{e^x} \overset{\text{L}}{=} \lim_{x \to \infty} \frac{(2.5)(1.5)x^{0.5}}{e^x} \overset{\text{L}}{=} \lim_{x \to \infty} \frac{(2.5)(1.5)(0.5)}{x^{0.5}e^x} = 0$$

A similar argument works for any $a > 0$. Let m denote the greatest integer less than a. Then $m + 1$ applications of l'Hôpital's Rule give

$$\lim_{x \to \infty} \frac{x^a}{e^x} \overset{\text{L}}{=} \lim_{x \to \infty} \frac{ax^{a-1}}{e^x} \overset{\text{L}}{=} \lim_{x \to \infty} \frac{a(a-1)x^{a-2}}{e^x} \overset{\text{L}}{=} \cdots \overset{\text{L}}{=} \lim_{x \to \infty} \frac{a(a-1)\cdots(a-m)}{x^{m+1-a}e^x} = 0$$

EXAMPLE 6 Find $\lim_{x \to 1^+} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right)$.

SOLUTION The first term is growing without bound; so is the second. We say that the limit is an $\infty - \infty$ indeterminate form. L'Hôpital's Rule will determine the result, but only after we rewrite the problem in a form for which the rule applies. In this case, the two fractions must be combined, a procedure that changes the problem to a 0/0 form. Two applications of l'Hôpital's Rule yield

$$\lim_{x \to 1^+} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right) = \lim_{x \to 1^+} \frac{x \ln x - x + 1}{(x-1)\ln x} \overset{\text{L}}{=} \lim_{x \to 1^+} \frac{x \cdot 1/x + \ln x - 1}{(x-1)(1/x) + \ln x}$$

$$= \lim_{x \to 1^+} \frac{x \ln x}{x - 1 + x \ln x} \overset{\text{L}}{=} \lim_{x \to 1^+} \frac{1 + \ln x}{2 + \ln x} = \frac{1}{2}$$

The Indeterminate Forms 0^0, ∞^0, 1^∞ We turn now to three indeterminate forms of exponential type. Here the trick is to consider not the original expression, but rather its logarithm. Usually, l'Hôpital's Rule will apply to the logarithm.

EXAMPLE 7 Find $\lim_{x \to 0^+} (x + 1)^{\cot x}$.

SOLUTION This takes the indeterminate form 1^∞. Let $y = (x + 1)^{\cot x}$, so

$$\ln y = \cot x \ln(x + 1) = \frac{\ln(x + 1)}{\tan x}$$

Using l'Hôpital's Rule for 0/0 forms, we obtain

$$\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} \frac{\ln(x + 1)}{\tan x} \overset{\text{L}}{=} \lim_{x \to 0^+} \frac{\frac{1}{x + 1}}{\sec^2 x} = 1$$

Now $y = e^{\ln y}$, and since the exponential function $f(x) = e^x$ is continuous,

$$\lim_{x \to 0^+} y = \lim_{x \to 0^+} \exp(\ln y) = \exp\left(\lim_{x \to 0^+} \ln y\right) = \exp 1 = e$$

Summary We have classified certain limit problems as indeterminate forms, using the seven symbols $0/0$, ∞/∞, $0 \cdot \infty$, $\infty - \infty$, 0^0, ∞^0, and 1^∞. Each involves a competition of opposing forces, which means that the result is not obvious. However, with the help of l'Hôpital's Rule, which applies directly only to the $0/0$ and ∞/∞ forms, we can usually determine the limit.

There are many other possibilities symbolized by, for example, $0/\infty$, $\infty/0$, $\infty + \infty$, $\infty \cdot \infty$, 0^∞, and ∞^∞. Why don't we call these indeterminate forms? Because, in each of these cases, the forces work together, not in competition.

Section 8.3

Improper Integrals: Infinite Limits of Integration

One Finite Limit

Definition

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a \to -\infty} \int_{a}^{b} f(x)\,dx$$

$$\int_{a}^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_{a}^{b} f(x)\,dx$$

If the limits on the right exist and have finite values, then we say that the corresponding improper integrals **converge** and have those values. Otherwise, the integrals are said to **diverge.**

EXAMPLE 1 Find, if possible, $\int_{-\infty}^{-1} xe^{-x^2}\,dx$.

SOLUTION

$$\int_a^{-1} xe^{-x^2}\,dx = -\frac{1}{2}\int_a^{-1} e^{-x^2}(-2x\,dx) = \left[-\frac{1}{2}e^{-x^2}\right]_a^{-1}$$

$$= -\frac{1}{2}e^{-1} + \frac{1}{2}e^{-a^2}$$

Thus,

$$\int_{-\infty}^{-1} xe^{-x^2}\,dx = \lim_{a\to-\infty}\left[-\frac{1}{2}e^{-1} + \frac{1}{2}e^{-a^2}\right] = -\frac{1}{2e}$$

We say the integral converges and has value $-1/2e$.

Both Limits Infinite

Definition

If both $\int_{-\infty}^{0} f(x)\,dx$ and $\int_{0}^{\infty} f(x)\,dx$ converge, then $\int_{-\infty}^{\infty} f(x)\,dx$ is said to converge and have value

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$

Otherwise, $\int_{-\infty}^{\infty} f(x)\,dx$ diverges.

EXAMPLE 4 Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$ or state that it diverges.

SOLUTION

$$\int_0^{\infty} \frac{1}{1+x^2}\,dx = \lim_{b\to\infty} \int_0^{b} \frac{1}{1+x^2}\,dx$$

$$= \lim_{b\to\infty} \left[\tan^{-1} x\right]_0^b$$

$$= \lim_{b\to\infty} \left[\tan^{-1} b - \tan^{-1} 0\right] = \frac{\pi}{2}$$

Since the integrand is an even function,

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \int_0^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2}$$

Therefore,

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_0^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

The Paradox of Gabriel's Horn Let the curve $y = 1/x$ on $[1, \infty)$ be revolved about the x-axis, thereby generating a surface called Gabriel's horn (Figure 6). We claim that

1. the volume V of this horn is finite;
2. the surface area A of the horn is infinite.

To put the results in practical terms, they seem to say that the horn can be filled with a finite amount of paint, and yet there is not enough to paint its inside surface. Before we try to unravel this paradox, let us establish (1) and (2). We use results for volume from Section 5.2 and for surface area from Section 5.4.

$$V = \int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx = \lim_{b\to\infty} \pi \int_1^b x^{-2}\,dx$$

$$= \lim_{b\to\infty}\left[-\frac{\pi}{x}\right]_1^b = \pi$$

$$A = \int_1^\infty 2\pi y\,ds = \int_1^\infty 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + \left(\frac{-1}{x^2}\right)^2}\,dx$$

$$= \lim_{b\to\infty} 2\pi \int_1^b \frac{\sqrt{x^4 + 1}}{x^3}\,dx$$

Now,

$$\frac{\sqrt{x^4 + 1}}{x^3} > \frac{\sqrt{x^4}}{x^3} = \frac{1}{x}$$

Thus,

$$\int_1^b \frac{\sqrt{x^4 + 1}}{x^3}\,dx > \int_1^b \frac{1}{x}\,dx = \ln b$$

and since $\ln b \to \infty$ as $b \to \infty$, we conclude that A is infinite.

EXAMPLE 7 Show that $\int_1^\infty 1/x^p\,dx$ diverges for $p \le 1$ and converges for $p > 1$.

SOLUTION We showed in our solution to Gabriel's horn that the integral diverges for $p = 1$. If $p \neq 1$,

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_1^b x^{-p}\,dx = \lim_{b\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^b$$

$$= \lim_{b\to\infty}\left[\frac{1}{1-p}\right]\left[\frac{1}{b^{p-1}} - 1\right] = \begin{cases} \infty & \text{if } p < 1 \\ \dfrac{1}{p-1} & \text{if } p > 1 \end{cases}$$

The conclusion follows. ∎

Section 8.4

Improper Integrals: Infinite Integrands

Integrands that are Infinite at an End Point

Definition

Let f be continuous on the half-open interval $[a, b)$ and suppose that $\lim_{x \to b^-} |f(x)| = \infty$. Then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

provided that this limit exists and is finite, in which case we say that the integral converges. Otherwise, we say that the integral diverges.

EXAMPLE 1 Evaluate, if possible, the improper integral $\int_0^2 \frac{dx}{\sqrt{4 - x^2}}$.

SOLUTION Note that the integrand tends to infinity at 2.

$$\int_0^2 \frac{dx}{\sqrt{4 - x^2}} = \lim_{t \to 2^-} \int_0^t \frac{dx}{\sqrt{4 - x^2}} = \lim_{t \to 2^-} \left[\sin^{-1}\left(\frac{x}{2}\right) \right]_0^t$$

$$= \lim_{t \to 2^-} \left[\sin^{-1}\left(\frac{t}{2}\right) - \sin^{-1}\left(\frac{0}{2}\right) \right] = \frac{\pi}{2}$$

EXAMPLE 4 Show that $\int_0^1 \frac{1}{x^p}\,dx$ converges if $p < 1$, but diverges if $p \geq 1$.

SOLUTION Example 3 took care of the case $p = 1$. If $p \neq 1$,

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{t\to 0^+} \int_t^1 x^{-p}\,dx = \lim_{t\to 0^+} \left[\frac{x^{-p+1}}{-p+1}\right]_t^1$$

$$= \lim_{t\to 0^+} \left[\frac{1}{1-p} - \frac{1}{1-p}\cdot\frac{1}{t^{p-1}}\right] = \begin{cases} \dfrac{1}{1-p} & \text{if } p < 1 \\ \infty & \text{if } p > 1 \end{cases}$$

Integrands that are Infinite at an Interior Point

Definition

Let f be continuous on $[a, b]$ except at a number c, where $a < c < b$, and suppose that $\lim_{x \to c} |f(x)| = \infty$. Then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

provided both integrals on the right converge. Otherwise, we say that $\int_a^b f(x)\,dx$ diverges.

EXAMPLE 6 Show that $\int_{-2}^{1} 1/x^2\,dx$ diverges.

SOLUTION

$$\int_{-2}^{1} \frac{1}{x^2}\,dx = \int_{-2}^{0} \frac{1}{x^2}\,dx + \int_{0}^{1} \frac{1}{x^2}\,dx$$

The second of the integrals on the right diverges by Example 4. This is enough to give the conclusion. ∎

EXAMPLE 7 Evaluate, if possible, the improper integral $\int_0^3 \frac{dx}{(x-1)^{2/3}}$.

SOLUTION The integrand tends to infinity at $x = 1$ (see Figure 4). Thus,

$$\begin{aligned}
\int_0^3 \frac{dx}{(x-1)^{2/3}} &= \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}} \\
&= \lim_{t\to 1^-} \int_0^t \frac{dx}{(x-1)^{2/3}} + \lim_{s\to 1^+} \int_s^3 \frac{dx}{(x-1)^{2/3}} \\
&= \lim_{t\to 1^-} [3(x-1)^{1/3}]_0^t + \lim_{s\to 1^+} [3(x-1)^{1/3}]_s^3 \\
&= 3\lim_{t\to 1^-} [(t-1)^{1/3} + 1] + 3\lim_{s\to 1^+} [2^{1/3} - (s-1)^{1/3}] \\
&= 3 + 3(2^{1/3}) \approx 6.78
\end{aligned}$$

End of Chapter 8