



COMPUTER SCIENCE & MATHEMATICS

Teknik Pengintegralan

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Sebelumnya ...

Pengenalan Integral Sebagai Anti-Turunan

Notasi:

$$\int f(x)dx = F(x) + C \quad \text{dengan kata lain} \quad \int dF(x) = F(x) + C$$

dengan $f(x)$ = integran

Sebelumnya ...

Anti-derivative (Integral Tak Tentu)

Kumpulan derivative:

f(x)	c	x^n	sin x	cos x	ln x	e^x
f(x) dx	0	nx^{n-1}	cos x	-sin x	1/x	e^x

anti-derivative ↑

Aturan yang dapat diaplikasikan pada anti-derivative:

1. $\int kf(x)dx = k \int f(x)dx$
2. $\int (f(x) \pm g(x)) dx = \int f(x)dx \pm \int g(x)dx$

Sebelumnya ...

Anti-derivative (Integral Tak Tentu)

Kumpulan fungsi, derivative, dan anti-derivative:

$f(x)$	c	x^n	$\sin x$	$\cos x$	$\ln x$	e^x
$f(x) dx$	0	nx^{n-1}	$\cos x$	$-\sin x$	$1/x$	e^x
$\int f(x) dx$	C	x^n	$\sin x$	$\cos x$	$\ln x$	e^x

*Contoh teorema anti-derivative lainnya:

$$1. \int a^x dx = \frac{a^x}{\ln a} + C$$

$$4. \int x^r dx = \frac{x^{r+1}}{r+1} + C$$

Power Rule

$$2. \int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \left(\frac{u}{a} \right) + C$$

$$5. \int [g(x)]^r g'(x) dx = \frac{[g(x)]^{r+1}}{r+1} + C$$

Generalized Power Rule

$$3. \int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \left(\frac{u}{a} \right) + C$$

1. Substitusi Sederhana

Teorema Substitusi Sederhana

Jika $t = g(x)$,

$$\int f(g(x))g'(x)dx = \int f(t)dt = F(t) + C = F(g(x)) + C$$

Contoh Soal

1. $\int -\frac{x}{\sin^2(x^2)}dx =$

3. $\int \frac{e^x}{4 + 9e^{2x}}dx =$

2. $\int \frac{e^{1/x}}{x^2}dx =$

4. $\int \sin z \cos z \sqrt{1 + \sin^2 z} dz =$

2. Pengintegralan Bagian demi Bagian

Integral Parsial

Dari aturan perkalian pada derivative [$(fg)' = fg' + gf'$], jika kedua sisi kita integralkan, maka:

$$fg = \int f'g + \int fg' \quad \rightarrow \quad fg - \int f'g = \int fg'$$

Contoh Soal

1. $\int \left(\ln x \sec x \tan x + \frac{\sec x}{x} \right) dx =$

2. $\int -2 \ln x \cos x \sin x dx =$

3. Integral Trigonometri

Substitusi + Aturan Trigonometri = Integral Trigonometri

Bentuk umum:

1. $\int \sin^m(x) \cos^n(x) dx$

a. m atau n ganjil

i. $(\cos x)' = -\sin x dx$ atau $(\sin x)' = \cos x dx$; dan

ii. $\cos^2 x = 1 - \sin^2 x$

b. m dan n genap

i. $\cos^2 x = (\cos(2x) + 1) / 2$; $\sin^2 x = (1 - \cos(2x)) / 2$

Contoh Soal

1. $\int \sin^5(x) dx =$

3. Integral Trigonometri

Substitusi + Aturan Trigonometri = Integral Trigonometri

Bentuk umum:

$$2. \int \sec^m(x) \tan^n(x) dx ; \int \csc^m(x) \cot^n(x) dx$$

a. m genap

i. $(\tan x)' = \sec^2 x \, dx$, $\sec^2 x = 1 + \tan^2 x$ **atau** $(\cot x)' = -\csc^2 x \, dx$, $\csc^2 x = 1 + \cot^2 x$

b. m dan n ganjil

i. $(\sec x)' = \sec x \tan x \, dx$

c. m ganjil dan n genap

i. bawa ke sec atau csc → diikuti rumus recursive untuk sec atau csc

Contoh Soal

1. $\int \csc^3(x) dx =$

3. Integral Trigonometri

Contoh Soal

1. $\int \frac{\sin^2 x}{\sec^2 x} dx =$

2. $\int \tan^5 x \sec x dx =$

3. Integral Trigonometri

Contoh Soal

1. $\int \frac{\sin^2 x}{\sec^2 x} dx =$



Bentuk $\int \sin^m(x) \cos^n(x) dx$, case: m dan n genap

2. $\int \tan^5 x \sec x dx =$



Bentuk $\int \sec^m(x) \tan^n(x) dx$, case: m dan n ganjil

3. Integral Trigonometri

Contoh Soal

$$\begin{aligned} 1. \int \frac{\sin^2 x}{\sec^2 x} dx &= \int \sin^2 x \cos^2 x dx = \int (1 - \cos^2 x) \cos^2 x dx = \int (\cos^2 x - \cos^4 x) dx \\ &= \int \cos^2 x dx - \int (\cos^2 x)^2 dx = \int \left(\frac{1 + \cos 2x}{2} \right) dx - \int \left(\frac{1 + \cos 2x}{2} \right)^2 dx \\ &= \frac{1}{2} \int (1 + \cos 2x) dx - \frac{1}{4} \int (1 + \cos 2x)^2 dx = \frac{1}{2} \int (1 + \cos 2x) dx - \frac{1}{4} \int (1 + 2 \cos 2x + \cos^2 2x) dx \\ &= \left(\frac{1}{2} - \frac{1}{4} \right) x + \left(\frac{1}{2} - \frac{2}{4} \right) \int \cos 2x dx - \frac{1}{4} \int \cos^2 2x dx = \frac{1}{4} \left(x - \int \cos^2 2x dx \right) \\ &= \frac{1}{4} \left(x - \int \left(\frac{1 + \cos 2(2x)}{2} \right) dx \right) = \frac{1}{4} x - \frac{1}{8} \left(x + \int \cos (4x) dx \right) \\ &= \frac{1}{8} x - \frac{1}{8} \left(\frac{1}{4} \sin (4x) \right) = \frac{1}{8} x - \frac{1}{32} \sin 4x + C \end{aligned}$$

3. Integral Trigonometri

Contoh Soal

$$\begin{aligned} 2. \int \tan^5 x \sec x dx &= \int \tan^4 x (\tan x \sec x dx) = \int (\sec^2 x - 1)^2 d(\sec x) \\ &= \int (\sec^4 x - 2 \sec^2 x + 1) d(\sec x) \\ &= \frac{1}{5} \sec^5 x - \frac{2}{3} \sec^3 x + \sec x + C \end{aligned}$$

3. Integral Trigonometri

Substitusi + Aturan Trigonometri = Integral Trigonometri

Bentuk umum:

$$3. \int \sin(Ax) \cos(Bx) dx = \frac{1}{2} \int (\sin((A+B)x) + \sin((A-B)x)) dx;$$

$$\int \sin(Ax) \sin(Bx) dx = \frac{1}{2} \int (\cos((A-B)x) - \cos((A+B)x)) dx;$$

$$\int \cos(Ax) \cos(Bx) dx = \frac{1}{2} \int (\cos((A+B)x) + \cos((A-B)x)) dx$$

$$a. \sin(\alpha \pm \beta) = \sin(\alpha) \cos(\beta) \pm \cos(\alpha) \sin(\beta)$$

$$b. \cos(\alpha \pm \beta) = \cos(\alpha) \cos(\beta) \mp \sin(\alpha) \sin(\beta)$$

Contoh Soal

$$1. \int \cos^2 3x \sin 5x dx =$$

3. Integral Trigonometri

Substitusi + Aturan Trigonometri = Integral Trigonometri

Bentuk umum:

$$3. \int \sin(Ax) \cos(Bx) dx = \frac{1}{2} \int (\sin((A+B)x) + \sin((A-B)x)) dx;$$

$$\int \sin(Ax) \sin(Bx) dx = \frac{1}{2} \int (\cos((A-B)x) - \cos((A+B)x)) dx;$$

$$\int \cos(Ax) \cos(Bx) dx = \frac{1}{2} \int (\cos((A+B)x) + \cos((A-B)x)) dx$$

$$a. \sin(\alpha \pm \beta) = \sin(\alpha) \cos(\beta) \pm \cos(\alpha) \sin(\beta)$$

$$b. \cos(\alpha \pm \beta) = \cos(\alpha) \cos(\beta) \mp \sin(\alpha) \sin(\beta)$$

Contoh Soal

$$1. \int \cos^2 3x \sin 5x dx = \int \left(\frac{1 + \cos 2(3x)}{2} \right) \sin 5x dx \quad \text{atau}$$

$$\int \cos^2 3x \sin 5x dx = \int \cos 3x \left(\frac{1}{2} (\sin 8x + \sin(-2x)) \right) dx = \frac{1}{2} \int \cos 3x (\sin 8x - \sin 2x) dx$$

4. Substitusi Trigonometri

Teorema Substitusi Trigonometri

1. Integral melibatkan bentuk $\sqrt[n]{ax+b}$, cukup lakukan substitusi $u = \sqrt[n]{ax+b}$
2. Integral melibatkan bentuk $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, $\sqrt{x^2 - a^2}$
Asumsikan a positif, maka dapat dilakukan substitusi:

Akar	Substitusi	Pembatasan pada t
$\sqrt{a^2 - x^2}$	$x = a \sin \theta$	$-\pi/2 \leq \theta \leq \pi/2$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta$	$-\pi/2 < \theta < \pi/2$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta$	$0 \leq \theta \leq \pi, \theta \neq \pi/2$

4. Substitusi Trigonometri

Latihan Soal

1. $\int \frac{dx}{x - \sqrt{x}} =$

2. $\int \sqrt{x^2 - 6x + 6} dx =$

3. $\int \frac{\sqrt{7 - 6x - x^2}}{x + 3} dx =$

4. $\int \frac{dx}{\sqrt{x^2 - 8x + 20}} =$

Akar	Substitusi
$\sqrt{a^2 - x^2}$	$x = a \sin \theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta$

5. Integral Fungsi Rasional

Dekomposisi Pecahan Parsial

Fungsi rasional $r(x) = (p(x))/(q(x))$, disebut proper jika derajat polinomial $p(x) <$ derajat polinomial $q(x)$.

Untuk menyelesaikan $\int [p(x)/(q(x))] dx$ digunakan cara berikut:

- Jika akar-akar $q(x)$ semua riil dan berbeda, misalkan $q(x) = (x-x_1) \dots (x-x_k)$
 - maka $\frac{p(x)}{q(x)} = \sum_{i=1}^k \frac{A_i}{(x-x_i)}$, sehingga $\int \frac{p(x)}{q(x)} dx = \sum_{i=1}^k A_i \ln(x-x_i) + C$
- Jika akar-akar $q(x)$ semua riil ada yg ganda
 - khusus untuk akar ganda faktornya disesuaikan, namun dengan prinsip yang sama.
- Jika ada akar kompleks
 - khusus untuk faktor yang memiliki akar kompleks, digunakan faktor pecahan berbentuk $\frac{Ax+B}{ax^2+bx+c}$, yang integralnya bisa berbentuk \ln atau \arctan

5. Integral Fungsi Rasional

Latihan Soal

1. $\int \frac{x}{(x-4)^2} dx =$

2. $\int \frac{6x^2 - 3x + 1}{(4x+1)(x^2+1)} dx =$

3. $\int \frac{2x^2 + x - 8}{x^3 + 4x} dx =$

4. $\int \frac{1}{(x-1)^2(x+4)^2} dx =$



End