1 Homogenous Heat Equation

This project implements and verifies a space-time finite element method (FEM) for solving the one-dimensional heat equation. We use bilinear rectangular elements over a structured spacetime grid, impose Dirichlet boundary and initial conditions, and compare the numerical solution to the known analytical one. We demonstrate second-order convergence in the L^2 norm and confirm solver accuracy and stability using PETSc.

1.1 Mathematical Model

We consider the classical one-dimensional heat equation:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \quad x \in (0,1),\ t > 0, \qquad (1) \\ u(0,t) = u(1,t) &= 0, \quad t \geq 0, \\ u(x,0) &= \sin(\pi x), \quad x \in [0,1]. \end{aligned}$$

The analytical solution is known and given by:

$$u(x,t) = \sin(\pi x)\, e^{-\pi^2 t}.$$

1.2 Space-Time Finite Element Discretization

1.2.1 Variational Form

We multiply the PDE by a test function ϕ and integrate over the spacetime domain to obtain the weak form equation:

$$\int\int \left(\frac{\partial u}{\partial t}\phi + \frac{\partial u}{\partial x}\frac{\partial \phi}{\partial x} \right) dx\, dt = 0. \qquad (2)$$

You may notice that in going from the strong from, Eq. 1, to the weak form, Eq. 2, we integrated by parts spatially. This moved a derivative over to the test function, and since the spatial BC specify no activity, we were able to drop the boundary term from the weak form.

We approximate u using basis functions over rectangular elements. With $\xi = x/hx$ and $\tau = t/ht$, we define the rectangular basis functions:

$$
\begin{aligned}
\phi_1(\xi,\tau) &= (1-\xi)(1-\tau),\\
\phi_2(\xi,\tau) &= \xi(1-\tau),\\
\phi_3(\xi,\tau) &= \xi\tau,\\
\phi_4(\xi,\tau) &= (1-\xi)\tau.
\end{aligned}
$$

1.2.2 Element Stiffness Matrix

The local element matrix is computed symbolically and hardcoded using expressions dependent on grid spacing h:

$$A_{ij}^{(e)} = \int_{K_e} \left(\frac{\partial \phi_j}{\partial t}\phi_i + \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial x} \right) dx\, dt.$$

The result is a 4×4 matrix with entries involving h and constants. This matrix is reused for all elements due to uniformity. The symbolic calculation of the element stiffness matrices utilizes Python's SymPy.

1.3 Global Assembly and Solution

We loop over all elements, assemble the global stiffness matrix and residual vector, and solve using PETSc's SNES nonlinear solver. Since the problem is linear, SNES converges in one Newton step. A direct LU solver is used via command-line options:

```
./Heat_Linear \
        -snes_rtol 1e-8 -snes_atol 1e-12  \
        -snes_type newtonls -snes_linesearch_type bt  \
        -ksp_type preonly -pc_type lu \
        -snes_converged_reason -snes_max_it 10000 \
        -nx 80 -nt 80
```

1.4 Boundary and Initial Condition Enforcement

We enforce Dirichlet boundary conditions at $x = 0$ and $x = 1$ by:

- Zeroing the corresponding rows of the Jacobian.

- Inserting a 1 on the diagonal.
- Setting the residual to zero at these entries.

We also enforce the initial condition $u(x,0) = \sin(\pi x)$ as a Dirichlet condition at $t = 0$ using the same technique.

1.5 Error Computation and Convergence Rate

The L^2 error norm is computed as:

$$E = \|u_h - u\|_{L^2} = \left(\sum_{i,j} \left(u_h(x_i, t_j) - u_{\text{exact}}(x_i, t_j)\right)^2 h^2 \right)^{1/2},$$

where $h = \Delta x = \Delta t$ and $u_{\text{exact}}(x,t) = \sin(\pi x) e^{-\pi^2 t}$.

Given two successively refined grids (h_i, E_i) and (h_{i+1}, E_{i+1}), we find the order of convergence,

$$p_i = \frac{\log\bigl(E_i / E_{i+1}\bigr)}{\log\bigl(h_i / h_{i+1}\bigr)}.$$

With finer refinements, we see the order of convergence approach 2.

1.6 Results

We ran the simulation for some values of $nx = nt$, and measured the L^2 error norm.

$nx = nt$	h	L^2 Error Norm	Rate
100	1.0101e-02	9.505×10^{-6}	-
200	5.0251e-03	2.341×10^{-6}	2.01
400	2.5062e-03	5.816×10^{-7}	2.00
800	1.2516e-03	1.450×10^{-7}	2.00

The convergence rate seems to be approaching 2nd order. Also note that numerical convergence was achieved in 1 SNES iteration for all mesh sizes which is to be expected for a linear PDE.

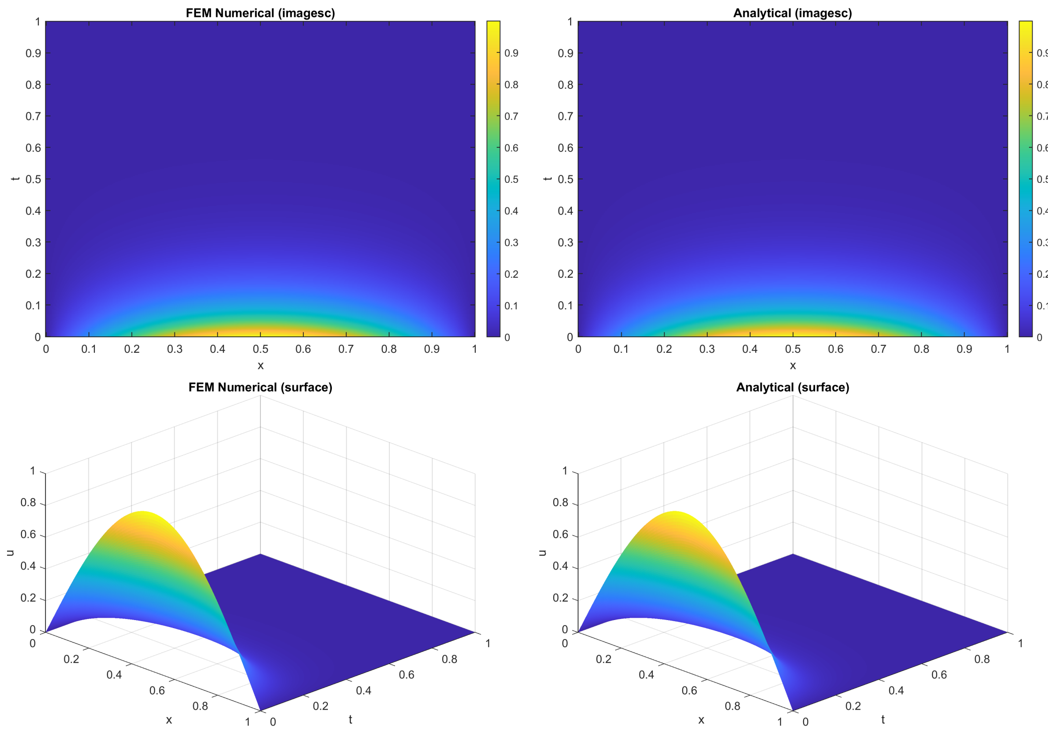

Figure 1: Visual comparison between numerical and analytical solutions of the heat equation for $nx = nt = 400$. Left: FEM solution. Right: Exact solution $u(x,t) = \sin(\pi x)e^{-\pi^2 t}$.

1.7 Conclusion

We implemented a space-time finite element method for the 1D heat equation using PETSc and verified it against an analytical solution. The method demonstrates second-order convergence in the L^2 norm. The use of PETSc allows flexibility in solver configuration and output. This framework will now be extended to solve the nonhomogenous wave equation.