Data Streams:

Probabilistic Counting

Mining Massive Datasets

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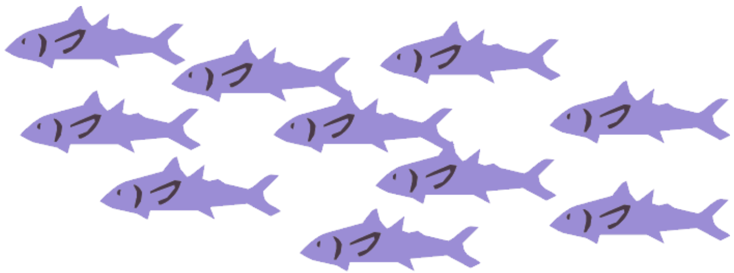
Sources

- Mining of Massive Datasets (2014) by Leskovec et al. (chapter 4)
  - Slides part 1, part 2
- Tutorial: Mining Massive Data Streams (2019) by Michael Hahsler

Probabilistic counting

Counting fishes with pebbles

- Normally, to count with pebbles, you add one pebble every time you see an event
- How do you extend this method to count up to 1000 fishes with 10 pebbles?
- Assume you have access to a random number generator but not to an abacus for ... reasons



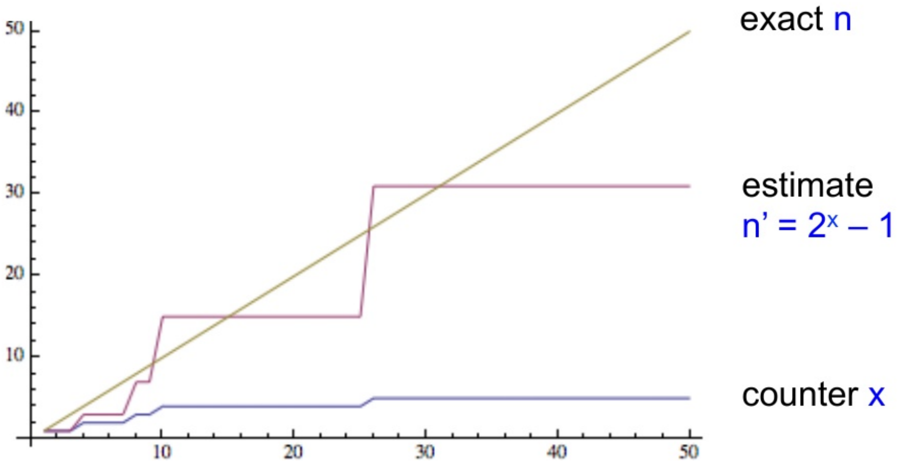
Morris' probabilistic counting (1977)

$x \leftarrow 0$

For each of the n events:

$x \leftarrow x + 1$ with probability $(1/2)^x$

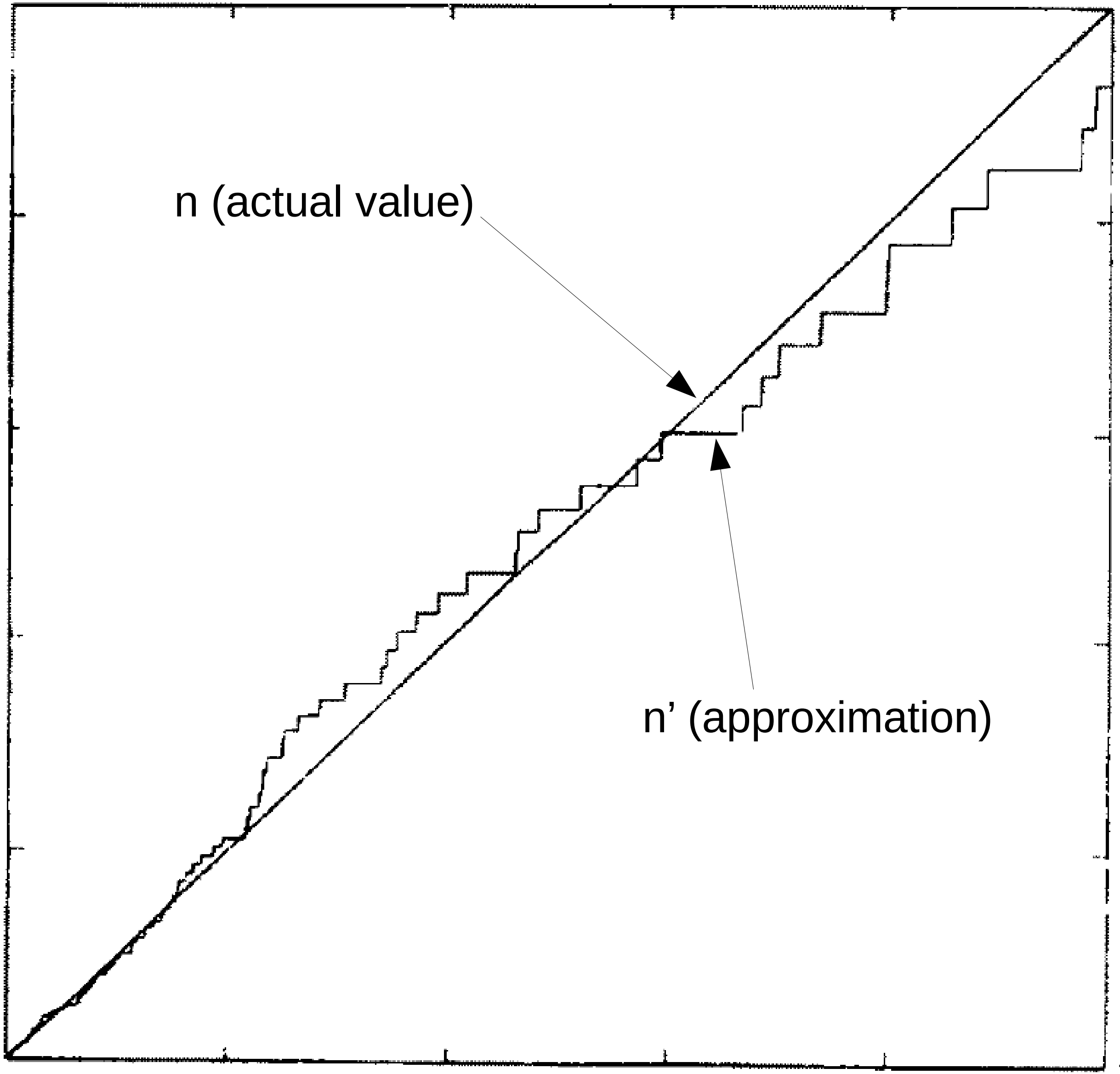
Return estimate $n' = 2^x + 1$



Counter x needs only $\log_2(n)$ bits

Source: Slides by Nick Duffield

Simulation results by Flajolet (1985)



Morris' algorithm provides an unbiased estimator

- Init $x=0$, let $p_x = 2^{-x}$, estimate $n' = 2^x - 1$
- $n = 1$
  - before: $x = 0$ $p_0 = 1$;
  - prob. 1: $x \to 1$
  - estimate $n' = 2^1 - 1 = 1 = n$
- $n = 2$
  - before: $x = 1$; $p_1 = ½$
  - prob. ½: x stays at 1; $n' = 2^1 - 1 = 1$
  - prob. ½: $x \to 2$. $n' = 2^2 - 1 = 3$
  - $E[n'] = ½ \times 1 + ½ \times 3 = 2 = n$

Source: Slides by Nick Duffield

Morris' algorithm provides an unbiased estimator (cont.)

Let $X(n)$ denote random counter x after n^{th} arrival

Initialize $X(0) = 0$; increment w.p. $p_x = 2^{-x}$

Estimate $n' = 2^{X(n)} - 1$

$$
\begin{aligned}
E[2^{X(n)}] &= \Sigma_{j=1,\ldots,n-1} \Pr[X(n-1) = j\,]\; E[2^{X(n)} \mid X(n-1) = j\,] \\
&= \Sigma_{j=1,\ldots,n-1} \Pr[X(n-1) = j\,]\; (\, p_j\, 2^{j+1} + (1 - p_j)\, 2^j) \\
&= \Sigma_{j=1,\ldots,n-1} \Pr[X(n-1) = j\,]\; (2^j + 1) \\
&= E[2^{X(n-1)}\,] + 1
\end{aligned}
$$

Iterating: $E[2^{X(n)}] = E[2^{X(0)}] + n = 1 + n$

Therefore: $E[2^{X(n)} - 1] = n$

urce: Slides by Nick Duffield

Flajolet-Martin algorithm for distinct counting

Motivating example
how many neighbors?

- Let $n(u,h)$ be the number of nodes reachable through a path of length up to h from node u
- Naïve method
  - Maintain a set for each node u, initialize $S(u) = \{u\}$
  - Repeat h times:

$$S(u) = S(u) \cup \bigcup_{v \text{ neighbor of } u} S(v)$$

  - Answer $n(u,h) = |S(u)|$

What is the problem with this method?

- Let $n(u,h)$ be the number of nodes reachable through a path of length up to h from node u
- Naïve method
  - Maintain a set for each node u, initialize $S(u) = \{u\}$
  - Repeat h times:

$$S(u) = S(u) \cup \bigcup_{v \text{ neighbor of } u} S(v)$$

  - Answer $n(u,h) = |S(u)|$

Let's look at each node

- We will receive a stream of items
  - Our neighbors at distance $<=$ h
  - Repeated many times because of loops
- We want to use a small amount of memory
- We don't care which items are in the stream
- We just want to know **how many are distinct**

Flajolet-Martin algorithm for counting distinct elements

- For every element u in the stream, compute hash $h(u)$
- Let $r(u)$ be the number of trailing zeros in hash value
  - Example: if $h(u)$ = `001011101000` then $\mathrm{r(u)} = 3$

What is the probability of having
r(u)=1? r(u)=2? r(u)=3?

Flajolet-Martin algorithm for counting distinct elements

- For every element u in the stream, compute hash $h(u)$
- Let $r(u)$ be the number of trailing zeros in hash value
  - Example: if $h(u) = 001011101\underline{000}$ then $r(u) = 3$
- Maintain $R = max\, r(u)$ seen so far
- Output 2^R as an estimator of the number of distinct elements seen so far

Flajolet-Martin algorithm

(intuition)

- Let $r(u)$ be the number of trailing zeros in hash value, keep $R = max\ r(u)$, output 2^R as estimate
- Repeated items don't change our estimates because their hashes are equal
- About ½ of distinct items hash to *******0
  - To actually see a *******0, we expect to wait until seeing 2 distinct items
- About ¼ of distinct items hash to ******00
  - To actually see a ******00, we expect to wait until seeing 4 items
- ...
- If we actually saw a hash value of ***000...0 (having R trailing zeros) then on expectation we saw 2^R different items

Flajolet-Martin, correctness proof

- Let m be the number of distinct elements
- Let $z(r)$ be the probability of finding a tail of r zeroes
- We will prove that

$$z(r) \to 1 \text{ if } m \gg 2^r$$

$$z(r) \to 0 \text{ if } m \ll 2^r$$

- Hence 2^r should be around m

Flajolet-Martin, correctness proof (cont.)

- Probability a hash value ends in r zeroes = $(1/2)^r$
  - Assuming $h(u)$ produces values at random
  - Prob. random binary ends in r zeroes = $(1/2)^r$
- Probability of seeing m distinct elements and NOT seeing a tail of r zeroes $= (1 - (½)^r)^m$

Flajolet-Martin, correctness proof (cont.)

- Probability of seeing m distinct elements and NOT seeing a tail of r zeroes $= (1 - (½)^r)^m$
- Remember $(1\text{-}\varepsilon)^{1/\varepsilon} \simeq 1/e$ for small ε
- Hence $$\left(1-\left(\frac{1}{2}\right)^r\right)^m = \left(1-\left(\frac{1}{2}\right)^r\right)^{\frac{m\left(\frac{1}{2}\right)^r}{\left(\frac{1}{2}\right)^r}} \approx \left(\frac{1}{e}\right)^{\left(\frac{m}{2^r}\right)}$$

Flajolet-Martin, correctness proof (cont.)

- Probability of seeing m distinct elements and NOT seeing a tail of r zeroes
- If $m \gg 2^r$, this tends to 0 $\approx (1/e)^{\left(\frac{m}{2^r}\right)}$
  - We almost certainly will see a tail of r zeroes
- If $m \ll 2^r$, this tends to 1
  - We almost certainly will not see a tail of r zeroes
- Hence, 2^r should be around m

Flajolet-Martin: increasing precision

- Idea: repeat many times or compute in parallel for multiple hash functions
- How to combine?
  - Average? $E[2^r]$ is infinite, extreme values will skew the number excessively
  - **Median?** 2^r is always a power of 2
- Solution: group hash functions, take median of values obtained in each group, then average across groups

Let's go back to counting neighbors

Naïve method:

Maintain a set for each node u, initialize $S(u) = \{u\}$

Repeat h times: $S(u) = S(u) \cup \bigcup_{v \text{ neighbor of } u} S(v)$

Answer $n(u,h) = |S(u)|$

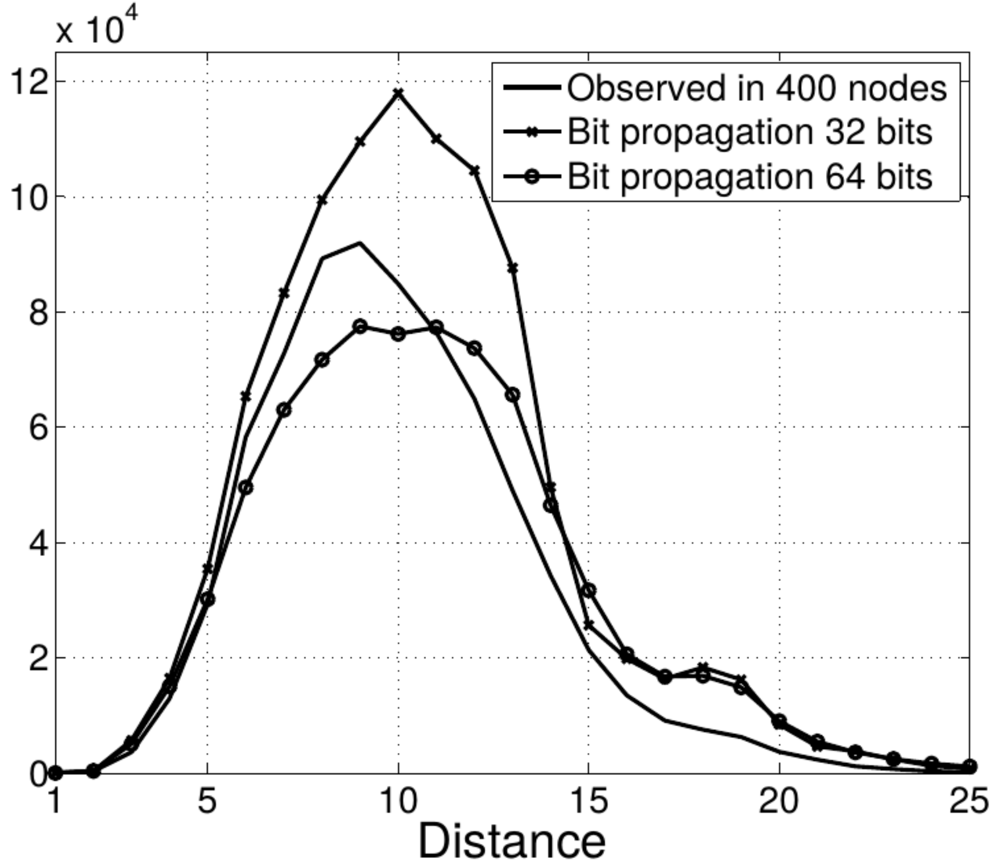
ANF method:

// Set $\mathcal{M}(x,0) = \{x\}$
FOR each node x DO
    $M(x,0) =$ concatenation of k bitmasks
        each with 1 bit set ($P(\text{bit } i) = .5^{i+1}$)
FOR each distance h starting with 1 DO
    FOR each node x DO $M(x,h) = M(x,h-1)$
    // Update $\mathcal{M}(x,h)$ by adding one step
    FOR each edge (x,y) DO
        $M(x,h) = (M(x,h)$ BITWISE-OR $M(y,h-1))$
    // Compute the estimates for this h
    FOR each node x DO
        Individual estimate $\hat{IN}(x,h) = (2^b)/.77351$
        where b is the average position of the least zero bits
        in the k bitmasks

Palmer, C. R., Gibbons, P. B., & Faloutsos, C. (2002, July). ANF: A fast and scalable tool for data mining in massive graphs. In Proc. KDD.

Example of another variant of the same type of algorithm

- More repetitions of the algorithm yield better precision



Becchetti, Luca, Carlos Castillo, Debora Donato, Stefano Leonardi, and Ricardo Baeza-Yates. "Using rank propagation and probabilistic counting for link-based spam detection." In Proc. of WebKDD, 2006.

Summary

Things to remember

- Probabilistic counting algorithms:
  - Morris
  - Flajolet-Martin

Exercises for TT22-T26

- Mining of Massive Datasets (2014) by Leskovec et al.
  - Exercises 4.2.5
  - Exercises 4.3.4
  - Exercises 4.4.5
  - Exercises 4.5.6