Appendix 2

The Null Geodesic

The path that a light beam, carrying negligible energy, follows in the curved spacetime of General Relativity is known as the null geodesic. It can be shown from the principle of equivalence that the null geodesic follows the form [32]

$$\frac{\partial^2 x^\lambda}{\partial \sigma^2} + \frac{1}{2} g^{\lambda\alpha} \left(\frac{\partial g_{\nu\alpha}}{\partial x^\mu} + \frac{\partial g_{\beta\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\beta} \right) \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \sigma} = 0,$$

where $x(\sigma)$ represents the trajectory of the light beam parameterized by σ, and g is the spacetime metric at location x. Recall from the main paper that $g^{\lambda\alpha}$ denotes the inverse metric. The principle of equivalence states that: for any point in space and any moment in time, we can always define a free-falling inertial frame of reference that the effect of gravity is locally absent. In such case, physical laws in this local region can be described by only the theory of Special Relativity, thus allowing us to analyze infinitesimal motions, including that of light, as inertial motions (i.e., motions of object in the absence of external forces) in flat spacetime. If we choose the flow of time in these local reference frames as our parameterization σ, and let x^α denote the coordinate position of the light beam at a moment in time, then the following equality must hold:

$$\frac{\partial^2 x^\alpha}{\partial \sigma^2} = 0$$

To constrain the speed of this beam, we shall impose

$$\eta_{AB} \frac{\partial x^A}{\partial \sigma} \frac{\partial x^B}{\partial \sigma} = 0,$$

where η_{AB} represents the Minkowski metric

$$\eta_{AB} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This ensures that the light beam's trajectory through spacetime stays on the null cone, as per the description of Special Relativity.

Now imagine that the light departs this local inertial frame as it travels forward. Meanwhile, there exist a varying gravitational field at every point in space. Then the light will enter an adjacent region of a different spacetime curvature. Since the new region can also be approximated by laws of Special Relativity locally, we can setup a new inertial reference frame where its metric is simply a linear transformation of η_{AB} (i.e., the product with Jacobian and its transpose, as the variation in space is thought as infinitesimal). This would hold as long as the change of spacetime curvature is continuous (e.g., no singularity) along the path of the light beam. Furthermore, we can see the new inertial frame as the consequence of a coordinate transform acting on the old, thus:

$$\frac{\partial \xi^\alpha}{\partial \sigma} = \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial x^\mu}{\partial \sigma},$$

where ξ^α represents the new inertial frame after coordinate change.

Again, by the equivalence principle,

$$\frac{\partial^2 \xi^\alpha}{\partial \sigma^2} = \frac{\partial}{\partial \sigma}\left(\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial x^\mu}{\partial \sigma}\right)$$

$$= \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial^2 x^\mu}{\partial \sigma^2} + \frac{\partial^2 \xi^\alpha}{\partial \sigma \partial x^\mu}\frac{\partial x^\mu}{\partial \sigma} = 0$$

Apply the chain rule on the second term, as indicated by the brackets:

$$0 = \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial^2 x^\mu}{\partial \sigma^2} + \frac{\partial^2 \xi^\alpha}{\underbrace{\partial x^\nu}_{} \partial x^\mu}\frac{\partial x^\mu}{\partial \sigma}\frac{\partial x^\nu}{\underbrace{\partial \sigma}_{}}$$

Then, multiply both side by the Jacobian $\partial x^\lambda / \partial \xi^\alpha$

$$0 = \underbrace{\frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial \xi^\alpha}{\partial x^\mu}}_{}\frac{\partial^2 x^\mu}{\partial \sigma^2} + \frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\nu \partial x^\mu}\frac{\partial x^\mu}{\partial \sigma}\frac{\partial x^\nu}{\partial \sigma},$$

and recognize that the annotated contraction results in the Kronecker delta δ^λ_μ, a tensor equivalence of the identity matrix. By observing the structure of δ^λ_μ (indexed component equals 1 when $\lambda = \mu$, and 0 otherwise), one shall realize its index-altering ability in Einstein summation notation [33]. For example:

$$V^\mu \delta^\lambda_\mu = V^\lambda$$

for an arbitrary vector V^μ. Using such property, we yield the equation for the light beam's motion:

$$0 = \frac{\partial^2 x^\lambda}{\partial \sigma^2} + \frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\nu \partial x^\mu}\frac{\partial x^\mu}{\partial \sigma}\frac{\partial x^\nu}{\partial \sigma} \tag{1}$$

which connects the ray's acceleration with its velocity with respect to coordinate time. Steven Weinberg's deduction in [32] ends here, followed by defining the factor in front of the second term as the affine connection (i.e., the Christoffel symbol in the main paper). However, we could proceed to show that this factor can by written entirely in terms of the metric tensor of coordinate system ξ, its derivative, and its inverse.

By the transformation rule of metric tensors (See Appendix 1), it can be shown that the metric tensor $g_{\alpha\beta}$ of the coordinate system of ξ^μ takes the form:

$$g_{\alpha\beta} = \eta_{AB}\frac{\partial \xi^A}{\partial x^\alpha}\frac{\partial \xi^B}{\partial x^\beta},$$

with its inverse

$$g^{\alpha\beta} = \eta^{AB}\frac{\partial x^\alpha}{\partial \xi^A}\frac{\partial x^\beta}{\partial \xi^B}$$

One can easily validate that contracting $g_{\alpha\beta}$ with $g_{\beta\gamma}$ by the above expansion indeed produces the Kronecker delta.

Now consider the derivative

$$\frac{\partial}{\partial x^\alpha} g_{\mu\nu} = \frac{\partial}{\partial x^\alpha}\eta_{AB}\frac{\partial \xi^A}{\partial x^\mu}\frac{\partial \xi^B}{\partial x^\nu}$$

By the product rule and the invariance of the Minkowski metric we obtain

$$\frac{\partial}{\partial x^\alpha} g_{\mu\nu} = \eta_{AB}\left(\frac{\partial^2 \xi^A}{\partial x^\alpha \partial x^\mu}\frac{\partial \xi^B}{\partial x^\nu} + \frac{\partial \xi^A}{\partial x^\mu}\frac{\partial^2 \xi^B}{\partial x^\alpha \partial x^\nu}\right)$$

The similar applies for the permutations of indices:

$$
\begin{aligned}
\frac{\partial}{\partial x^\nu} g_{\alpha\mu} &= \eta_{AB}\left(\frac{\partial^2 \xi^A}{\partial x^\nu \partial x^\alpha}\frac{\partial \xi^B}{\partial x^\mu} + \frac{\partial \xi^A}{\partial x^\alpha}\frac{\partial^2 \xi^B}{\partial x^\nu \partial x^\mu}\right) \\
\frac{\partial}{\partial x^\mu} g_{\nu\alpha} &= \eta_{AB}\left(\frac{\partial^2 \xi^A}{\partial x^\mu \partial x^\nu}\frac{\partial \xi^B}{\partial x^\alpha} + \frac{\partial \xi^A}{\partial x^\nu}\frac{\partial^2 \xi^B}{\partial x^\mu \partial x^\alpha}\right)
\end{aligned}
$$

By the symmetry of the Minkowski metric (i.e., $\eta_{AB} = \eta_{BA}$), and the symmetry of second derivatives (i.e., $\partial/(\partial x^\mu \partial x^\nu) = \partial/(\partial x^\nu \partial x^\mu)$), we can always rewrite any of the above derivtives with any pairs of ξ^A and ξ^B swaped. For example,

$$
\begin{aligned}
\frac{\partial}{\partial x^\nu} g_{\alpha\mu} &= \eta_{AB}\left(\frac{\partial^2 \xi^A}{\partial x^\nu \partial x^\alpha}\frac{\partial \xi^B}{\partial x^\mu} + \frac{\partial \xi^A}{\partial x^\alpha}\frac{\partial^2 \xi^B}{\partial x^\nu \partial x^\mu}\right) \\
&= \eta_{AB}\frac{\partial^2 \xi^A}{\partial x^\nu \partial x^\alpha}\frac{\partial \xi^B}{\partial x^\mu} + \eta_{AB}\frac{\partial \xi^A}{\partial x^\alpha}\frac{\partial^2 \xi^B}{\partial x^\nu \partial x^\mu}
\end{aligned}
$$

Since the order of summation does not matter for A and B,

$$
\begin{aligned}
\frac{\partial}{\partial x^\nu} g_{\alpha\mu} &= \eta_{BA}\frac{\partial^2 \xi^B}{\partial x^\nu \partial x^\alpha}\frac{\partial \xi^A}{\partial x^\mu} + \eta_{AB}\frac{\partial \xi^A}{\partial x^\alpha}\frac{\partial^2 \xi^B}{\partial x^\nu \partial x^\mu} \\
&= \eta_{AB}\frac{\partial^2 \xi^B}{\partial x^\nu \partial x^\alpha}\frac{\partial \xi^A}{\partial x^\mu} + \eta_{AB}\frac{\partial \xi^A}{\partial x^\alpha}\frac{\partial^2 \xi^B}{\partial x^\nu \partial x^\mu} \\
&= \eta_{AB}\left(\frac{\partial^2 \xi^B}{\partial x^\nu \partial x^\alpha}\frac{\partial \xi^A}{\partial x^\mu} + \frac{\partial \xi^A}{\partial x^\alpha}\frac{\partial^2 \xi^B}{\partial x^\nu \partial x^\mu}\right) \qquad (2)
\end{aligned}
$$

At this point, one may conjecture that a connection exist between a linear combination of the three derivatives and the factor in eq. (1). This is indeed correct. Consider the sum

$$
\begin{aligned}
\frac{\partial g_{\nu\alpha}}{\partial x^\mu} + \frac{\partial g_{\alpha\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} = \eta_{AB}\Bigg[&\left(\frac{\partial^2 \xi^A}{\partial x^\mu \partial x^\nu}\frac{\partial \xi^B}{\partial x^\alpha} + \frac{\partial \xi^A}{\partial x^\alpha}\frac{\partial^2 \xi^B}{\partial x^\nu \partial x^\mu}\right) \\
&+ \left(\frac{\partial^2 \xi^A}{\partial x^\nu \partial x^\alpha}\frac{\partial \xi^B}{\partial x^\mu} - \frac{\partial \xi^A}{\partial x^\mu}\frac{\partial^2 \xi^B}{\partial x^\alpha \partial x^\nu}\right) \\
&+ \left(\frac{\partial \xi^A}{\partial x^\nu}\frac{\partial^2 \xi^B}{\partial x^\mu \partial x^\alpha} - \frac{\partial^2 \xi^A}{\partial x^\alpha \partial x^\mu}\frac{\partial \xi^B}{\partial x^\nu}\right)\Bigg]
\end{aligned}
$$

All grouped terms in $(\cdot)$ can be either combined or canceled via manipulations like that of (2). Resulting in a simpler form:

$$
\frac{\partial g_{\nu\alpha}}{\partial x^\mu} + \frac{\partial g_{\alpha\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} = 2\eta_{AB}\frac{\partial^2 \xi^A}{\partial x^\mu \partial x^\nu}\frac{\partial \xi^B}{\partial x^\alpha}
$$

We see that this is already very close to the factor in (1). To continue the simplification, we contract the sum of derivatives with the inverse metric, specifically, summing over α:

$$
\begin{aligned}
g^{\lambda\alpha}\left(\frac{\partial g_{\nu\alpha}}{\partial x^\mu} + \frac{\partial g_{\alpha\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right) &= 2g^{\lambda\alpha}\eta_{AB}\frac{\partial^2 \xi^A}{\partial x^\mu \partial x^\nu}\frac{\partial \xi^B}{\partial x^\alpha} \\
&= 2\eta^{CD}\frac{\partial x^\lambda}{\partial \xi^C}\underbrace{\frac{\partial x^\alpha}{\partial \xi^D}}\eta_{AB}\frac{\partial^2 \xi^A}{\partial x^\mu \partial x^\nu}\underbrace{\frac{\partial \xi^B}{\partial x^\alpha}} \\
&= 2\eta^{CD}\frac{\partial x^\lambda}{\partial \xi^C}\eta_{AB}\frac{\partial^2 \xi^A}{\partial x^\mu \partial x^\nu}\delta^B_D \\
&= 2\eta^{CD}\delta^B_D\eta_{AB}\frac{\partial x^\lambda}{\partial \xi^C}\frac{\partial^2 \xi^A}{\partial x^\mu \partial x^\nu} \\
&= 2\delta^C_A\frac{\partial x^\lambda}{\partial \xi^C}\frac{\partial^2 \xi^A}{\partial x^\mu \partial x^\nu} \\
&= 2\frac{\partial x^\lambda}{\partial \xi^A}\frac{\partial^2 \xi^A}{\partial x^\mu \partial x^\nu}
\end{aligned}
$$

Compare to (1), we see that this is exactly 2 times the factor in front of $\frac{\partial x^\mu}{\partial \sigma}\frac{\partial x^\nu}{\partial \sigma}$. Thus we have proven that the null geodesic equation can be written as

$$\frac{\partial^2 x^\lambda}{\partial \sigma^2} + \frac{1}{2} g^{\lambda\alpha} \left(\frac{\partial g_{\nu\alpha}}{\partial x^\mu} + \frac{\partial g_{\beta\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\beta} \right) \frac{\partial x^\mu}{\partial \sigma}\frac{\partial x^\nu}{\partial \sigma} = 0$$