# Appendix 1

## Transformation Law of the Metric

In this section, we deduce the transformation law for each individual component of the metric from a fully linear algebraic perspective. Recall that from the main paper, under transformation from coordinate $x$ to $x'$, the metric $M$ transforms according to

$$M' = J_{x'\mapsto x}^{\top} M J_{x'\mapsto x}, \tag{1}$$

where $M'$ is the metric after transformation, and $J_{x'\mapsto x}$ represents the Jacobian matrix for the inverse transformation $x' \mapsto x$. Taking $x$ as a function of $x'$: $x(x')$, we can write out the components of $J_{x'\mapsto x}$ as

$$J_{x'\mapsto x} = \begin{bmatrix} \frac{\partial x^1}{\partial x'^1} & \frac{\partial x^1}{\partial x'^2} & \cdots & \frac{\partial x^1}{\partial x'^n} \\ \frac{\partial x^2}{\partial x'^1} & \frac{\partial x^2}{\partial x'^2} & \cdots & \frac{\partial x^2}{\partial x'^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x^n}{\partial x'^1} & \frac{\partial x^n}{\partial x'^2} & \cdots & \frac{\partial x^n}{\partial x'^n} \end{bmatrix}$$

where $x^i$ and $x'^j$ denote the $i$-th coordinate function of $x$ and the $j$-th coordinate function of $x'$, respectively. Writing the component on the $\mu$-th row and $\nu$-th column as $g_{\mu\nu}$ for (1), we yield

$$M' = \begin{bmatrix} \frac{\partial x^1}{\partial x'^1} & \frac{\partial x^2}{\partial x'^1} & \cdots & \frac{\partial x^n}{\partial x'^1} \\ \frac{\partial x^1}{\partial x'^2} & \frac{\partial x^2}{\partial x'^2} & \cdots & \frac{\partial x^n}{\partial x'^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x^1}{\partial x'^n} & \frac{\partial x^2}{\partial x'^n} & \cdots & \frac{\partial x^n}{\partial x'^n} \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ g_{n1} & g_{n2} & \cdots & g_{nn} \end{bmatrix} \begin{bmatrix} \frac{\partial x^1}{\partial x'^1} & \frac{\partial x^1}{\partial x'^2} & \cdots & \frac{\partial x^1}{\partial x'^n} \\ \frac{\partial x^2}{\partial x'^1} & \frac{\partial x^2}{\partial x'^2} & \cdots & \frac{\partial x^2}{\partial x'^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x^n}{\partial x'^1} & \frac{\partial x^n}{\partial x'^2} & \cdots & \frac{\partial x^n}{\partial x'^n} \end{bmatrix}$$

Now focus on the the matrix product $MJ_{x'\mapsto x}$.

$$MJ_{x'\mapsto x} = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ g_{n1} & g_{n2} & \cdots & g_{nn} \end{bmatrix} \begin{bmatrix} \frac{\partial x^1}{\partial x'^1} & \frac{\partial x^1}{\partial x'^2} & \cdots & \frac{\partial x^1}{\partial x'^n} \\ \frac{\partial x^2}{\partial x'^1} & \frac{\partial x^2}{\partial x'^2} & \cdots & \frac{\partial x^2}{\partial x'^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x^n}{\partial x'^1} & \frac{\partial x^n}{\partial x'^2} & \cdots & \frac{\partial x^n}{\partial x'^n} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{\lambda=1}^n g_{1\lambda}\frac{\partial x^\lambda}{\partial x'^1} & \sum_{\lambda=1}^n g_{1\lambda}\frac{\partial x^\lambda}{\partial x'^2} & \cdots & \sum_{\lambda=1}^n g_{1\lambda}\frac{\partial x^\lambda}{\partial x'^n} \\ \sum_{\lambda=1}^n g_{2\lambda}\frac{\partial x^\lambda}{\partial x'^1} & \sum_{\lambda=1}^n g_{2\lambda}\frac{\partial x^\lambda}{\partial x'^2} & \cdots & \sum_{\lambda=1}^n g_{2\lambda}\frac{\partial x^\lambda}{\partial x'^n} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{\lambda=1}^n g_{n\lambda}\frac{\partial x^\lambda}{\partial x'^1} & \sum_{\lambda=1}^n g_{n\lambda}\frac{\partial x^\lambda}{\partial x'^2} & \cdots & \sum_{\lambda=1}^n g_{n\lambda}\frac{\partial x^\lambda}{\partial x'^n} \end{bmatrix}$$

Thus, denoting the entry at the $\mu$-th row, $\nu$-th column as $\tilde{g}_{\mu\nu}$, we have

$$\tilde{g}_{\mu\nu} = \sum_{\lambda=1}^{n} g_{\mu\lambda}\frac{\partial x^\lambda}{\partial x'^\nu} \tag{2}$$

Similarly, for the entire $M'$,

$$M' = \begin{bmatrix} \frac{\partial x^1}{\partial x'^1} & \frac{\partial x^2}{\partial x'^1} & \cdots & \frac{\partial x^n}{\partial x'^1} \\ \frac{\partial x^1}{\partial x'^2} & \frac{\partial x^2}{\partial x'^2} & \cdots & \frac{\partial x^n}{\partial x'^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x^1}{\partial x'^n} & \frac{\partial x^2}{\partial x'^n} & \cdots & \frac{\partial x^n}{\partial x'^n} \end{bmatrix} M J_{x' \mapsto x}$$

$$= \begin{bmatrix} \frac{\partial x^1}{\partial x'^1} & \frac{\partial x^2}{\partial x'^1} & \cdots & \frac{\partial x^n}{\partial x'^1} \\ \frac{\partial x^1}{\partial x'^2} & \frac{\partial x^2}{\partial x'^2} & \cdots & \frac{\partial x^n}{\partial x'^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x^1}{\partial x'^n} & \frac{\partial x^2}{\partial x'^n} & \cdots & \frac{\partial x^n}{\partial x'^n} \end{bmatrix} \begin{bmatrix} \tilde{g}_{11} & \tilde{g}_{12} & \cdots & \tilde{g}_{1n} \\ \tilde{g}_{21} & \tilde{g}_{22} & \cdots & \tilde{g}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{g}_{n1} & \tilde{g}_{n2} & \cdots & \tilde{g}_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{\lambda=1}^n \tilde{g}_{\lambda 1} \frac{\partial x^\lambda}{\partial x'^1} & \sum_{\lambda=1}^n \tilde{g}_{\lambda 2} \frac{\partial x^\lambda}{\partial x'^1} & \cdots & \sum_{\lambda=1}^n \tilde{g}_{\lambda n} \frac{\partial x^\lambda}{\partial x'^1} \\ \sum_{\lambda=1}^n \tilde{g}_{\lambda 1} \frac{\partial x^\lambda}{\partial x'^2} & \sum_{\lambda=1}^n \tilde{g}_{\lambda 2} \frac{\partial x^\lambda}{\partial x'^2} & \cdots & \sum_{\lambda=1}^n \tilde{g}_{\lambda n} \frac{\partial x^\lambda}{\partial x'^2} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{\lambda=1}^n \tilde{g}_{\lambda 1} \frac{\partial x^\lambda}{\partial x'^n} & \sum_{\lambda=1}^n \tilde{g}_{\lambda 2} \frac{\partial x^\lambda}{\partial x'^n} & \cdots & \sum_{\lambda=1}^n \tilde{g}_{\lambda n} \frac{\partial x^\lambda}{\partial x'^n} \end{bmatrix},$$

it would have entries

$$g'_{\mu\nu} = \sum_{\lambda=1}^{n} \tilde{g}_{\lambda\nu} \frac{\partial x^\lambda}{\partial x'^\mu}$$

Now substitute in the expression for $\tilde{g}_{\lambda\nu}$, we obtain the components of $M'$:

$$M' = \begin{bmatrix} g'_{11} & g'_{12} & \cdots & g'_{1n} \\ g'_{21} & g'_{22} & \cdots & g'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ g'_{n1} & g'_{n2} & \cdots & g'_{nn} \end{bmatrix},$$

where

$$g'_{\mu\nu} = \sum_{\beta=1}^{n} \sum_{\alpha=1}^{n} g_{\alpha\beta} \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu}$$