Question 9

a) If $\frac{d\nu}{d\mu}$ and $\frac{d\eta}{d\nu}$ exist then $\eta \ll \nu \ll \mu$. For any $A \subseteq \Omega$ we have:

$$\begin{aligned}
\mu(I_A fg) &= \int_A fg \, d\mu \\
&= \int_A \frac{d\nu}{d\mu} g \, d\mu && (f = \tfrac{d\nu}{d\mu}) \\
&= \int_A g \, d\nu \\
&= \int_A \frac{d\eta}{d\nu} \, d\nu && (g = \tfrac{d\eta}{d\nu}) \\
&= \int_A 1 \, d\eta \\
&= \eta(I_A)
\end{aligned}$$

But notice that $\eta \ll \mu$. So we also have

$$\begin{aligned}
\mu(I_A f') &= \int_A f' \, d\mu \\
&= \int_A \frac{d\eta}{d\mu} \, d\mu && (f' = \tfrac{d\eta}{d\mu}) \\
&= \int_A 1 \, d\eta \\
&= \eta(I_A)
\end{aligned}$$

$\mu(I_A \frac{d\eta}{d\nu}\frac{d\nu}{d\mu}) = \mu(I_A \frac{d\eta}{d\mu})$ μ-almost everywhere.

$$\frac{d\eta}{d\mu} = \frac{d\eta}{d\nu}\frac{d\nu}{d\mu} \qquad (\mu\text{-almost everywhere})$$

Need to check with prof that this is an almost everywhere result

b) Existence of $\frac{d\nu}{d\mu} \to \nu \ll \mu$. Therefore $\mu(A) = 0 \to \nu(A) = 0$.
If $\frac{d\nu}{d\mu} > 0$ μ-a.e. then $\nu(A) = 0 \to \mu(A) = 0$ and $\mu \ll \nu$.
ν and μ are equivalent measures.

For any $A \subseteq \Omega$

$$\begin{aligned}
\mu(I_A fg) &= \int_A fg \, d\mu \\
&= \int_A \frac{d\nu}{d\mu} g \, d\mu && (f = \tfrac{d\nu}{d\mu}) \\
&= \int_A g \, d\nu \\
&= \int_A \frac{d\mu}{d\nu} \, d\nu && (g = \tfrac{d\mu}{d\nu}) \\
&= \int_A 1 \, d\mu \\
&= \mu(I_A)
\end{aligned}$$

$\mu(I_A) = \mu(\frac{d\mu}{d\nu}\frac{d\nu}{d\mu} I_A)$ μ-almost everywhere.
This implies that $\frac{d\mu}{d\nu}\frac{d\nu}{d\mu}$ is the same as the ghost function of 1 under μ.

$$\begin{aligned}
\frac{d\mu}{d\nu}\frac{d\nu}{d\mu} &= 1 && (\mu\text{-almost everywhere}) \\
\frac{d\mu}{d\nu}\frac{d\nu}{d\mu}(\frac{d\nu}{d\mu})^{-1} &= (\frac{d\nu}{d\mu})^{-1} && (\mu\text{-almost everywhere}) \\
\frac{d\mu}{d\nu} &= \frac{1}{\frac{d\nu}{d\mu}} && (\mu\text{-almost everywhere})
\end{aligned}$$