Question 6

a) $\bar{f} : \mathbb{R} \to \mathbb{R}$ is a Borel function if the measure space is a Borel measure space and $\bar{f}$ is $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable. This requires that:
$\{a \in \mathbb{R} : \bar{f}(a) \in B\} = \bar{f}^{-1}[B] \in \mathcal{B}(\mathbb{R}) \quad \forall B \in \mathcal{B}(\mathbb{R})$.

$\bar{f} = \mathbb{E}[f(x, Y)] = \int f(x, y)\mathbb{P}_Y(dy)$
By Fubini we have:

$$\begin{aligned}
&\int\int f(x,y)d\mathbb{P}_Y(dy)\mathbb{P}_X(dx) \\
&= \int \mathbb{E}[f(x,Y)]\mathbb{P}_X(dx) \\
&= \int \bar{f}(x)\ \mathbb{P}_x(dx) \\
&< \infty && (f \text{ is a bounded function})
\end{aligned}$$

This shows that the pullback of $\bar{f}$ is $\mathcal{B}(\mathbb{R})$-measurable

b)

$$\mathcal{H} = \left\{ f : (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2)) \xrightarrow{f} (\mathbb{R}, \mathcal{B}(\mathbb{R})) \wedge \mathbb{E}[f(X,Y)|\mathcal{G}] = \mathbb{E}[f(X,Y)] \right\}$$

We are required to show that $\mathcal{H}$ is closed under addition and scalar multiplication.

i) for $f, g \in \mathcal{H}$ we have:
$h = f(B_1 \times B_2) + g(B_1 \times B_2) = x + y$ for $x, y \in \mathbb{R}$ and $B_1, B_2 \in \mathcal{B}(\mathbb{R})$.
Which is just a mapping $h : (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$.
So it suffices to show that $\mathbb{E}[h|\mathcal{G}] = \mathbb{E}[h]$

$$\begin{aligned}
\mathbb{E}[f+g|\mathcal{G}] &= \mathbb{E}[f|\mathcal{G}] + \mathbb{E}[g|\mathcal{G}] \\
&= \mathbb{E}[f] + \mathbb{E}[g] \\
&= \mathbb{E}[f+g] \\
&= \mathbb{E}[h]
\end{aligned}$$

Therefore $h \in \mathcal{H}$

ii) for $\alpha \in \mathbb{R}$:
$h = \alpha \cdot f(B_1 \times B_2) = \alpha \times x \quad$ where $\alpha, x \in \mathbb{R}$.
Which is just a mapping $h : (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

$\mathbb{E}[h|\mathcal{G}] = \mathbb{E}[\alpha \cdot f|\mathcal{G}] = \alpha \times \mathbb{E}[f|\mathcal{G}] = \alpha\mathbb{E}[f] = \mathbb{E}[\alpha \cdot f] = \mathbb{E}[h]$

Therefore $h \in \mathcal{H}$

We have shown $\mathcal{H}$ is a vector space.

c)

$$\begin{aligned}
f(B_1 \times B_2) &= h(B_1) \times k(B_2) && (B_1, B_2 \in \mathcal{B}(\mathbb{R})) \\
&= x \times y && (x, y \in \mathbb{R}) \\
&= c && (c \in \mathbb{R})
\end{aligned}$$

Therefore f is clearly a mapping $f : (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

It remains to show $\mathbb{E}[f|\mathcal{G}] = \mathbb{E}[f]$

$$\begin{aligned}
\mathbb{E}[h(X)k(Y); \Omega \times G] &= \int_\Omega \int_G h(X)k(Y)\mathbb{P}_X(dx)\mathbb{P}_Y(dy) && (\forall G \in \mathcal{G}) \\
&= \int_\Omega k(Y)\mathbb{P}_Y(dy) \int_G h(X)\mathbb{P}_X(dx) && (\text{independent } Y) \\
&= \mathbb{E}[k(Y)]\mathbb{E}[h(X)] && (w \in \mathcal{G}) \\
&= \mathbb{E}[k(Y)h(X)]
\end{aligned}$$

d) $I_{A\times B} : (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2)) \to (\{0,1\}, \mathcal{P}(\{0,1\})) \subset (\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

$$\begin{aligned}
\mathbb{E}[I_{A\times B}|\mathcal{G}] &= \mathbb{E}[I_{A\times\Omega}|\mathcal{G}]\mathbb{E}[I_{\Omega\times B}] && (\text{by independence}) \\
&= \mathbb{E}[I_{A\times\Omega}]\mathbb{E}[I_{\Omega\times B}] && (\mathcal{G}\text{-measurable}) \\
&= \mathbb{E}[I_{A\times B}]
\end{aligned}$$

Therefore $I_{A\times B} \in \mathcal{H}$ by having the requisite properties.

e) $f_n(x,y) = \sum_{i=0}^{2^n n-1} \sum_{j=0}^{2^n n-1} a_{i,j} I_{\{\frac{i}{2^n} < x \le \frac{i+1}{2^n}\} \cap \{\frac{j}{2^n} < y \le \frac{j+1}{2^n}\}}(x,y)$
$\underset{n}{lim} f_n = f$, and $f_n \uparrow f$ if we set $a_{i,j} = f(\frac{i}{2^n}, \frac{j}{2^n})$.

We need to show that $f_n \in \mathcal{H}$. Clearly $f_n : (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

$$\begin{aligned}
&\mathbb{E}[f_n; G \times \Omega] \\
&= \int_\Omega \int_G \sum_{i=0}^{2^n n-1} \sum_{j=0}^{2^n n-1} a_{i,j} I_{\{\frac{i}{2^n} < x \le \frac{i+1}{2^n}\} \cap \{\frac{j}{2^n} < y \le \frac{j+1}{2^n}\}}(x,y) \mathbb{P}_x(dx) \mathbb{P}_y(dy) \\
&= \int_\Omega \sum_{j=0}^{2^n n-1} I_{\{\frac{j}{2^n} < y \le \frac{j+1}{2^n}\}}(y) \int_G \sum_{i=0}^{2^n n-1} a_{i,j} I_{\{\frac{i}{2^n} < x \le \frac{i+1}{2^n}\}}(x) \mathbb{P}_x(dx) \mathbb{P}_y(dy) \\
&= \int_\Omega \sum_{j=0}^{2^n n-1} I_{\{\frac{j}{2^n} < y \le \frac{j+1}{2^n}\}}(y) \int_\Omega \sum_{i=0}^{2^n n-1} a_{i,j} I_{\{\frac{i}{2^n} < x \le \frac{i+1}{2^n}\}}(x) \mathbb{P}_x(dx) \mathbb{P}_y(dy) \qquad (x \in G) \\
&= \mathbb{E}[f_n; \Omega \times \Omega] \\
&= \mathbb{E}[f_n]
\end{aligned}$$

Now we can show that $\mathbb{E}[f|\mathcal{G}] = \mathbb{E}[f]$ by taking limits of f_n.

$$\begin{aligned}
\mathbb{E}[f; G \times \Omega] &= \int_\Omega \int_G \lim_n f_n(x,y) \mathbb{P}_x(dx) \mathbb{P}_y(dy) \\
&\overset{\text{DCT}}{=} \lim_n \int_\Omega \int_G f_n(x,y) \mathbb{P}_x(dx) \mathbb{P}_y(dy) \\
&= \lim_n \mathbb{E}[f_n] \\
&= \mathbb{E}[\lim_n f_n] \\
&= \mathbb{E}[f]
\end{aligned}$$

Therefore $f \in \mathcal{H}$

f) We have shown that $\mathcal{H}$ satisfies the following conditions:

i) $\mathcal{H}$ is a vector space

ii) The constant function $\mathbf{1} \in \mathcal{H}$. Consider (d) and set $A = B = \Omega$

iii) We have shown that if $f_n \in \mathcal{H}$ and $f_n \uparrow f$ where f is bounded then $f \in \mathcal{H}$.

We can apply monotone class theorem and every bounded $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable function is in $\mathcal{H}$. This is exactly the Borel functions which includes $\bar{f}$.