Exercise Sheet 6

Question 5

(a) Show that X is not attainable.

Solution:

If X is attainable, then there exist $V_t(\theta)$ such that $V_t(\theta) = X_t$. Hence $\bar{V}_t(\theta) = \bar{X}_t$

$$\mathbb{Q}(\omega_k) = \mathbb{E}_{\mathbb{Q}}[\bar{X}] = \mathbb{E}_{\mathbb{Q}}[\bar{V}_T(\theta)] = \bar{V}_0(\theta)$$

$$\mathbb{Q}'(\omega_k) = \mathbb{E}_{\mathbb{Q}'}[\bar{X}] = \mathbb{E}_{\mathbb{Q}'}[\bar{V}_T(\theta)] = \bar{V}_0(\theta)$$

but

$$\mathbb{Q}(\omega_k) \neq \mathbb{Q}'(\omega_k)$$

Therefore under no abitrage, there is no portfolio $V(\theta)$ which attains X

(b) i Show that there is a vector $\xi = (\xi_1, ..., \xi_K)$ with property that $\xi.\bar{V}_T(\theta) = 0$ for each trading strategy , but has $\xi.X > 0$

Solution:

Define $L = \{\bar{V}_T(\theta) : \text{all } \theta\}$ and $K = \{\bar{X}\}$ then $K \cap L = \emptyset$ by (a). By the hyperplane separation theorem, there exists $\xi \in \mathbb{R}^n$ such that.

$$\xi.\bar{V}_T(\theta) = 0$$
$$\xi.\bar{X} > 0$$

hence $\xi.X > 0$ by definition of X

ii Show that $\mathbb{Q}(\omega\neg) = q'_k$ defines a probability measure Ω which is equivalent to $\mathbb{P}$

Solution:

$$q'.\bar{V}_T(\theta) = (q + \lambda\xi)\bar{V}_T(\theta) \text{ (but } \xi.\bar{V}_T(\theta) = 0)$$
$$q'.\bar{V}_T(\theta) = q.\bar{V}_T(\theta)$$

If $q'_k = 1$, then $q'_j = 0$ for all $j \neq k$, take $\bar{V}_T(\theta) = 1$ for all ω. Then $q_k = 1$ and $q_j = 0$ for all $j \neq k$. Thus $\mathbb{Q}'$ is equivalent to $\mathbb{Q}$ and hence it is also equivalent to $\mathbb{P}$.

iii Show that $\mathbb{Q}'$ is in fact a risk-neutral measure.

Solution:

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}'}[\bar{V}_T(\theta)] &= \sum_{k=1}^{K} \mathbb{Q}'(\omega_k)\bar{V}_T(\omega_k) \\
&= \sum_{k=1}^{K} q'_k \bar{V}_T(\omega_k) \\
&= \sum_{k=1}^{K} q_k \bar{V}_T(\omega_k) + \lambda \sum_{k=1}^{K} \xi_k \bar{V}_T(\omega_k) \\
&= \sum_{k=1}^{K} \mathbb{Q}(\omega_k)\bar{V}_T(\omega_k) + \lambda \sum_{k=1}^{K} \xi_k \bar{V}_T(\omega_k) \\
&= \mathbb{E}_{\mathbb{Q}}[\bar{V}_T(\theta)] + \lambda \xi . \bar{V}_T(\theta) \text{ (but } \xi . \bar{V}_T(\theta) = 0) \\
&= \mathbb{E}_{\mathbb{Q}}[\bar{V}_T(\theta)] \\
&= \bar{V}_0(\theta)
\end{aligned}
$$

Thus $\mathbb{Q}'$ is in fact a risk-neutral measure.

iv Show that $\mathbb{E}_{\mathbb{Q}'}[\bar{X}] \neq \mathbb{E}_{\mathbb{Q}}[\bar{X}]$

Solution:

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}'}[\bar{X}] &= \sum_{k=1}^{K} \mathbb{Q}'(\omega_k)[\bar{X}] \\
&= \sum_{k=1}^{K} q_k \bar{X} + \lambda \sum_{k=1}^{K} \xi_k \bar{X} \\
&= \sum_{k=1}^{K} \mathbb{Q}(\omega_k)\bar{X} + \lambda \sum_{k=1}^{K} \xi_k \bar{X} \\
&= \mathbb{E}_{\mathbb{Q}}[\bar{X}] + \lambda \xi . \bar{X} \text{ (but } \xi . \bar{X} > 0) \\
&\neq \mathbb{E}_{\mathbb{Q}}[\bar{X}]
\end{aligned}
$$