Question 8

Suppose $f(X_n) \overset{P}{\nrightarrow} f(X)$.
Choose $\langle f(X_{n_k})\rangle_k$ such that $\mathbb{E}[|f(X_{n_k}) - f(X)| \wedge 1] > \epsilon \, \forall k$.
If $\langle f(X_{n_{j_k}})\rangle_j$ is a subsubsequence we cannot have $f(X_{n_{k_j}}) \to f(X)$ almost surely.
Otherwise $|f(X_{n_{k_j}}) - f(X)| \wedge 1 \to 0$ and by DCT $\mathbb{E}[|f(X_{n_{k_j}}) - f(X)| \wedge 1] \to 0$.

The contrapositive statement implies $f(X_n) \xrightarrow{P} f(X)$ if every subsequence $\langle f(X_{n_k})\rangle_k$ of $\langle f(X_n)\rangle$ has a subsubsequence $\langle f(X_{n_{k_j}})\rangle_j$ that converges to $f(X)$ almost surely.

Given that $X_n \xrightarrow{P} X$ we can fix a subsequence $\langle X_{n_k}\rangle_k$ of $\langle X_n\rangle_n$.
It will be true that $X_{n_k} \xrightarrow{P} X$.
We may choose a subsubsequence $\langle X_{n_{k_j}}\rangle_j$ such that $\mathbb{E}[|X_{n_{k_j}} - X| \wedge 1] < 2^{-j}$
$\sum_{j=1}^{\infty} \mathbb{E}[|Xn_{k_j} - X| \wedge 1] < \infty$. Therefore $\sum_{j=1}^{\infty} |Xn_{k_j} - X| \wedge 1$ converges almost surely.
$|X_{n_{k_j}} - X| \to 0$ almost surely.

Now we know that the subsequences $\langle X_{n_k}\rangle_k$ of $\langle X_n\rangle$ have subsubsequences $\langle X_{n_{k_j}}\rangle_j$ that converge to X almost surely. Similarly, $f(X_{n_{k_j}}) \to f(X)$ because f is almost surely continuous at X. This proves the result.