Question 8

Suppose $f(X_n) \stackrel{P}{\not\to} f(X)$.
Choose $\langle f(X_{n_k}) \rangle_k$ such that $\mathbb{E}[|f(X_{n_k}) - f(X)| \wedge 1] > \epsilon \;\forall k$.
If $\langle f(X_{n_{j_k}}) \rangle_j$ is a subsubsequence we cannot have $f(X_{n_{k_j}}) \to f(X)$ almost surely.
Otherwise $|f(X_{n_{k_j}}) - f(X)| \wedge 1 \to 0$ and by DCT $\mathbb{E}[|f(X_{n_{k_j}}) - f(X)| \wedge 1] \to 0$.

The contrapositive result implies $f(X_n) \xrightarrow{P} f(X)$ if every subsequence $\langle f(X_{n_k}) \rangle_k$ of $\langle f(X_n) \rangle$ has a subsubsequence $\langle f(X_{n_{k_j}}) \rangle_j$ that converges to $f(X)$ almost surely.

We are given that subsequences $\langle X_{n_k} \rangle_k$ of $\langle X_n \rangle$ have subsubsequences $\langle X_{n_{k_j}} \rangle_j$ that converge to X almost surely. Similarly, $f(X_{n_{k_j}}) \to f(X)$ because f is almost surely continuous at X.