AIFRM

Stochastic Calculus

Assignment 3

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Execise sheet 6 Solutions

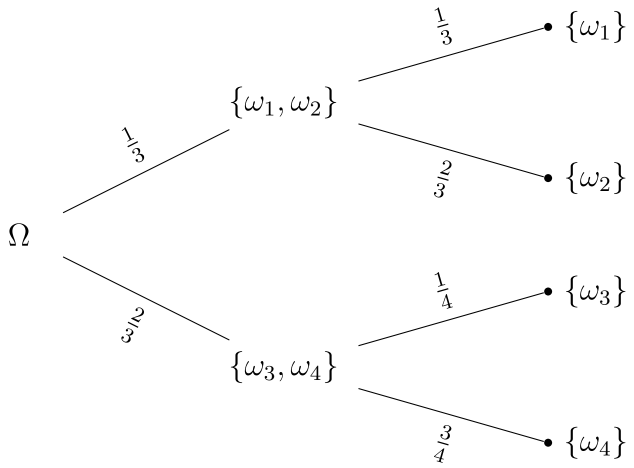
Question 13

a) $\mathcal{F}_0 = \sigma(\{\omega_1, \omega_2, \omega_3, \omega_4\})$
$\mathcal{F}_1 = \sigma(\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\})$
$\mathcal{F}_2 = \sigma(\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\})$

b) $Z_0 = \mathbb{E}[Y|\mathcal{F}_0] = \frac{1}{9} \times 12 + \frac{2}{9} \times 15 + \frac{1}{6} \times 16 + \frac{1}{2} \times 8 = \frac{34}{3}$
$Z_1 = \mathbb{E}[Y|\mathcal{F}_1]$
Since $\mathbb{E}[Y; \{\omega_1, \omega_2\}] = \frac{1}{9} \times 12 + \frac{2}{9} \times 15 = \frac{14}{3}$
therefore, $\mathbb{E}[Y|\{\omega_1, \omega_2\}] = \frac{14}{3}/(\frac{1}{9} + \frac{2}{9}) = 14$
Since $\mathbb{E}[Y; \{\omega_3, \omega_4\}] = \frac{1}{6} \times 16 + \frac{1}{2} \times 8 = \frac{20}{3}$
therefore, $\mathbb{E}[Y|\{\omega_3, \omega_4\}] = \frac{20}{3}/(\frac{1}{6} + \frac{1}{2}) = 10$
therefore, $Z_1 = 14I_{\{\omega_1,\omega_2\}} + 10I_{\{\omega_3,\omega_4\}}$
$Z_2 = \mathbb{E}[Y|\mathcal{F}_2] = Y$ (since Y is $\mathcal{F}_2$-measurable)

c) If it can be shown that $\mathbb{E}[Z_{n+1}|\mathcal{F}_n] = Z_n$ for $n = 0, 1$, then $(Z_n)_{n\leq 2}$ is a martingale.
By tower property, we know
$\mathbb{E}[Z_{n+1}|\mathcal{F}_n] = \mathbb{E}[\mathbb{E}[Y|\mathcal{F}_{n+1}]|\mathcal{F}_n] = \mathbb{E}[Y|\mathcal{F}_n] = Z_n$ for $n = 0, 1$
therefore, $(Z_n)_{n\leq 2}$ is a martingale.

d) The probabilities are calculated as follows:
$\mathbb{P}(\{\omega_1, \omega_2\}|\Omega) = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}$
$\mathbb{P}(\{\omega_3, \omega_4\}|\Omega) = \frac{1}{6} + \frac{1}{2} = \frac{2}{3}$
$\mathbb{P}(\{\omega_1\}|\{\omega_1, \omega_2\}) = \frac{1}{9}/\frac{1}{3} = \frac{1}{3}$
$\mathbb{P}(\{\omega_2\}|\{\omega_1, \omega_2\}) = \frac{2}{9}/\frac{1}{3} = \frac{2}{3}$
$\mathbb{P}(\{\omega_3\}|\{\omega_3, \omega_4\}) = \frac{1}{6}/\frac{2}{3} = \frac{1}{4}$
$\mathbb{P}(\{\omega_4\}|\{\omega_3, \omega_4\}) = \frac{1}{2}/\frac{2}{3} = \frac{3}{4}$
Therefore, the tree diagram looks as follows:



e) If it can be shown that $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n$ for $n = 0, 1$, then $(X_n)_{n\leq 2}$ is a martingale.

Now,

$$\mathbb{E}[X_2|\mathcal{F}_1](\omega) = \begin{cases} \frac{1}{3} \times 36 + \frac{2}{3} \times 9 = 18 & \text{if } \omega \in \{\omega_1, \omega_2\} \\ \frac{1}{4} \times 12 + \frac{3}{4} \times 8 = 9 & \text{if } \omega \in \{\omega_3, \omega_4\} \end{cases}$$

So, $\mathbb{E}[X_2|\mathcal{F}_1](\omega) = X_1(\omega)$ for $\omega \in \Omega$

Similarly,
$\mathbb{E}[X_1|\mathcal{F}_0](\omega) = \frac{1}{3} \times 18 + \frac{2}{3} \times 9 = 12$ if $\omega \in \{\omega_1, \omega_2, \omega_3, \omega_4\}$
So, $\mathbb{E}[X_1|\mathcal{F}_0](\omega) = X_0(\omega)$ for $\omega \in \Omega$

Therefore, $(X_n)_{n\leq 2}$ is a martingale.

f) $\mathbb{E}[X_1^2|\mathcal{F}_0](\omega) = \frac{1}{3} \times 18^2 + \frac{2}{3} \times 9^2 = 162$
$X_0^2(\omega) = 12^2 = 144$
Therefore, $\mathbb{E}[X_1^2|\mathcal{F}_0](\omega) \geq X_0^2(\omega)$.

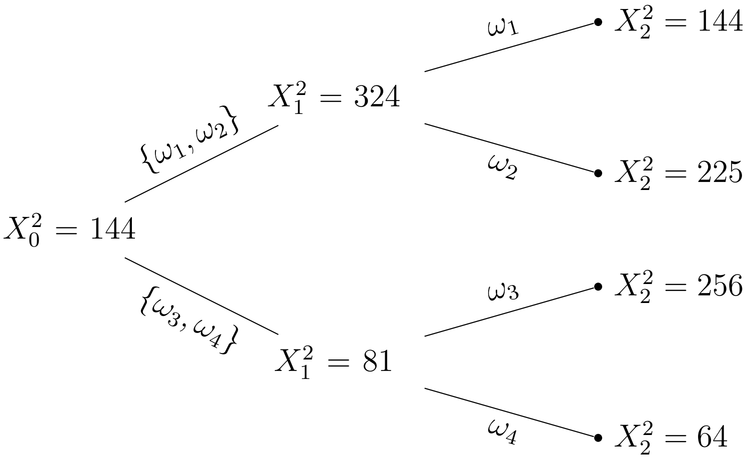
Now,

$$\mathbb{E}[X_2^2|\mathcal{F}_1](\omega) = \begin{cases} \frac{1}{3} \times 36^2 + \frac{2}{3} \times 9^2 = 486 & \text{if } \omega \in \{\omega_1, \omega_2\} \\ \frac{1}{4} \times 12^2 + \frac{3}{4} \times 8^2 = 84 & \text{if } \omega \in \{\omega_3, \omega_4\} \end{cases}$$

$$X_1^2(\omega) = \begin{cases} 18^2 = 324 & \text{if } \omega \in \{\omega_1, \omega_2\} \\ 9^2 = 81 & \text{if } \omega \in \{\omega_3, \omega_4\} \end{cases}$$

Therefore, $\mathbb{E}[X_2^2|\mathcal{F}_1](\omega) \geq X_1^2(\omega)$.
Therefore, $\mathbb{E}[X_n^2|\mathcal{F}_{n-1}](\omega) \geq X_{n-1}^2(\omega)$ for $n = 1, 2$.
Therefore, $(X_n^2)_n$ is a submartingale.

g) The tree diagram for $(X_n^2)_n$ looks as follows:



We can decompose X_n^2 as follows:
$X_n^2 = X_0^2 + M_n + A_n$
where $A_0 = 0$ and $A_n = A_{n-1} + \mathbb{E}[X_n^2|\mathcal{F}_{n-1}] - X_{n-1}^2$

Therefore,
$A_1 = 0 + 162 - 144 = 18$
$A_2 = A_1 + \mathbb{E}[X_2^2|\mathcal{F}_1] - X_1^2$

$$A_2 = \begin{cases} 18 + 486 - 324 = 180 & \text{if } \omega \in \{\omega_1, \omega_2\} \\ 18 + 84 - 81 = 21 & \text{if } \omega \in \{\omega_3, \omega_4\} \end{cases}$$

Since $M_n = X_n^2 - X_0^2 - A_n$ and $M_0 = 0$, it follows that
$M_1 = X_1^2 - X_0^2 - A_1$

$$M_1 = \begin{cases} 324 - 144 - 18 = 162 & \text{if } \omega \in \{\omega_1, \omega_2\} \\ 81 - 144 - 18 = -81 & \text{if } \omega \in \{\omega_3, \omega_4\} \end{cases}$$

and $M_2 = X_2^2 - X_0^2 - A_2$

$$M_2 = \begin{cases} 144 - 144 - 180 = -180 & \text{if } \omega = \omega_1 \\ 225 - 144 - 180 = -99 & \text{if } \omega = \omega_2 \\ 256 - 144 - 21 = 91 & \text{if } \omega = \omega_3 \\ 64 - 144 - 21 = -101 & \text{if } \omega = \omega_4 \end{cases}$$