Core of a Polytope and Rich Cells in Random Arrangement of Hyperplanes

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1 Introduction

These notes are the beginning of an analysis of the distribution of polytope facet complexity within a random hyperplane arrangement. The notion of the Core of a polytope is introduced as a new analysis tool. With this notion formalized, hopefully it can be used to show that the proportion of rich cells in a random hyperplane arrangement is low. NOTE: the region described below is actually not correctly characterized for polytopes for d ≥ 3 (ask Justin for more details)

2 Notation

First, some standard definitions as well as an overview of notation:

- Hyperplane h is defined as $\{x | \langle c, x \rangle + d = 0\}$ for $x \in \mathbb{R}^d$
- Halfspace H is similarly defined as $\{x | \langle c, x \rangle + d \leq 0\}$
- Polytope P is the intersection of n halfspaces. $P = \bigcap_{i=1}^{n} H_i$ or equivalently $P = \{x | \; C^\intercal x + \vec{d} \leq 0 \,\}$
- Vertex set $\mathcal{V}(P)$ denotes the set of vertices of polytope P.
- Vertex neighborhood $N(v)$ takes vertex v of a polytope and returns the 1-hop neighborhood of v on the 1-skeleton of the polytope.

3 Polytope Core

In words, the Core of a polytope $C(P)$ is a region such that any hyperplane with nonempty intersection with $C(P)$ is guaranteed to not increase the polytope's number of facets. Equivalently, such an intersecting hyperplane is guaranteed to not cleanly separate any vertex $v \in \mathcal{V}(P)$ of polytope P. A vertex v is considered cleanly separated by hyperplane h if v is strictly separated from every neighbor

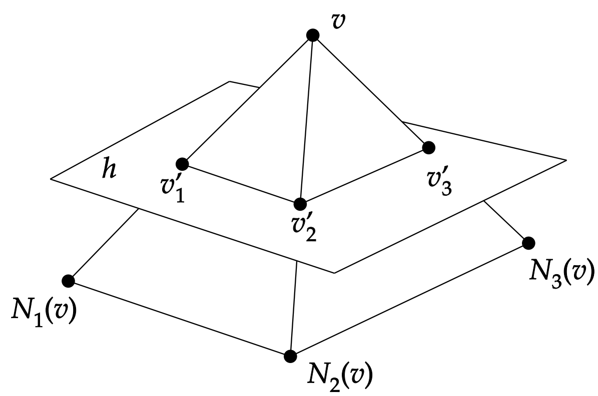

Figure 1: Example of a cleanly cut vertex in $\mathbb{R}^3$

$N_j \in N(v)$. More specifically, hyperplane $h = \{x | \langle c, x \rangle + d = 0\}$ cleanly separates v if it satisfies:

$$sign(\langle c, v \rangle + d) \neq sign(\langle c, N_j(v) \rangle + d) \quad \forall j$$
$$\langle c, v \rangle + d \neq 0, \ \langle c, N_j(v) \rangle + d \neq 0 \quad \forall j$$

To mathematically define $C(P)$, first consider a single vertex $v \in \mathcal{V}(P)$. One can define a region $R(v)$ such that any hyperplane with nonempty intersection with $R(v)$ is guaranteed to not cleanly separate v. To specify $R(v)$ consider its complement $R(v)^C$ which is the region that can be reached by the space of hyperplanes cleanly separating v. In general position, each vertex v has d neighbors (i.e. the cardinality $|N(v)| = d$). Thus, the space of hyperplanes cleanly separating v is characterized by d independent choices of points $\{v_1', v_2', ..., v_d'\}$ s.t:

$$v_j' = \alpha_j v + (1 - \alpha_j) N_j(v) \quad \forall j \tag{1}$$
$$\alpha_j \in (0, 1) \tag{2}$$

There are two regions, $S_1(v)$ and $S_2(v)$, which cannot be reached by the space of cleanly separating hyperplanes (Proof → section 5.1) (???). Let $\tilde{h}(v)_j \ \forall j \in \{1, 2, ..., d\}$ denote one of the d hyperplanes which pass through v. Then, let $\tilde{H}(v)_j$ denote the corresponding negative halfspace associated with each of these hyperplanes. Lastly, let $\hat{H}(v)$ denote the negative halfspace supported by the hyperplane which passes through the points $N(v)$ and whose normal vector has positive inner product with v, then we have:

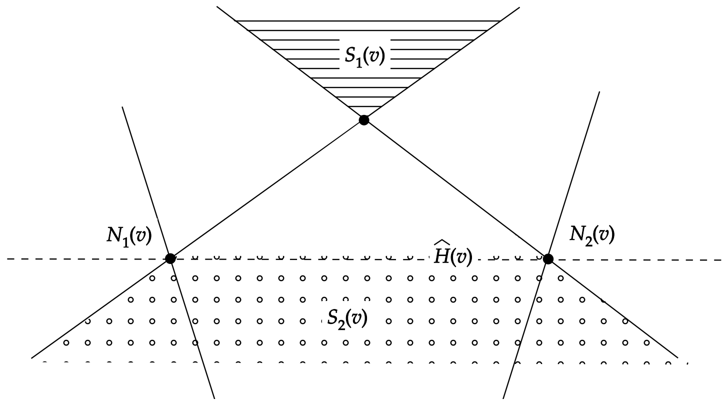

Figure 2: Example of $R(v)$ for a vertex of a polytope in $\mathbb{R}^2$

$$S_1(v) = \bigcap_{j=1}^{d} \tilde{H}(v)_j^C \tag{3}$$

$$S_2(v) = \{\bigcap_{j=1}^{d} \tilde{H}(v)_j\} \bigcap \hat{H}(v) \tag{4}$$

$$R(v) = S_1(v) \bigcup S_2(v) \tag{5}$$

Now, we have all of the components to define the Core of a polytope. For each vertex, we have defined $R(v)$ which is the region which guarantees non-clean separation for a single vertex. The Core of P is the region which possesses this property for all vertices, therefore:

$$\begin{aligned} C(P) = &\bigcap_{j=1}^{|\mathcal{V}(P)|} R(v_j) \\ &= \{\bigcap_{j=1}^{|\mathcal{V}(P)|} \hat{H}(v_j)\} \bigcap P \end{aligned} \tag{6}$$

The above equality in (6) holds as every non-redundant hyperplane of the polytope passes through at least one vertex and $S_2(v) \notin C(P)\ \forall\, v$ of P (Proof $\rightarrow$ section 5.2) (???). Thus, there is a simple mathematical expression for $C(P)$, it is the intersection of $|\mathcal{V}(P)|$ neighbor-defined halfspaces in turn intersected with the polytope P. Note, this rearrangement shows that $C(P) \subset P$.

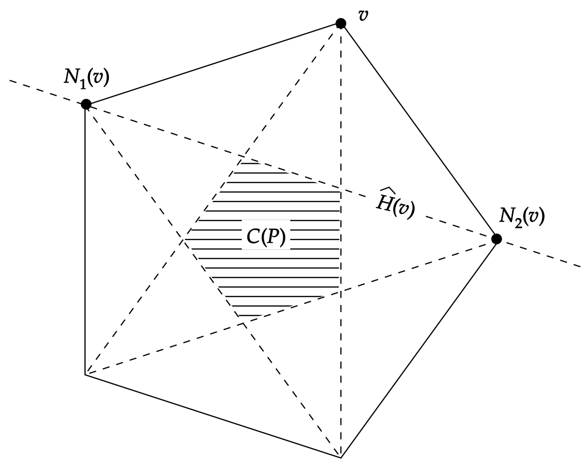

Figure 3: Example of $C(P)$ for a polytope in $\mathbb{R}^2$

4 Polytope Core Expansion

In this section, we begin to consider the evolution of $C(P)$ as the number of facets of P increases. Let P' represent the new polytope after increasing the facets of P by one. With Lemmas **??** and **??** we will prove Theorem 4.2, that is, the volume of the Core always increases with each added facet. To understand the change from $C(P)$ to $C(P')$, first we understand how the polytope changes. Note, the new facet added via new hyperplane h' cleanly separates v from P, thus:

$$\mathcal{V}(P') = \{\mathcal{V}(P) \setminus v\} \bigcup \{v_1', v_2', ..., v_d'\} \tag{7}$$

That is, the vertex v is removed and d new vertices are added. Note that these new vertices satisfy equations (1) and (2) (i.e. each is a strict convex combination of v and $N_j(v)$ for some j). Now, we consider the changes to the Core due to the vertex set change. The changes from $C(P)$ to $C(P')$ are as follows:

1) The halfspace $\hat{H}(v)$ is removed

2) d new halfspaces $\{\hat{H}(v_1'), ..., \hat{H}(v_d')\}$ are added

3) d old halfspaces $\{\hat{H}(N_1(v)), ..., \hat{H}(N_d(v))\}$ are modified

To show that $C(P) \subset C(P')$ first we show that $\hat{H}(v_i')$ is redundant to $R(v)$ $\forall\, i$. This takes care of changes 1) and 2). Then we show that $\hat{H}(N_j(v))'$

is redundant to $\hat{H}(N_j(v))\ \forall\, j$. This takes care of change 3). We start with the first redundancy claim, thus we wish to prove:

Lemma 4.1. *Given hyperplane h' which cleanly separates v, d new vertices are added to P, and thus d new hyperplanes $\hat{h}_i = \{x | \langle \hat{c}_i, x\rangle + \hat{d}_i = 0\}\ \forall\, i$ are added to $C(P)$. Furthermore, $\hat{h}_i$ is redundant to $R(v)\ \forall\, i$:*

$$\langle \hat{c}_i, x\rangle + \hat{d}_i \leq 0 \qquad \forall\, x \in R(v) \tag{8}$$

$$\forall\, i \tag{9}$$

Proof. □

Theorem 4.2. *Given P', the polytope which results from adding one new facet to an arbitrary polytope P, we have that: $C(P) \subset C(P')$*

5 Appendix

5.1 Proof of Expression of $R(v)$

5.2 Proof of Simplified Expression of $C(P)$

6 Random Notes

6.1 Minor Details

- perhaps somewhere it should be mentioned that the beginning version of the proofs assume general position everywhere and later we will consider the case outside of general position.

- Much of the discussion about the Core of a polytope runs into confusion when the polytope is unbounded. (E.g: for unbounded polytope not every vertex has d neighbors, thus how is $\hat{H}(v)$ defined?)

- I think that the boundaries of the definition of $R(v)$ might be improperly defined. (E.g: $S_1(v)$ is the complement plus the boundary?)

- Does "negative halfspace" include the boundary of the halfspace

- Figure 2 and Figure 3 should be changed from dashed lines to solid lines