Assignment 2

Jay Deshmukh

1) i) $\frac{dy}{dx} = -9y \qquad 0 \le x \le 1, \quad y(0) = e$

$$y_{i+1} = y_i + (-9y_{i+1})h$$

$$\Rightarrow y_{i+1}(1+9h) = y_i \Rightarrow \boxed{y_{i+1} = \frac{y_i}{1+9h}}$$

ii) $\frac{dy}{dx} = -20(y-x)^2 + 2x \qquad 0 \le x \le 1, \quad y(0) = 1/3$

$$\Rightarrow y_{i+1} = y_i + \left[-20(y_{i+1} - x_{i+1})^2 + 2x_{i+1}\right]h$$

$$\Rightarrow 20h\left(y_{i+1}^2 - 2y_{i+1}x_{i+1} + x_{i+1}^2\right) + y_{i+1} - y_i - 2hx_{i+1} = 0$$

$$\Rightarrow y_{i+1}^2 + y_{i+1}\left(\frac{1}{20h} - 2x_{i+1}\right) + x_{i+1}^2 - \frac{y_i}{20h} - \frac{x_{i+1}}{10}$$

$$\Rightarrow y_{i+1} = \frac{2x_{i+1} - \frac{1}{20h} \pm \sqrt{\left(\frac{1}{20h} - 2x_{i+1}\right)^2 - 4\left(x_{i+1}^2 - \frac{y_i}{20h} - \frac{x_{i+1}}{10}\right)}}{2}$$

$$= x_{i+1} - \frac{1}{40h} \pm \frac{1}{2}\sqrt{\frac{1}{400h^2} - \frac{x_{i+1}}{5h} + \frac{y_i}{5h} + \frac{4.x_{i+1}}{10}}$$

$$= x_{i+1} - \frac{1}{40h} \pm \frac{1}{40h}\sqrt{1 - 80h(x_{i+1} - y_i) + 160h^2 x_{i+1}}$$

$$\boxed{y_{i+1} = x_{i+1} - \frac{1}{40h}\left(1 - \sqrt{1 - 80h(x_{i+1} - y_i) + 160h^2 x_{i+1}}\right)}$$

↳ Determinant will become -ve for other sol^n

2)

$$\frac{dy}{dt} = \frac{y}{t} - \left(\frac{y}{t}\right)^2 \qquad 1 \le t \le 2 \quad ; \quad y(1) = 1$$

$h = 0.1$

Analytical solution - $y(t) = \dfrac{t}{1 + \ln t}$

4) Stiff differential equations are numerically unstable unless the step-size chosen is extremely small. Kinetic processes in chemistry are often described by stiff equations.

We use backward integration / implicit integration techniques to solve such equations.

scipy.integrate.ode can be used to solve stiff DEs as well.

3)

$$y'' - 2y' + y = xe^x - x$$

$$y' = z = f_1(y,z,x)$$
$$z' = 2z - y + xe^x - x = f_2(y,z,x)$$

Let $\begin{pmatrix} y \\ z \end{pmatrix} = Y$

$$\Rightarrow \quad \frac{d}{dx} Y = F(y,z,x) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \quad ; \quad Y(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Euler : $\quad Y_{i+1} = Y_i + F(Y_i, x_i)\,h$

RK4 :
$$K1 = h \cdot F(Y_i, x_i)$$
$$K2 = h \cdot F\left(Y_i + \frac{K1}{2}, x_i + h/2\right)$$
$$K3 = h \cdot F\left(Y_i + \frac{K2}{2}, x_i + h/2\right)$$
$$K4 = h \cdot F(Y_i + K3, x_i + h)$$
$$Y_{i+1} = Y_i + \frac{1}{6} \cdot (K1 + 2K2 + 2K3 + K4)$$

7) • $\frac{dy}{dt} = 1 + \frac{y}{t}$

$\frac{y}{t} = u \qquad y = tu \qquad \frac{dy}{dt} = u + t\frac{du}{dt}$

$\cancel{u} + t\frac{du}{dt} = 1 + \cancel{u} \Rightarrow \frac{du}{dt} = \frac{1}{t} \Rightarrow u = \ln t + C$

$\Rightarrow y = t\ln t + Ct$

$\frac{dy}{dt} = 1 + \ln t + C = 1 + \frac{y}{t}$

$y(1) = 2 \Rightarrow \boxed{y(t) = t(\ln t + 2)}$

• $\frac{dy}{dt} = te^{3t} - 2y \qquad y(0) = 0$

Let $y(t) = Bte^{3t} + Ce^{3t} + f(t)$ s.t $f'(t) = -2f(t)$

$\Rightarrow f(t) = ce^{-2t}$

$\Rightarrow 3\cancel{e^{3t}}(Bt + C) + \cancel{e^{3t}}B + 2\cancel{e^{3t}}(Bt + C) = t\cancel{e^{3t}}$

$\Rightarrow 5Bt + 5C + B = t \qquad \Rightarrow y(t) = \frac{e^{3t}}{5}\left(t - \frac{1}{5}\right) + ce^{-2t}$

$\Rightarrow B = \frac{1}{5}, \quad C = -\frac{1}{25} \qquad y(0) = 0 \Rightarrow c = \frac{1}{25}$

$\Rightarrow \boxed{y(t) = \frac{e^{3t}}{5}\left(t - \frac{1}{5}\right) + \frac{1}{25}e^{-2t}}$

- $\frac{dy}{dt} = \cos 2t + \sin 3t$

$\Rightarrow y(t) = \frac{1}{2}\sin 2t - \frac{1}{3}\cos 3t + C$

$y(0) = 1 \Rightarrow \boxed{y(t) = \frac{1}{2}\sin 2t - \frac{1}{3}\cos 3t + \frac{4}{3}}$

- $\frac{dy}{dt} = 1 - (t-y)^2$

$t - y = u \Rightarrow 1 - \frac{dy}{dt} = \frac{du}{dt}$

$\Rightarrow 1 - \frac{du}{dt} = 1 - u^2 \Rightarrow \frac{du}{dt} = u^2 \Rightarrow -\frac{1}{u} = t + C$

OR $u = 0$

$\Rightarrow y(t) = t + \frac{1}{t+C}$; $y(2) = 2 + \frac{1}{2+C} = 0 \Rightarrow \boxed{C = -\frac{5}{2}}$

$\Rightarrow \boxed{y(t) = t + \frac{1}{t - 5/2}}$

10)

$$\frac{dx}{dt} = \frac{1}{x^2+t^2} \qquad 0<t<\infty$$

$$x(0) = 1$$

We can perform a variable change for the interval $1<t<\infty$

$$u = \frac{1}{t} \Rightarrow \frac{du}{dt} = \frac{-1}{t^2}$$

$$\Rightarrow \frac{dx}{dt} = \frac{dx}{du}\frac{dk}{dt} = \frac{dx}{du}(-u^2) = \frac{1}{x^2+\frac{1}{u^2}}$$

$$\Rightarrow \frac{dx}{du} = \frac{-1}{u^2x^2+1} \qquad 0<u<1$$

We can determine $x(u=1)$ by solving the original differential equation in the region $0 \leq t \leq 1$

Once we have $u(1)$, we can evaluate the derivative at the point and use the RK4 algorithm in the backward direction. Since we know $u(1)$, we can do this explicitly. We will choose a suitable step size, depending on the maximum value of t we are interested in.

According to my program, value of $x(t)$ at $t = 3.5\times 10^6$ is 2.129896

12) The idea behind RK methods is to achieve higher orders of precision without having to evaluate the derivatives of the function. For RK, the error will be $O(h^7)$

$$f(x_0+h) = f(x_0) + h f^1(x)\Big|_{x_0} + \frac{h^2}{2} f^2(x)\Big|_{x_0} + \frac{h^3}{6} f^3(x)\Big|_{x_0} + \frac{h^4}{24} f^4(x)\Big|_{x_0}$$
$$+ \frac{h^5}{120} f^5(x)\Big|_{x_0} + \frac{h^6}{720} f^6(x)\Big|_{x_0} + O(h^7)$$

$$y(t_i + h) = y\left(t_i + \frac{h}{2}\right) + \frac{h}{2} \frac{dy}{dt}\Big|_{t_i+\frac{h}{2}} + \frac{1}{2}\left(\frac{h}{2}\right)^2 \frac{d^2y}{dt^2}\Big| + \frac{1}{3!}\left(\frac{h}{2}\right)^3 \frac{d^3y}{dt^3}\Big|$$

13)

$$t^2y'' - 2ty' + 2y = t^3 \ln t \qquad 1 \le t \le 2; \quad y(1) = 1, \ y'(1) = 0$$

$$y' = z \qquad\qquad f_1(y, z, t) = z$$

$$t^2 z' - 2tz + 2y = t^3 \ln t$$

$$\Rightarrow \ z' = \frac{2z}{t} - \frac{2y}{t^2} + t \ln t \qquad\qquad f_2(y, z, t) = \frac{2z}{t} - \frac{2y}{t^2} + t \ln t$$

$$Y = \begin{pmatrix} y \\ z \end{pmatrix} \Rightarrow \frac{d}{dt} Y = F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$Y_{i+1} = Y_i + F \cdot h \quad ; \quad Y_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$y(t) = \frac{7t}{4} + \frac{t^3}{2} \ln t - \frac{3t^3}{4}$$

$$y'(t) = \frac{7}{4} + \frac{3t^2}{2} \ln t + \frac{t^2}{2} - \frac{9t^2}{4} = \frac{7}{4} + \frac{3t^2}{2} \ln t - \frac{7t^2}{4}$$

$$y''(t) = 3t \ln t + \frac{3t}{2} + t - \frac{9t}{2}$$

$$= 3t \ln t - 2t$$

$$t^2y'' - 2ty' + 2y$$

$$= \cancel{3t^3 \ln t} - 2t^3 - 2t\left(\frac{7}{\cancel{4}} + \cancel{\frac{3t^2}{2} \ln t} - \frac{7t^2}{4}\right) + 2\left(\frac{7\cancel{t}}{\cancel{4}} + \frac{t^3}{2} \ln t - \frac{3t^3}{4}\right)$$

$$= -2t^3 + \frac{7t^3}{2} - \frac{3t^3}{2} + t^3 \ln t = t^3 \ln t$$

14) GSL functions for solving initial value problems,

gsl_odeiv2_step_rk4 $\rightsquigarrow$ Forward RK4

gsl_odeiv2_step_rk4imp $\rightsquigarrow$ Backward RK4

gsl_odeiv2_step_msadams $\rightsquigarrow$ Variable-coefficient linear multistep Adam's method in Nordsieck form

15)

$$y' = y - t^2 + 1 \quad ; \quad 0 \le t \le 2, \; y(0) = 0.5$$
$$= f(y,t) \qquad\qquad h = 0.2$$

$D \equiv \{(t,y) \mid t \in [0,2], \; y \in (-\infty, \infty)\} \rightsquigarrow$ convex

$$\frac{\partial f}{\partial y} = 1 \Rightarrow \left|\frac{\partial f}{\partial y}\right|\Big|_t \le 1 \quad \forall (t,y) \in D$$

$\Rightarrow f$ satisfies Lifshitz condition with $L = 1$

$$y(t) = (t+1)^2 - 0.5e^t$$
$$y'(t) = 2(t+1) - 0.5e^t$$
$$y''(t) = 2 - 0.5e^t$$

$$|y''(t)| = |2 - 0.5e^t| \le \underbrace{0.5e^2 - 2}_{M} \quad \forall t \in [0, 2]$$

$$\Rightarrow |y_i - w_i| \le \frac{h}{2}\frac{M}{L}\left[e^{L(t_i - a)} - 1\right] \le \frac{0.5e^2 - 2}{10}\left[e^{t_i} - 1\right]$$