Analytical Inverse Kinematics for the manipulator with UR-like configuration

SOFT SERVO

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Contents

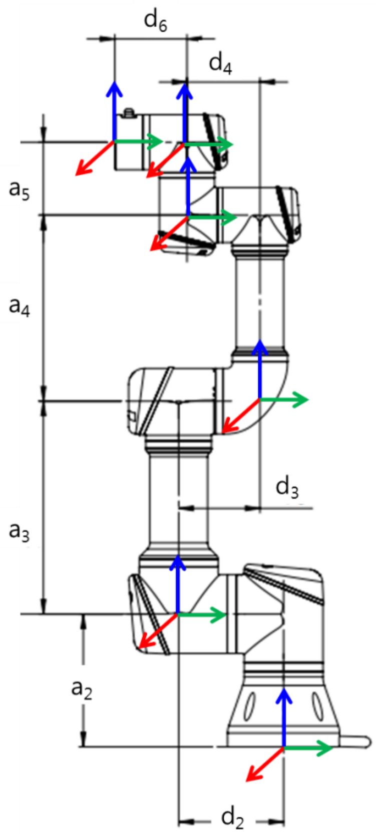
1 Introduction

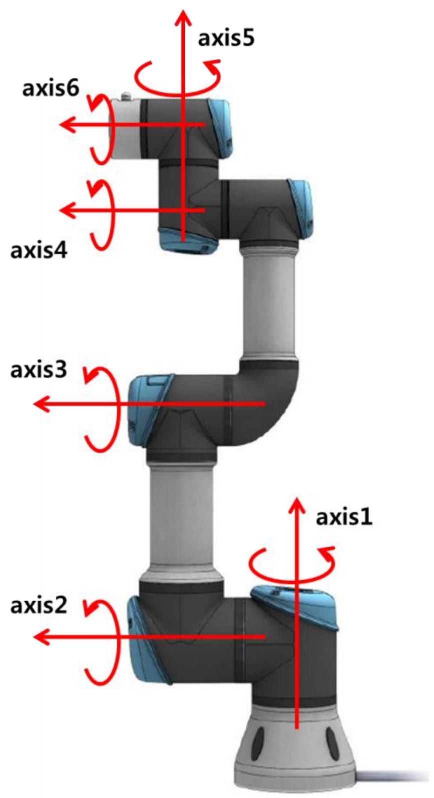
This document includes how to derive inverse kinematics of UR type manipulator. This IK is tested on MATLAB. The MATLAB code is uploaded into github. Due to DH notation's disadvantage, homogeneous transformation which is a common transformation method is used for the manipulator's coordinate system.

2 Forward Kinematics

2.1 Coordinate

The below figure shows the coordinate system and joint axis we used. The red arrows are x-axis, the arrows are y-axis, and the blue arrows are z-axis. When the 1st joint angle is zero, the 1st coordinate on the 1st joint is the same as the world coordinate.





2.2 Link Parameters

The link parameters are defined as the followed table according to the above figure. Unlike D-H notation, there are more variables but they are more intuitive than those of D-H notation. The zero values in the table are not considered when deriving inverse kinematics.

Joint Number	Pose From Previous			Joint Axis
	x	y	z	
1	0	0	0	Z-Axis
2	0	d_2	a_2	-Y-Axis
3	0	d_3	a_3	-Y-Axis
4	0	d_4	a_4	-Y-Axis
5	0	0	a_5	Z-Axis
6	0	d_6	0	-Y-Axis

2.3 Forward Kinematics

The following formula is used to calculate forward kinematics. When calculating forward kinematics, following formula is called 6 times. t_0 is $[0 \quad 0 \quad 0]$, and R_0 is identity matrix.

$$
\begin{aligned}
{}^0t_i &= {}^0t_{i-1} + {}^0R_{i-1}{}^{i-1}t_i \\
{}^0R_i &= {}^0R_{i-1}\,{}^{i-1}R_i
\end{aligned}
$$

3 Inverse Kinematics

The following variables will be used to calculate inverse kinematics.

$$R_d = \begin{pmatrix} x_{d,x} & y_{d,x} & z_{d,x} \\ x_{d,y} & y_{d,y} & z_{d,y} \\ x_{d,z} & y_{d,z} & z_{d,z} \end{pmatrix} \text{ : the desired rotation of end-effector}$$

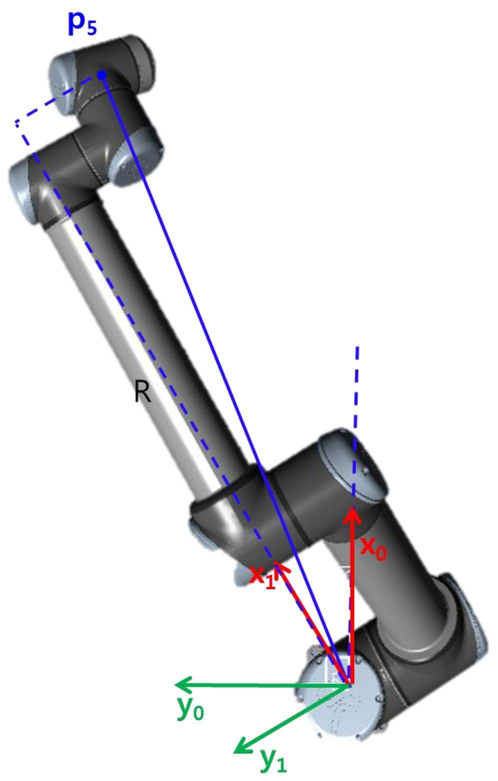
$$t_d = \begin{pmatrix} X_d \\ Y_d \\ Z_d \end{pmatrix} \text{ : the desired positon of end-effector}$$

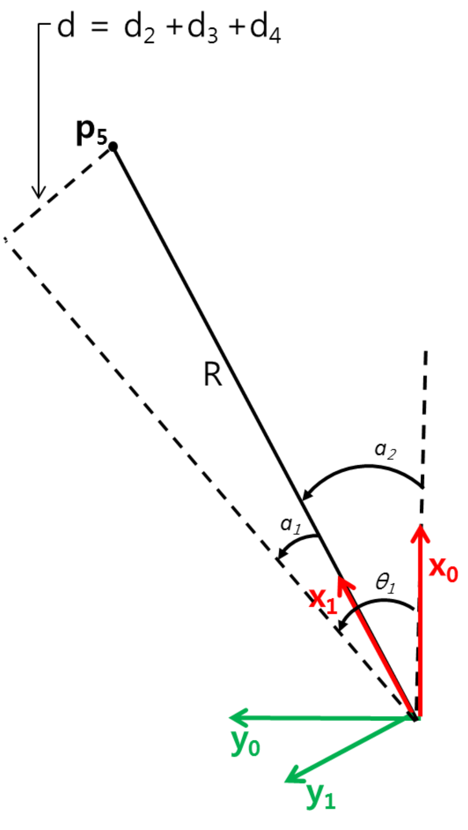
3.1 Get θ_1

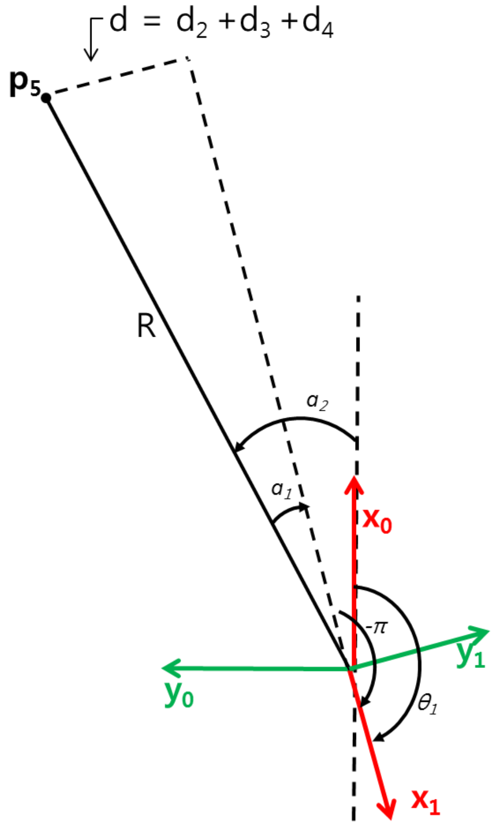
The 1st angle can be found by using 5th joint position. We can find 5th joins position like as follows.

$$p_5 = p_d + R_d \begin{bmatrix} 0 \\ -d_6 \\ 0 \end{bmatrix}$$

The below figures illustrate how to find the 1st joint angle geometrically.







We can derive 1st joint angle from the below equations.

$$\theta_1 = \alpha_1 + \alpha_2 = \mathrm{atan2}\left(p_{5,y}, p_{5,x}\right) + \mathrm{asin}(|d|/R)$$
$$\theta_1 = \alpha_1 + \alpha_2 - \pi = \mathrm{atan2}\left(p_{5,y}, p_{5,x}\right) - \mathrm{asin}(|d|/R) - \pi$$

The first solution is the left of the above figure and the second solution is the right of the above figure.

3.2 Get θ_5

The fifth angle can be found by using the 1st joint angle we already know. Also, it is used that the y-axis of fifth coordinate is the same as the y-axis of end-effector's coordinate. Therefore, we can derive the below equations.

$$\left({}^{1}R_5 \right)_y = \left({}^{1}R_6 \right)_y$$
$$\left(R_y(\theta_2)R_y(\theta_3)R_y(\theta_4)R_z(\theta_5)\right)_y = \left(R_y\,(-\theta_1)R_d\right)_y$$

Then,

$$\begin{pmatrix} c_{234}s_5 \\ c_5 \\ s_{234}s_5 \end{pmatrix} = \begin{pmatrix} y_{d,x}c_1 + y_{d,y}s_1 \\ -y_{d,x}s_1 + y_{d,y}c_1 \\ y_z \end{pmatrix}$$

We do not know θ_2, θ_3, and θ_4. This means that we can use only the second elements of the above equation.

Thus, θ_5 can be calculated by the below equation.

$$\theta_5 = \pm\mathrm{acos}(-y_{d,x}s_1 + y_{d,y}c_1)$$

3.3 Get θ_6

To get θ_6, ${}^{6}y_4$ which is the fifth y-axis with respect to the end-effector's coordinate is used. We can derive the below equations.

$$\left({}^{6}R_4 \right)_y = \left(R_y(-\theta_6)R_z(-\theta_5)\right)_y = \left(R_d^T R_z(\theta_1)R_y(\theta_2)R_y(\theta_3)R_y(\theta_4)\right)_y$$

Then,

$$\begin{pmatrix} c_6 s_5 \\ c_5 \\ s_6 s_5 \end{pmatrix} = \begin{pmatrix} -x_{d,x}s_1 + x_{d,y}c_1 \\ -y_{d,x}s_1 + y_{d,y}c_1 \\ -z_{d,x}s_1 + z_{d,y}c_1 \end{pmatrix}$$

Then,

$$c_6 = \frac{-x_{d,x}s_1 + x_{d,y}c_1}{s_5}, \quad s_6 = \frac{-z_{d,x}s_1 + z_{d,y}c_1}{s_5}$$

Therefore,

$$\theta_6 = \text{atan2}(s_6, c_6)$$

If θ_5 is zero, s_5 is zero. This derives infinite solutions for θ_6. This is because 2nd, 3rd, 4th and 6th axes are the same joint axis when θ_5 is zero. In this case, we have to set the 6th or 4th joint angle as arbitrary value we previously defined. In this document and my MATLAB code, a predefined 6th joint angle is used for this case.

3.4 Get θ_2 and θ_3

The 2nd, 3rd, and 4th joint angle can be found easily because it forms classical 3R planar arm. To get θ_2 and θ_3, the position of the 4th joint is used. The 4th joint position with respect to 2nd coordinate can be found by the below equations.

$$p_4 = t_d - d_6 \begin{pmatrix} y_{d,x} \\ y_{d,y} \\ y_{d,z} \end{pmatrix} - a_5(R_5)_z$$

where,

$$R_5 = R_d R_y(-\theta_6)$$

Then,

$$^1p_4 = R_z(-\theta_1)p_4$$

Next step is the same as inverse kinematics for 2-link manipulator. There are two solutions for pair of θ_2 and θ_3 as the same as 2-link manipulator. We set the upright positon as the robot's pose at zero angle, so θ_2 and θ_3 can be found by using the below equations.

The first pair of solutions is

$$\theta_{3,1} = \pi - \text{acos}\left(\frac{a_3{}^2 + a_4{}^2 - \left({}^1p_{4,x}{}^2 + {}^1p_{4,z}{}^2\right)}{2a_3a_4}\right)$$

$$\theta_{2,1} = \frac{\pi}{2} - \text{atan2}\left({}^1p_{4,x}, {}^1p_{4,z}\right) + \text{atan2}(a_4\sin(-\theta_{3,1}), a_3 + a_4\cos(-\theta_{3,1}))$$

The second pair of solution is

$$\theta_{3,2} = -\pi + \cos(\frac{{a_3}^2 + {a_4}^2 - ({^1p_{4,x}}^2 + {^1p_{4,z}}^2)}{2a_3a_4})$$
$$\theta_{2,2} = \frac{\pi}{2} - \mathrm{atan2}({^1p_{4,x}}, {^1p_{4,z}}) + \mathrm{atan2}(a_4\sin(-\theta_{3,2}), a_3 + a_4\cos(-\theta_{3,2}))$$

3.5 Get θ_4

We already know five joints angles. The 4th joint angle can be found very easily by using other angles. We can get the 4th angle by using the below equations.

$$^3R_4 = R_y(-\theta_3)R_y(-\theta_2)R_z(-\theta_1)R_dR_y(-\theta_6)R_z(-\theta_5)$$

Then,

$$\theta_4 = atan2(\,^3R_4(1,3),\ ^3R_4(1,1))$$

3.6 The number of solutions

The number of solutions is 8. This is because we have 2 solutions for θ_1, 2 solutions for θ_5, and 2 pairs of solutions for θ_2 and θ_3.