TAUTOLOGY

TAUTOLOGY: A statement which is always true is known **as Tautology**. It is denoted by t. It is also known as logical truth.

There are two methods to check whether the given proposition is a Tautology.

1. Using truth table method.
2. without using truth table.(using Identities or algebra of proposition)

Example: Show $\left(P \rightarrow (Q \rightarrow R)\right) \rightarrow \left((P \rightarrow Q) \rightarrow (P \rightarrow R)\right)$

Solution: Let $\alpha = \left(P \rightarrow (Q \rightarrow R)\right) \rightarrow \left((P \rightarrow Q) \rightarrow (P \rightarrow R)\right)$

P	Q	R	$Q \rightarrow R$	$\left(P \rightarrow (Q \rightarrow R)\right)$	$P \rightarrow Q$	$P \rightarrow R$	$(P \rightarrow Q) \rightarrow (P \rightarrow R)$	α
T	T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F	T
T	F	T	T	T	F	T	T	T
T	F	F	T	T	F	F	T	T
F	T	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

Since the truth value is true for all possible values of the propositional variables which can be seen in the last column of the table, the given proposition is a tautology.

Example: Using identities prove that it is a Tautology:

$$\boldsymbol{Q} \vee (\boldsymbol{P} \wedge \sim\boldsymbol{Q}) \vee (\sim\boldsymbol{P} \wedge \sim\boldsymbol{Q})$$

Solution:

$$Q \vee (P \wedge \sim Q) \vee (\sim P \wedge \sim Q) \equiv [(Q \vee P) \wedge (Q \vee \sim Q)] \vee (\sim P \wedge \sim Q)$$

(Distributive Law)

$$\equiv [(Q \vee P) \wedge T] \vee (\sim P \wedge \sim Q)$$

(Complement Law)

$$\equiv [(Q \vee P)] \vee (\sim P \wedge \sim Q)$$

(Identity Law)

$$\equiv [(Q \vee P)] \vee \sim(P \vee Q)$$

(Commutative Law)

$$\equiv [(Q \vee P)] \vee \sim(Q \vee P)$$

$$\equiv T$$

<u>CONTRADICATION</u>: A statement which is always false is known **as contradiction** or **Fallacy**. It is denoted by c or F. It is also known as Contradiction.

<u>Example</u>: Verify that the proposition $p \wedge (q \wedge \sim p)$ is a contradiction.

<u>Solution:</u>

p	q	$\sim p$	$(q \wedge \sim p)$	$p \wedge (q \wedge \sim p)$
T	T	F	F	F
T	F	F	F	F
F	T	T	T	F
F	F	T	F	F

It is clear from the truth table that for any possible value of p and q , the given proposition is false which establish that given proposition is a contradiction.

PRACTICE EXAMPLE

Example: **Show that** $\left(p \vee q\right) \wedge \left(\sim p \wedge \sim \mathrm{q}\right)$ **is a contradiction.**

CONTINGENCY: A statement which is neither tautology nor contradiction (fallacy) is known **as Contingency**.

FUNCTIONALLY COMPLETE SET OF CONNECTIVES:

Any set of connective in which every formula can be expressed in terms of an equivalence formula containing the connectives from the set is called functionally complete set of connectives.

Remark: A minimal functionally complete **set of Connectives does not contain a connective** which can be expressed in terms of other connectives.

<u>Result</u>: Prove that {∧, ~} and {∨, ~} is a Minimal functionally complete set of connectives.

Proof: We five basic connectives {∧, ∨, →, ↔, ~ }.

To eliminate the conditional, one uses the following logical equivalence:

$$p \rightarrow q \equiv \sim p \vee q ----------- (1)$$

There are two ways to express the biconditional as given by

$$p \leftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q), ---------- (2)$$

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p). ----------- (3)$$

Using (1) in (3), we get

$$p \leftrightarrow q \equiv (\sim p \vee q) \wedge (\sim q \vee p).$$

Thus all the conditional and biconditional can be replaced by the three connectives ∧, ∨ and ~ .

Again note that from De Morgan's Law , we have

$$p \wedge q \equiv \sim (\sim p \vee \sim q),$$

$$p \vee q \equiv \sim (\sim p \wedge \sim q).$$

The first equivalence mans that it is also possible to obtain a formula which is equivalent to a given formula in which conjunction is eliminated. Similarly, it is possible to remove disjunction.

Thus, we can replace first all the biconditional, then the conditional and finally all the conjunctions or all the disjunctions to obtain equivalent formula. This alternative form contain either the negation and disjunction or the negation and conjunction. Thus that set of connectives $\{\wedge, \sim\}$ and $\{\vee, \sim\}$ is a Minimal functionally complete set of connectives.

NORMAL FORM: By comparing truth tables, one determines whether two logical expression P and Q are equivalent. But the process is very tedious when the number of variables increases. A better method is to transform the expression P and Q to some standard forms of expression P' and Q' such that a simple comparison of P' and Q' shows whether.

The standard forms are called normal forms or canonical forms.

ELEMENTARY SUM: In logical expression, a sum of the variables and their negation is called elementary sum. Example of elementary sum : $p, \ \sim p, \ p \vee q, \ p \vee q, \ \sim p \vee q, \ p \vee \sim q.$

ELEMENTARY PRODUCT: In logical expression, a product of variables and their negation is called an elementary product. Example: p, $\sim p, p \wedge q$, $p \wedge q$, $\sim p \wedge q$, $p \wedge \sim q$, $p \wedge q \wedge \sim r$ is elementary products in two variables p and q.

Disjunctive Normal Form (DNF): A logical expression is said to be in disjunctive normal form if it is the **sum of elementary products**.

PROCEDURE TO OBTAIN DNF OF A GIVEN LOGICAL EXPRESSION:

1) Remove Conditional and Bi conditional using an equivalent expression which contains negation, disjunction and conjunction only.

2) Eliminate negation before the sum and products using De Morgan's Law.

3) Apply distributive law until a sum of elementary product is obtained.

EXAMPLE: Obtain the disjunctive normal form of the following:

1) $p \wedge (p \rightarrow q) \equiv p \wedge (\sim p \vee q)$

Solution:

Step 1: Remove by logically equivalent statement

$$p \wedge (p \rightarrow q)$$

Step 2: Apply distributive law

$$p \wedge (p \rightarrow q) \equiv (p \wedge \sim p) \vee (p \wedge q)$$

which is the required disjunctive normal form (DNF).

2) $p \vee \left(\sim p \to \left(q \vee \left(q \to \sim r \right) \right) \right)$

Solution:

$$
\begin{aligned}
p \vee \left(\sim p \to \left(q \vee \left(q \to \sim r \right) \right) \right) &\equiv p \vee \left(\sim p \to \left(q \vee \left(\sim q \vee \sim r \right) \right) \right) \\
&\equiv p \vee \left(p \vee \left(q \vee \left(\sim q \vee \sim r \right) \right) \right) \\
&\equiv p \vee p \vee q \vee \sim q \vee \sim r \\
&\equiv p \vee q \vee \sim q \vee \sim r
\end{aligned}
$$

$3)\ \left(p\wedge\sim(q\wedge r)\right)\vee(p\to q)$

<u>Solution:</u>

$$\left(p\wedge\sim(q\wedge r)\right)\vee(p\to q)\equiv\left(p\wedge\sim(q\wedge r)\right)\vee(\sim p\vee q)$$

$$(Remove\ by\ \to\ logically\ equivalent\ statement)$$

$$\equiv\left(p\wedge(\sim q\vee\sim r)\right)\vee(\sim p\vee q)(\ De\ Morgan\text{'}s\ Law)$$

$$\equiv\left((p\wedge\sim q)\vee(p\wedge\sim r)\right)\vee(\sim p\vee q)(Distributive\ law)$$

which is the required disjunctive normal form.

<u>Remark</u>: Disjunctive normal form (DNF)of logical expression is not unique.

<u>CONJUNCTIVE NORMAL FORM (CNF)</u>: A logical expression is said to be in disjunctive normal form if it is the **product of elementary sum**.

<u>PROCEDURE TO OBTAIN CNF OF A GIVEN LOGICAL EXPRESSION</u>:

1) Remove Conditional and Bi conditional using an equivalent expression which contains negation, disjunction and conjunction only.

2) Eliminate negation before the sum and products using De Morgan's Law.

3) Apply distributive law until a sum of elementary product is obtained.

EXAMPLE: Obtain the conjunctive normal form of the following:

1) $p \wedge (p \to q)$

Solution:

$$p \wedge (p \to q) \equiv p \wedge (\sim p \vee q).$$

(*Remove conditional by logical connective*)

2) $\left[q\vee(p\wedge q)\right]\wedge\sim\left[(p\vee r)\wedge q\right]$

<u>Solution</u>:

$$\left[q\vee(p\wedge r)\right]\wedge\sim\left[(p\vee r)\wedge q\right]\equiv\left[q\vee(p\wedge r)\right]\wedge\left[\sim(p\vee r)\vee\sim q\right]$$

$$(\,De\ Morgan\text{'}s\ Law)$$

$$\equiv\left[q\vee(p\wedge r)\right]\wedge\left[(\sim p\wedge\sim r)\vee\sim q\right]$$

$$(\,De\ Morgan\text{'}s\ Law)$$

$$\equiv(q\vee p)\wedge(q\vee r)\wedge(\sim p\vee\sim q)\wedge(\sim r\vee\sim q),$$

Which is required CNF.

Remark: Conjunctive normal form (CNF) of logical expression is not unique.

Point to remember for DNF and CNF

1) $p \to q \equiv \sim p \vee q$

2) $p \leftrightarrow q \equiv (\sim p \vee q) \wedge (p \vee \sim q) \quad OR \quad p \leftrightarrow q \equiv (p \wedge q) \vee (p \wedge q)$

3) *De Morgan's Law*

$$\sim (p \vee q) \equiv \sim p \wedge \sim q$$

$$\sim (p \wedge q) \equiv \sim p \vee \sim q$$

4) *Distributive Law*

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$