PRINCIPAL DISJUNCTIVE NORMAL FORM (PDNF)

MINTERMS: Let p and q be two statement variables then are $p \wedge q$, $\sim p \wedge q$, $p \wedge \sim q$, $\sim p \wedge \sim q$ **called minterms of p and q.**

The number of minterms in **n** variables **is 2^n.**

For example the minterms for three variables p , q and r are

$p \wedge q \wedge r$, $\sim p \wedge q \wedge r$, $p \wedge \sim q \wedge r$, $p \wedge q \wedge \sim r$, $\sim p \wedge \sim q \wedge r$,

$p \wedge \sim q \wedge \sim r$, $\sim p \wedge q \wedge \sim r$, $\sim p \wedge \sim q \wedge \sim r$.

The truth table for the minterms of p and q are given below:

p	**q**	$p \wedge q$	$\sim p \wedge q$	$p \wedge \sim q$	$\sim p \wedge \sim q$
T	T	T	F	F	F
T	F	F	F	T	F
F	T	F	T	F	F
F	F	F	F	F	T

REMARKS:

1) From the truth table it is clear that no minterms are equivalent.
2) Each minterms has truth value T for exactly one combination of the truth values of the variables p and q.

Principal Disjunctive Norm Form (PDNF): PDNF of a given formula can be defined as an equivalent formula consisting of disjunctives of minterms (elementary product which contains the entire variables) only. This is also known as sum of products canonical form. There are two ways to obtain the principal disjunctive normal form.

Method 1: Using Truth Table.

Method 2: Without Using Truth Table.

Method 1: Using Truth Table.

Step 1: Construct a truth table for the given compound statement.

Step 2: For every truth value T of the given proposition, select the minterms which has also true value T for the same combination of truth value of the statement variable.

Step 3: The disjunctive of minterms selected in step 2 is the required principal disjunctive normal form.

Remark: In PDNF, the repeated minterms are mention only once.

Example: Obtain the principal disjunctive normal form using the truth table of $p \rightarrow q$.

Solution: The truth table $p \rightarrow q$ for is given by

p	**q**	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

The column containing $p \to q$ has truth value T for three combination of the truth values of p and q.

T in the first row of $p \to q$ corresponds to the minterms

$p \wedge q$, (**refer TABLE: Minterms**)

T in the third row of $p \to q$ corresponds to the minterms

$\sim p \wedge q$, (**refer TABLE: Minterms**)

T in the fourth row of $p \to q$ corresponds to the minterms

$\sim p \wedge \sim q$, (**refer TABLE: Minterms**)

Thus, PDNF of $p \to q$ is

$$(p \wedge q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q)$$

Method 2: Without Using Truth Table.

Step 1: Obtain the disjunctive normal form.

Step 2: Drop elementary products which are contradictions.

Step 3: If p_i and $\sim p_i$ are missing in an elementary product , replace α by $(\alpha \wedge p_i) \vee (\alpha \wedge \sim p_i)$.

Step 4: Repeat step 3 until all elementary products are reduced to sum of minterms. Identical minterms appearing in the disjunction are deleted.

<u>POINTS TO BE REMEMBER FOR PDNF WITHOUT USING TRUTH TABLE</u>

After finding DNF, we apply identity law, complement law and Distributive Law.

Identity law for DNF is $q \equiv (q \wedge T)$.

Complement Law is $\sim q \vee q \equiv T$.

Example: Obtain the Principal Disjunctive Normal Form of the following without using the truth table.

1) $p \to q$

Solution: From DNF, We have $p \to q \equiv \sim p \vee q$.

$\sim p \equiv (\sim \mathrm{p} \wedge T)$ $(Identity\ Law)$

$\equiv (\sim p \wedge (\sim q \vee q))$ $(\ Complement\ Law)$

$\equiv ((\sim p \wedge \sim q) \vee (\sim \mathrm{p} \wedge q))$ $(Distributive\ Law)$

$q \equiv (q \wedge T)$ $(\ Identity\ Law)$

$\equiv (q \wedge (\sim p \vee p))$ $(\ Complement\ Law)$

$\equiv ((q \wedge \sim p) \vee (q \wedge p))$ $(\ Distributive\ Law)$

Thus, the required PDNF is

$$p \to q \equiv\equiv ((\sim p \wedge \sim q) \vee (\sim \mathrm{p} \wedge q)) \vee ((q \wedge \sim p) \vee (q \wedge p)).$$

2) $q \vee (p \vee \sim q)$

Solution:

$$q \vee (p \vee \sim q) \equiv (q \wedge T) \vee [(p \wedge T) \vee (\sim q \wedge T)] \; (\textit{Identity Law})$$

$$\equiv (q \wedge (\sim p \vee p)) \vee [(p \wedge (\sim q \vee q)) \vee (\sim q \wedge (\sim p \vee p))]$$

$$(\textit{Complement Law})$$

$$\equiv [(q \wedge \sim p) \vee (q \wedge p)] \vee [(p \wedge \sim q) \vee (p \wedge q) \vee (\sim q \wedge \sim p) \vee (\sim q \wedge p)]$$

$$(\textit{Distributive Law})$$

$$\equiv (q \wedge \sim p) \vee [(p \wedge q) \vee (\sim q \wedge \sim p) \vee (\sim q \wedge p)]$$

(In PDNF each minterms appear once so we mention the repeated minterms only once.)

3) $\left(\sim p \wedge q \wedge \sim r\right) \vee \left(q \wedge r\right)$

Solution:

In the given formula $\sim p \wedge q \wedge \sim r$ is already a minterm, so we need to only convert $\left(q \wedge r\right)$ into minterms.

Now,

$$\left(q \wedge r\right) \equiv \left(q \wedge r\right) \wedge T \; \left(Identity\ Law\right)$$

$$\equiv \left(q \wedge r\right) \wedge \left(\sim p \vee p\right) \; \left(Complement\ Law\right)$$

$$\left(q \wedge r\right) \equiv \left[\left(q \wedge r\right) \wedge \sim p\right] \vee \left[\left(q \wedge r\right) \wedge p\right]$$

$$\left(Distributive\ Law\right) \qquad \left(1\right)$$

The required PDNF form using (1) is

$$\left(\sim p \wedge q \wedge \sim r\right) \vee \left[\left(q \wedge r\right) \wedge \sim p\right] \vee \left[\left(q \wedge r\right) \wedge p\right].$$

4) $(p \wedge q) \vee (\sim p \wedge r) \vee (q \wedge r)$

Solution:

$(p \wedge q) \vee (\sim p \wedge r) \vee (q \wedge r)$ is in DNF.

Now we convert it into PDNF form as follows:

In order to find the PDNF we first find the minterms for 1st term $(p \wedge q)$ (since minterms contain entire variable so we introduce the missing variable r in the 1st term using identity, complement and distributive law)

$$(p \wedge q) \equiv (p \wedge q) \wedge T \,(\,Identity\ Law)$$

$$\equiv (p \wedge q) \wedge (\sim r \vee r) (\,Complement\ Law)$$

$$\equiv [(p \wedge q) \wedge \sim r] \vee [(p \wedge q) \wedge r] (\,Distributive\ Law)$$

Similarly $(\sim p \wedge r)$ and $(q \wedge r)$ can be converted into following minterms

$$\sim p \wedge r \equiv (\sim p \wedge r \wedge \sim q) \vee (\sim p \wedge r \wedge q)$$

$$(q \wedge r) \equiv (q \wedge r \wedge p) \vee (q \wedge r \wedge \sim p).$$

Therefore PDNF form of $(p \wedge q) \vee (\sim p \wedge r) \vee (q \wedge r)$ is

$$(p \wedge q \wedge \sim r) \vee (\sim p \wedge r \wedge q) \vee (q \wedge r \wedge p) \vee (q \wedge r \wedge \sim p).$$

5) $p \leftrightarrow q$

Solution:

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

$$(Logical\ equivalent\ statement\ of\ biconditional)$$

$$\equiv (\sim p \vee q) \wedge (\sim q \vee p)$$

$$(Logical\ equivalent\ statement\ of\ conditional\ statement)$$

$$\equiv \left[(\sim p \vee q) \wedge \sim q\right] \vee \left[(\sim p \vee q) \wedge p\right] (Distributive\ Law)$$

$$\equiv (\sim p \wedge \sim q) \vee (q \wedge \sim q) \vee (\sim p \wedge p) \vee (q \wedge p) (Distributive\ Law)$$

$$\equiv (\sim p \wedge \sim q) \vee (q \wedge p),$$

Which is required PDNF.

6) $p \vee (\sim p \to (\sim q \to r))$

Solution:

$$p \vee (\sim p \to (\sim q \to r))$$

$$\equiv p \vee (\sim p \to (q \vee r))(Remove\ conditional\ using\ equivalentstatement)$$

$$\equiv p \vee (p \vee (q \vee r))(Remove\ conditional\ using\ equivalent\ statement)$$

$$\equiv (p \vee p) \vee q \vee r\ (Associative\ Law)$$

$$\equiv p \vee q \vee r (Idempotent\ Law)$$

$$DNF\ FORM\ \ (Now\ we\ convert\ it\ into\ PDNF\ form)$$

(Remark: PDNF contain minterms and, minterms include entire variable of compound statement so we introduce the missing terms using identity and complement law in 1st term p , 2nd term q and 3rd r)

$$p\vee q\vee r\equiv\left[\left(p\wedge T\right)\wedge T\right]\vee\left[\left(q\wedge T\right)\wedge T\right]\vee\left[\left(r\wedge T\right)\wedge T\right]$$

$$\left(\textit{Identity Law}\right)$$

$$\equiv\left[\left(p\wedge(\sim q\vee q)\right)\wedge(\sim r\vee r)\right]\vee\left[\left(q\wedge(\sim p\vee p)\right)\wedge(\sim \mathrm{r}\vee r)\right]$$

$$\vee\left[\left(r\wedge(\sim p\vee p)\right)\wedge(\sim q\vee q)\right]$$

$$\left(\textit{Complement Law}\right)$$

Now we simplify 1st bracket $\left(p\wedge(\sim q\vee q)\right)\wedge(\sim r\vee r)$ using distributive law we get

$$\left[\left(p\wedge(\sim q\vee q)\right)\wedge(\sim r\vee r)\right]\equiv\left[\left(p\wedge\sim q\right)\vee\left(p\wedge q\right)\right]\wedge(\sim r\vee r)$$

$$\equiv\left[\left(p\wedge\sim q\right)\wedge(\sim r\vee r)\right]\vee\left[\left(p\wedge q\right)\wedge(\sim r\vee r)\right]$$

$$\left[\left(p\wedge(\sim q\vee q)\right)\wedge(\sim r\vee r)\right]\equiv\left[\left(p\wedge\sim q\wedge\sim r\right)\vee\left(p\wedge\sim q\wedge r\right)\right]$$

$$\vee\left[\left(p\wedge q\wedge\sim r\right)\vee\left(p\wedge q\wedge r\right)\right]$$

Similarly we can simplify 2nd $\left[\left(q\wedge(\sim p\vee p)\right)\wedge(\sim \mathrm{r}\vee r)\right]$ and 3rd bracket $\left[\left(r\wedge(\sim p\vee p)\right)\wedge(\sim q\vee q)\right]$ are given by

$$\left[\left(q\wedge(\sim p\vee p)\right)\wedge(\sim \mathrm{r}\vee r)\right]\equiv(\sim p\wedge q\wedge \sim r)\vee(\sim p\wedge q\wedge r)$$
$$\vee(p\wedge q\wedge \sim r)\vee(p\wedge q\wedge r)$$

$$\left[\left(r\wedge(\sim p\vee p)\right)\wedge(\sim q\vee q)\right]\equiv(\sim p\wedge q\wedge r)\vee(\sim p\wedge \sim q\wedge r)$$
$$\vee(p\wedge \sim q\wedge r)\vee(p\wedge q\wedge r).$$

Therefore required PDNF is

$$(\sim p\wedge q\wedge r)\vee(\sim p\wedge \sim q\wedge r)\vee(p\wedge \sim q\wedge r)\vee(p\wedge q\wedge r)\vee$$
$$(\sim p\wedge q\wedge \sim r)\vee(p\wedge q\wedge \sim r)\vee(p\wedge \sim q\wedge \sim r).$$

Advantages of obtaining principal disjunctive normal form are as below.

1) The principal disjunctive normal of a given formula is unique.
2) Two formulas are equivalent if and only if their principal disjunctive normal forms coincide.
3) If the given compound proposition is a tautology, then its principal disjunctive normal form will contain all possible minterms of its components.

PRINCIPAL CONJUNCTIVE NORMAL FORM

MAXTERMS: Let p and q be two statement variables then $p \vee q,\ \sim p \vee q,\ p \vee \sim q,\ \sim p \vee \sim q$ are called maxterms of p and q. The number of maxterms in n variables is 2^n.

For example the maxterms for three variables p, q and r are

$$p \vee q \vee r,\ \sim p \vee q \vee r,\ p \vee \sim q \vee r,\ p \vee q \vee \sim r,$$

$$\sim p \vee \sim q \vee r,\ p \vee \sim q \vee \sim r,\ \sim p \vee q \vee \sim r,\ \sim p \vee \sim q \vee \sim r.$$

The truth table for the maxterms of p and q are given below:

Table: Maxterms

p	q	$p \vee q$	$\sim p \vee q$	$p \vee \sim q$	$\sim p \vee \sim q$
T	T	T	T	T	F
T	F	T	F	T	T
F	T	T	T	F	T
F	F	F	T	T	T

REMARKS:

- From the truth table it is clear that no max terms are equivalent.
- Each max term has truth value F for exactly one combination of the truth values of the variables p and q.

PRINCIPAL CONJUNCTIVE NORMAL FORM:

It is defined as an equivalent formula consists of conjunctive of max terms only. It is also called the **product of sums canonical form.** There are two ways to obtain the principal disjunctive normal form.

Method 1: Using Truth Table.

Method 2: Without Using Truth Table.

Method 1: Using Truth Table

Step 1: Construct a truth table for the given compound statement.

Step 2: For every truth value F of the given proposition, select the maxterms which has also true value F for the same combination of truth value of the statement variable.

Step 3: The conjunctive of maxterms selected in step 2 is the required principal conjunctive normal form.

Remark: In PCNF, the repeated maxterms are mention only once.

Example: Obtain the Principal Conjunctive Normal Form of

1) $p \wedge q$.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

The column containing $p \wedge q$ truth value F for the three combination of the truth value of p and q. Now use following using truth table.

F in the second row of $\boldsymbol{p} \wedge \boldsymbol{q}$ corresponds to the maxterms $\sim\boldsymbol{p} \vee \boldsymbol{q}$, (**refer TABLE: Maxterms**).

F in the third row of $\boldsymbol{p} \wedge \boldsymbol{q}$ corresponds to the maxterms $\boldsymbol{p} \vee \sim\boldsymbol{q}$, (**refer TABLE: Maxterms**).

F in the fourth row of $\boldsymbol{p} \wedge \boldsymbol{q}$ corresponds to the maxterms $\sim\boldsymbol{p} \vee \sim\boldsymbol{q}$, (**refer TABLE: Maxterms**).

Thus, PDNF of $\boldsymbol{p} \wedge \boldsymbol{q}$ is

$$(\sim p \vee \sim q) \wedge (\sim p \vee q) \wedge (p \vee \sim q)$$

Method 2: Without Using Truth Table

POINTS TO BE REMEMBER FOR PCNF WITHOUT USING TRUTH TABLE

After finding CNF, we apply identity law, complement law and Distributive Law.

Identity law for CNF is $q \equiv (q \vee F)$.

Complement Law: $\sim q \wedge q \equiv F$.

Example: **Obtain the Principal Conjunctive Normal Form of the following without using truth table.**

1) $\boldsymbol{p \wedge q}$.

Solution: $p \wedge q$ is the CNF form.

[Remark: Since p and q are individual variable so we introduce the missing variable q in the 1st term p and missing variable p in the 2nd term q to obtain the maxterms.]

$$p \wedge q \equiv (p \vee F) \wedge (q \vee F) \quad (\mathit{Identity\ Law})$$

$$\equiv (p \vee (\sim q \wedge q)) \wedge (q \vee (\sim \mathrm{p} \wedge p)) \quad (\mathit{Complement\ Law})$$

$$\equiv ((p \vee \sim q) \wedge (p \vee q)) \wedge ((q \vee \sim \mathrm{p}) \wedge (q \vee p)) \quad (\mathit{Distributive\ Law})$$

The required PCNF form is

$$p \wedge q \equiv (p \vee \sim q) \wedge (p \vee q) \wedge (q \vee \sim \mathrm{p}) \wedge (q \vee p).$$

2) $\left(\sim p \rightarrow r\right) \wedge \left(p \leftrightarrow q\right)$

Solution: First we find CNF form using logical equivalent statement of

$$p \rightarrow q \equiv \sim p \vee q \ \ and \ \ p \leftrightarrow q \equiv \left(p \rightarrow q\right) \wedge \left(q \rightarrow p\right).$$

$$\left(\sim p \rightarrow r\right) \wedge \left(p \leftrightarrow q\right) \equiv \left(p \vee r\right) \wedge \left[\left(p \rightarrow q\right) \wedge \left(q \rightarrow p\right)\right]$$

$$\equiv \left(p \vee r\right) \wedge \left[\left(\sim p \vee q\right) \wedge \left(\sim q \vee p\right)\right].(\text{Re}\,quired \ \ \textbf{CNF form})$$

[Remark: From CNF we find PCNF by introducing the missing variable q, r and r in 1st , 2nd , 3rd term respectively using identity , complement to obtain the maxterms and apply distributive law to find the PCNF]

$$(\sim p \to r) \wedge (q \leftrightarrow p) \equiv (p \vee r) \wedge \left[(\sim p \vee q) \wedge (\sim q \vee p)\right] ----(1)$$

Now,

$$(p \vee r) \equiv (p \vee r) \vee F$$

$$\equiv (p \vee r) \vee (q \wedge \sim q)$$

Thus,

$$(p \vee r) \equiv (p \vee r \vee q) \wedge (p \vee r \sim q) --------(A)$$

Similarly, we have

$$(\sim p \vee q) \equiv (\sim \mathrm{p} \vee q \vee r) \wedge (\sim \mathrm{p} \vee q \vee \sim r) ---------(B)$$

$$(\sim q \vee p) \equiv (\sim q \vee p \vee r) \wedge (\sim q \vee p \vee \sim r) ---------(C)$$

Putting the value from (A), (B), (C) in (1) we get PCNF as

$$(\sim p \to r) \wedge (q \leftrightarrow p) \equiv (p \vee r \vee q) \wedge (\sim \mathrm{p} \vee q \vee r) \wedge (\sim \mathrm{p} \vee q \vee \sim r)$$

$$\wedge (\sim q \vee p \vee r) \wedge (\sim q \vee p \vee \sim r).$$

Advantages of obtaining principal conjunctive normal form are

1) The principal conjunctive normal of a given formula is unique.
2) Every compound proposition, which is not a tautology, has an equivalent principal conjunctive normal form.
3) If the given compound proposition is a contradiction, then its principal conjunctive normal form will contain all possible maxterms of its components.

<u>PRATICE EXAMPLE</u>

EX: Obtain the PDNF and PCNF form of the following without using truth table:

1) $(\sim p \vee \sim q) \to (p \leftrightarrow \sim q)$
2) $q \wedge (p \vee \sim \mathrm{q})$
3) $p \vee (\sim p \to (q \vee (\sim q \to r)))$
4) $(p \to (q \wedge r)) \wedge (\sim p \to (\sim q \wedge \sim r))$
5) $p \to ((p \to q) \wedge \sim (\sim q \vee \sim \mathrm{p}))$