Directed Acyclic Graphs & Topological Order

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COMPCSI220: WEEK 10

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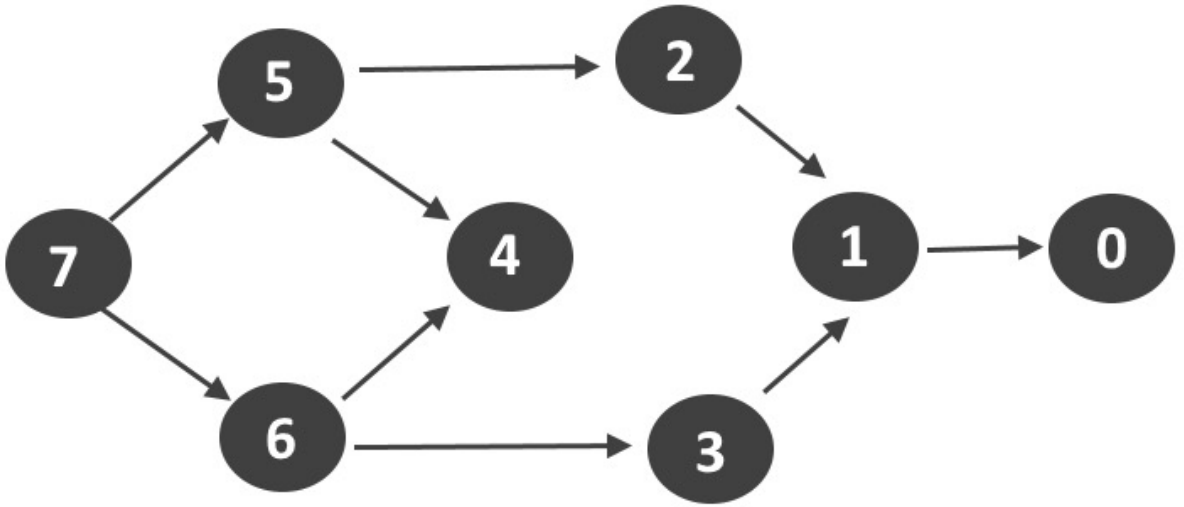
NEW ZEALAND

OUTLINE

- Directed Acyclic Graphs (DAGs)
- Topological Orders
  - Definition
  - Illustration
- Topological Sorting
  - Illustrative Examples

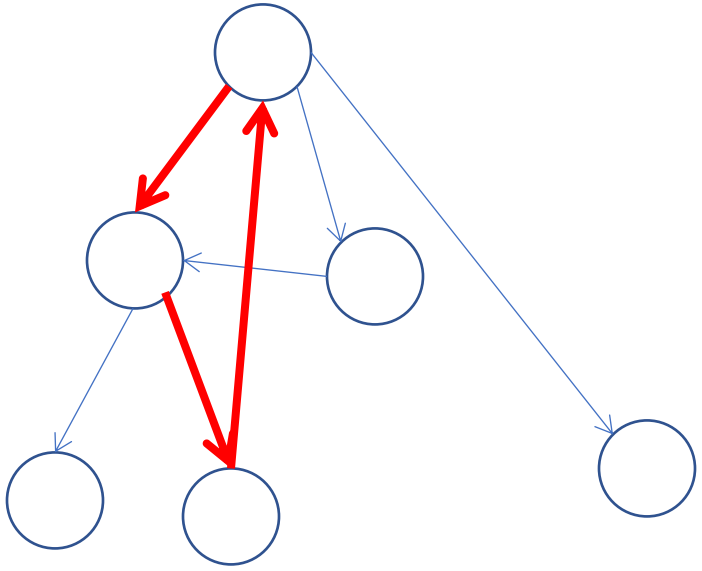


Topological Sort : 7 6 5 4 3 2 1 0

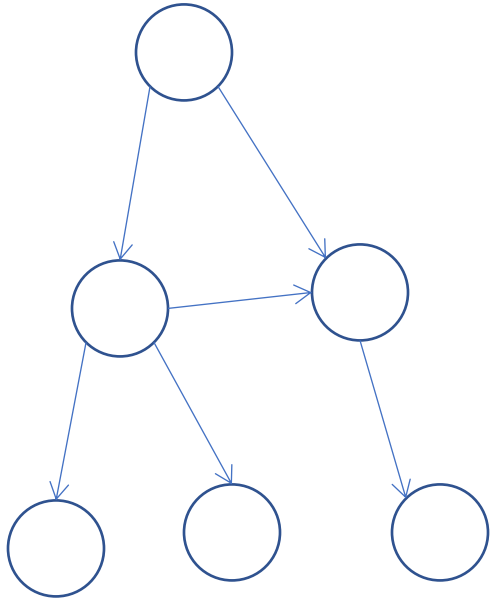
Cycle Detection

- Suppose that there is a cycle in G and let v be the node in the cycle visited first by DFS. If (u, v) is an arc in the cycle then it must be a back arc.
- Conversely if there is a back arc, we must have a cycle.
- Suppose that **DFS** is run on a digraph G. Then G is acyclic if and only if G does not contain a back arc.
- A digraph with no cycle is called a directed acyclic graph (DAG).

Example: Directed Acyclic Graphs (DAGs)



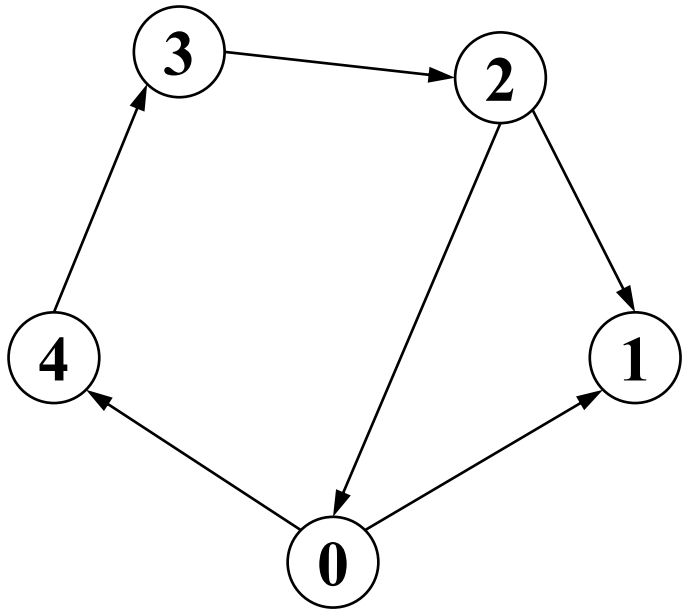
G

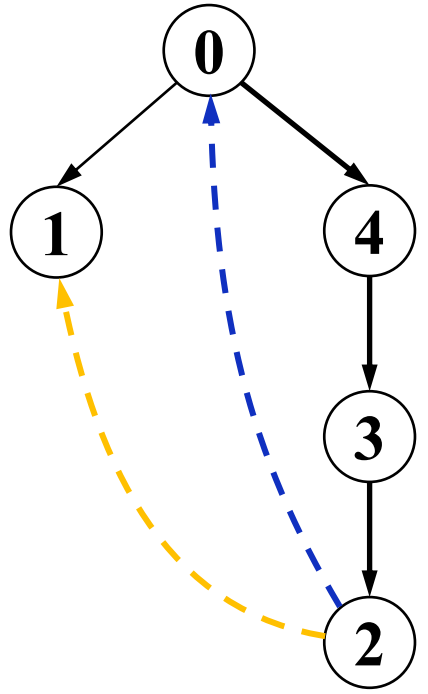


G': Acyclic

Using DFS to Find Cycles in Digraphs

- Once DFS finds a cycle, the stack contains the nodes that form the cycle



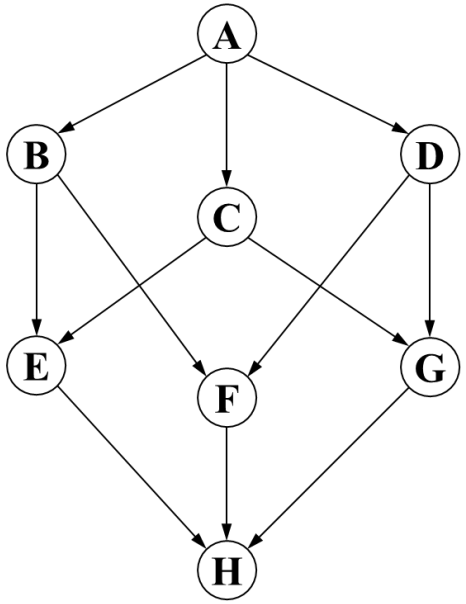


Properties: Directed Acyclic Graphs (DAGs)

- Any walk on a DAG G is limited in length
- A DAG must have at least one source and one sink (why?)
  - A graph that has no source or no sink must have a cycle
- Although a DAG has no cycles the underlying graph could have cycles.

Topological Sorting

- A topological sorting of a digraph G is an ordering on its vertices such that, for each arc (u, v) of G, u appears before v in the ordering.
- To place nodes of a digraph on a line so all arcs go in one direction.
- Main application: scheduling events (arithmetic expressions, university prerequisites, etc).



- A D B E C F G H is NOT a topological sorting, because there is an arc (C,E), but E comes prior to C in the ordering.
- A D B C E F G H is a topological sorting.

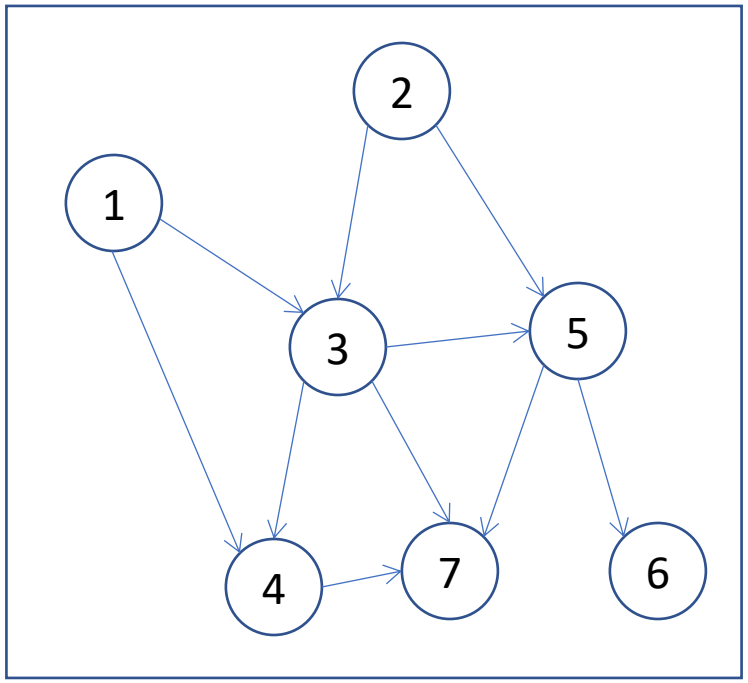
Topological Sorting (Contd.)

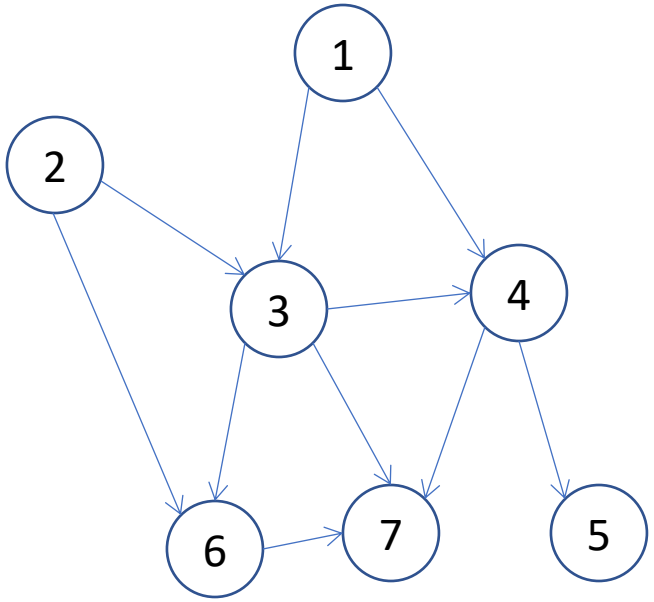
- If G is a DAG, then it is possible to find a topological order of the vertices.
- A topological order is a numbering of the vertices such that an arc (u,v) in the digraph means that u has a smaller number than v.
- Only DAGs have a topological ordering

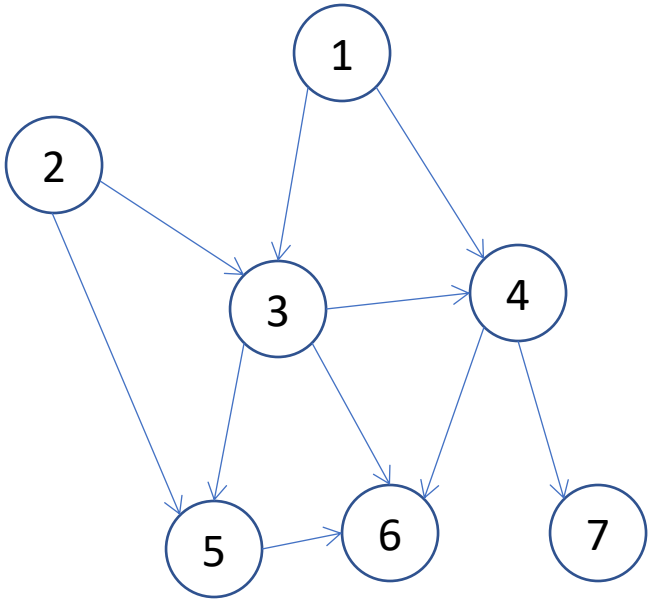
Topological Orders (Contd.)

- There is often more than one topological order for a digraph (see examples on previous slides)
- A topological order is also called a topological sort or a linear order.
- Why "linear" order? Because you can order the vertices in a line and make all arcs point the same way!

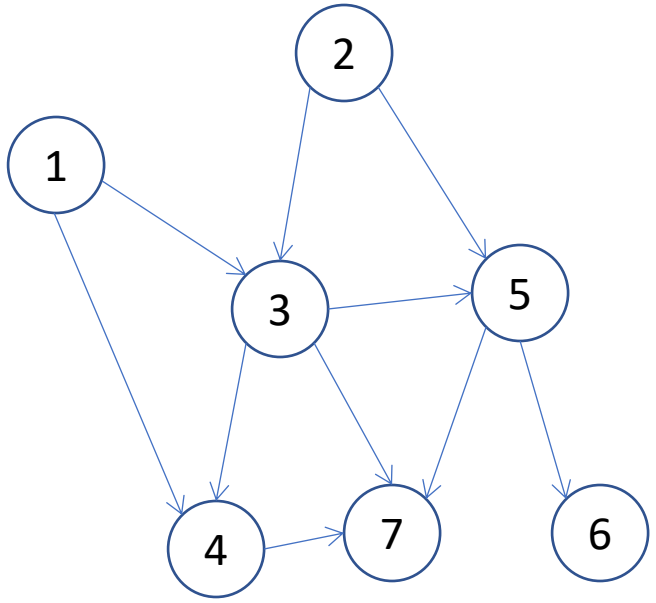
Example: Topological Sorting 1



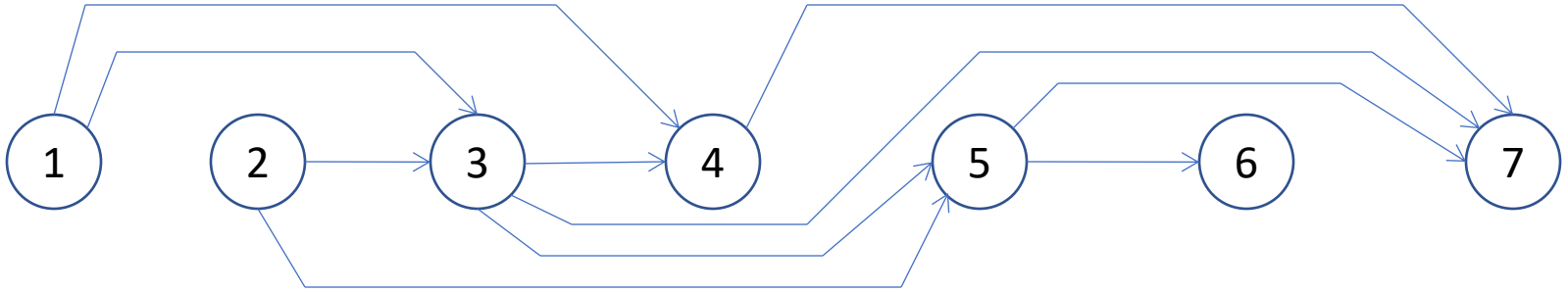




Topological Order: to Linear



These two digraphs
are exactly the same!



In linear sort order, all arcs run from left to right!

Topological Sorting

- Topological sorting is possible if and only if digraph is a DAG.
  - If there is a topological order of a digraph, then there is no cycle
    - Suppose there is a cycle $u \to \cdots \to v \to \cdots \to u$, u and v are both ancestors of each other
  - A DAG always has at least one topological order
    - Every DAG has a source

      Suppose there is no source. Let $u_1, u_2, \ldots, u_n$ be a directed path of maximal length.
      Then there is no longer directed path that contains $u_1, u_2, \ldots, u_n$. But then u_1 is a source.
      → A contradiction.

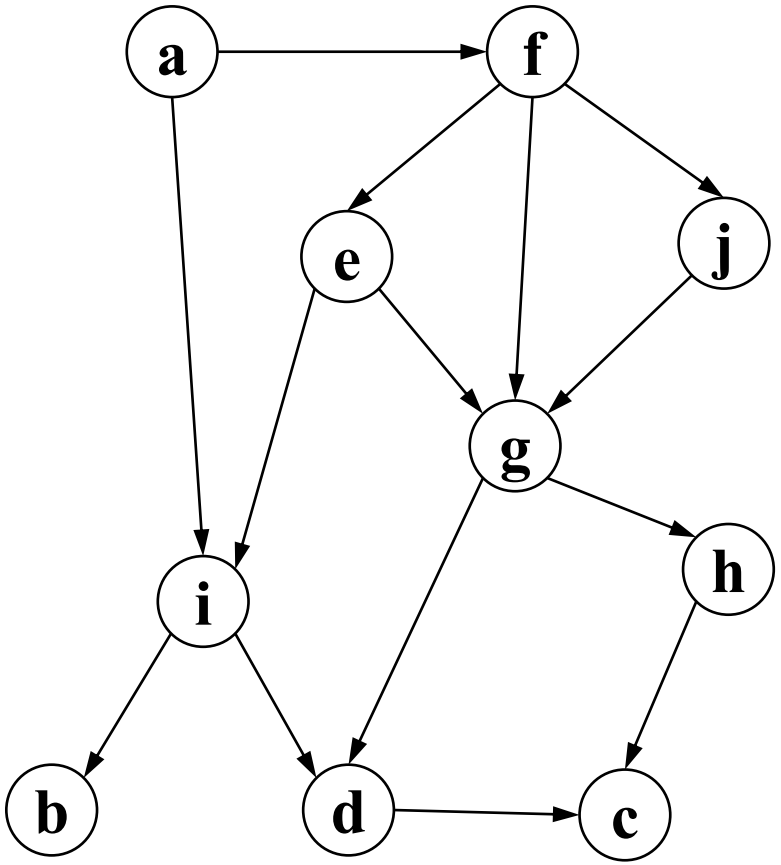
    - By removing the source and any out-arcs from the source we still have a DAG.
    - The order we remove the sources forms a topological order

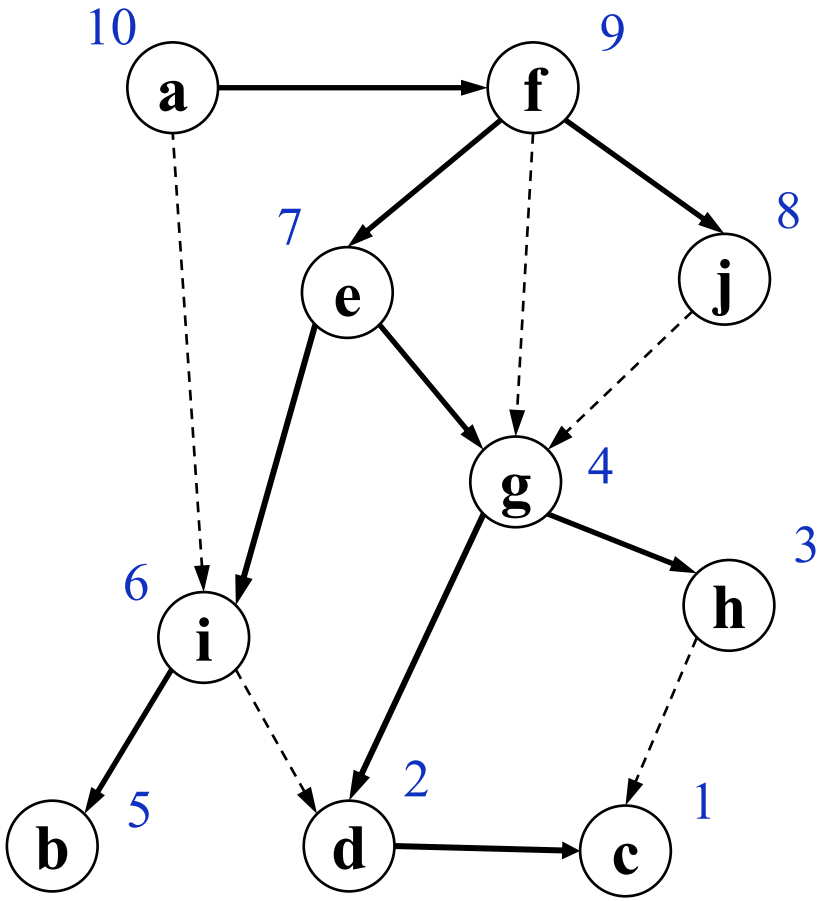
Topological Sorting

- Two solutions:

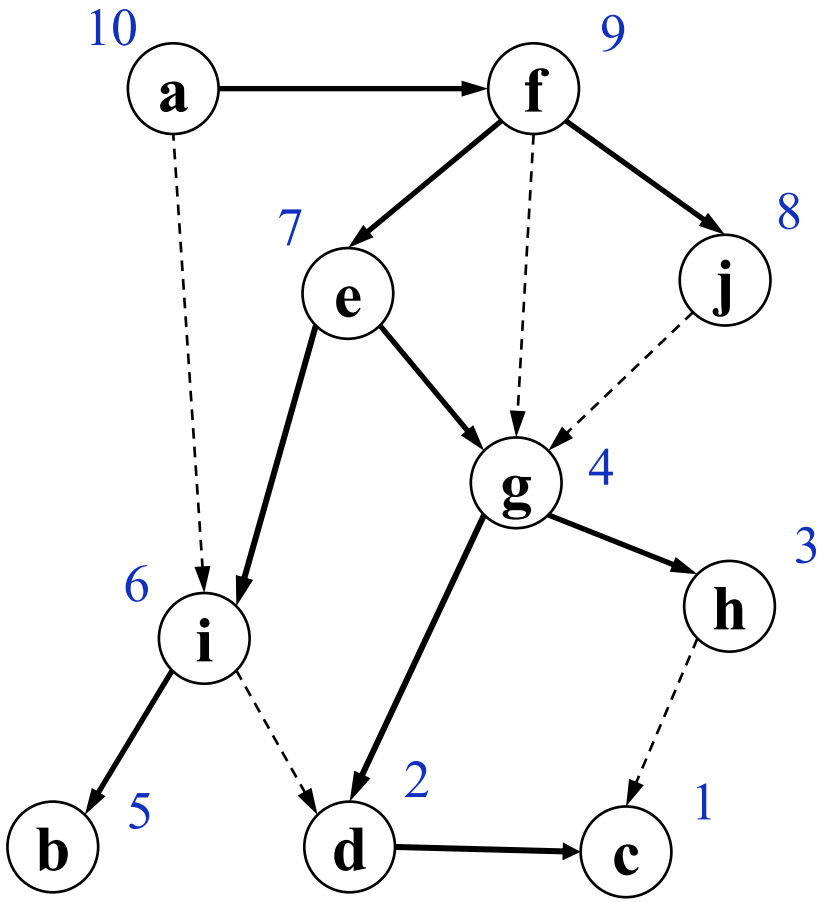
1. List of finishing times by DFS, in reverse order (since there are no back arcs, each node finishes before anything pointing to it).
2. Zero in-degree sorting – Find a node of in-degree zero, delete it and repeat until all nodes listed.

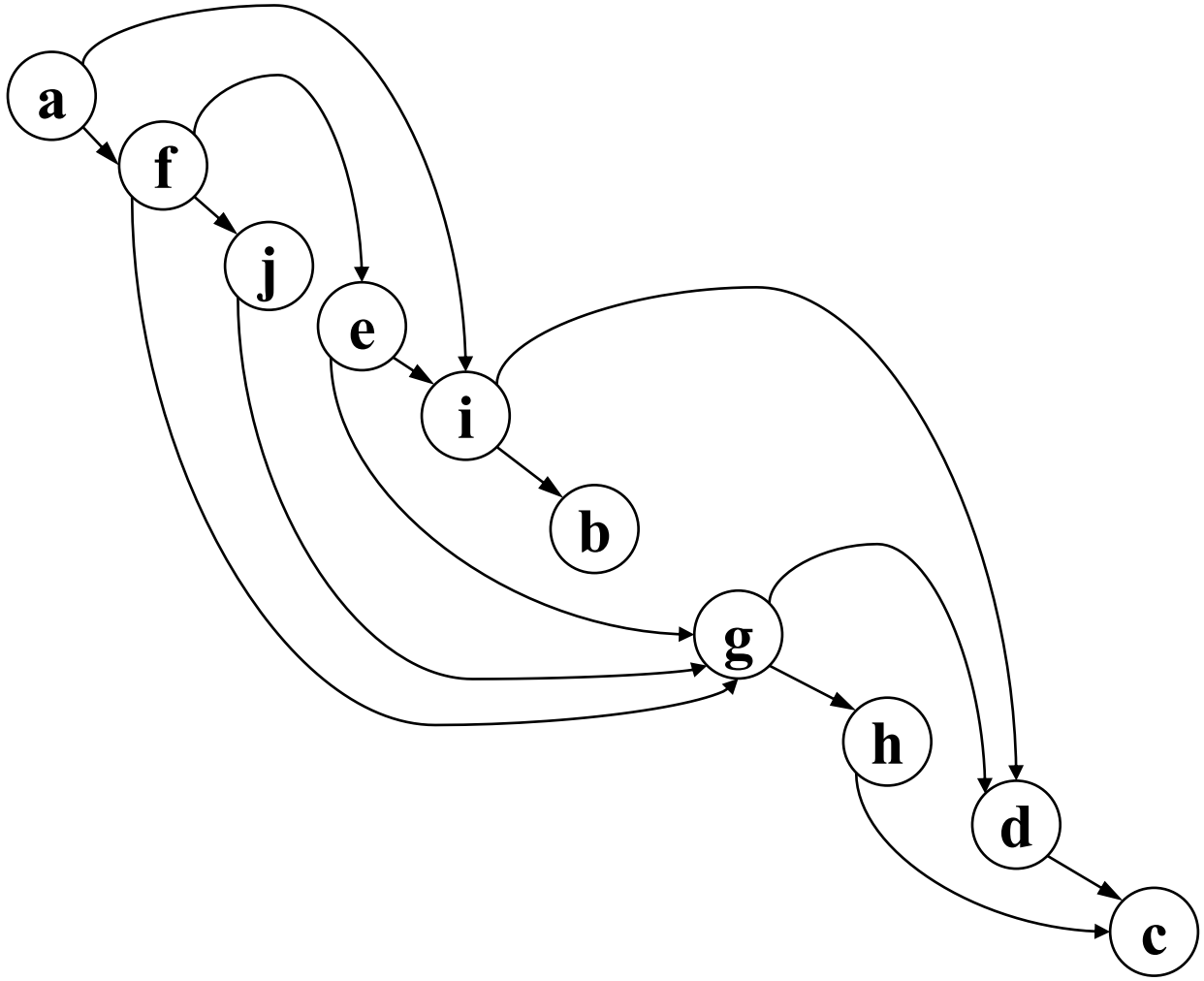
Example: Topological Order





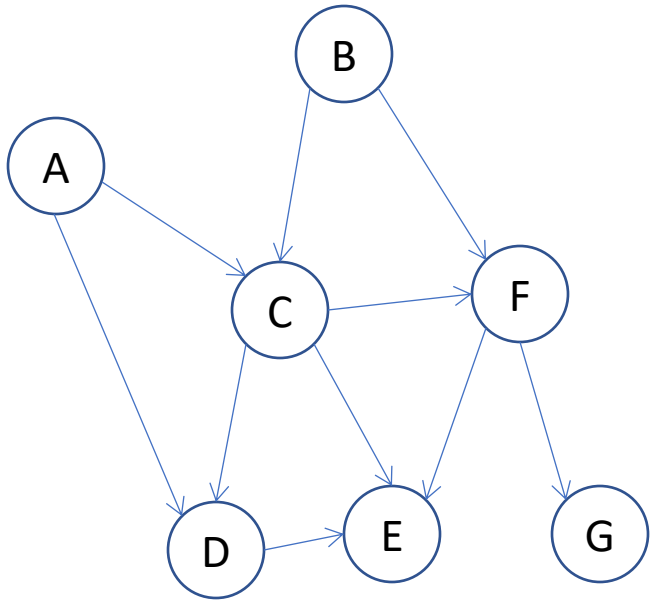
Example: Topological Order





Example: Topological Order by DFS

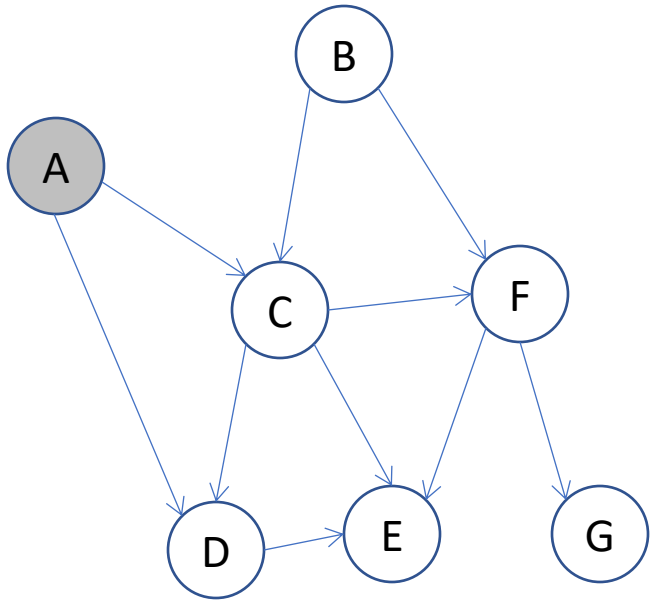
- Starting at the node A:



Example: Topological Order by DFS

- Starting at the node A:

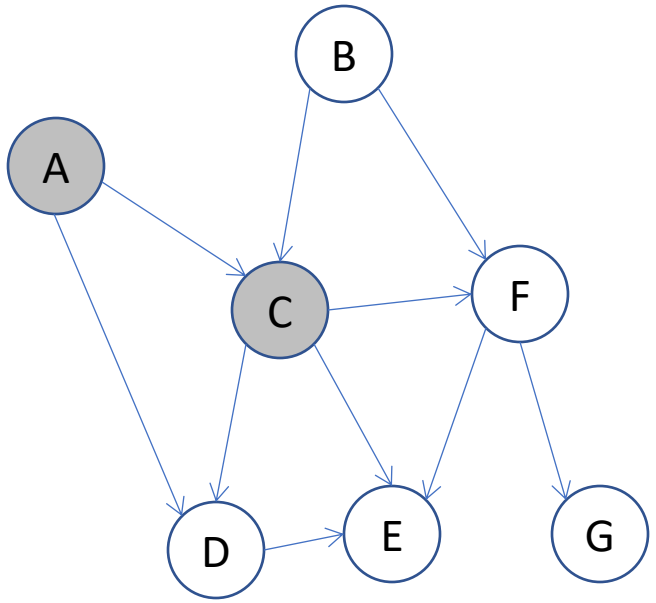
- push A



Example: Topological Order by DFS

- Starting at the node A:

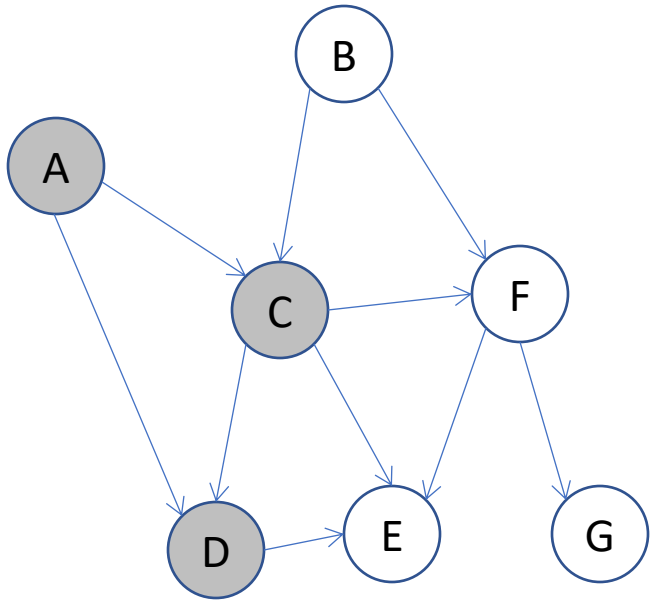
- push A, push C



Example: Topological Order by DFS

- Starting at the node A:

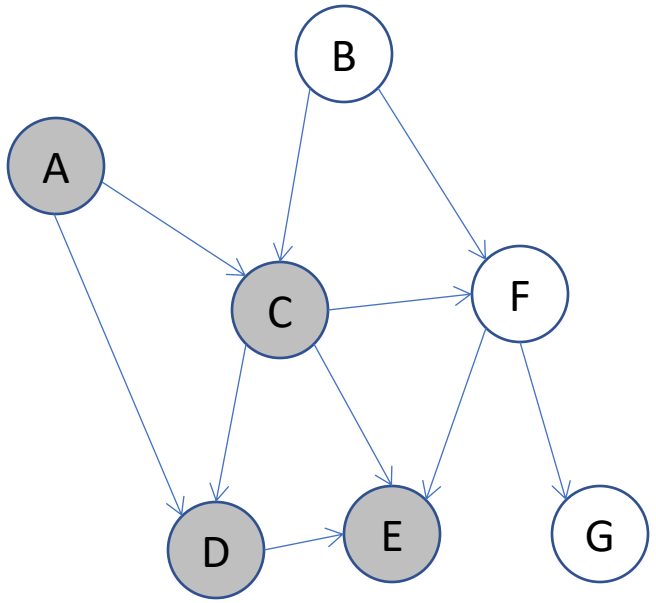
- push A, push C, push D



Example: Topological Order by DFS

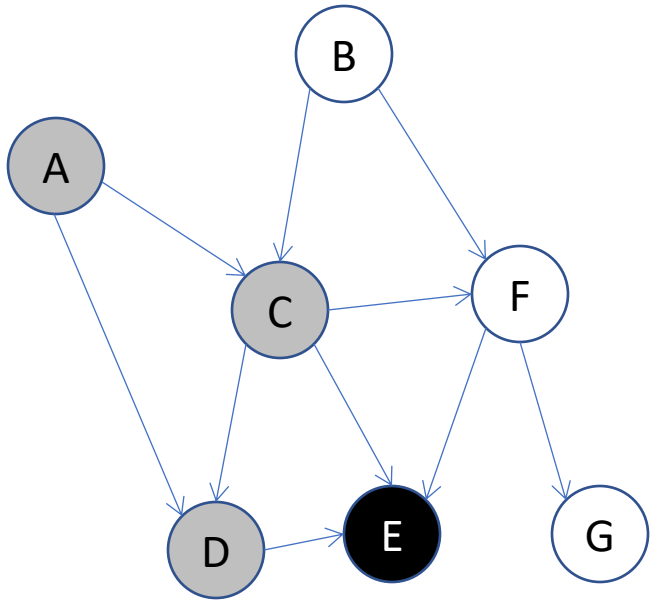
- Starting at the node A:

- push A, push C, push D, push E



Example: Topological Order by DFS

- Starting at the node A:

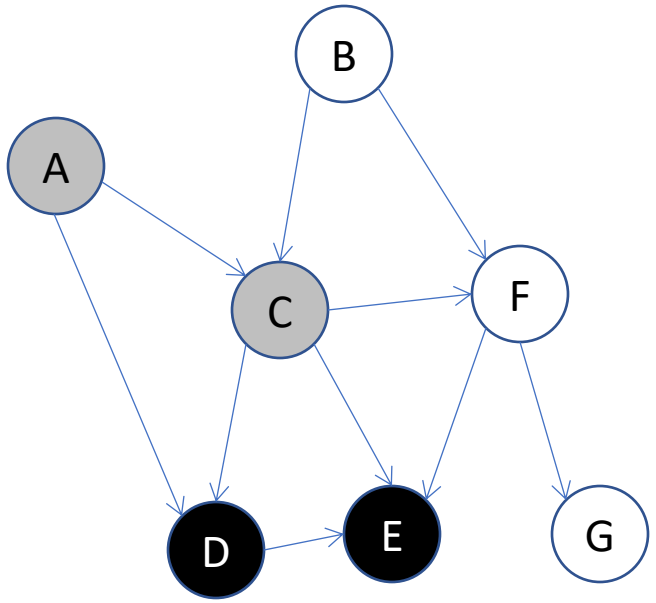


- push A, push C, push D, push E,
- pop E,

- And store E in the last position in the array

Example: Topological Order by DFS

- Starting at the node A:

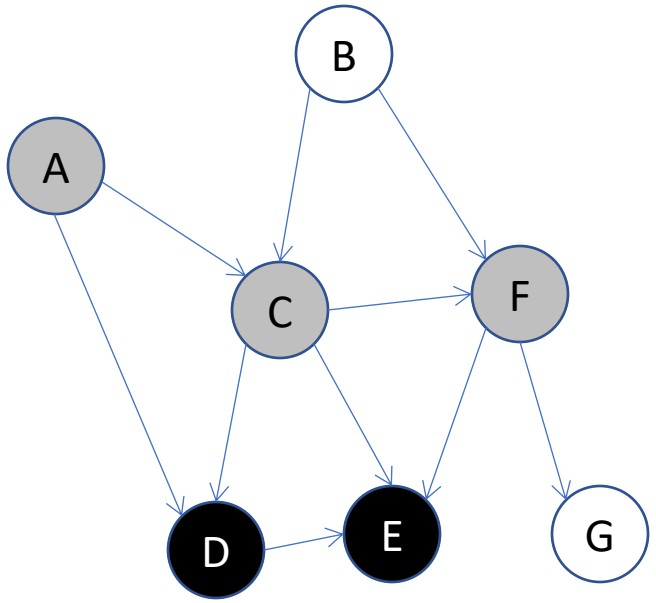


- push A, push C, push D, push E,
- pop E, pop D

- And store D in position array.length()-2

Example: Topological Order by DFS

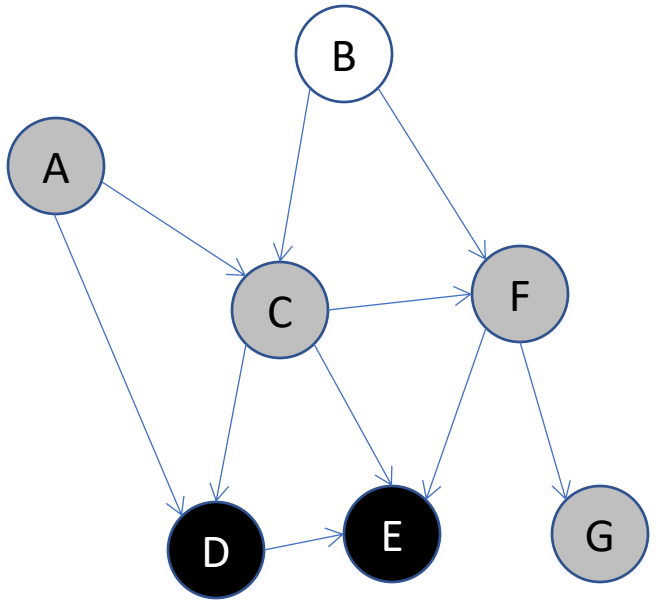
- Starting at the node A:



- push A, push C, push D, push E,
- pop E, pop D, push F

Example: Topological Order by DFS

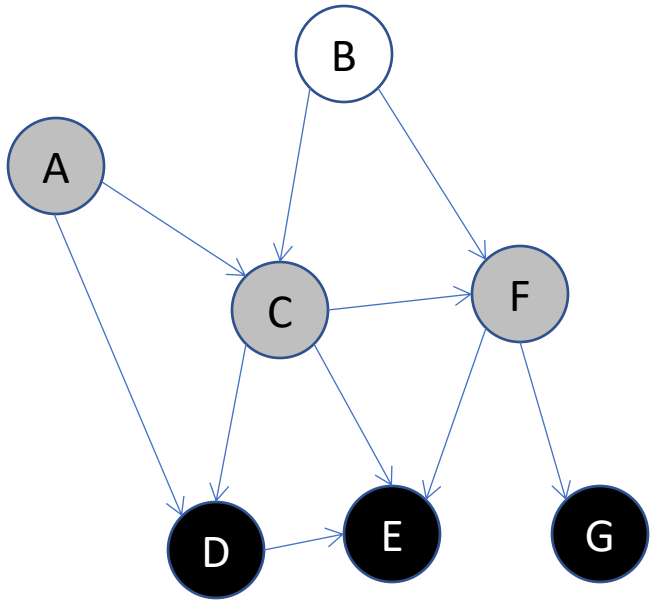
- Starting at the node A:



- push A, push C, push D, push E,
- pop E, pop D, push F, push G

Example: Topological Order by DFS

- Starting at the node A:

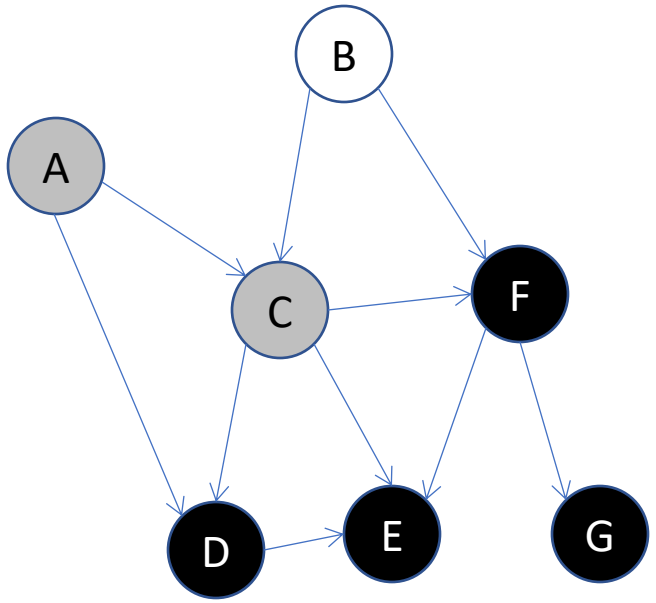


- push A, push C, push D, push E,
- pop E, pop D, push F, push G, pop G

- And store G in position array.length()-3

Example: Topological Order by DFS

- Starting at the node A:

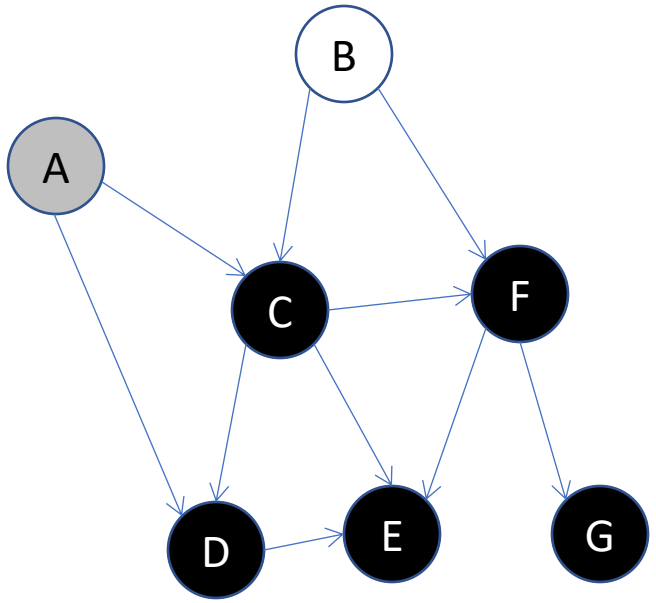


- push A, push C, push D, push E,
- pop E, pop D, push F, push G, pop G
- pop F

- And store F in position array.length()-4

Example: Topological Order by DFS

- Starting at the node A:

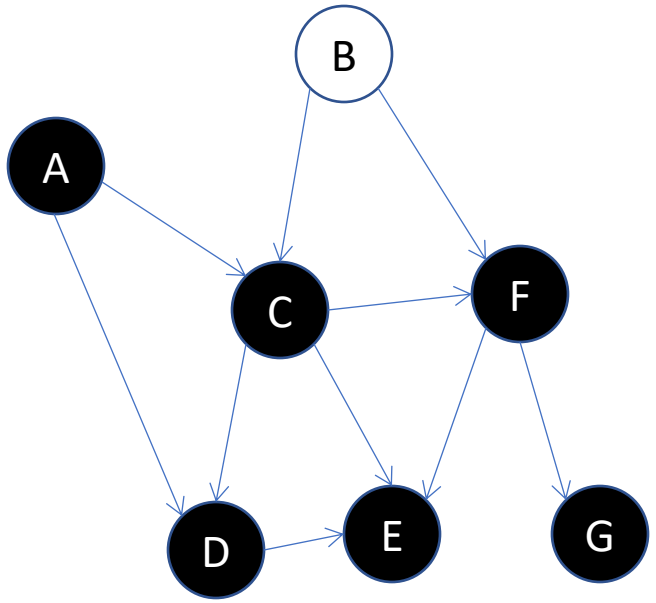


- push A, push C, push D, push E,
- pop E, pop D, push F, push G, pop G
- pop F, pop C

- And store C in position array.length()-5

Example: Topological Order by DFS

- Starting at the node A:



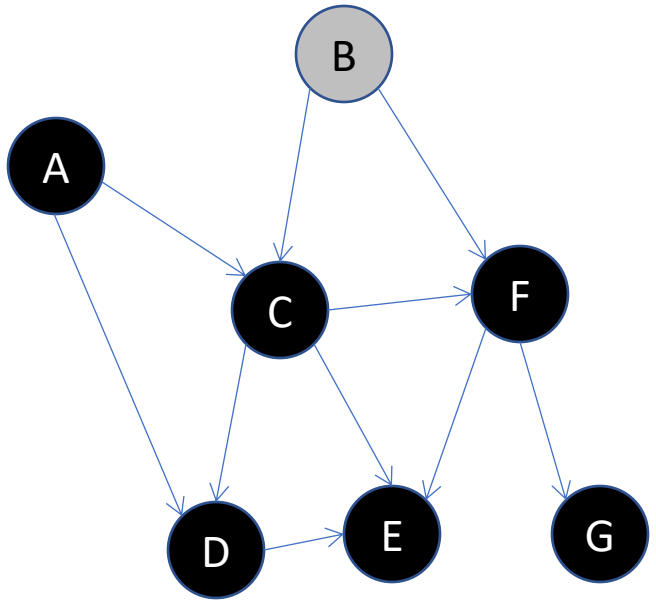
- push A, push C, push D, push E,
- pop E, pop D, push F, push G, pop G
- pop F, pop C, pop A,

- And store A in position array.length()-6

	A	C	F	G	D	E

Example: Topological Order by DFS

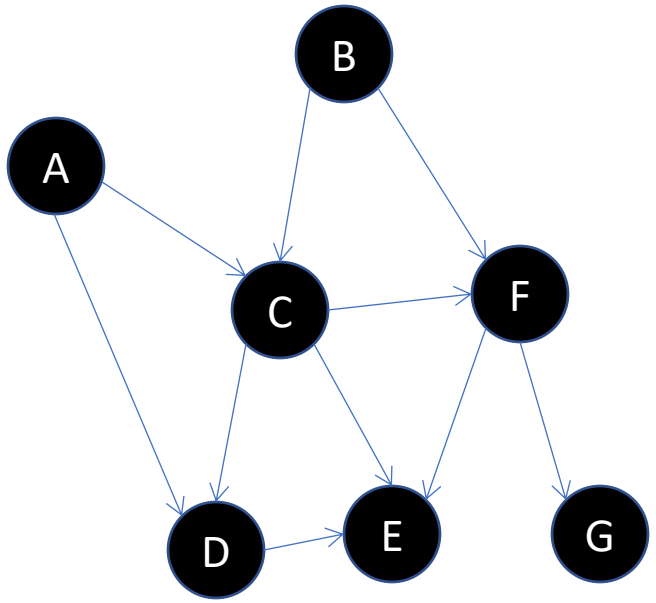
- Starting at the node A:



- push A, push C, push D, push E,
- pop E, pop D, push F, push G, pop G
- pop F, pop C, pop A, push B

Example: Topological Order by DFS

- Starting at the node A:



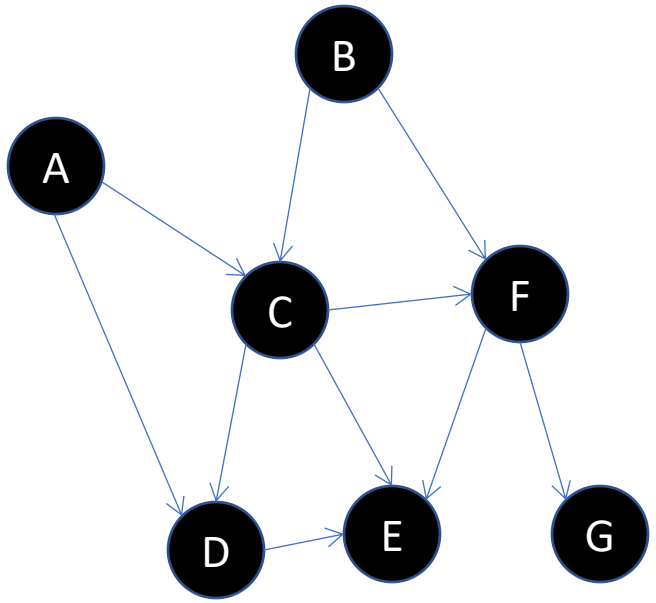
- push A, push C, push D, push E,
- pop E, pop D, push F, push G, pop G
- pop F, pop C, pop A, push B, pop B

- And store B in position array.length()-7

B	A	C	F	G	D	E

Example: Topological Order by DFS

- Starting at the node A:

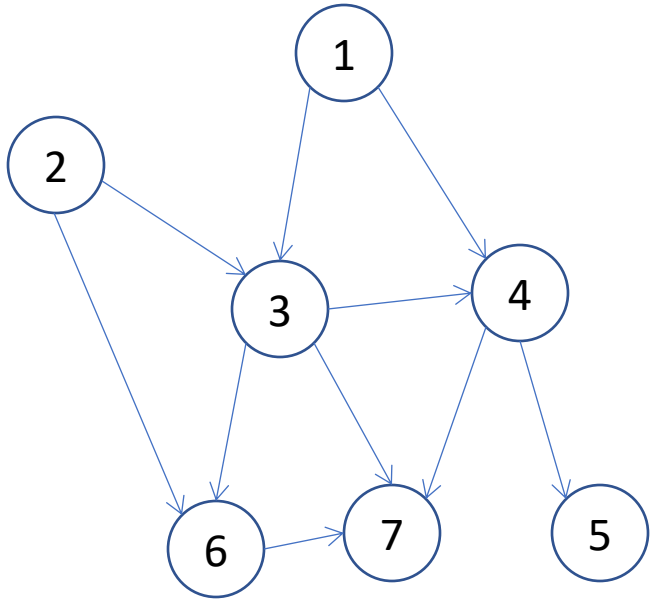


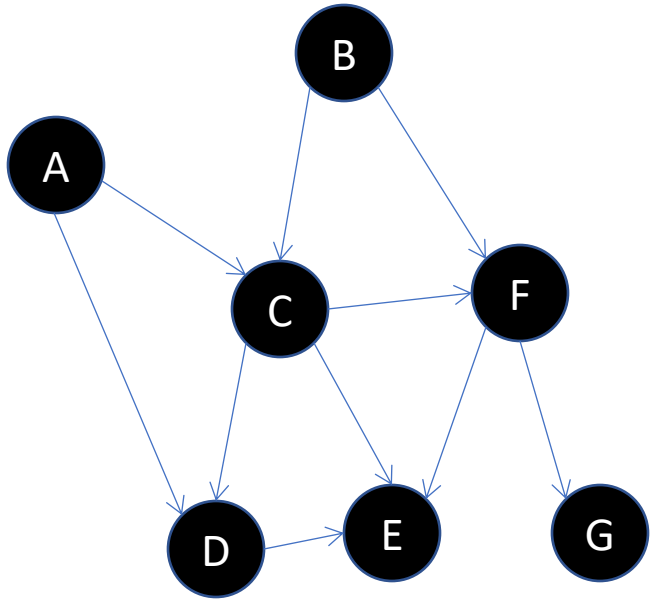
- push A, push C, push D, push E,
- pop E, pop D, push F, push G, pop G
- pop F, pop C, pop A, push B, pop B

- So, we get B=1, A=2, C=3, F=4, G=5, D=6, E=7

B	A	C	F	G	D	E

Example: Topological Order 2





B=1, A=2, C=3, F=4, G=5, D=6, E=7

Zero in-degree Sorting

- **Zero indegree sorting**: Start with DAG, say G. Initialize the order list L as empty.

1. Find a source vertex v.

2. Delete v and all its out-going arcs. Append node v to the ordered list L. Since we only delete arcs and vertices, this process does not create a cycle. Hence, the resulting graph is a DAG and has at least a source node.

3. Repeat 1 and 2 under all the vertices have been deleted. Then, we will get a topological order L of G.

Zero in-degree Sorting

- What is the running time of a naive implementation of zero-indegree sorting where a source is found and then removed at each step? How could this idea be made more efficient?
- For each iteration, we need to find zero indegree vertex v, delete v and its out-going arcs. We calculate the in-degree over every node each time, it takes

  Adjacency Matrix: $O(n^2)$ Adjacency List: $O(m)$

- For all n iterations:

  Adjacency Matrix: $O(n^3)$ Adjacency List: $O(nm)$

- We don't have to calculate the in-degree every time, we can use an array to track the in-degree of every node!

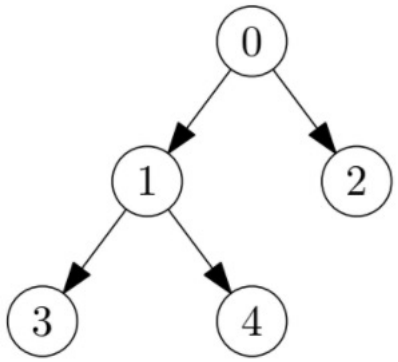
Zero In-degree Sorting: A Faster Algorithm

Algorithm 1 TopSort.

```
1: function TopSort(digraph G)
2:         for u ∈ V(G) do
3:             indegree[u] ← indegree of u
4:         queue Q, count ← 0, order = []
5:         for u ∈ V(G) do
6:             if indegree[u] = 0 then Q.enqueue(u)
7:         while Q is not empty do
8:             u ← Q.dequeue(), order.append(u), count ← count + 1
9:             for v as out-neighbor of u do
10:                indegree[v] ← indegree[v] − 1
11:                if indegree[v] = 0 then Q.enqueue(v)
12:        if count != |V(G)| then
13:            return NULL
14:        else
15:            return order
```

Exercise: Topological Orders

- Topological orders are not unique. List all possible topological orders of the following digraph. We can use zero in-degree sorting.



8 topological orders

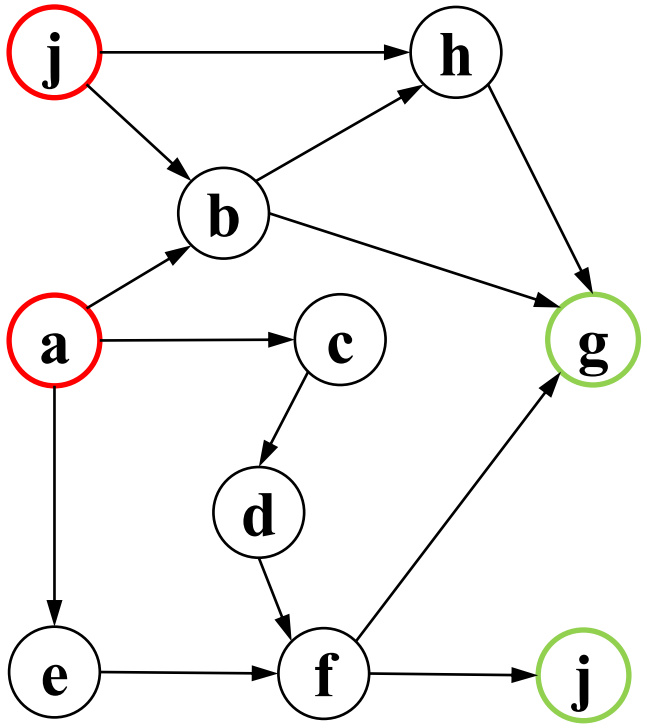
Ordering: 0 1 2 3 4
Ordering: 0 1 2 4 3
Ordering: 0 1 3 2 4
Ordering: 0 1 3 4 2
Ordering: 0 1 4 2 3
Ordering: 0 1 4 3 2
Ordering: 0 2 1 4 3
Ordering: 0 2 1 3 4

Exercise: Topological Sorting

For each arc (u, v), u appears before v in a topological sorting.

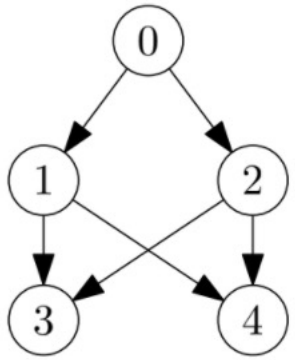
- a e j b c d f i h g.
- Usually not unique.

source



sink

Example 25.3. A digraph with all possible topological orders and drawn with topological order 0, 1, 2, 3, 4. Draw the digraph for the topological order 0, 2, 1, 4, 3.

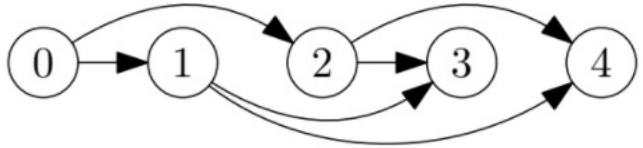


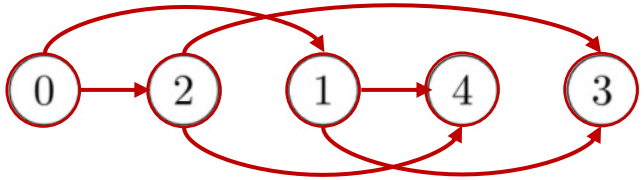
0, 1, 2, 3, 4

0, 1, 2, 4, 3

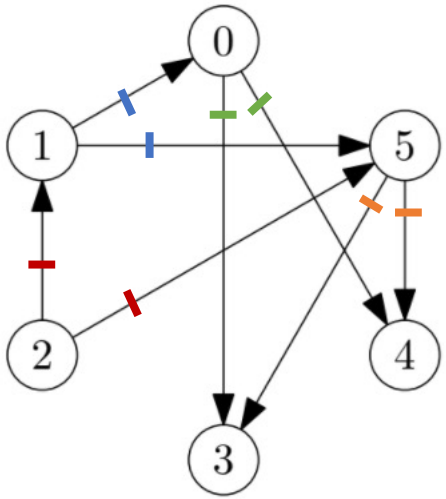
0, 2, 1, 3, 4

0, 2, 1, 4, 3





Find a topological order of the following digraph and draw it.



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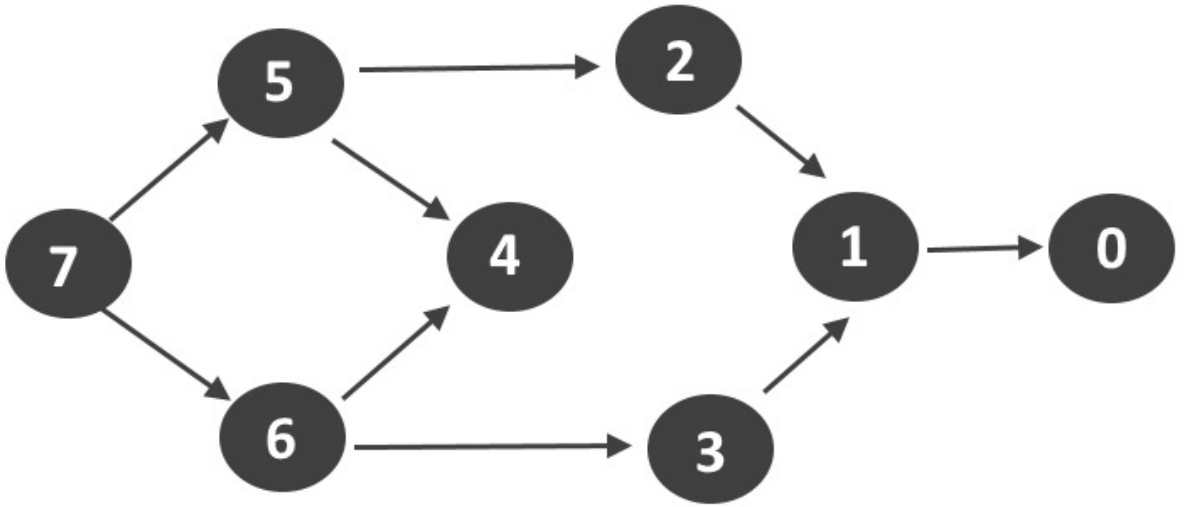
SUMMARY

- Directed Acyclic Graphs (DAGs)
- Topological Orders
  - Illustrative Examples
- Topological Sorting
  - DFS
  - Zero-indegree Sorting



Topological Sort : 7 6 5 4 3 2 1 0