# Graph Traversals I

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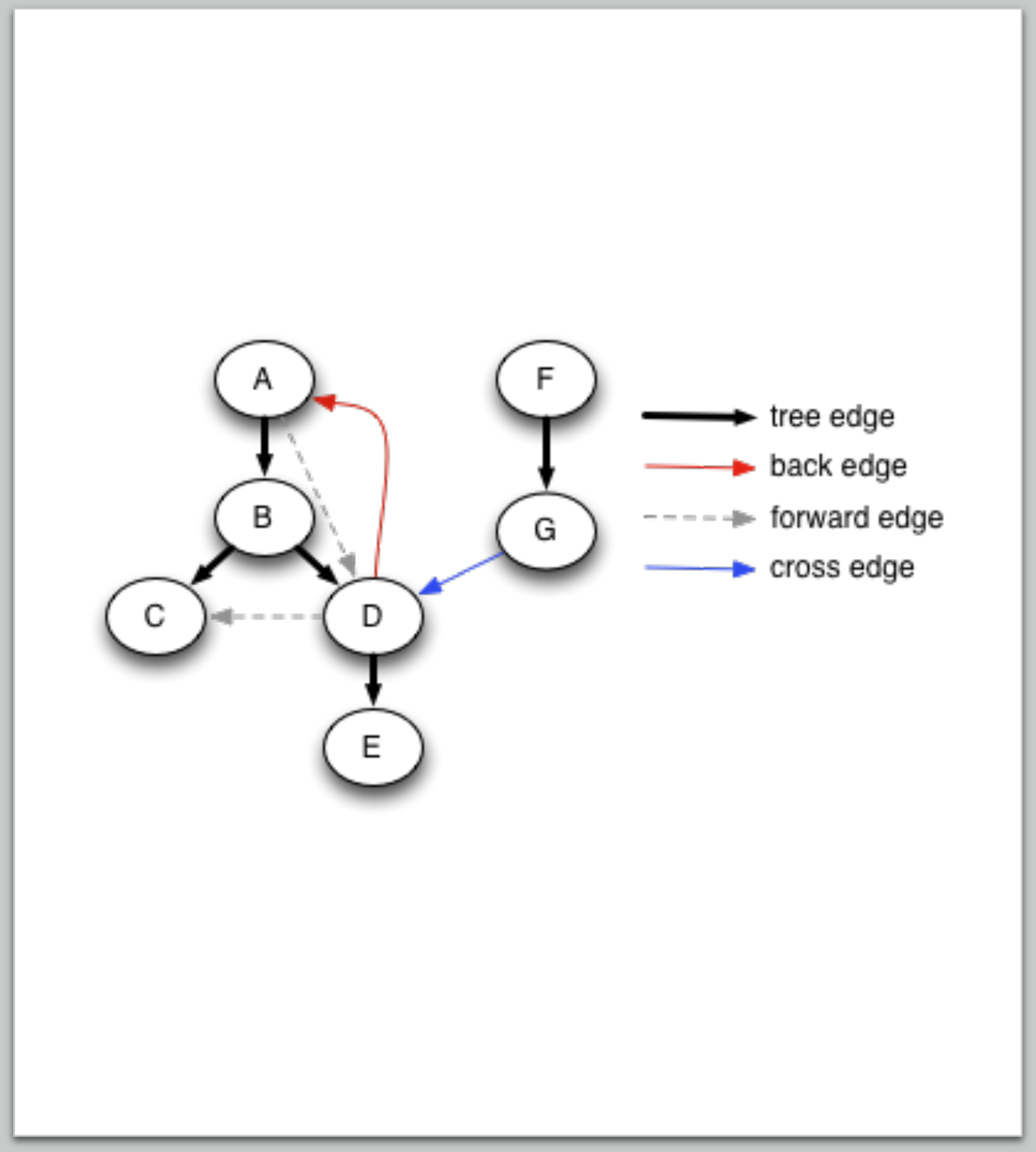
COMPCSI220: WEEK 10

THE UNIVERSITY OF AUCKLAND
Te Whare Wananga o Tamaki Makaurau
NEW ZEALAND

Slides adapted from Mark Wilson, Georgy Gimel'farb, Simone Linz and Tanya Gvozdeva

# OUTLINE

- Graph Traversal Algorithm
- Facts about Traversal Trees
- Complexity Analysis
- Illustrative Example



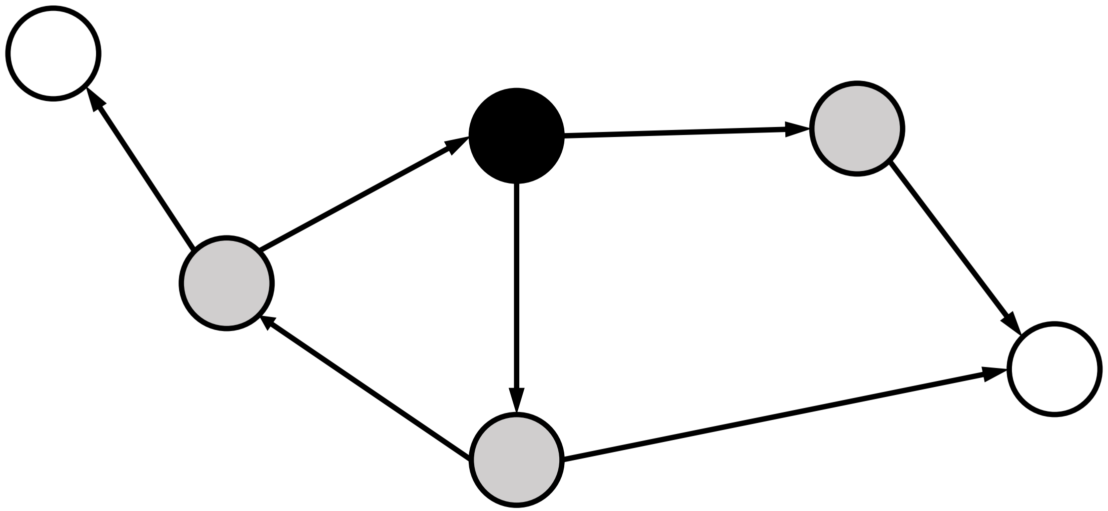
# Motivation: Graph Traversals

- Want to visit each node of a digraph in a systematic and efficient way (e.g. to search a graph).

- We can walk only on arcs following their direction

# General Graph Traversal: Colour Scheme

- All graph traversal algorithm follow the same structure which is called the The general graph traversal algorithm. This algorithm uses three types of nodes:
  - **White nodes**: have not yet been visited.
  - **Grey (frontier) nodes**: have been visited but may have adjacent nodes that are white.
  - **Black nodes**: have been visited and all their (out-)neighbors have been visited as well.



# Graph Traversal Algorithm

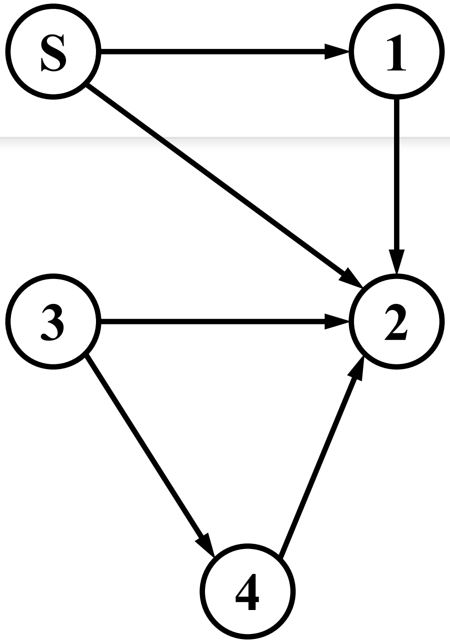
- All nodes are **white** to begin with.

- A starting white node is chosen and turned grey.
- A **grey** node is chosen and its out-neighbours explored.
- If any out-neighbour is white, it is visited and turned **grey**. If no out-neighbours are white, the grey node is turned **black**.
- The process of choosing grey nodes and exploring neighbours is continued until all nodes reachable from the initial node are black.

- If any white nodes remain in the digraph, a new starting node is chosen and the process continues until all nodes are **black**.

# General Graph Traversal: Visit(s)

1. $s$ is coloured grey and $pred[s]$=$null$.
2. choose a grey node $u$.
3. if $u$ has a white (out)-neighbour $v$ then colour $v$ grey and $pred[v]$=$u$ else colour $u$ black.
4. if we have grey nodes go to 2).




* *pred* - *predecessor*

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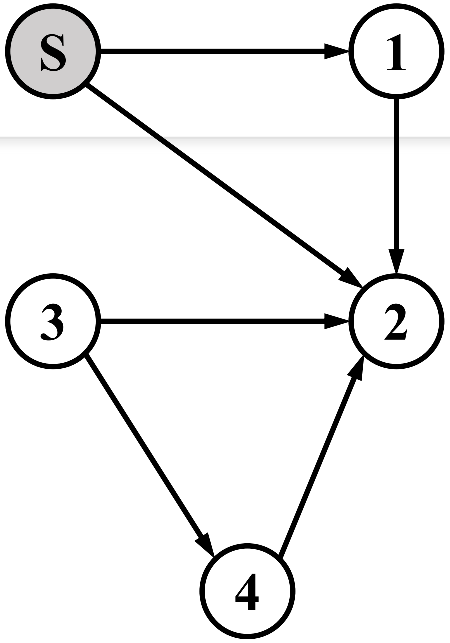


$\boldsymbol{Pred[s] = null}$

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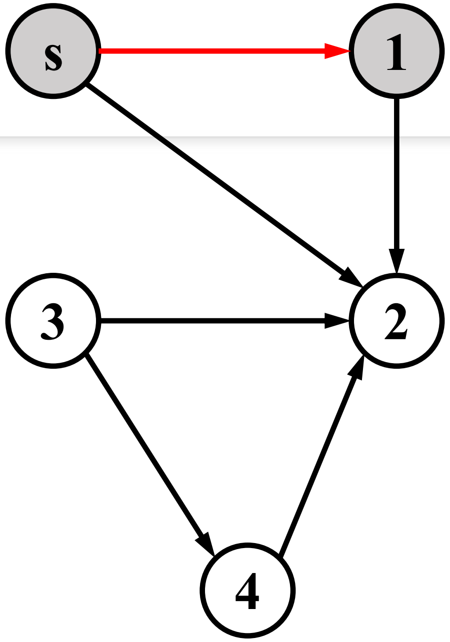


$\boldsymbol{Pred[1] = s}$

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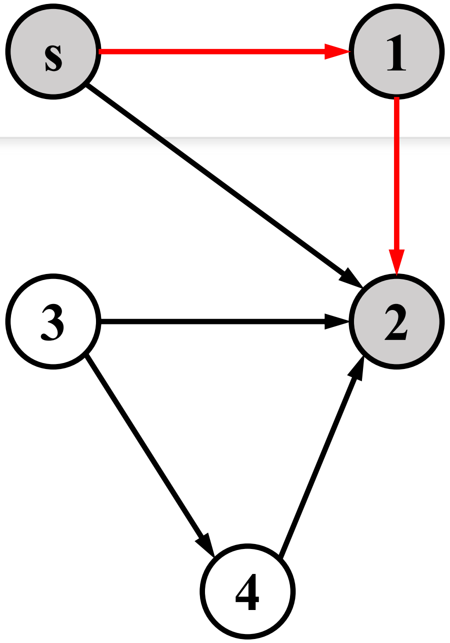
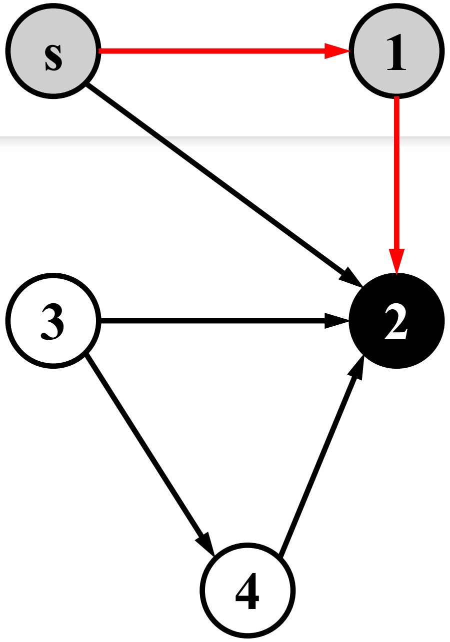



$\boldsymbol{Pred[2] = 1}$

# General Graph Traversal: Visit(s)

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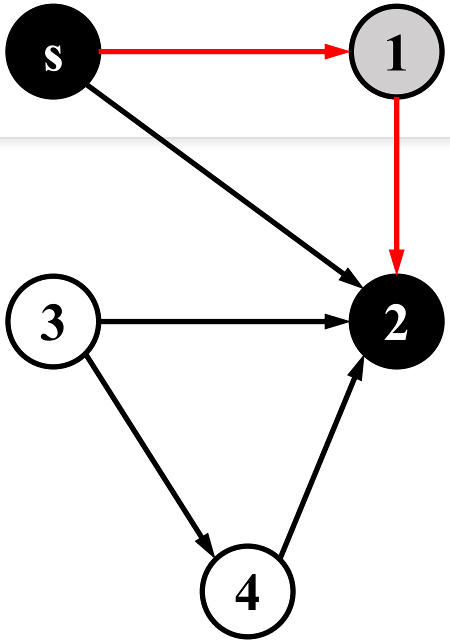

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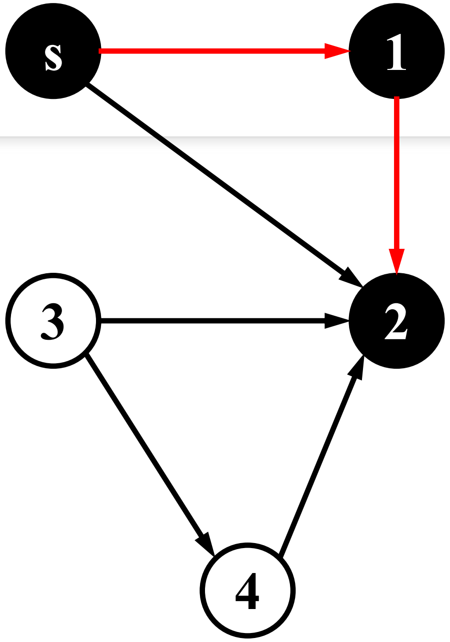


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* *pred – predecessor*

- $Visit(s)$ visits all nodes reachable from $s$.
- After the run of $visit(s)$ all reachable nodes are coloured black.

# General Graph Traversal: Visit(s)

**Algorithm 1** Visit.

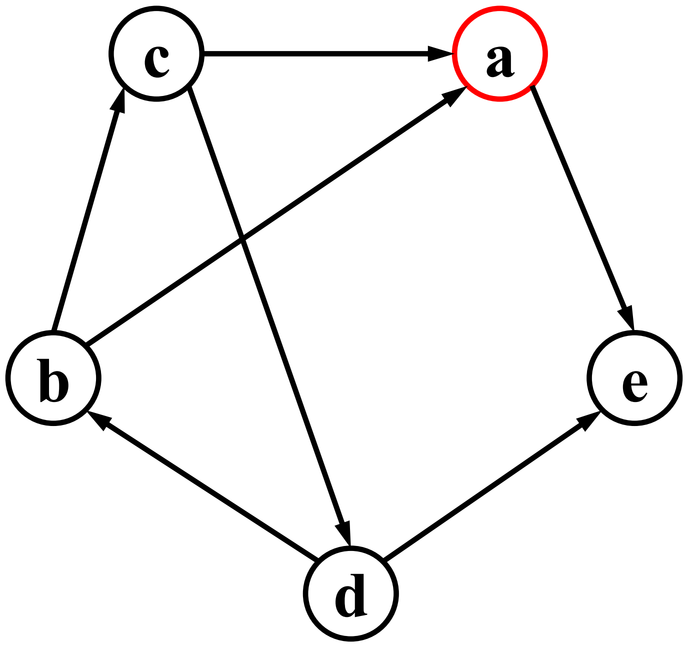
**function** VISIT(node $s$ of digraph $G$)
    $color[s] \leftarrow$ Grey
    $pred[s] \leftarrow$ Null
    **while** there is a Grey node **do**
        choose a Grey node $u$
        **if** $u$ has a WHITE (out-)neighbour **then**
            choose such a white (out-)neighbour $v$
            $color[v] \leftarrow$ Grey
            $pred[v] \leftarrow u$
        **else**
            $color[u] \leftarrow$ Black

# General Graph Traversal Algorithm: Main

**Algorithm 2** Traverse.

**function** TRAVERSE(digraph $G$)<br>
    array $color[0..n-1]$<br>
    array $pred[0..n-1]$<br>
    **for** $u \in V(G)$ **do**<br>
        $color[u] \leftarrow$ WHITE<br>
    **end for**<br>
    **for** $s \in V(G)$ **do**<br>
        **if** $color[s] =$ WHITE **then**<br>
            VISIT$(s)$<br>
    **return** $pred$

# Illustrating the general traversal algorithm



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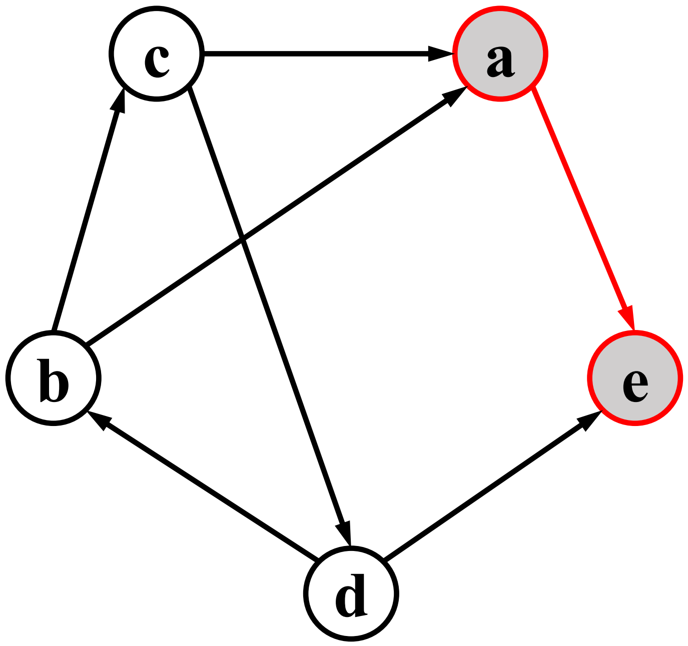


- VISIT(a)

  e is the white neighbour of a

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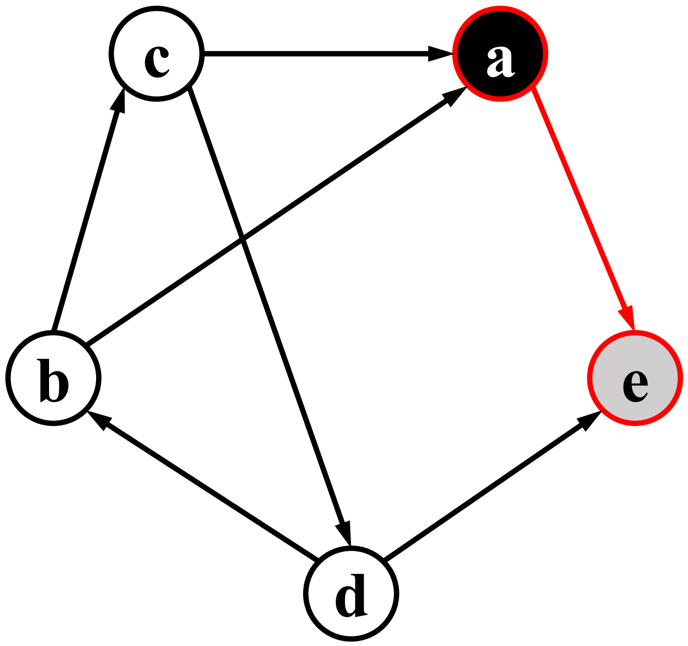


- VISIT(a)

  e is the white neighbour of a

  choose grey a; no white neighbour; colour black

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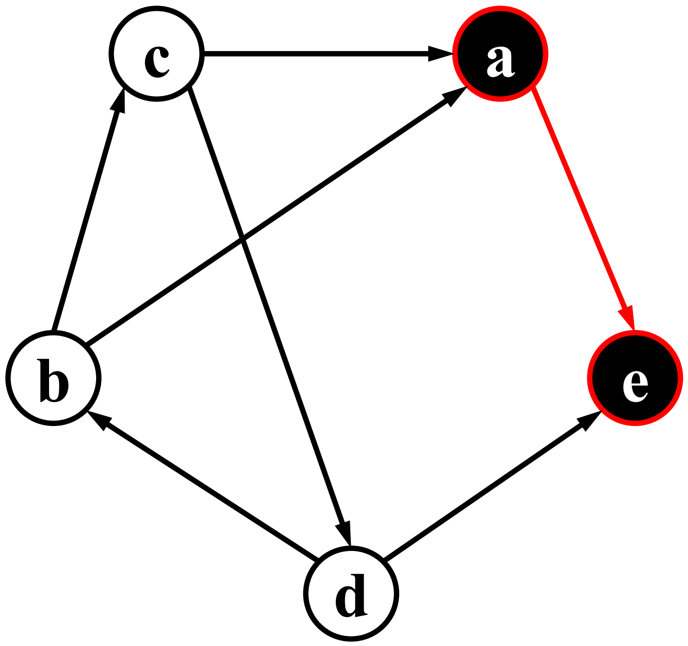


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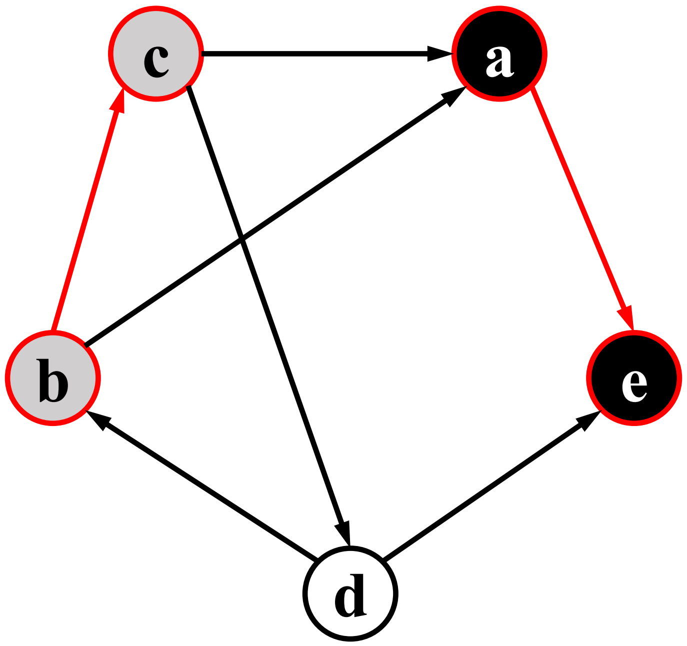



- VISIT(a)

  e is the white neighbour of a

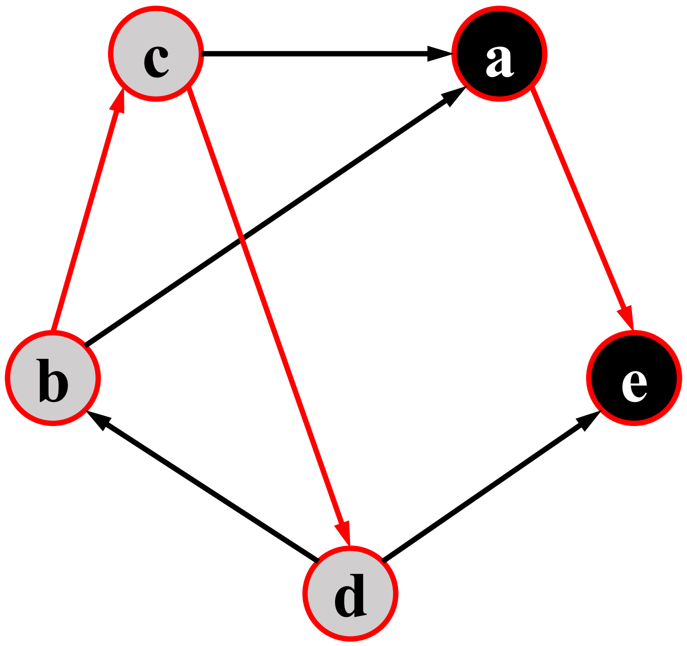
  choose grey a; no white neighbour; colour black

  choose grey e; no white neighbour; colour black

- VISIT(b)

  c is the white neighbour of b

# Illustrating the general traversal algorithm




- VISIT(a)

  e is the white neighbour of a

  choose grey a; no white neighbour; colour black

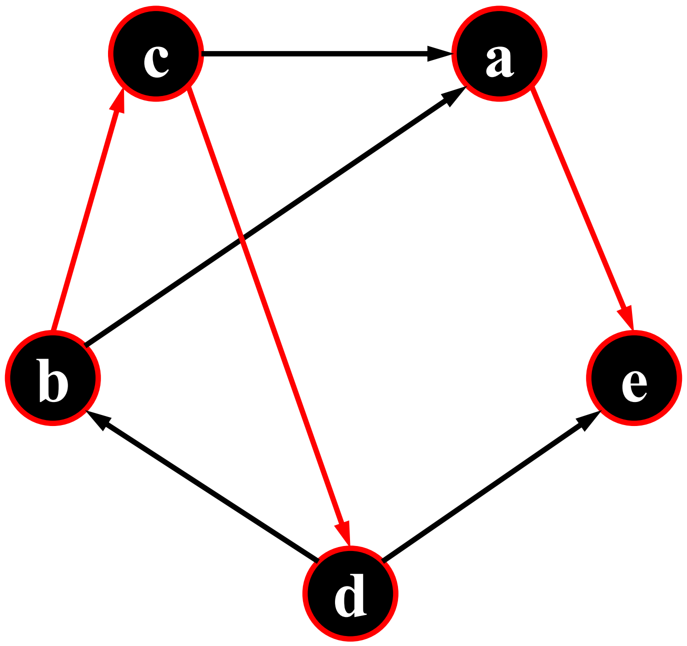
  choose grey e; no white neighbour; colour black

- VISIT(b)

  c is the white neighbour of b

  choose grey c; d is white neighbour

# Illustrating the general traversal algorithm

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  choose grey a; no white neighbour; colour black

  choose grey e; no white neighbour; colour black

- VISIT(b)

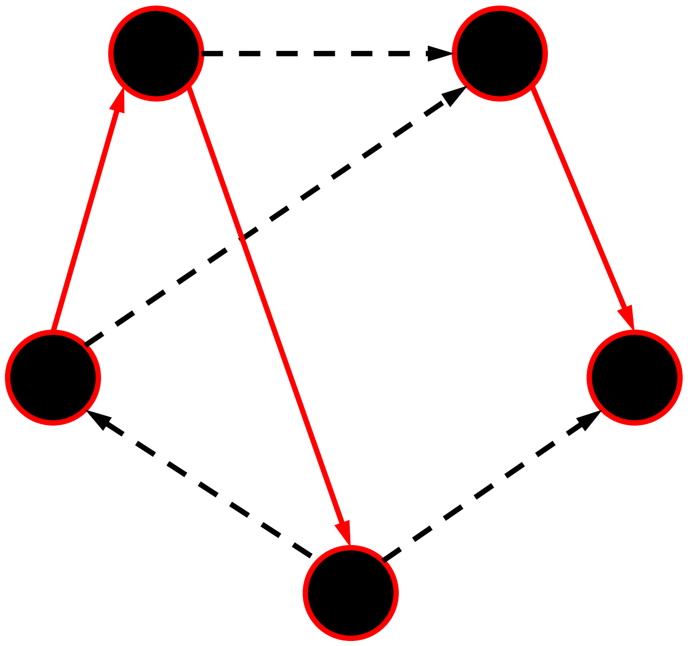
  c is the white neighbour of b

  choose grey c; d is white neighbour

  no more white nodes; all nodes turn black

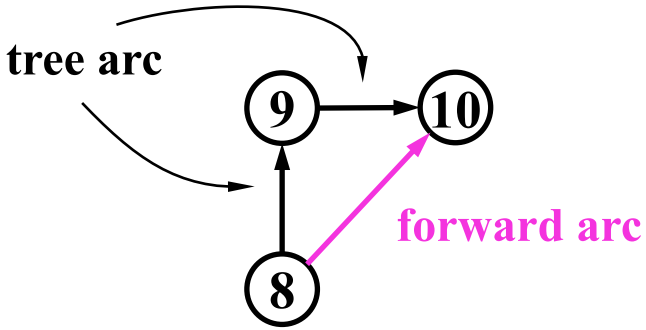
# A search forest

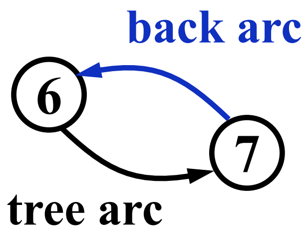
A search forest is a collection of node-disjoint trees that span the digraph and contain, for each node $u$ with $pred[u] \neq NULL$, the arc $(pred[u], u)$.

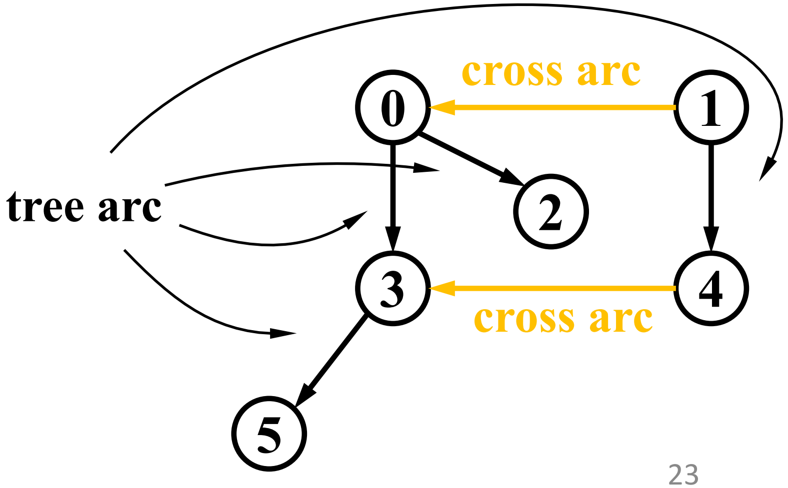


# Traversal Arc Classifications

- Suppose we have performed a traversal of a digraph $G$, resulting in a search forest $F$. Let $(u, v) \in E(G)$ be an arc.
- The arc is called a tree arc if it belongs to one of the trees of $F$. If the arc is not a tree arc, there are three possibilities:
  - a forward arc if $u$ is an ancestor of $v$ in $F$,
  - a back arc if $u$ is a descendant of $v$ in $F$, and
  - a cross arc if neither $u$ nor $v$ is an ancestor of the other in $F$.



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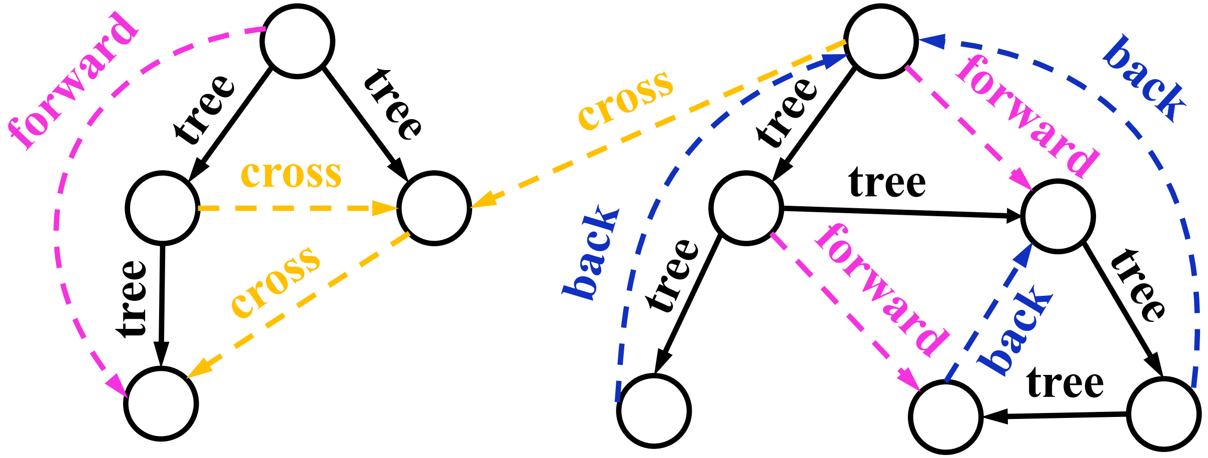
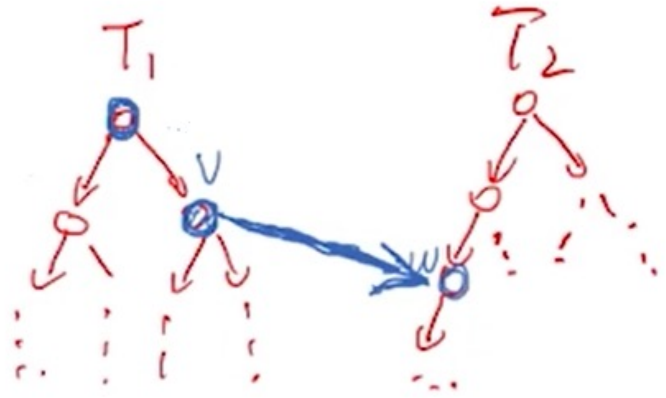


# Facts about Traversal Trees

- **Theorem**: Suppose we run algorithm traverse on $G$, resulting in a search forest $F$. Let $v, w \in V(G)$.

Let $T_1$ and $T_2$ be different trees in $F$ and suppose that $T_1$ was explored before $T_2$. Then there are no arcs from $T_1$ to $T_2$.

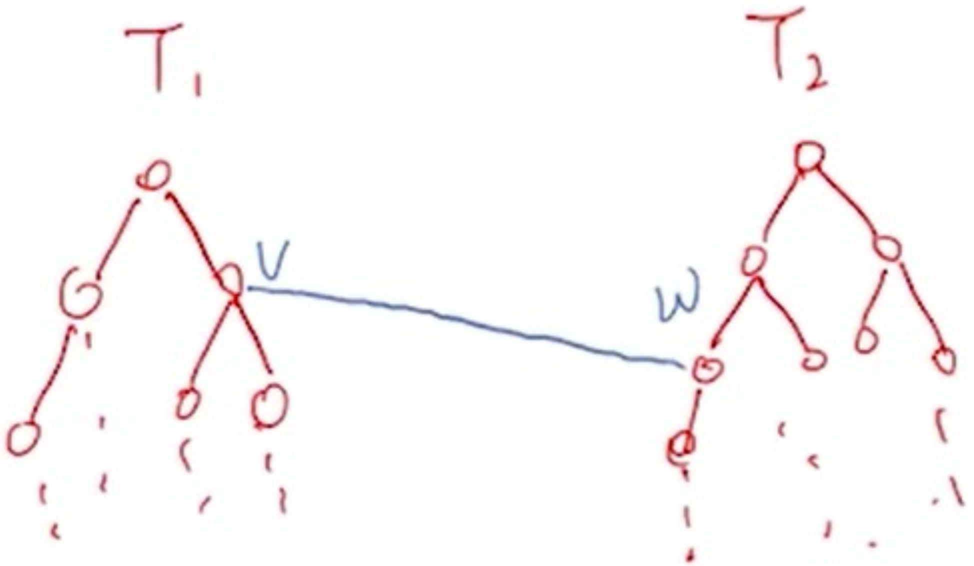
- **Proof**:
  - Assume $(v, w) \in E(G)$, $v \in T_1, w \in T_2$
  - VIST(s) $\left\{ \right.$
    1. A single run of VISIT generates a tree
    2. All nodes reachable from s will be visited
  - With 1 and 2, we have $w \in T_1 \Rightarrow$ contradiction



# Facts about Traversal Trees (Contd.)

- **Theorem**: Suppose we run algorithm traverse on $G$, resulting in a search forest $F$. Let $v,w \in V(G)$.

Then there can be no edges joining different trees of $F$.



# Facts about Traversal Trees (Contd.)

- **Theorem**: Suppose we run algorithm traverse on $G$, resulting in a search forest $F$. Let $v, w \in V(G)$.

Suppose that $v$ is visited before $w$ and $w$ is reachable from $v$ in $G$. Then $v$ and $w$ belong to the same tree of $F$.

- **Proof**.
  - Let $v \in T$ and $s$ be the root of $T$.
  - Because $w$ is reachable from $v$, $v$ is reachable from $s$, then $w$ is reachable from $s$
  - Then, $w$ should be in the same tree as $v$.

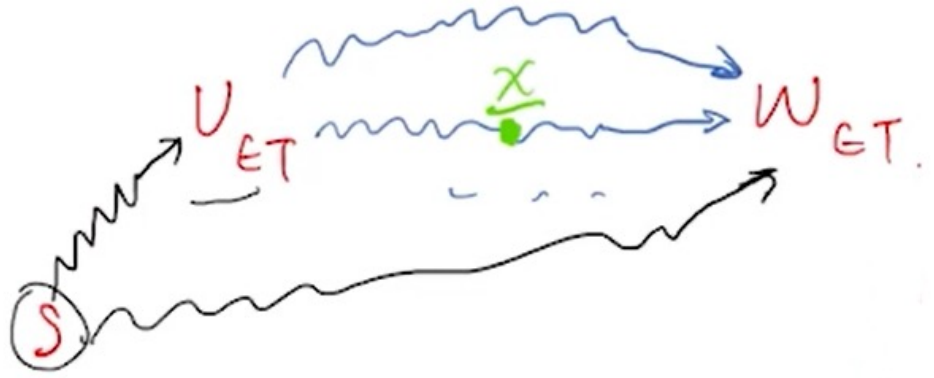
# Facts about Traversal Trees (Contd.)

- **Theorem**: Suppose we run algorithm traverse on $G$, resulting in a search forest $F$. Let $v$,$w$∈$V(G)$.

Suppose that $v$ and $w$ belong to the same tree $T$ in $F$. Then any path from $v$ to $w$ in $G$ must have all nodes in $T$.

- **Proof.** For any node $x$ in any path from $v$ to $w$

1. v, w ∈ $T$
2. v is reachable from $s$ the root of $T$, then $x$ is reachable from $s$
3. By 2, $x$ ∈ $T$



# Complexity Analysis: General Graph Traversal

**Algorithm 2** Traverse.

**function** TRAVERSE(digraph $G$)
    array $color[0..n-1]$
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    **for** $u \in V(G)$ **do**
        $color[u] \leftarrow$ WHITE
    **end for**
    **for** $s \in V(G)$ **do**
        **if** $color[s] =$ WHITE **then**
            VISIT($s$)
    **return** $pred$

# Complexity Analysis: Visit(s)

**Algorithm 1** Visit.

**function** VISIT(node $s$ of digraph $G$)
    $color[s] \leftarrow$ Grey
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        choose a Grey node $u$
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        **else**
            $color[u] \leftarrow$ Black

# Runtime Analysis of Traverse

- The initialization of the array $colour$ takes time $\Theta(n)$ so $traverse$ is in $\Theta(n + t)$, where $t$ is the total time taken by all the calls to $visit$.
- We execute the while-loop of $visit$ in total $\Theta(n)$ times since every node must eventually move from white through grey to black. In each loop:
  - The time taken in choosing grey nodes is $\Theta(1)$ each time.
  - The time taken to find a white neighbour involves examining each neighbour of $u$ and checking whether it is white, then applying a selection rule.
    - If adjacency matrix is used, we need to scan the whole row, which takes $\Theta(n)$
    - If adjacency lists are used, we only need $\Theta(|L_i|)$ for finding white nodes in the adjacency list of node $i$.
- So the running time of $traverse$ is $\Theta(n + (n + \sum_i |L_i|)) = \Theta(n + m)$ if adjacency lists are used, and $\Theta(n + n^2) = \Theta(n^2)$ if the adjacency matrix format is used.

# Runtime Analysis of Traverse (Contd.)

- So, for simple selection rules and assuming a sparse input digraph, the adjacency list format seems preferable.

- If more complex selection rules are used, for example, rules that choose a single grey node $\Theta(n)$ time by scanning the whole list of grey nodes, then the running time is asymptotically $\Theta(n^2)$ regardless of the data structure.

# Graph Traversal

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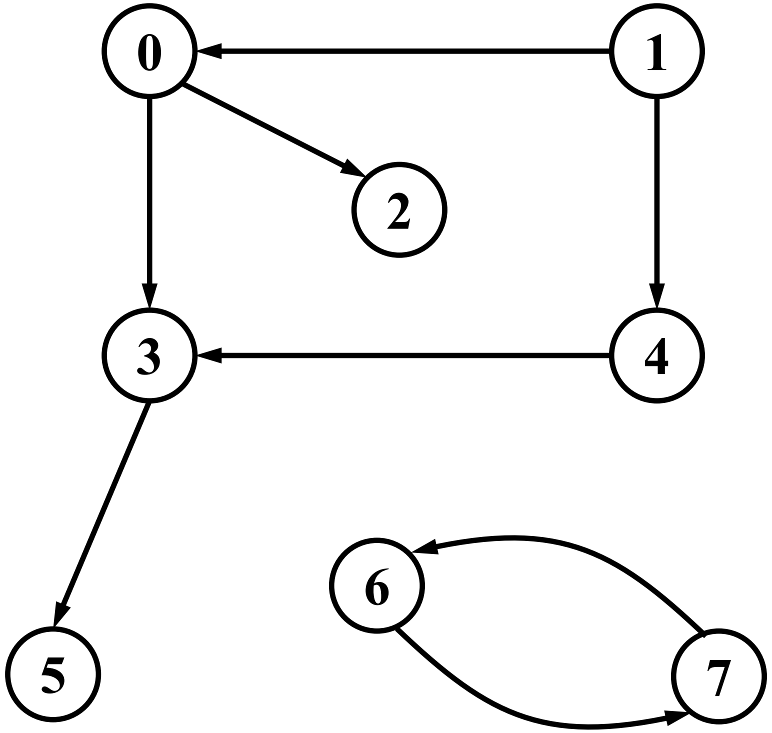
how to choose?

# General graph traversal: example

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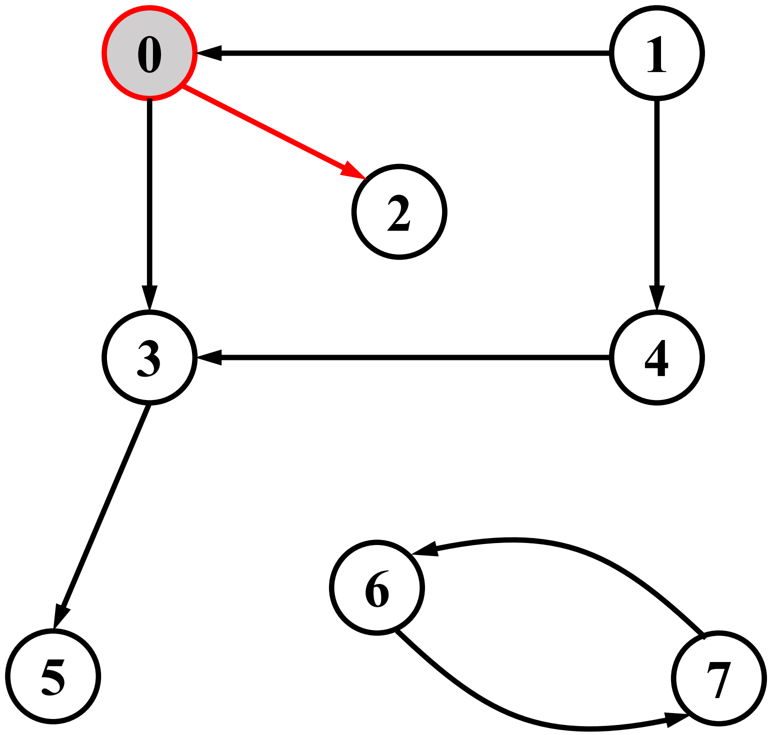


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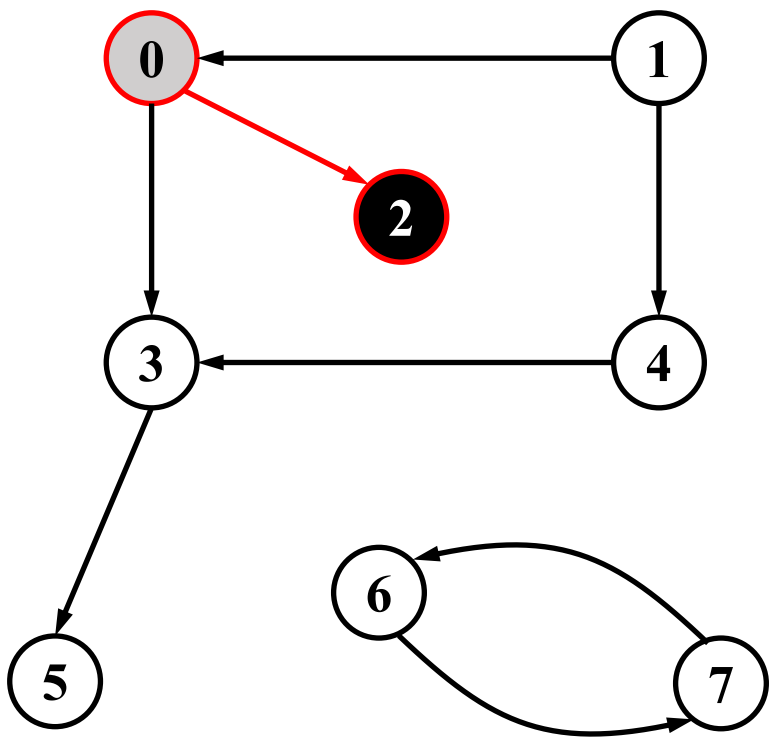


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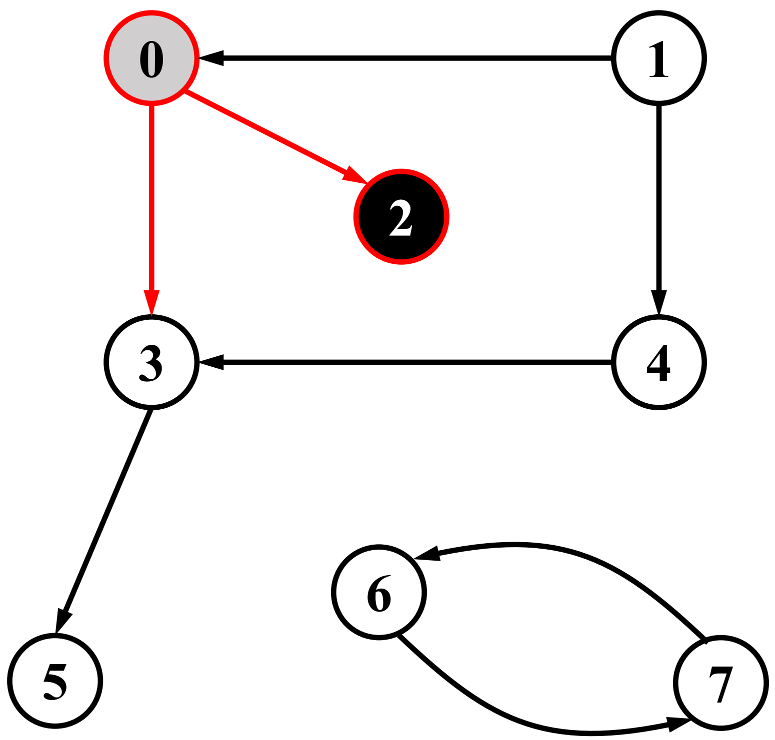


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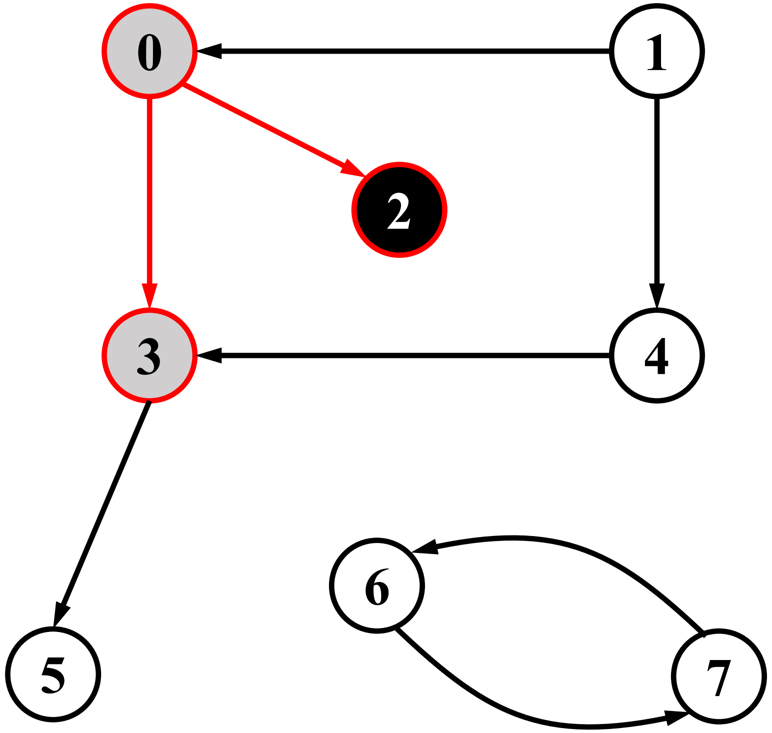


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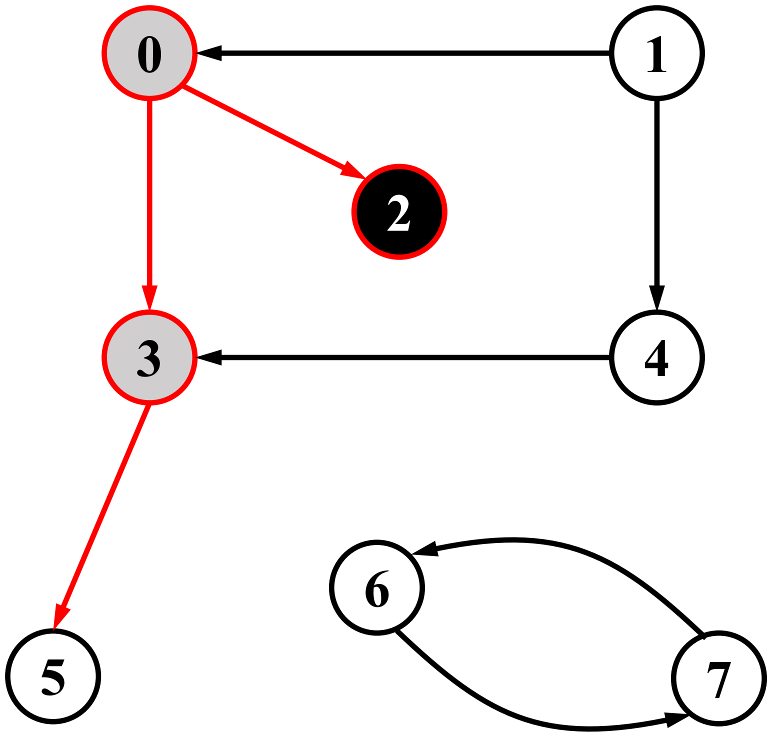


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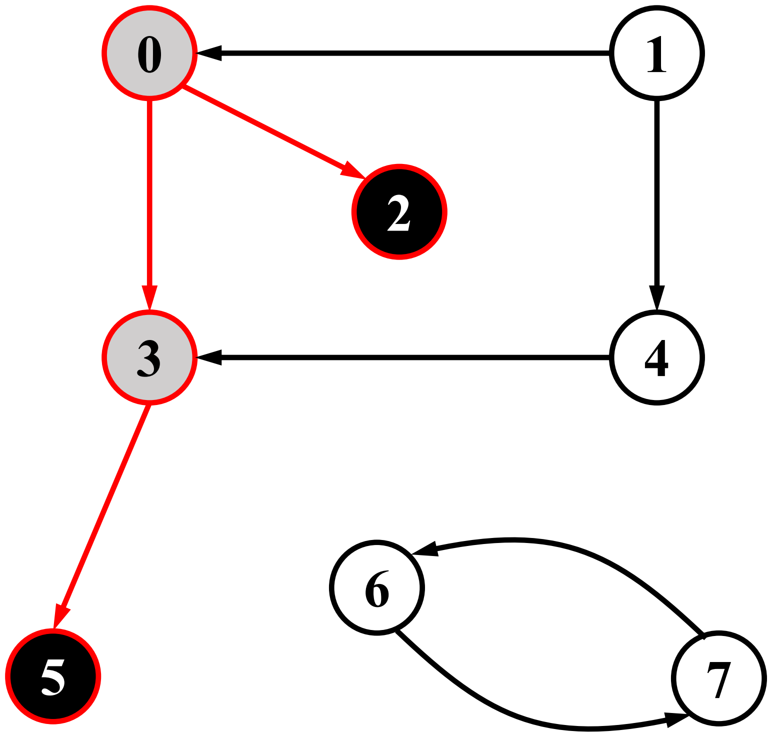


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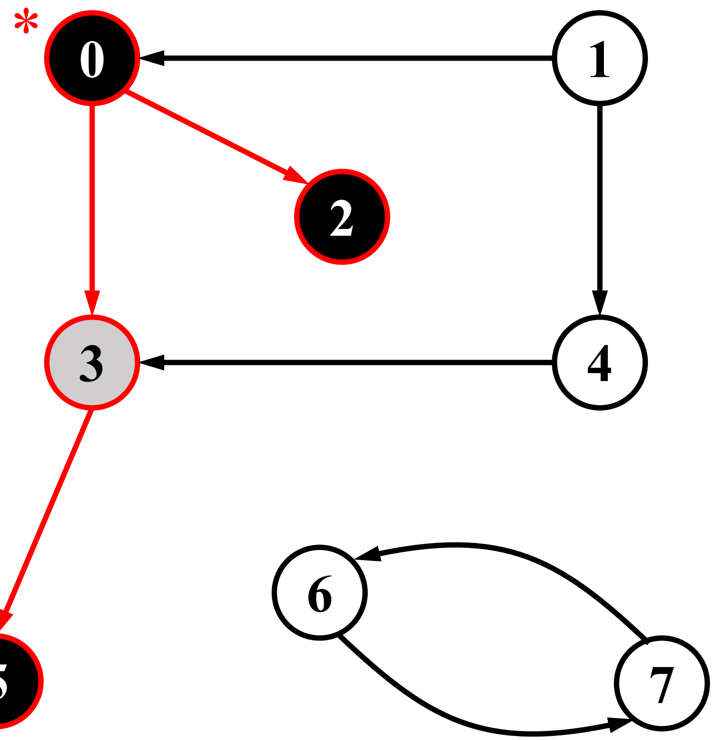


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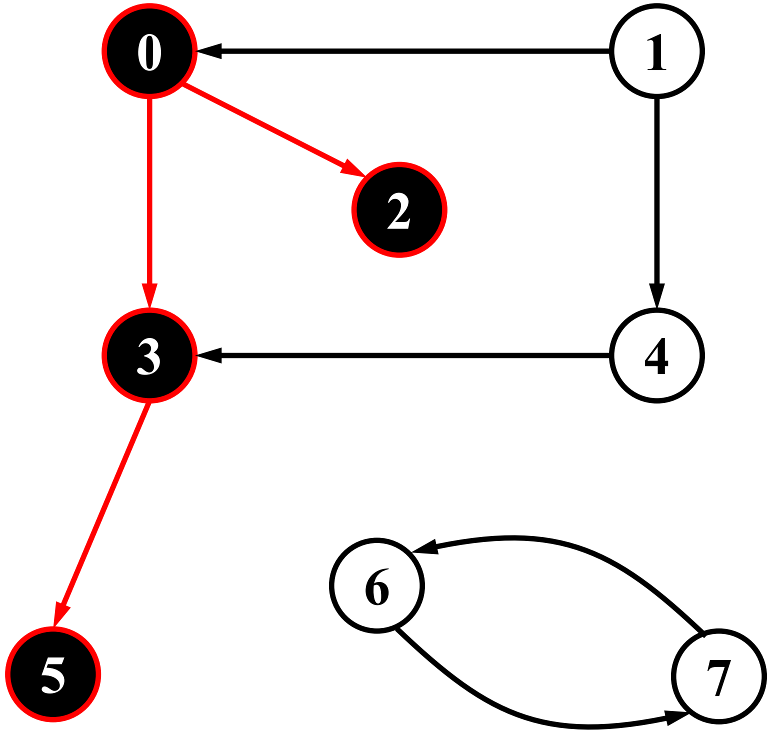


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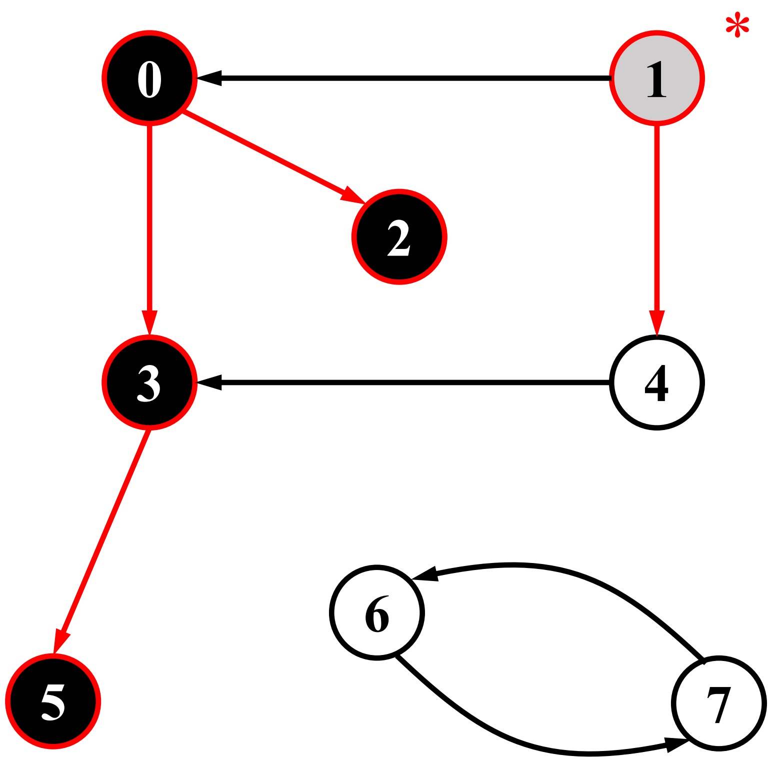


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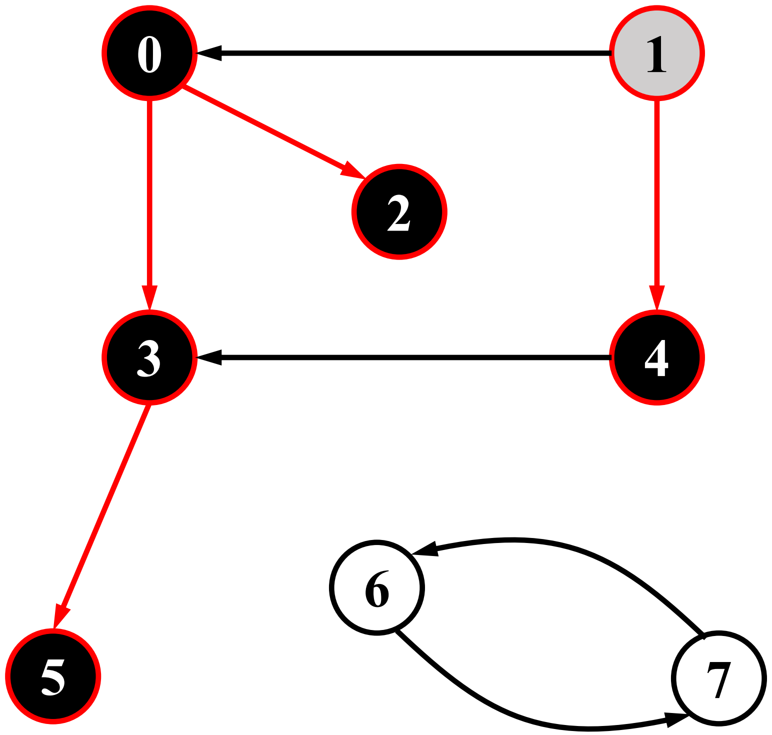


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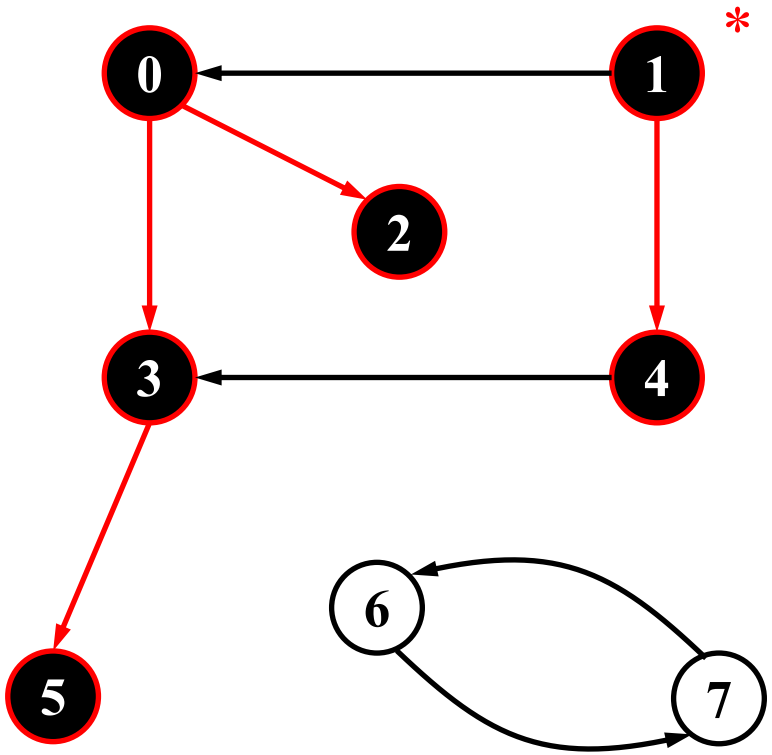


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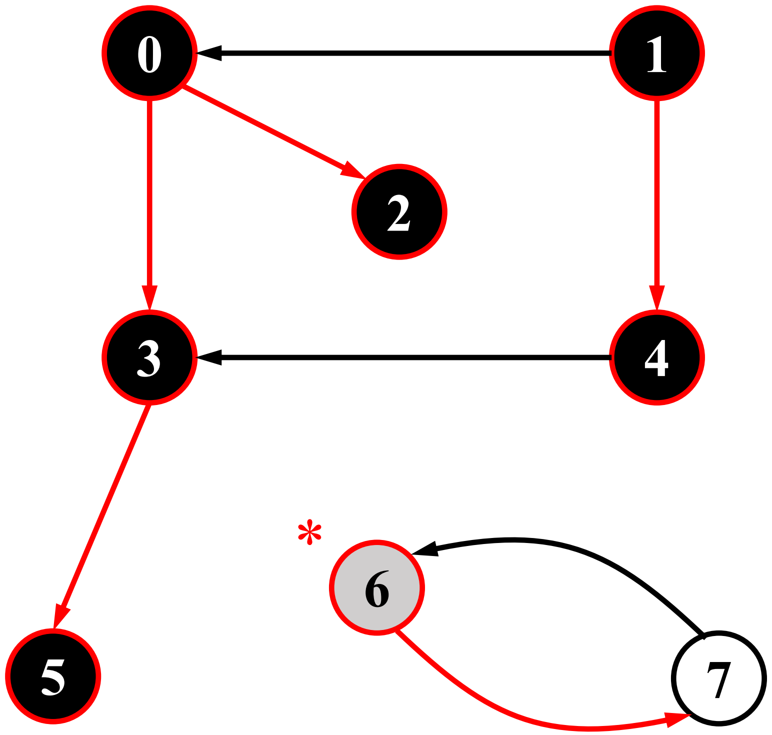


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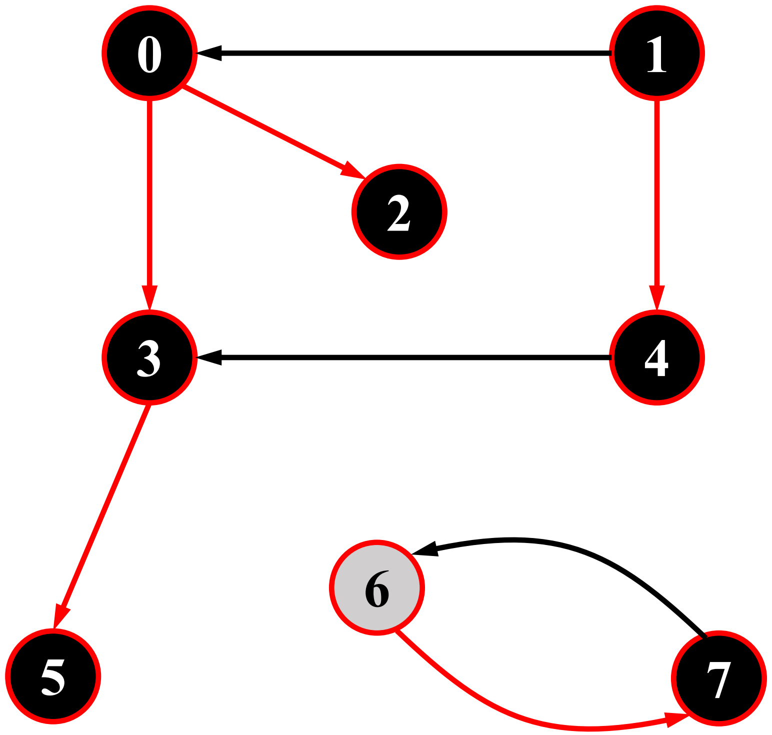


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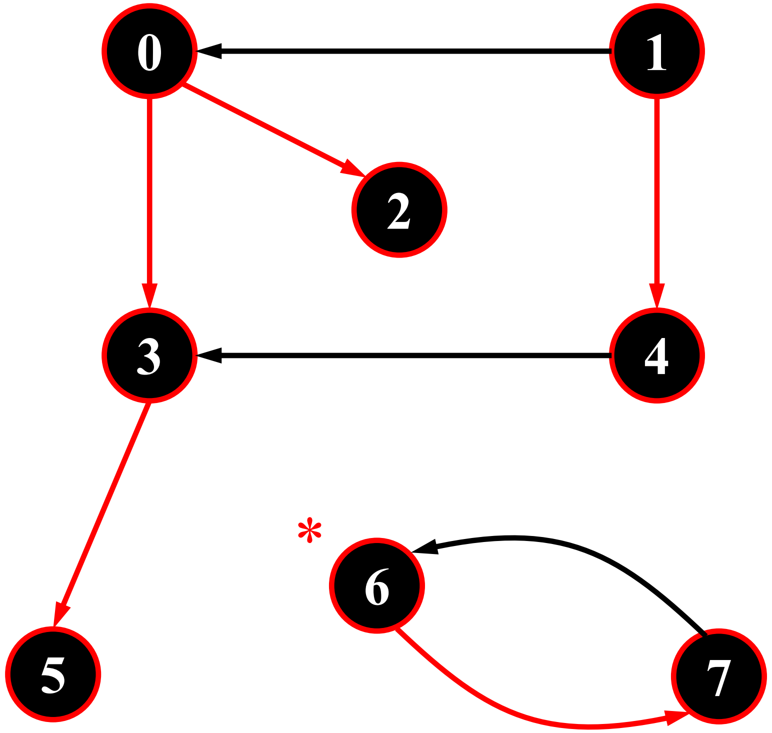
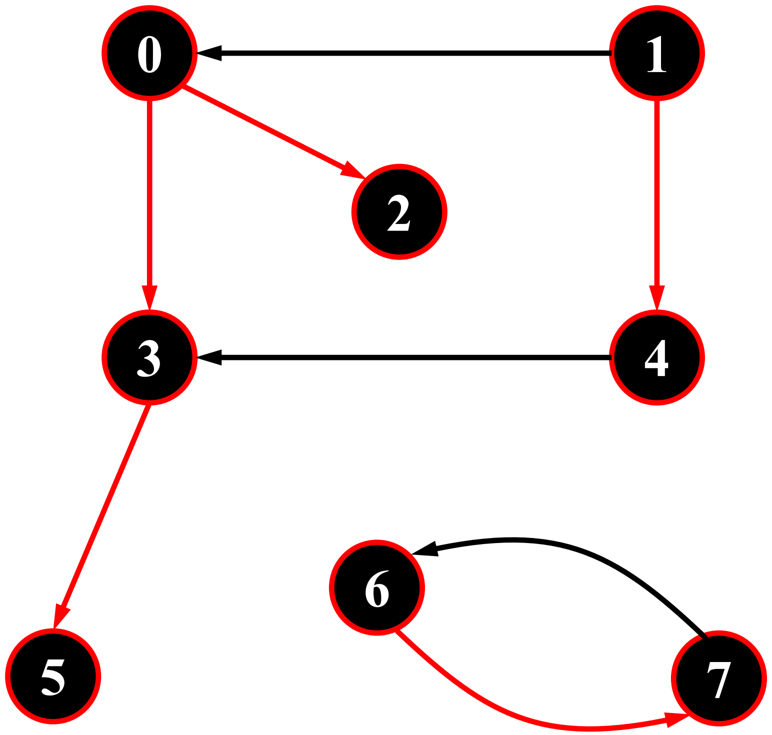



- Emmmm……

# General graph traversal: example




**Red line: search forest**

# SUMMARY

- Graph Traversal Algorithm
- Facts about Traversal Trees
- Complexity Analysis
- Illustrative Example