Binary Trees and Lower Bound of Sorting

CPMPSCI 220: WEEK 8.3

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RECAP: Sorting Algorithm Summary

- We have looked at several different sorting algorithms so far.

Algorithm	**Best**	**Worst**	**Average**
Selection Sort	n^2	n^2	n^2
Insertion Sort	n	n^2	n^2
Mergesort	$n \log n$	$n \log n$	$n \log n$
QUICKSORT	$n \log n$	n^2	$n \log n$
Heapsort	$n \log n$	$n \log n$	$n \log n$

- The best worst/average case time complexity we have seen so far is $n \log n$.

Can we do better?

- In other words, does there exist a sorting algorithm that can sort an input list of n items with time complexity $\Theta(f(n))$ in the worst/average case s.t.

$$\lim_{n\to\infty} \frac{f(n)}{n \log n} = 0$$

- Unfortunately, no such algorithm exists. We prove by constructive proof rather than the non-existence of such things.

Decision Tree

- Suppose we are given a list of n distinct keys, there are $n!$ possible permutations. A correct sorting algorithm must be able to sort all $n!$ permutations correctly.

- Any correct sorting algorithm must be able to distinguish between all $n!$ permutations.

- If it can't distinguish between two permutations, say $S1$ and $S2$, it would rearrange $S1$ and $S2$ in the same way and we get at least one incorrect output.

- Any sorting algorithm must have gained some information about the input permutation S and acted based on these information. We can model this by using a **decision tree.**

Definitions

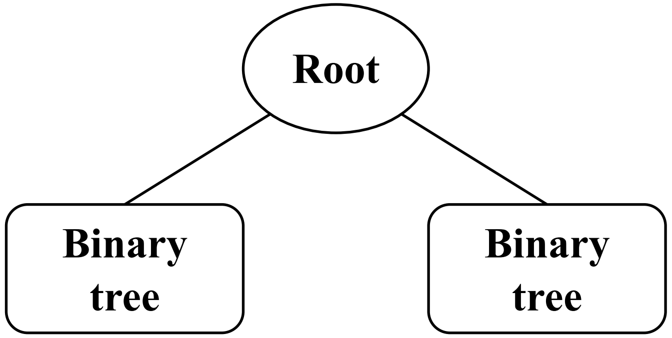
- **Definition (Very-very recursively)**

A binary tree is an object that is either empty or consists of a root node connected to an ordered pair of binary trees.

Possibility 1

Empty tree

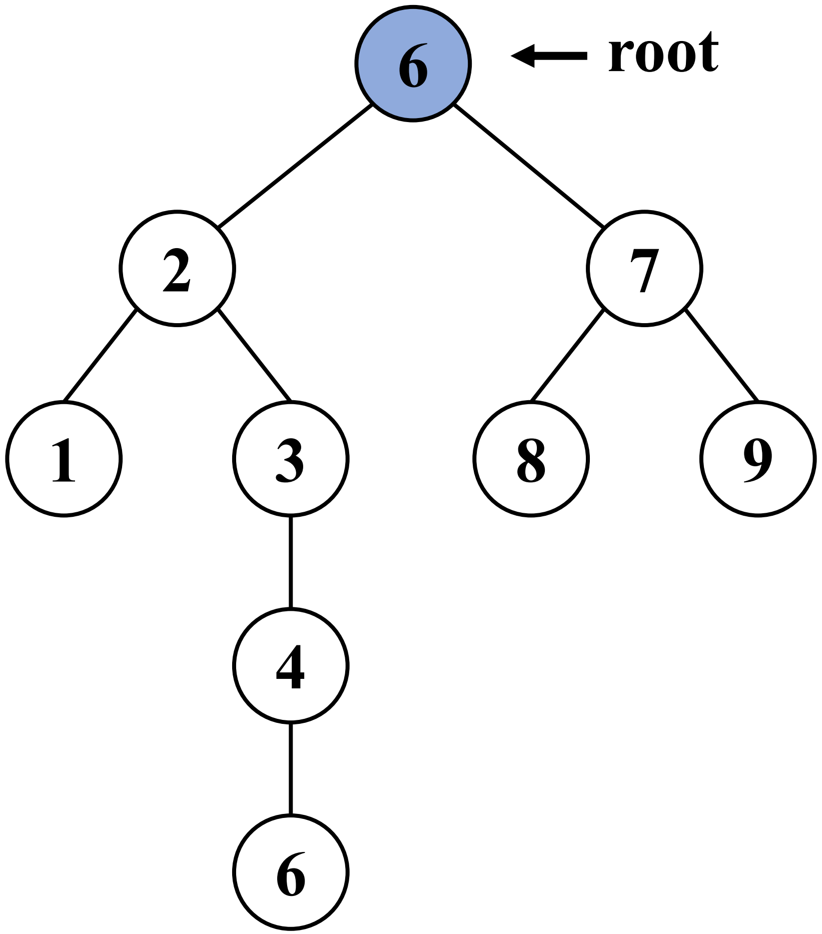
Possibility 2



Definitions (Contd.)

- **Definition (Intuitive definition)**

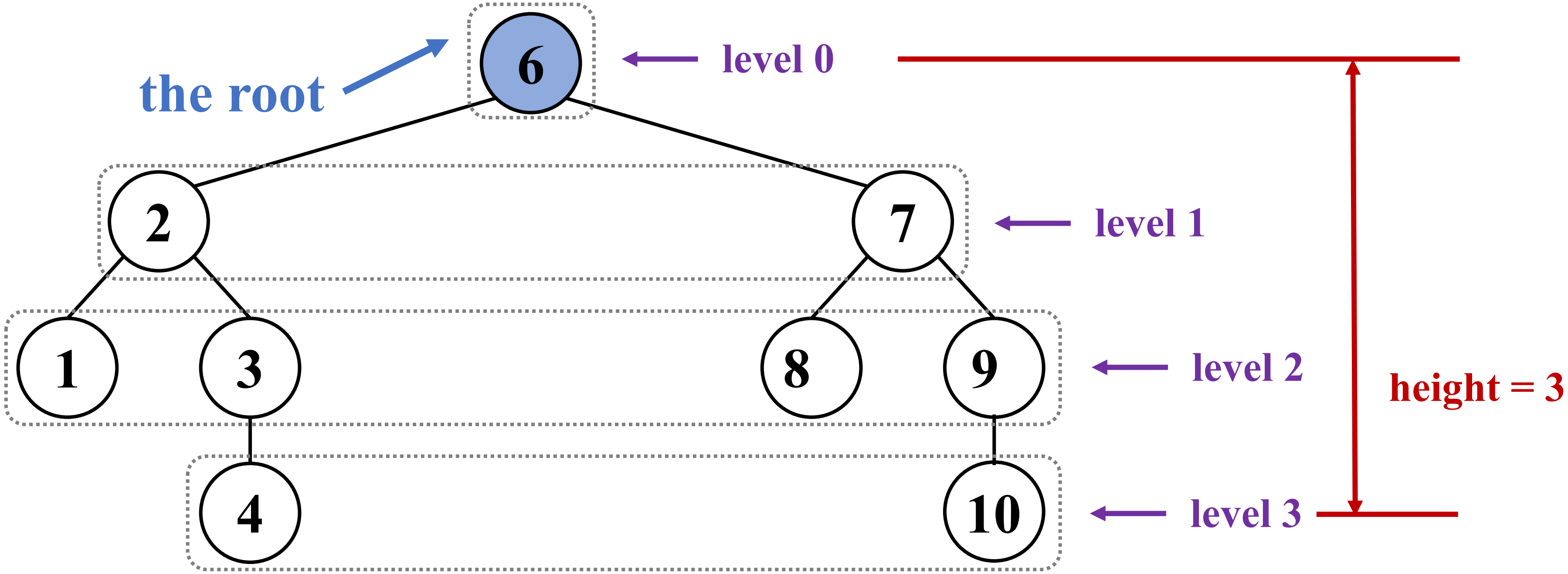
A binary tree is an object that is either empty or consists of nodes, which are connected to either 0,1, or 2 nodes under it. There is one special node called the root.



Definitions (Contd.)

- **Definition (Root and Height)**

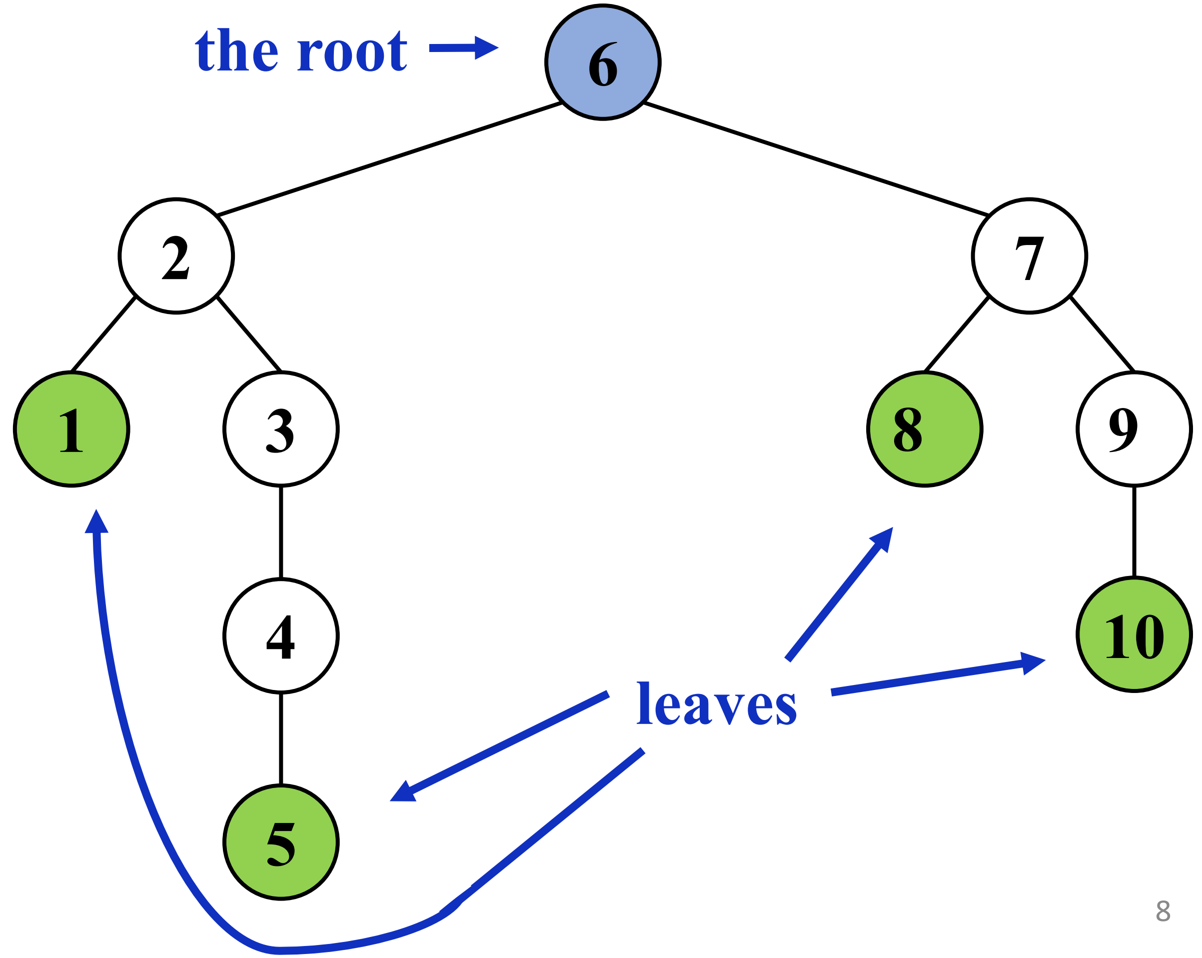
The level of a node is the length of the (unique) path from the root to that node. The height of a binary tree is the maximum level of its nodes.



Definitions (Contd.)

- **Definition (Leaves)**

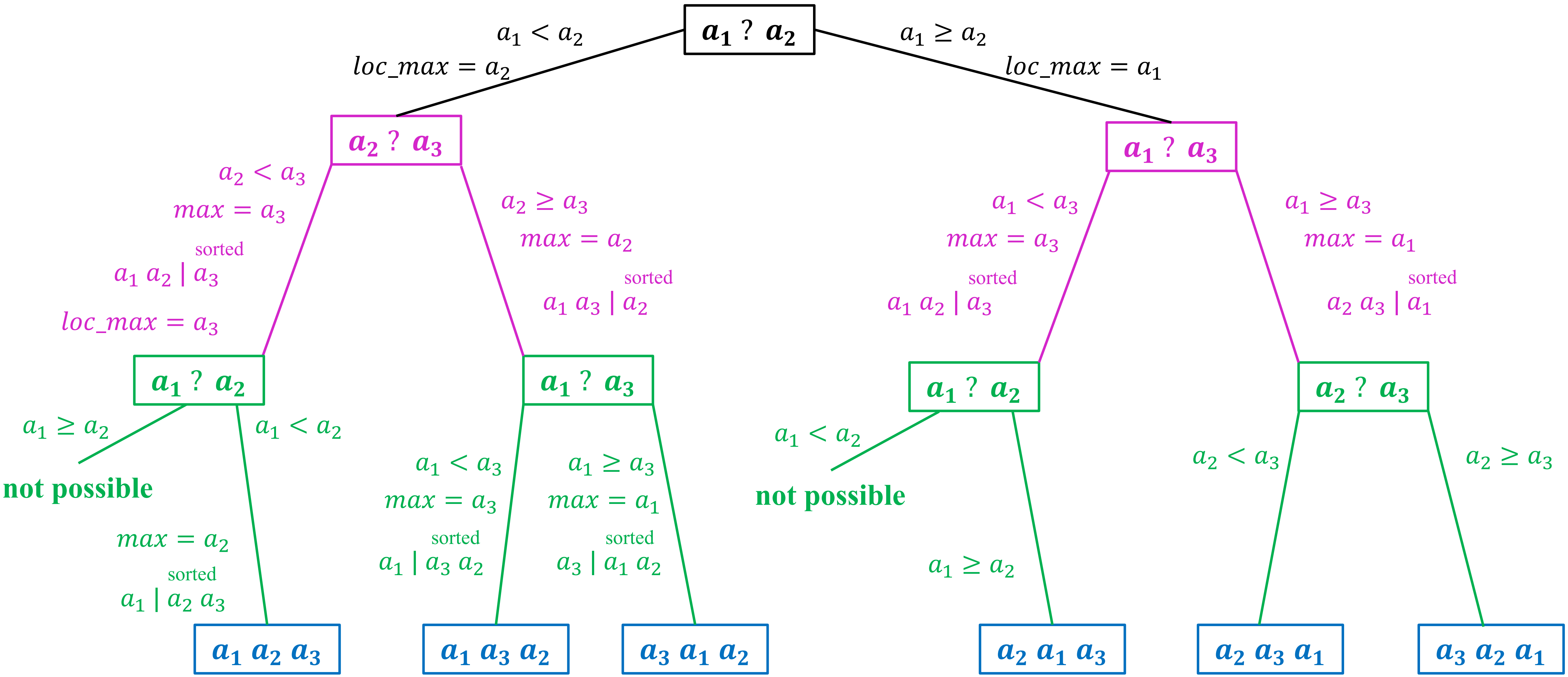
A leaf is a node, which is not connected to any other nodes on the next level.



Decision Tree: Selection Sort

1. Split the input list into sorted and unsorted sublists.
2. Sorted sublist is initially empty, and the unsorted sublist is the whole list.
3. Find a maximal element of the unsorted part by sequential scan.
4. Move the maximal element to the head of the sorted part.
5. If the unsorted sublist is empty, then terminate else go to step 3

Selection sort: $a = [a_1, a_2, a_3]$ max of a: $loc\_max = a_1$



Selection sort: $a = [a_0, \ldots, a_{n-1}]$

- The number of leaves is the number of all possible sorted orders on n elements. There are $n!$ ordered lists on n elements.

→ Thus, there are $\boldsymbol{n}!$ leaves

- The level of a leaf is the number of comparisons the algorithm needs to do to achieve this leaf.

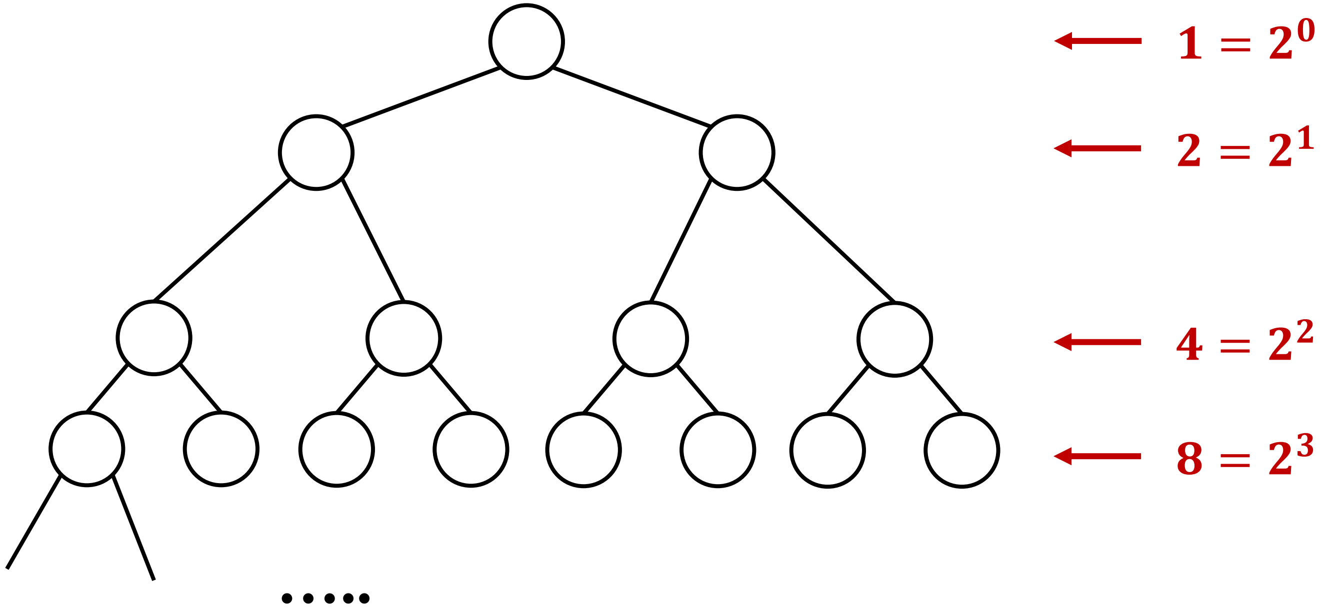
- The height of a decision tree is the max possible number of comparisons we may need to do using this algorithm.

→ The height is the number of comparisons in the worst case

- The runtime in the worst case is at least $\Omega(n \log_2 n)$

The Lower Bound of Sorting

- Any decision tree is a binary tree. Thus, if we want to find the lower bound on the number of comparisons in the worst case, we need to find the smallest possible height of a binary tree with $n!$ leaves.



The Lower Bound of Sorting (Contd.)

- The height is the smallest if every level except for the last one is full.
- If the level i is full then there are 2^i elements.
- The last level contains all leaves
- h is the last level. All leaves could not fit into (h−1) level.

$$2^{h-1} < n!$$

- On the h−th level, there are 2^h nodes and all leaves can be on this level:

$$n! \leq 2^h$$
$$\log_2 n! \leq h$$

- The smallest value for h is $\log_2(n!)$ → $h \in \Omega(n \log_2 n)$

SUMMARY

- Decision Tree
- Lower Bound of Comparison-based Sorting
- Every comparison-based sorting algorithm takes $\Theta(n \log n)$ in the worst case.