

Complexity

Instructor: Meng-Fen Chiang

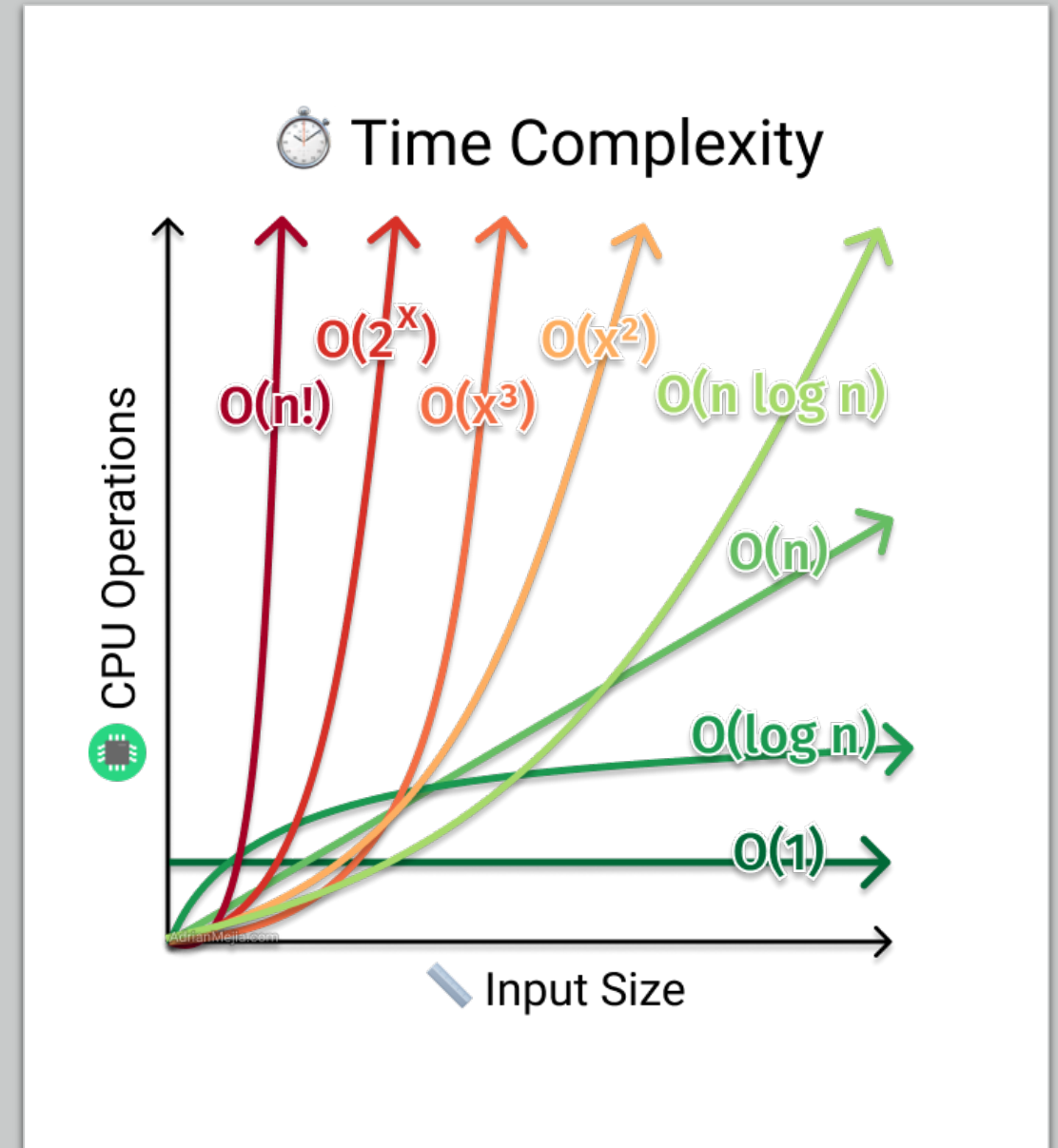
COMPCSI220: WEEK 8



Slides adapted from Mark Wilson, Georgy Gimel'farb, Simone Linz and Tanya Gvozdeva

OUTLINE

- Analysis of Algorithm
 - Illustrative examples
- Time Complexity
 - How to measure running time?
 - Illustrative examples
- Asymptotic Notation



Analysis of Algorithms

- What to analyze
 - Domain of definition – what inputs are legal?
 - Correctness – does it solve the problem for all legal inputs?
- Efficiency: its **maximum** or **average** requirements to resources:
 - Runtime
 - Memory space
 - Other resources
- There could be different **implementations** of the same algorithm: different programs, programming languages, computer platforms, operating systems, etc.
- The analysis should be isolated from a particular implementation.

Complexity

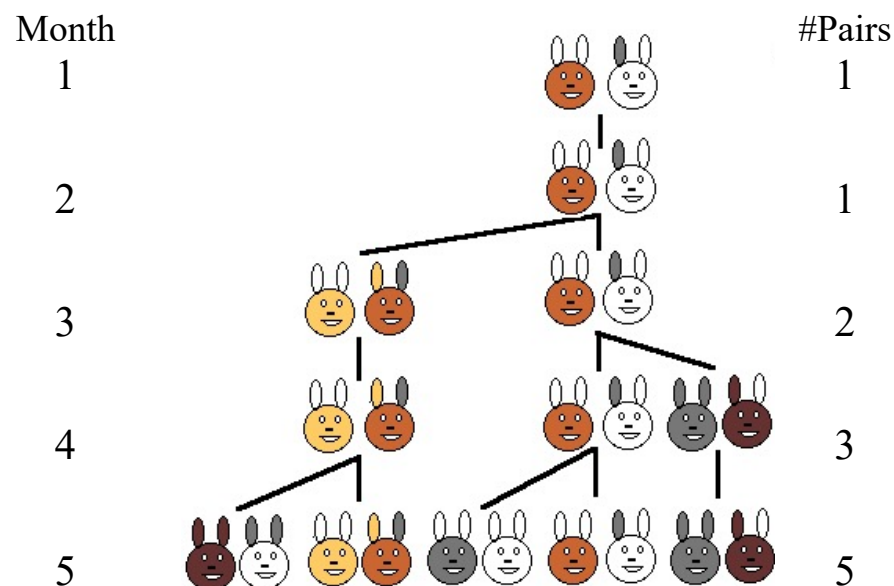
- In a practical sense, complexity is an estimate of resource requirement
 - How long would it take to drive from Auckland to Wellington?
 - How much petrol would we need to drive from Auckland to Wellington?

Complexity (Contd.)

- Time complexity
 - An estimate of how long it would take to complete a task
- Space complexity
 - An estimate of how much memory the task might require for completion

Example: Fibonacci Numbers

- Italian mathematician, Leonardo Fibonacci (1170–1250). A problem of breeding rabbits.
 - A pair of rabbits takes a month to become mature and start to have pairs of baby rabbits
 - Each pair of newly born rabbits also take a month to reach maturity.
 - How many pairs of rabbits, $F(n)$ would there be after n months?



Fibonacci numbers:

$$F(n) = F(n - 1) + F(n - 2)$$

$$F(0) = 0, \quad F(1) = 1$$

This immediately suggests a **recursive algorithm**

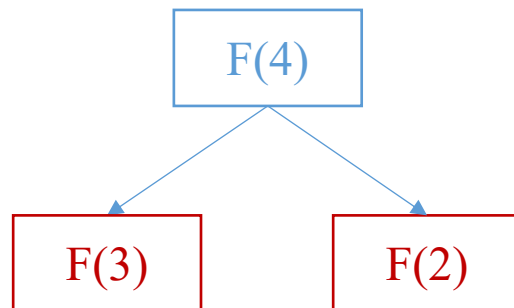
Algorithm 1 Slow method for computing Fibonacci numbers

```
1: function SLOWFIB(integer n)
2:   if  $n < 0$  then return 0
3:   else if  $n = 0$  then return 0
4:   else if  $n = 1$  then return 1
5:   else return SLOWFIB( $n - 1$ ) + SLOWFIB( $n - 2$ )
```

Correctness: The algorithm SLOWFIB is obviously correct

Efficiency: **This algorithm is not efficient!** It does a lot of **repeated computation**

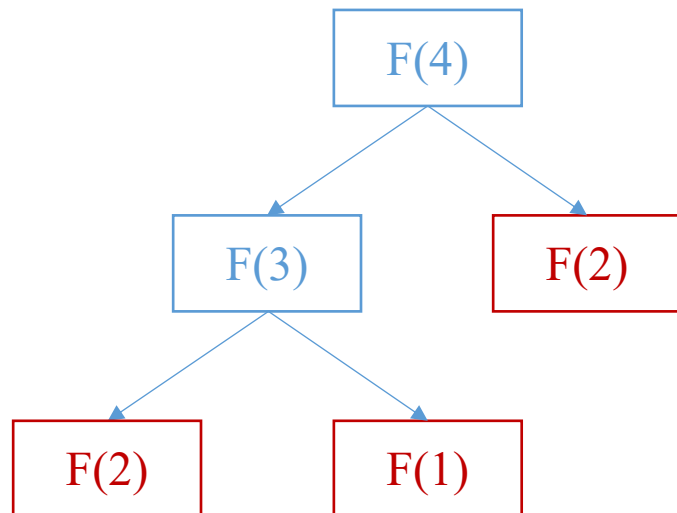
Example – $F(4)$



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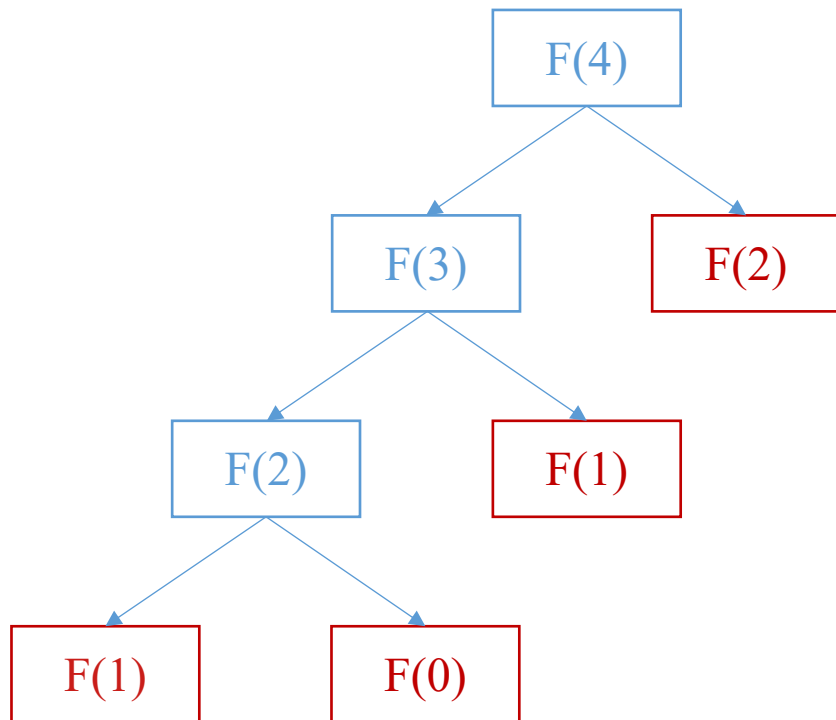
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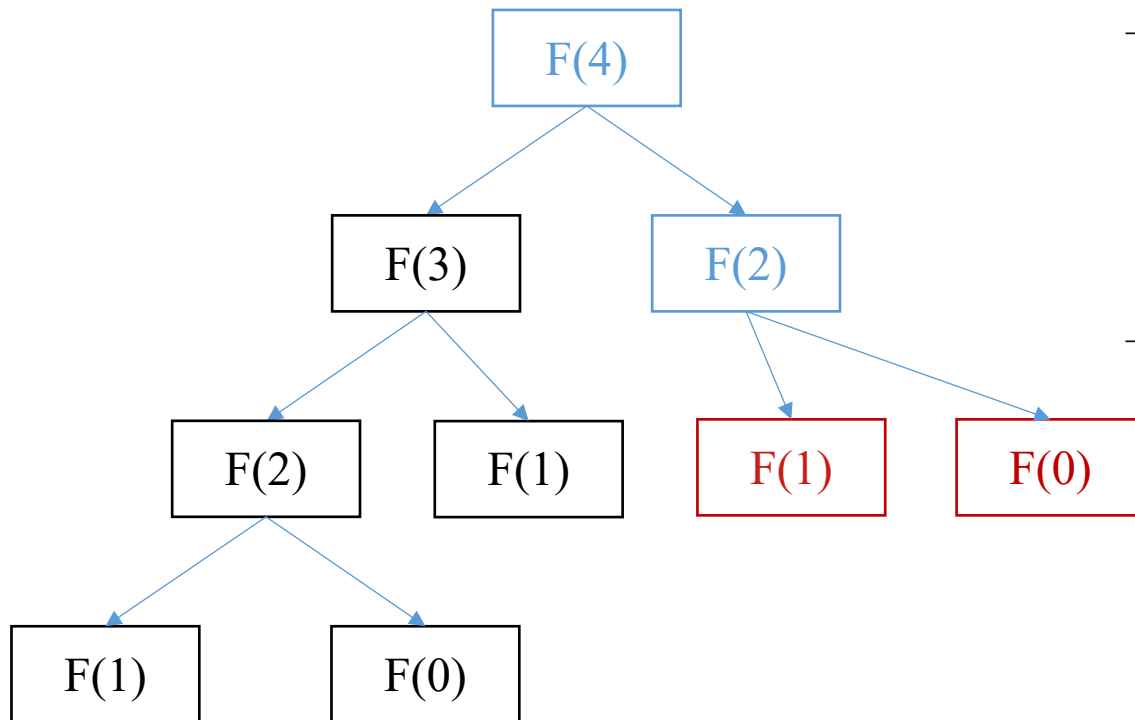
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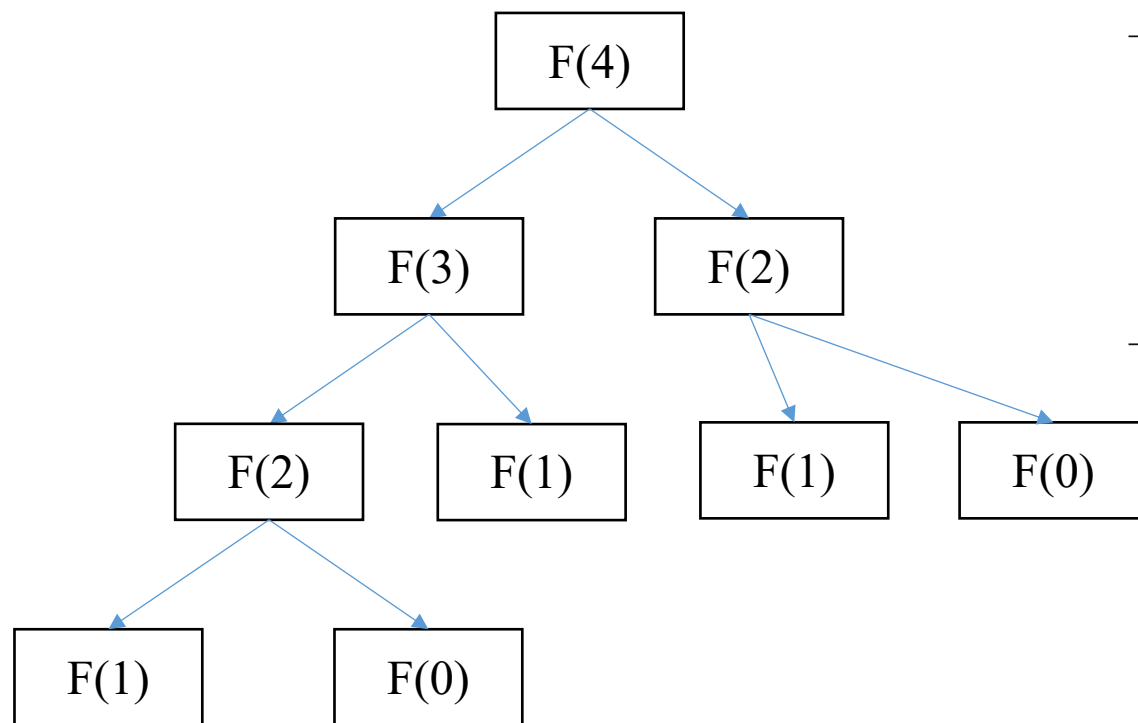
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Algorithm 1 Slow method for computing Fibonacci numbers

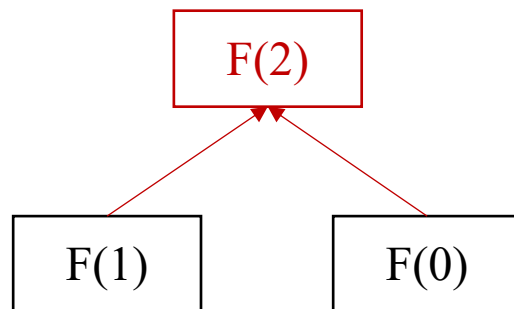
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```

- With a small (fixed) amount of extra space, we can do better, by working from the **bottom up** instead of from the top down.

Algorithm 2 Fast method for computing Fibonacci numbers

```
1: function FASTFIB(integer n)
2:     if  $n < 0$  then return 0
3:     else if  $n = 0$  then return 0
4:     else if  $n = 1$  then return 1
5:     else
6:          $a \leftarrow 1$                 ▷ stores  $F(i - 1)$  at bottom of loop
7:          $b \leftarrow 0$                 ▷ stores  $F(i - 2)$  at bottom of loop
8:         for  $i \leftarrow 2$  to  $n$  do
9:              $t \leftarrow a$ 
10:             $a \leftarrow a + b$ 
11:             $b \leftarrow t$ 
12:     return  $a$ 
```

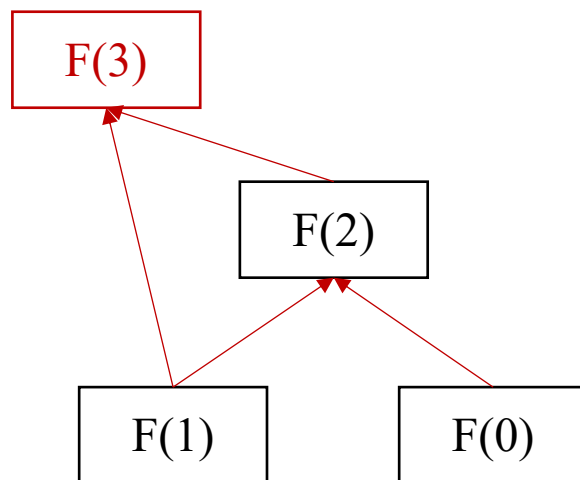
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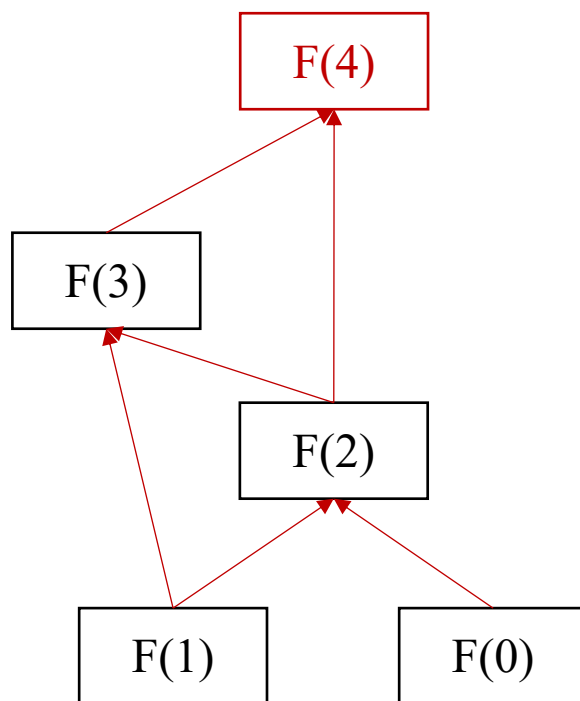
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Example – $F(4)$



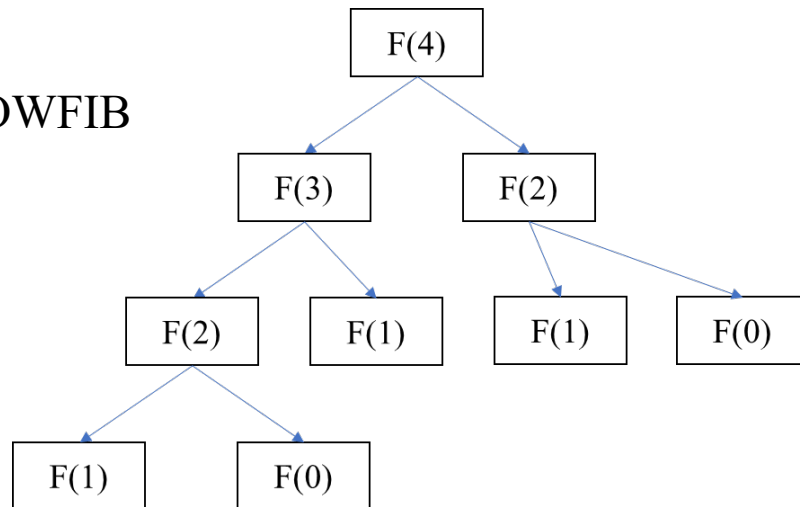
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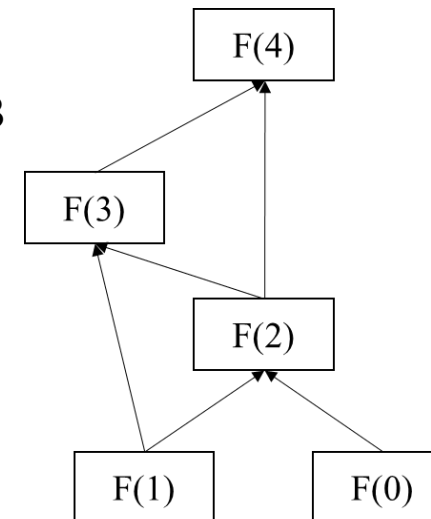
Analysis of the Fast Algorithm

- It is easy to see that the number of additions, function calls, etc needed by **FASTFIB** to compute $F(n)$ has the form $An + B$ for some constants A, B .

SLOWFIB

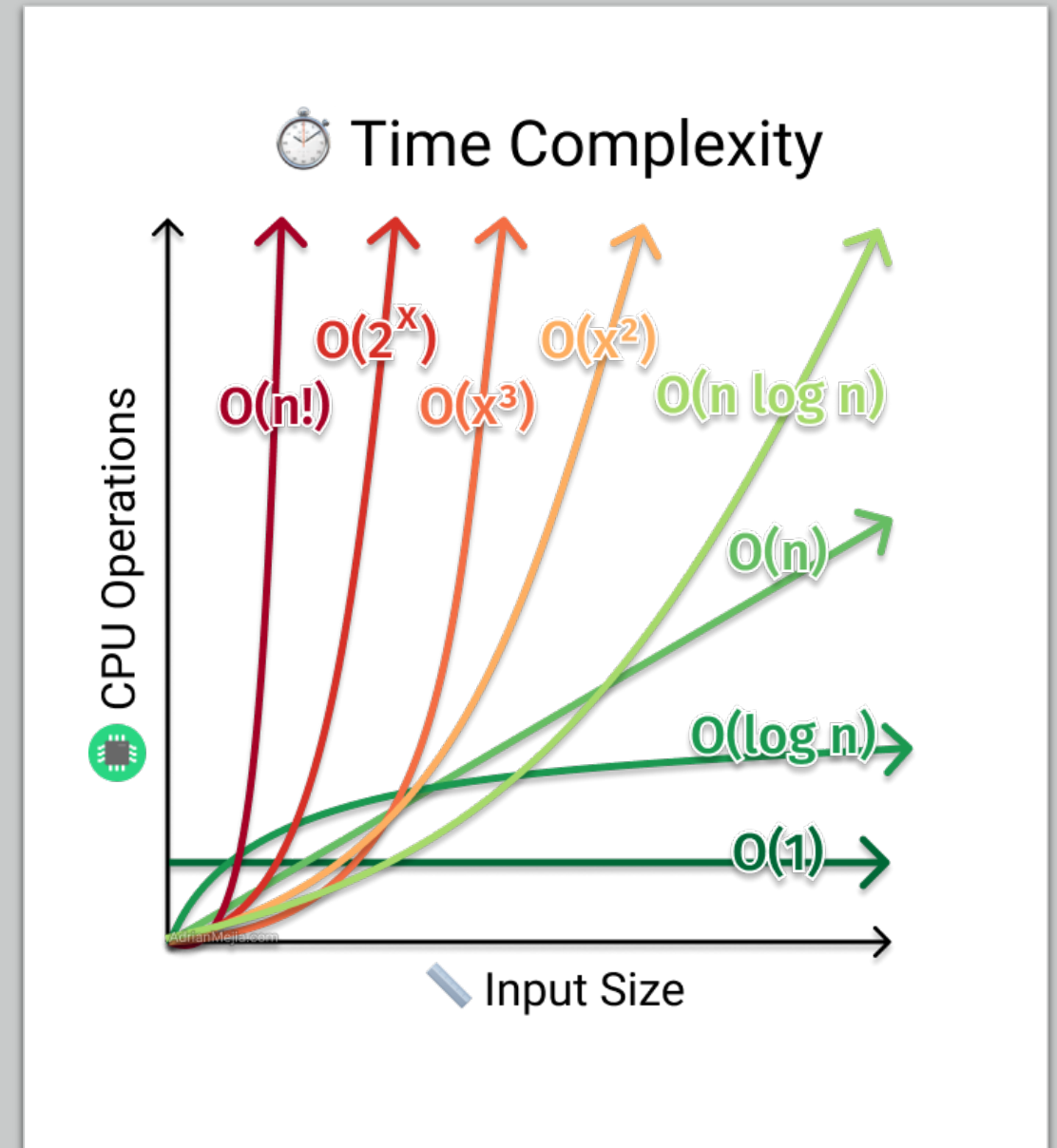


FASTFIB



SUMMARY

- Analysis of Algorithm: Fibonacci numbers
 - Slow implementation
 - Fast implementation
- **Time Complexity**
 - How to measure running time?
 - Illustrative examples
- Asymptotic Notation



Time Complexity

- The actual running time of an algorithm \mathcal{A} on a given input X depends on implementation, such as programming languages used, operating systems and hardware.
- The **running time usually grows with the size of the input**. Running time for very small inputs is not usually important; it is large inputs that cause problems if the algorithm is inefficient.

Elementary Operations

- We use the concept of **elementary operation** as our basic measuring unit of running time. This is any operation whose execution time does NOT depend on the size of the input.
- Typically, a standard unary or binary arithmetic operation:
 - Negation (-5)
 - Addition / subtraction ($5 + 37$; $350 - 75$)
 - Multiplication / division / modulo (67×89 ; $399/54$; $399\%54$)
 - Boolean operations ($x \text{ AND } y$; $x \text{ OR } y$, etc.)
 - Binary comparisons ($x == y$; $x \leq y$; $x < y$; $x \geq y$; etc.)
 - variable assignments, function calls.
- The running time $T(n)$ of algorithm \mathcal{A} on input X of size n is the **number of elementary operations** used when X is fed into \mathcal{A} .

Time Complexity

Running time $T(n)$		Input size of n			
<i>Function</i>	<i>Notation</i>	10	100	1000	10^7
Constant	1	1	1	1	1
Logarithmic	$\log n$	1	2	3	7
Linear	n	1	10	100	10^6
“Linearithmic”	$n \log n$	1	20	300	7×10^6
Quadratic	n^2	1	100	10000	10^{12}
Cubic	n^3	1	1000	10^6	10^{18}
Exponential	2^n	1	10^{27}	10^{298}	$10^{3010296}$

Constant Time Complexity

- Algorithm: Swapping two elements
- This is a **constant time** algorithm

Algorithm 1 Swapping two elements in an array

```
1: Require:  $0 \leq i \leq j \leq n - 1$ 
2:   function SWAP(array  $a[0 \dots n - 1]$ , integer  $i$ , integer  $j$ )
3:      $t \leftarrow a[i]$ 
4:      $a[i] \leftarrow a[j]$ 
5:      $a[j] \leftarrow t$ 
6:   return  $a$ 
```

Linear Time Complexity

- Algorithm: finding the maximum value
- This is a **linear time** algorithm, since it makes one pass through the array and does a constant amount of work each time.

Algorithm 2 Finding the maximum in an array

```
1: function FINDMAX(array  $a[0 \dots n - 1]$ )  
2:    $k \leftarrow 0$                                 ▷ location of maximum so far  
3:   for  $j \leftarrow 1$  to  $n - 1$  do  
4:     if  $a[k] < a[j]$  then  
5:        $k = j$   
6:   return  $k$ 
```

Logarithmic Time Complexity

- Algorithm: : loop increments
- This runs in **logarithmic time** because i doubles about $\log n$ times until reaching n .

Algorithm 3 Example: exponential change of variable in loop

```
1:  $i \leftarrow 1$   
2: while  $i \leq n$  do  
3:    $i \leftarrow 2 * i$   
4:   print  $i$ 
```

Quadratic Time Complexity

- Algorithm: nested loops
- This runs in **quadratic time**, because the inner loop runs $n-i+1$ times, while the outer loop runs n times. That is:

$$n + (n - 1) + (n - 2) + \dots + 1 = \frac{(n + 1)n}{2} = \frac{1}{2}n^2 + \frac{1}{2}n$$

Algorithm 4 Snippet: Nested loops

```
1: for  $i \leftarrow 1$  to  $n$  do  
2:   for  $j \leftarrow i$  to  $n$  do  
3:     print  $i + j$ 
```

Exponential Time Complexity

- Algorithm: SLOWFIB
- SLOWFIB makes $F(n)$ function calls each of which involves a constant number of elementary operations. It turns out that $F(n)$ grows exponentially in n , so this is an exponential time algorithm.

Algorithm 5 Slow method for computing Fibonacci numbers

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```

Example: Time Complexity Analysis

- Algorithm: Sum of an array $s = \sum_{i=0}^{n-1} a[i]$
- Summing n elements of the array a repeats elementary fetch/add operations n times.

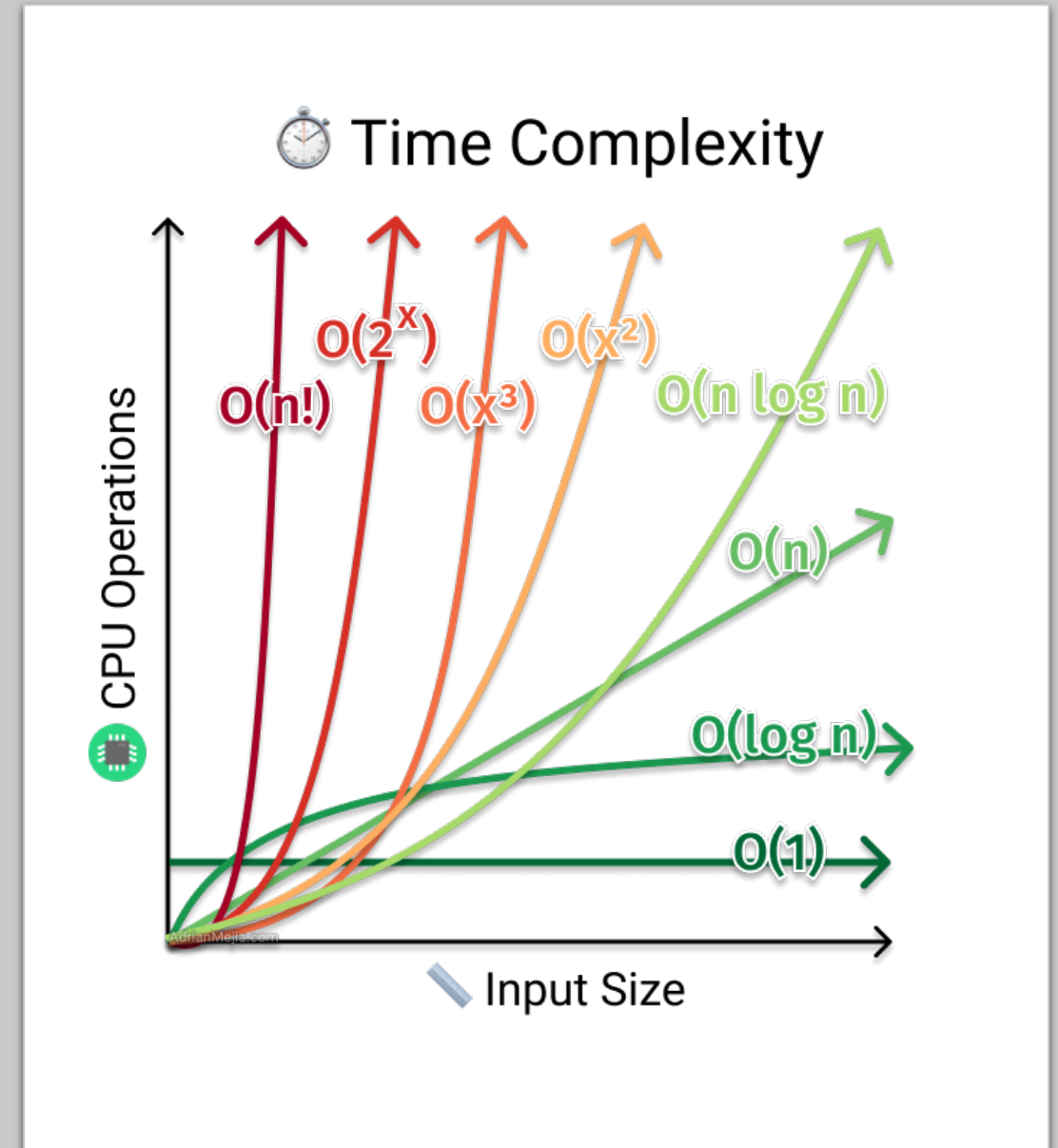
Algorithm 6 Summing elements of an array $a[0 \dots n - 1]$

```
1: Require: array  $a$ 
2:   function SUM( $a[0 \dots n - 1]$ )
3:      $s \leftarrow 0$ 
4:     for  $i \leftarrow 0$  to  $n - 1$  do
5:        $s \leftarrow s + a[i]$ 
6:     return  $s$ 
```

Running time is **linear** in n , i.e., $T(n) = cn$

SUMMARY

- Analysis of Algorithm: Fibonacci numbers
 - Slow implementation
 - Fast implementation
- Time Complexity
 - How to measure running time?
 - Illustrative examples
- **Asymptotic Notation (O , Ω , Θ)**
 - **Asymptotic Comparison**
 - **Asymptotic Bound**



Asymptotic Notation

- In order to compare running times of algorithms we want a way of comparing the **growth rates of functions**.
- We want to see what happens for **large values of n** – small ones are not relevant.
- We are not usually interested in constant factors and only want to consider the **dominant term**.
- The standard mathematical approach is to use **asymptotic notation O, Ω, Θ** which we will now describe.

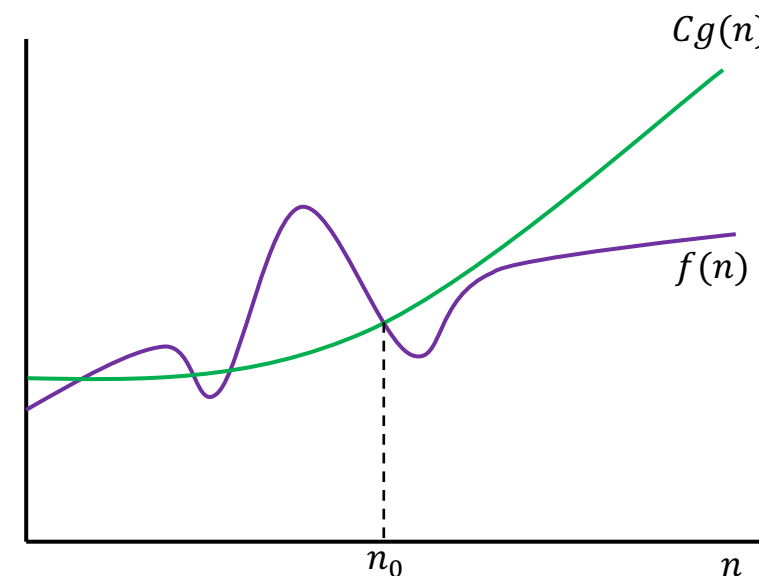
Asymptotic Notation (Contd.)

- Suppose that f and g are functions from \mathbb{N} to \mathbb{R} , which take on nonnegative values.

Say f is $O(g)$ (“ f is **big-Oh** of g ”) if there is some $C > 0$ and some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $f(n) \leq Cg(n)$. **Informally, f grows at most as fast as g .**

Say f is $\Omega(g)$ (“ f is **big-Omega** of g ”) if g is $O(f)$.
Or more formally if there is some $C > 0$ and some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $f(n) \geq Cg(n)$.

Informally, f grows at least as fast as g .



$f(n)$ is $O(g(n))$

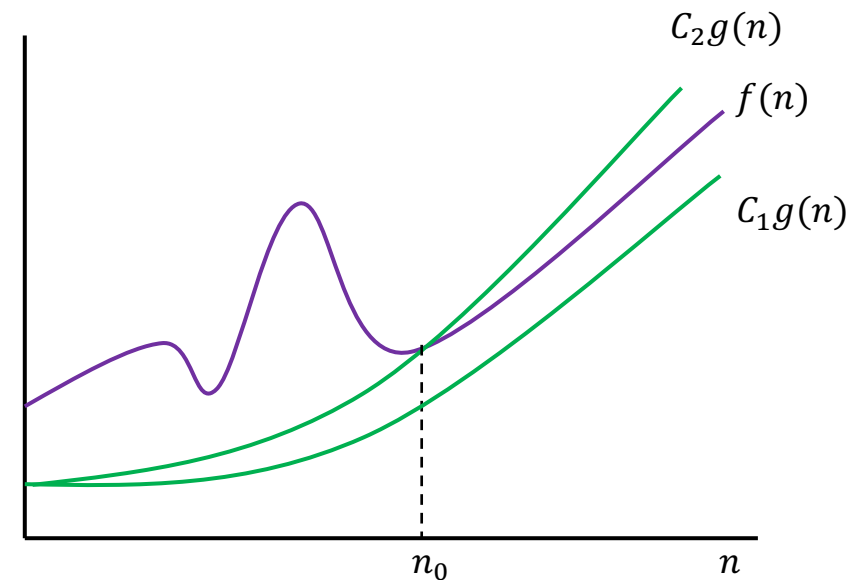
Asymptotic Notation (Contd.)

- Suppose that f and g are functions from \mathbb{N} to \mathbb{R} , which take on nonnegative values.

Say f is $\Theta(g)$ (“ f is **big-Theta** of g ”) if f is $O(g)$ and g is $O(f)$. Specifically, if there is some $C_1 > 0, C_2 > 0$, and some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $C_1 g(n) \leq f(n) \leq C_2 g(n)$.

Informally, f/g is bounded away from zero and infinity, and f grows **at the same rate** as g

Note that we could always reduce n_0 at the expense of a bigger C but it is often easier not to.



$f(n)$ is $\Theta(g(n))$

Asymptotic Comparison

- Every linear function $f(n) = an + b, a > 0$, is $O(n)$.
- **Proof**: $an + b \leq an + |b| \leq (a + |b|)n$ for $n \geq 0$.
- If $f(n) = n; g(n) = n^2/2$, then f is $O(g)$ and g is not $O(f)$, so g grows asymptotically faster than f .
- **Proof**: $f(n) \leq 2 \cdot g(n)$ for $n \geq 1$ (or $f(n) \leq 1 \cdot g(n)$ for $n \geq 2$); conversely, suppose that eventually $\frac{1}{2}n^2 \leq Cn$. Then $n \leq 2C$ for all sufficiently large n , a contradiction.

Exercises: Asymptotic Comparison

- I have seen the description “ $f = O(g)$ ” before. Is this correct?

No. We can't say a function equals to a set of function! $O(g)$ describes a set of functions that grows no faster than g . For instance, both $f_1(n) = n$ and $f_2(n) = 2n$ are in $O(n)$.

But we can say $f \in O(g)$

Rules for Asymptotic Notation

- **(irrelevance of constant factors)** if $c > 0$ is constant then cf is $O(f)$
- **(sum rule)** If f_1 is $O(g_1)$ and f_2 is $O(g_2)$ then $f_1 + f_2$ is $O(\max\{g_1, g_2\})$
- **(product rule)** If f_1 is $O(g_1)$ and f_2 is $O(g_2)$ then f_1f_2 is $O(g_1g_2)$
- **(limit rule)** Suppose that $L := \lim_{n \rightarrow \infty} f(n)/g(n)$ exists. Then,
 - if $L = 0$ then f is $O(g)$ and f is not $\Omega(g)$
 - if $0 < L < 1$ then f is $\Theta(g)$
 - if $L = 1$ then f is $\Omega(g)$ and f is not $O(g)$.
- **(L'Hôpital's rule)** is often useful for the application of the limit rule. Note that the limit may not exist at all.

Limit Rule

Suppose that $L := \lim_{n \rightarrow \infty} f(n)/g(n)$ exists. Then,

- if $L = 0$ then f is $O(g)$ and f is not $\Omega(g)$;
- if $0 < L < \infty$ then f is $\Theta(g)$;
- if $L = \infty$ then f is $\Omega(g)$ and f is not $O(g)$.

- When f and g are **positive** and **differentiable** functions for $n > 0$, one of the following satisfies:
 - $\lim_{n \rightarrow \infty} f(n) = \infty$ and $\lim_{n \rightarrow \infty} g(n) = \infty$
 - $\lim_{n \rightarrow \infty} f(n) = 0$ and $\lim_{n \rightarrow \infty} g(n) = 0$

L'Hopital rule of calculus can be applied:

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \lim_{n \rightarrow \infty} \frac{f'(n)}{g'(n)}$$

Exercises: Asymptotic Comparison

Prove that exponential functions grow faster than powers:


n^k is $O(a^n)$ for all $a > 1$ and $k \geq 0$

- **Proof:**

- First derivatives of $f(x) = n^k$ by x is $f'(x) = kx^{k-1}$
- First derivatives of $g(x) = a^x$ by x is $g'(x) = a^x \ln a$
- For $k = 0$, $n^k = 1 \in O(a^n)$
- For $k > 0$, by limit rule, and successive application of L'Hopital rule :

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{n^k}{a^n} &= \lim_{n \rightarrow \infty} \frac{kn^{k-1}}{a^n \ln a} = \lim_{n \rightarrow \infty} \frac{k(k-1)n^{k-2}}{a^n (\ln a)^2} \\ &\quad \dots \\ &= \lim_{n \rightarrow \infty} \frac{k!}{a^n (\ln a)^k} = 0 \end{aligned}$$

When $a > 1$

 $n^k \in O(a^n)$

Exercises: Asymptotic Comparison

- Logarithmic functions grow slower than powers?

$\log_a n$ is $O(n^k)$ for all $a > 1$ and $k \geq 1$

- Proof:**

- First derivatives of $f(x) = \log_a x$ when $a > 1$ is $f'(x) = \frac{1}{x \ln a}$
- First derivatives of $g(x) = x^k$ by x is $g'(x) = kx^{k-1}$
- By limit rule and L'Hopital rule :

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\log_a n}{n^k} &= \lim_{n \rightarrow \infty} \frac{\frac{1}{n \ln a}}{kn^{k-1}} \\ &= \lim_{n \rightarrow \infty} \frac{1}{k \ln a n^k} = 0 \end{aligned} \quad \begin{array}{c} \text{When } k > 0 \\ \longrightarrow \end{array} \quad \log_a n \in O(n^k)$$

Time Complexity

- **Informal definition:** A function $f(n)$ such that the running time $T(n)$ of a given algorithm is $\Theta(f(n))$ measures the time complexity of the algorithm.
- An algorithm is called **polynomial time algorithm** if its runtime $T(n)$ is $O(n^k)$, where k is a constant positive integer.
- A computational problem is called **intractable** if and only if no deterministic polynomial time algorithm can solve it.

Asymptotic Bounds

- Asymptotic notation measures the running time of the algorithm in terms of elementary operations, i.e., asymptotic bounds are independent of implementation:
- For a given problem, the running time varies NOT ONLY according to the size of the input, BUT ALSO the input itself.
 - E.g., sorting an already sorted array takes almost no time for some sorting algorithms
- Common measures:
 - the **worst-case** running time
 - the **average-case** running time

Example: “Find the position of an element x in array”

- **Worst Case**: x is not in the array
- **Average Case**: treat all possible positions equally distributed

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