Bipartite Graphs and K-Colouring

Instructor: Meng-Fen Chiang

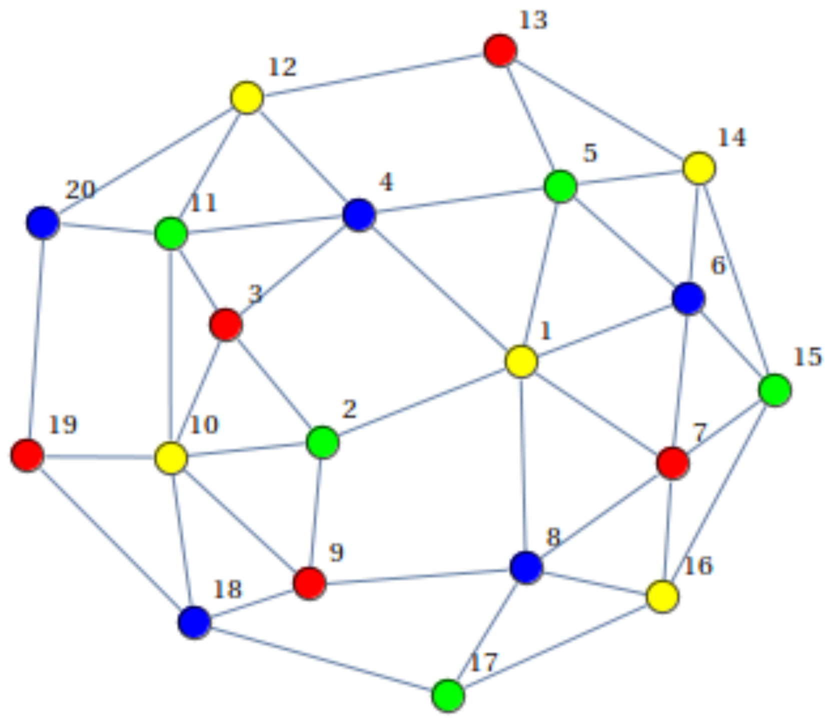
COMPSCI: WEEK 11.2

THE UNIVERSITY OF AUCKLAND
Te Whare Wānanga o Tāmaki Makaurau
NEW ZEALAND

Slides adapted from Mark Wilson, Georgy Gimel'farb, Simone Linz, Tanya Gvozdeva, and Kaiqi Zhao

OUTLINE

- Bipartite Graph
- Colouring Problem
  - K-Colour Mapping
  - K-coulourings

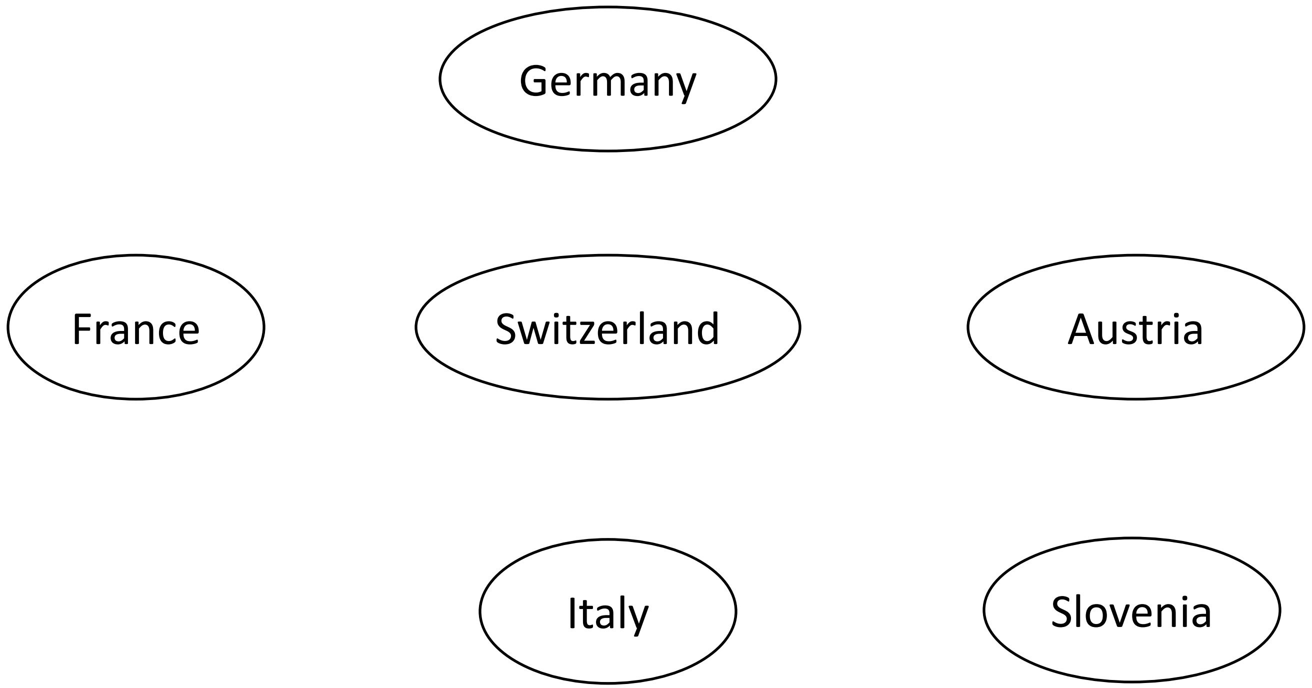


Problem

- Colour the map of Europe with k colours such that no two adjacent countries have the same colour

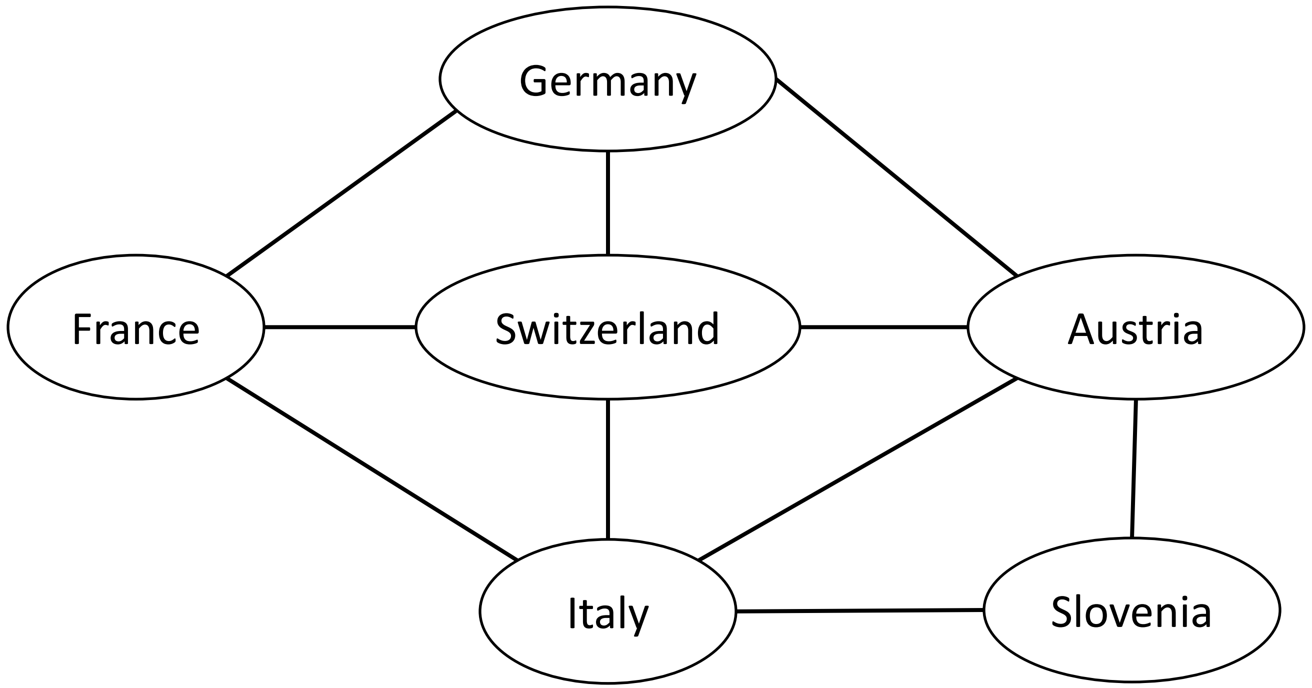
K-Colour Mapping

Reducing to a Graph Problem



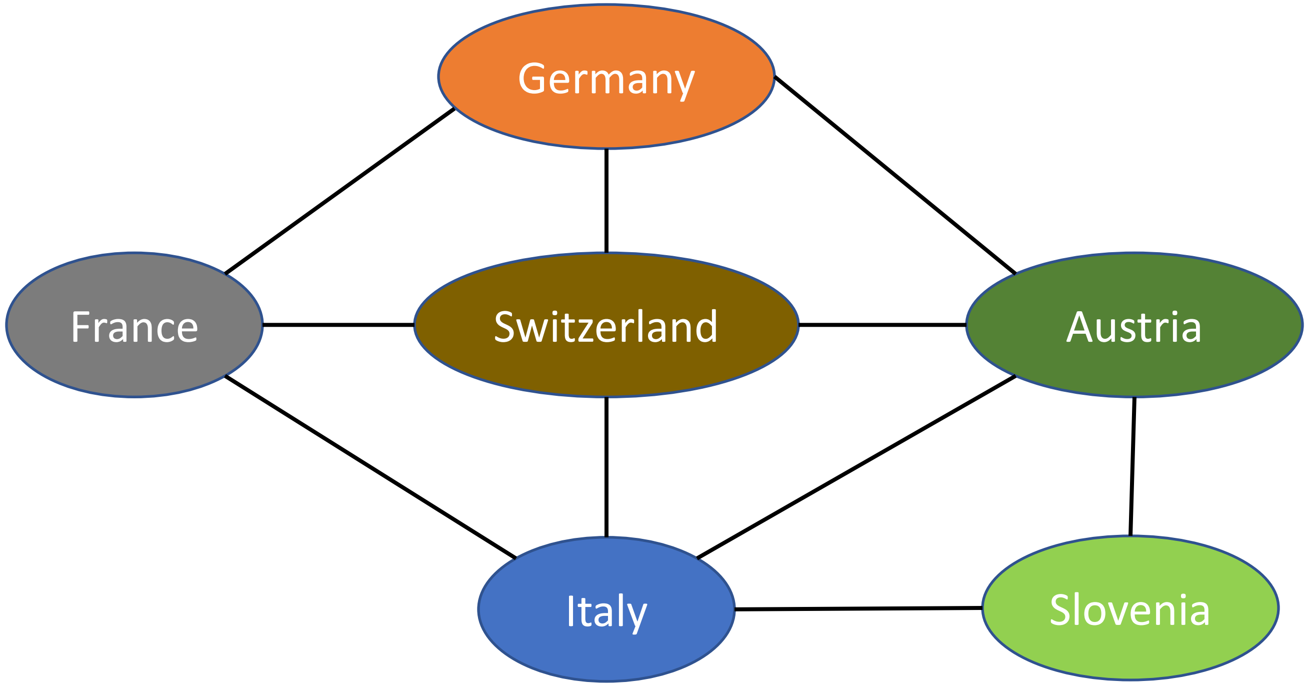
Represent each node as a country

Reducing to a Graph Problem



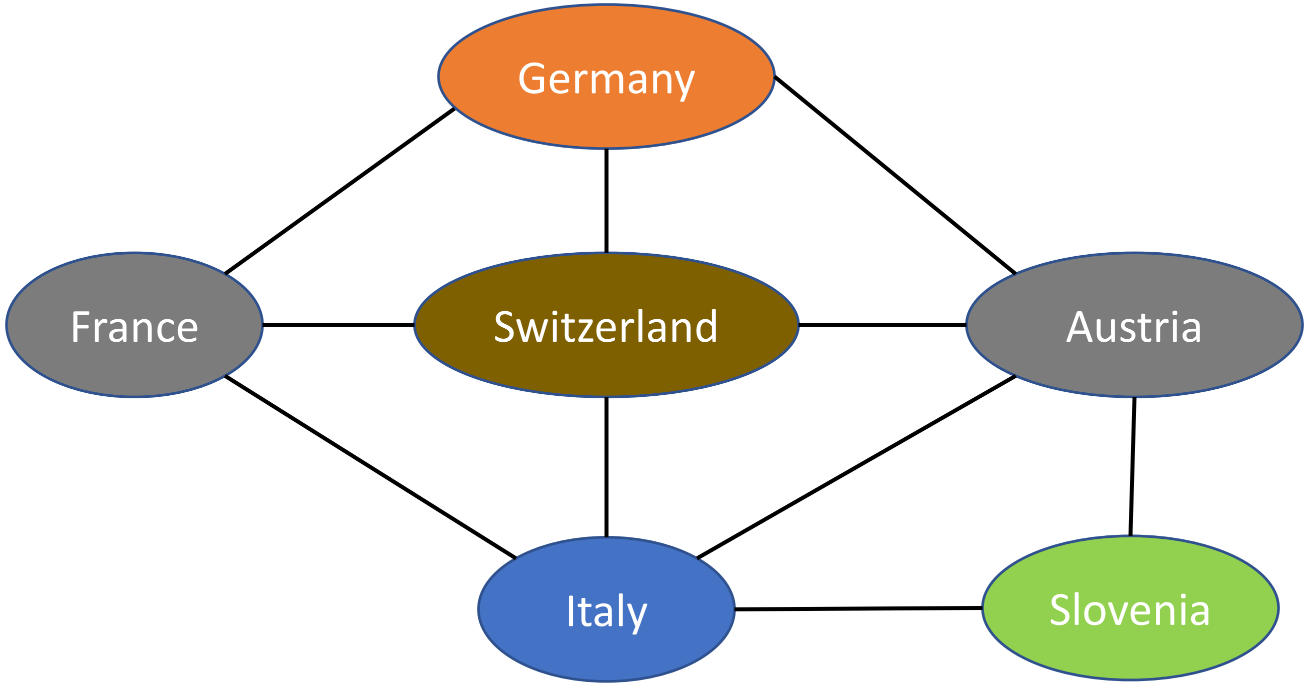
Add an edge (x,y) if x and y are neighbouring countries

Reducing to a Graph Problem



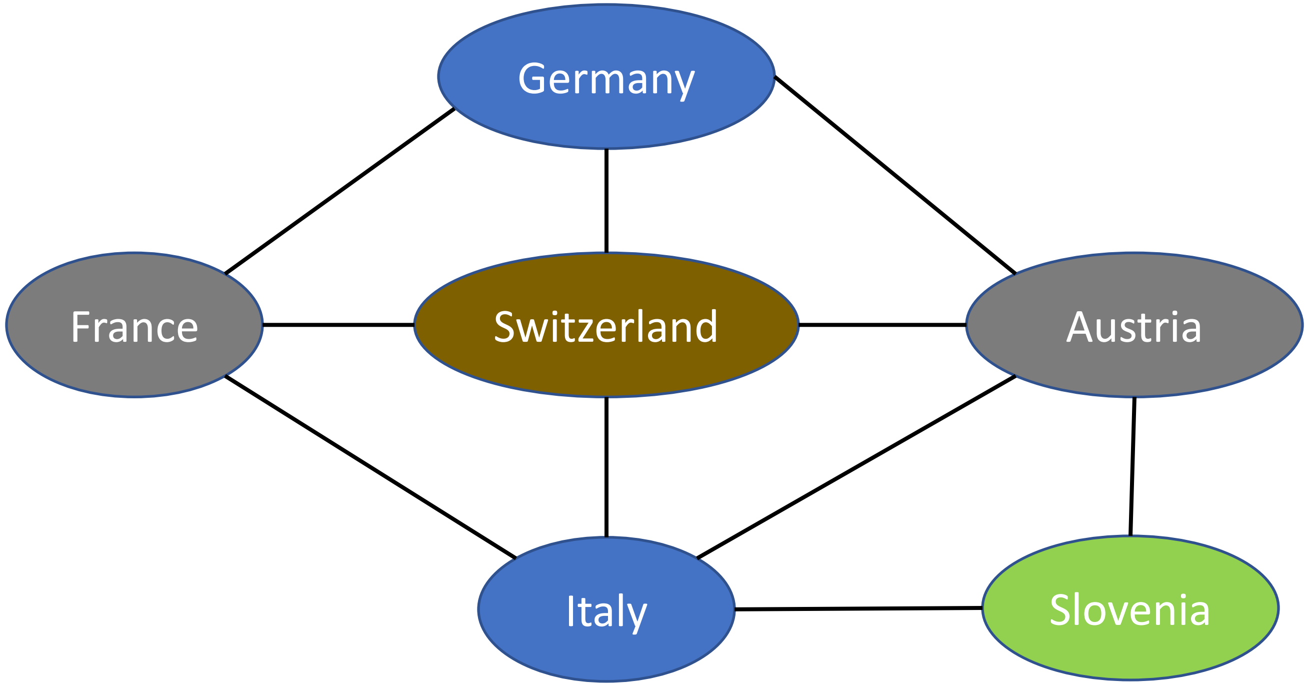
The naïve approach is to have one colour for each country
6-Colouring graph…Can we do better?

Reducing to a Graph Problem



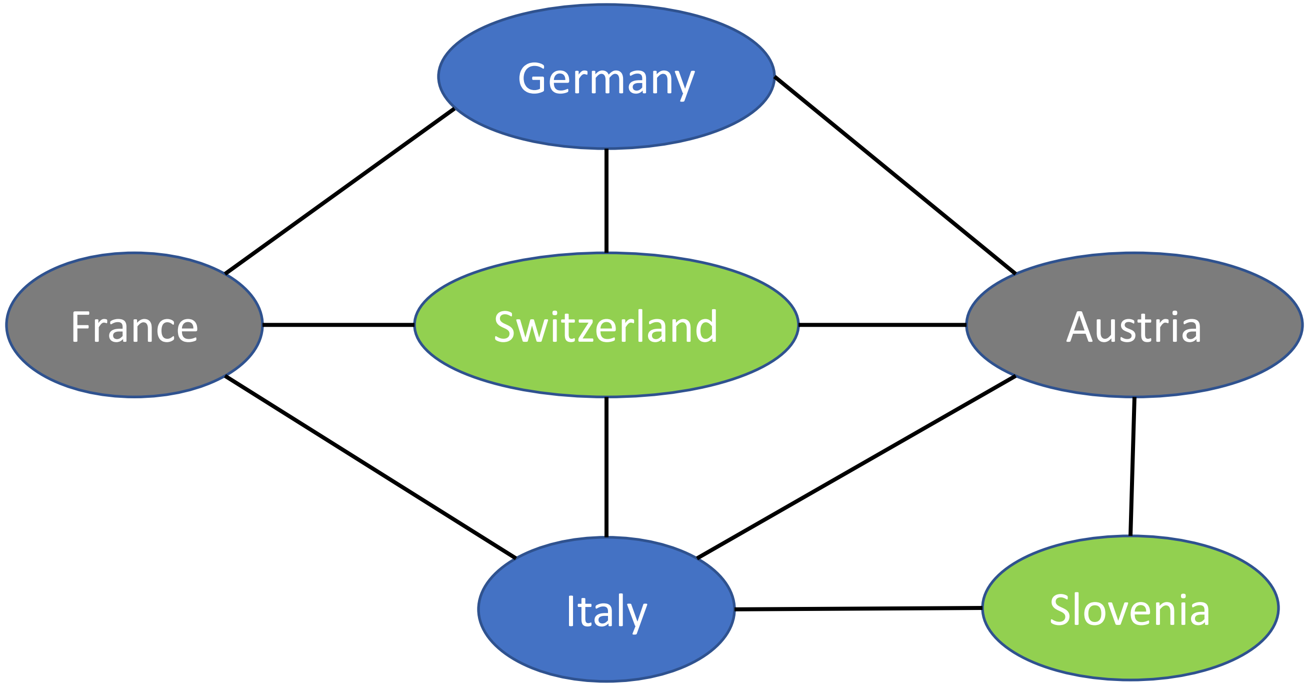
5-Colouring graph…Can we do better?

Reducing to a Graph Problem



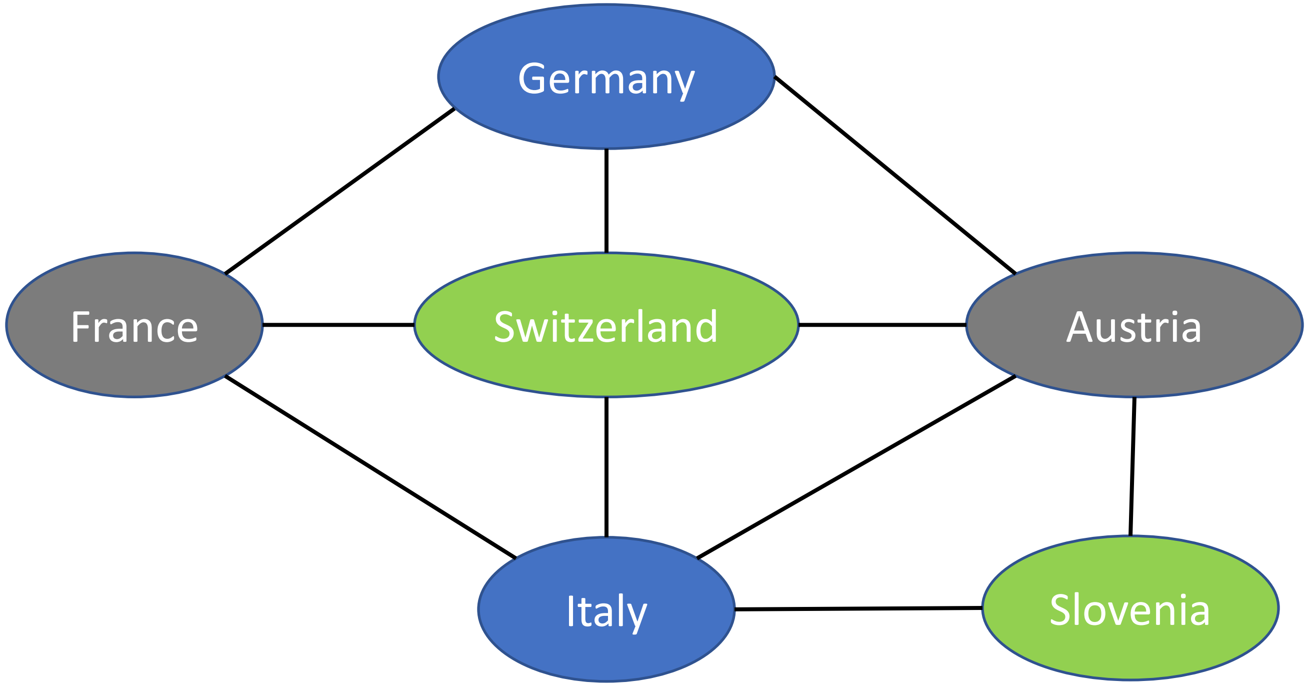
4-Colouring graph…Can we do better?

Reducing to a Graph Problem



3-Colouring graph…Can we do better?

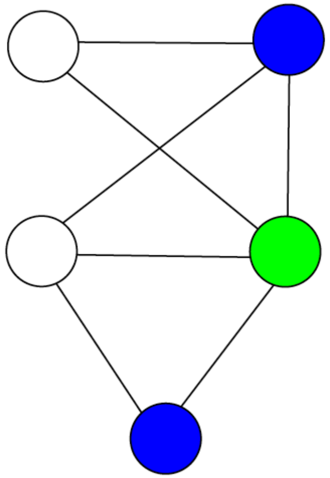
Reducing to a Graph Problem

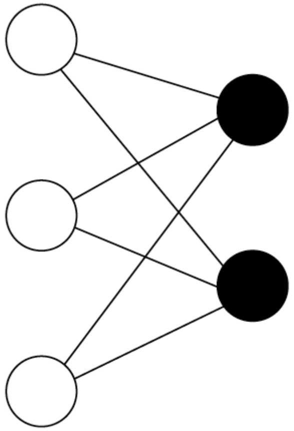


3-Colouring graph…Can we do better? Nope!

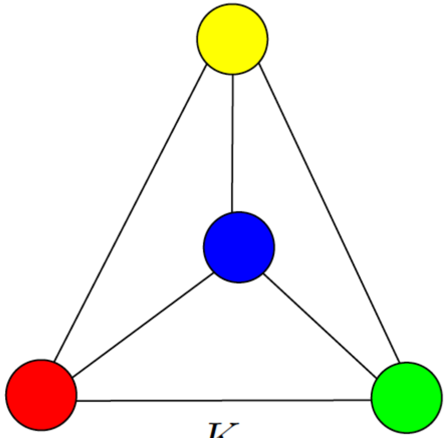
k-colorable Graphs

- Definition. Let k be a positive integer. A graph G has a k-coloring if $V(G)$ can be partitioned into k nonempty disjoint subsets such that each edge of G joins two vertices in different subsets (colors). The smallest number of colors needed to color a graph is called chromatic number.





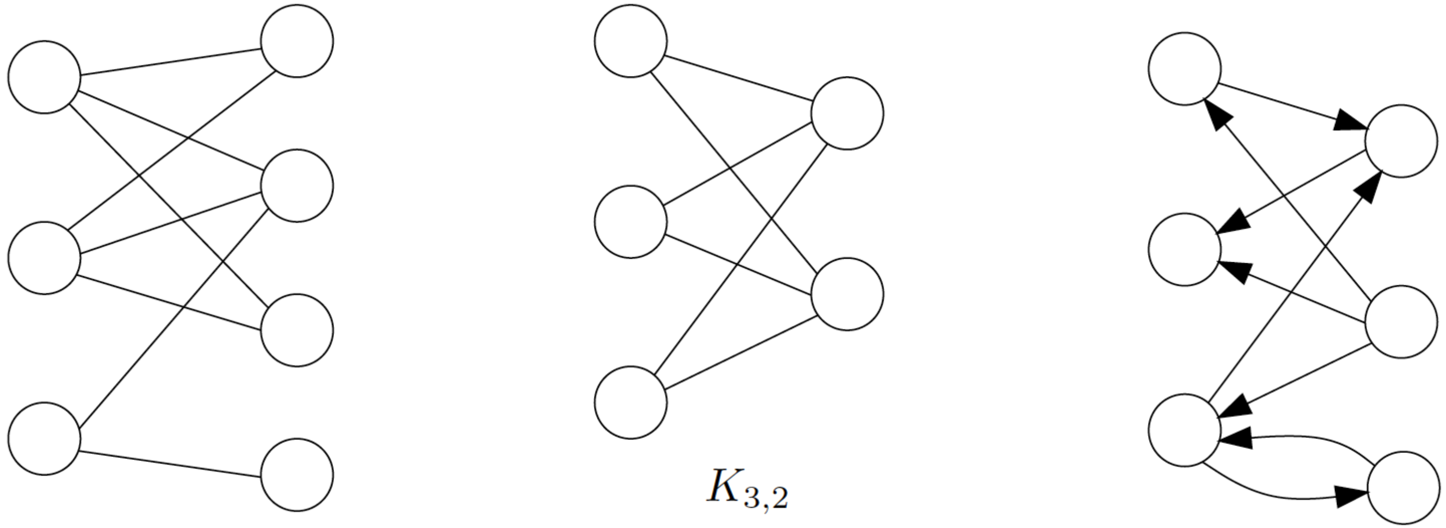
$K_{3,2}$



K_4

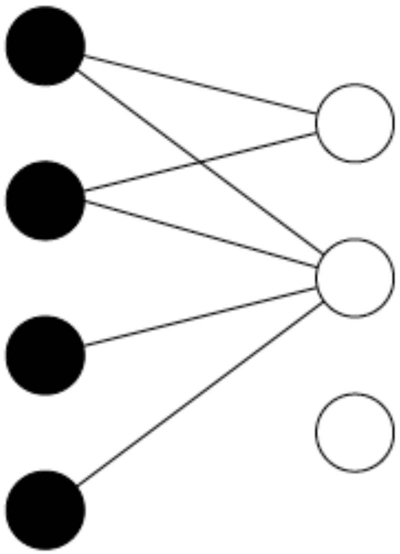
Bipartite Graphs (digraphs)

- Definition. A graph G is **bipartite** if V(G) can be partitioned into two nonempty disjoint subsets V_1, V_2 such that each edge of G has one endpoint in V_1 and one in V_2 . (Similar for digraphs)



Example 27.3

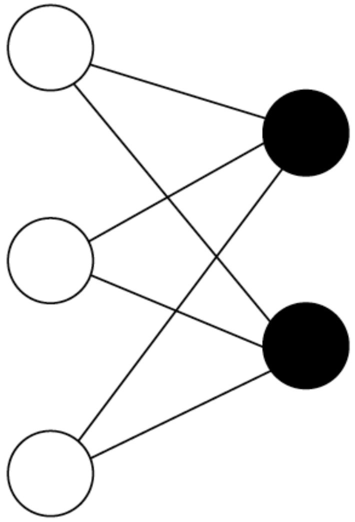
- Q: Is this graph bipartite?
- A: Yes, it is a bipartite. The isolated vertex could be placed on either side.



Bipartite Graphs

- **Theorem**. The following conditions on a graph G are equivalent.
    1. G has a 2-coloring;
    2. G is bipartite;
    3. G does not contain an odd length cycle.

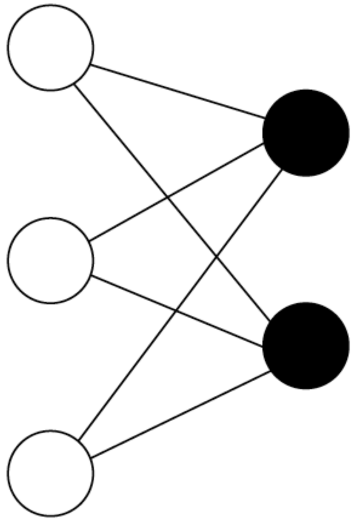
- Suppose G has a 2-coloring. Let V_1 be the set of vertices with color c_1, and let V_2 be the set of vertices with color c_2. Then each edges joins a vertex in V_1 with a vertex in V_2 . By definition, $G = (V_1 \cup V_2, E)$ is bipartite.



Bipartite Graphs (Contd.)

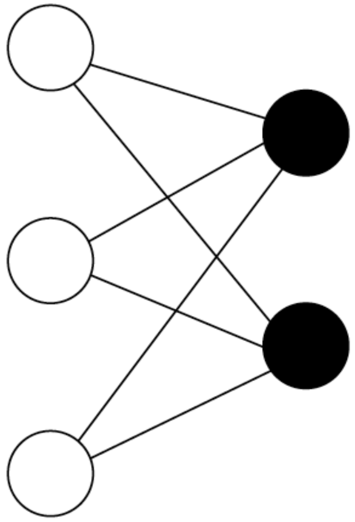
- **Theorem**. The following conditions on a graph G are equivalent.
  1. G has a 2-coloring;
  2. G is bipartite;
  3. G does not contain an odd length cycle.

- Suppose $G = (V_1 \cup V_2, E)$ is bipartite. Each edges joins a vertex in V_1 with a vertex in V_2. Color each vertex in V_1 with color c_1 and each vertex in V_2 with color c_2. Since G is bipartite, this induces a 2-coloring of G.



Bipartite Graphs (Contd.)

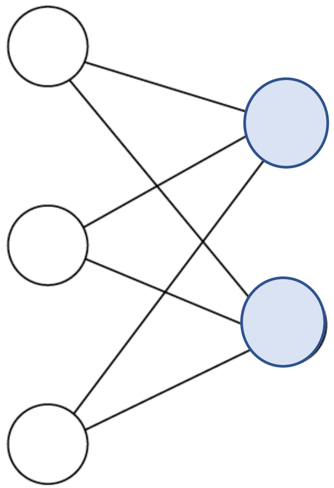
- **Theorem**. The following conditions on a graph G are equivalent.
  1. G has a 2-coloring;
  2. G is bipartite;
  3. G does not contain an odd length cycle.



- Suppose G is bipartite. Let C be a cycle in G. Then, since G is 2-colorable, C has even length (for any path, $v_1 \ldots, v_n, v_1$, the start node v_1 and end node v_n have different colors).
- Hence, G does not contain a cycle of odd length.

Bipartite Graphs (Contd.)

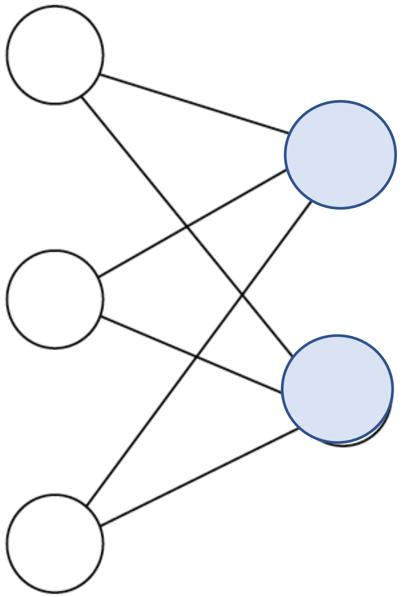
- **Theorem**. The following conditions on a graph G are equivalent.
  1. G has a 2-coloring;
  2. G is bipartite;
  3. G does not contain an odd length cycle.



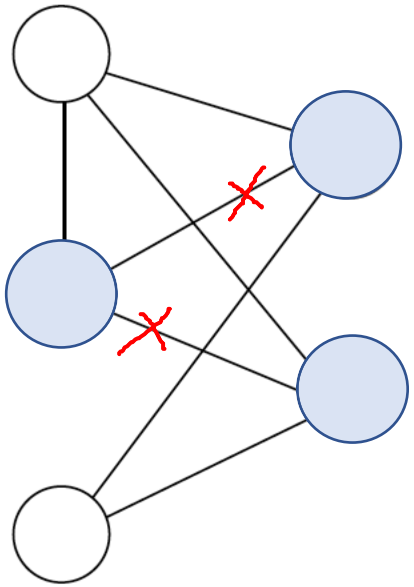
- Suppose G has no cycle of odd length. Obtain a 2-coloring as follows: Start BFS at v, assign v to color c_1, assign all neighbors of v to color c_2, assign all neighbors of neighbors of v to color c_1 and continue in this way until all vertices are colored. Since there is no odd cycle, each cross edge joins vertices of different color. (Why?)

Deciding if a Graph is Bipartite

- **Fact**. A version of BFS can be used to check if a graph is bipartite(e.g. 2-colorable).



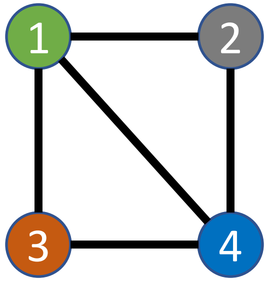
Bipartite



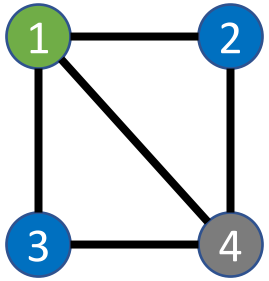
Not Bipartite

k-Colourings (Contd.)

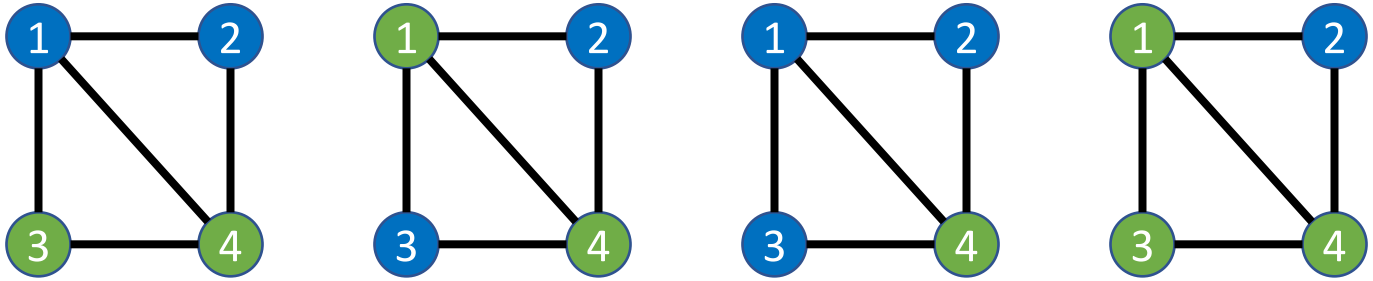
- If a graph has a k-colouring, then it also has a (k+1)-colouring. The reverse does not apply!



This graph has a 4-colouring



… and a 3-colouring…



… but no 2-colouring!

SUMMARY

- Bipartite Graph
- Colouring Problem
  - K-Colour Mapping
  - K-coulourings