

Bipartite Graphs and K-Colouring

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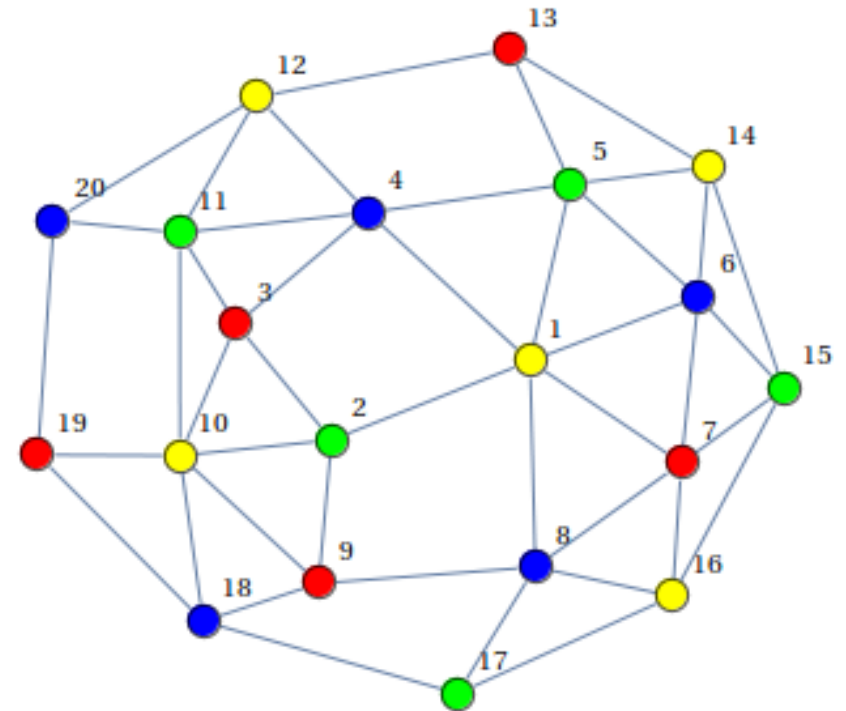
COMPSCI: WEEK 10.6





OUTLINE

- Bipartite Graph
- Colouring Problem
 - K-Colour Mapping
 - K-colourings



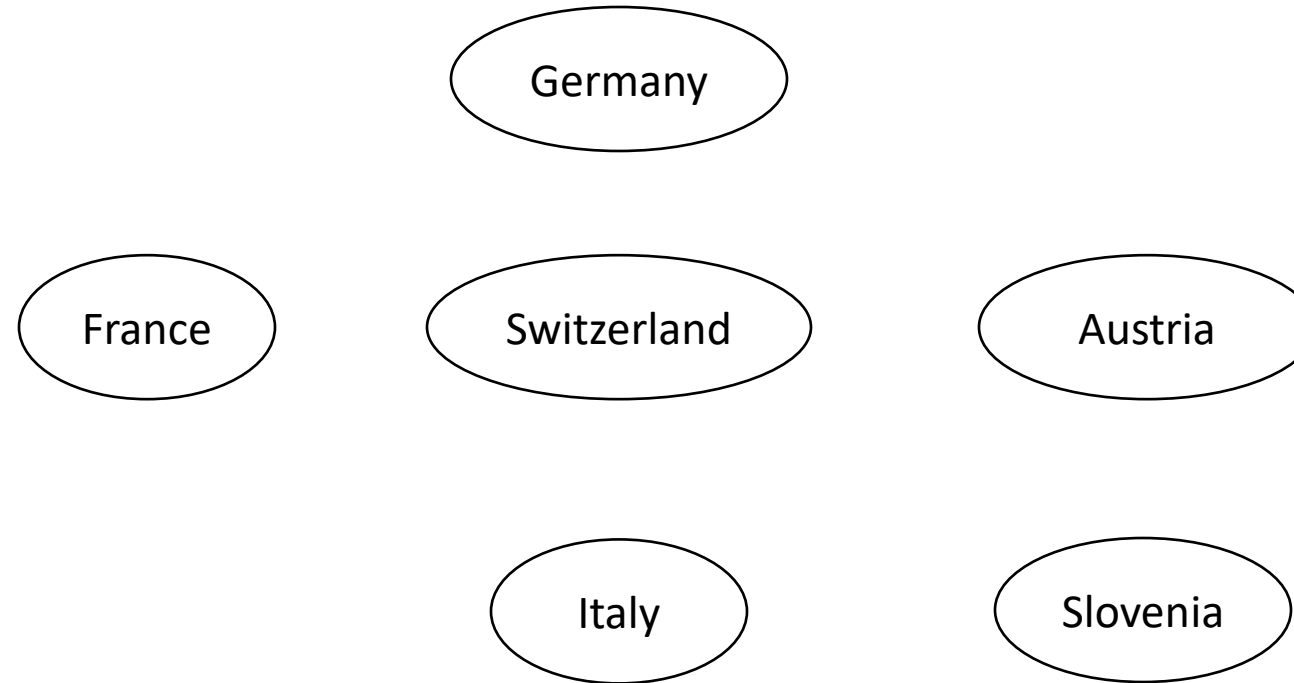
Problem

- Colour the map of Europe with k colours such that no two adjacent countries have the same colour



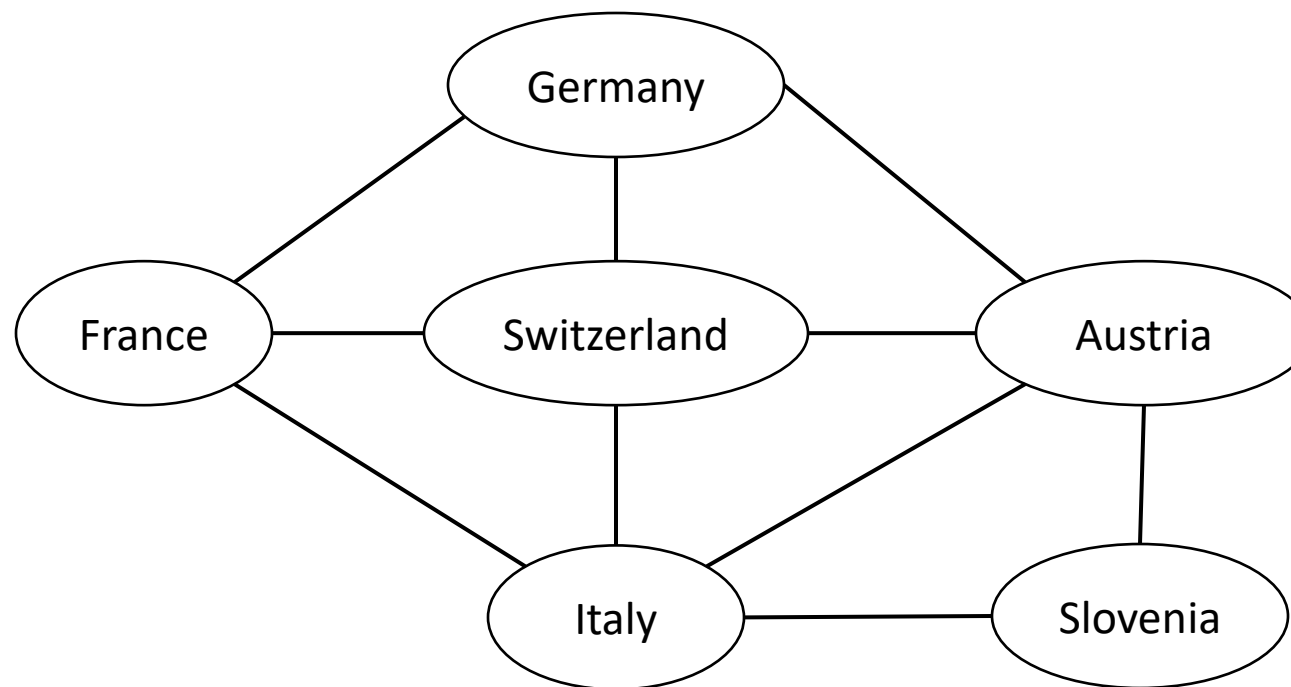


Reducing to a Graph Problem



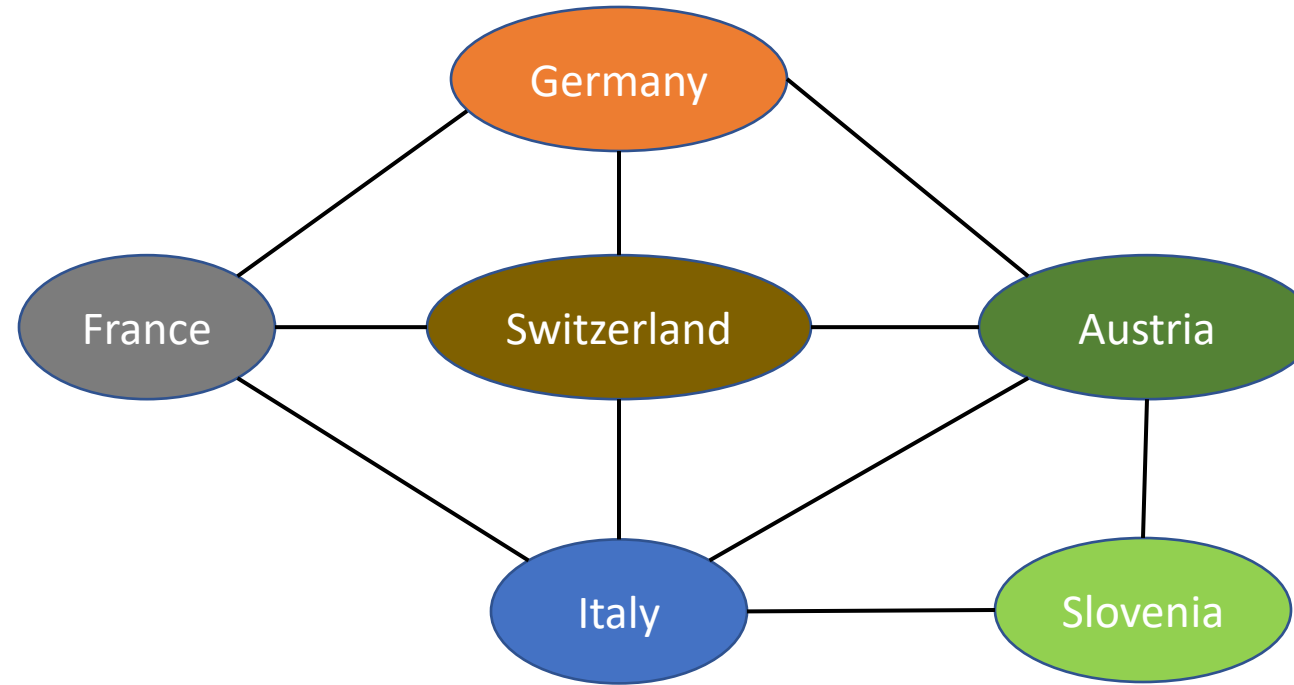
Represent each node as a country

Reducing to a Graph Problem



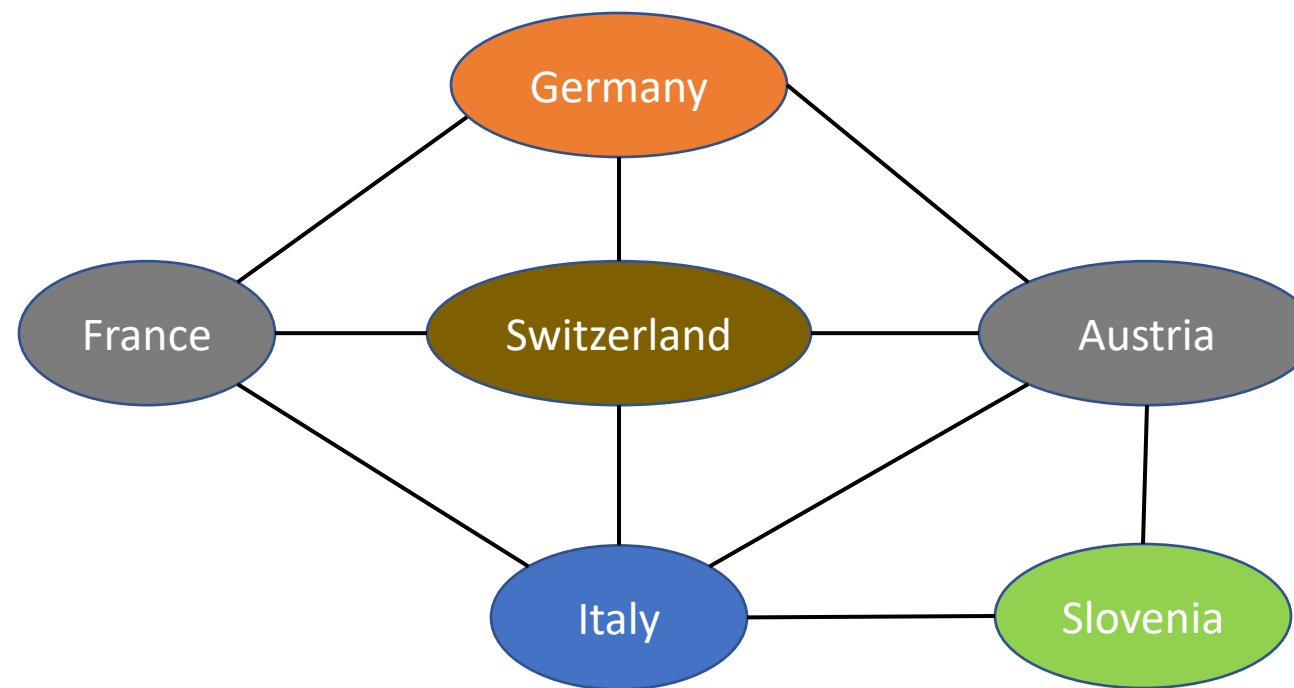
Add an edge (x,y) if x and y are neighbouring countries

Reducing to a Graph Problem



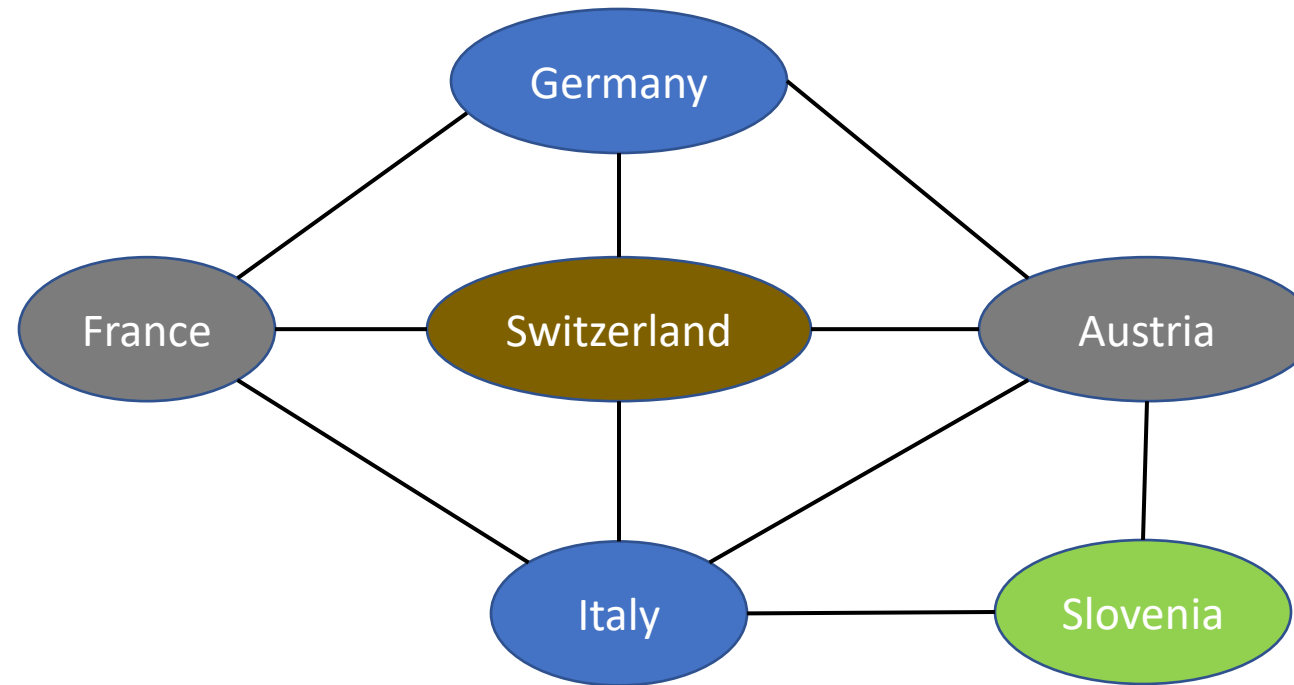
The naïve approach is to have one colour for each country
6-Colouring graph...Can we do better?

Reducing to a Graph Problem



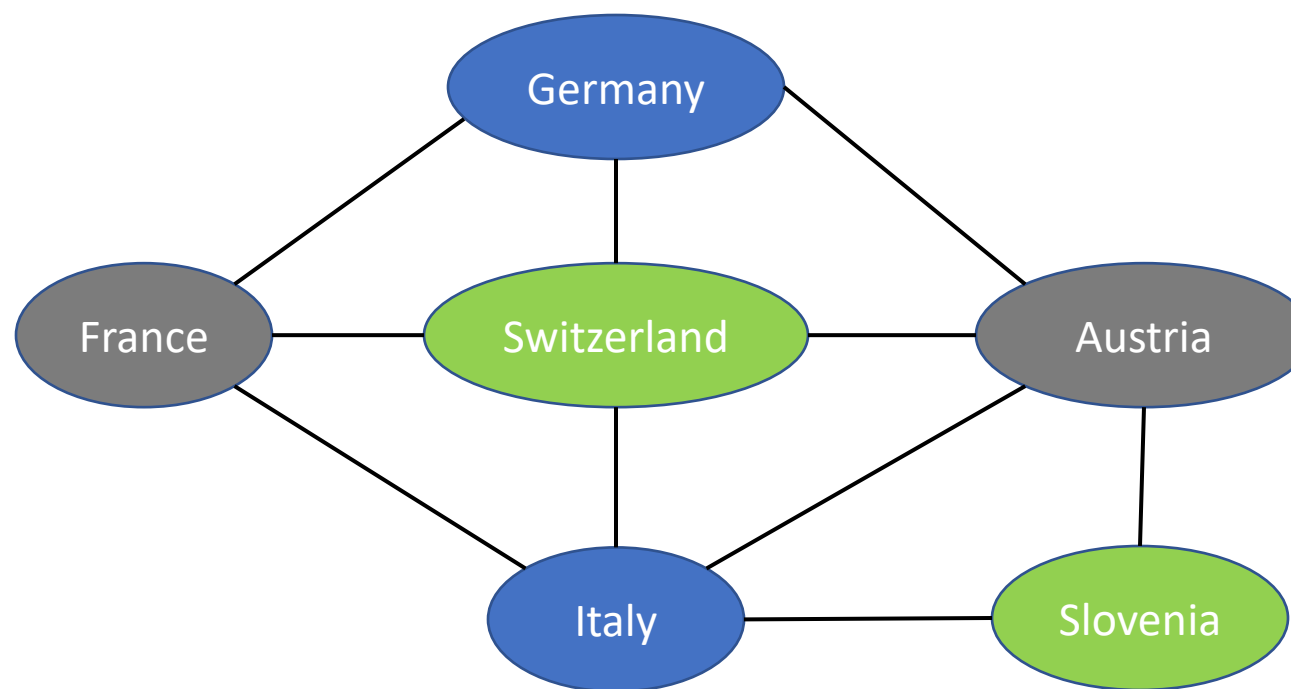
5-Colouring graph...Can we do better?

Reducing to a Graph Problem



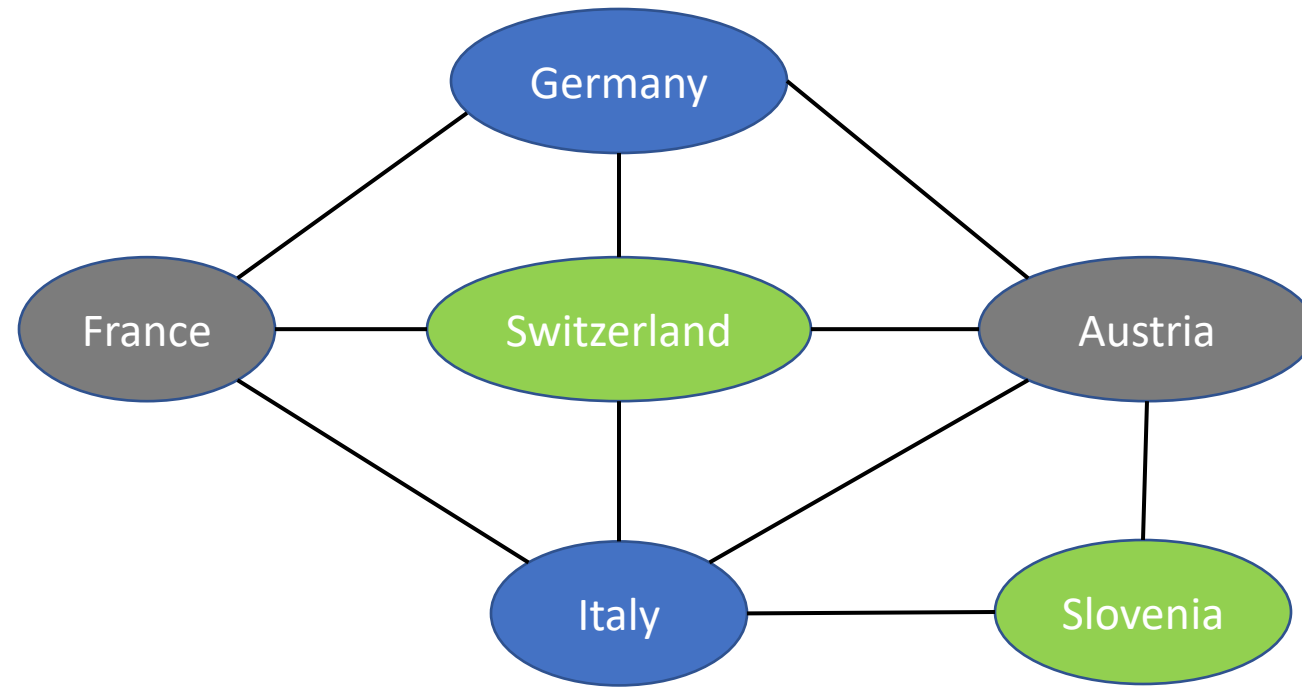
4-Colouring graph...Can we do better?

Reducing to a Graph Problem



3-Colouring graph...Can we do better?

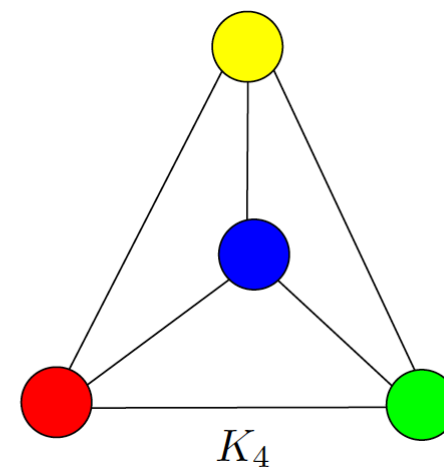
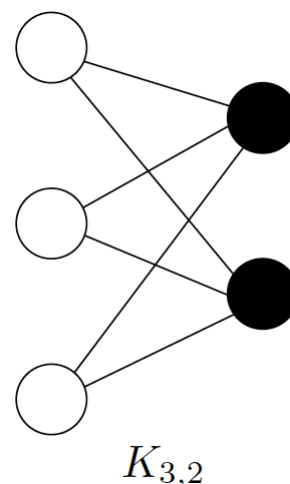
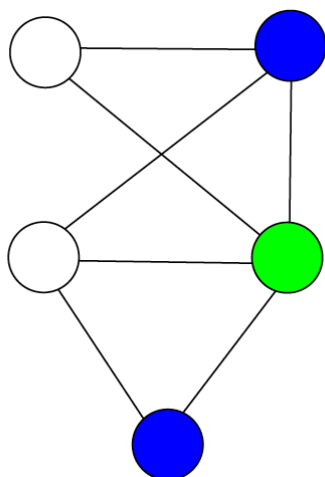
Reducing to a Graph Problem



3-Colouring graph...Can we do better? Nope!

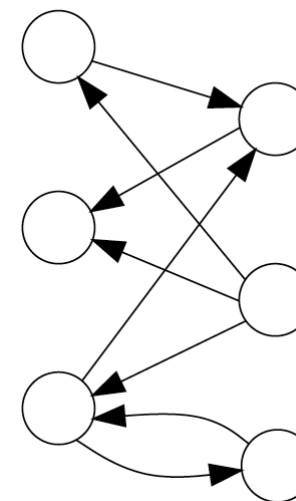
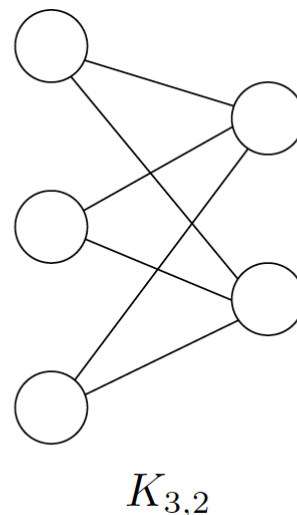
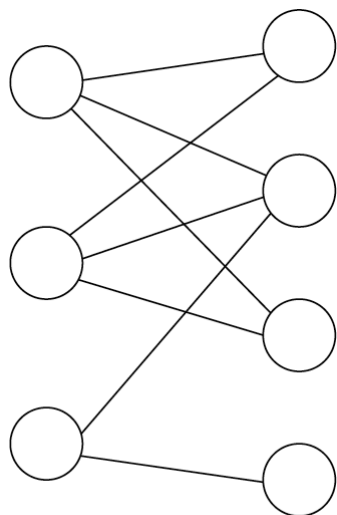
k-colorable Graphs

- Definition. Let k be a positive integer. A graph G has a **k-coloring** if $V(G)$ can be partitioned into k nonempty disjoint subsets such that each edge of G joins two vertices in different subsets (colors). The smallest number of colors needed to color a graph is called **chromatic number**.



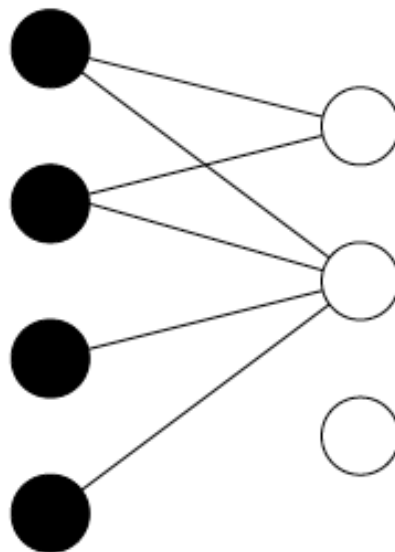
Bipartite Graphs (digraphs)

- Definition. A graph G is **bipartite** if $V(G)$ can be partitioned into two nonempty disjoint subsets V_1, V_2 such that each edge of G has one endpoint in V_1 and one in V_2 . (Similar for digraphs)



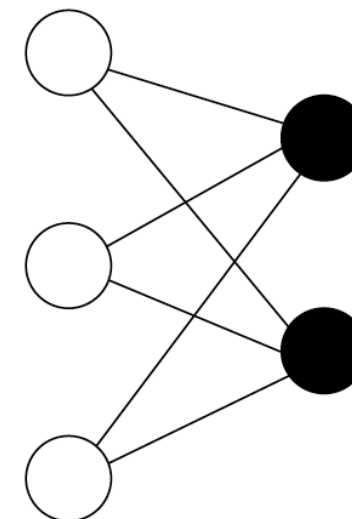
Example 27.3

- Q: Is this graph bipartite?
- A: Yes, it is a bipartite. The isolated vertex could be placed on either side.



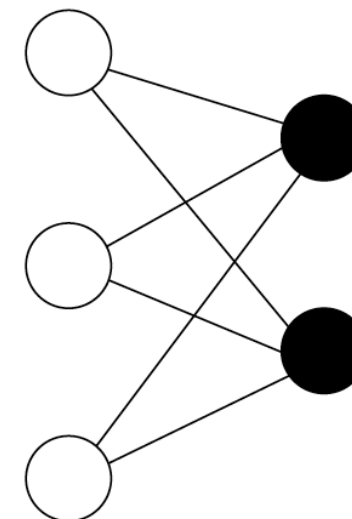
Bipartite Graphs

- **Theorem.** The following conditions on a graph G are equivalent.
 1. G has a 2-coloring;
 2. G is bipartite;
 3. G does not contain an odd length cycle.
- Suppose G has a 2-coloring. Let V_1 be the set of vertices with color c_1 , and let V_2 be the set of vertices with color c_2 . Then each edges joins a vertex in V_1 with a vertex in V_2 . By definition, $G = (V_1 \cup V_2, E)$ is bipartite.



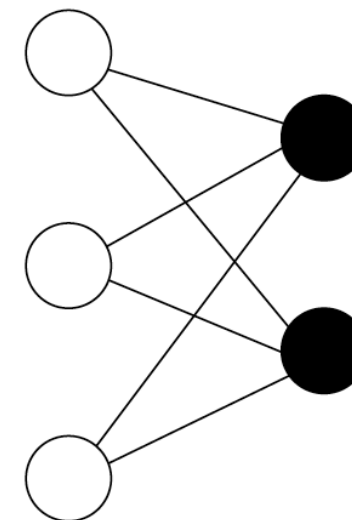
Bipartite Graphs (Contd.)

- **Theorem.** The following conditions on a graph G are equivalent.
 1. G has a 2-coloring;
 2. G is bipartite;
 3. G does not contain an odd length cycle.
- Suppose $G = (V_1 \cup V_2, E)$ is bipartite. Each edge joins a vertex in V_1 with a vertex in V_2 . Color each vertex in V_1 with color c_1 and each vertex in V_2 with color c_2 . Since G is bipartite, this induces a 2-coloring of G .



Bipartite Graphs (Contd.)

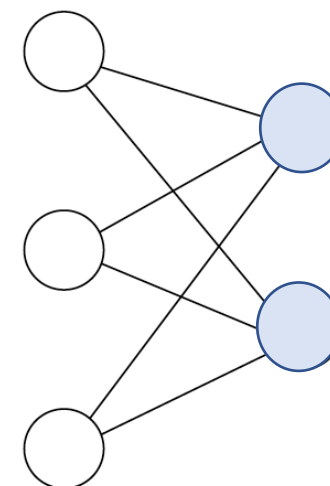
- **Theorem.** The following conditions on a graph G are equivalent.
 1. G has a 2-coloring;
 2. G is bipartite;
 3. G does not contain an odd length cycle.
- Suppose G is bipartite. Let C be a cycle in G . Then, since G is 2-colorable, C has even length (for any path, $v_1 \dots, v_n, v_1$, the start node v_1 and end node v_n have different colors).
- Hence, G does not contain a cycle of odd length.



Bipartite Graphs (Contd.)

- **Theorem.** The following conditions on a graph G are equivalent.

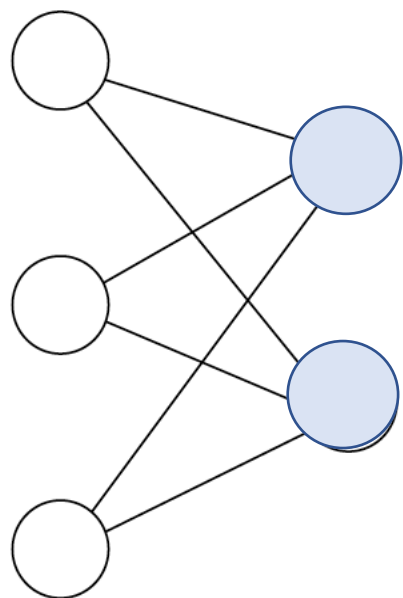
1. G has a 2-coloring;
2. G is bipartite;
3. G does not contain an odd length cycle.



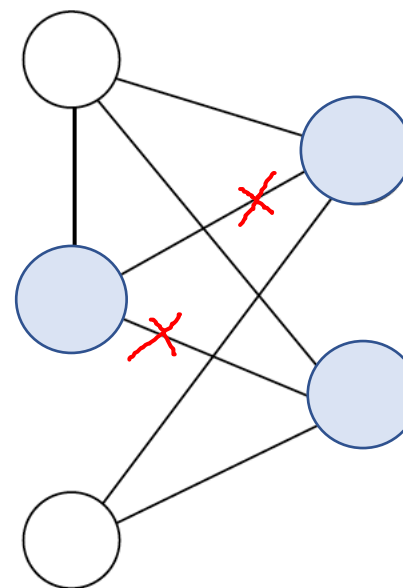
- Suppose G has no cycle of odd length. Obtain a 2-coloring as follows: Start BFS at v , assign v to color c_1 , assign all neighbors of v to color c_2 , assign all neighbors of neighbors of v to color c_1 and continue in this way until all vertices are colored. Since there is no odd cycle, each cross edge joins vertices of different color. (Why?)

Deciding if a Graph is Bipartite

- **Fact.** A version of BFS can be used to check if a graph is bipartite(e.g. 2-colorable).



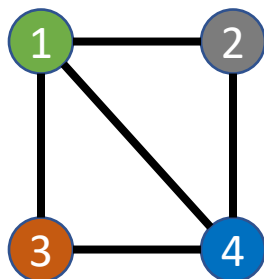
Bipartite



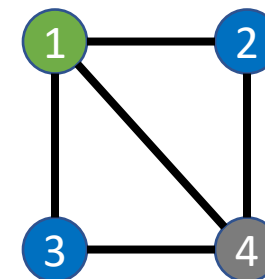
Not Bipartite

k-Colourings (Contd.)

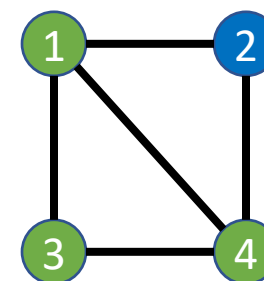
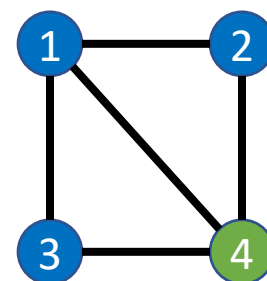
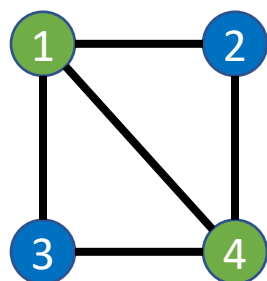
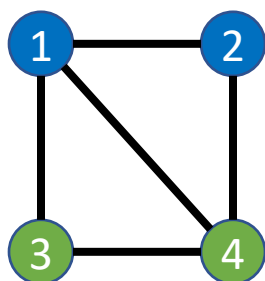
- If a graph has a k -colouring, then it also has a $(k+1)$ -colouring. The reverse does not apply!



This graph has a 4-colouring



... and a 3-colouring...



... but no 2-colouring!

SUMMARY

- Bipartite Graph
- Colouring Problem
 - K-Colour Mapping
 - K-colourings

