Sorting III: Heapsort

COMPSCI 220: WEEK 8.2

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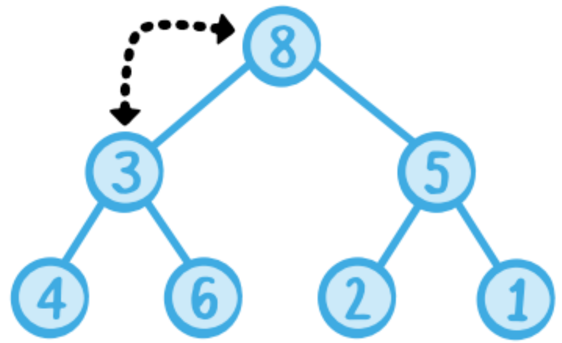
Slides adapted from Mark Wilson, Georgy Gimel'farb, Simone Linz and Tanya Gvozdeva

OUTLINE

- Heapsort
  - Algorithms
  - Illustrating Example

- Time Complexity Analysis



Heapsort

Given an array:

1. Build the heap
2. Delete the maximum repeatedly (arranging the elements in the output list in reverse order of deletion) until the heap is empty.

Step1: Building a Heap

- A straightforward method: sequentially insert nodes to the heap and maintain the heap property
- Inserting a node to a heap of height h, we need $h = \lfloor \log_2 n \rfloor$ comparisons at most. Then the worst-case running time complexity of building a heap of size n is:

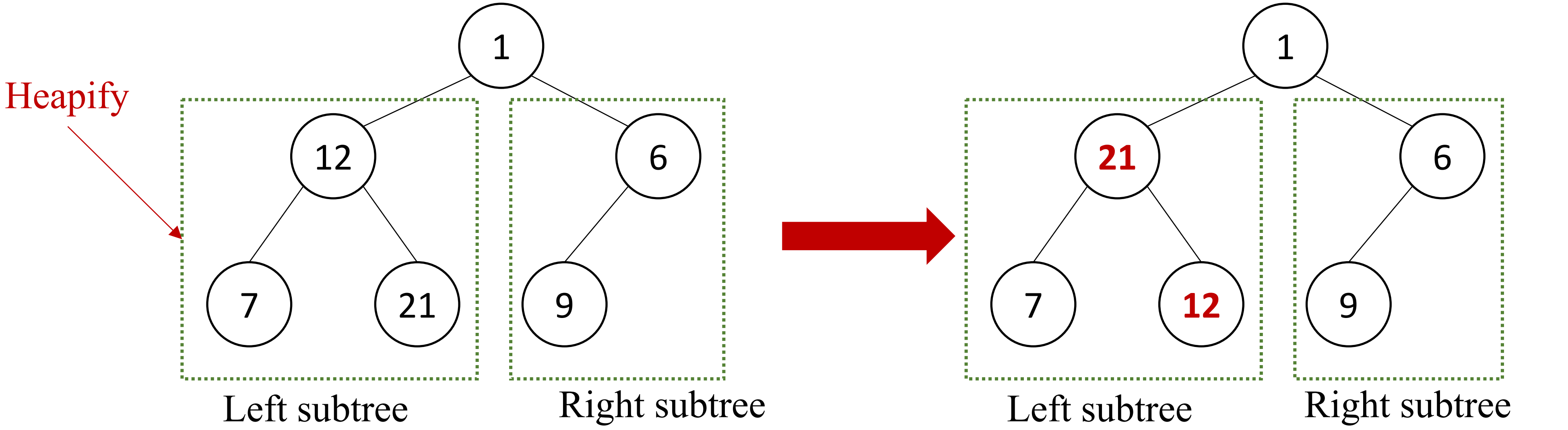
$$\bullet T(n) = \lfloor \log_2 1 \rfloor + \lfloor \log_2 2 \rfloor + \lfloor \log_2 3 \rfloor + \cdots + \lfloor \log_2 n \rfloor$$
$$\leq \log_2 n + \log_2 n + \log_2 n + \cdots + \log_2 n = n \log_2 n$$

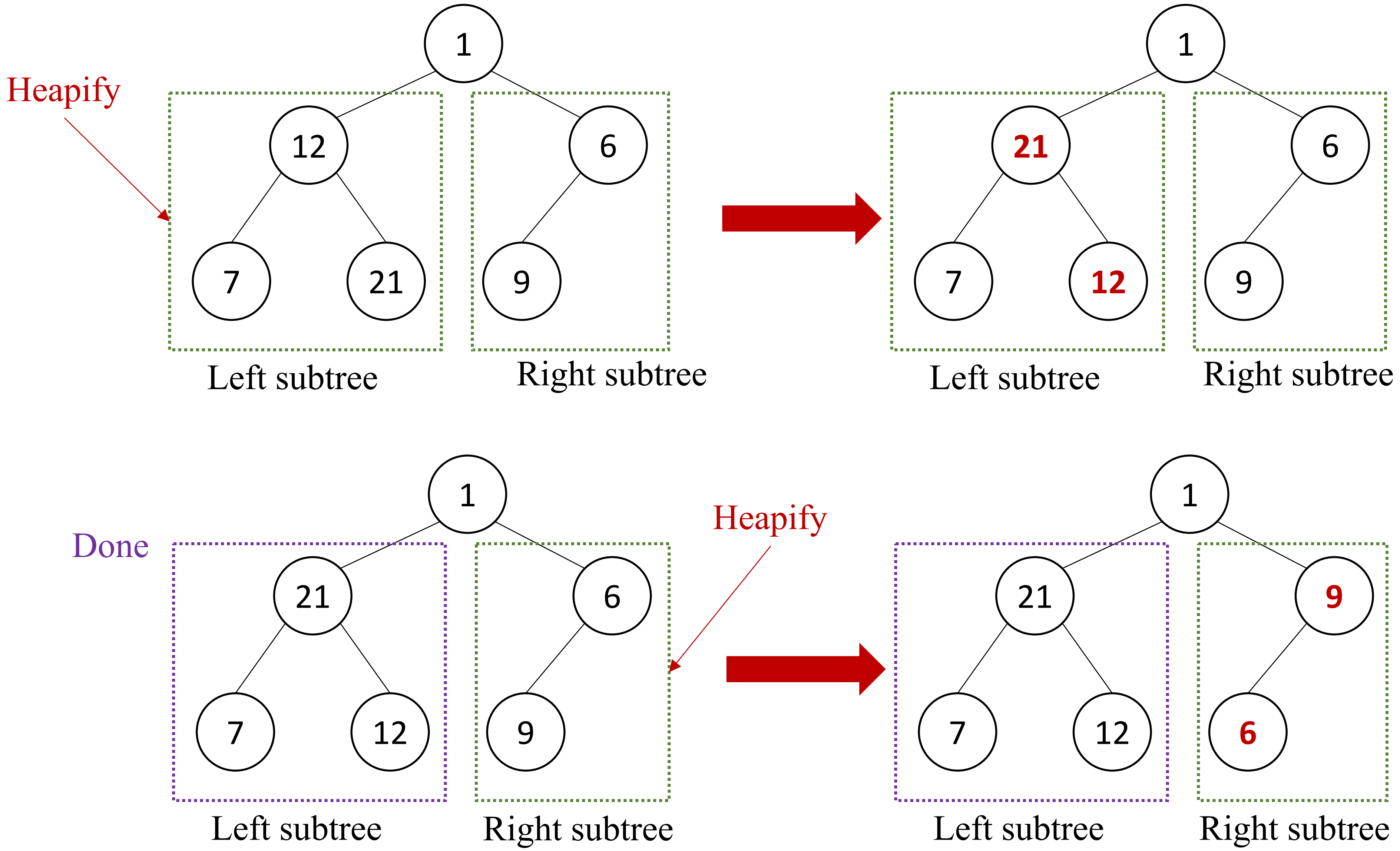
$$\bullet T(n) = \lfloor \log_2 1 \rfloor + \lfloor \log_2 2 \rfloor + \cdots + \left\lfloor \log_2 \left\lfloor \frac{n}{2} \right\rfloor \right\rfloor + \left\lfloor \log_2 \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right) \right\rfloor + \cdots + \lfloor \log_2 n \rfloor$$
$$\geq \left\lfloor \log_2 \left(\left\lceil \frac{n}{2} \right\rceil \right) \right\rfloor + \left\lfloor \log_2 \left(\left\lceil \frac{n}{2} \right\rceil \right) \right\rfloor + \cdots + \left\lfloor \log_2 \left(\left\lceil \frac{n}{2} \right\rceil \right) \right\rfloor$$
$$= \left\lceil \frac{n}{2} \right\rceil \left\lfloor \log_2 \left(\left\lceil \frac{n}{2} \right\rceil \right) \right\rfloor = \left\lceil \frac{n}{2} \right\rceil \left(\left\lceil \log_2 (\left\lfloor \frac{n}{2} \right\rfloor + 1) \right\rceil - 1 \right) \geq \frac{n}{2} \log_2 (\left\lceil \frac{n+2}{2} \right\rceil) \geq \frac{n}{2} \log_2 (\frac{n}{2})$$

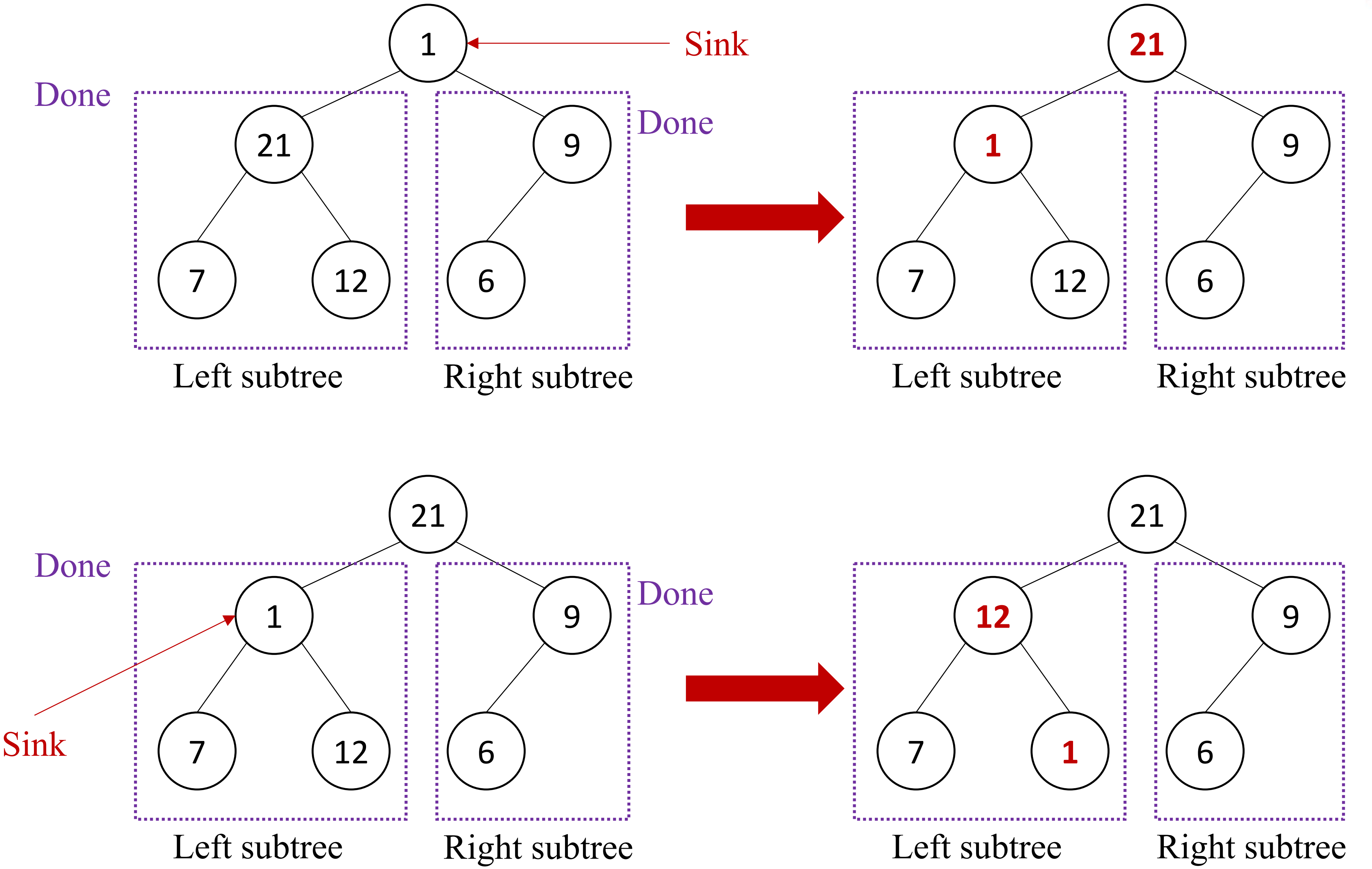
The above gives: $T(n) \in \Theta(n \log n)$

Can we do better?

- **Motivation of an alternative method.** We only need the heap property at the end, no need to keep it whenever an element is inserted.
  1. Build a complete binary tree without keeping the heap property – this is especially easy for array implementation which only needs to insert all elements to a list.
  2. After all elements are inserted, reorganize the positions of the elements to follow heap property, this operation is called heapify – recursively heapify the left and right subtrees, then put the root down to the right position.

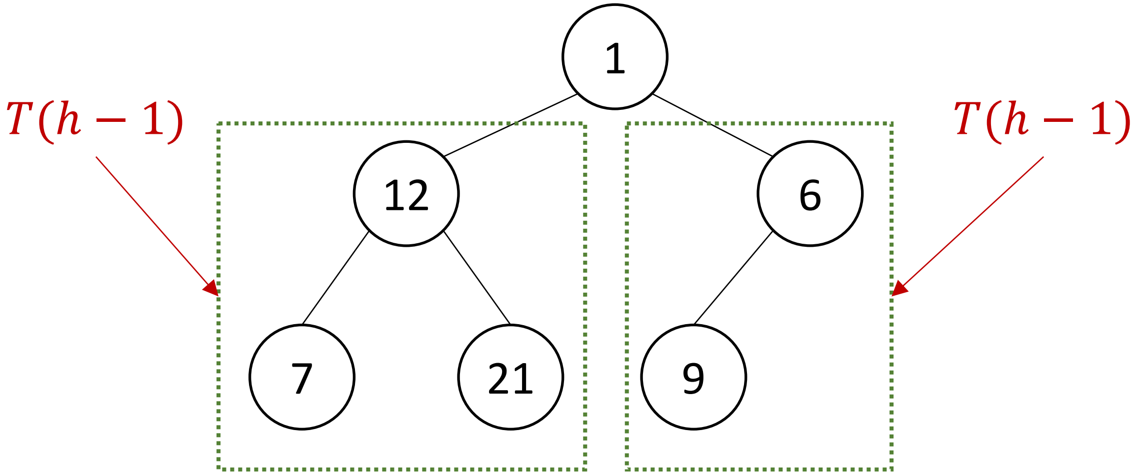



Heapify
1
12
6
7
21
9
Left subtree
Right subtree
1
21
6
7
12
9
Left subtree
Right subtree
Done
1
21
6
7
12
9
Left subtree
Right subtree
Heapify
1
21
9
7
12
6
Left subtree
Right subtree


Sink
Done
Done
Left subtree
Right subtree
Left subtree
Right subtree
Done
Done
Sink
Left subtree
Right subtree
Left subtree
Right subtree

Time Complexity Analysis: Heapifying

- Let $T(h)$ is the running time complexity for building a heap with height at most h. It comprises of
  - Running time of heapifying the two subtrees recursively $T(h-1)$
  - Running time of sinking the root to the right position needs at most h comparisons/swaps
  - Thus, $T(h) = 2T(h-1) + h,\ T(0) = 0$



We can show that $T(h)$ is $\Theta(2^h)$

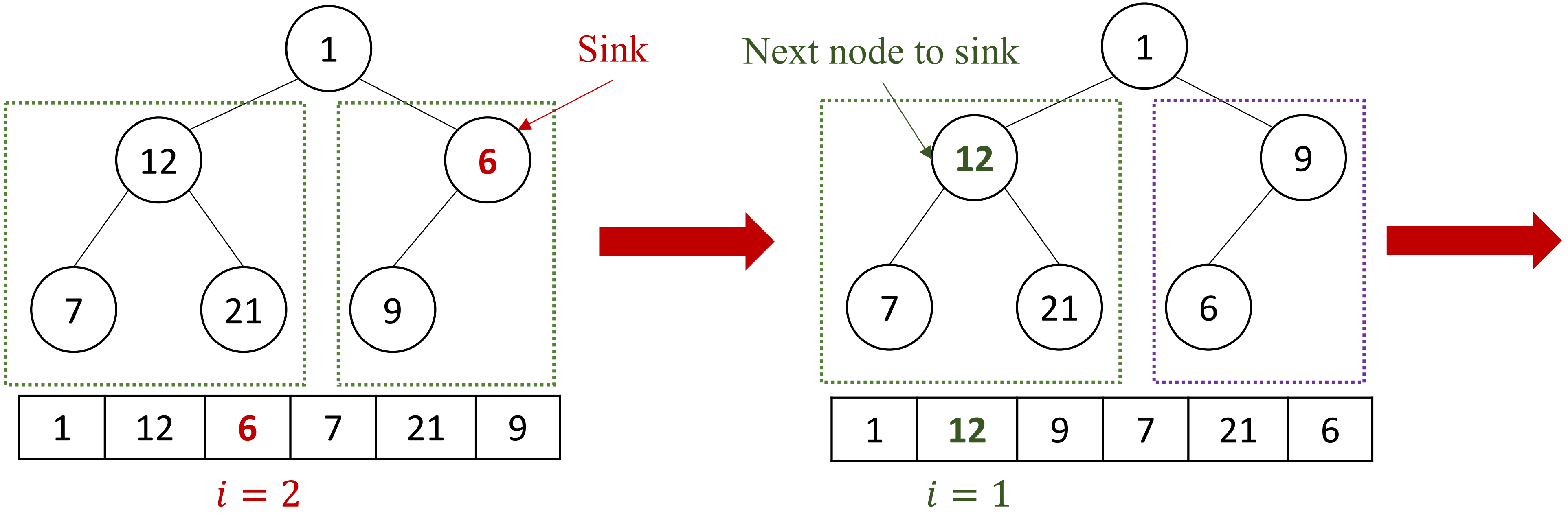
Because $h = \lfloor \log_2 n \rfloor$, then $T(n)$ is $\Theta(n)$

$$T(h) - 2T(h-1) = h$$
$$2T(h-1) - 4T(h-2) = 2(h-1)$$
$$4T(h-2) - 8T(h-3) = 4\,(h-2)$$
$$\cdots\cdots$$
$$2^{h-1}T(1) - 2^h T(0) = 2^{h-1} \cdot 1$$

$$T(h) = \left(2^{h-1} \cdot 1 + 2^{h-2} \cdot 2 + \cdots + 2^0 \cdot h\right)$$

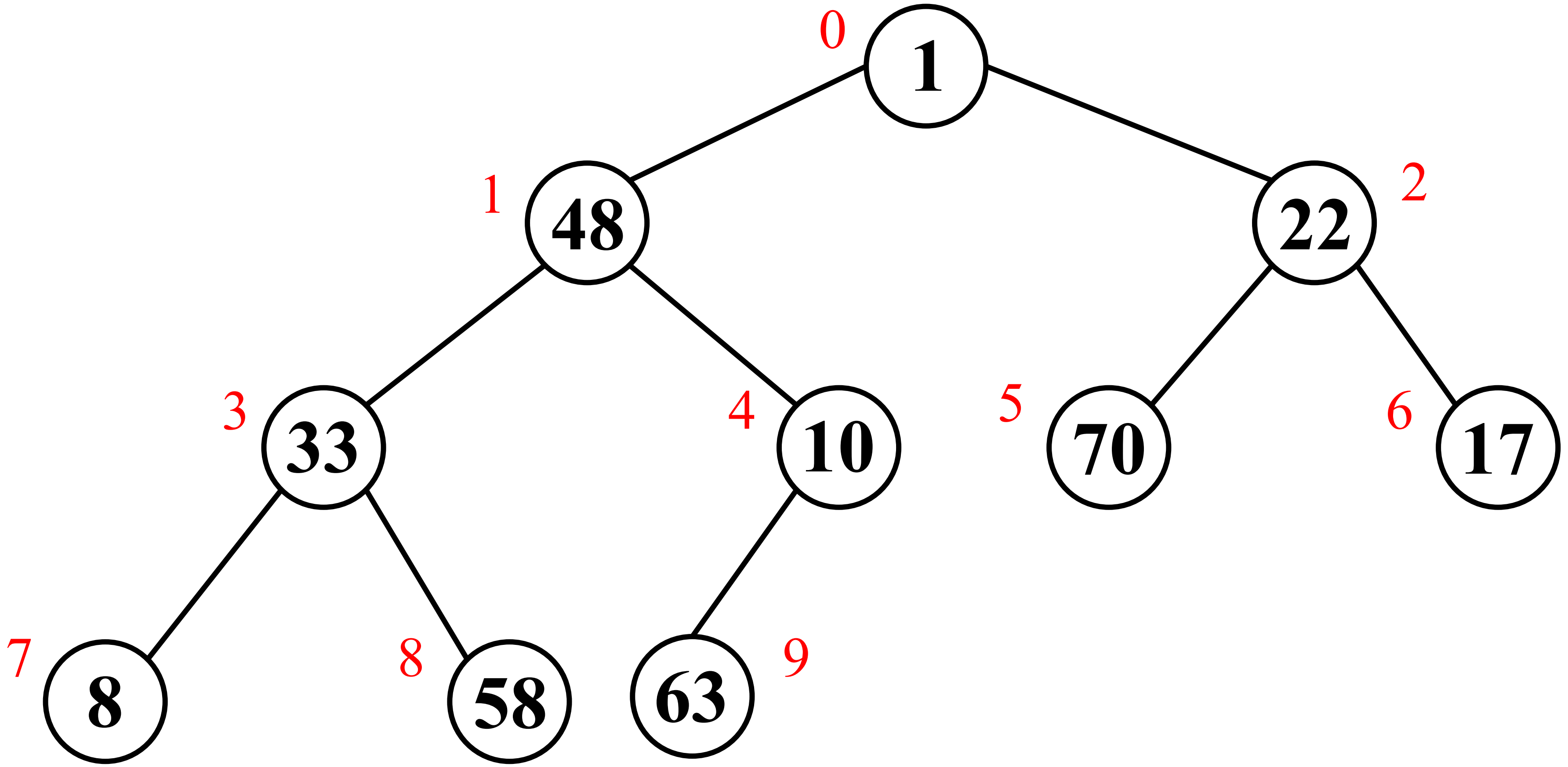
Heapifying without Recursions

- Sink the nodes in a bottom-up manner. Consider the array implementation of heap.
    1. Find the lowest non-leaf node, which is at position $\lfloor (n-1)/2 \rfloor$. Set the current node index $i = \lfloor (n-1)/2 \rfloor$
    2. Sink the current node at position i to the correct position.
    3. Find the next non-leaf node, that is the node at position $i-1$, go to step 2.



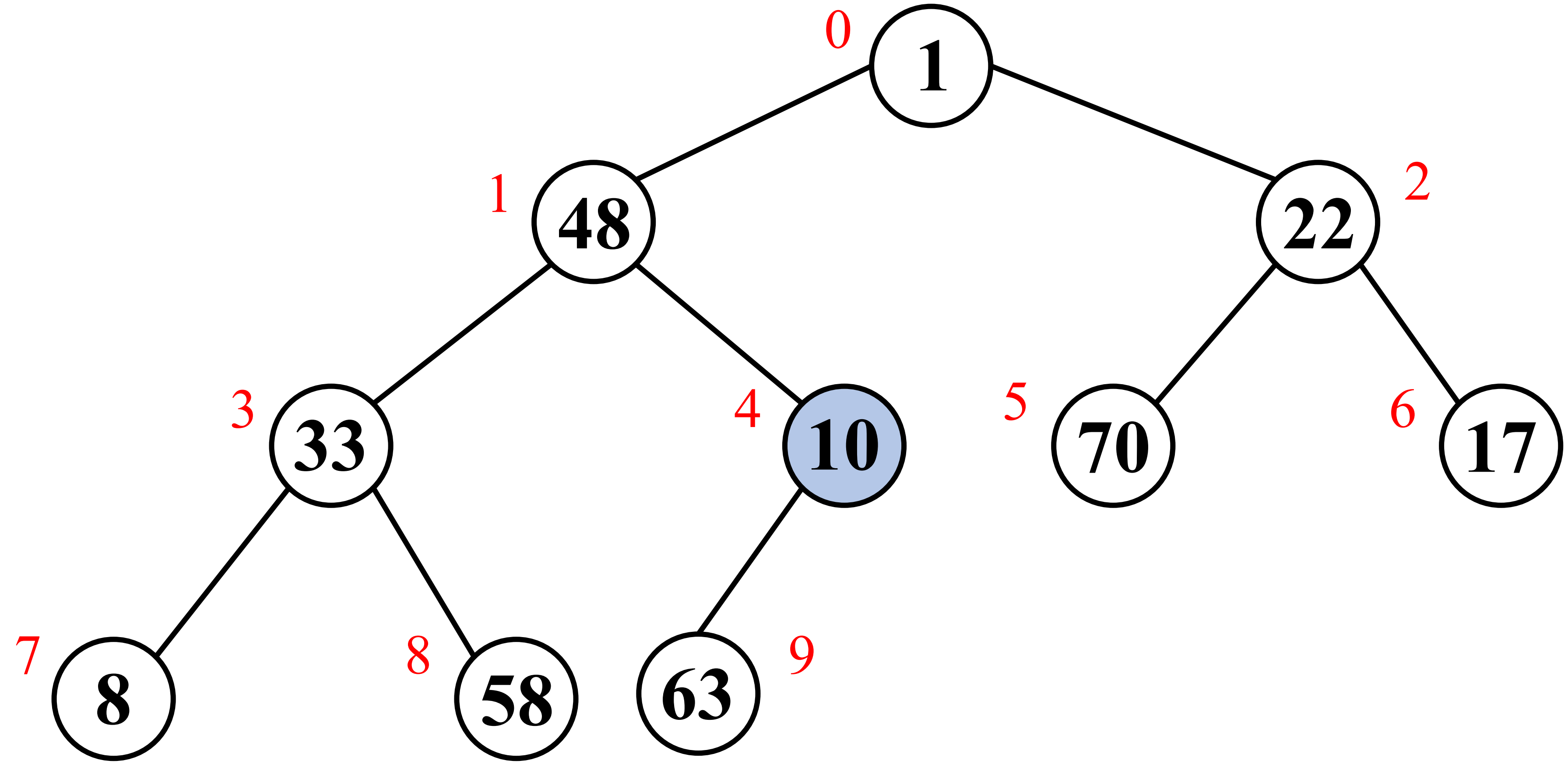
Repeat until we complete sinking the root

Heapsort algorithms: array implementation



1	48	22	33	10	70	17	8	58	63
0	1	2	3	4	5	6	7	8	9

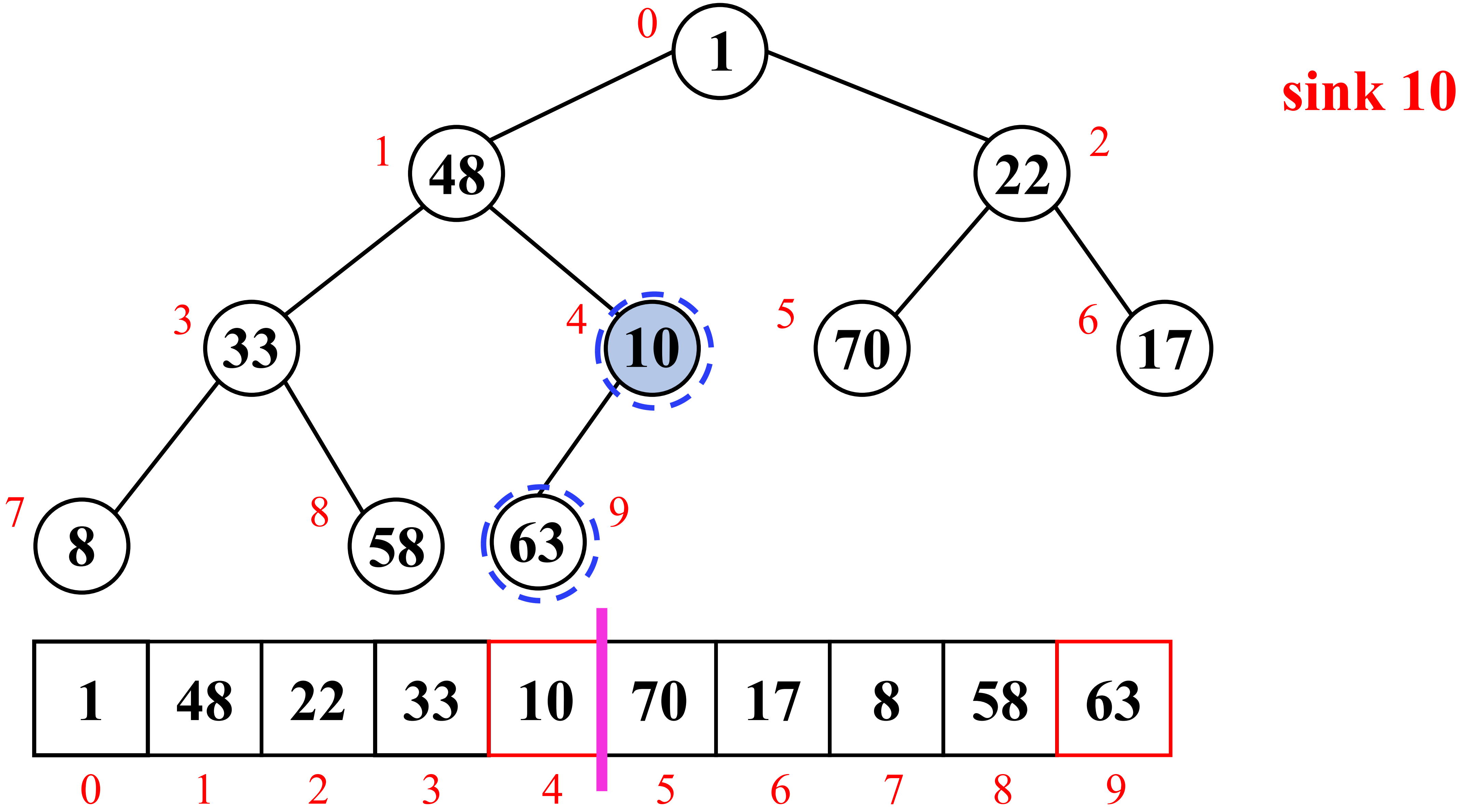
Heapsort algorithms: array implementation



sink 10

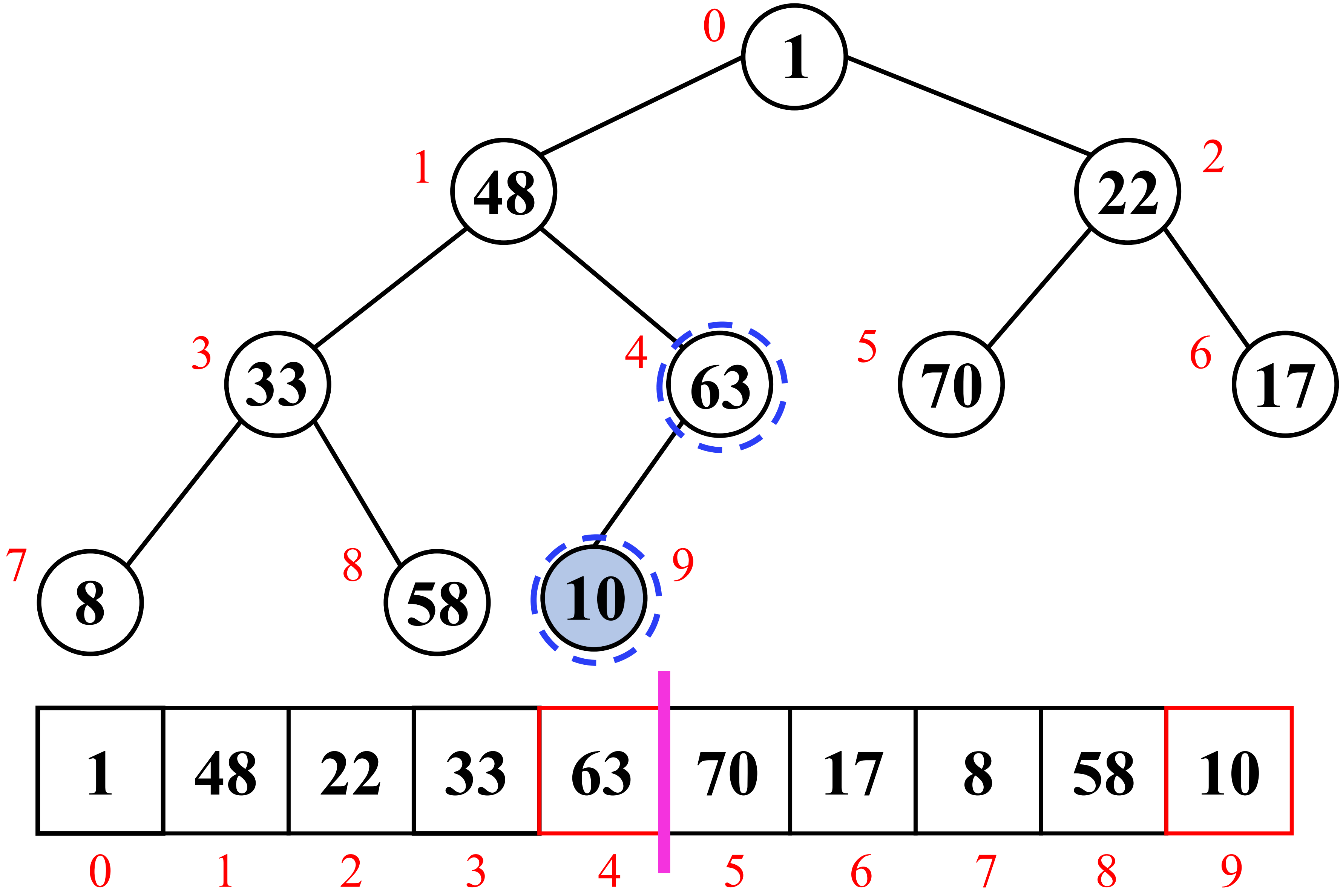
1	48	22	33	10	70	17	8	58	63
0	1	2	3	4	5	6	7	8	9

Heapsort algorithms: array implementation

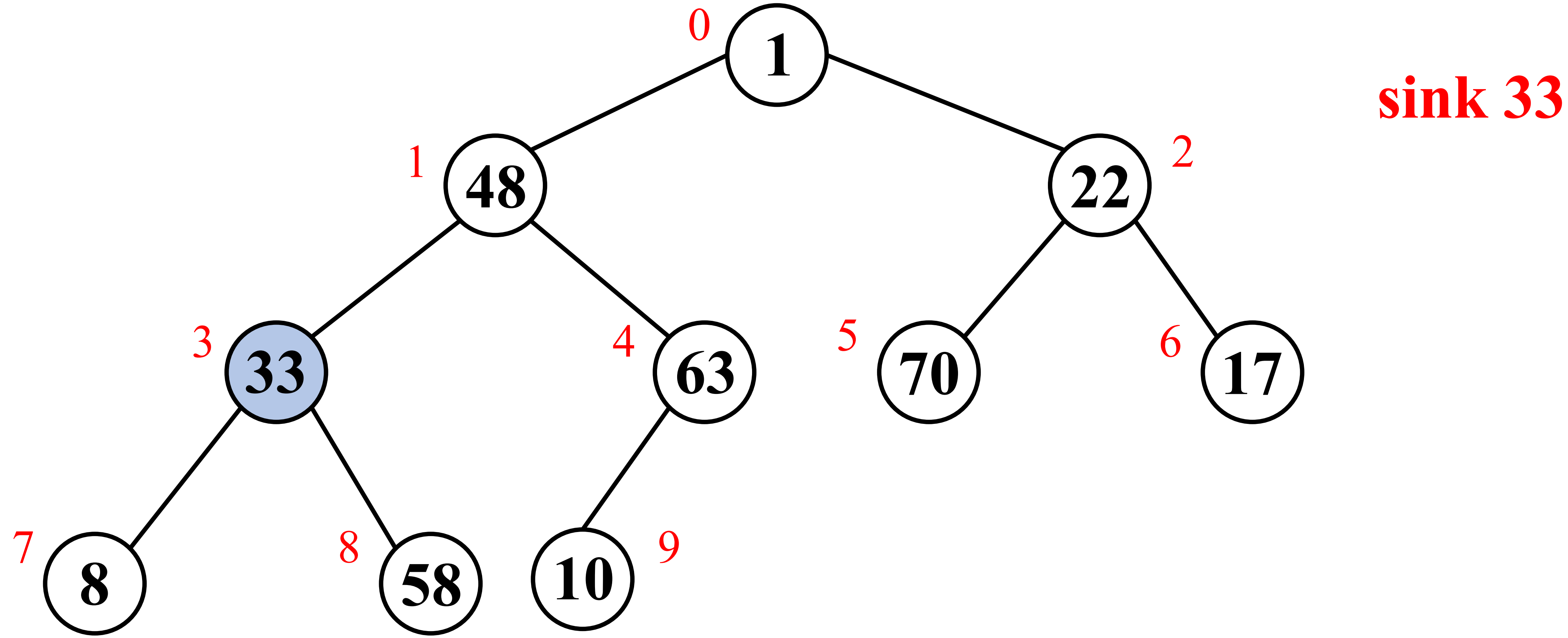


Heapsort algorithms: array implementation

sink 10



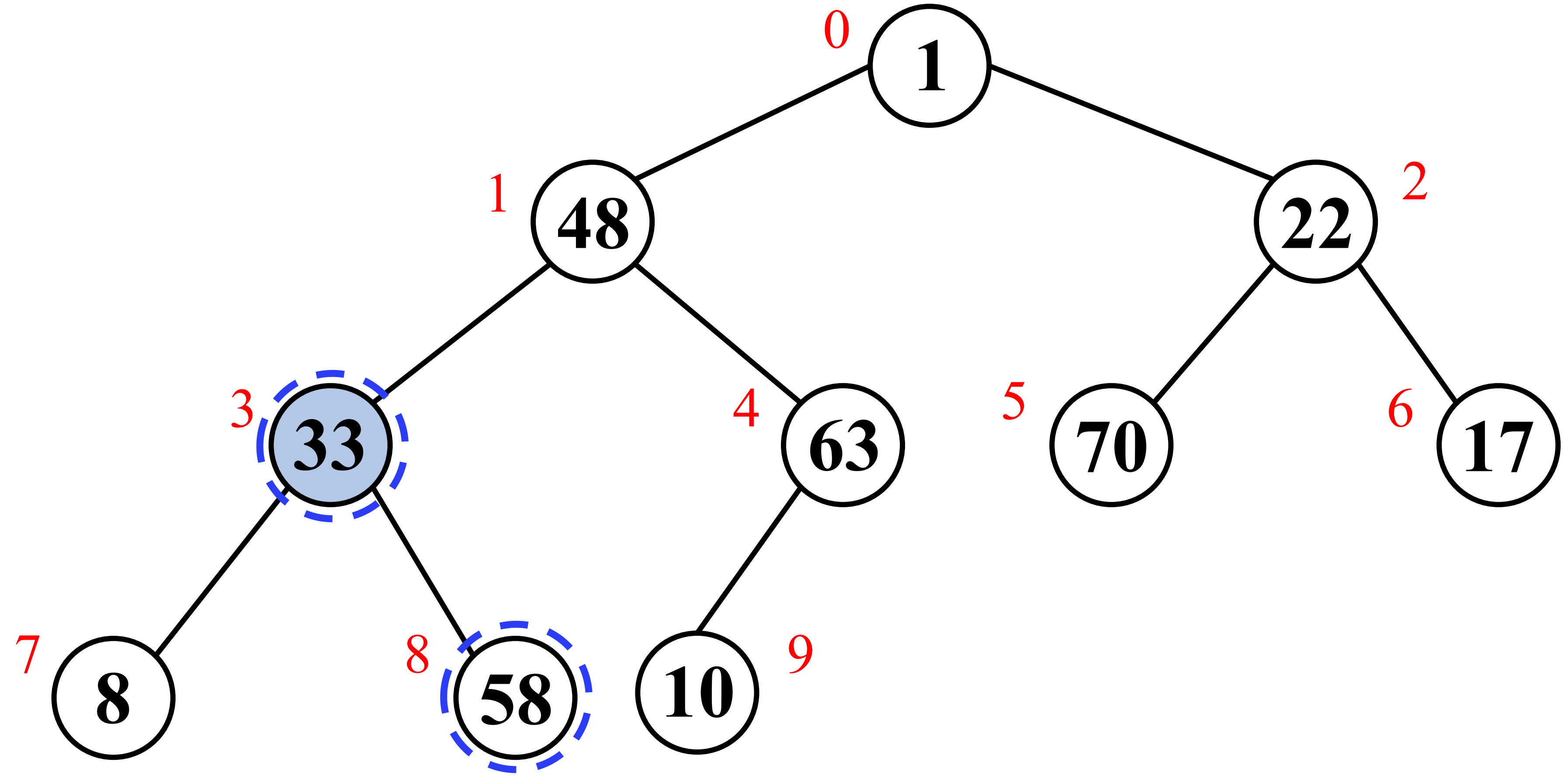
Heapsort algorithms: array implementation



1	48	22	33	63	70	17	8	58	10
0	1	2	3	4	5	6	7	8	9

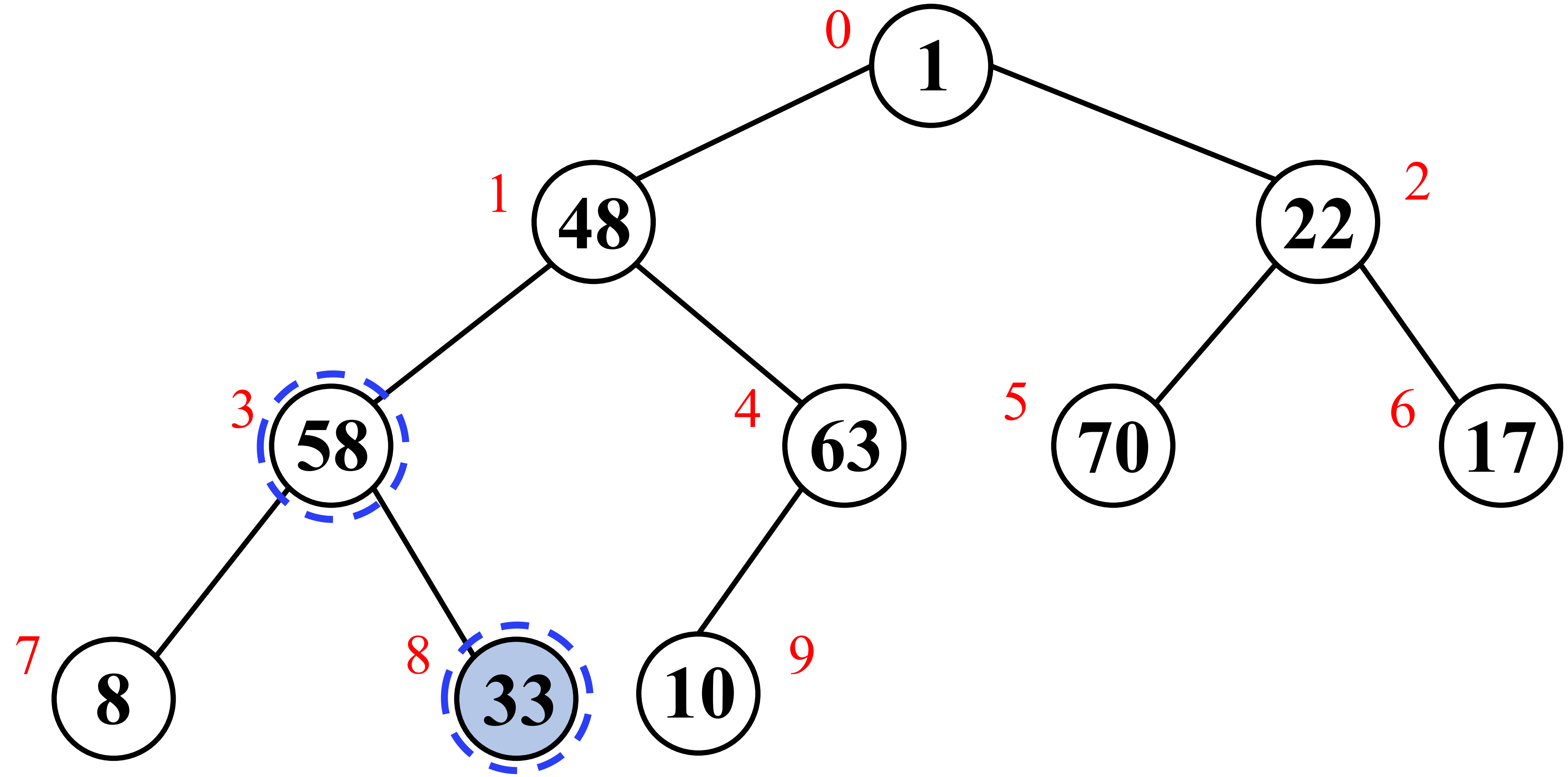
Heapsort algorithms: array implementation

sink 33



1	48	22	33	63	70	17	8	58	10
0	1	2	3	4	5	6	7	8	9

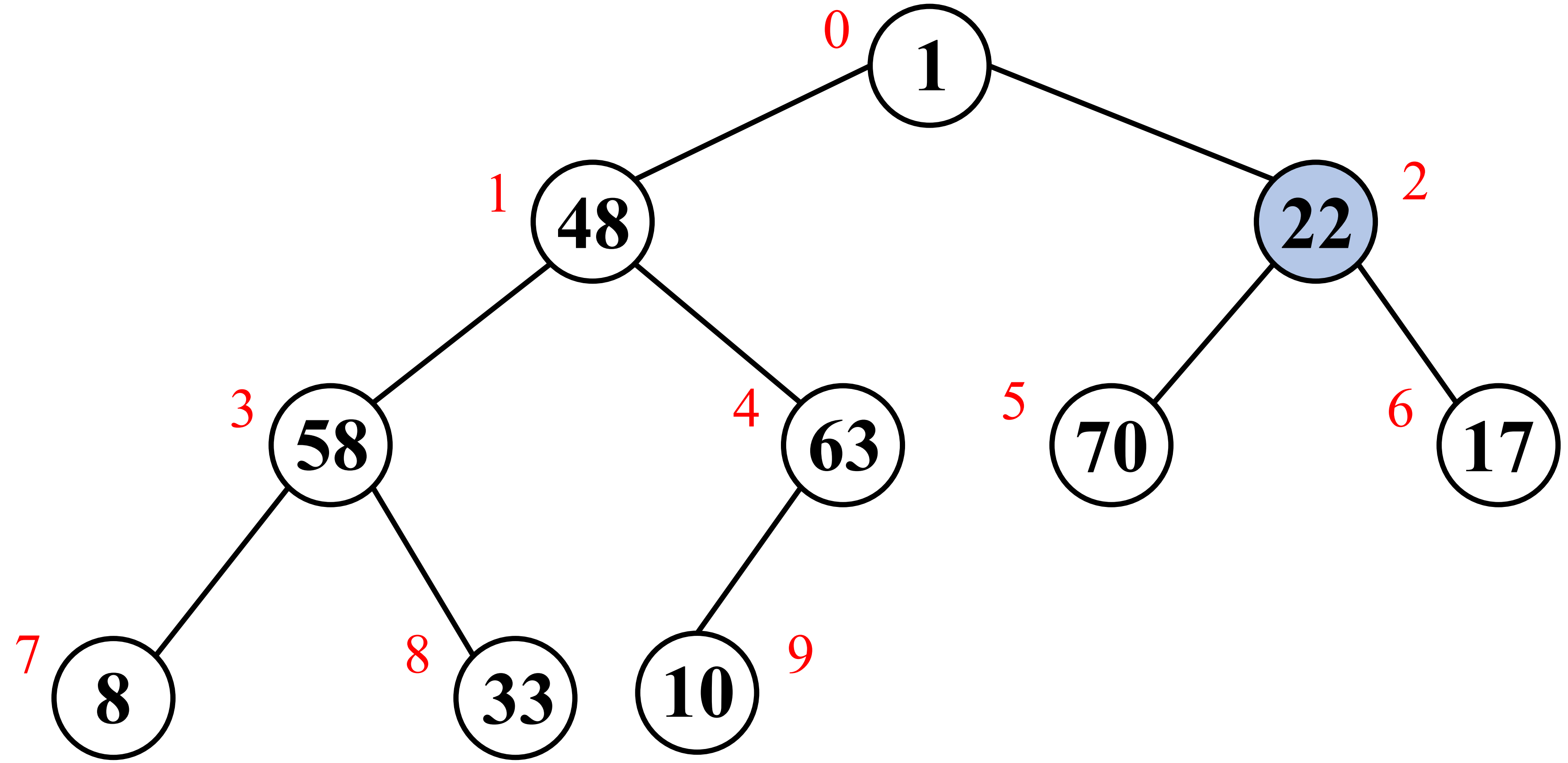
Heapsort algorithms: array implementation



sink 33

1	**48**	**22**	**58**	**63**	**70**	**17**	**8**	**33**	**10**
0	1	2	3	4	5	6	7	8	9

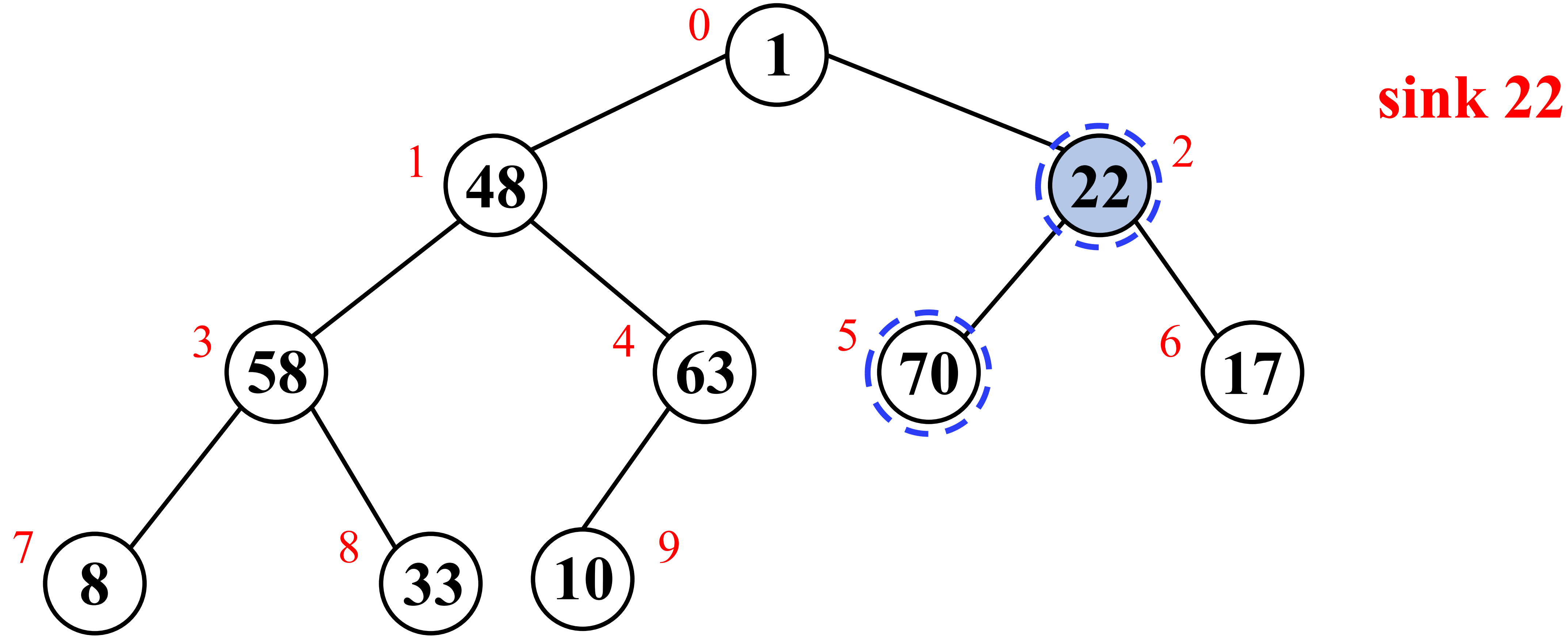
Heapsort algorithms: array implementation



sink 22

1	48	22	58	63	70	17	8	33	10
0	1	2	3	4	5	6	7	8	9

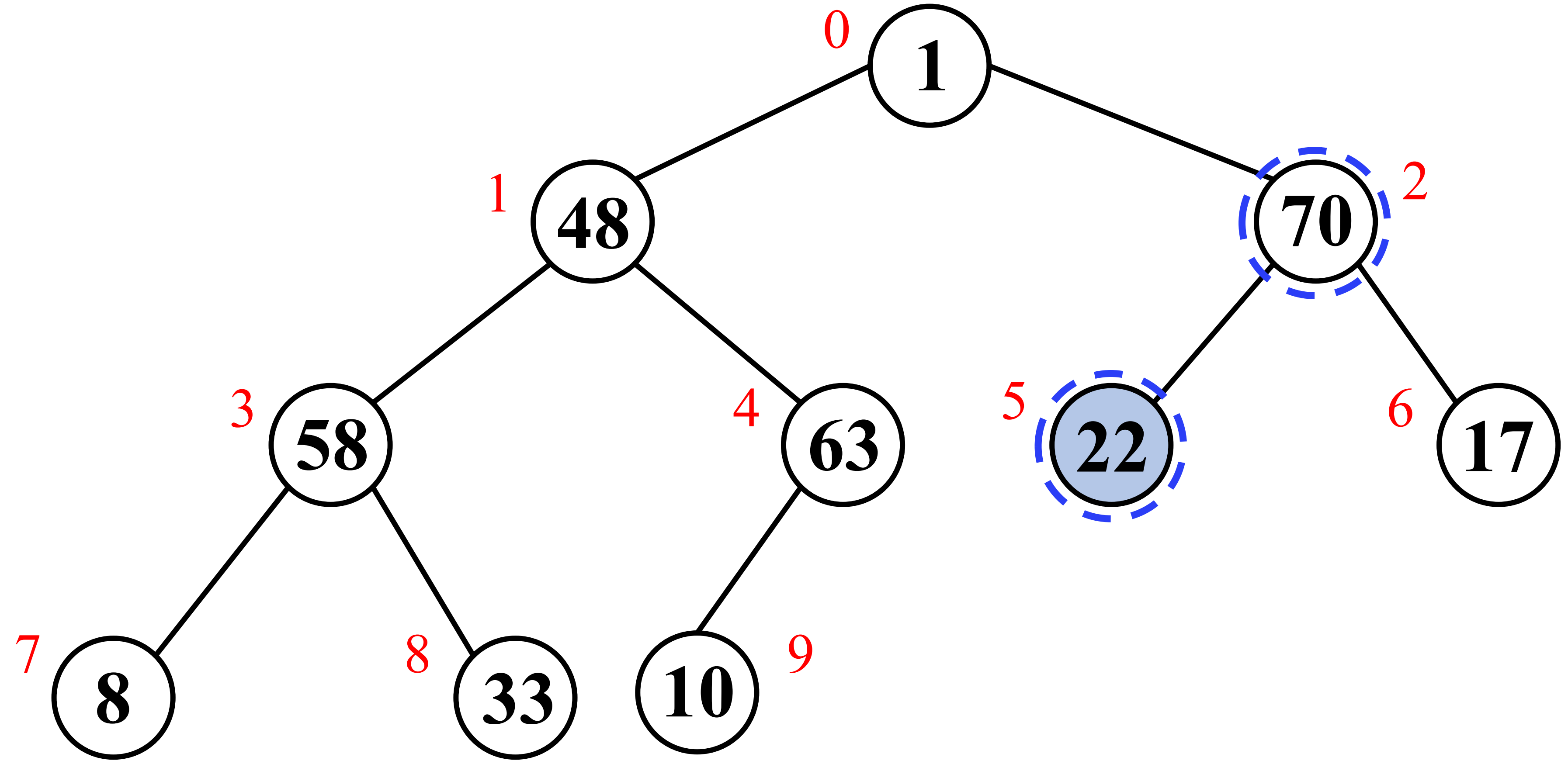
Heapsort algorithms: array implementation



1	48	22	58	63	70	17	8	33	10
0	1	2	3	4	5	6	7	8	9

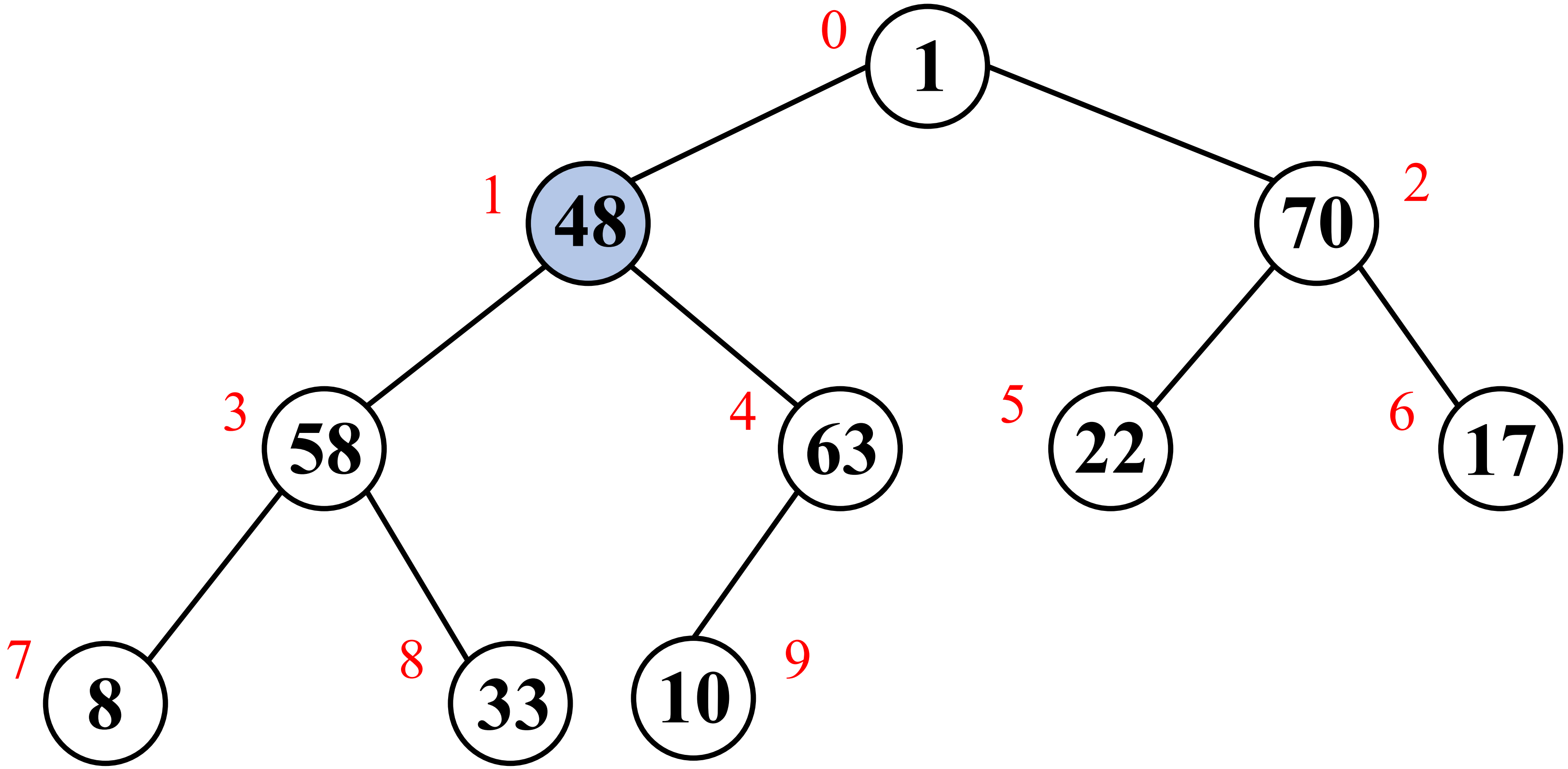
Heapsort algorithms: array implementation

sink 22



1	48	70	58	63	22	17	8	33	10
0	1	2	3	4	5	6	7	8	9

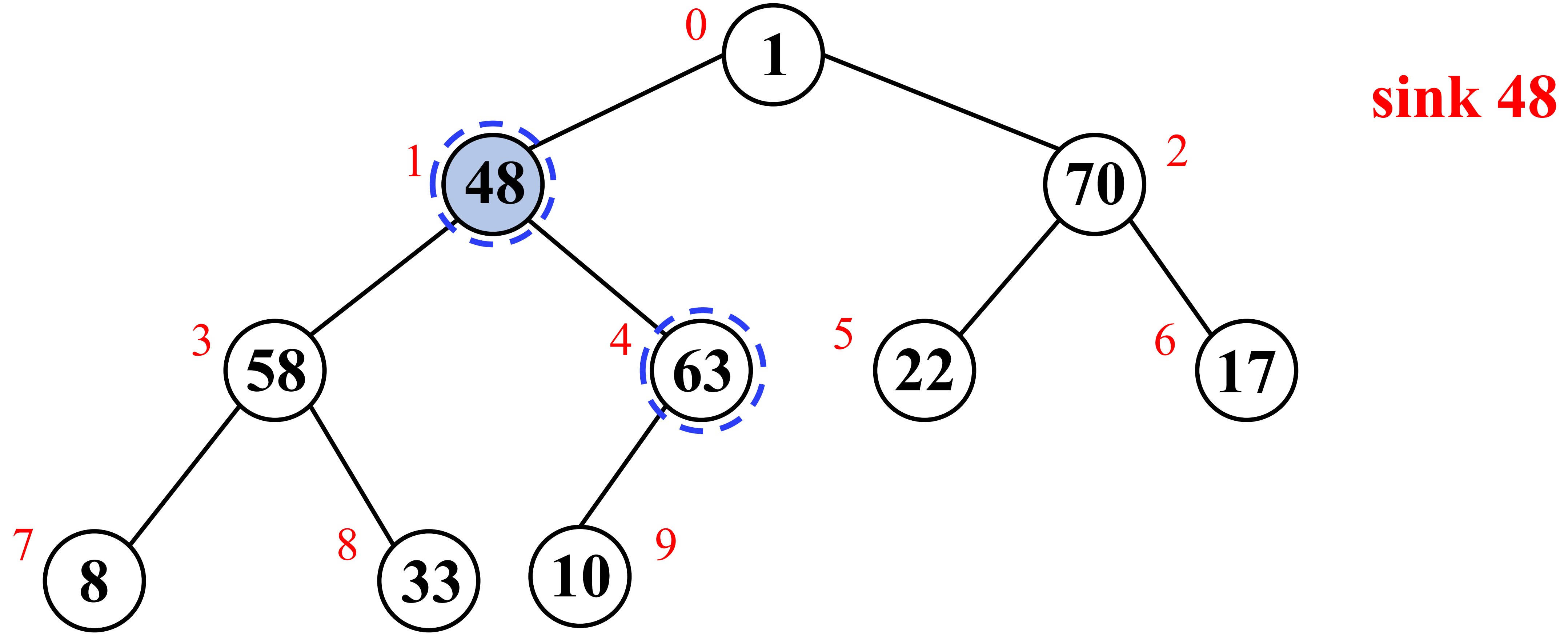
Heapsort algorithms: array implementation



sink 48

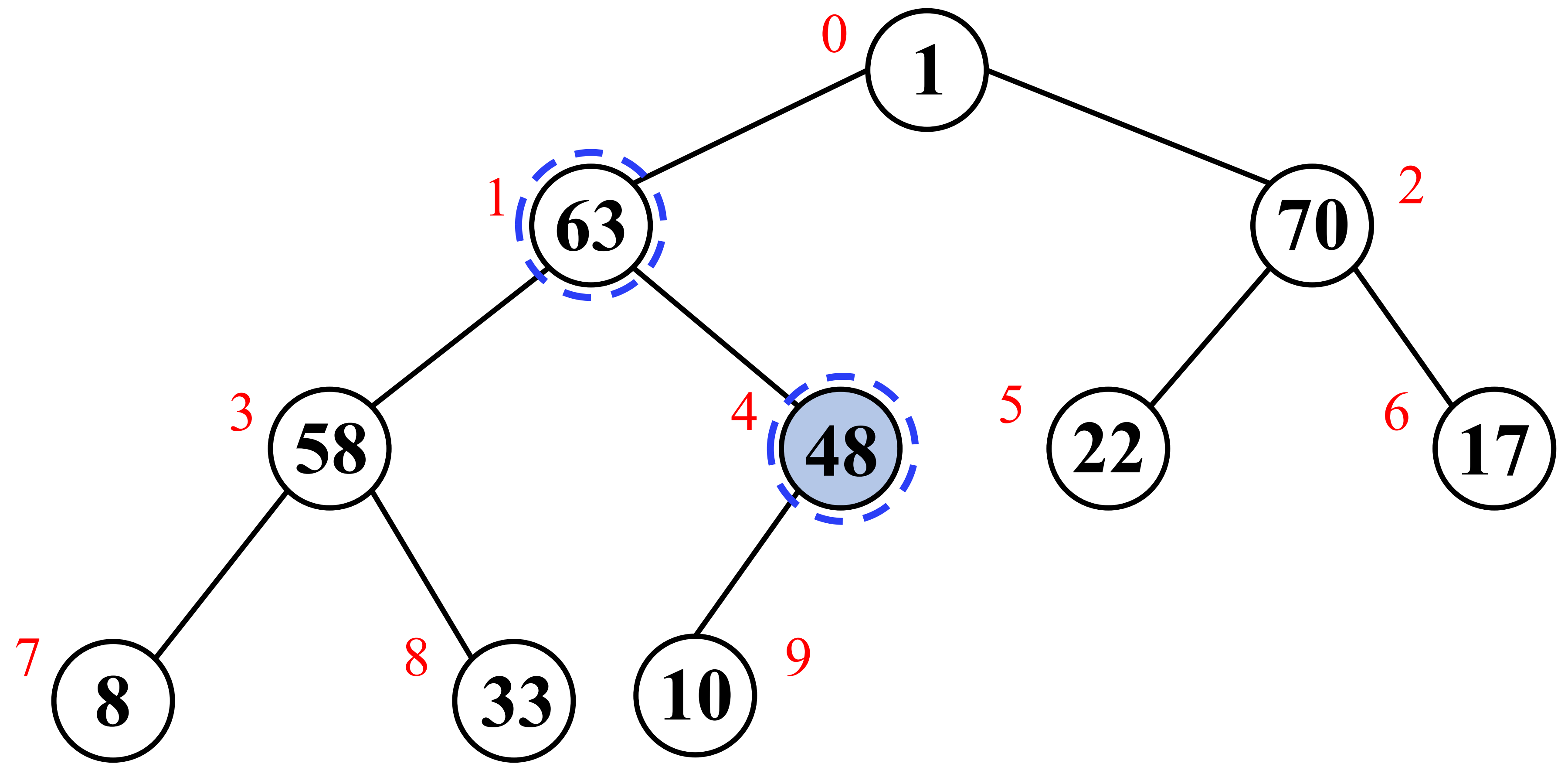
1	48	70	58	63	22	17	8	33	10
0	1	2	3	4	5	6	7	8	9

Heapsort algorithms: array implementation



1	48	70	58	63	22	17	8	33	10
0	1	2	3	4	5	6	7	8	9

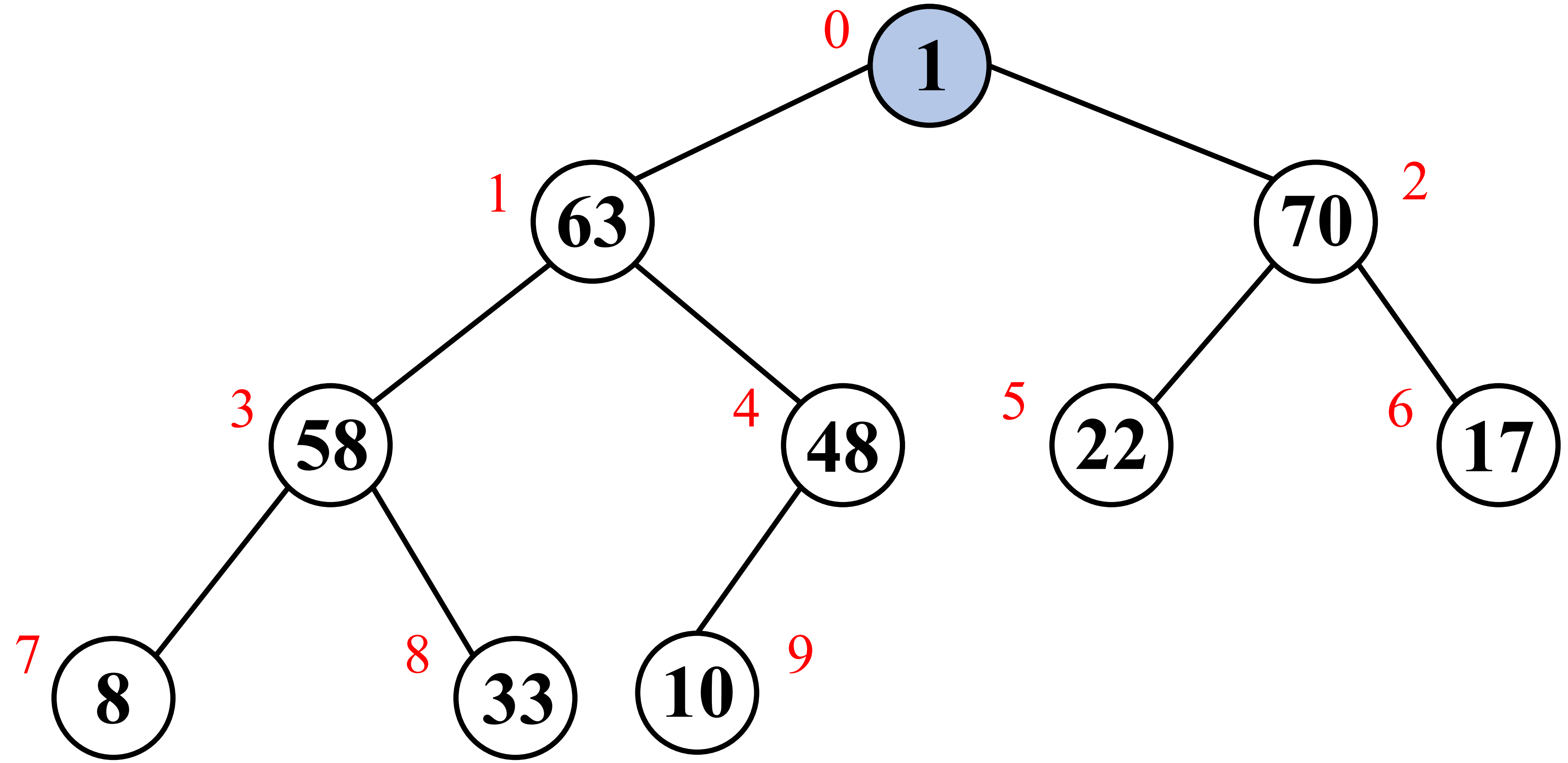
Heapsort algorithms: array implementation



sink 48

1	63	70	58	48	22	17	8	33	10
0	1	2	3	4	5	6	7	8	9

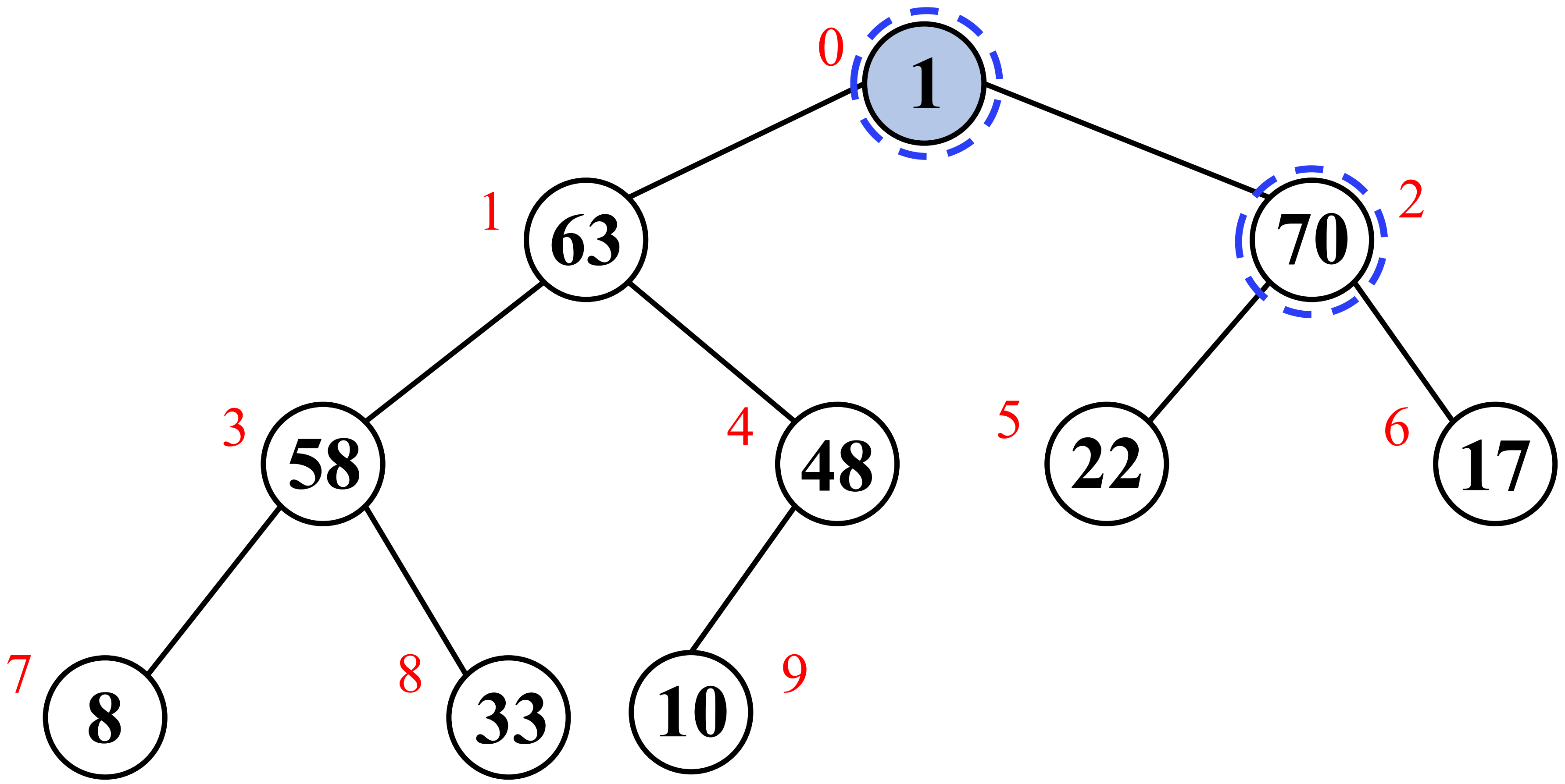
Heapsort algorithms: array implementation



sink 1

1	**63**	**70**	**58**	**48**	**22**	**17**	**8**	**33**	**10**
0	1	2	3	4	5	6	7	8	9

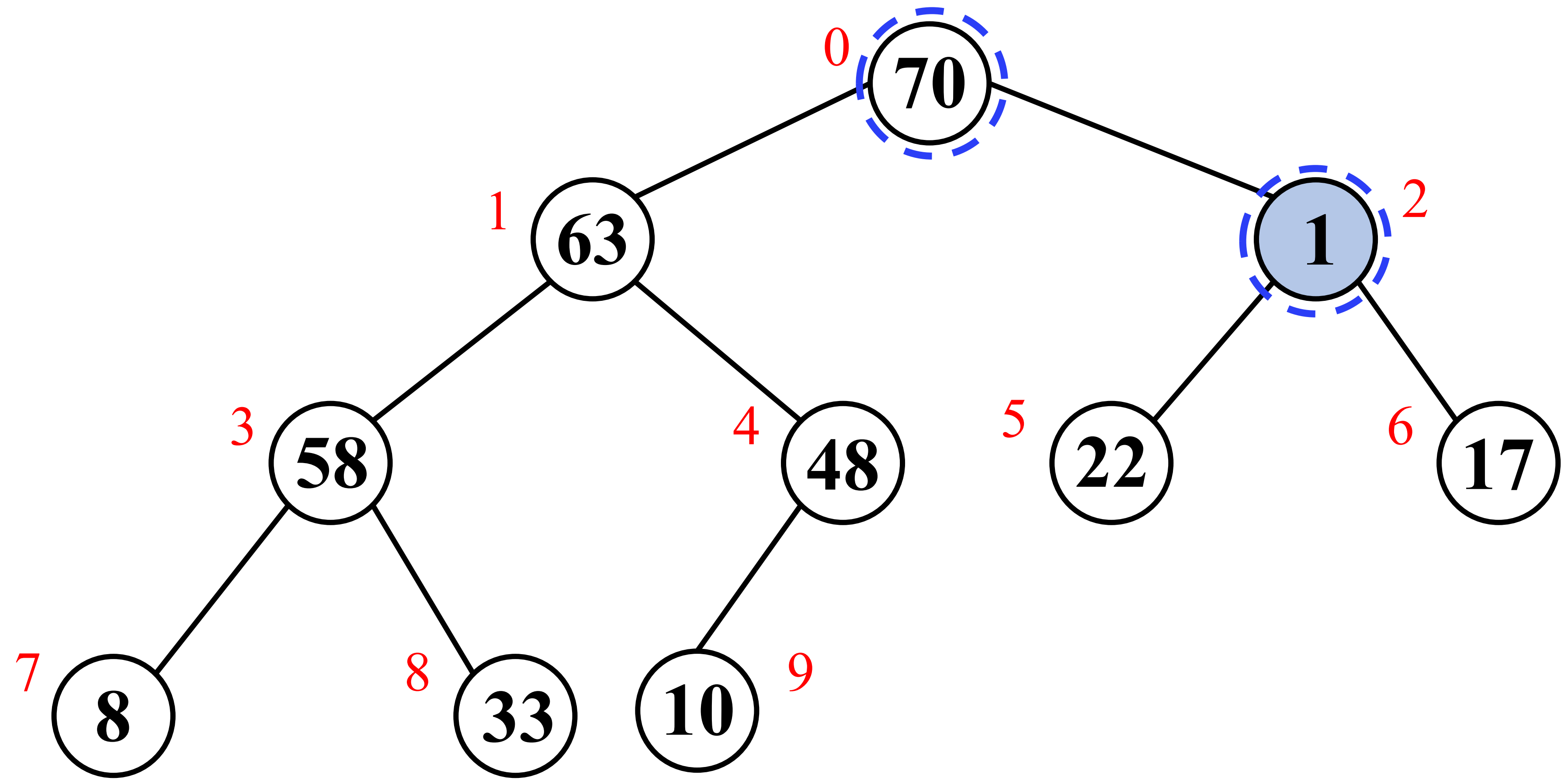
Heapsort algorithms: array implementation



sink 1

1	**63**	**70**	**58**	**48**	**22**	**17**	**8**	**33**	**10**
0	1	2	3	4	5	6	7	8	9

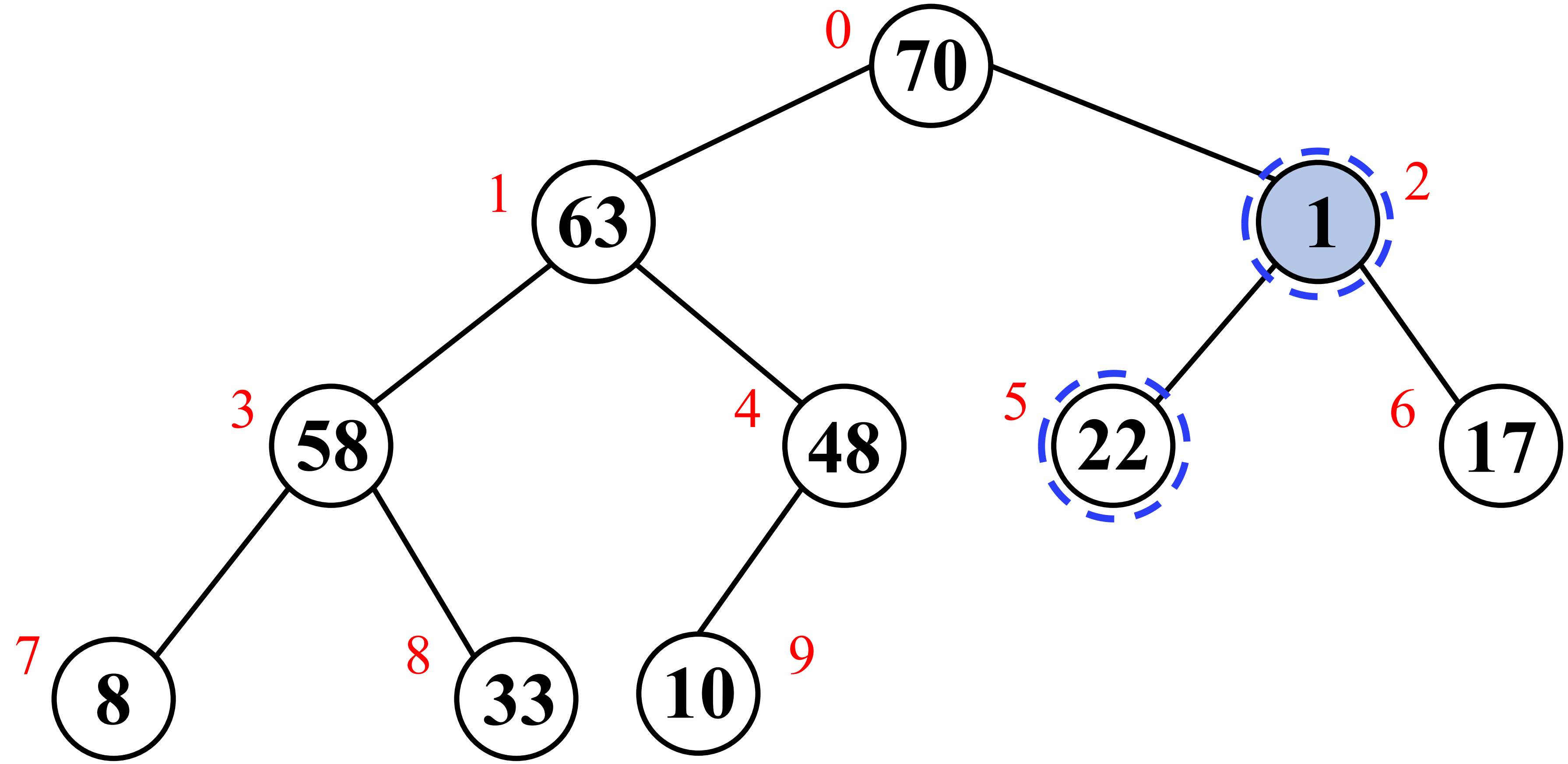
Heapsort algorithms: array implementation



sink 1

70	63	1	58	48	22	17	8	33	10
0	1	2	3	4	5	6	7	8	9

Heapsort algorithms: array implementation

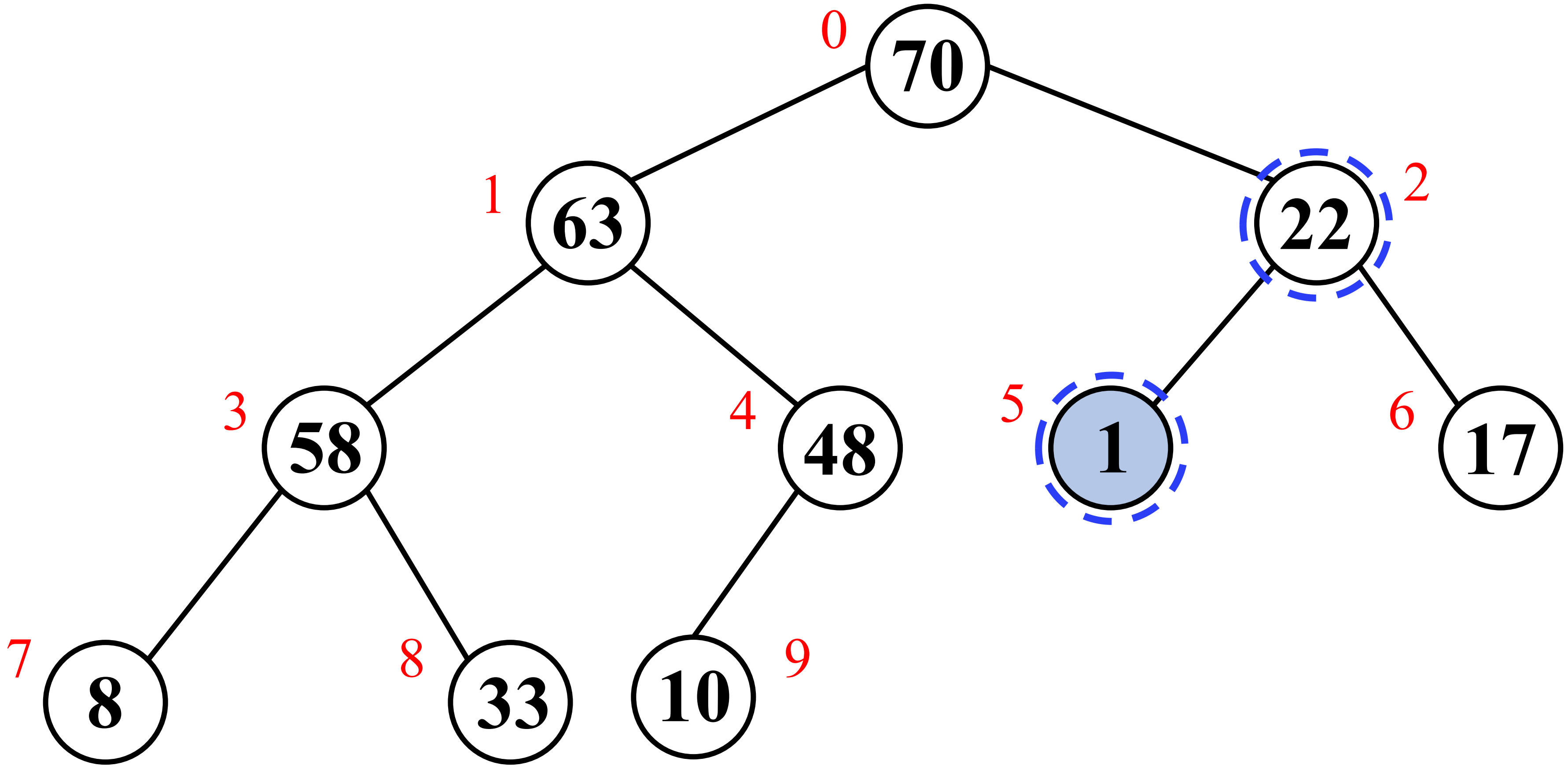


sink 1

70	**63**	**1**	**58**	**48**	**22**	**17**	**8**	**33**	**10**
0	1	2	3	4	5	6	7	8	9

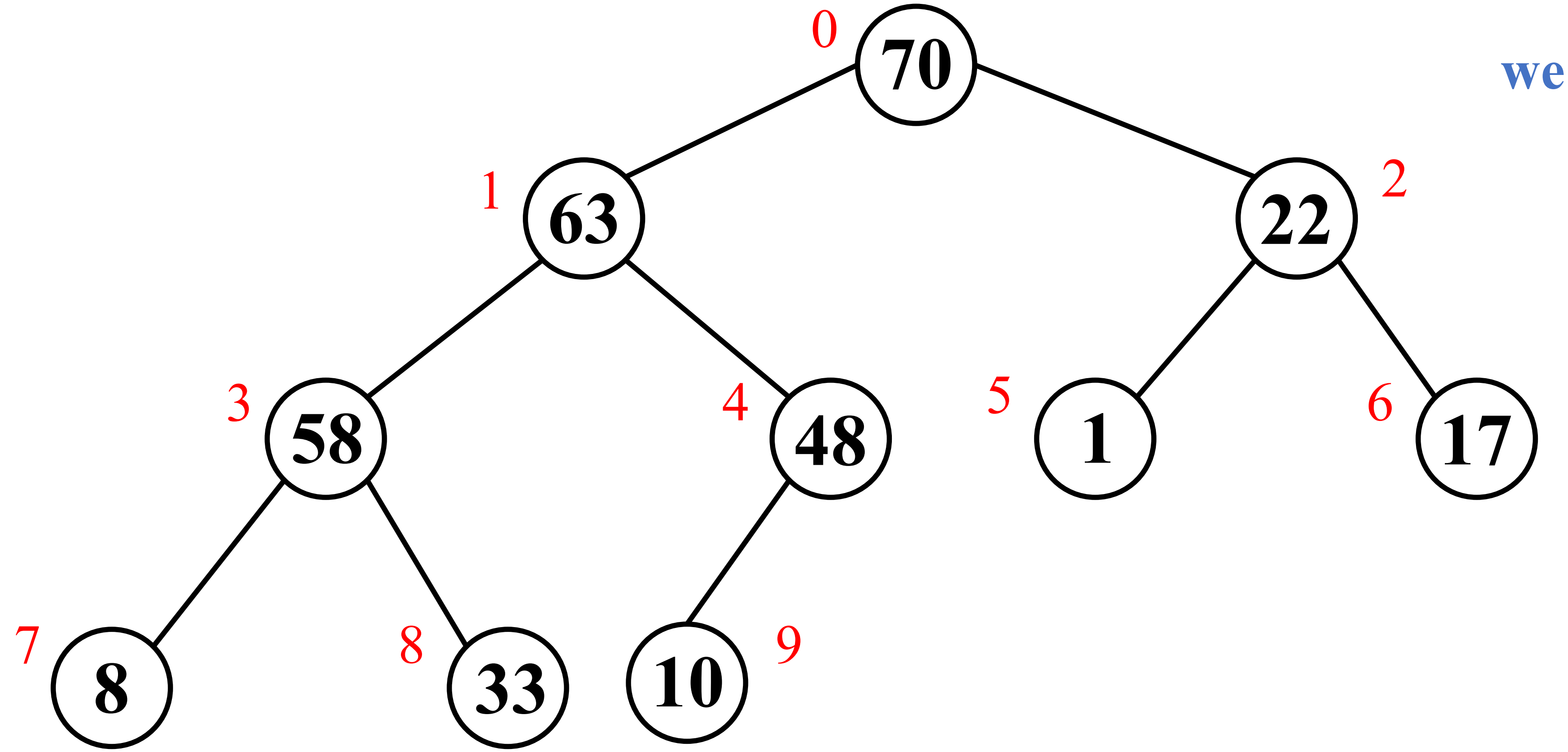
Heapsort algorithms: array implementation

sink 1



70	63	22	58	48	1	17	8	33	10
0	1	2	3	4	5	6	7	8	9

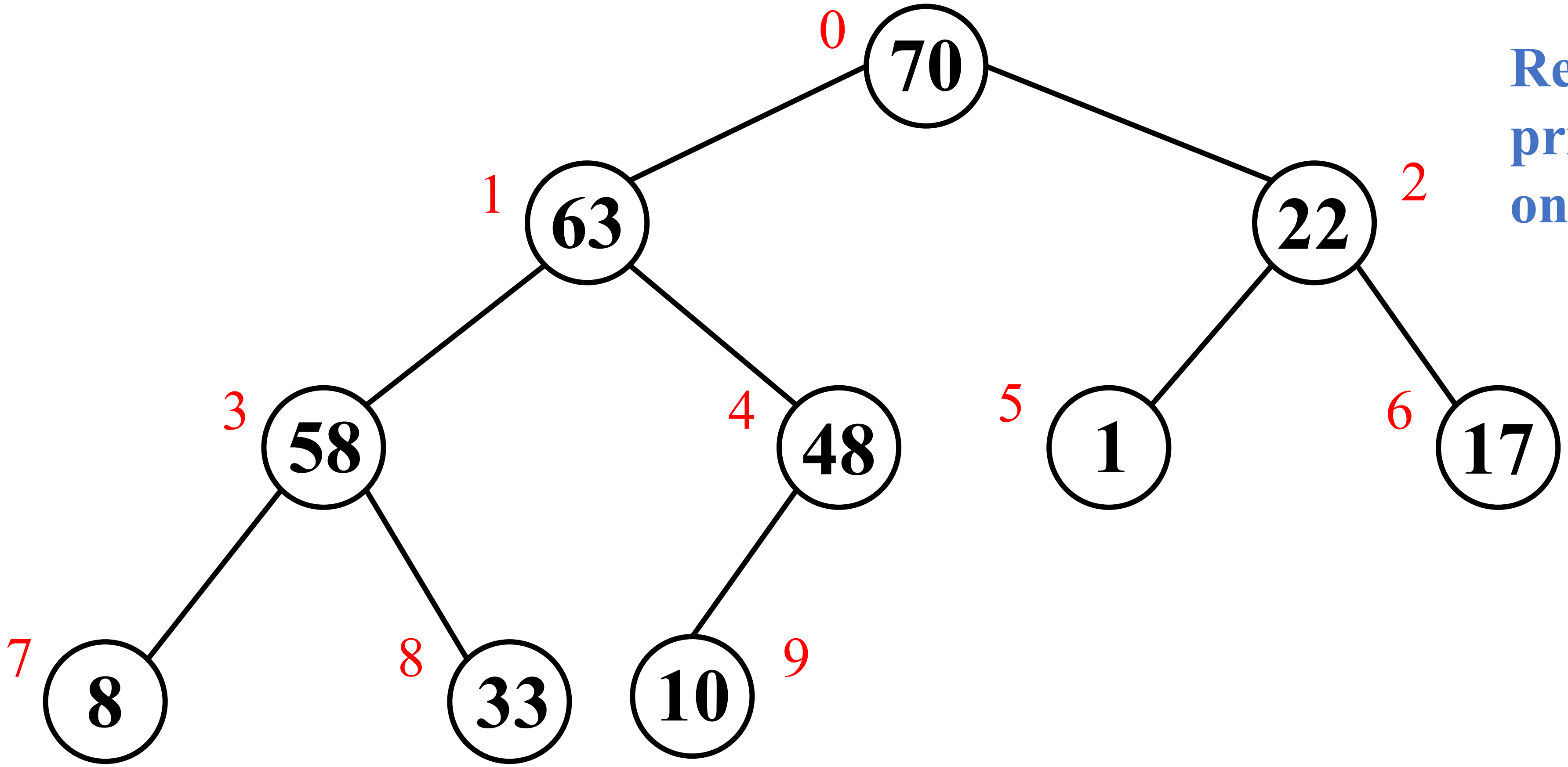
Heapsort algorithms: array implementation



we have built the heap

70	63	22	58	48	1	17	8	33	10
0	1	2	3	4	5	6	7	8	9

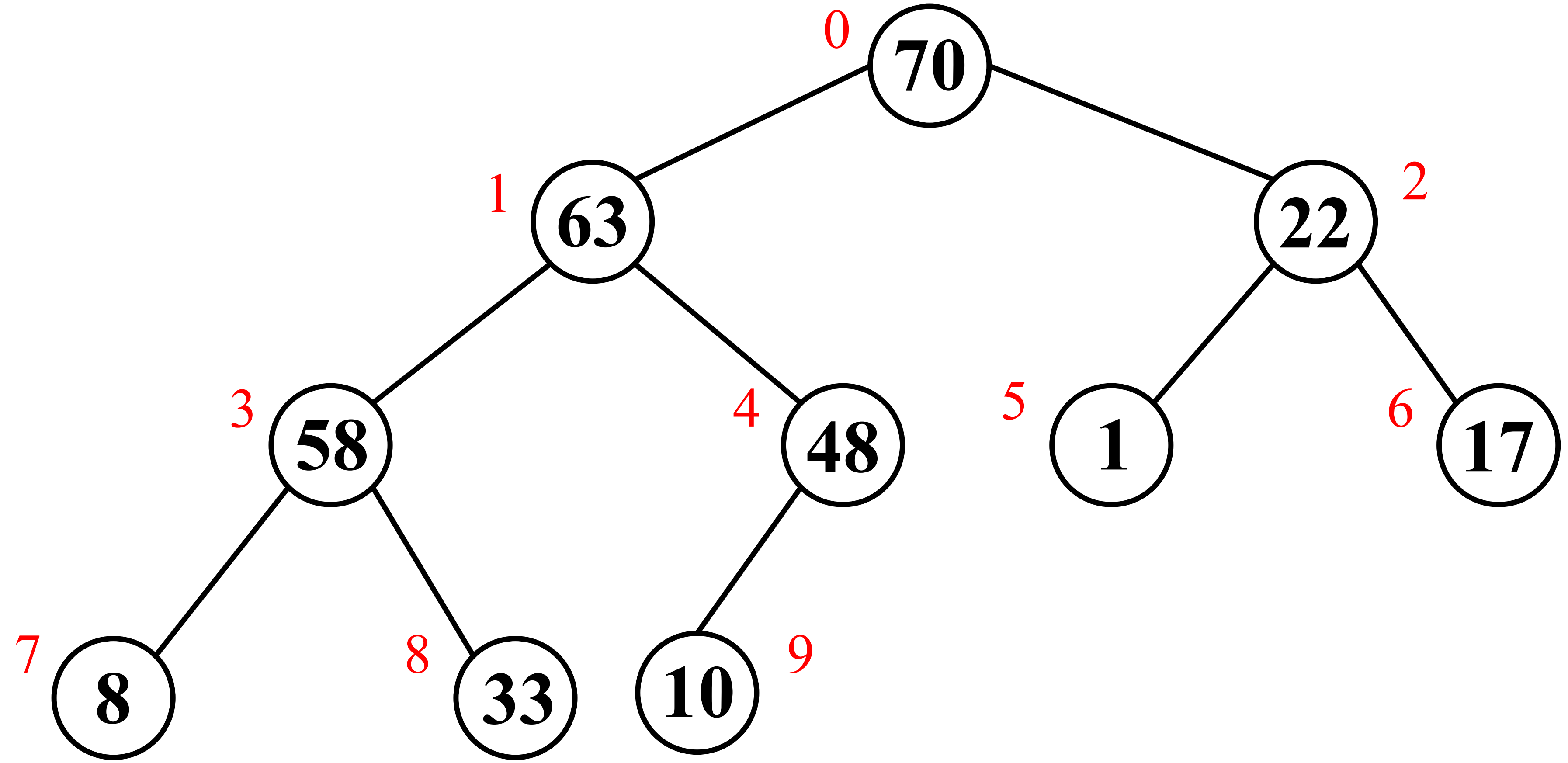
Heapsort algorithms: array implementation



70	63	22	58	48	1	17	8	33	10
0	1	2	3	4	5	6	7	8	9

Remove the highest priority element one by one.

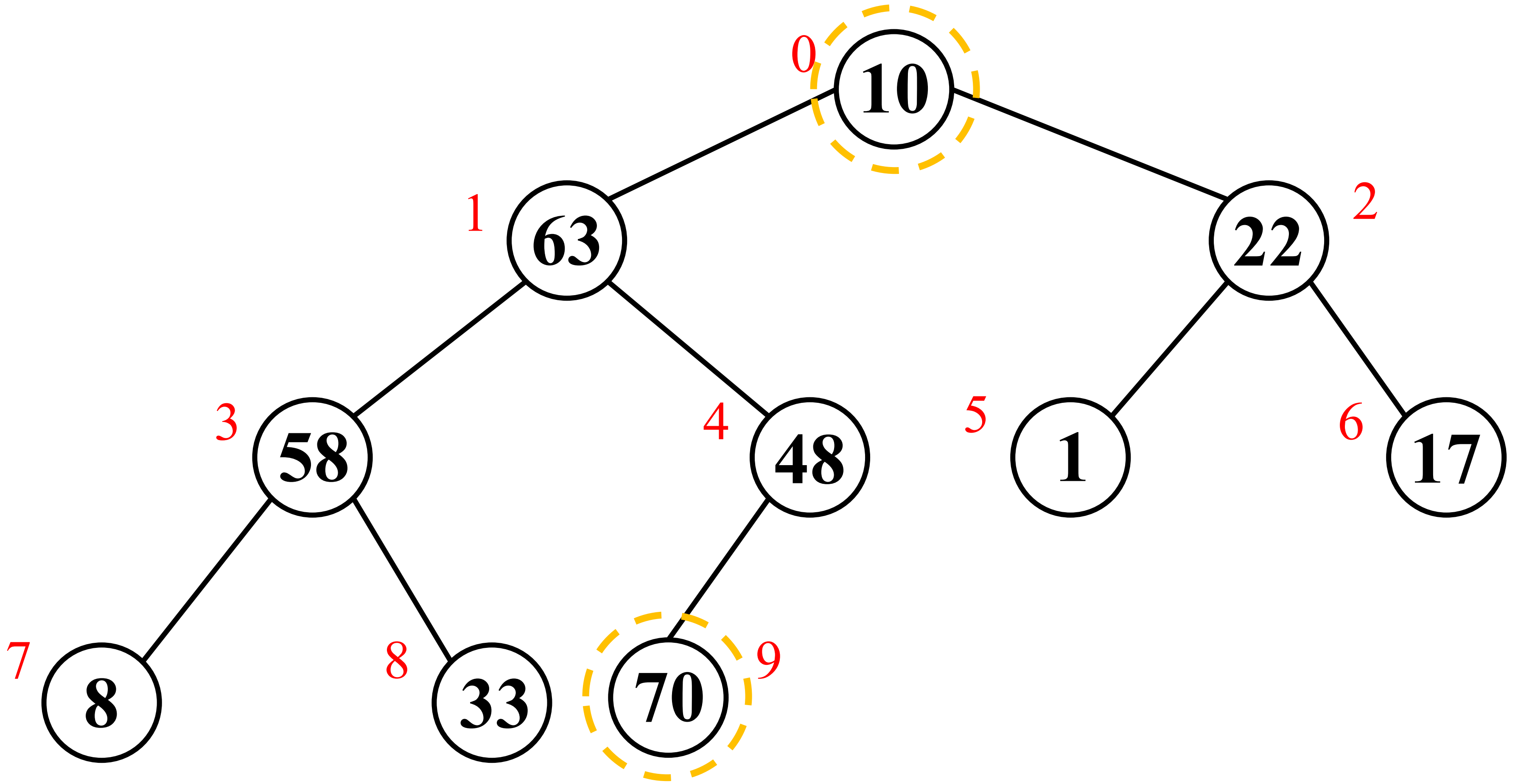
Heapsort algorithms: array implementation



swap

70	**63**	**22**	**58**	**48**	**1**	**17**	**8**	**33**	**10**
0	1	2	3	4	5	6	7	8	9

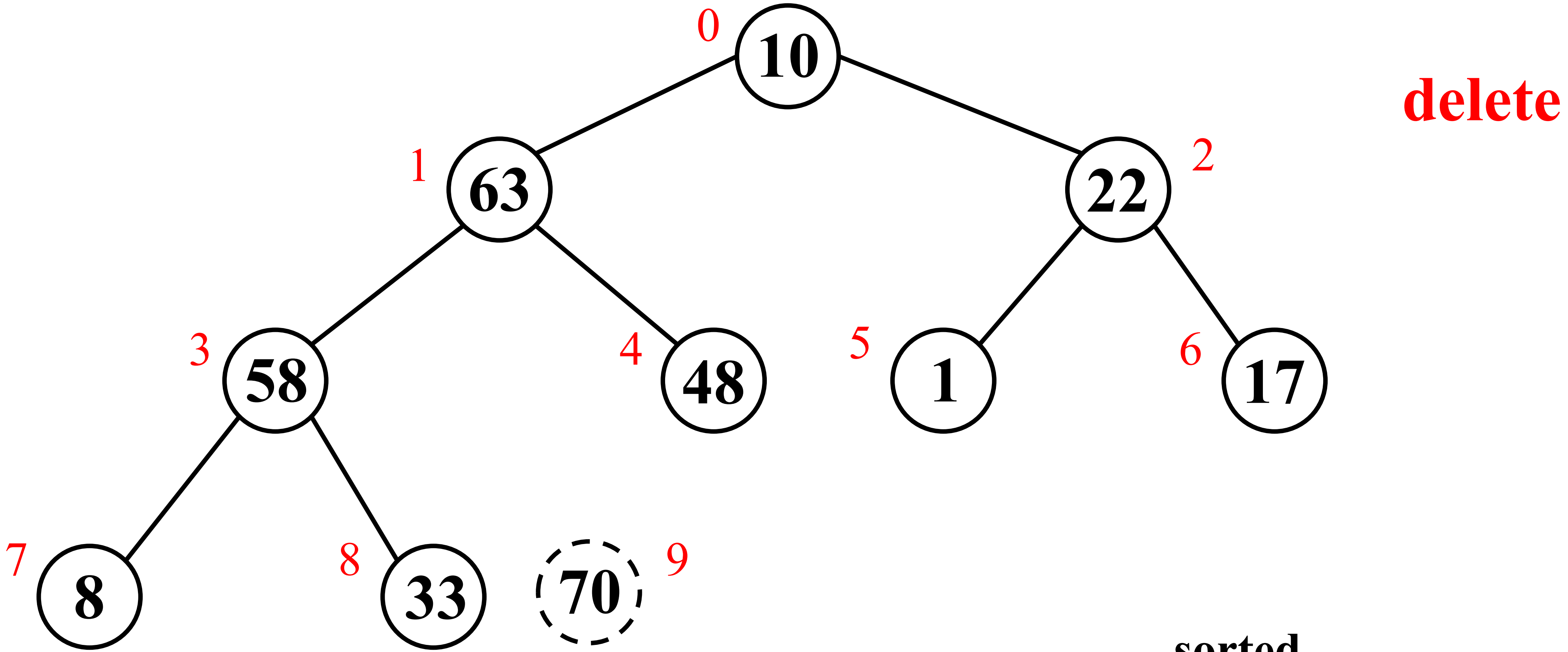
Heapsort algorithms: array implementation



swap

10	63	22	58	48	1	17	8	33	70
0	1	2	3	4	5	6	7	8	9

Heapsort algorithms: array implementation

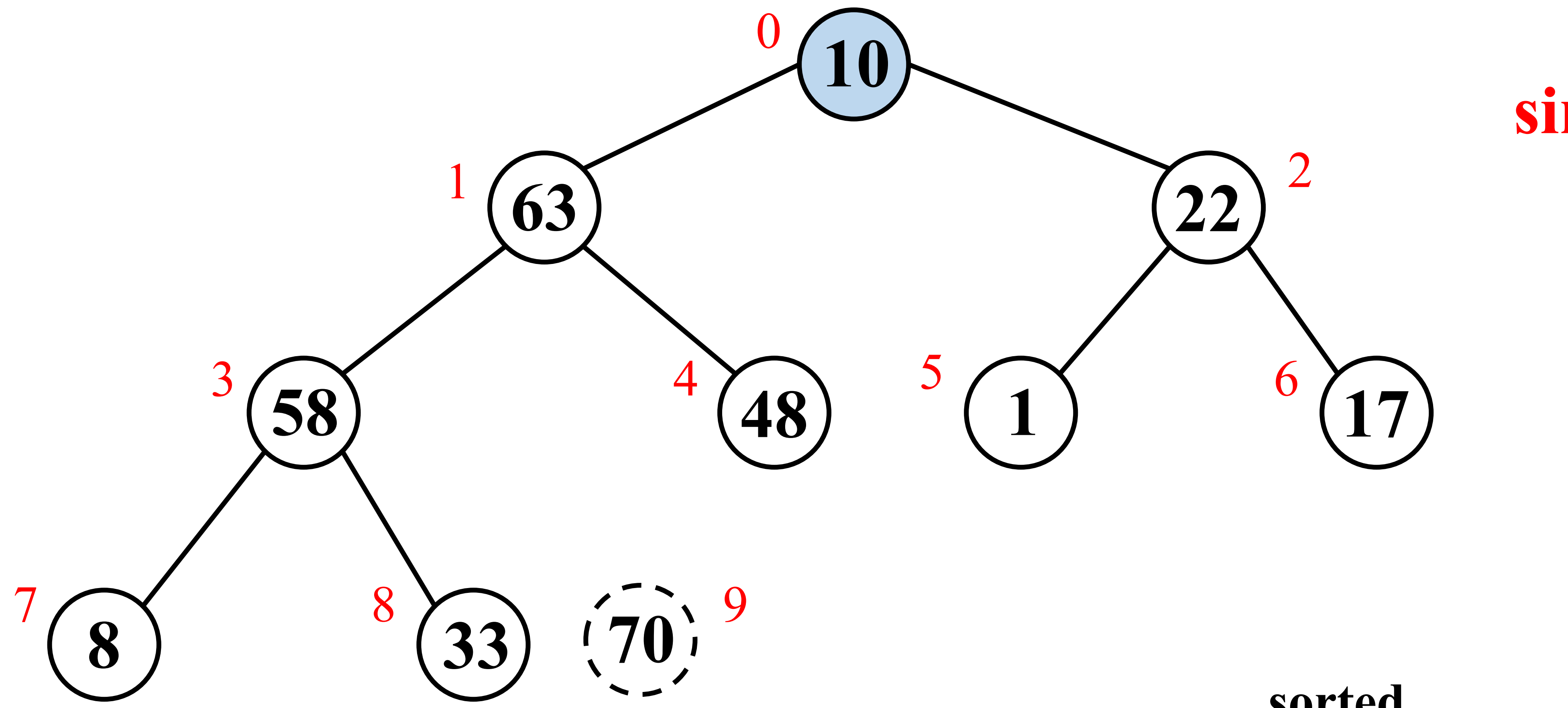


sorted

10	63	22	58	48	1	17	8	33	70
0	1	2	3	4	5	6	7	8	9

Heapsort algorithms: array implementation

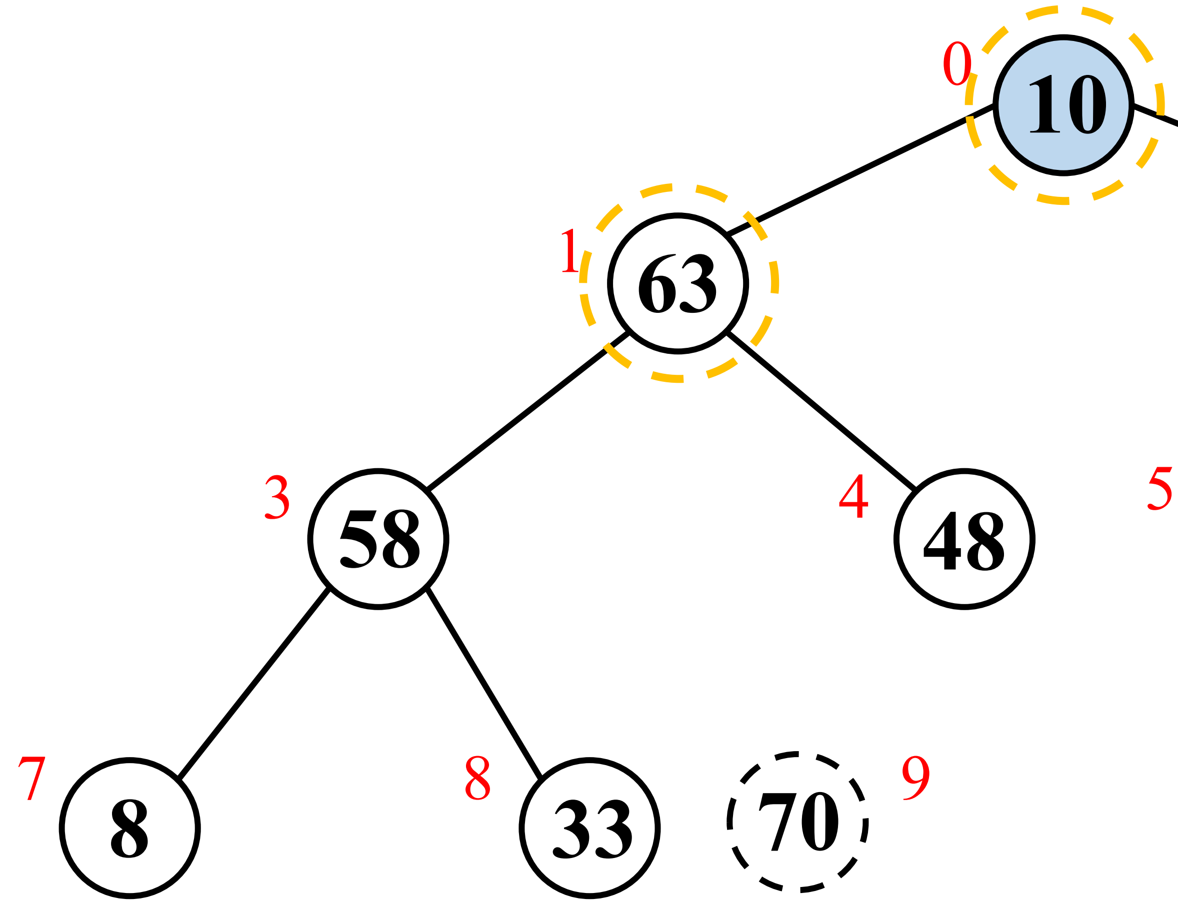
sink 10



sorted

10	63	22	58	48	1	17	8	33	70
0	1	2	3	4	5	6	7	8	9

Heapsort algorithms: array implementation

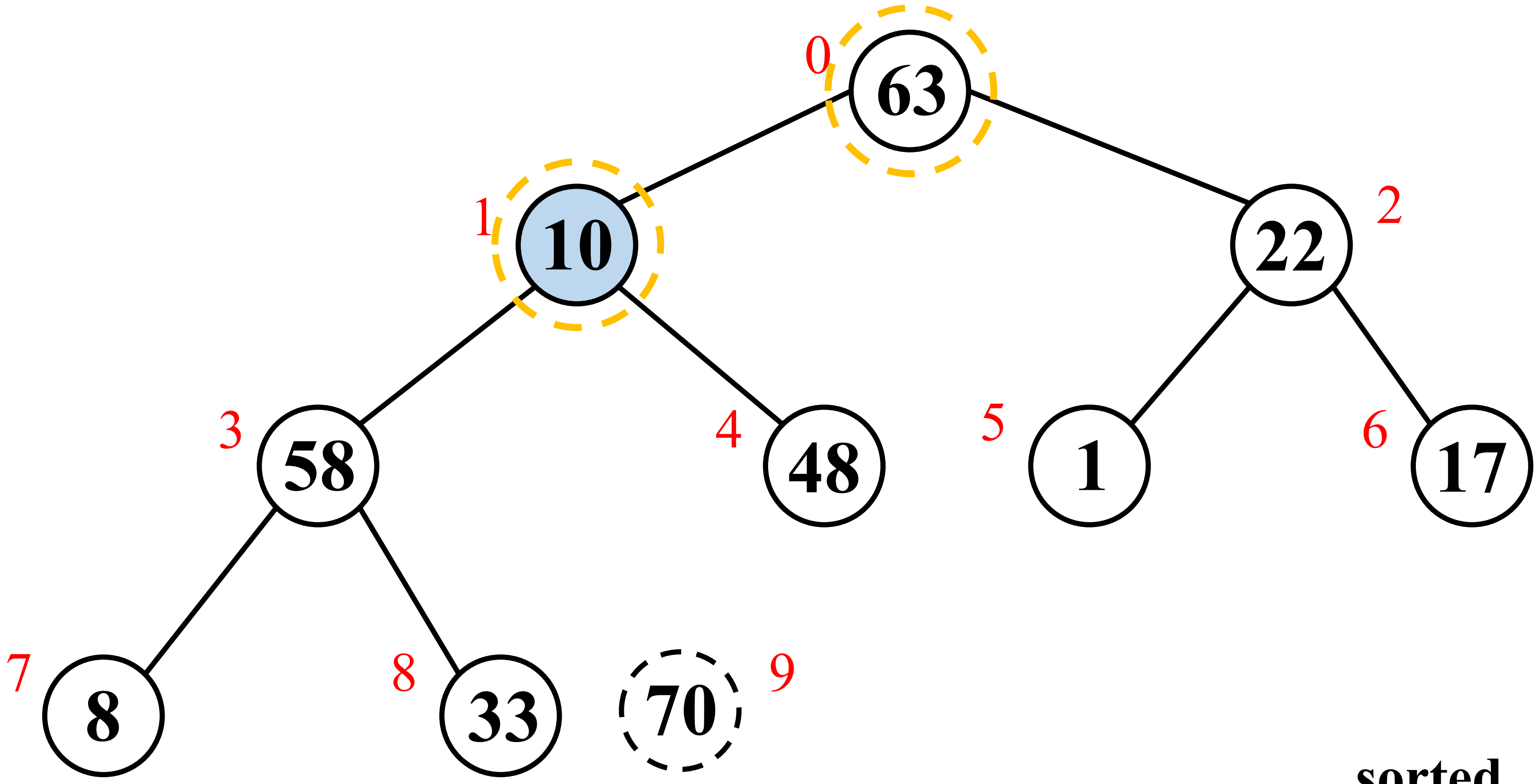


sink 10

sorted

10	63	22	58	48	1	17	8	33	70
0	1	2	3	4	5	6	7	8	9

Heapsort algorithms: array implementation



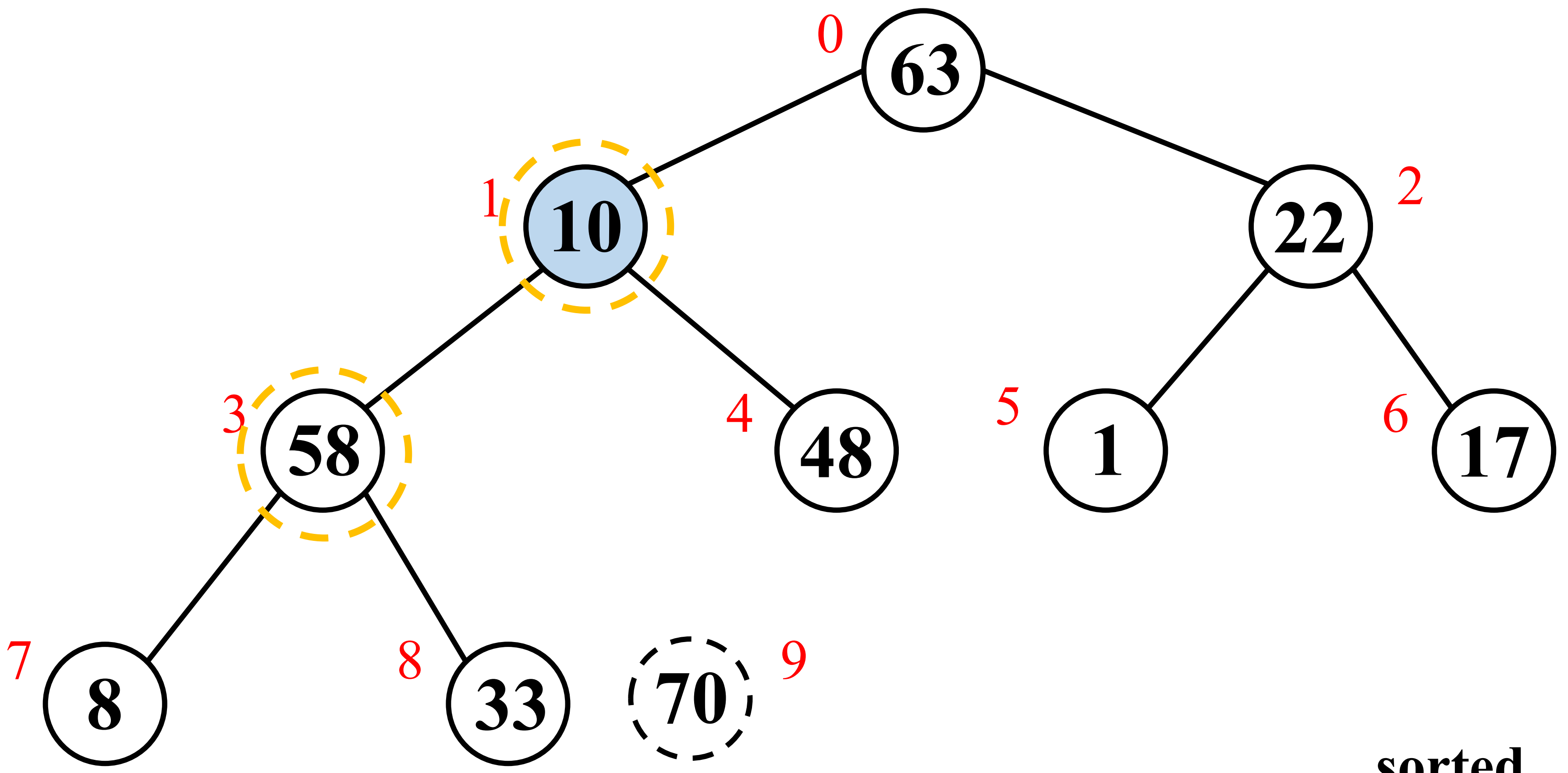
sink 10

sorted

63	**10**	**22**	**58**	**48**	**1**	**17**	**8**	**33**	**70**
0	1	2	3	4	5	6	7	8	9

Heapsort algorithms: array implementation

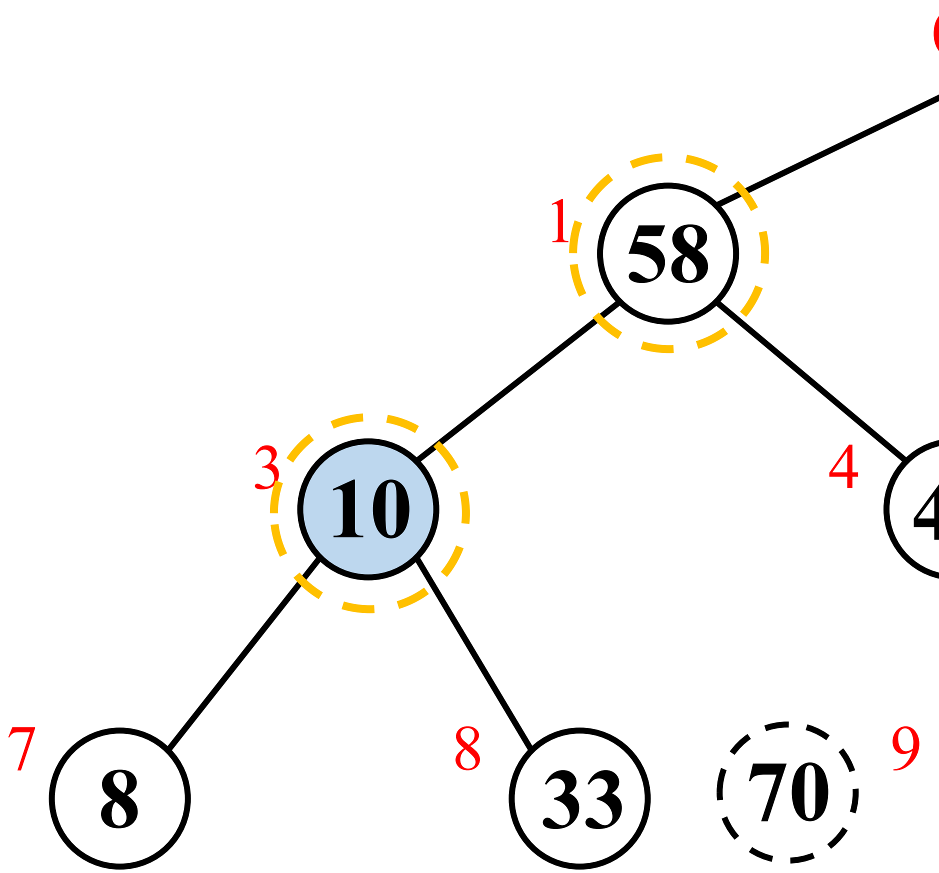
sink 10



sorted

63	10	22	58	48	1	17	8	33	70
0	1	2	3	4	5	6	7	8	9

Heapsort algorithms: array implementation



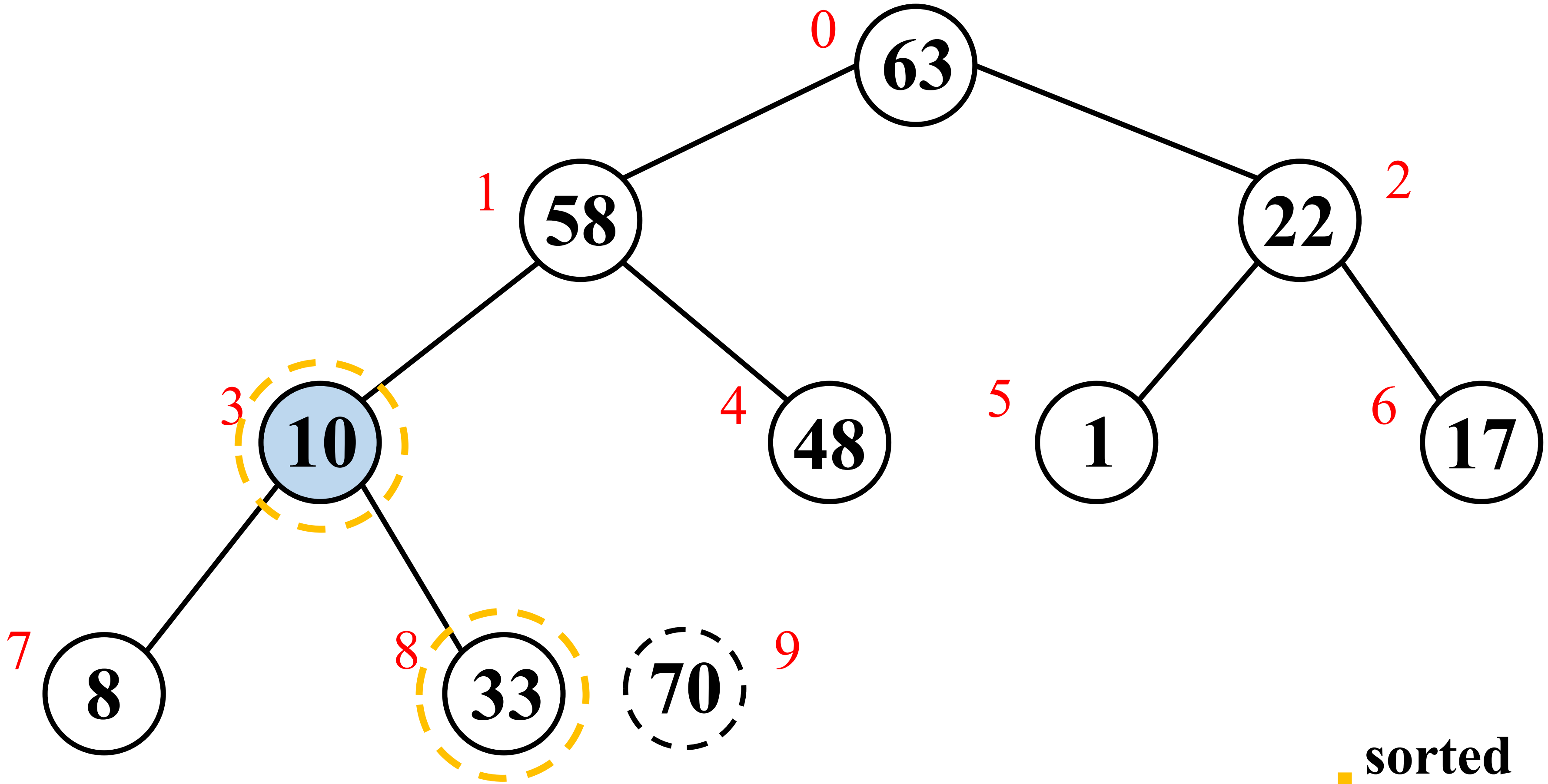
sink 10

sorted

63	**58**	**22**	**10**	**48**	**1**	**17**	**8**	**33**	**70**
0	1	2	3	4	5	6	7	8	9

Heapsort algorithms: array implementation

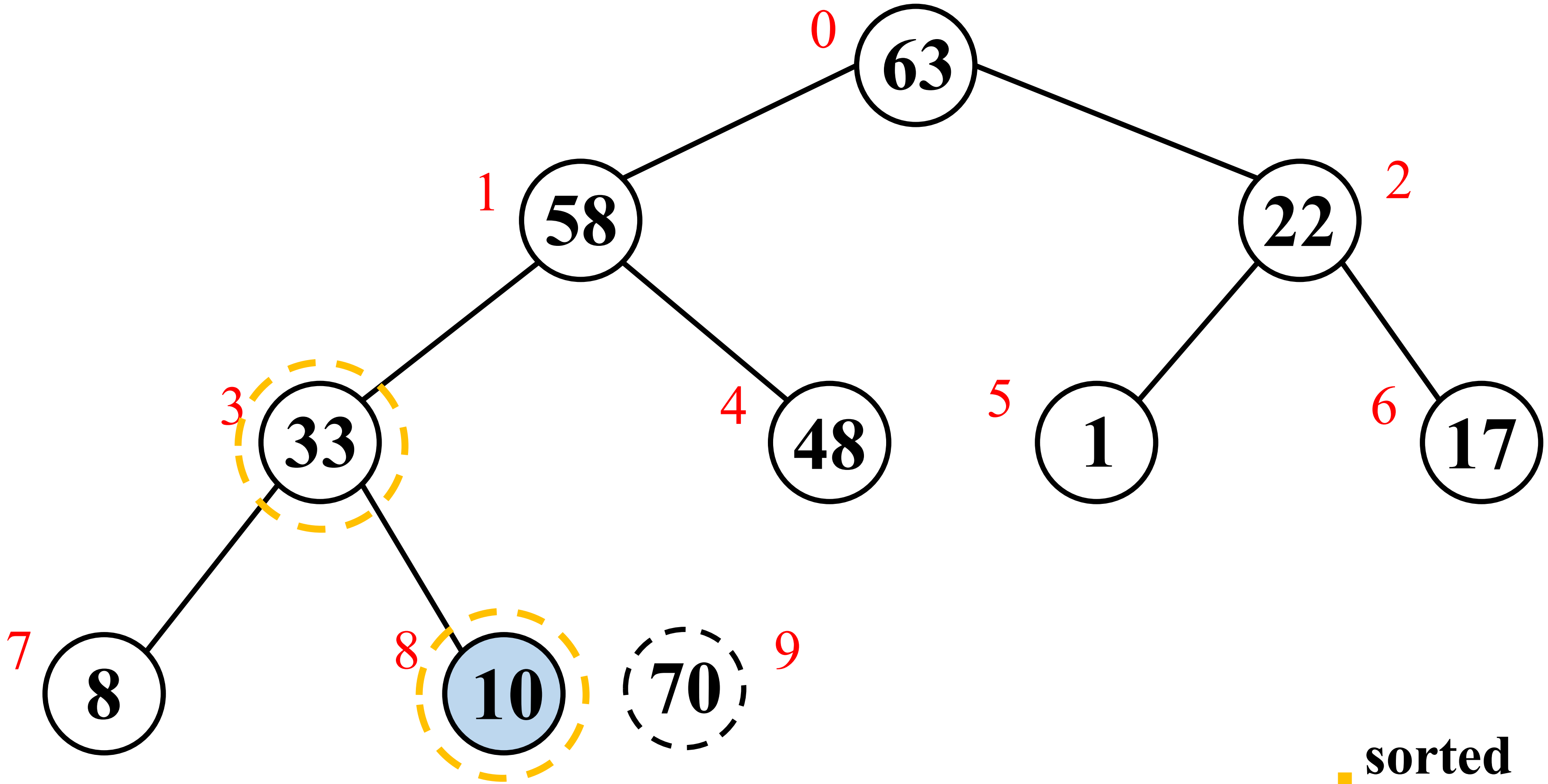
sink 10



sorted

63	58	22	10	48	1	17	8	33	70
0	1	2	3	4	5	6	7	8	9

Heapsort algorithms: array implementation



sink 10

sorted

63	**58**	**22**	**33**	**48**	**1**	**17**	**8**	**10**	**70**
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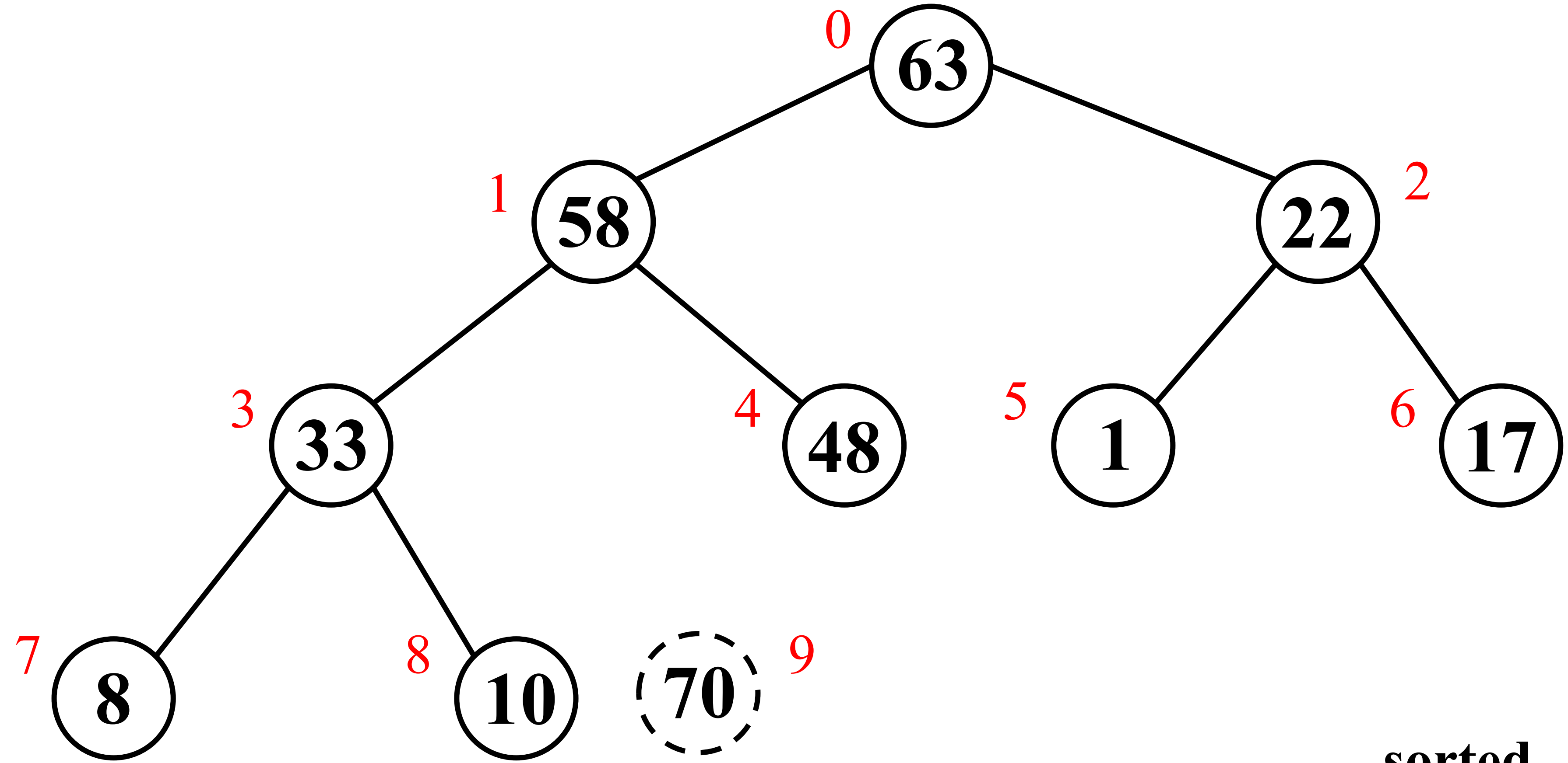
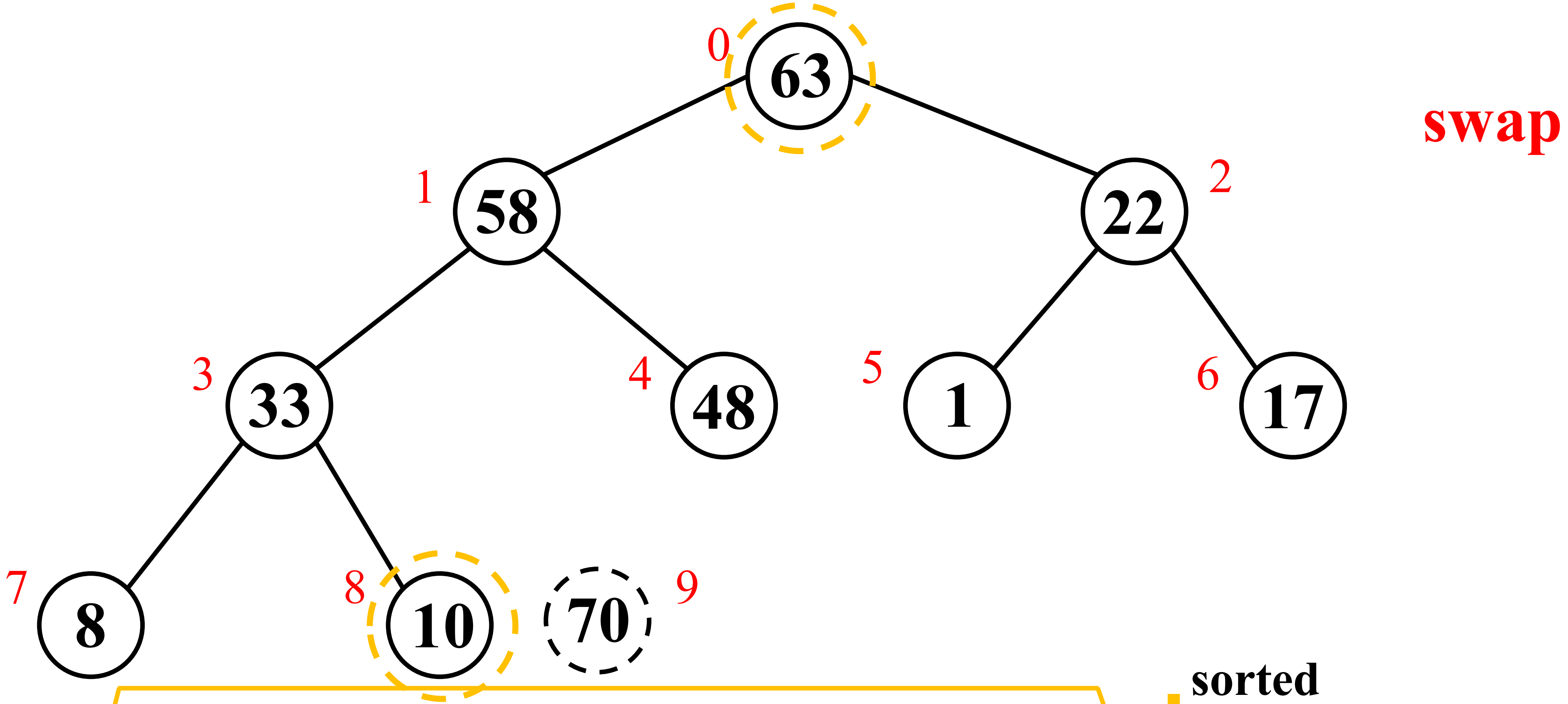
Heapsort algorithms: array implementation



sorted

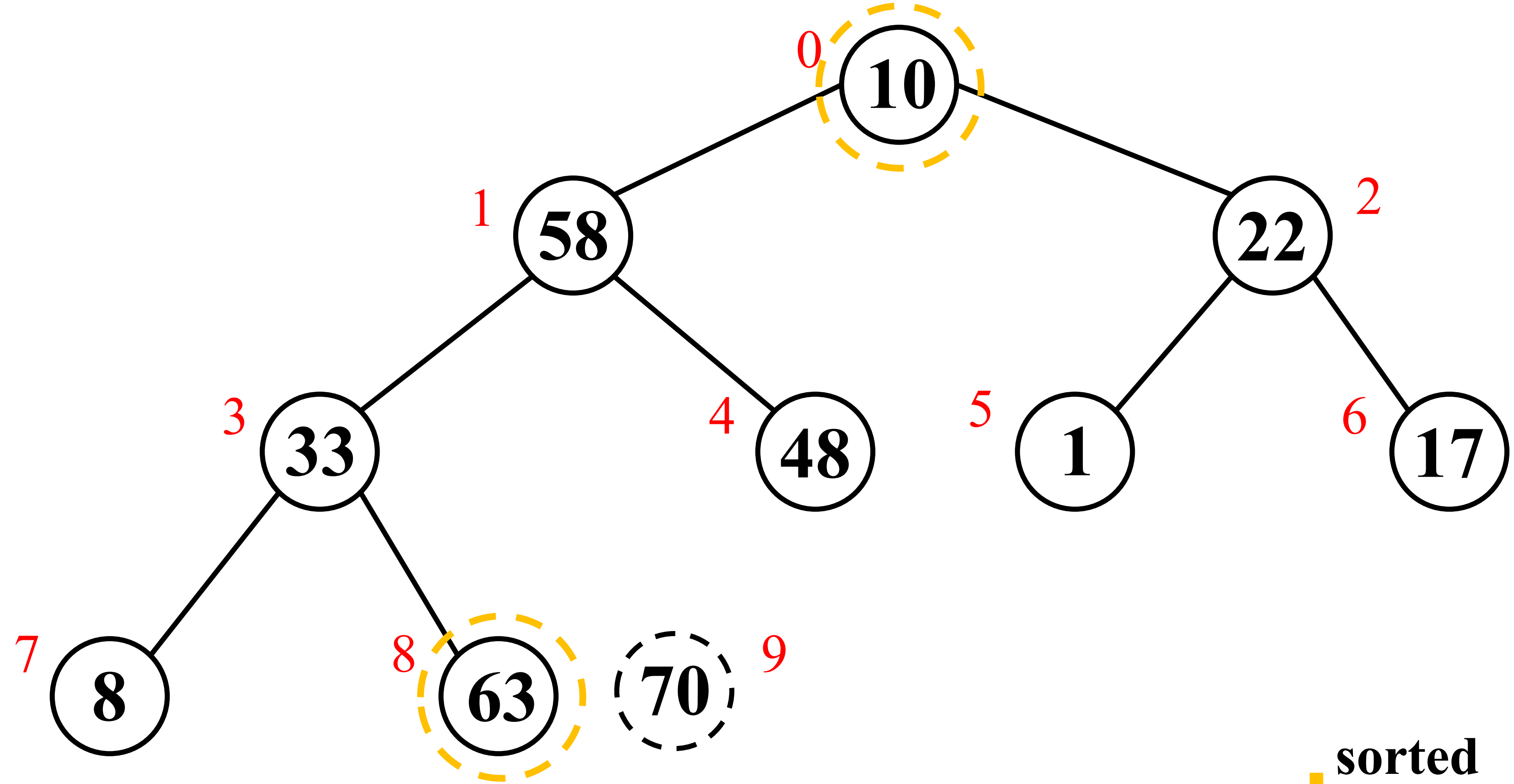
63	**58**	**22**	**33**	**48**	**1**	**17**	**8**	**10**	**70**
0	1	2	3	4	5	6	7	8	9

Heapsort algorithms: array implementation



63	58	22	33	48	1	17	8	10	70
0	1	2	3	4	5	6	7	8	9

Heapsort algorithms: array implementation

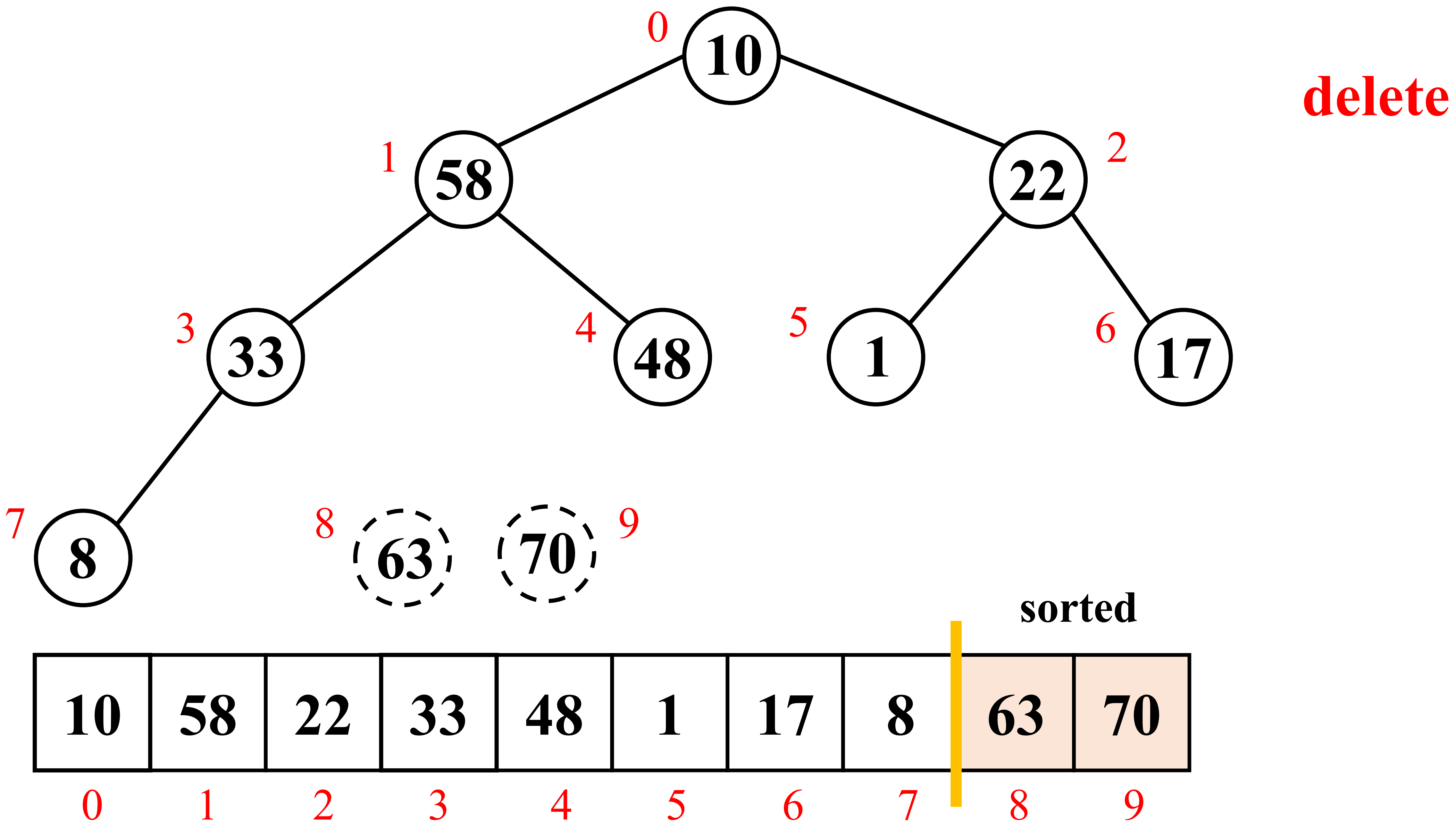


swap

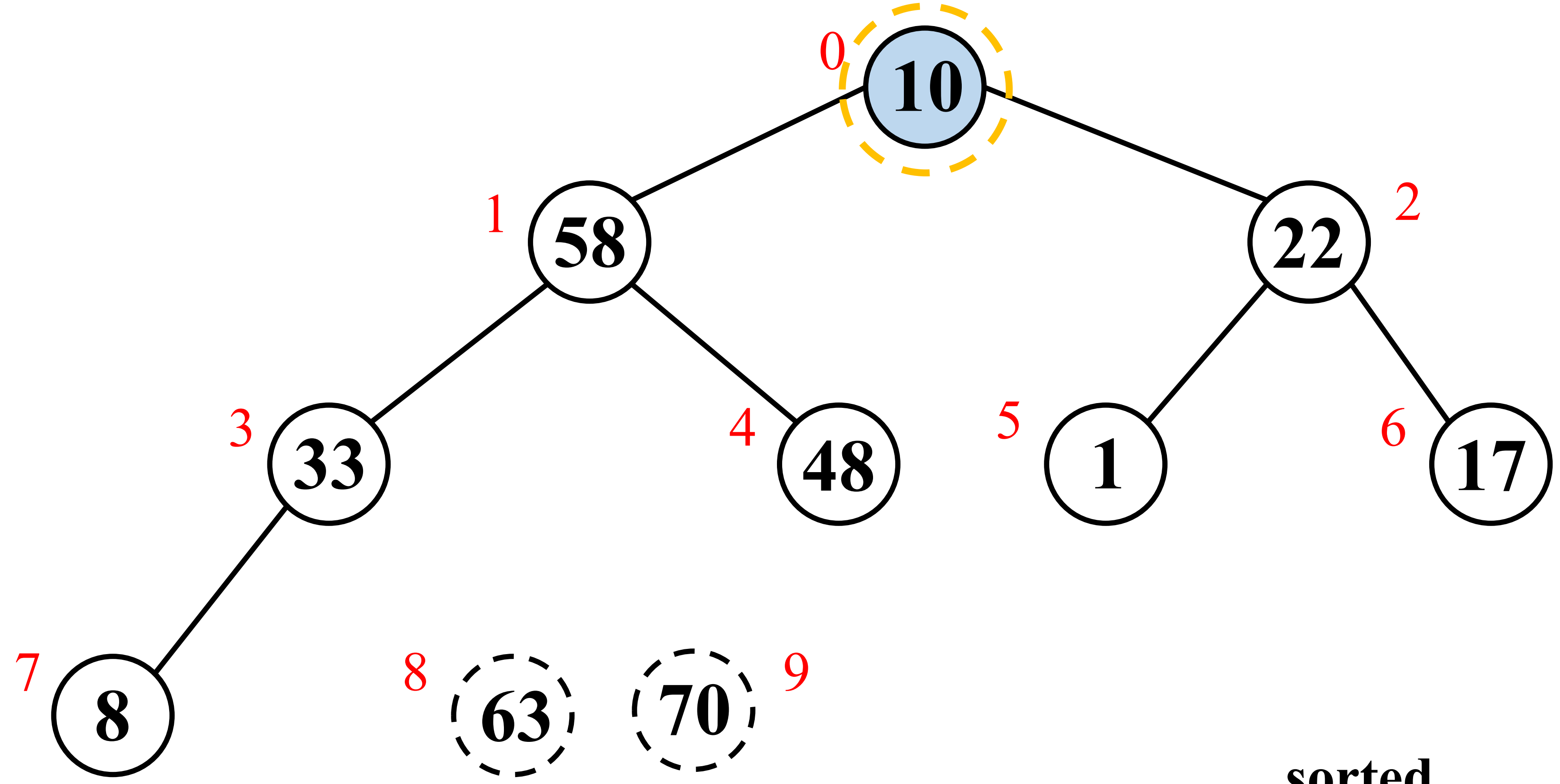
sorted

10	**58**	**22**	**33**	**48**	**1**	**17**	**8**	**63**	**70**
0	1	2	3	4	5	6	7	8	9

Heapsort algorithms: array implementation



Heapsort algorithms: array implementation

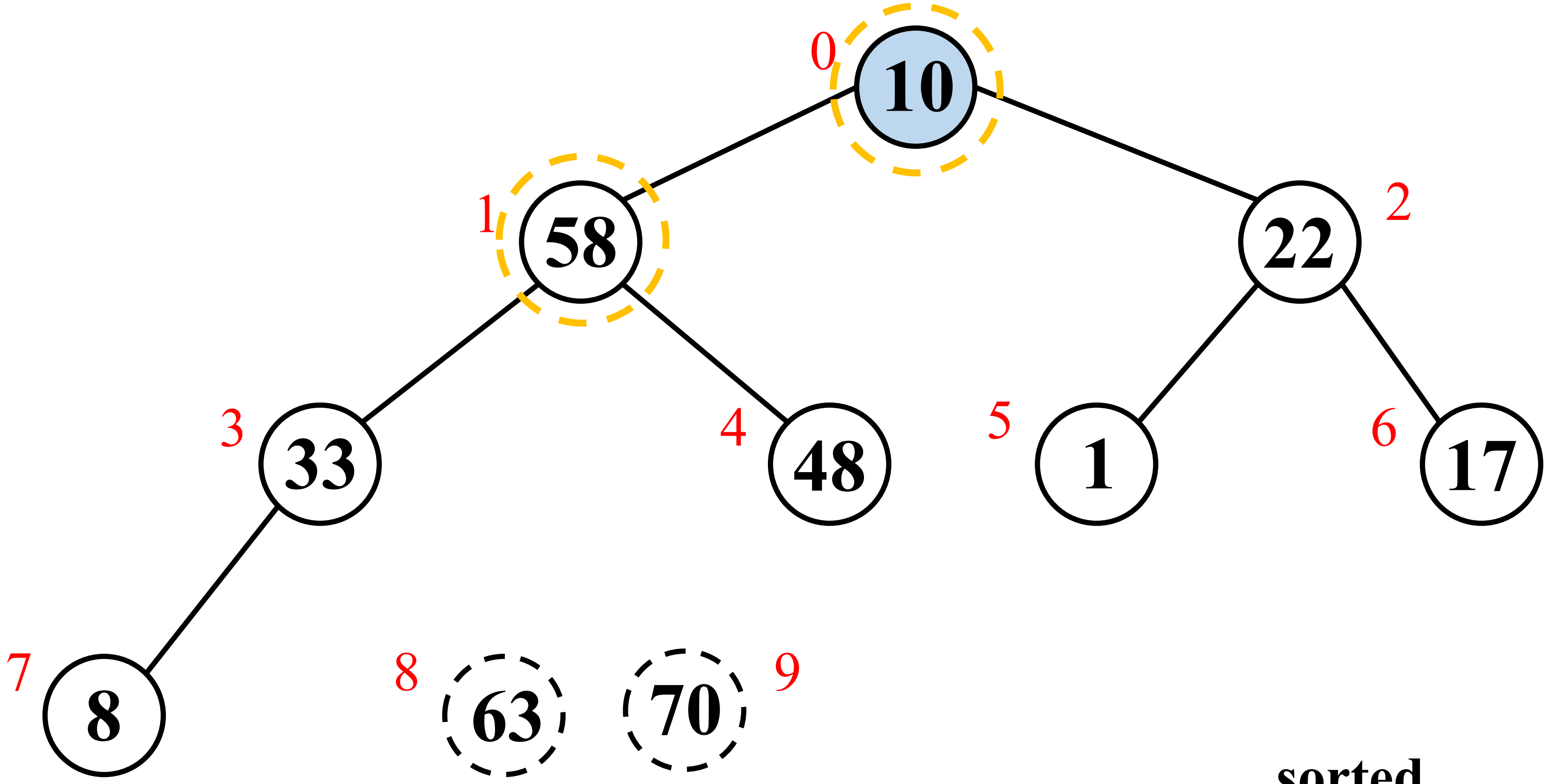


sink 10

sorted

10	58	22	33	48	1	17	8	63	70
0	1	2	3	4	5	6	7	8	9

Heapsort algorithms: array implementation

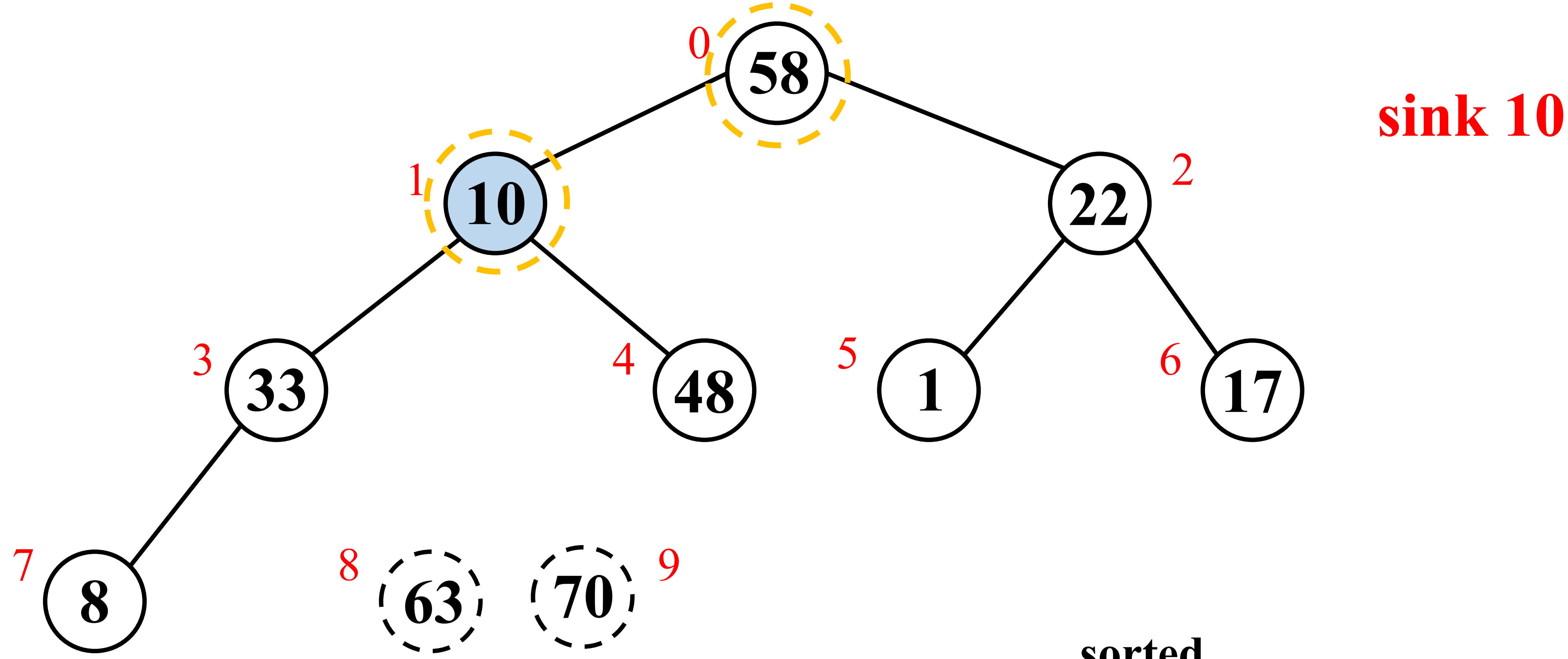


sink 10

sorted

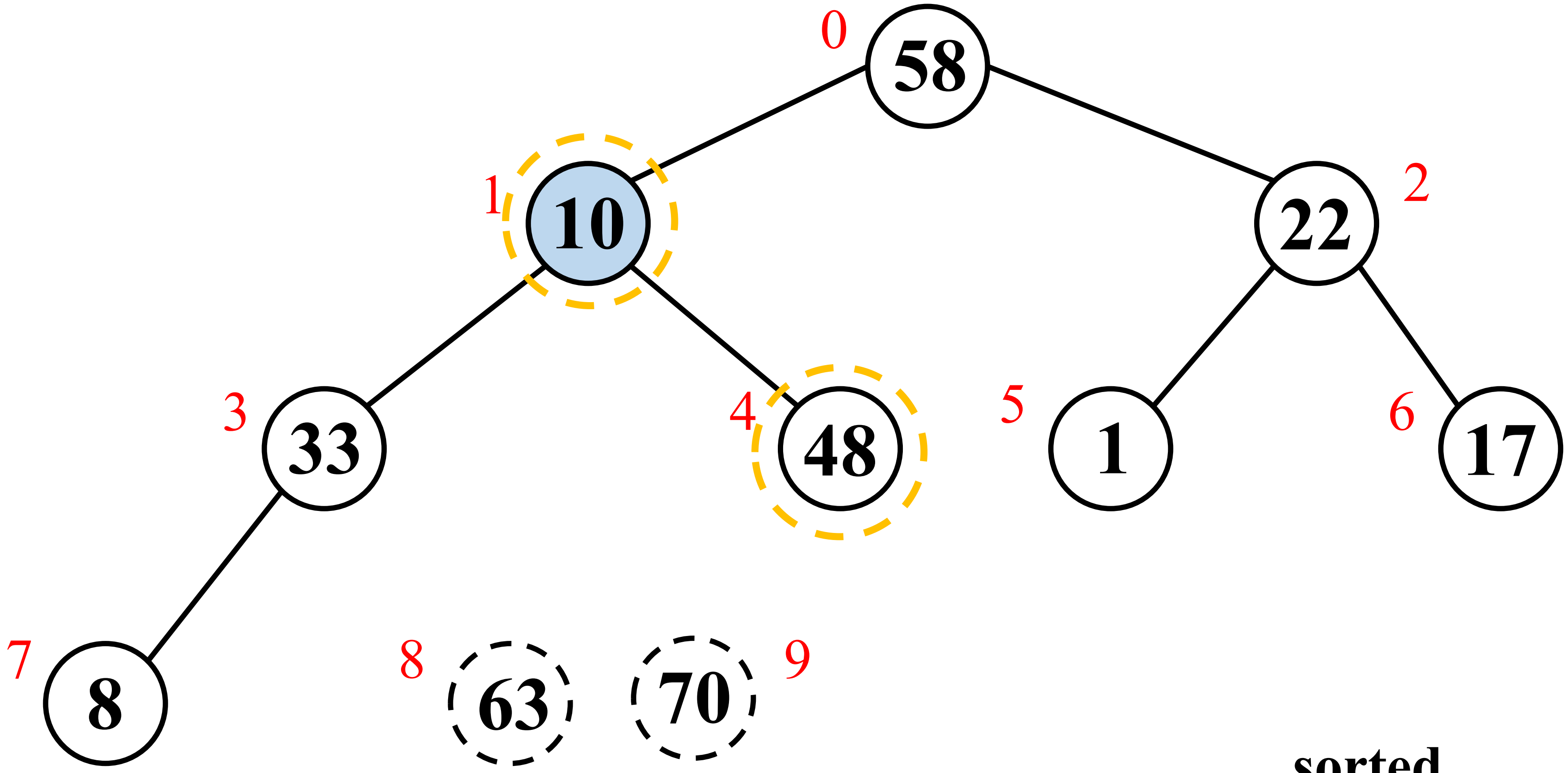
10	58	22	33	48	1	17	8	63	70
0	1	2	3	4	5	6	7	8	9

Heapsort algorithms: array implementation



58	10	22	33	48	1	17	8	63	70
0	1	2	3	4	5	6	7	8	9

Heapsort algorithms: array implementation

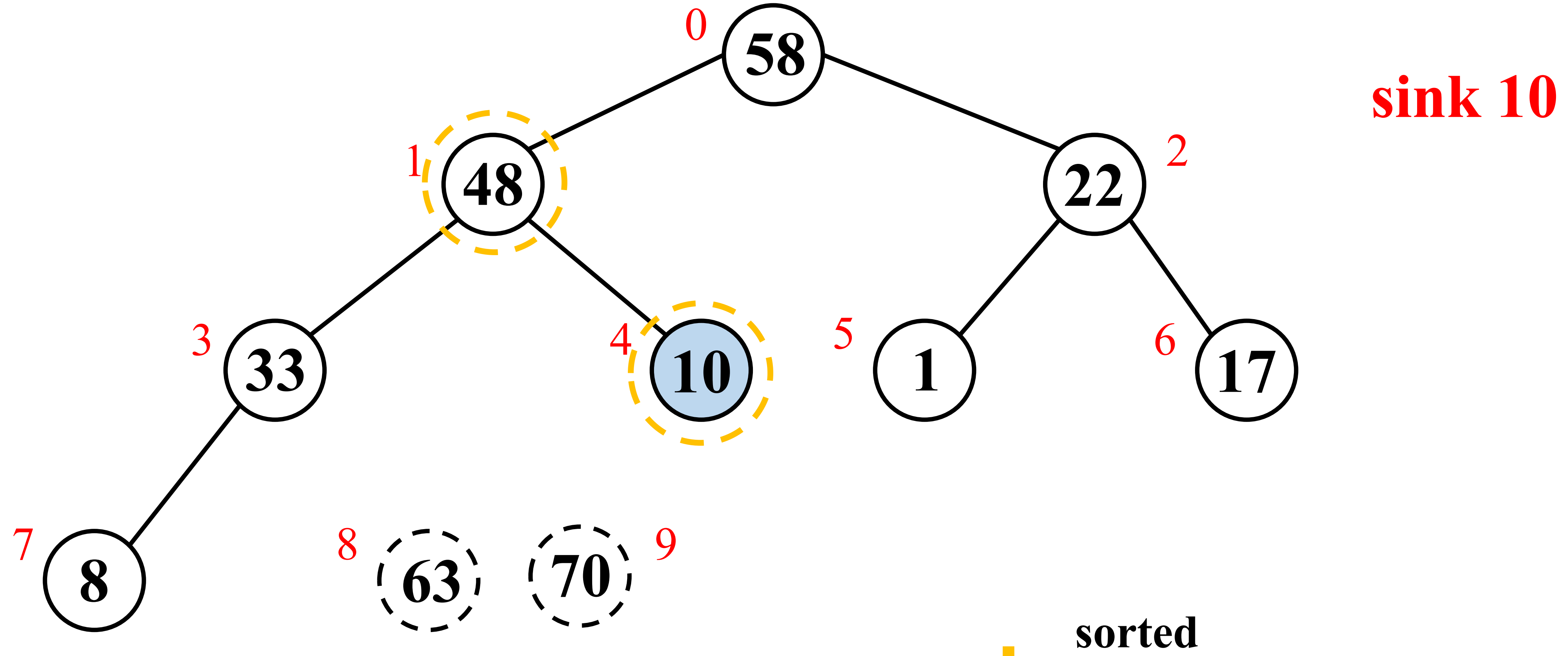


sink 10

sorted

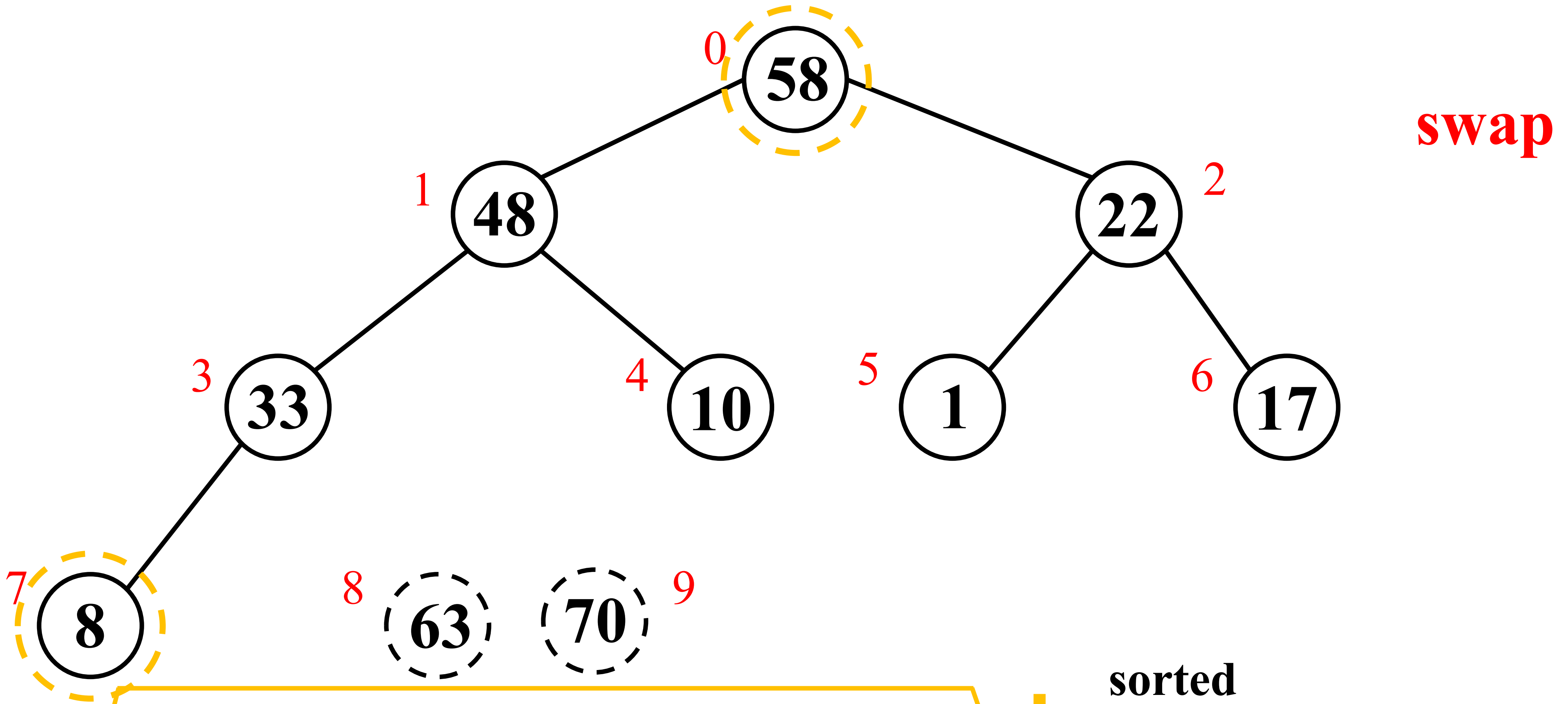
58	10	22	33	48	1	17	8	63	70
0	1	2	3	4	5	6	7	8	9

Heapsort algorithms: array implementation



0	1	2	3	4	5	6	7	8	9
58	48	22	33	10	1	17	8	63	70

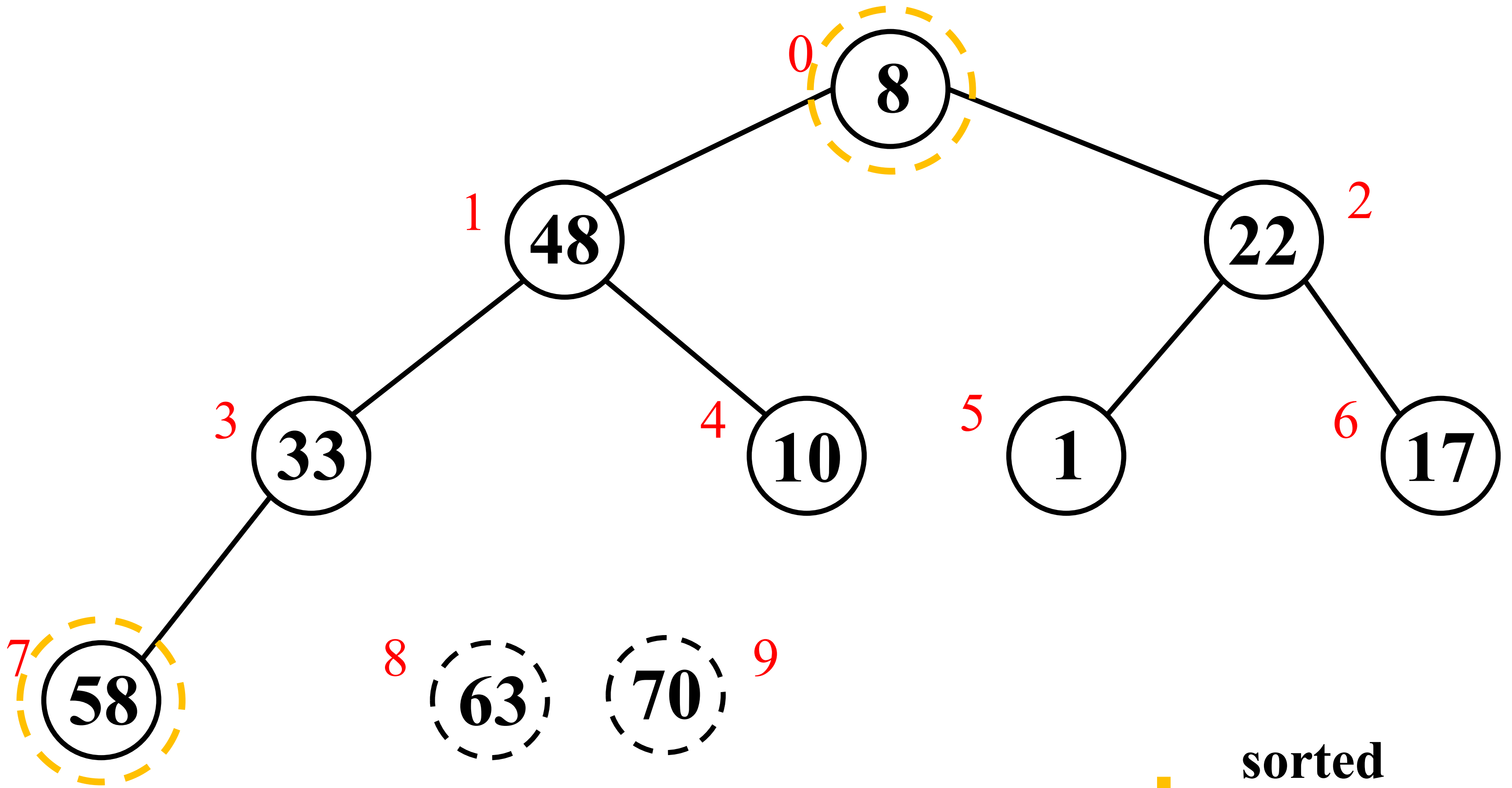
Heapsort algorithms: array implementation



sorted

58	48	22	33	10	1	17	8	63	70
0	1	2	3	4	5	6	7	8	9

Heapsort algorithms: array implementation

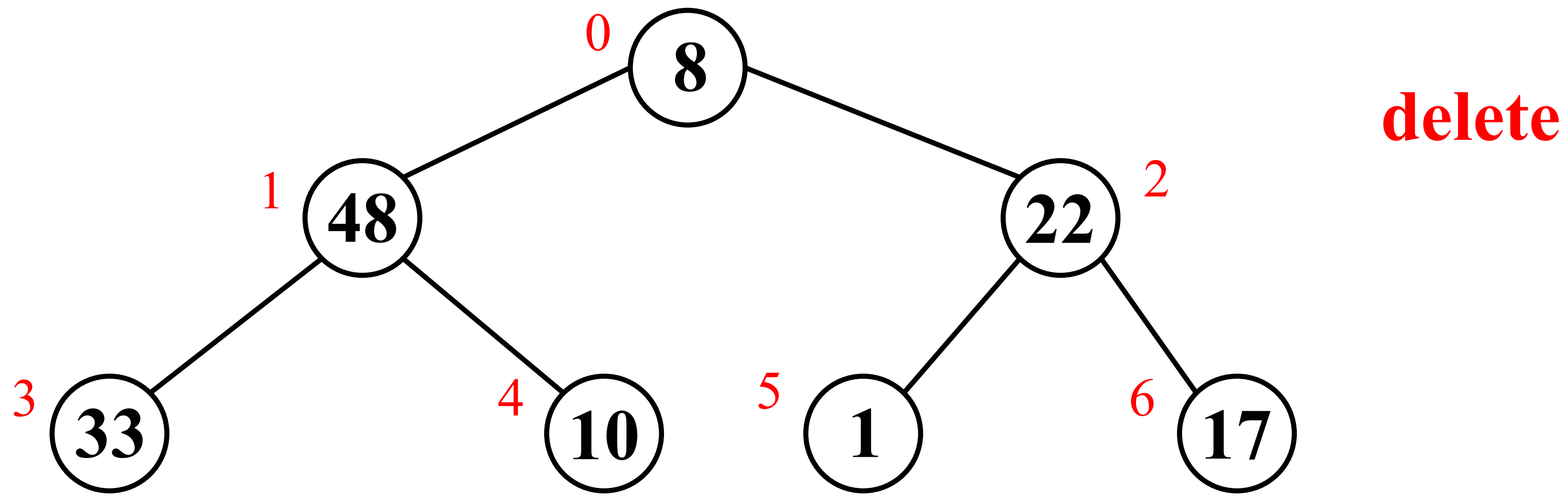


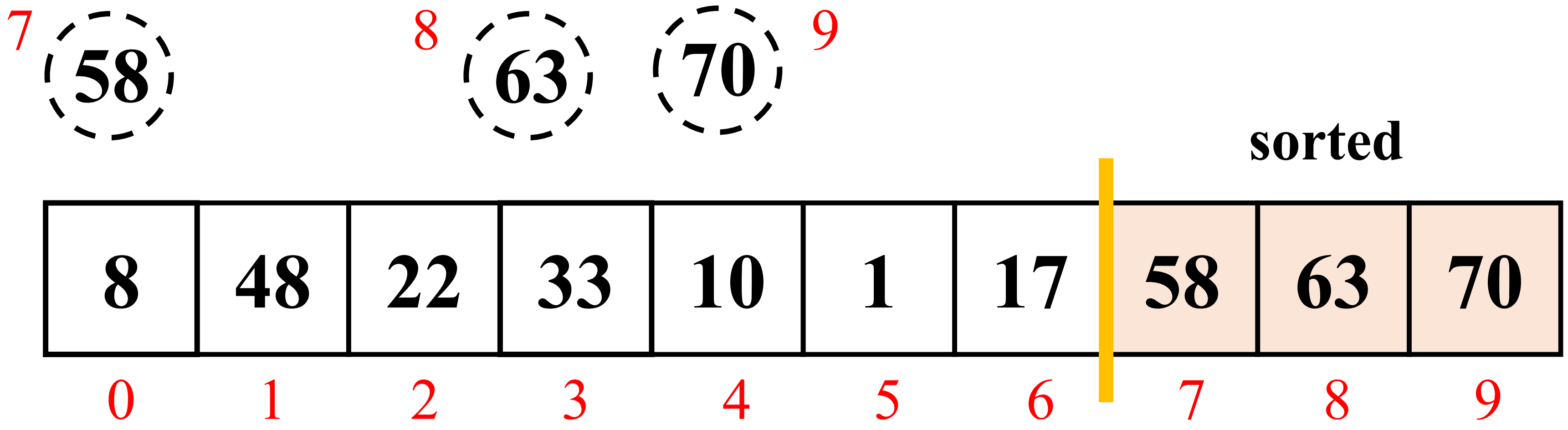
swap

sorted

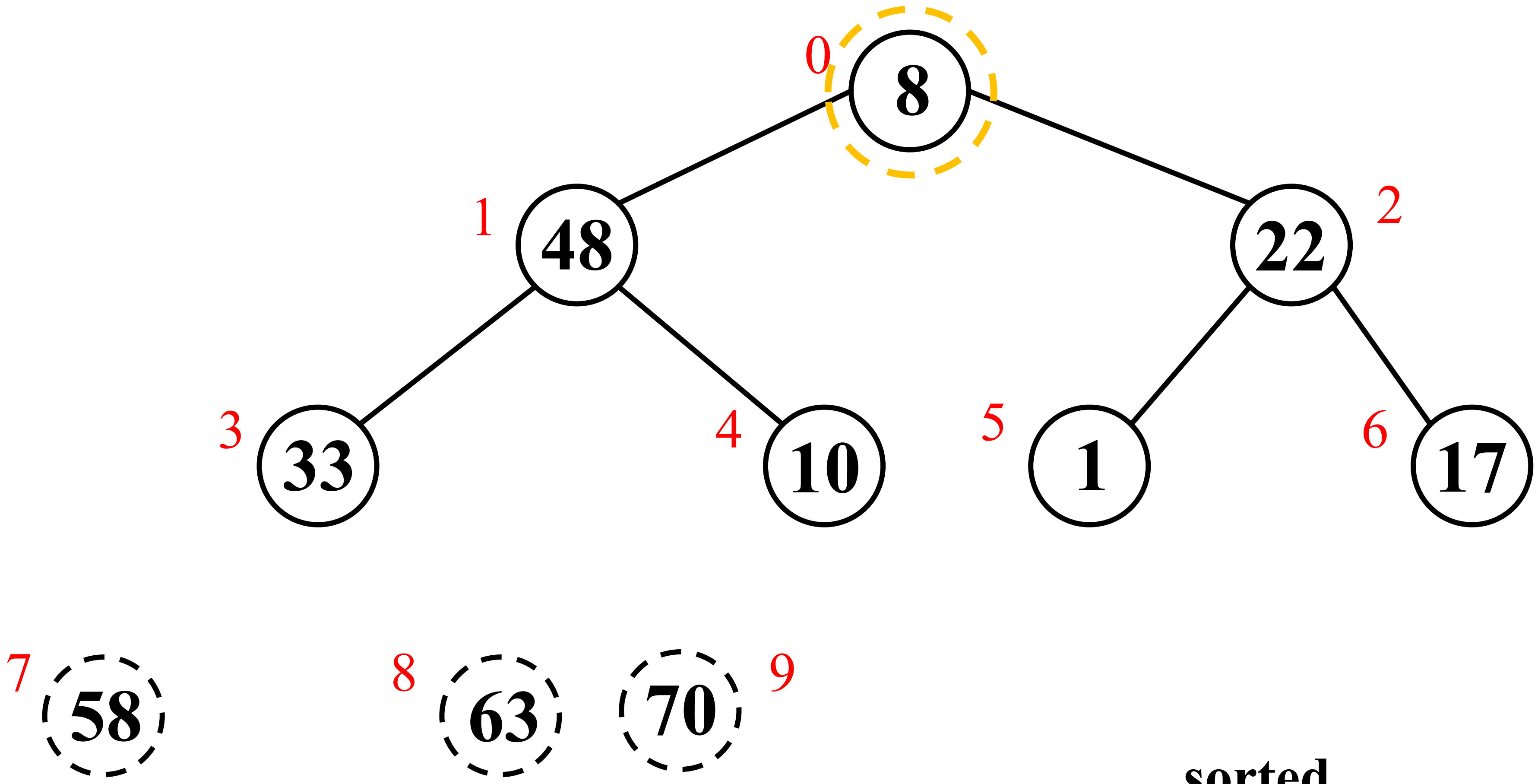
8	48	22	33	10	1	17	58	63	70
0	1	2	3	4	5	6	7	8	9

Heapsort algorithms: array implementation





Heapsort algorithms: array implementation

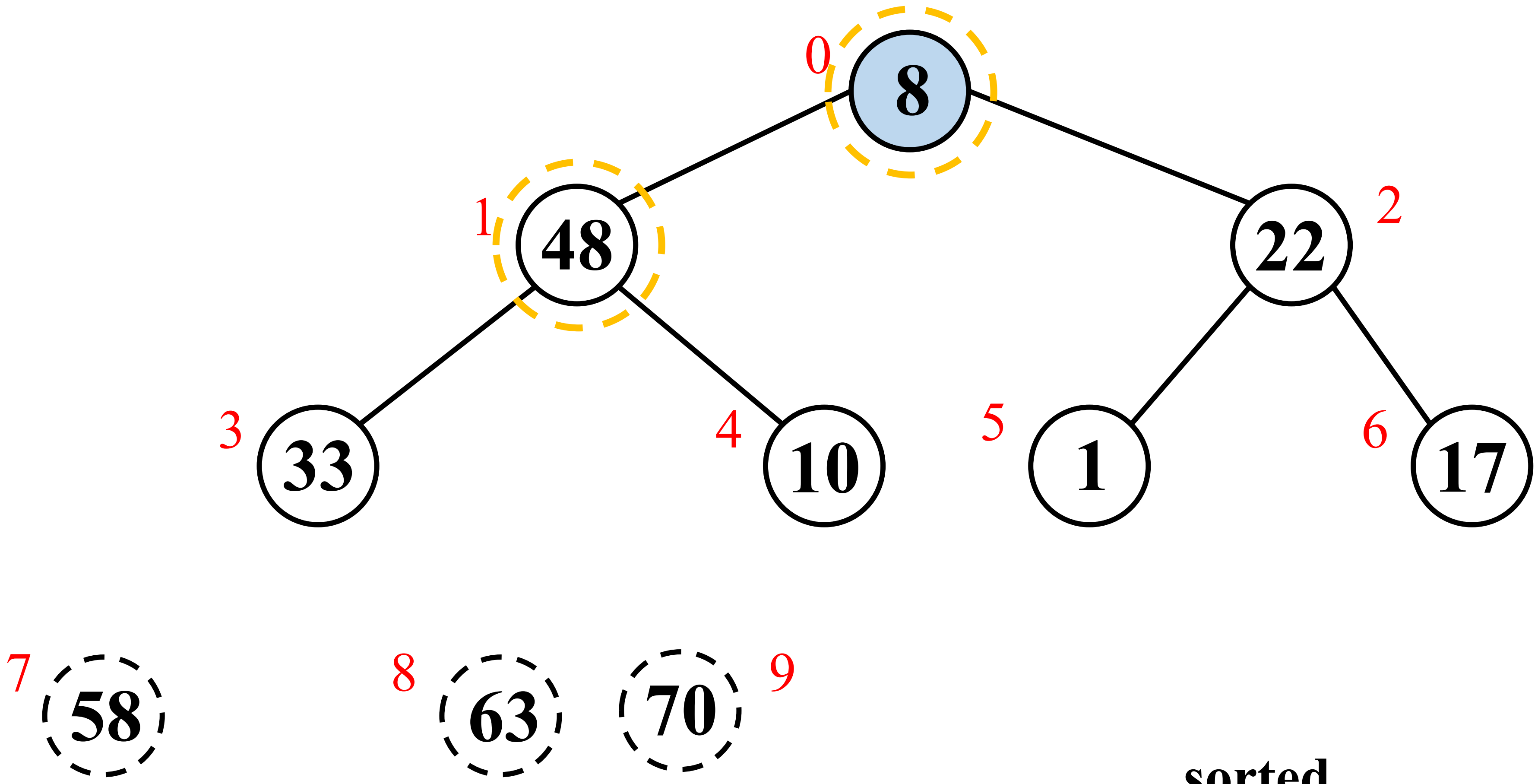


sink 8

sorted

8	**48**	**22**	**33**	**10**	**1**	**17**	**58**	**63**	**70**
0	1	2	3	4	5	6	7	8	9

Heapsort algorithms: array implementation

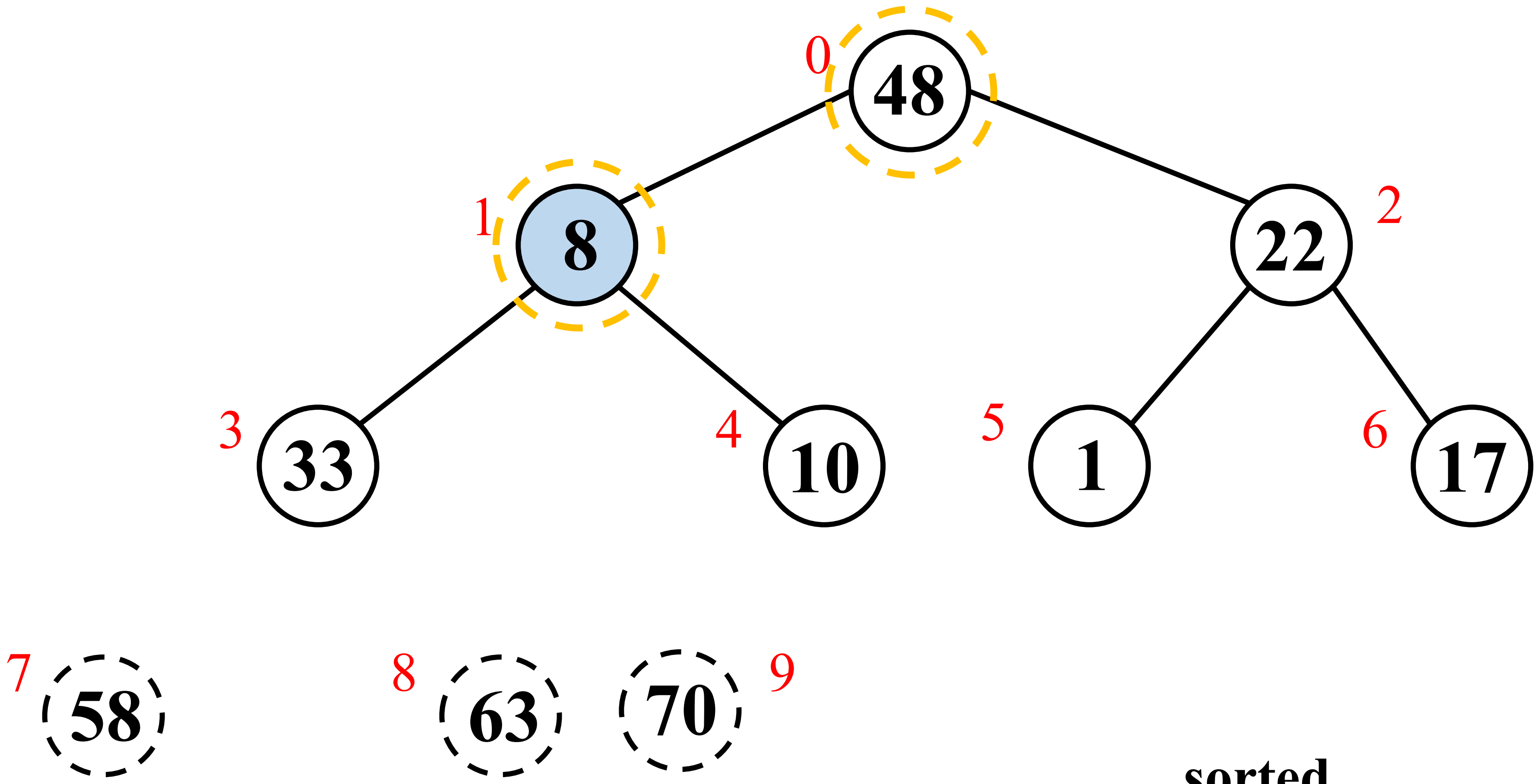


sink 8

sorted

8	**48**	**22**	**33**	**10**	**1**	**17**	**58**	**63**	**70**
0	1	2	3	4	5	6	7	8	9

Heapsort algorithms: array implementation



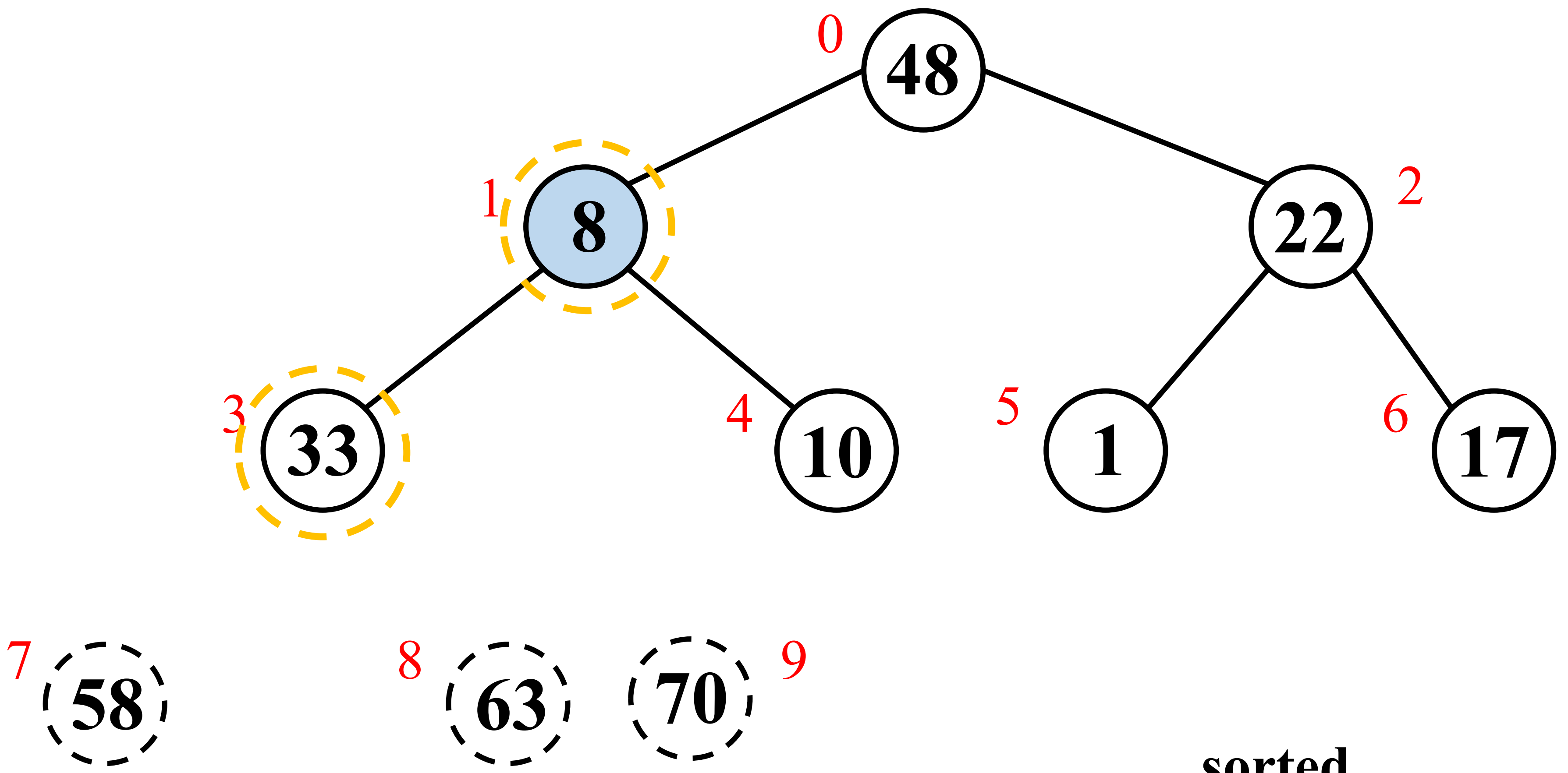
sink 8

sorted

48	**8**	**22**	**33**	**10**	**1**	**17**	**58**	**63**	**70**
0	1	2	3	4	5	6	7	8	9

Heapsort algorithms: array implementation

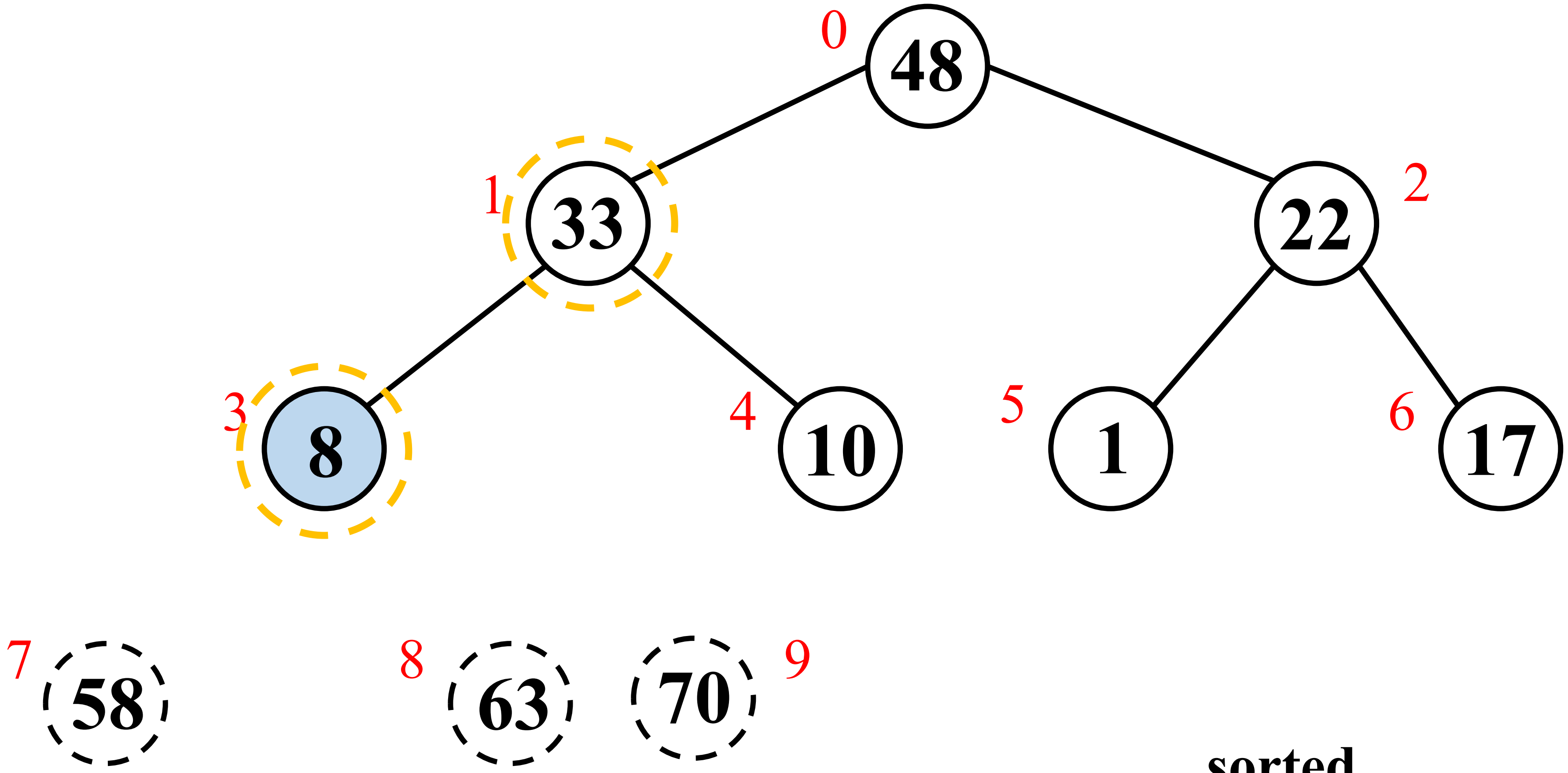
sink 8



sorted

48	**8**	**22**	**33**	**10**	**1**	**17**	**58**	**63**	**70**
0	1	2	3	4	5	6	7	8	9

Heapsort algorithms: array implementation

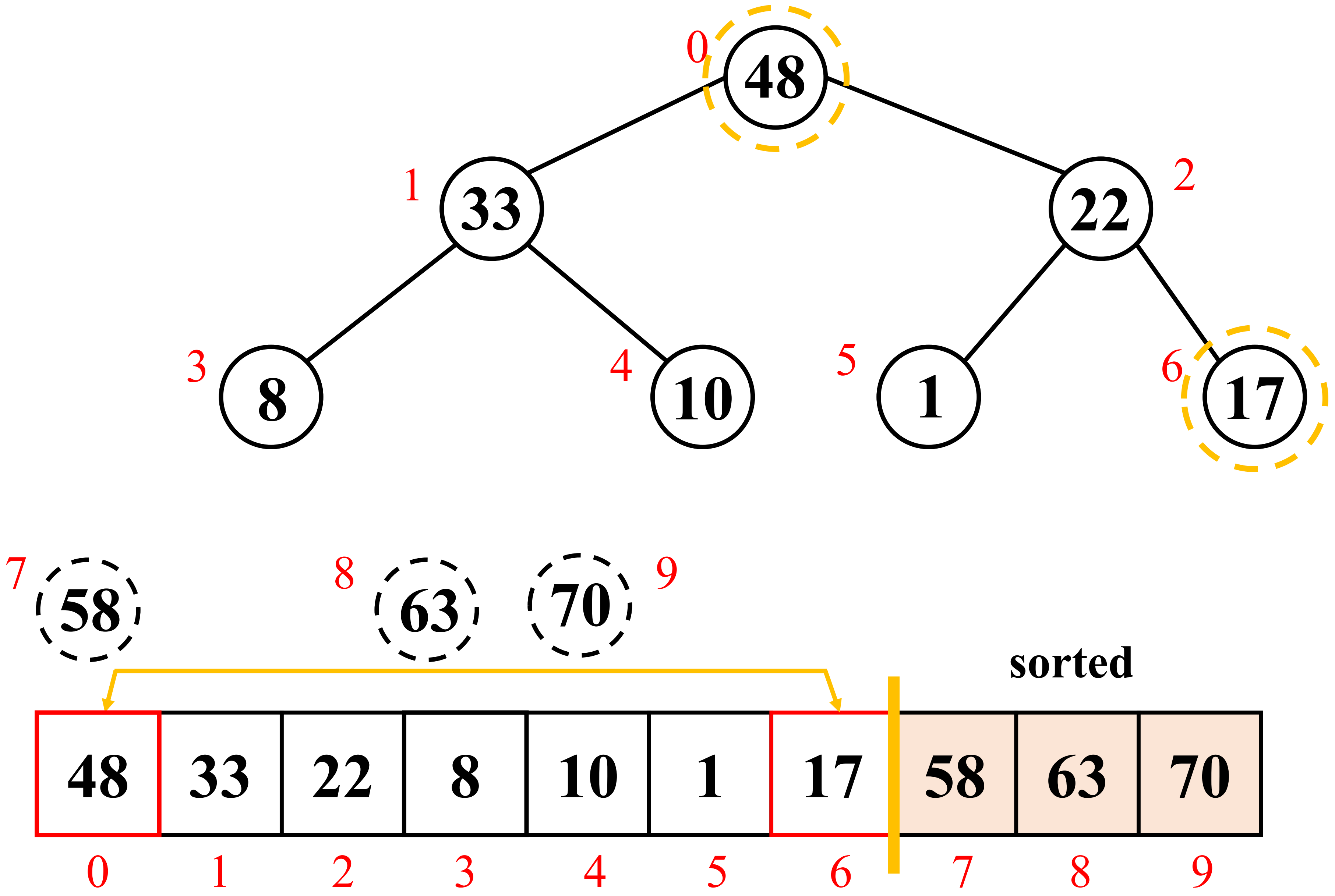


sink 8

sorted

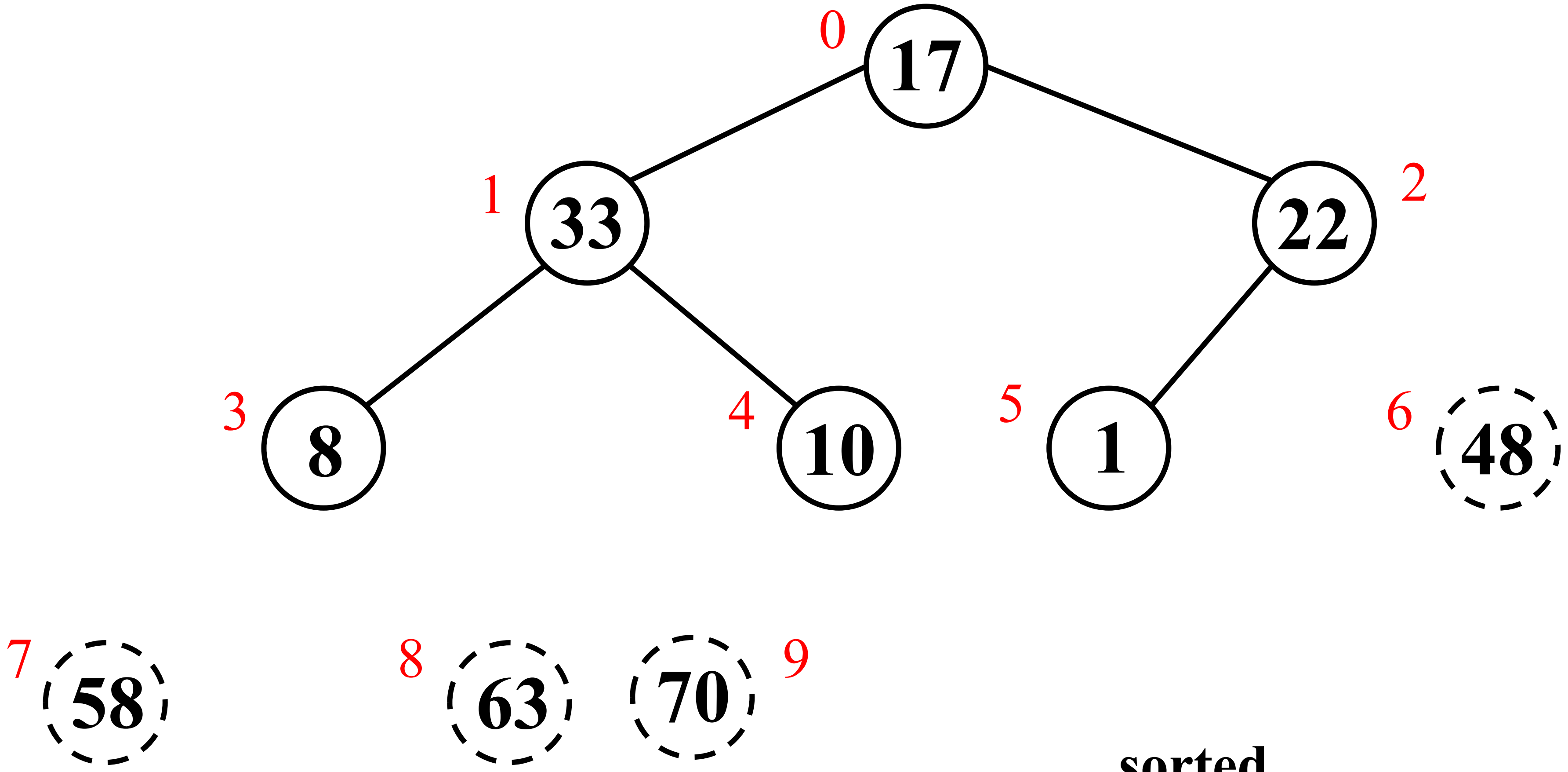
48	**33**	**22**	**8**	**10**	**1**	**17**	**58**	**63**	**70**
0	1	2	3	4	5	6	7	8	9

Heapsort algorithms: array implementation



swap

Heapsort algorithms: array implementation

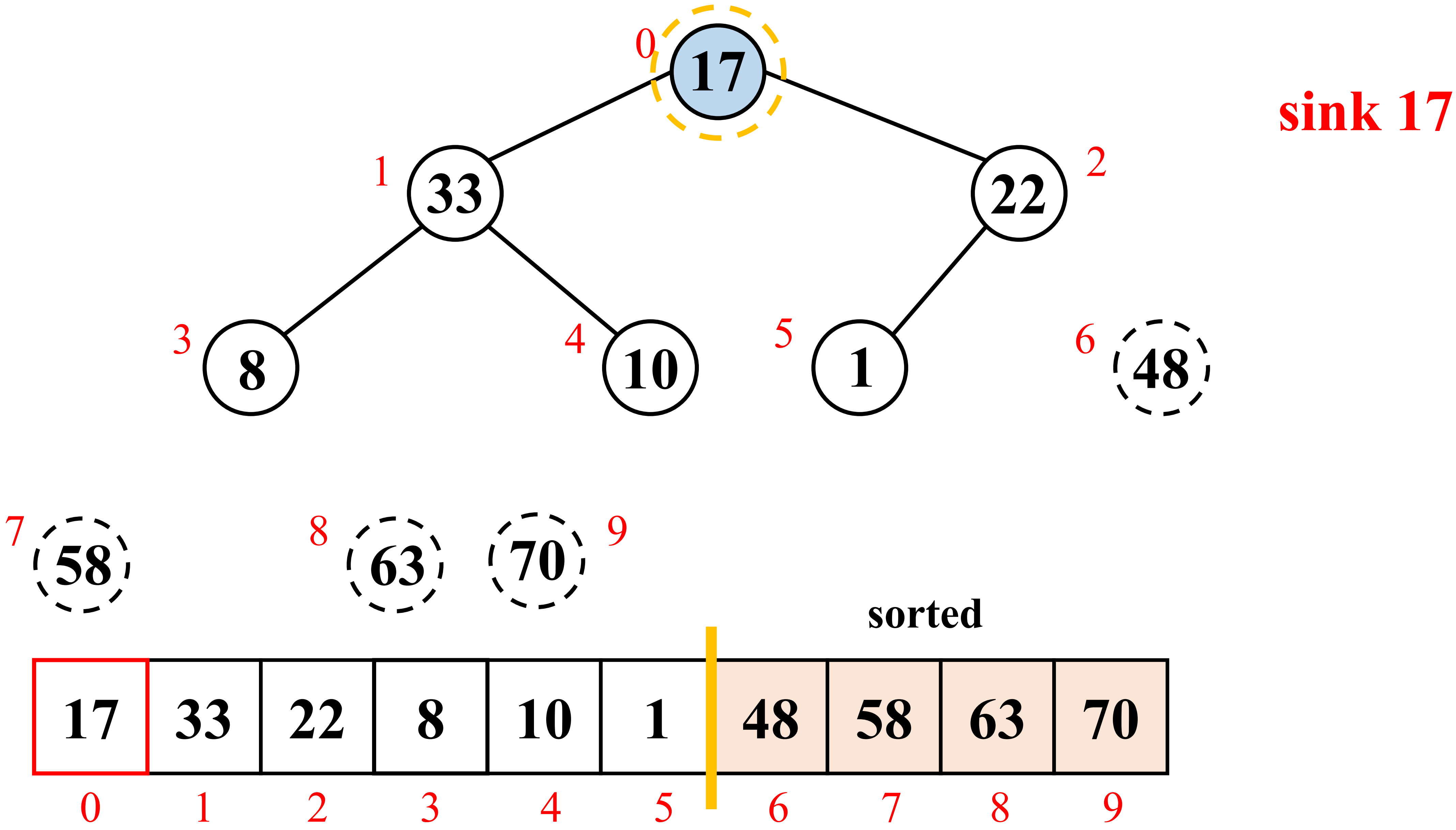


delete

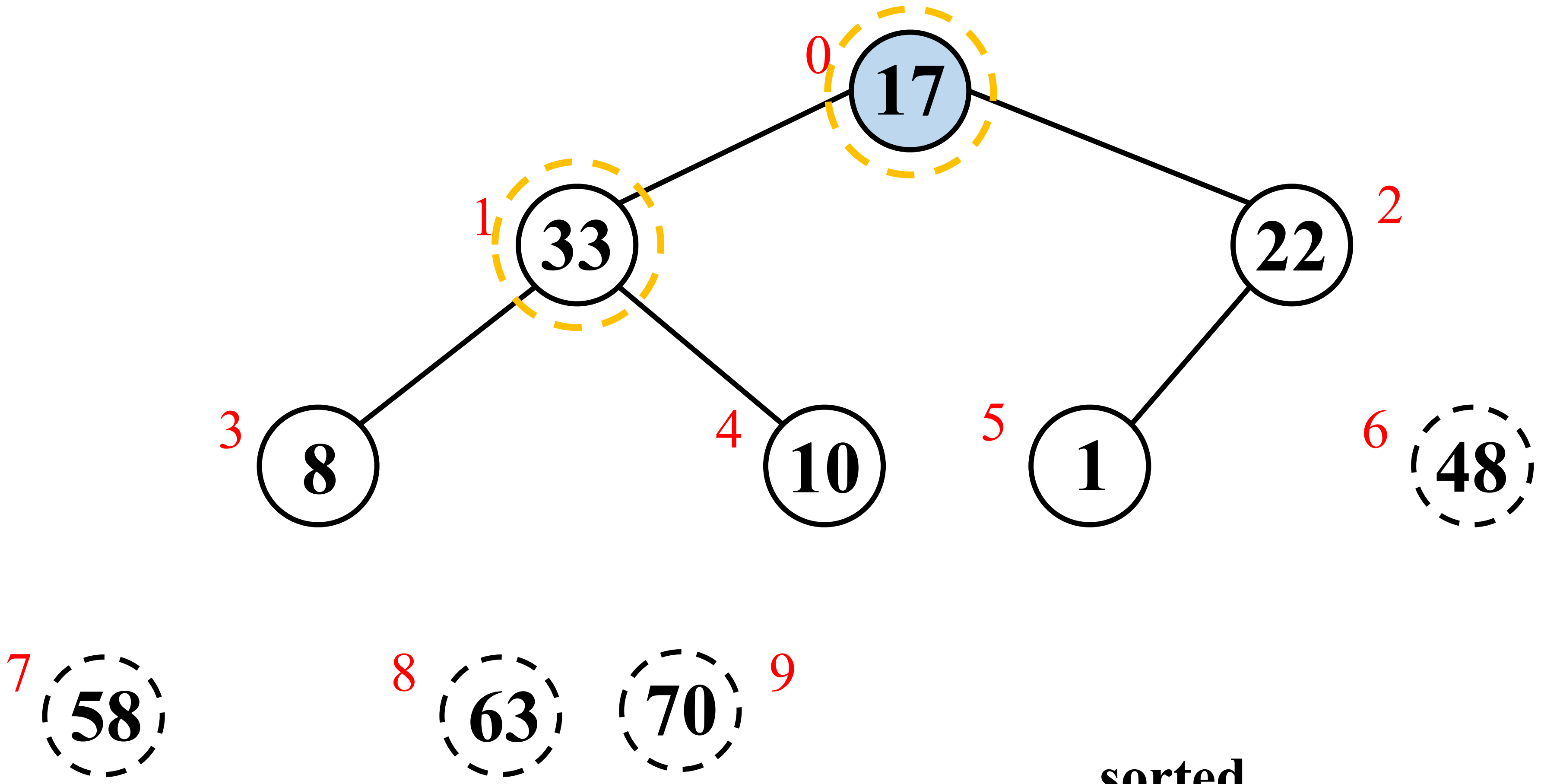
sorted

0	1	2	3	4	5	6	7	8	9
17	**33**	**22**	**8**	**10**	**1**	**48**	**58**	**63**	**70**

Heapsort algorithms: array implementation



Heapsort algorithms: array implementation



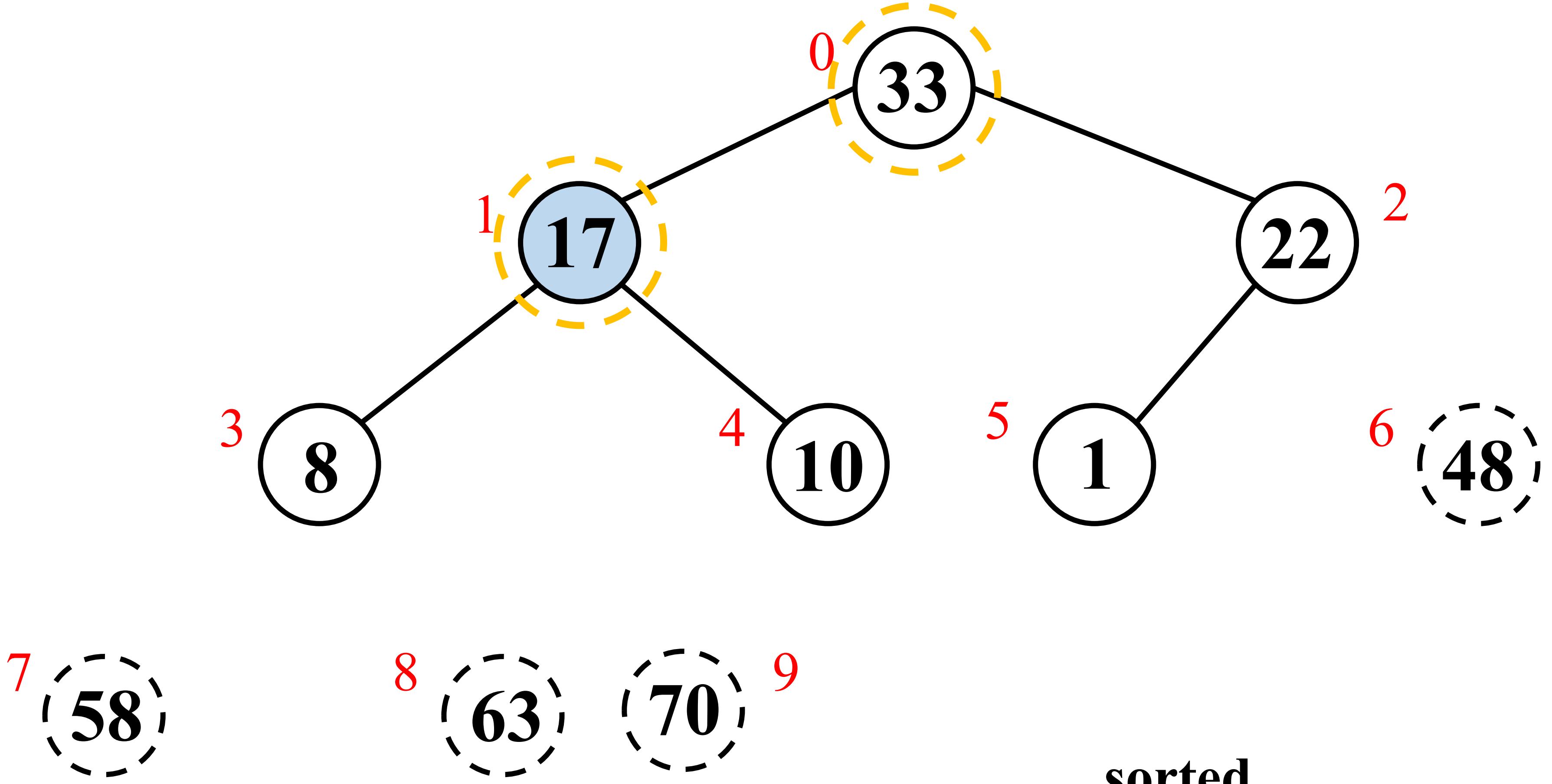
sink 17

sorted

17	33	22	8	10	1	48	58	63	70
0	1	2	3	4	5	6	7	8	9

Heapsort algorithms: array implementation

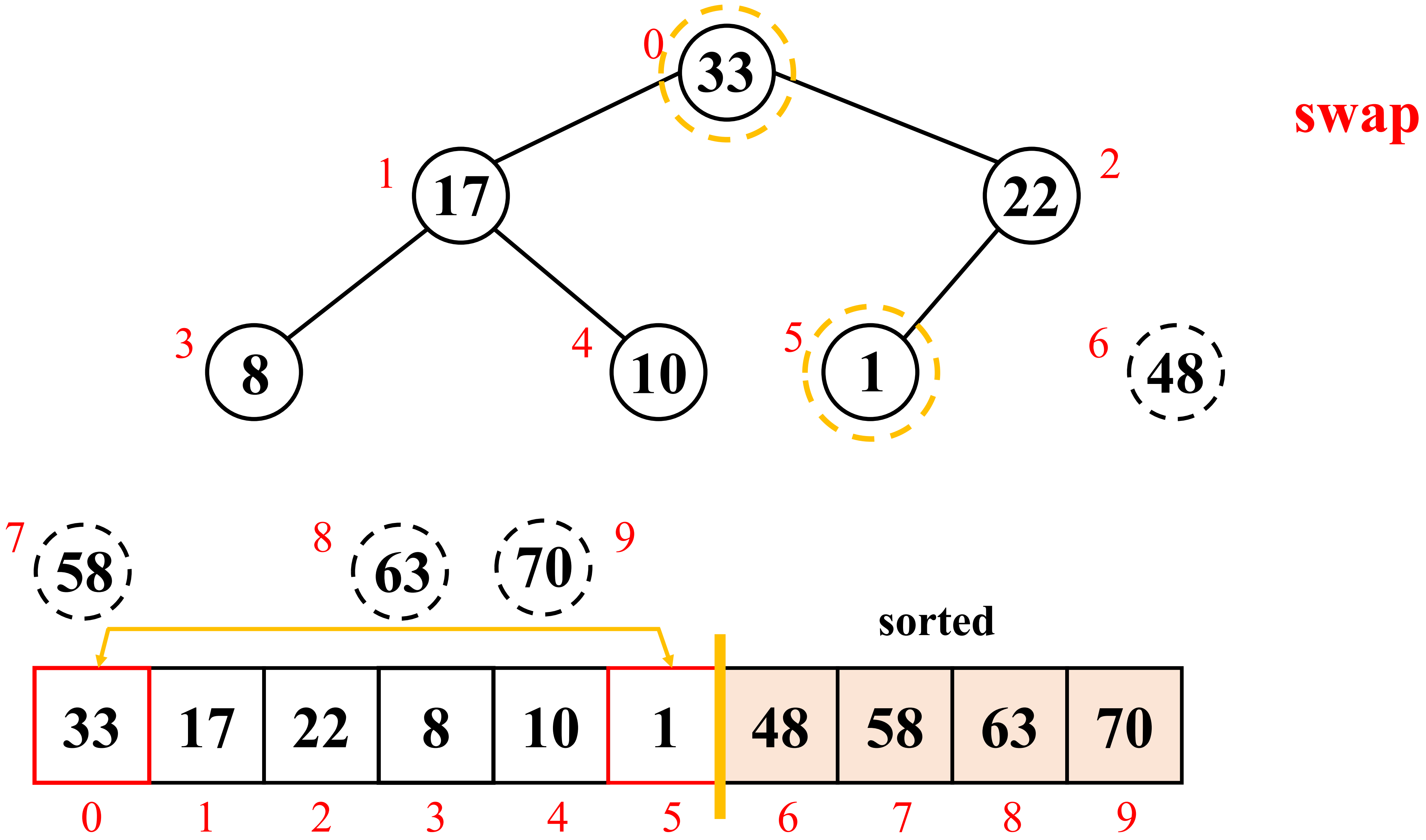
sink 17



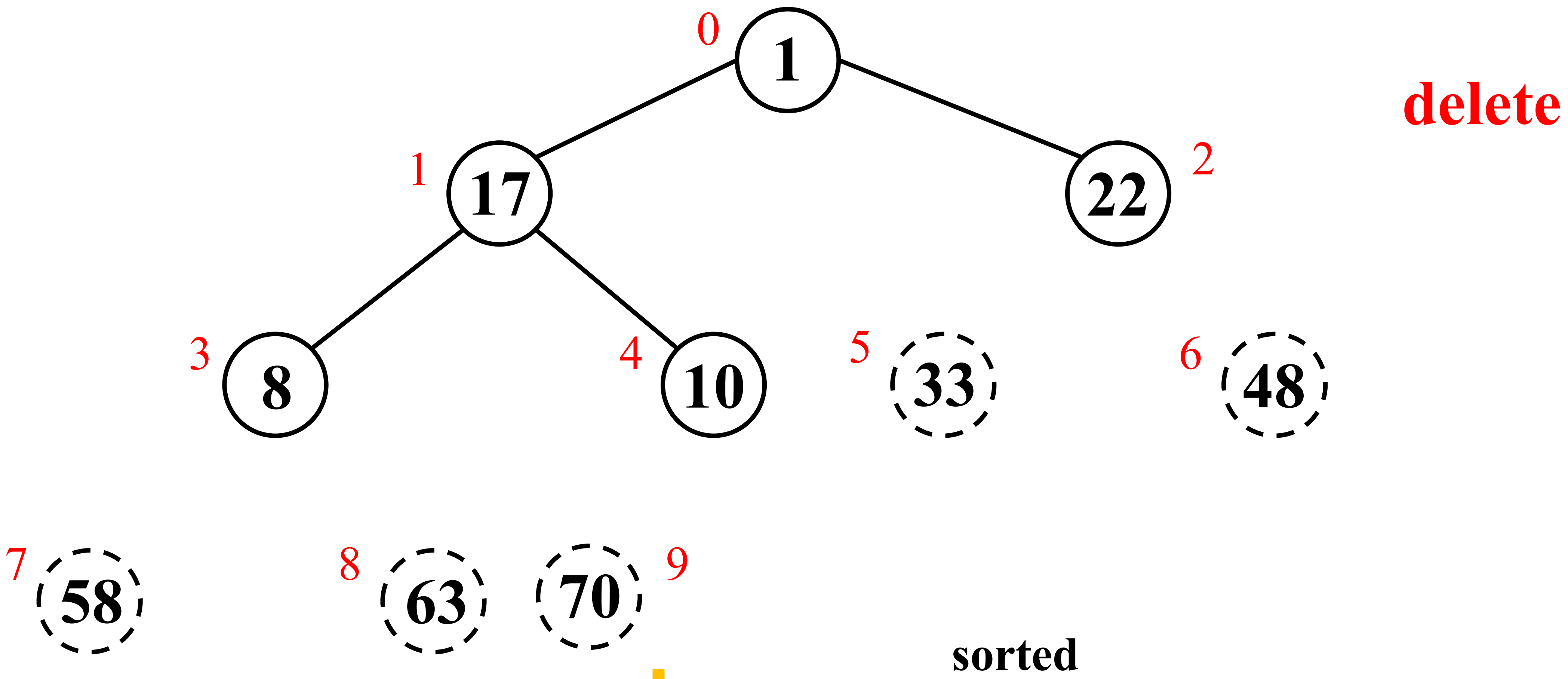
sorted

0	1	2	3	4	5	6	7	8	9
33	**17**	**22**	**8**	**10**	**1**	**48**	**58**	**63**	**70**

Heapsort algorithms: array implementation

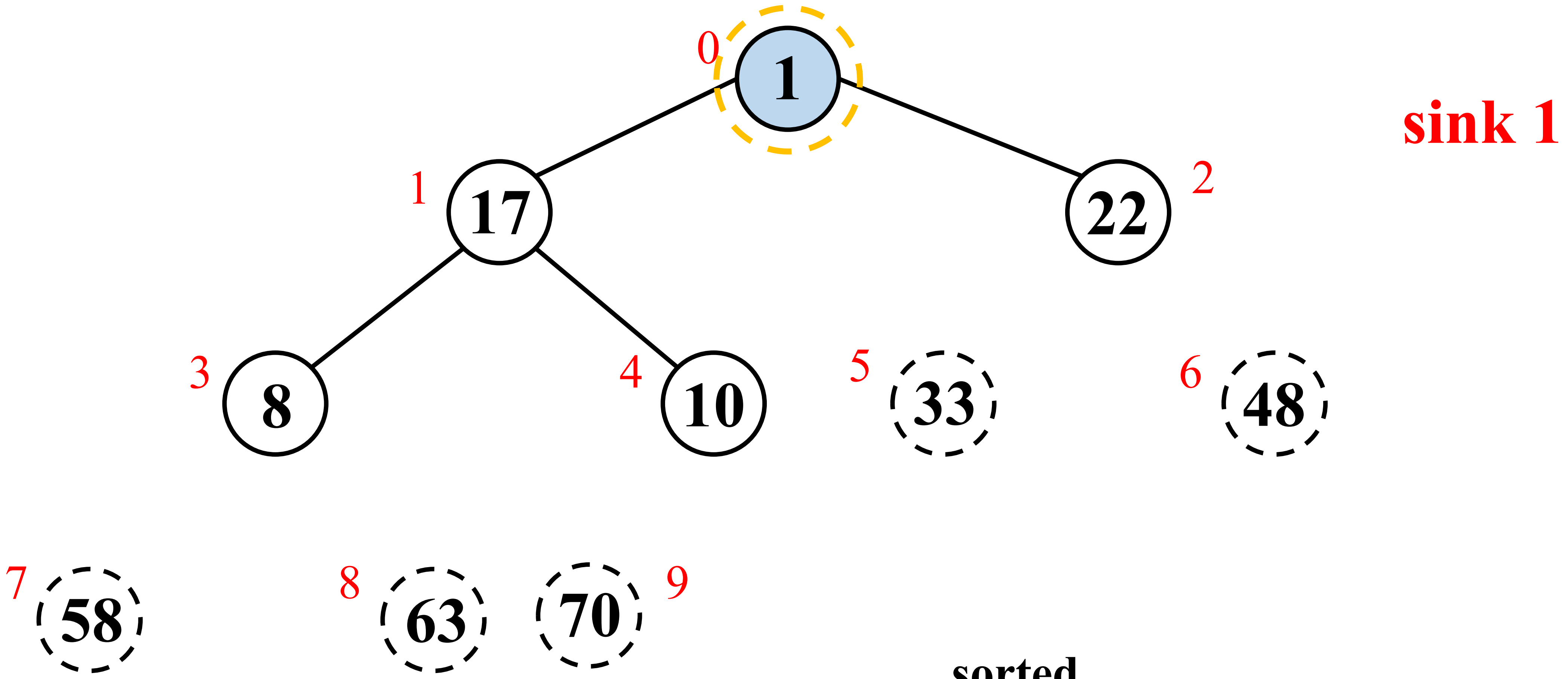


Heapsort algorithms: array implementation



1	17	22	8	10	33	48	58	63	70
0	1	2	3	4	5	6	7	8	9

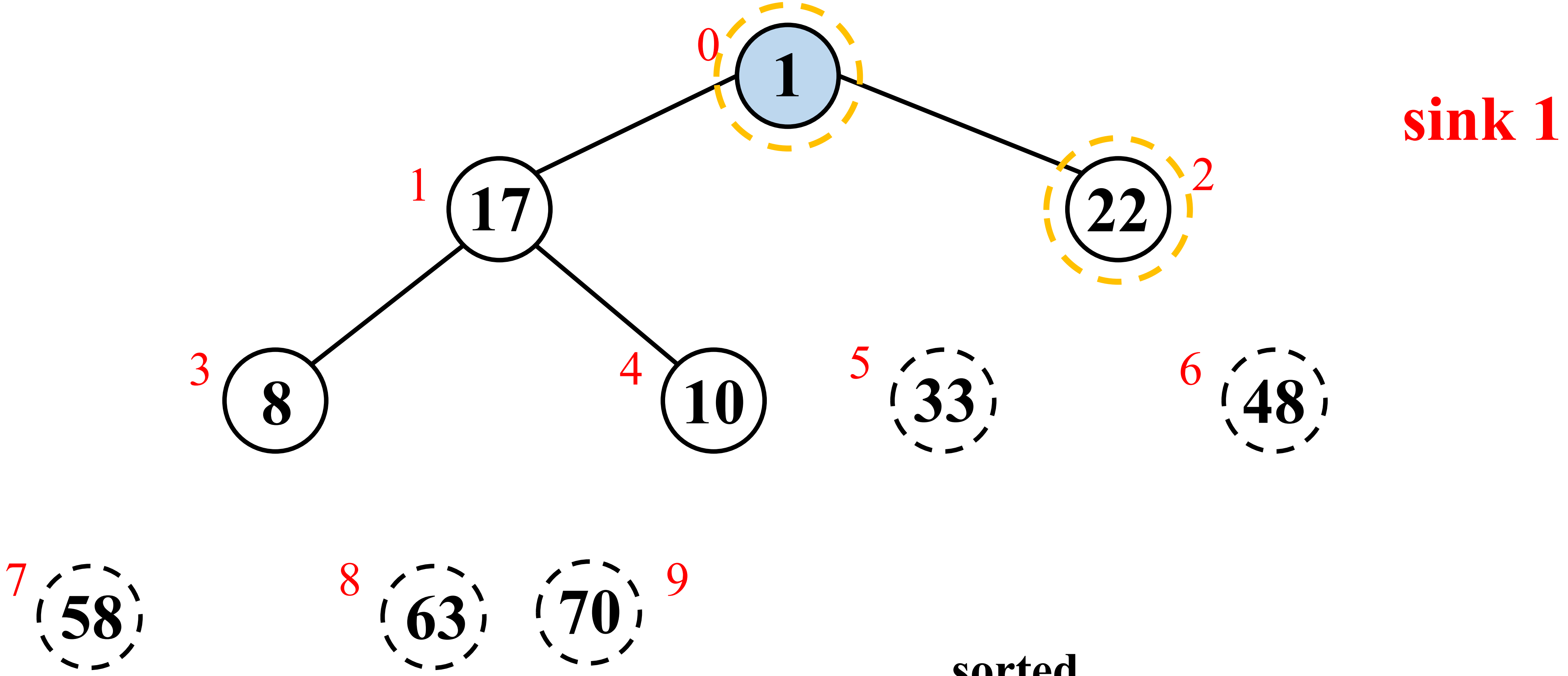
Heapsort algorithms: array implementation



sorted

1	17	22	8	10	33	48	58	63	70
0	1	2	3	4	5	6	7	8	9

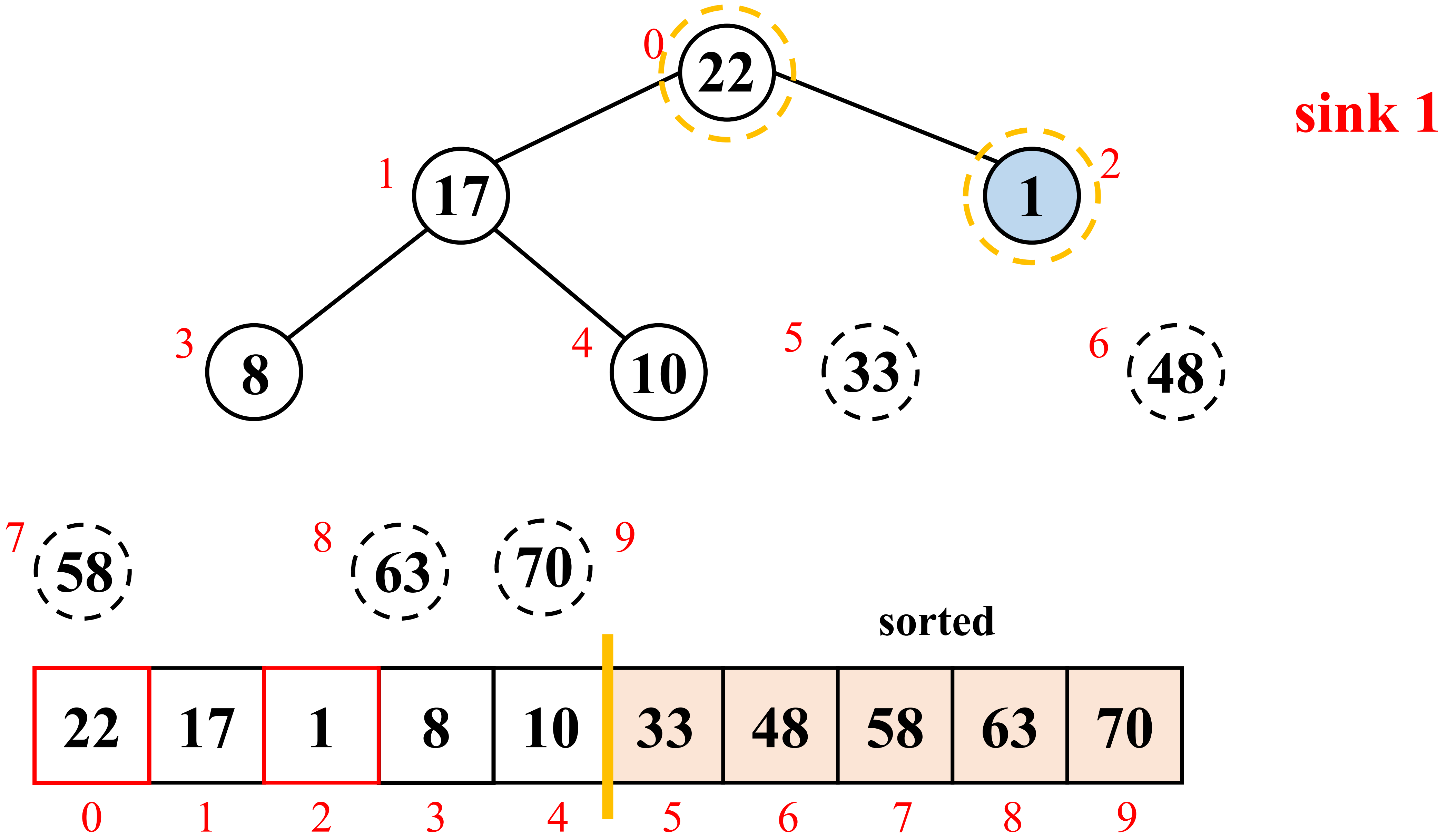
Heapsort algorithms: array implementation



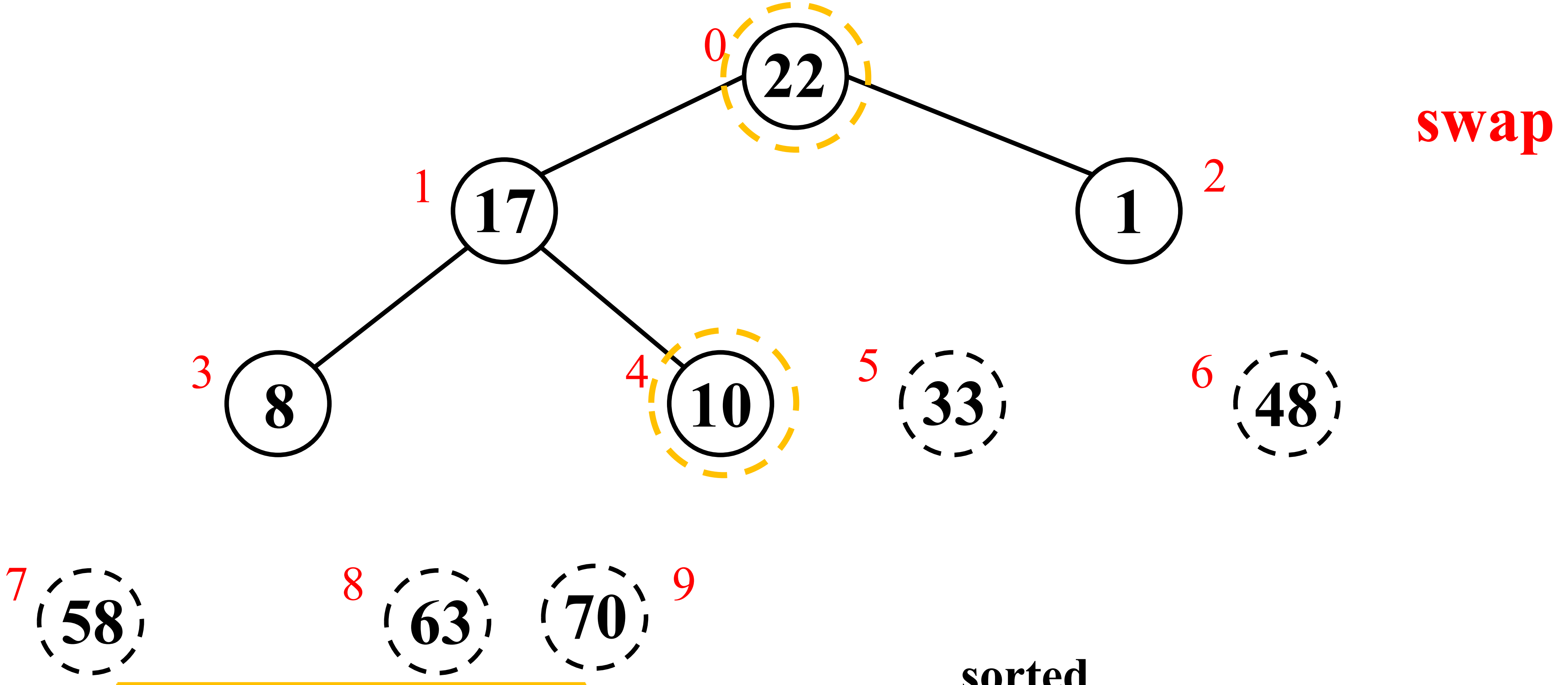
sorted

1	17	22	8	10	33	48	58	63	70
0	1	2	3	4	5	6	7	8	9

Heapsort algorithms: array implementation

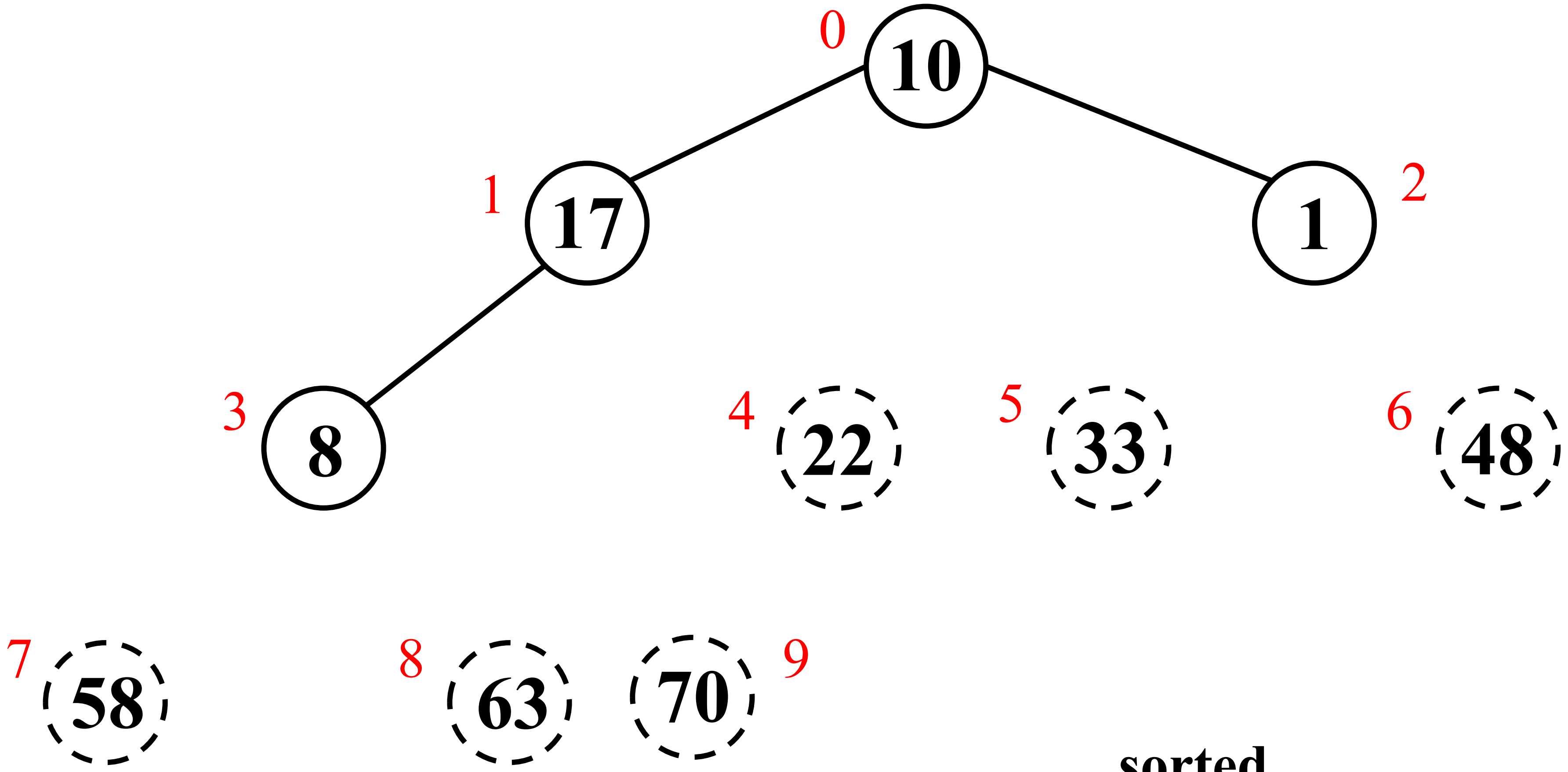


Heapsort algorithms: array implementation



22	17	1	8	10	33	48	58	63	70
0	1	2	3	4	5	6	7	8	9

Heapsort algorithms: array implementation

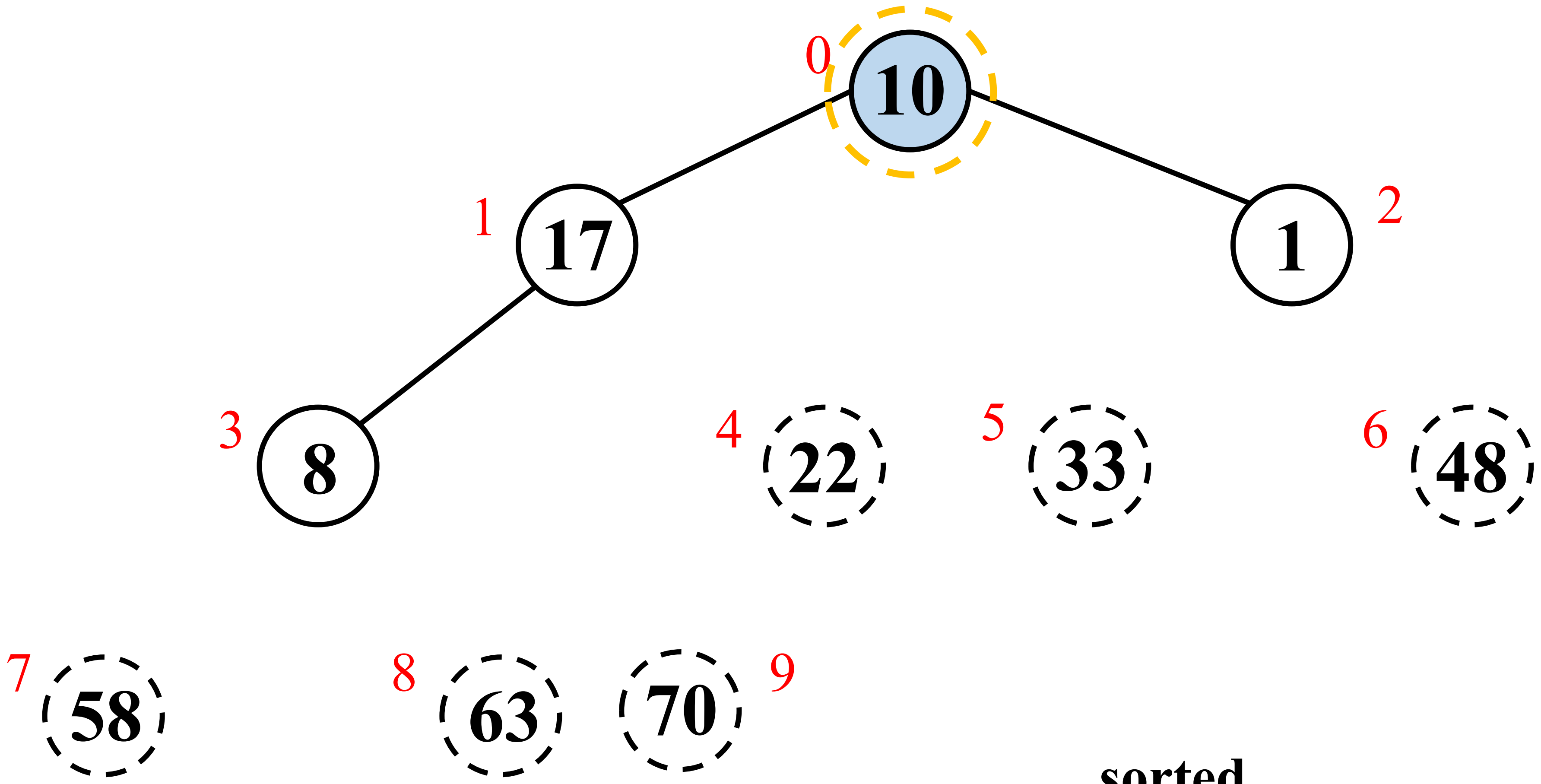


delete

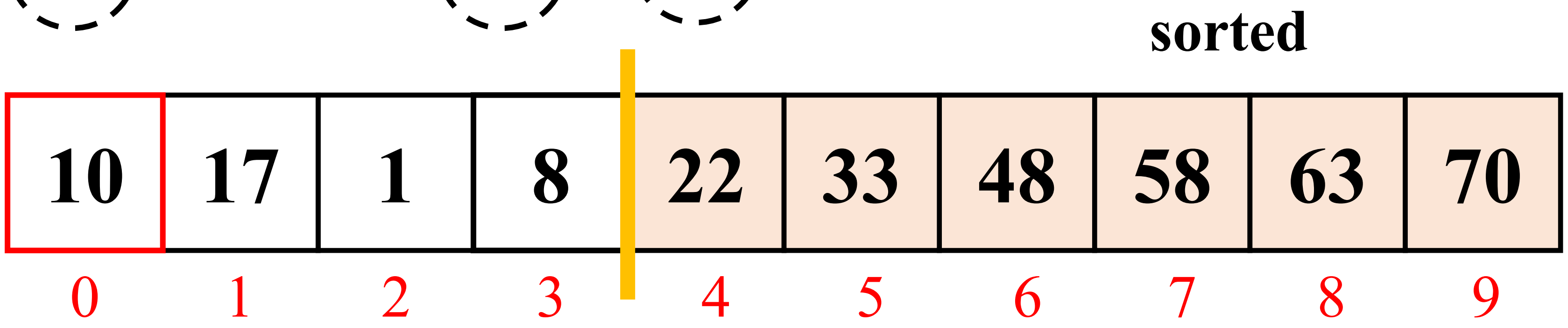
sorted

10	17	1	8	22	33	48	58	63	70
0	1	2	3	4	5	6	7	8	9

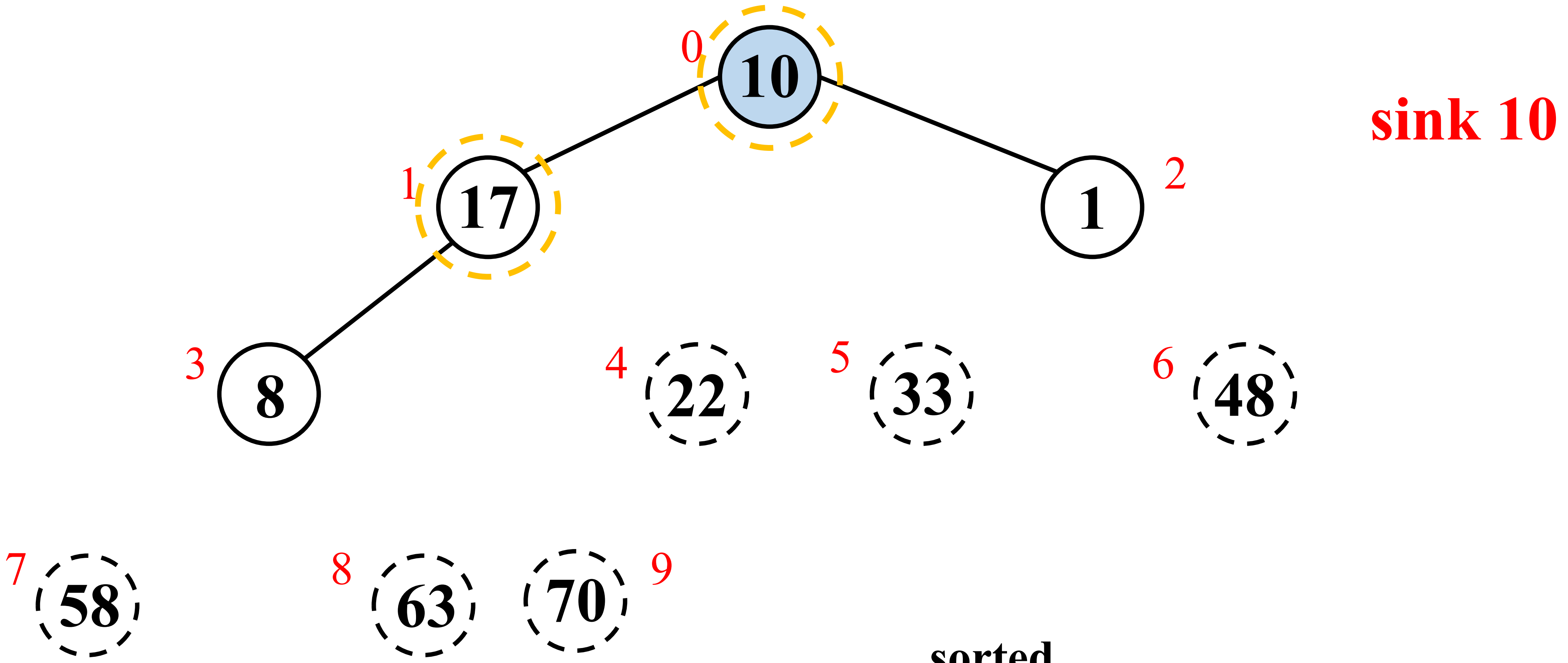
Heapsort algorithms: array implementation



sink 10



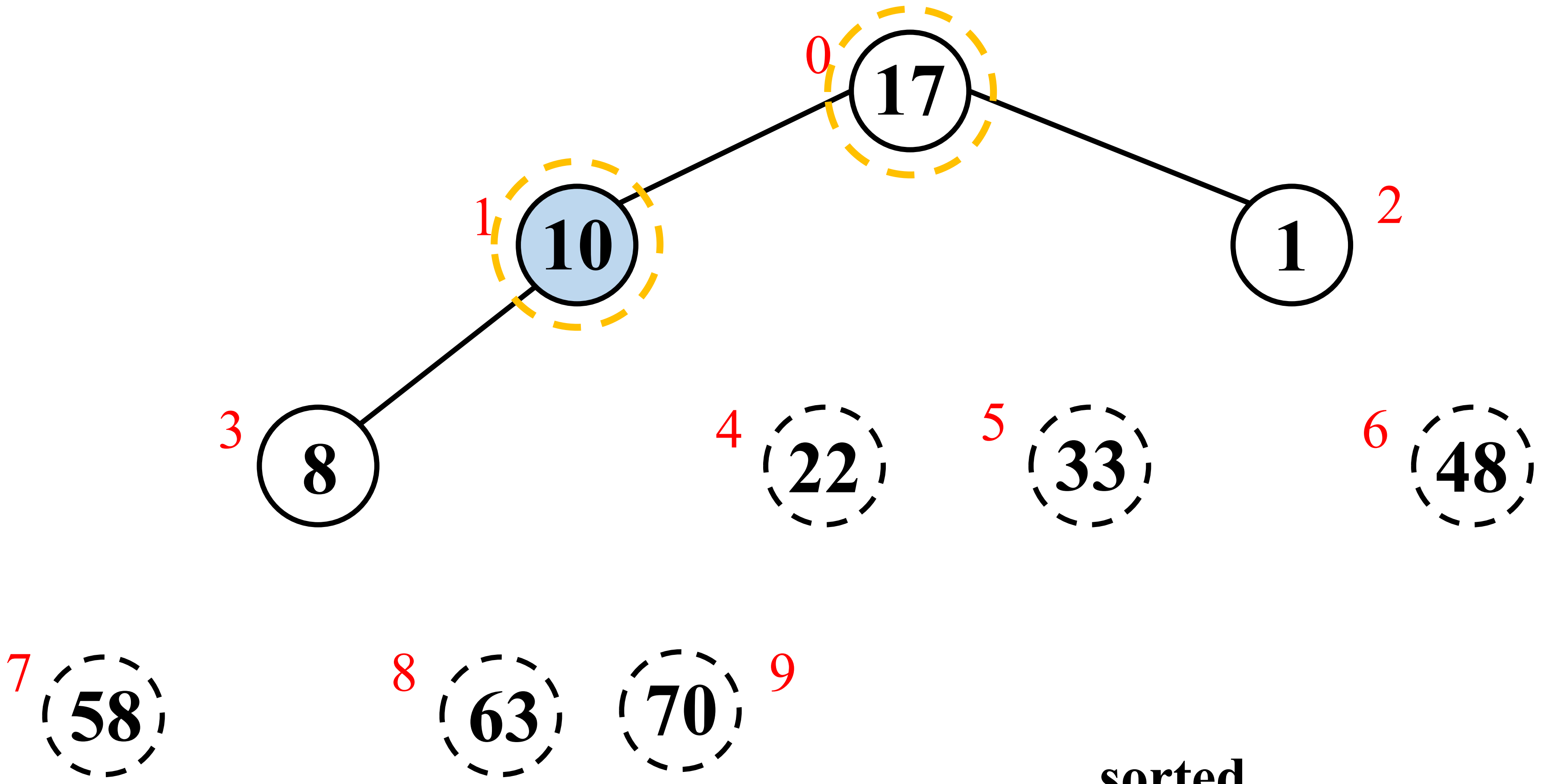
Heapsort algorithms: array implementation



sorted

0	1	2	3	4	5	6	7	8	9
10	17	1	8	22	33	48	58	63	70

Heapsort algorithms: array implementation

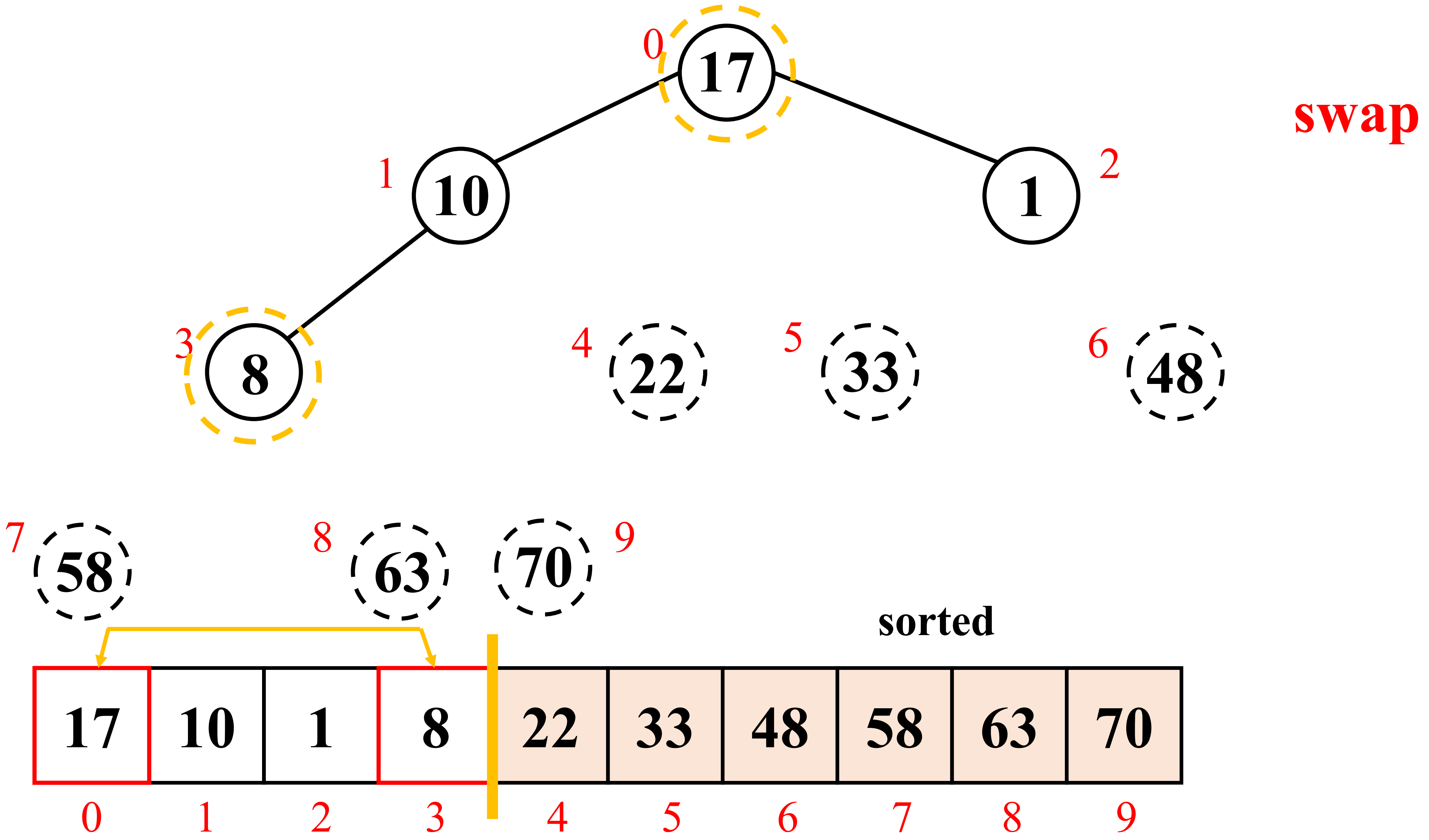


sink 10

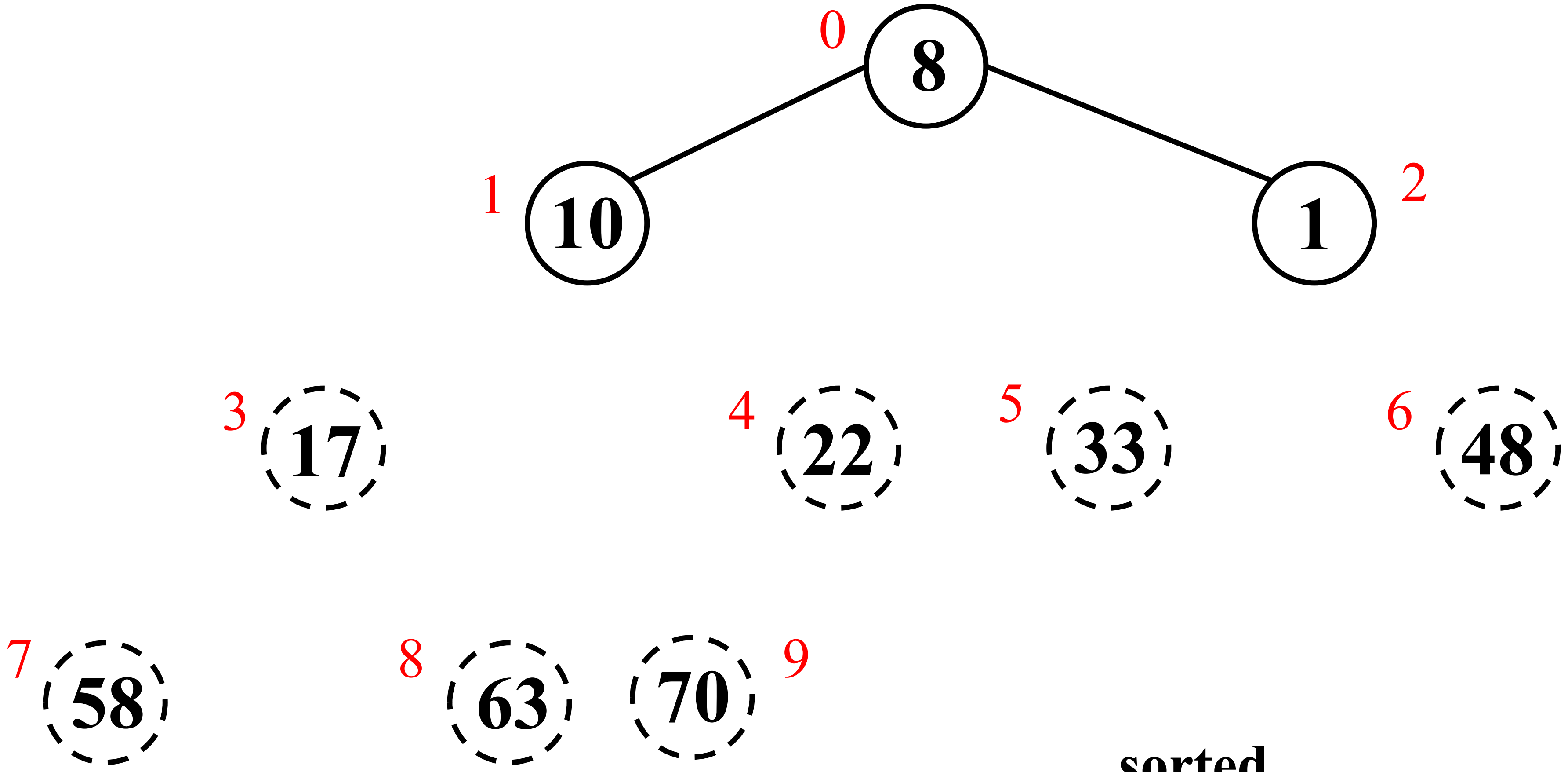
sorted

17	**10**	**1**	**8**	**22**	**33**	**48**	**58**	**63**	**70**
0	1	2	3	4	5	6	7	8	9

Heapsort algorithms: array implementation



Heapsort algorithms: array implementation

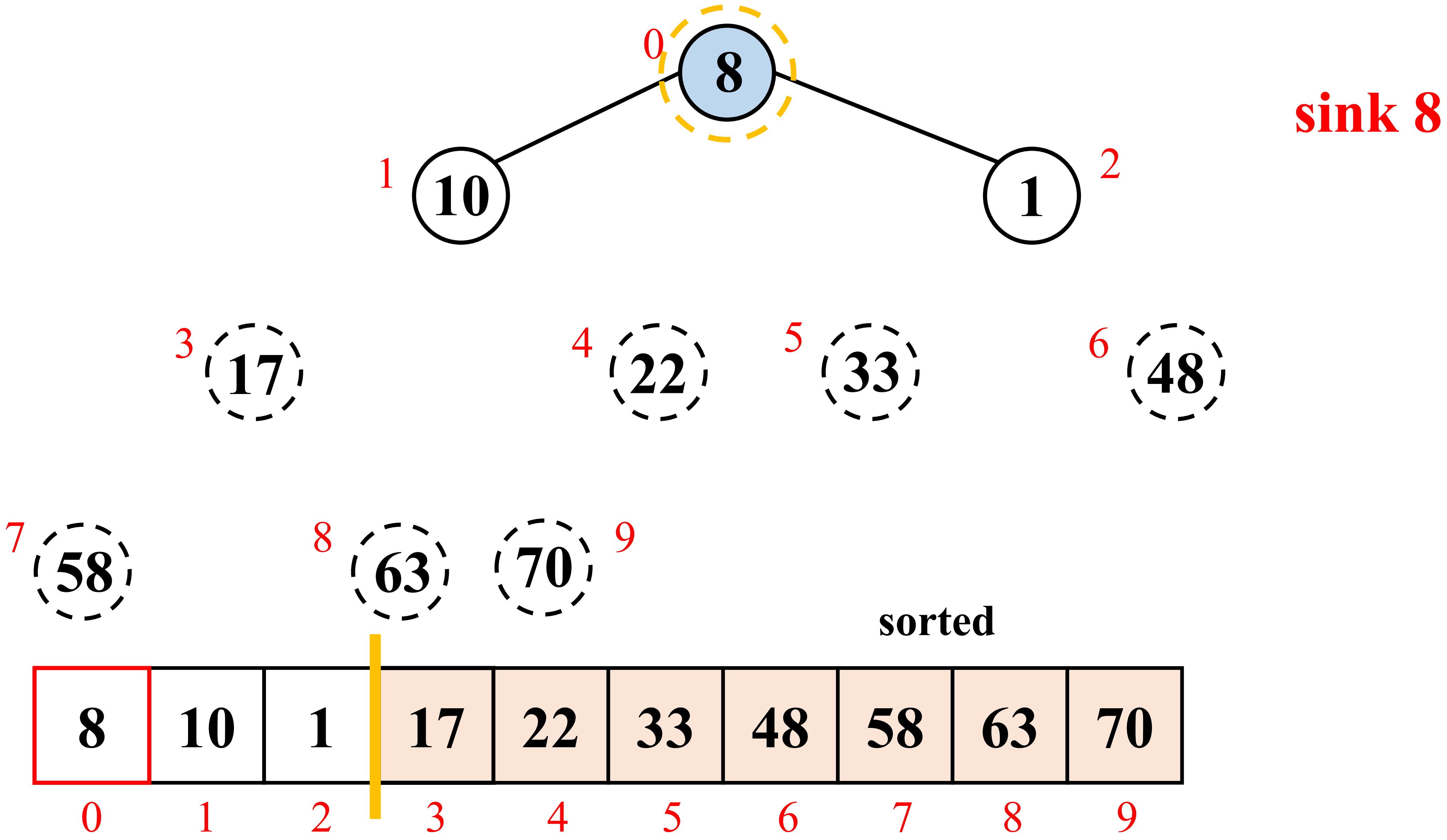


delete

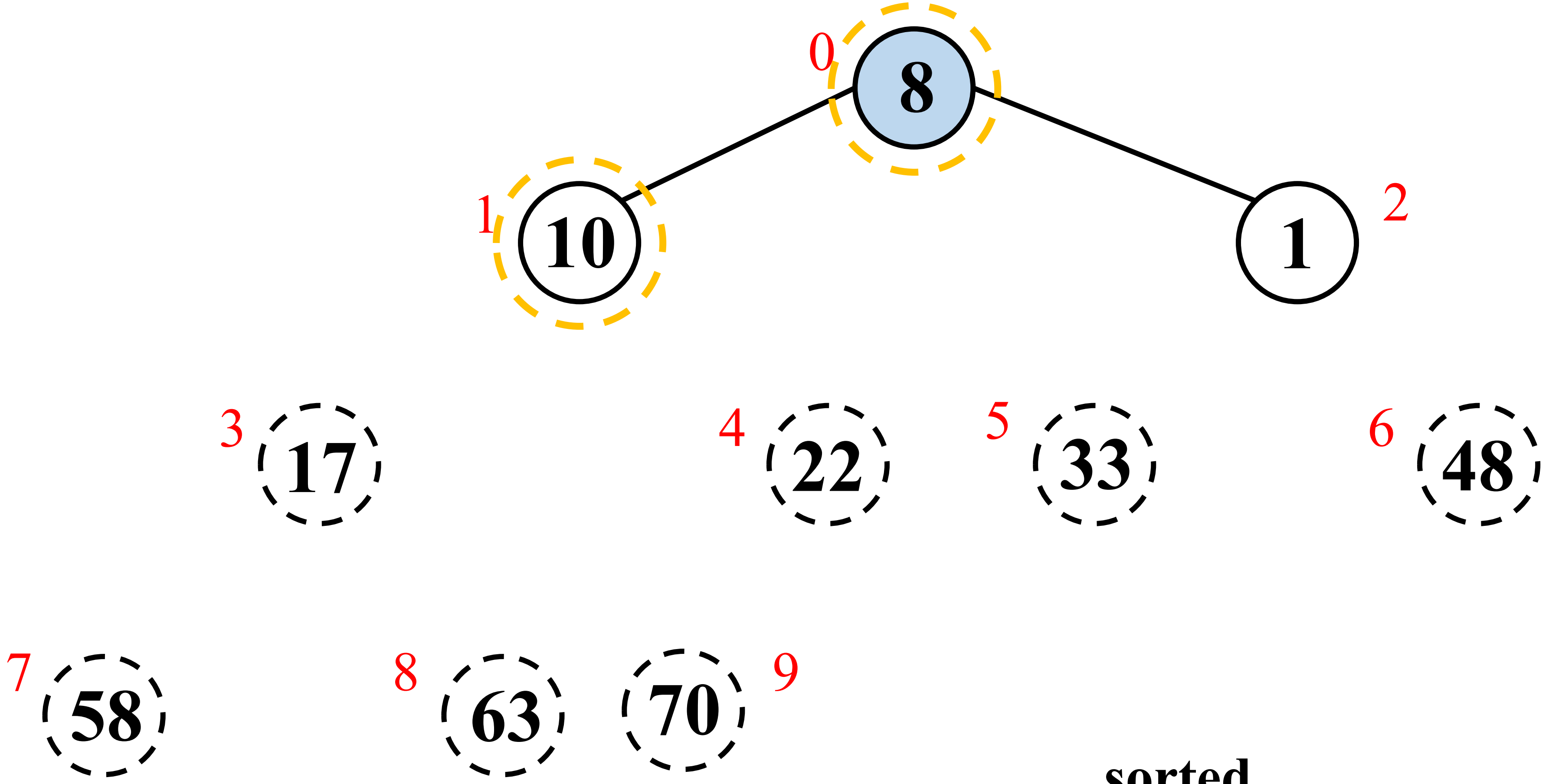
sorted

0	1	2	3	4	5	6	7	8	9
8	**10**	**1**	**17**	**22**	**33**	**48**	**58**	**63**	**70**

Heapsort algorithms: array implementation



Heapsort algorithms: array implementation

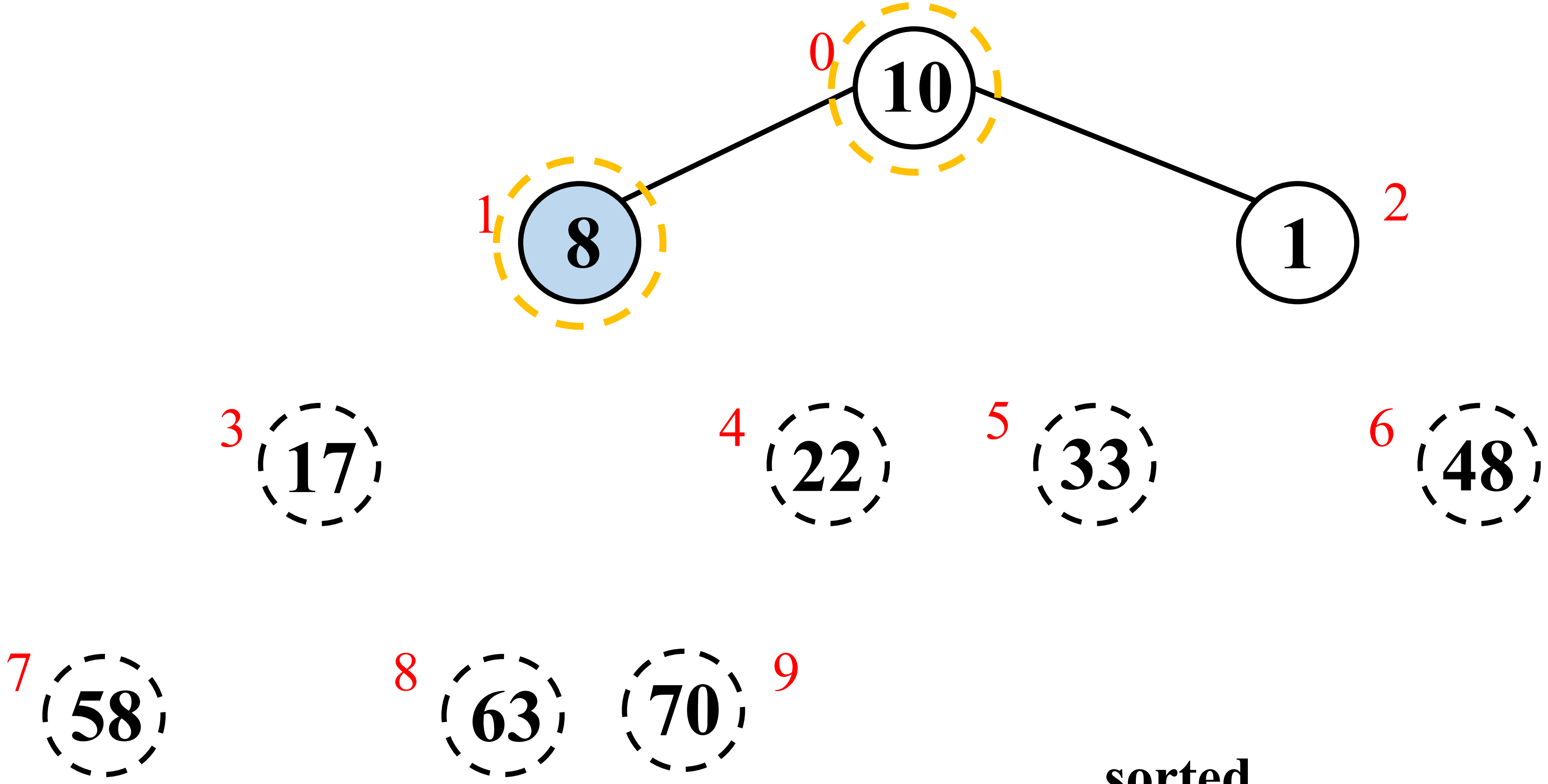


sink 8

sorted

0	1	2	3	4	5	6	7	8	9
8	10	1	17	22	33	48	58	63	70

Heapsort algorithms: array implementation

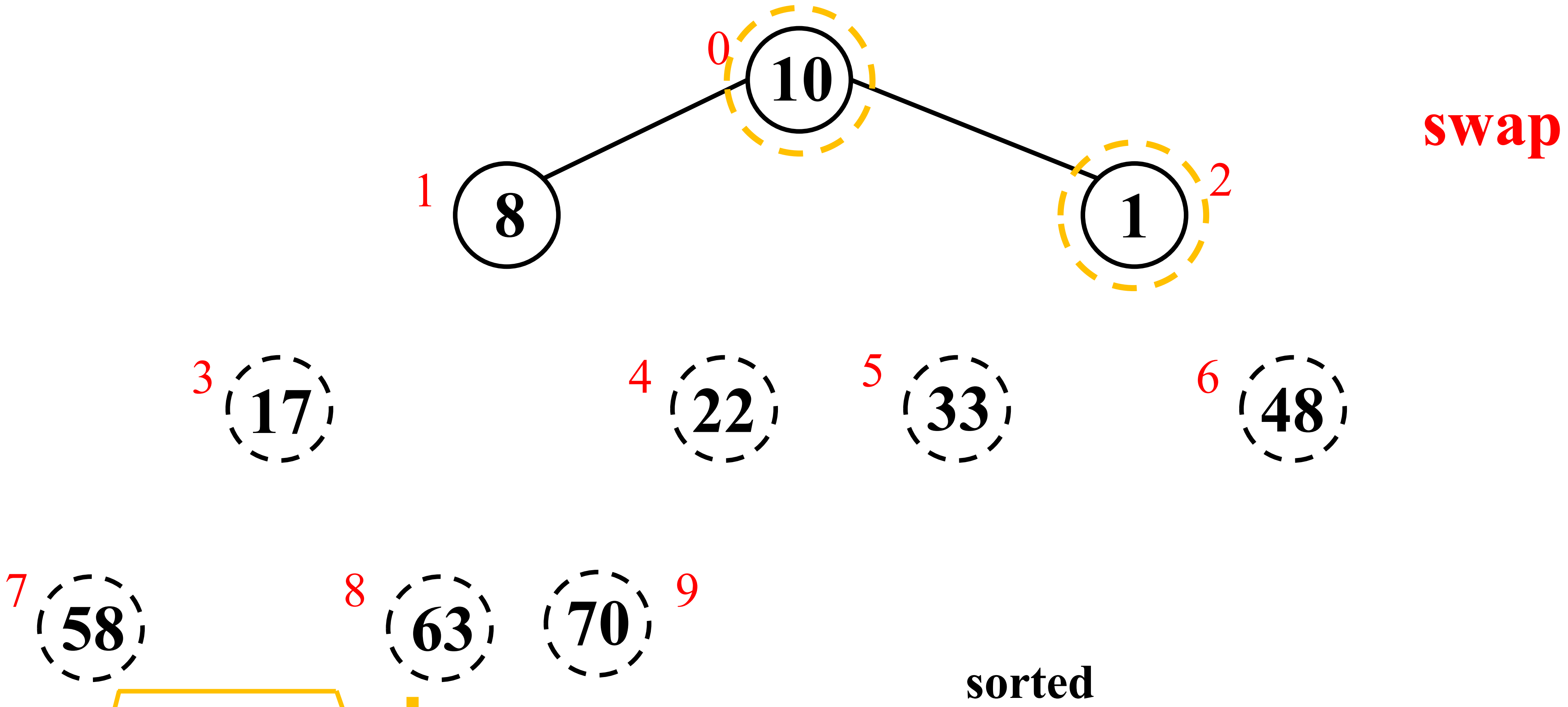


sink 8

sorted

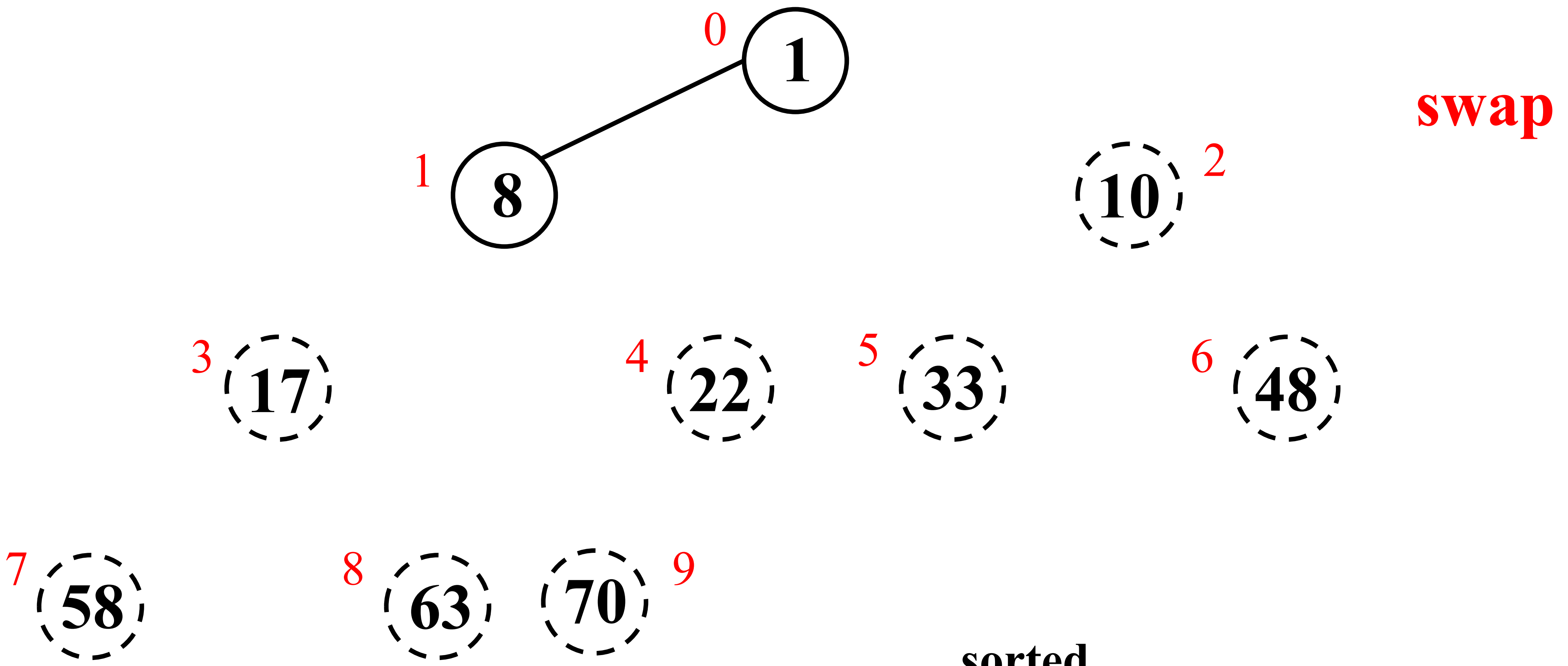
10	8	1	17	22	33	48	58	63	70
0	1	2	3	4	5	6	7	8	9

Heapsort algorithms: array implementation



10	8	1	17	22	33	48	58	63	70
0	1	2	3	4	5	6	7	8	9

Heapsort algorithms: array implementation

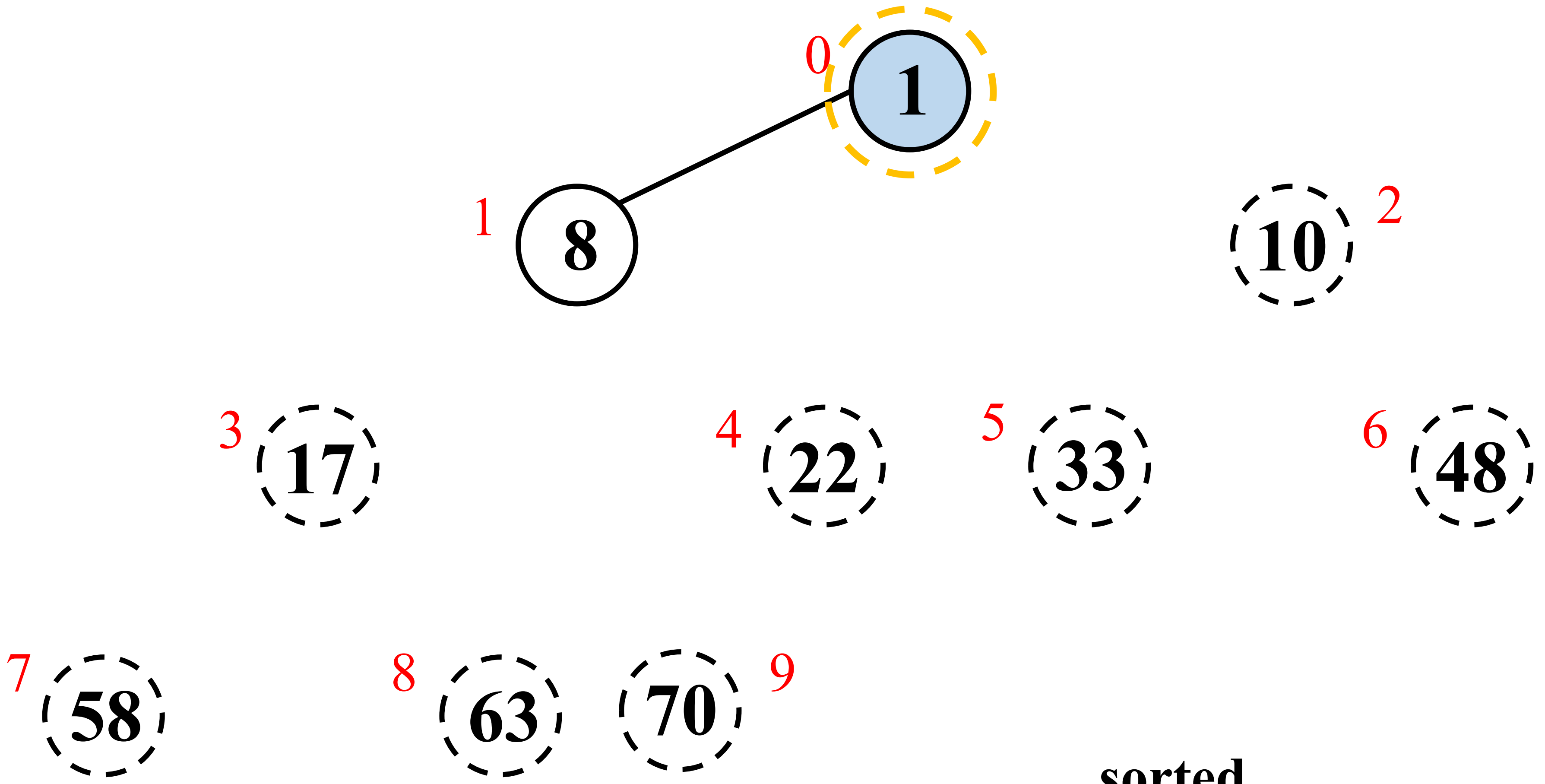


sorted

1	8	10	17	22	33	48	58	63	70
0	1	2	3	4	5	6	7	8	9

Heapsort algorithms: array implementation

sink 1

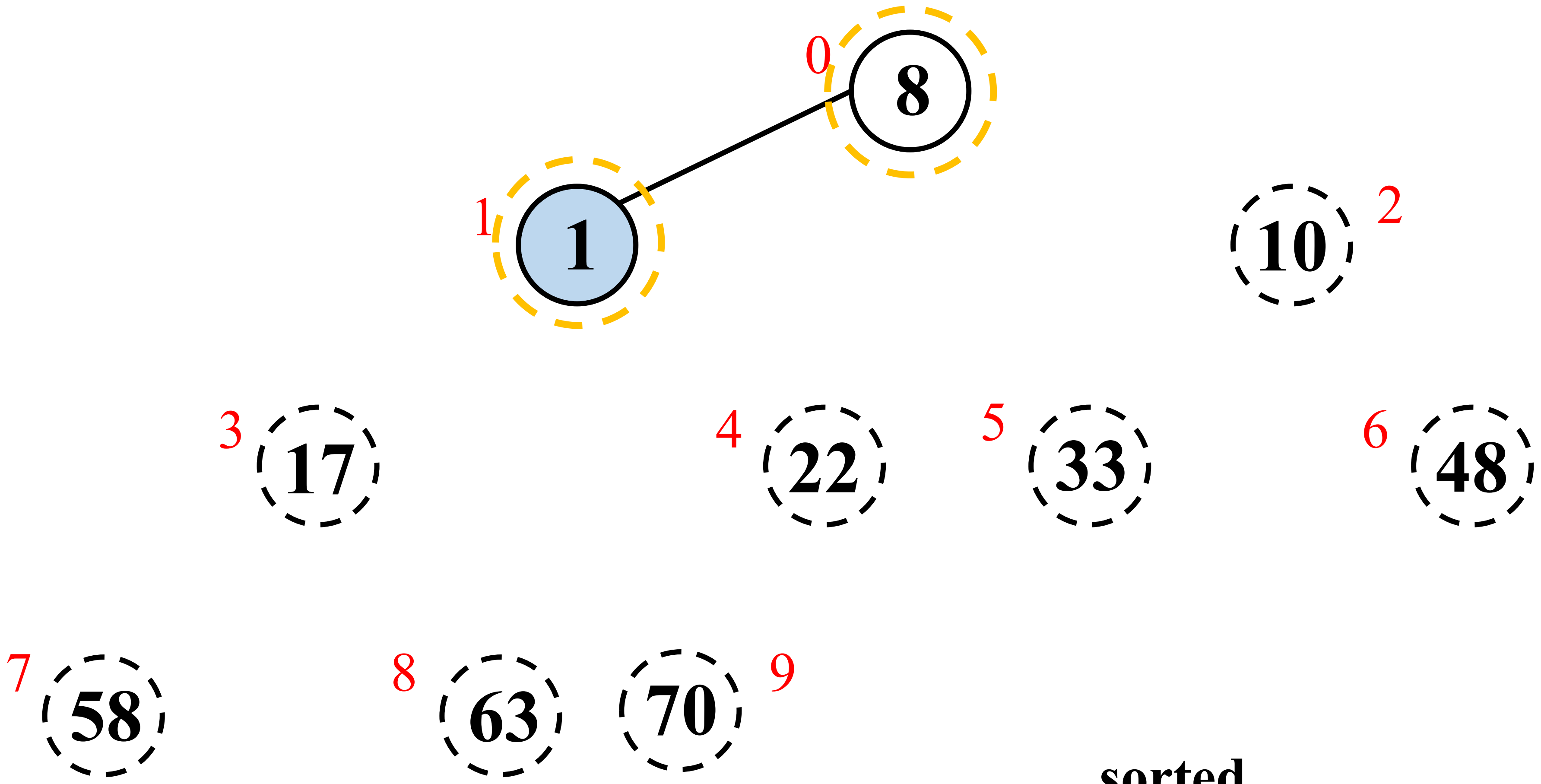


sorted

1	8	10	17	22	33	48	58	63	70
0	1	2	3	4	5	6	7	8	9

Heapsort algorithms: array implementation

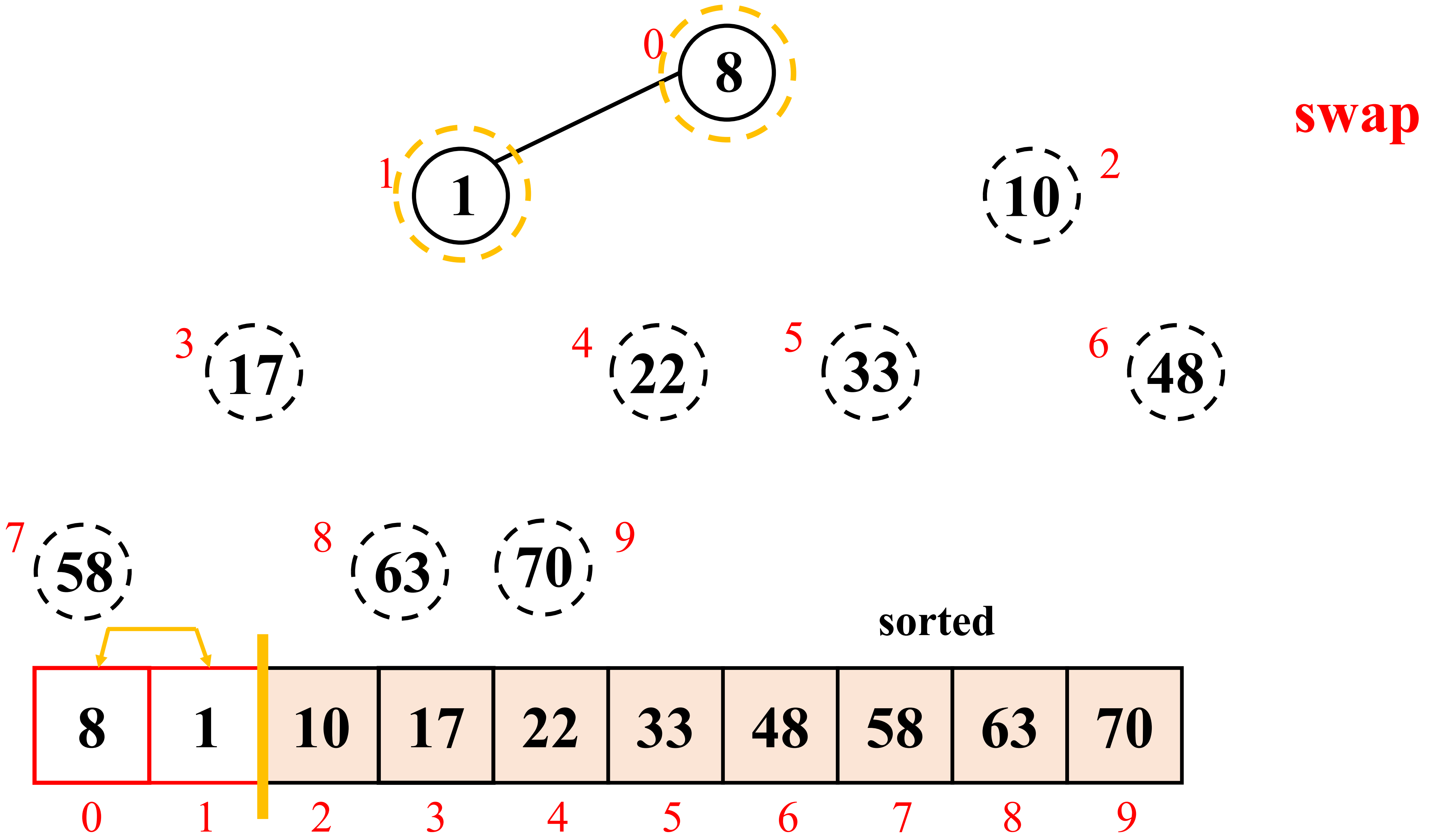
sink 1



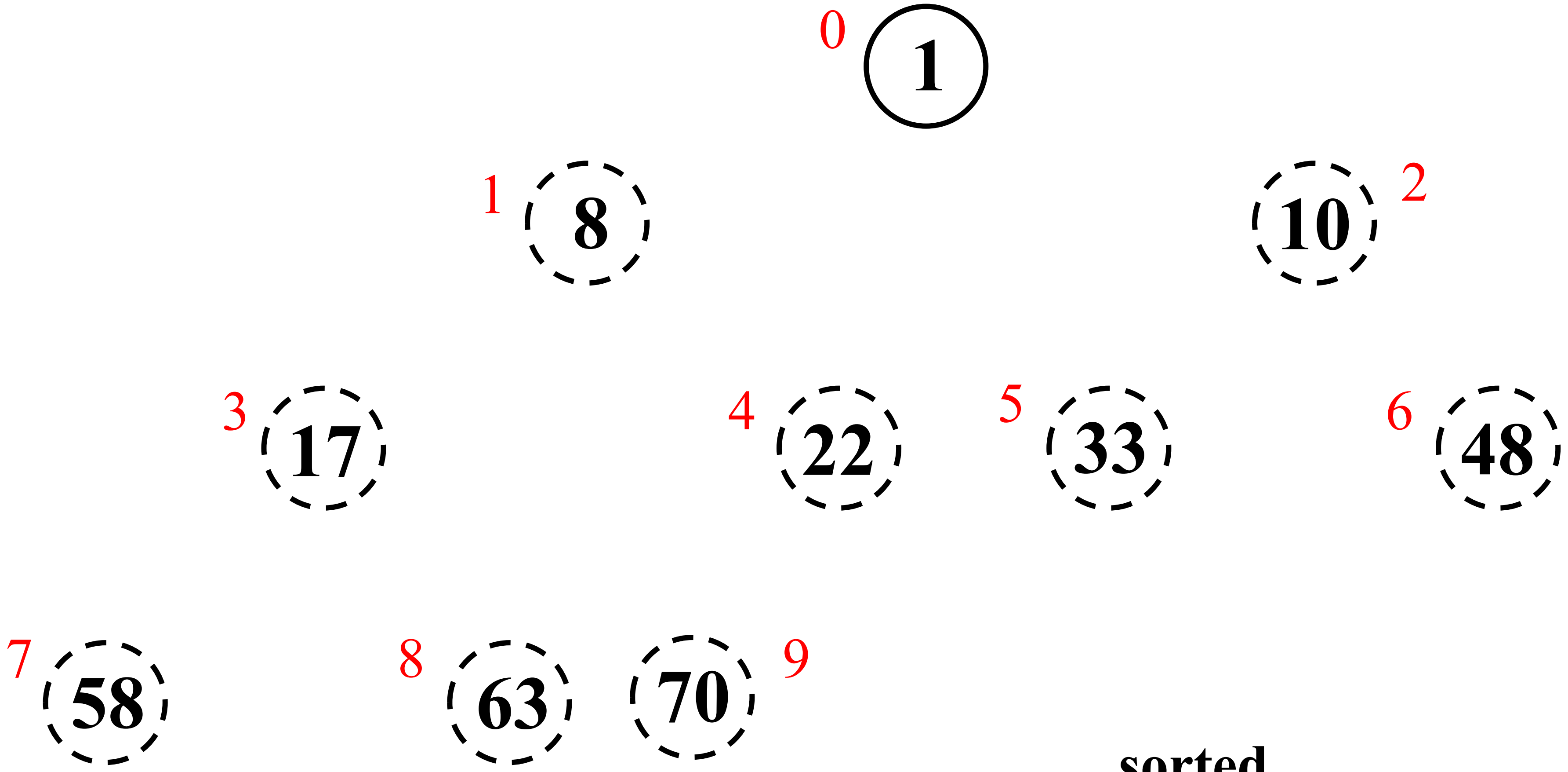
sorted

8	**1**	**10**	**17**	**22**	**33**	**48**	**58**	**63**	**70**
0	1	2	3	4	5	6	7	8	9

Heapsort algorithms: array implementation



Heapsort algorithms: array implementation

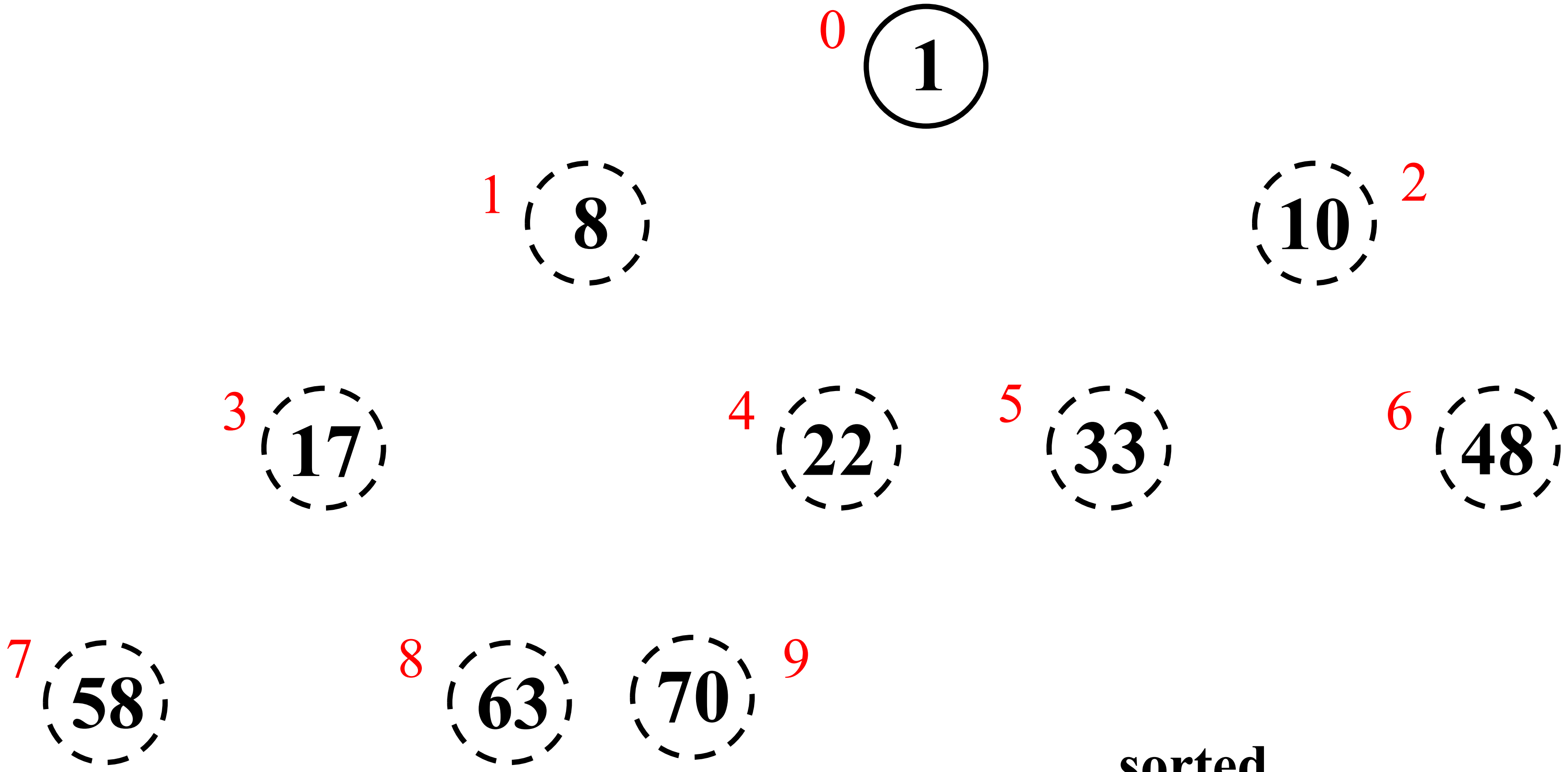


delete

sorted

1	**8**	**10**	**17**	**22**	**33**	**48**	**58**	**63**	**70**
0	1	2	3	4	5	6	7	8	9

Heapsort algorithms: array implementation



sorted

1	8	10	17	22	33	48	58	63	70
0	1	2	3	4	5	6	7	8	9

Heapsort Algorithms: Array Implementation

Algorithm 1 Heapsort.

function HEAPSORT(array $a[0..n-1]$)

  $k \leftarrow n-1$ ▷ index of last leaf in heap

  $i \leftarrow \left\lfloor \frac{n-1}{2} \right\rfloor$

  while $i \geq 0$ **do**

    $sink(a, i, k)$ ▷ let element sink to its correct place

    $i \leftarrow i - 1$ ▷ heap now build

(Build the heap)

  while $k > 0$ **do**

    $swap(a[k], a[0])$

    $k \leftarrow k - 1$

    $sink(a, 0, k)$ ▷ let element at root sink to its correct place

  return a ▷ Array is now sorted

Heapsort Algorithms: Array Implementation (Contd.)

Algorithm 1 Heapsort.

function HEAPSORT(array $a[0..n-1]$)
  $k \leftarrow n - 1$ ▷ index of last leaf in heap
  $i \leftarrow \left\lfloor \frac{n-1}{2} \right\rfloor$
  while $i \geq 0$ **do**
    $sink(a, i, k)$ ▷ let element sink to its correct place
    $i \leftarrow i - 1$ ▷ heap now build
  while $k > 0$ **do**
    $swap(a[k], a[0])$
    $k \leftarrow k - 1$ **(Delete the maximum repeatedly)**
    $sink(a, 0, k)$ ▷ let element at root sink to its correct place
  return a ▷ Array is now sorted

Time Complexity Analysis

- Heapsort runs in time in $\Theta(n \log n)$ in the worst and average case.
- **Proof**:
  1. Building the heap:
     - The heap can be constructed in time $\Theta(n \log n)$ with straightforward method. We can do better using the alternative bottom-up method with $\Theta(n)$ time.
  2. Removing the maximum key:
     - Then heapsort repeats n times the deletion of the maximum key and restoration of the heap property (each restoration is logarithmic in the worst and average case).
     - The running time is $\log(n) + \log(n-1) + \log(n-2) + \cdots + 1 = \log(n!) \in \Theta(n \log n)$.
  - Thus, the total running time is $\Theta(n \log n)$

Notes on Heapsort

- Heapsort is not stable. Find an example that shows Heapsort is not stable.

- Heapsort is not in-place. we need extra space for the heap. Can we make it in-place?

SUMMARY

- Heapsort
  - Algorithms
  - Illustrating Example
- Time Complexity Analysis