Complexity

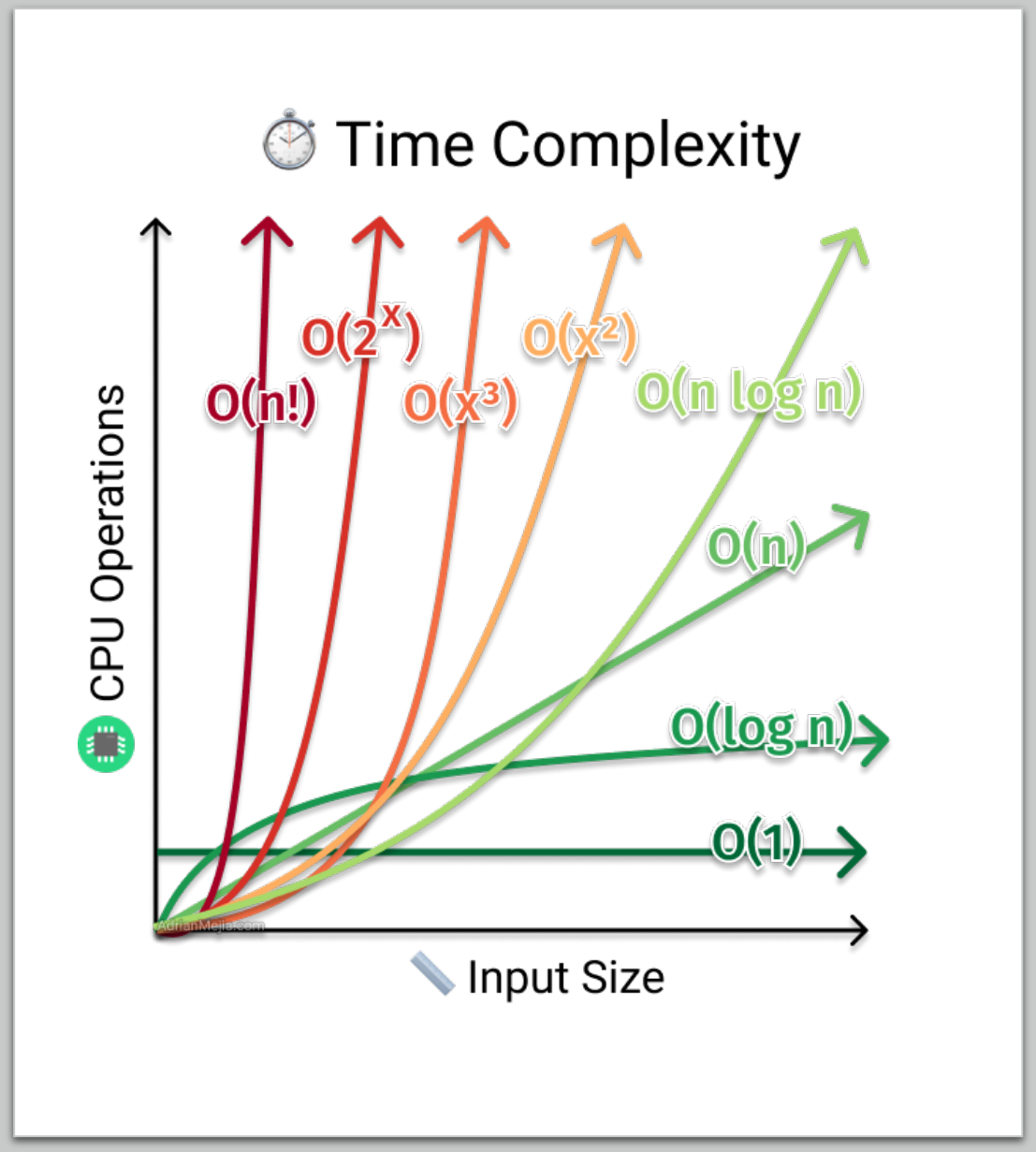
Instructor: Meng-Fen Chiang

COMPCSI220: WEEK 8

Slides adapted from Mark Wilson, Georgy Gimel'farb, Simone Linz and Tanya Gvozdeva

OUTLINE

- Analysis of Algorithm
  - Illustrative examples
- Time Complexity
  - How to measure running time?
  - Illustrative examples
- Asymptotic Notation



Analysis of Algorithms

- What to analyze
  - Domain of definition – what inputs are legal?
  - Correctness – does it solve the problem for all legal inputs?

- Efficiency: its maximum or average requirements to resources:
  - Runtime
  - Memory space
  - Other resources

- There could be different implementations of the same algorithm: different programs, programming languages, computer platforms, operating systems, etc.
- The analysis should be isolated from a particular implementation.

Complexity

- In a practical sense, complexity is an estimate of resource requirement
  - How long would it take to drive from Auckland to Wellington?
  - How much petrol would we need to drive from Auckland to Wellington?

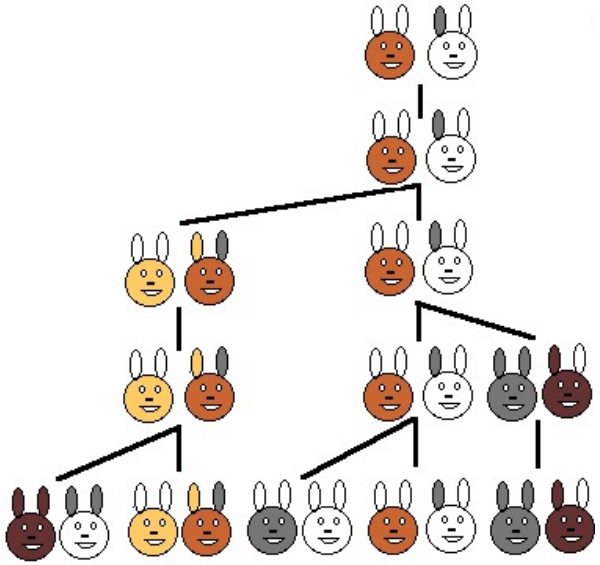
Complexity (Contd.)

- Time complexity
  - An estimate of how long it would take to complete a task
- Space complexity
  - An estimate of how much memory the task might require for completion

Example: Fibonacci Numbers

- Italian mathematician, Leonardo Fibonacci (1170–1250). A problem of breeding rabbits.
  - A pair of rabbits takes a month to become mature and start to have pairs of baby rabbits
  - Each pair of newly born rabbits also take a month to reach maturity.
  - How many pairs of rabbits, $F(n)$ would there be after n months?

Month	#Pairs
1	1
2	1
3	2
4	3
5	5



Fibonacci numbers:

$$F(n) = F(n-1) + F(n-2)$$
$$F(0) = 0, \quad F(1) = 1$$

This immediately suggests a recursive algorithm

http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibnat.html

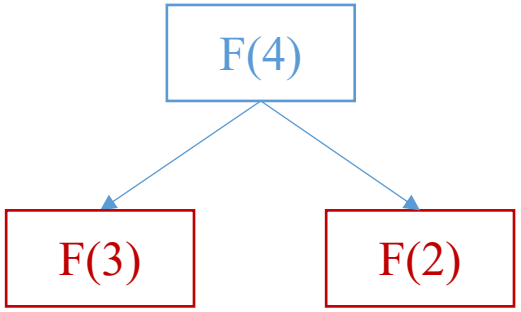
Algorithm 1 Slow method for computing Fibonacci numbers

```
1: function SLOWFIB(integer n)
2:         if n < 0 then return 0
3:         else if n = 0 then return 0
4:         else if n = 1 then return 1
5:         else return SLOWFIB(n − 1) + SLOWFIB(n − 2)
```

Correctness: The algorithm SLOWFIB is obviously correct

Efficiency: This algorithm is not efficient! It does a lot of repeated computation

Example – F(4)



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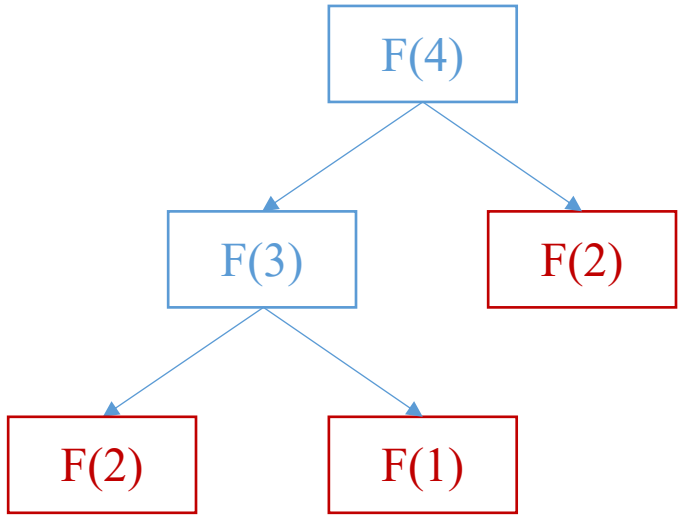

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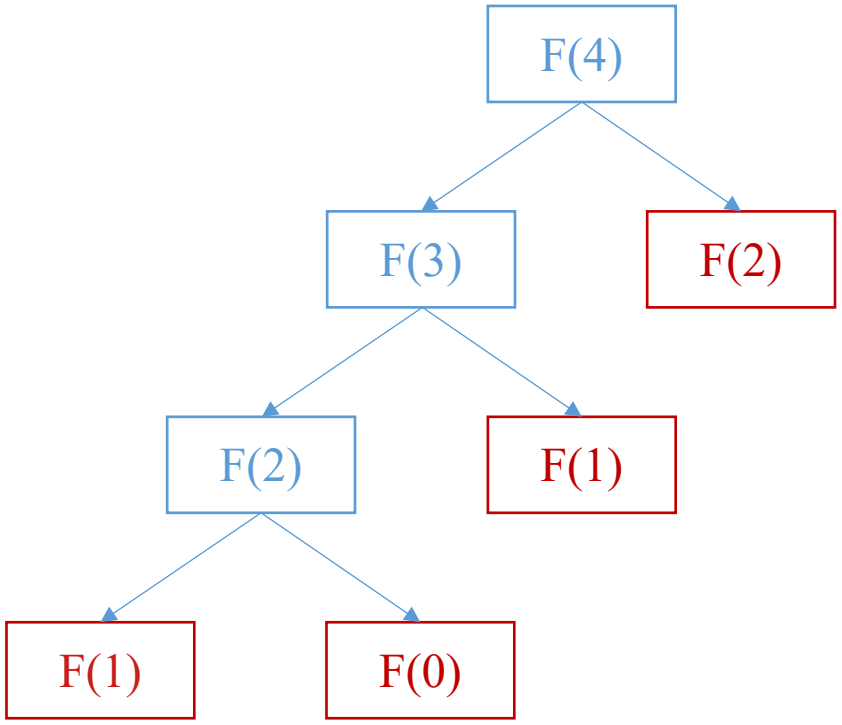

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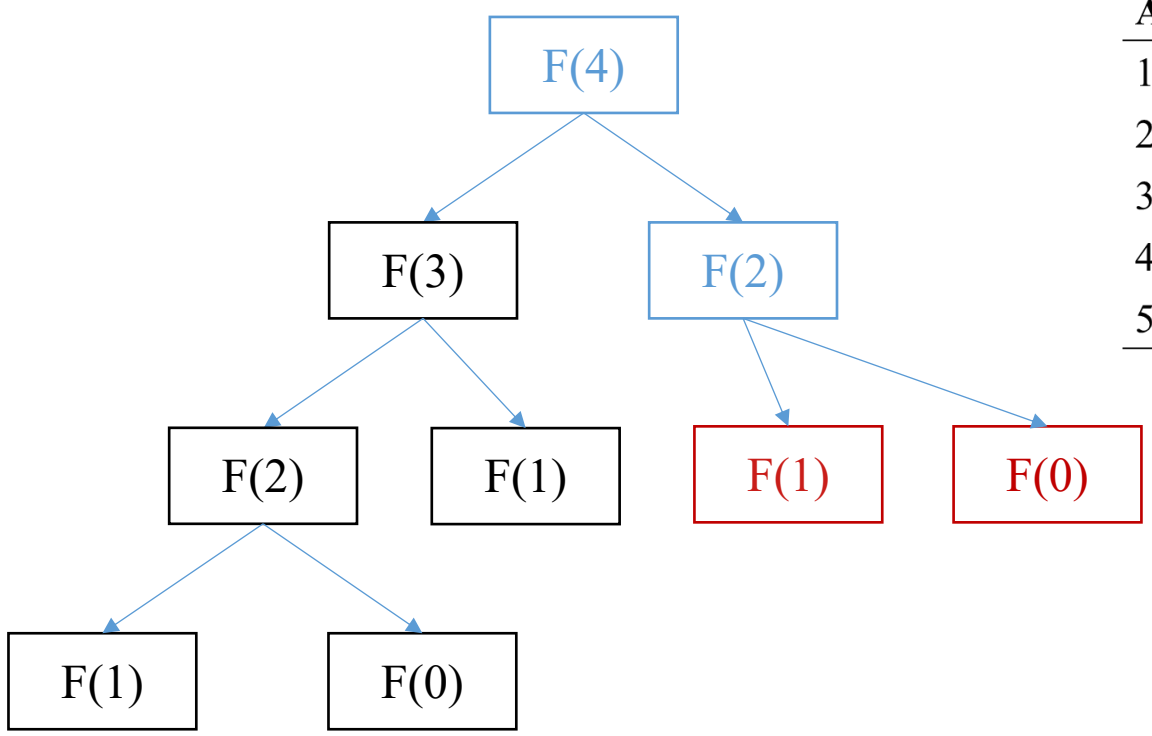

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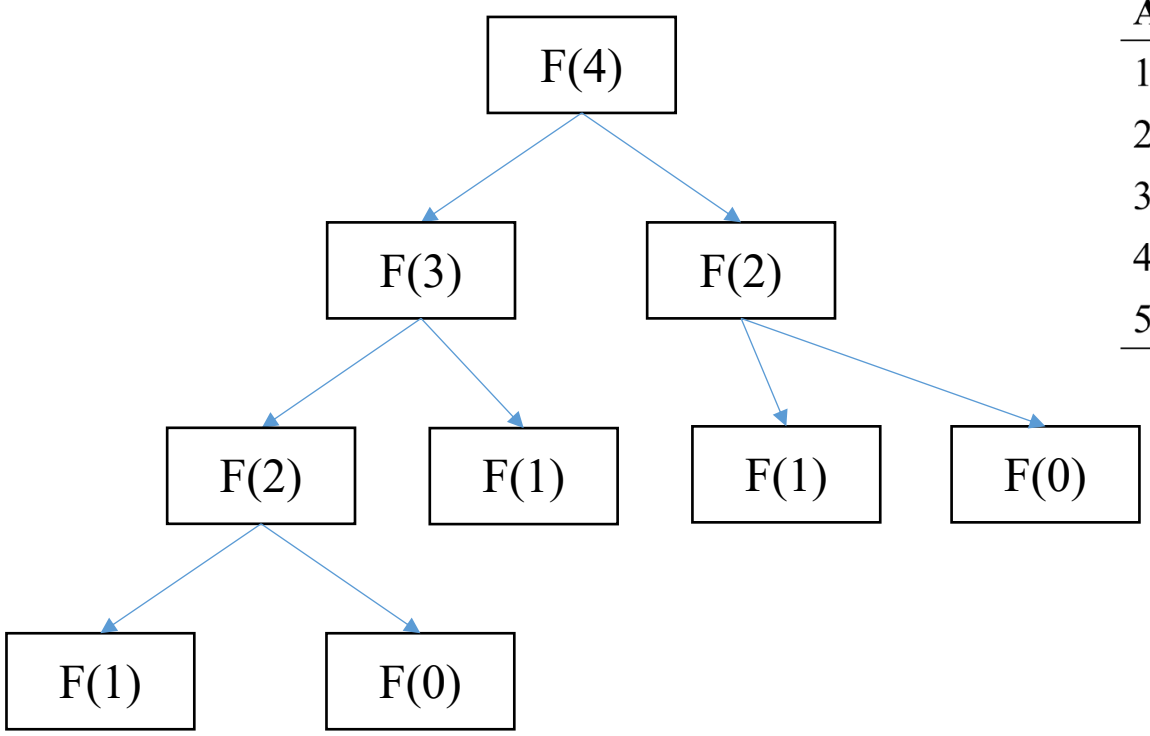

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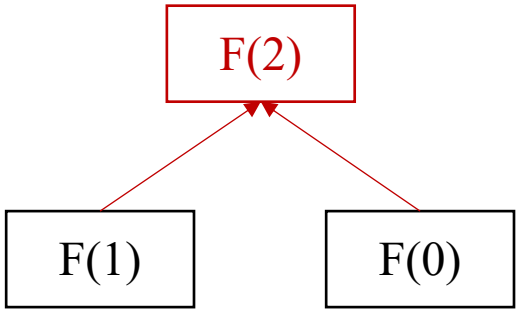
- With a small (fixed) amount of extra space, we can do better, by working from the bottom up instead of from the top down.

Algorithm 2 Fast method for computing Fibonacci numbers

```
1: function FASTFIB(integer n)
2:         if n < 0 then return 0
3:         else if n = 0 then return 0
4:         else if n = 1 then return 1
5:         else
6:             a ← 1                    ▷ stores F(i − 1) at bottom of loop
7:             b ← 0                    ▷ stores F(i − 2) at bottom of loop
8:             for i ← 2 to n do
9:                 t ← a
10:                a ← a + b
11:                b ← t
12:        return a
```

Example – F(4)



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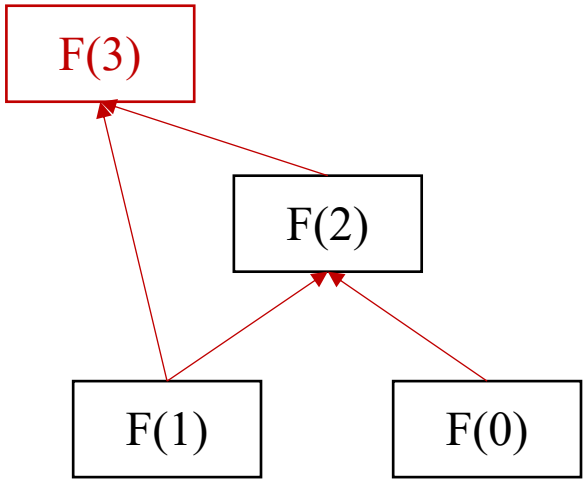

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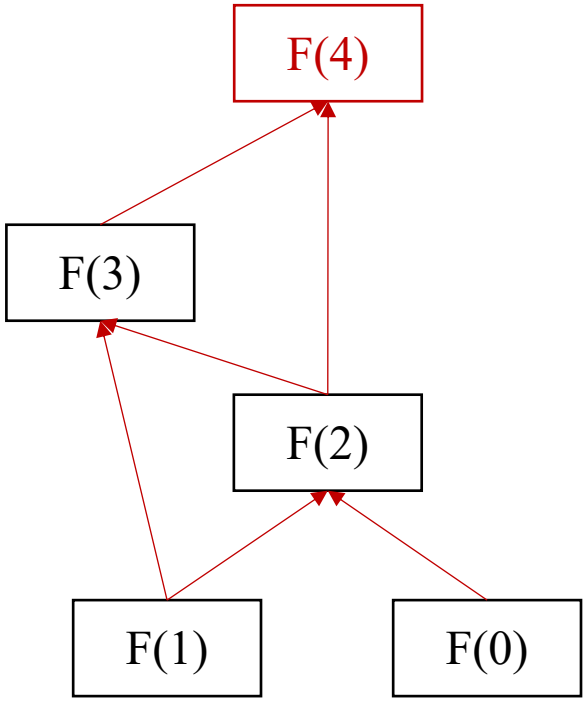

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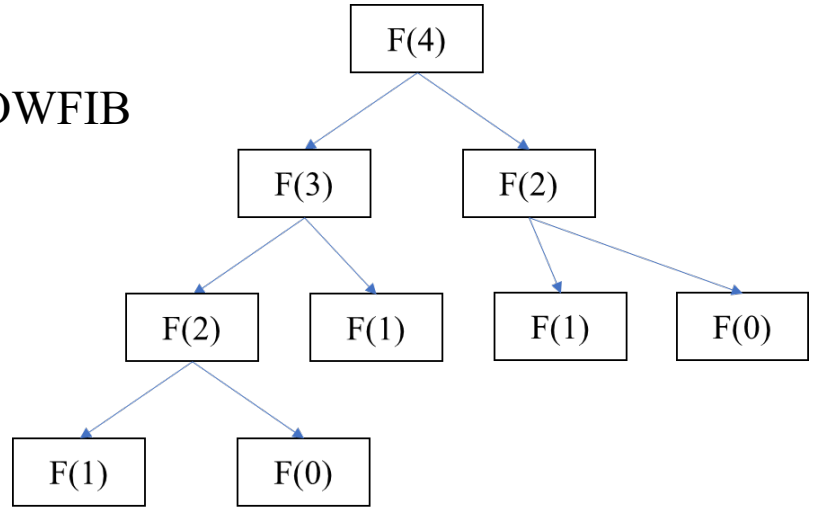
```

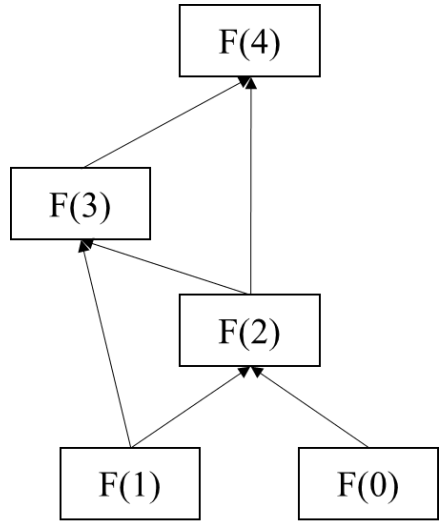
Analysis of the Fast Algorithm

- It is easy to see that the number of additions, function calls, etc needed by **FASTFIB** to compute $F(n)$ has the form $An + B$ for some constants A,B.

SLOWFIB



FASTFIB

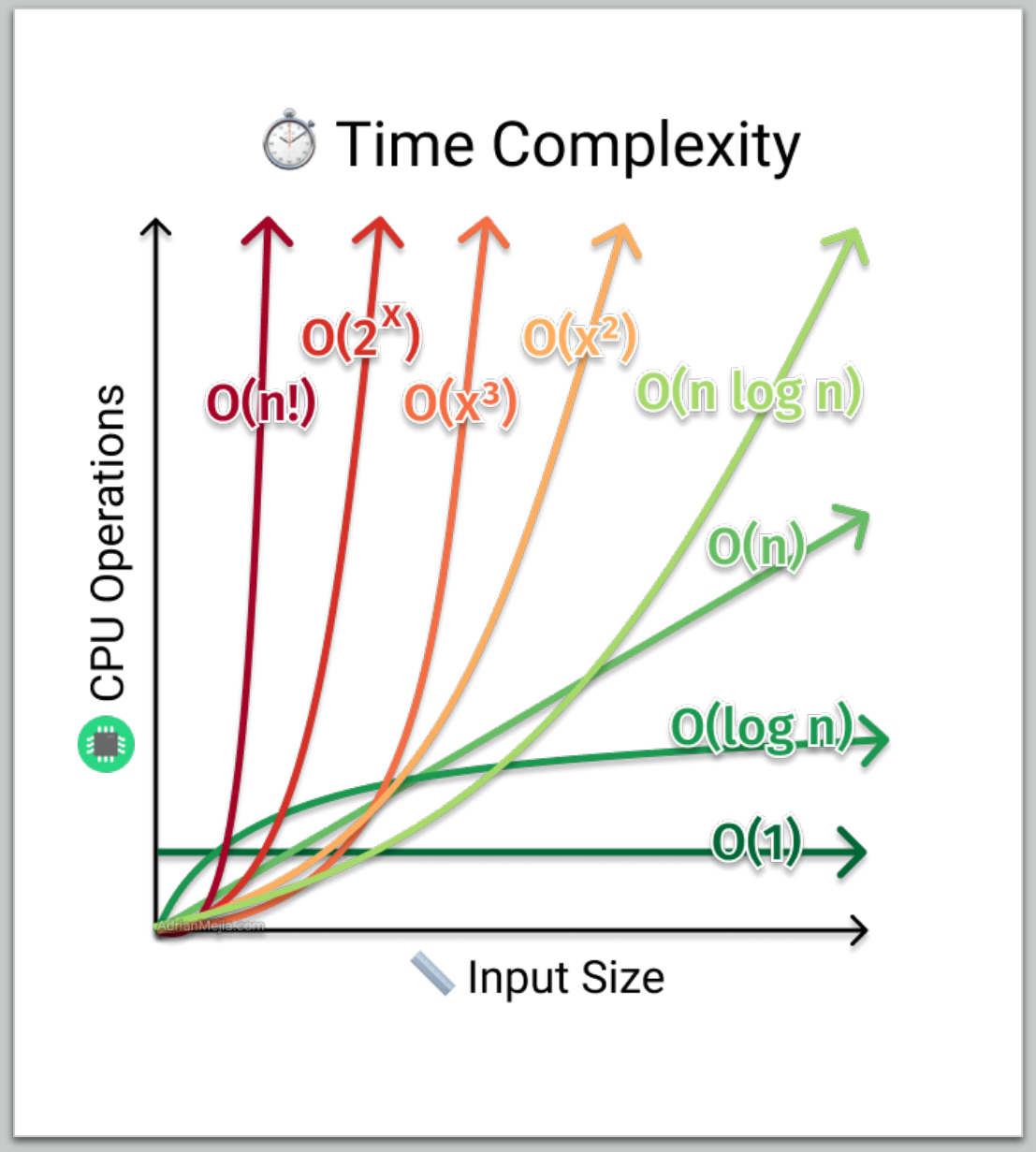


SUMMARY

- Analysis of Algorithm: Fibonacci numbers
  - Slow implementation
  - Fast implementation

- **Time Complexity**
  - **How to measure running time?**
  - **Illustrative examples**

- Asymptotic Notation



Time Complexity

- The actual running time of an algorithm $\mathcal{A}$ on a given input X depends on implementation, such as programming languages used, operating systems and hardware.

- The running time usually grows with the size of the input. Running time for very small inputs is not usually important; it is large inputs that cause problems if the algorithm is inefficient.

Elementary Operations

- We use the concept of elementary operation as our basic measuring unit of running time. This is any operation whose execution time does NOT depend on the size of the input.

- Typically, a standard unary or binary arithmetic operation:
  - Negation (−5)
  - Addition / subtraction (5 + 37; 350 − 75)
  - Multiplication / division / modulo (67 × 89; 399/54; 399%54)
  - Boolean operations (x AND y; x OR y, etc.)
  - Binary comparisons (x == y; $x \leq y$; $x < y$; $x \geq y$; etc.)
  - variable assignments, function calls.

- The running time $T(n)$ of algorithm $\mathcal{A}$ on input X of size n is the number of elementary operations used when X is fed into $\mathcal{A}$.

Time Complexity

Running time $T(n)$		**Input size of n**			
Function	*Notation*	10	100	1000	10^7
Constant	1	1	1	1	1
Logarithmic	$\log n$	1	2	3	7
Linear	n	1	10	100	10^6
"Linearithmic"	$n \log n$	1	20	300	7×10^6
Quadratic	n^2	1	100	10000	10^{12}
Cubic	n^3	1	1000	10^6	10^{18}
Exponential	2^n	1	10^{27}	10^{298}	$10^{3010296}$

Constant Time Complexity

- Algorithm: Swapping two elements
- This is a constant time algorithm

Algorithm 1 Swapping two elements in an array

1: **Require:** $0 \leq i \leq j \leq n-1$
2: **function** SWAP(array $a[0 \ldots n-1]$, integer i, integer j)
3: $t \leftarrow a[i]$
4: $a[i] \leftarrow a[j]$
5: $a[j] \leftarrow t$
6: **return** a

Linear Time Complexity

- Algorithm: finding the maximum value
- This is a linear time algorithm, since it makes one pass through the array and does a constant amount of work each time.

Algorithm 2 Finding the maximum in an array

```
1: function FINDMAX(array a[0 ... n − 1])
2:     k ← 0                                  ▷ location of maximum so far
3:     for j ← 1 to n − 1 do
4:         if a[k] < a[j] then
5:             k = j
6:     return k
```

Logarithmic Time Complexity

- Algorithm: : loop increments
- This runs in logarithmic time because i doubles about log n times until reaching n.

Algorithm 3 Example: exponential change of variable in loop

```
1: i ← 1
2: while i ≤ n do
3:         i ← 2 * i
4:         print i
```

Quadratic Time Complexity

- Algorithm: nested loops
- This runs in quadratic time, because the inner loop runs $n-i+1$ times, while the outer loop runs n times. That is:

$$n + (n-1) + (n-2) + \cdots + 1 = \frac{(n+1)n}{2} = \frac{1}{2}n^2 + \frac{1}{2}n$$

Algorithm 4 Snippet: Nested loops

```
1: for i ← 1 to n do
2:     for j ← i to n do
3:         print i + j
```

Exponential Time Complexity

- Algorithm: SLOWFIB
- SLOWFIB makes $F(n)$ function calls each of which involves a constant number of elementary operations. It turns out that $F(n)$ grows exponentially in n, so this is an exponential time algorithm.

Algorithm 5 Slow method for computing Fibonacci numbers

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3:     else if n = 0 then return 0
4:     else if n = 1 then return 1
5:     else return SLOWFIB(n − 1) + SLOWFIB(n − 2)
```

Example: Time Complexity Analysis

- Algorithm: Sum of an array $s = \sum_{i=0}^{n-1} a[i]$
- Summing n elements of the array a repeats elementary fetch/add operations n times.

Algorithm 6 Summing elements of an array $a[0 \ldots n-1]$

```
1: Require: array a
2:     function SUM(a[0 ... n − 1])
3:         s ← 0
4:         for i ← 0 to n − 1 do
5:             s ← s + a[i]
6:         return s
```

Running time is linear in n, i.e., $T(n) = cn$

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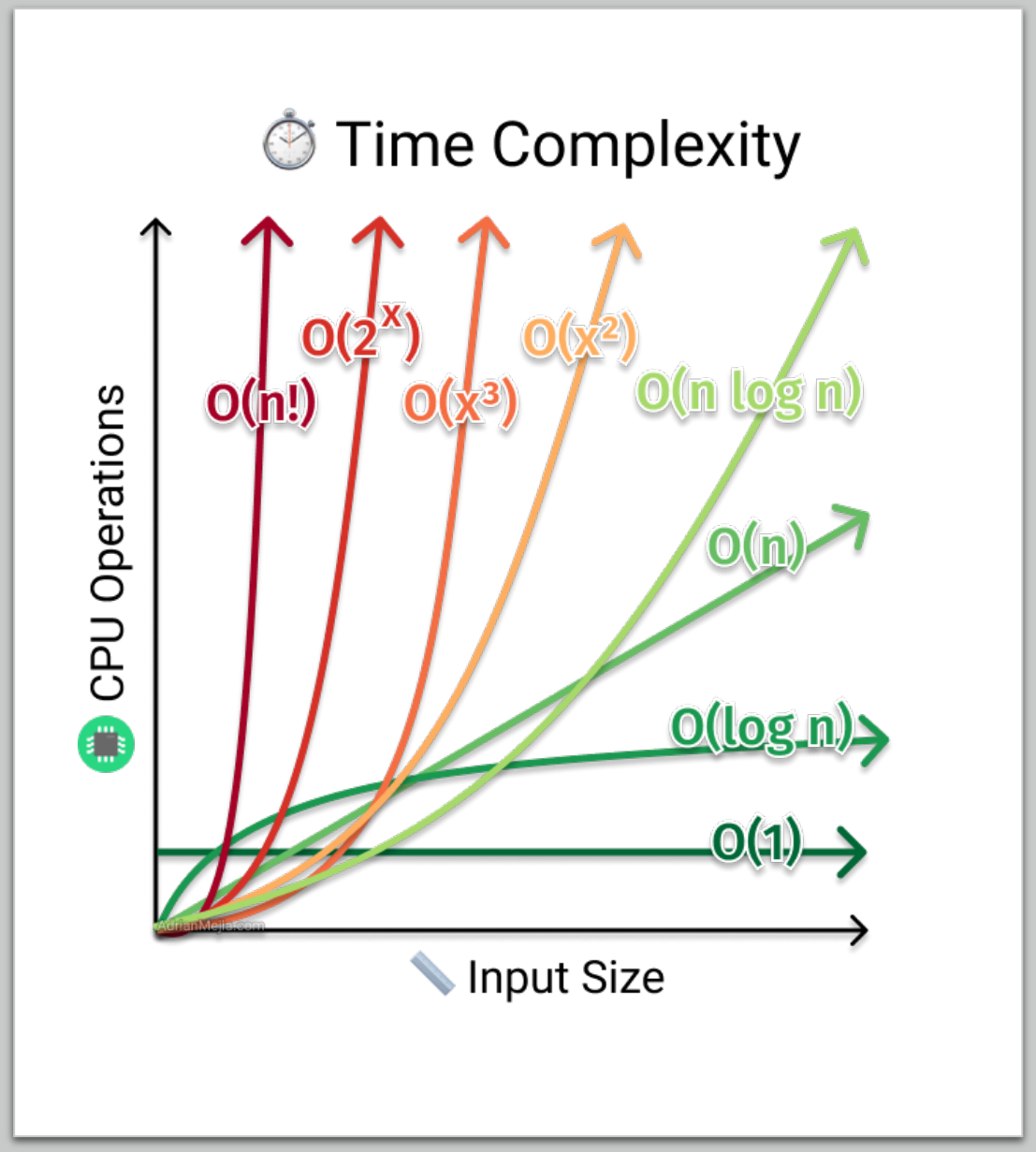
SUMMARY

- Analysis of Algorithm: Fibonacci numbers
  - Slow implementation
  - Fast implementation

- Time Complexity
  - How to measure running time?
  - Illustrative examples

- **Asymptotic Notation (O, Ω, Θ)**
  - **Asymptotic Comparison**
  - **Asymptotic Bound**



Asymptotic Notation

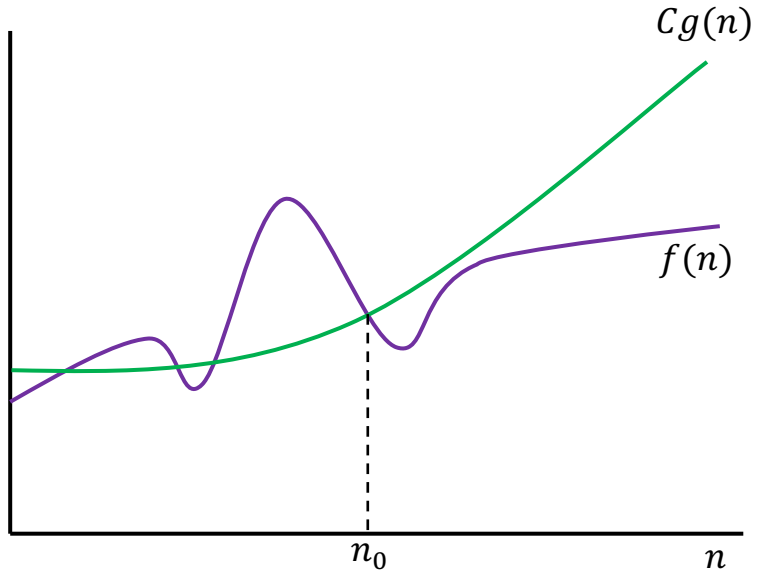
- In order to compare running times of algorithms we want a way of comparing the growth rates of functions.
- We want to see what happens for large values of n– small ones are not relevant.
- We are not usually interested in constant factors and only want to consider the dominant term.
- The standard mathematical approach is to use asymptotic notation $\boldsymbol{O}, \boldsymbol{\Omega}, \boldsymbol{\Theta}$ which we will now describe.

Asymptotic Notation (Contd.)

- Suppose that f and g are functions from $\mathbb{N}$ to $\mathbb{R}$, which take on nonnegative values.

Say f is $O(g)$ ("f is **big-Oh** of g)" if there is some $C > 0$ and some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $f(n) \leq Cg(n)$. Informally, f grows **at most** as fast as g.

Say f is $\Omega(g)$ ("f is **big-Omega** of g") if g is $O(f)$. Or more formally if there is some $C > 0$ and some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $f(n) \geq Cg(n)$.
Informally, f grows **at least** as fast as g.



$f(n)$ is $O(g(n))$

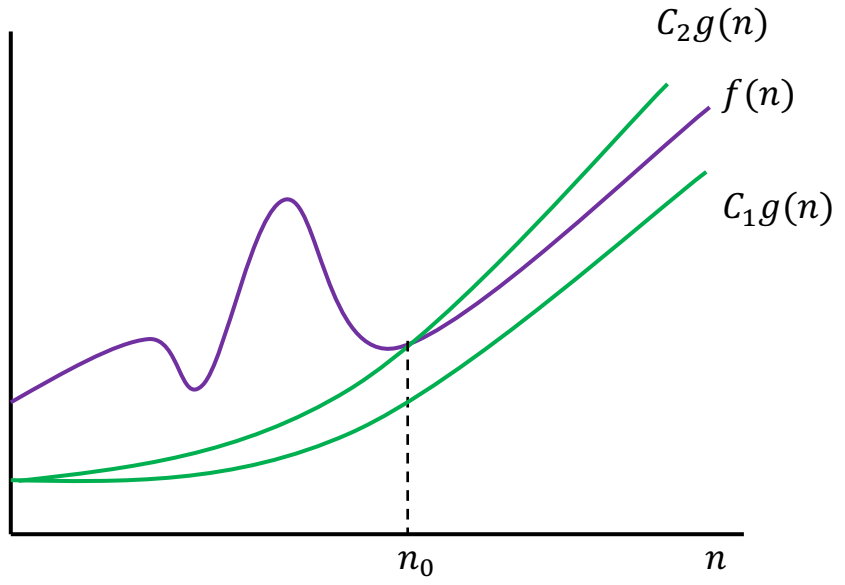
Asymptotic Notation (Contd.)

- Suppose that f and g are functions from $\mathbb{N}$ to $\mathbb{R}$, which take on nonnegative values.

Say f is $\Theta(g)$ ("f is **big-Theta** of g") if f is $O(g)$ and g is $O(f)$. Specifically, if there is some $C_1 > 0, C_2 > 0$, and some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $C_1 g(n) \leq f(n) \leq C_2 g(n)$.

Informally, f/g is bounded away from zero and infinity, and f grows **at the same rate** as g

Note that we could always reduce n_0 at the expense of a bigger C but it is often easier not to.



$f(n)$ is $\Theta\big(g(n)\big)$

Asymptotic Comparison

- Every linear function $f(n) = an + b, a > 0$, is $O(n)$.
- **Proof**: $an + b \leq an + |b| \leq (a + |b|)n$ for $n \geq 0$.

- If $f(n) = n; g(n) = n^2/2$, then f is $O(g)$ and g is not $O(f)$, so g grows asymptotically faster than f.
- **Proof**: $f(n) \leq 2 \cdot g(n)$ for $n \geq 1$ (or $f(n) \leq 1 \cdot g(n)$ for $n \geq 2$); conversely, suppose that eventually $\frac{1}{2}n^2 \leq Cn$. Then $n \leq 2C$ for all sufficiently large n, a contradiction.

Exercises: Asymptotic Comparison

- I have seen the description "$f = O(g)$ " before. Is this correct?

No. We can't say a function equals to a set of function! $O(g)$ describes a set of functions that grows no faster than g. For instance, both $f_1(n) = n$ and $f_2(n) = 2n$ are in $O(n)$.

But we can say $f \in O(g)$

Asymptotic Comparison (Contd.)

- Logarithmic functions grow slower than powers?

$$\log_a n \text{ is } O(n^k) \text{ for all } a > 1 \text{ and } k \geq 1$$

- **Proof:**
- First derivatives of $f(x) = \log_a x$ when $a > 1$ is $f'(x) = \frac{1}{x \ln a}$
- First derivatives of $g(x) = x^k$ by x is $g'(x) = kx^{k-1}$
- By limit rule and L'Hopital rule :

$$\lim_{n\to\infty} \frac{\log_a n}{n^k} = \lim_{n\to\infty} \frac{\frac{1}{n \ln a}}{kn^{k-1}}$$

$$= \lim_{n\to\infty} \frac{1}{k \ln a \, n^k} = 0$$

When $k > 0$ ⟹ $\log_a n \in O(n^k)$

Time Complexity

- **Informal definition**: A function $f(n)$ such that the running time $T(n)$ of a given algorithm is $\Theta(f(n))$ measures the time complexity of the algorithm.

- An algorithm is called polynomial time algorithm if its runtime $T(n)$ is $O(n^k)$, where k is a constant positive integer.

- A computational problem is called intractable if and only if no deterministic polynomial time algorithm can solve it.

Asymptotic Bounds

- Asymptotic notation measures the running time of the algorithm in terms of elementary operations, i.e., asymptotic bounds are independent of implementation:

- For a given problem, the running time varies NOT ONLY according to the size of the input, BUT ALSO the input itself.
  - E.g., sorting an already sorted array takes almost no time for some sorting algorithms

- Common measures:
  - the worst-case running time
  - the average-case running time

Example: "Find the position of an element x in array"
- Worst Case: x is not in the array
- Average Case: treat all possible positions equally distributed

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