Graph Data Structures

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COMPCSI220: WEEK 9

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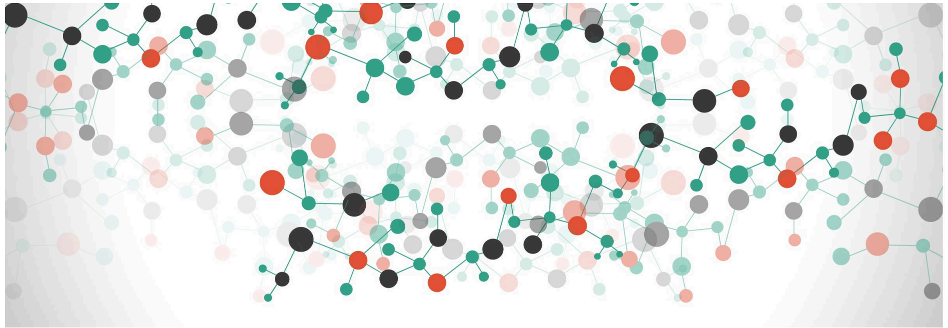
OUTLINE

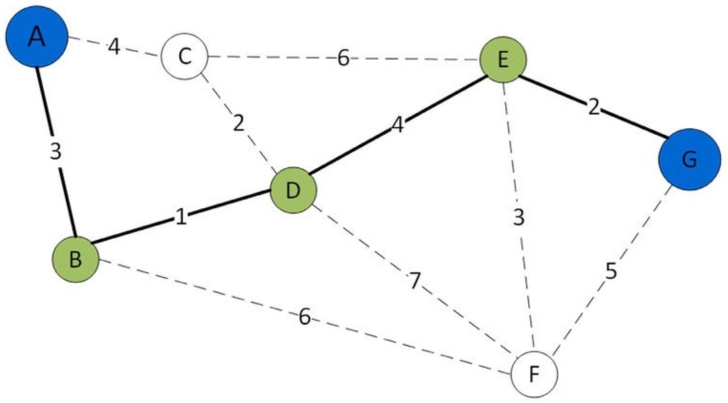
- Graph Data Structures
- Which to use?
- Graph Operations



Solving Graph Problems

There are lots of different problems defined on graphs we might like a computer to solve.

- List all nodes in order of their distance to a fixed node v.
- What is the nearest node to v with some property?
- Is there a path from u to v.
- What is the shortest path between u and v?
- How many different paths are there between u and v.



Storing a (Di)graph

- Fundamentally, computers don’t know what graphs are.
- All computers really know is a list of numbers (e.g. array).
- How would you write a graph algorithm in your favourite programming language?
- We need some ways to represent a graph which computers can make sense of.
- There are basically two types of representations.

Digraphs: Computer Representation

CONVENTION: For a digraph G of order n we label nodes $0,1,\ldots,n-1$:

The adjacency matrix of G:

The $n \times n$ boolean matrix (often encoded with 0's and 1's) such that its entry (i,j) is true if and only if there is an arc (i,j) from the node i to node j.

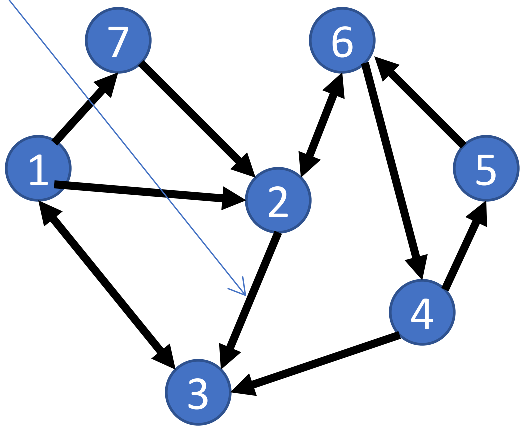
An adjacency list of G:

A list of n lists, $L_0,\ldots,L_{n-1}$, such that the list L_i contains all nodes of G that are adjacent to the node i.

Example: Adjacency Matrix of a Digraph

Arc from vertex 2 to 3

$$G = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & \boxed{1} & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



Arcs to out-neighbours of vertex 6 (vertices 2 and 4)

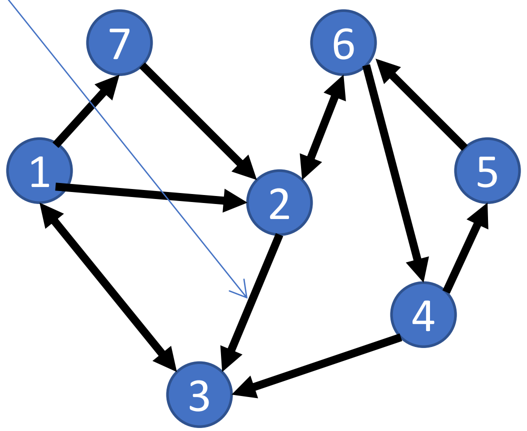
0 – a non-adjacent pair of vertices: $(i, j) \notin E$

1 – an adjacent pair of vertices: $(i, j) \in E$

Example: Adjacency List of a Digraph

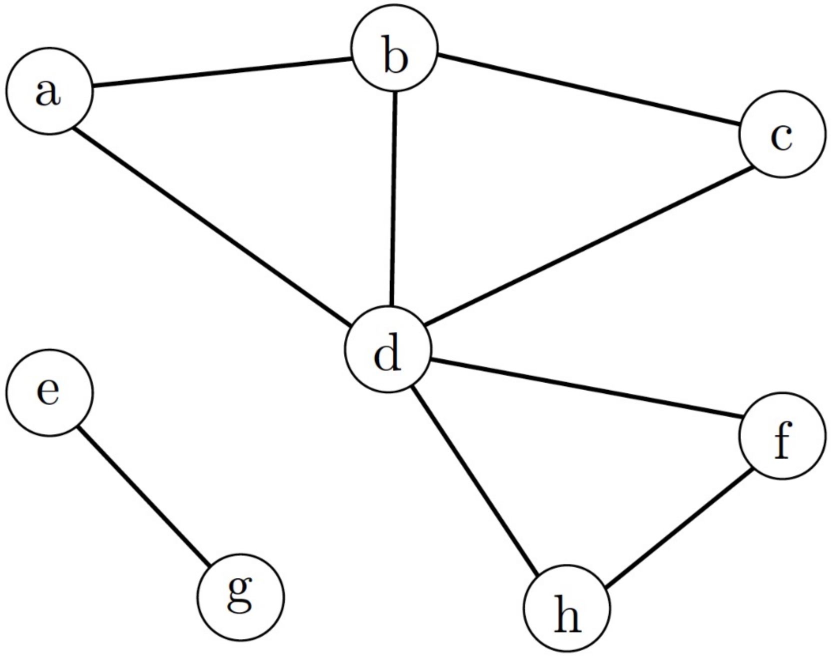
Arc from vertex 2 to 3

$$
\begin{aligned}
& L_0 = && 2\ (L_1), 3(L_2), 7 \\
& L_1 = && \boxed{3}, 6 \\
& L_2 = && 1 \\
G = {} & L_3 = && 3,5 \\
& L_4 = && 6 \\
& L_5 = && \boxed{2,4} \\
& L_6 = && 2
\end{aligned}
$$



Arcs to out-neighbours of vertex 6
(vertices 2 and 4)

Example: Adjacency List of a Graph



Graph $G = (V, E)$

symbolic	numeric
$0 = a$: $b\ d$	1 3
$1 = b$: $a\ c\ d$	0 2 3
$2 = c$: $b\ d$	1 3
$3 = d$: $a\ b\ c\ f\ h$	0 1 2 5 7
$4 = e$: g	6
$5 = f$: $d\ h$	3 7
$6 = g$: e	4
$7 = h$: $d\ f$	3 5

Numeric node labels in adjacency lists can omitted.

Representing multiple graphs in a single file

We can store several digraphs one after the other in a single file as follows:

- Use one line to store the order at the beginning of each digraph.
- If the order is n then the next n lines give the adjacency matrix or adjacency lists representation of the digraph. If we use numeric labels, node labels can be omitted.
- The end of the file is marked with a line denoting a digraph of order 0.

Example: Representing multiple graphs in a single file

- The two digraphs on the left could be put in a single file:

```
0 | 2
1 | 0
2 | 1
```

```
0 | 1  2
1 | 3
2 |
3 | 0  1  2
```

```
3
2
0
1
4
1  2
3

0  1  2
0
```

Other structures to represent graphs

We have already seen a way to represent a binary heap as the array. Here is a way of storing a general tree in an array.

- A general rooted tree of n nodes can be stored in array $pred$ of size n.
- $\boldsymbol{pred}[\boldsymbol{i}]$ is the parent of node i.
- The root has no parent, so assign it $null$ or 1 if we number nodes from 0 to n−1 in the usual way.
- This is a form of adjacency lists, using in-neighbours instead of out-neighbours.

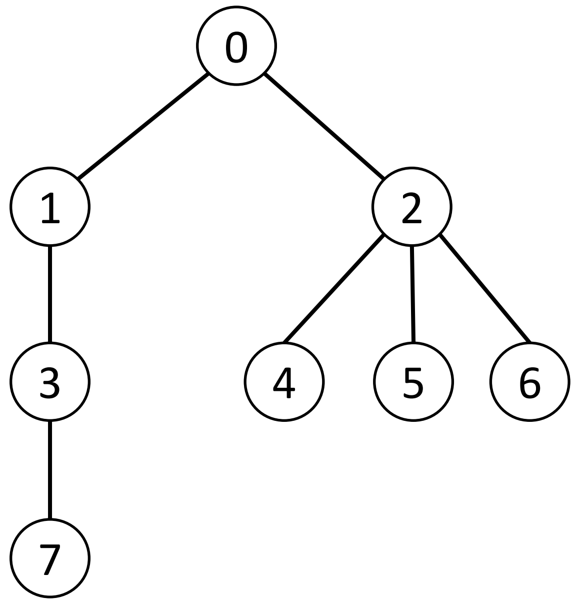
Example 21.8

- Draw the tree represented by the array

pred = [−1, 0, 0, 1, 2, 2, 2, 3]

0 1 2 3 4 5 6 7

The parent of 1



Implementation of digraph ADT

- An **adjacency matrix** is simply a matrix which is an array of arrays.
- **Adjacency lists** are a list of lists.
- There are several ways in which a list can be implemented, for example by an array, or singly- or doubly-linked lists using pointers.
- Depending on an implementation certain operations may have different running time.

Adjacency List or Matrix?

- Depends on three factors governing storage requirement and performance:

- The order of the (di)graph $|V(G)|$: the storage space we require for an adjacency matrix is $\Theta(\mathrm{n}^2)$.

- The size of the (di)graph $|E(G)|$: the storage space we require for an adjacency list is $\Theta(\mathrm{n}^2)$. However, in most practical cases, the graphs are sparse. E.g., if the in-/out-degree is limited, the space requirement is only $\Theta(n+e)$, where e is the number of edges.

- What we want to do with the graph (operations). This requires a bit more discussion.

Elementary Graph Operations

1. Does the arc (u,v) exist?
2. What is the outdegree of vertex u?
3. What is the indegree of vertex u?
4. Add an arc between two vertices u and v.
5. Delete the arc between vertices u and v.
6. Add a node to the (di)graph.
7. Delete a node from the (di)graph.

Q1: Does the arc (i, j) exist?

- In the adjacency matrix, we need to find the entry in row i, column j. This operation is equivalent to accessing an element in an array: $\Theta(1)$.

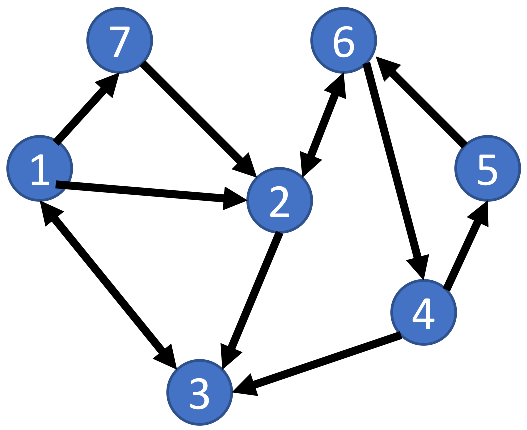
- In the adjacency list, the requires looking up the list ($\Theta(1)$) and then checking whether j is on that list. If d is the out-degree of vertex i, this will take $\Theta(d)$ in the worst case. The overall worst case scenario is thus $\Theta(d)$.

- The matrix wins!

Q1: Matrix Example

Arc from vertex 6 to 4

$$G = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

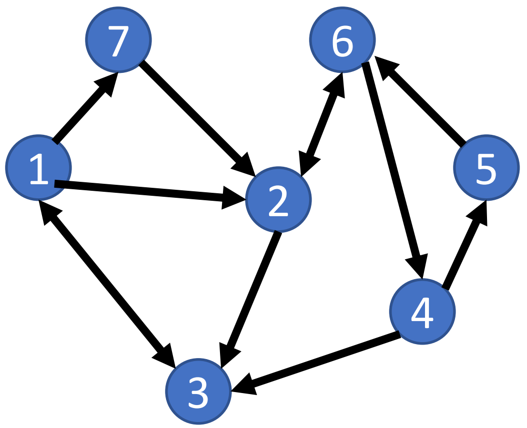


Arcs to out-neighbours of vertex 6
(vertices 2 and 4)

Q1: List Example

Arc from vertex 6 to 4

$$
G = \begin{array}{rl}
L_0 = & 2\ (L_1), 3(L_2), 7 \\
L_1 = & 3,6 \\
L_2 = & 1 \\
L_3 = & 3,5 \\
L_4 = & 6 \\
L_5 = & \boxed{2,4} \\
L_6 = & 2
\end{array}
$$



Arcs to out-neighbours of vertex 6
(vertices 2 and 4)

Q2: What is the out-degree of vertex i?

- In the adjacency matrix, we need to scan row i , and count the number of 1's we find. This operation is $\Theta(n)$.

- In the adjacency list, this requires looking up the size of the list L_i, which takes $\Theta(1)$ (if the list is properly implemented).

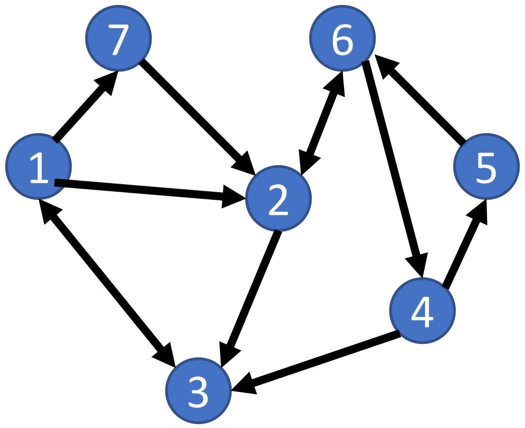
- The list wins!

- Note: This problem is equivalent to finding all the out-neighbours of i.

Q2: Matrix Example

Out-degree vertex 2

$$G = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ \boxed{0 & 0 & 1 & 0 & 0 & 1 & 0} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

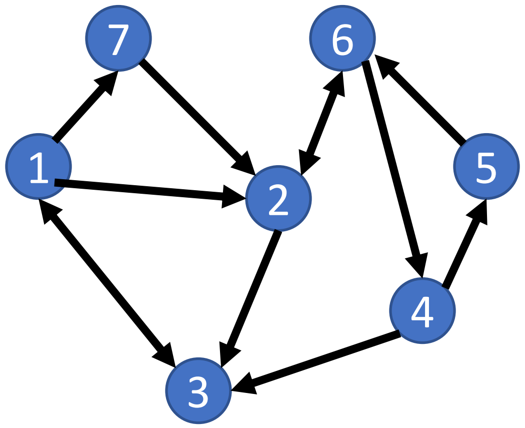


Scan array in row 2 and count “1”

Q2: List Example

Out-degree vertex 2

$$
G = \begin{aligned}
L_0 &= 2\,(L_1), 3(L_2), 7 \\
L_1 &= \boxed{3,6} \\
L_2 &= 1 \\
L_3 &= 3,5 \\
L_4 &= 6 \\
L_5 &= 2,4 \\
L_6 &= 2
\end{aligned}
$$



Get the size of these sequence

Q3: What is the in-degree of vertex i?

- In the adjacency matrix: Essentially the same operation as for the out-degree, except that we scan a column for u rather than a row: $\Theta(n)$.

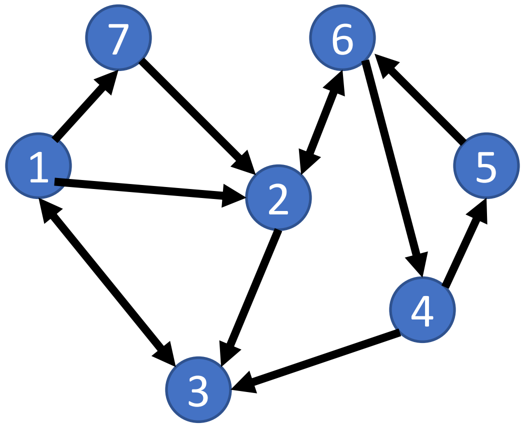
- In the adjacency list, this requires checking for all $j \neq i$ whether i is in the sequence: L_j. which takes $\Theta(n+e)$. Note: need to check each L_j even if the sequence is empty.

- The matrix wins – at least for dense (di)graphs!

Q3: Matrix Example

In-degree vertex 2

$$G = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

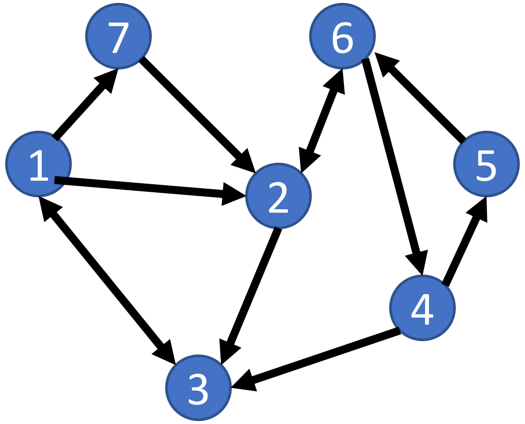


Scan array for column 2 and count “1”

Q3: List Example

In-degree vertex 2

$$
G = \begin{array}{rcc}
L_0 & = & 2\ (L_1), 3(L_2), 7 \\
L_1 & = & 3,6 \\
L_2 & = & 1 \\
L_3 & = & 3,5 \\
L_4 & = & 6 \\
L_5 & = & 2,4 \\
L_6 & = & 2
\end{array}
$$



Scan all the sequences and count the occurrence of “2”

Q4: Adding an arc from i to j

- For the adjacency matrix: change value of matrix element (i, j). This is $\Theta(1)$.

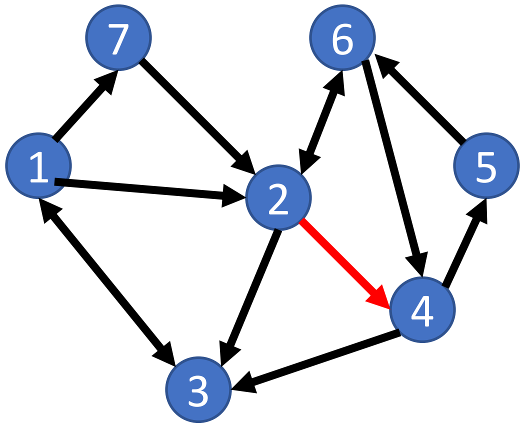
- For the adjacency list: Insert j into list i. This is also $\Theta(1)$.

- Matrix and list perform equally well here.

Q4: Matrix Example

Add arc (2,4)

$$G = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

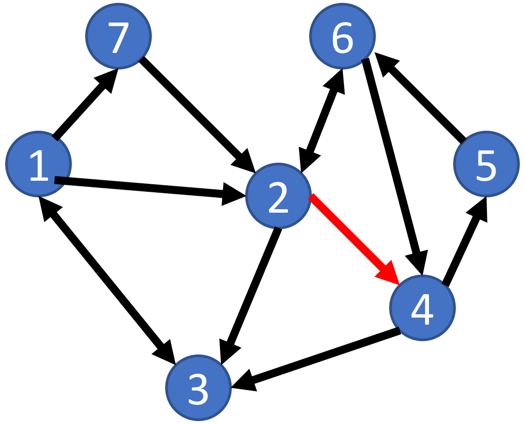


Put “1” in the second array in position 3

Q4: List Example

Adding an arc (2,4)

$$G = \begin{array}{rcl} L_0 &=& 2\,(L_1), 3(L_2), 7 \\ L_1 &=& \boxed{3,6,4} \\ L_2 &=& 1 \\ L_3 &=& 3,5 \\ L_4 &=& 6 \\ L_5 &=& 2,4 \\ L_6 &=& 2 \end{array}$$



Add 4 at the end of the sequence for node 2

Q5: Deleting an arc from i to j

- This problem is pretty much identical to the that of answering the question whether the arc exists

- In the adjacency matrix, rather than returning the value of (i, j), we set it to 0 (false). So we have $\Theta(1)$.

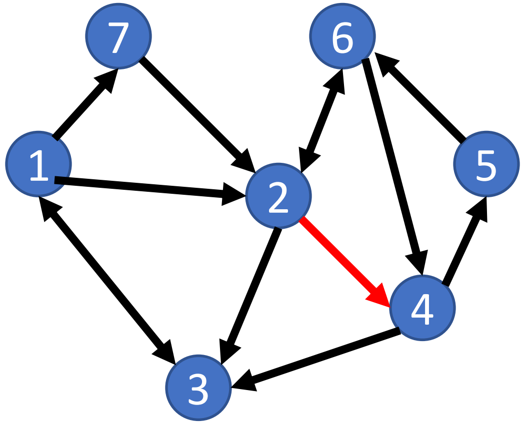
- In the adjacency list, we delete the appropriate list entry – constant time to locate the node in the list and up to d searches in the sub-list containing the arcs. Therefore, $\Theta(d)$.

- Matrix wins!

Q5: Matrix Example

Delete arc (2,4)

$$G = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

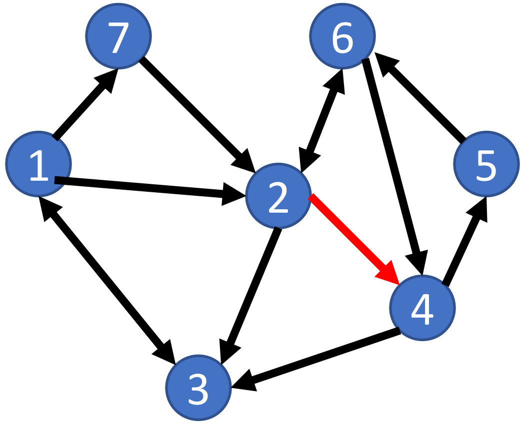


Put “0” in the second array in position 3

Q5: List Example

Delete arc (2,4)

$$G = \begin{array}{rcl} L_0 &=& 2\ (L_1), 3(L_2), 7 \\ L_1 &=& \boxed{3,6,4} \\ L_2 &=& 1 \\ L_3 &=& 3,5 \\ L_4 &=& 6 \\ L_5 &=& 2,4 \\ L_6 &=& 2 \end{array}$$



Scan the sequence for node 2 until you find node 4 and then delete it

Q6: Adding a vertex to the (di)graph

- In the adjacency matrix: Need to add $2n + 1$ entries (one row with n entries at the bottom and then one column with $n + 1$ entries at the end): $\Theta(n)$

- Think of a C/C++ implementation of this; and see if you can achieve $\Theta(n)$.

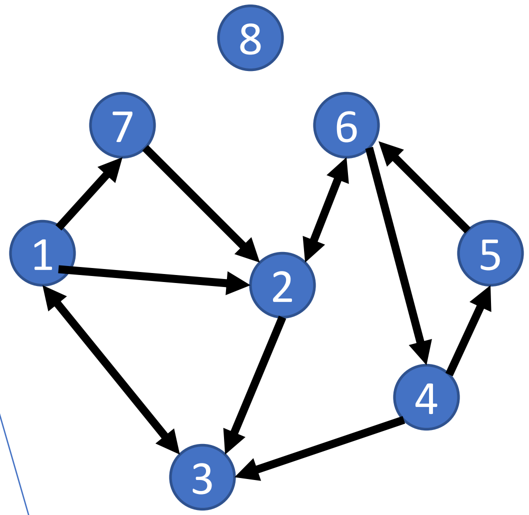
- In the adjacency list: Add one entry: $\Theta(1)$

- List wins!

Q6: Matrix Example

Adding vertex 8

$$G = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



Add 8 entries for column 8

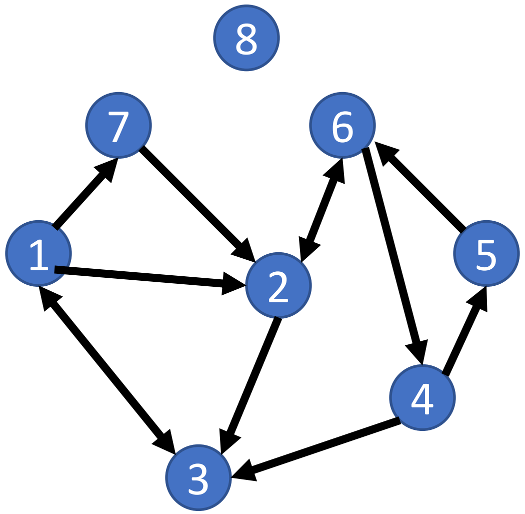
Add 7 entries for row 8

Q6: List Example

Adding vertex 8

$$
G = \begin{array}{rcc}
L_0 & = & 2\ (L_1), 3(L_2), 7 \\
L_1 & = & 3,6,4 \\
L_2 & = & 1 \\
L_3 & = & 3,5 \\
L_4 & = & 6 \\
L_5 & = & 2,4 \\
L_6 & = & 2
\end{array}
$$

$\boxed{L_7 = \quad}$



Create a new sequence for node 8

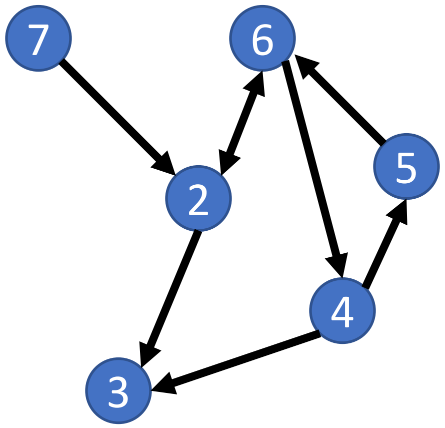
Q7: Deleting a vertex from the (di)graph

- Note that the vertex may have arcs attached!
- Matrix: Remove a row and a column, shifting up to $(n-1)$ elements up and, up to $(n-1)$ elements to the left: $\Theta(n^2)$. A simpler way to do this is to copy each required element of the original matrix to a new matrix of size $(n-1)^2$.
- List: Need to look at all n entries and need to check their sequences for the presence of the node that needs to be removed. The total number of sequence entries is the number of arcs, e, so the total time complexity here is $\Theta(n+e)$.
- In sufficiently sparse graphs, the list wins, but as $e \leq n(n-1)$ for digraphs and $e \leq n(n-1)/2$ for graphs, this isn't necessarily always so!

Q7: Matrix Example

Delete vertex 1 (worst case)

$$G = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

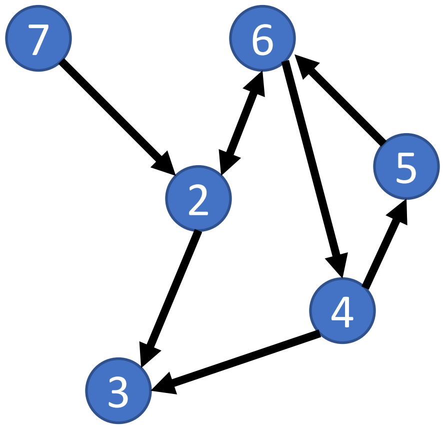


Need to shift 6 rows up (n-1)

Q7: Matrix Example

Delete vertex 1 (worst case)

$$G = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

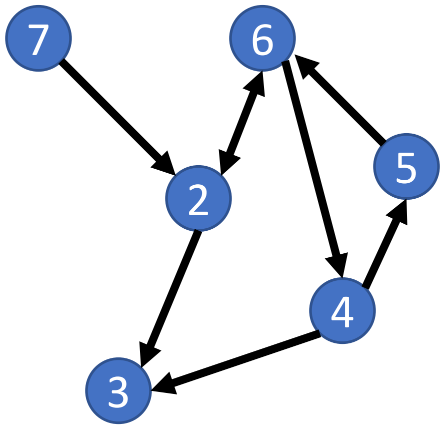


Need to shift 6 columns left (n-1)

Q7: List Example

Delete vertex 1

$$G = \begin{array}{l} L_0 = \\ L_1 = \\ L_2 = \\ L_3 = \\ L_4 = \\ L_5 = \\ L_6 = \end{array} \begin{array}{c} \boxed{2\ (L_1), 3(L_2), 7} \\ 3,6,4 \\ \cancel{1} \\ 3,5 \\ 6 \\ 2,4 \\ 2 \end{array}$$



Drop sequence for node 1; 3 no longer has an edge to 1

Digraph Operations w.r.t. Data Structures

Operation	**Adjacency Matrix**	**Adjacency Lists**
arc (i, j) exists?	is entry (i, j) 0 or 1	find j in list i
out-degree of i	scan row and sum 1's	size of list i
in-degree of i	scan column and sum 1's	for $j \neq i$, find i in list j
add arc (i, j)	change entry (i, j)	insert j in list i
delete arc (i, j)	change entry (i, j)	delete j from list i
add node	create new row/column	add new list at end
delete node i	Delete row/column i and shuffle other entries	delete list i and for $j \neq i$, delete i from list j

Adjacency Lists / Matrices: Comparative Performance

Operation	array/array	list/list
arc (i, j) exists?	$\Theta(1)$	$\Theta(\alpha)$*
out-degree of i	$\Theta(n)$	$\Theta(1)$
in-degree of i	$\Theta(n)$	$\Theta(n + m)$
add arc (i, j)	$\Theta(1)$	$\Theta(1)$
delete (i, j)	$\Theta(1)$	$\Theta(\alpha)$
add node	$\Theta(n)$	$\Theta(1)$
delete node i	$\Theta(n^2)$	$\Theta(n + m)$

Space Requirements

- The adjacency matrix representation requires $\Theta(n^2)$ storage as we simply need a matrix of n^2 bits.

- The adjacency list space requirement is $\Theta(n + m \log n)$.

Rule of Thumb

- Use adjacency matrix for **small, dense** (di)graphs for which we wish to test for the **existence of arcs, find the in-degree of vertices**, and/or delete arcs.

- Use adjacency list for **large, sparse** (di)graphs for which we need to **compute out-degree** and **add** and/or **delete vertices**.

SUMMARY

- Graph Data Structures
  - Adjacency matrix
  - Adjacency list
- 7 Elementary Graph Operations
  - Does the arc (u,v) exist?
  - What is the outdegree of vertex u?
  - What is the indegree of vertex u?
  - Add an arc between two vertices u and v.
  - Delete the arc between vertices u and v.
  - Add a node to the (di)graph.
  - Delete a node from the (di)graph
- Performance Comparison