Graphs and Directed Graphs (Digraphs)

Instructor: Meng-Fen Chiang

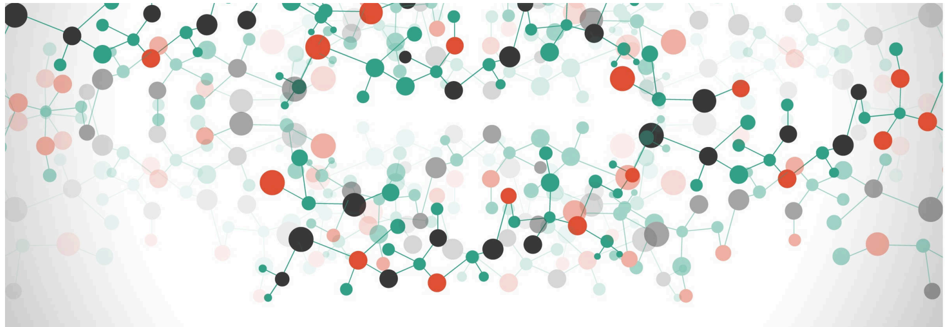
COMPCSI220: WEEK 9

THE UNIVERSITY OF AUCKLAND
Te Whare Wananga o Tamaki Makaurau
NEW ZEALAND

Slides adapted from Mark Wilson, Georgy Gimel'farb, Simone Linz, Tanya Gvozdeva, and Kaiqi Zhao

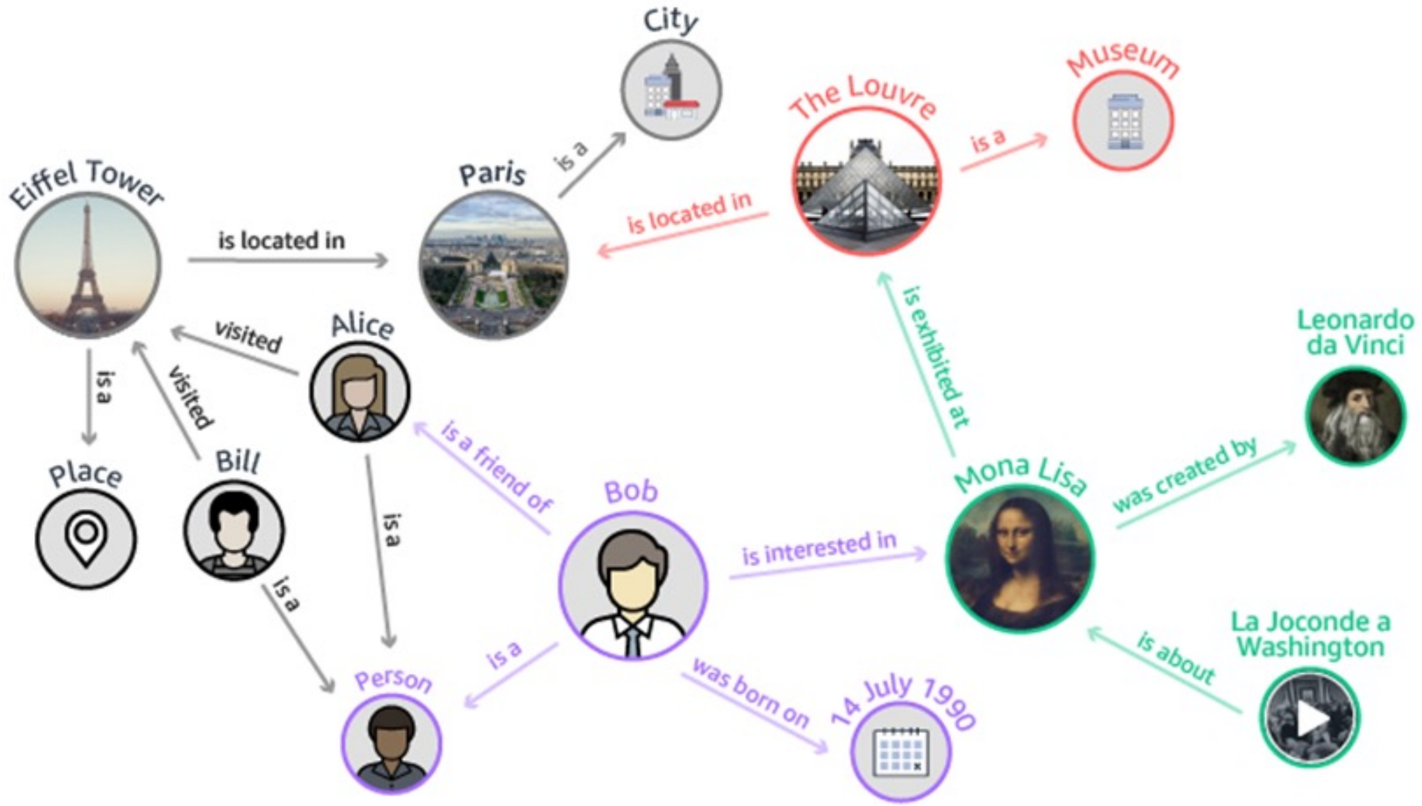
OUTLINE

- Graphs and Di-Graphs
- Preliminaries
- Basic operations



Graphs are Everywhere …

- Knowledge graphs: Commonsense reasoning



Yago

10M nodes

120M facts

DBPedia

4.58M nodes

3B facts

ConceptNet

300K nodes

1.6M facts

Graphs are Everywhere …

- Social networks: community detection, advertisement

Monthly Active Users (MAU)

Twitter

300 millions

Facebook

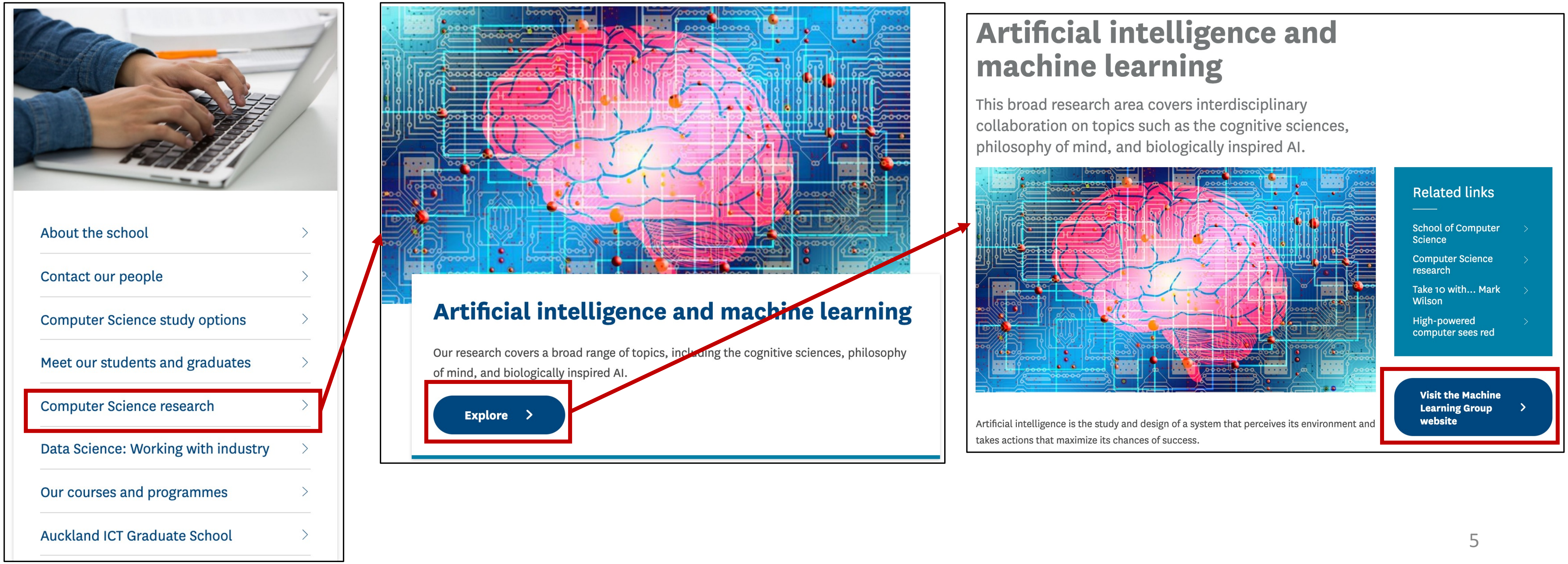
2.6 billions

Tiktok

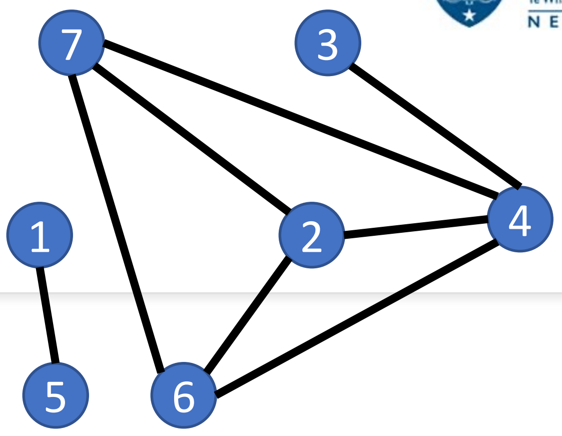
800 millions

Graphs are Everywhere …

- Web as a graph (4.2 billion webpages) – information retrieval
- Nodes: Webpages, Edges: Hyperlinks



Graphs

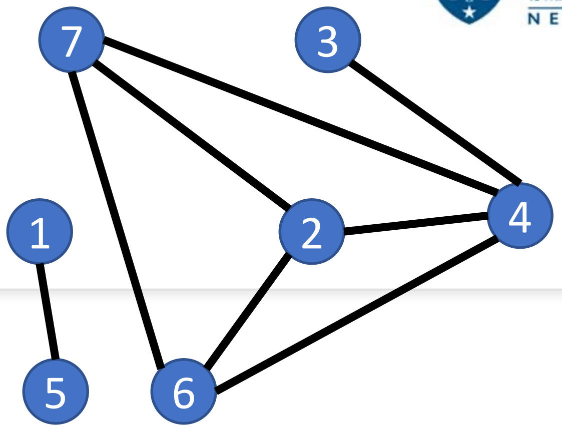


- A graph G consists of two sets V and E.

- V is the set of vertices (also called nodes). A vertex (not "vertice"!) is often graphically displayed as a point or junction point (like the seven little circles here).

- E is the set of edges. Each edge connects two vertices u and v in V. Edges have no direction – they equally go from u to v and from v to u. We denote an edge as (u, v) with the understanding that $(u, v) = (v, u)$. The graph here contains eight edges (the black lines).

Graphs (Contd.)



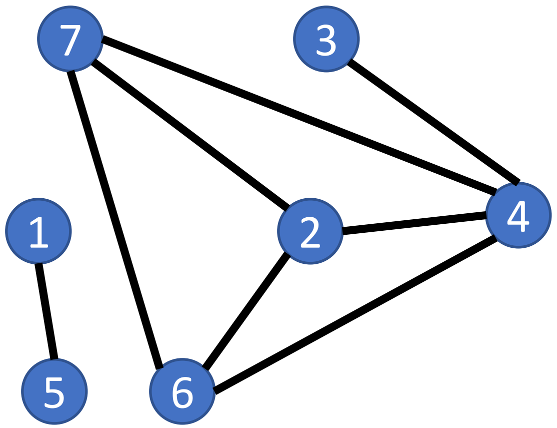
- If two vertices u and v are connected by an edge, they are said to be adjacent. We also say that u is a neighbour of v.

- It is not uncommon to number the vertices or otherwise label them uniquely. This is then called a labelled graph.

Graphs (Contd.)

- We also write $G(V, E)$ for the graph itself, or $E(G)$ for the edges, or $V(G)$ for the vertices.

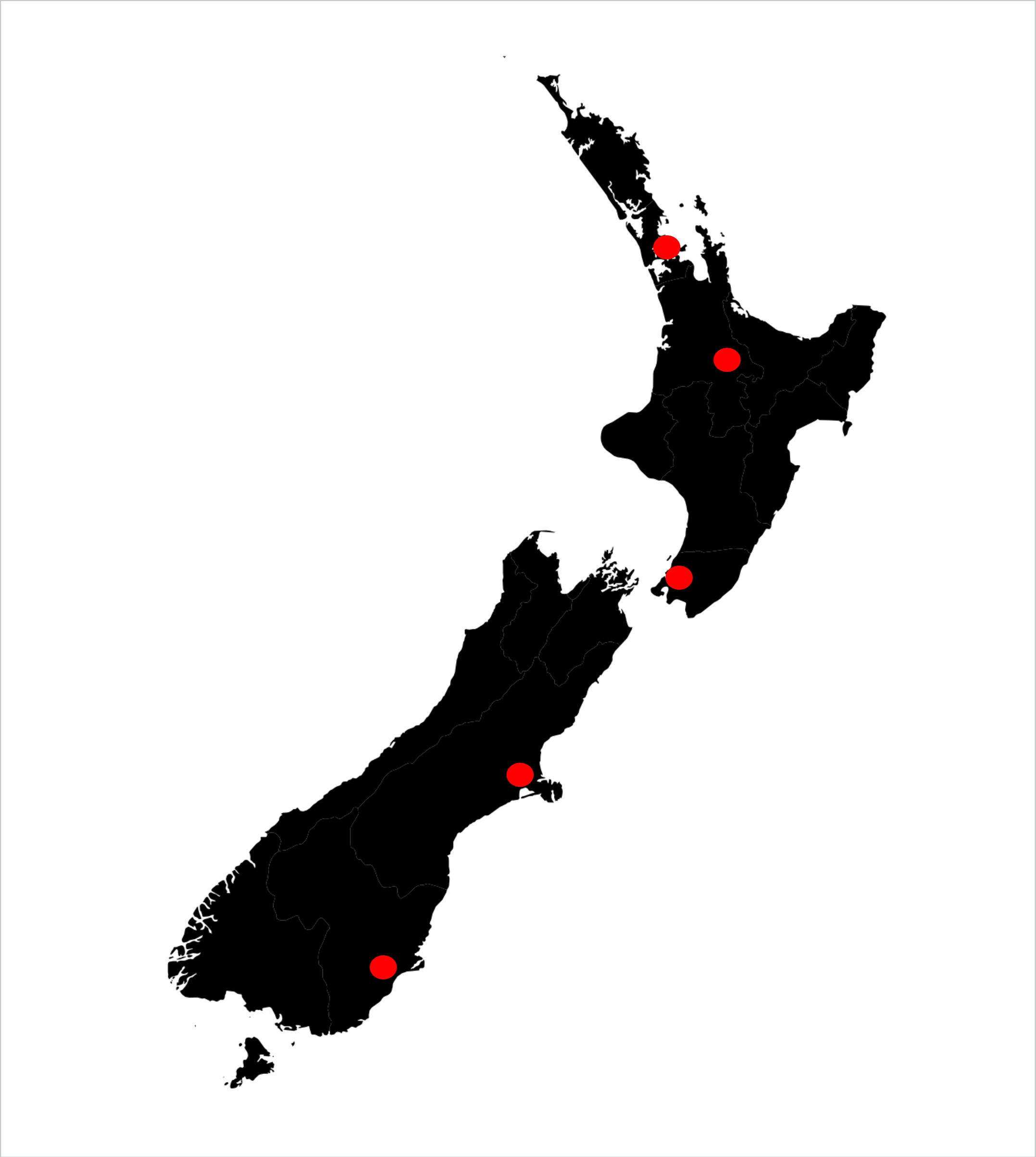
- Note
  - not every vertex of a graph needs to have an edge attached;
  - not every pair of vertices needs to be connected by an edge, and it's perfectly fine to have a graph with many vertices but no edges at all!



This is one graph, and it is a disconnected graph.

Example: Graph

- Road network does not connect North island cities with South island cities. We have a disconnected graph.

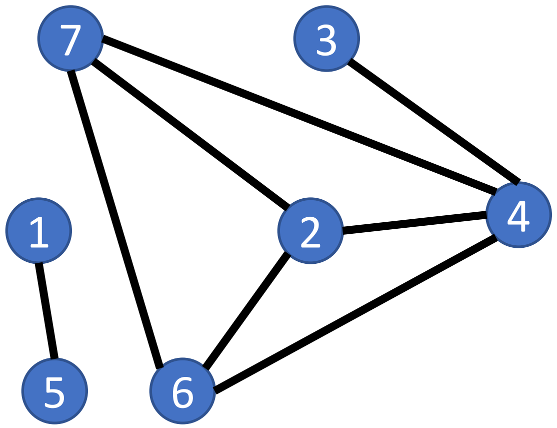


Graph: Order and Size

- The order of a graph G is the number of its vertices: $|\ V(G)\ |$

- The size of a graph G is the number of its edges: $|\ E(G)\ |$

- Remember that $|\ X\ |$ denotes the number of elements of a set X.



Graph: Order and Size (Contd.)

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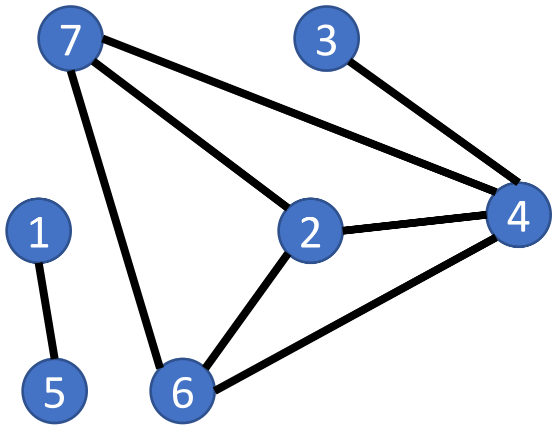


In this example:

$|\ V(G)\ | = ?$

$|\ E(G)\ | = ?$

How many edges can a graph have?

- **Observation 1**: At most one edge per pair of different vertices.
- **Observation 2**: There are $|V(G)|(|V(G)|-1)$ possible such vertex pairs (u, v), but there can be only one such edge for (u,v) and (v,u).
- So we need to divide this number by two:
- $0 \leq |E(G)| \leq |V(G)|(|V(G)|-1)/2$.

Sparse and Dense Graphs

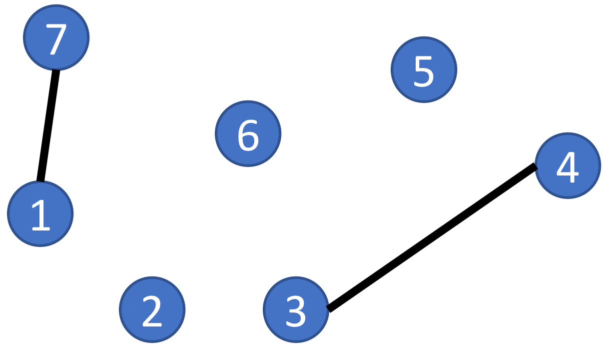
- If $|E(G)|$ approaches the high end of the scale $0 \leq |E(G)| \leq |V(G)|(|V(G)| - 1)/2$, we say that the graph is dense

- If $|E(G)|$ approaches the low end of the scale $0 \leq |E(G)| \leq |V(G)|(|V(G)| - 1)/2$, we say that the graph is sparse

- Knowing whether a graph that we are dealing with is dense or sparse makes a difference in how it is best stored when space efficiency is important.

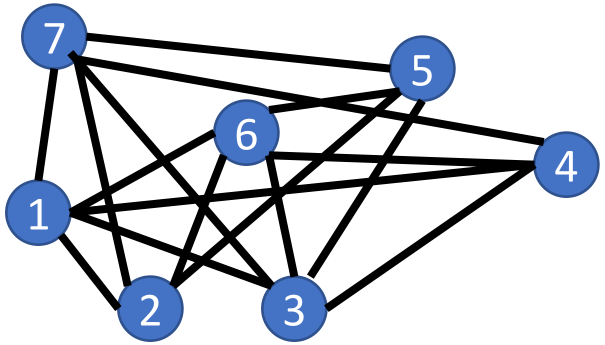
- There is no defined boundary between when we would call a graph sparse or dense – but the criterion of storage could help us decide if we wanted such a boundary.

Sparse and Dense Graphs



A sparse graph
(2 out of 21 possible edges)

Generally more efficient to store edges by recording between which vertices they occur

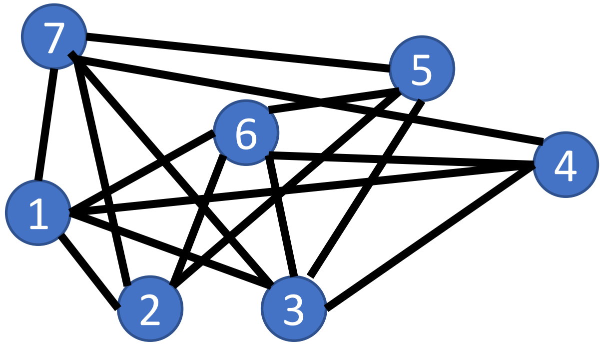


A dense graph
(16 out of 21 possible edges)

Generally more efficient to store edges by listing vertex pairs and recording between which pairs there are edges

Degree of a Node in a Graph

- The degree of a vertex v is the number of edges that terminate in this vertex
- **Observation 1**: The number of edges is the sum of the degrees of all vertices divide by 2
- **Observation 2**: In sparse graphs, the nodes tend to have a lower degree than in dense graphs.



Node 1: degree 5
Node 2: degree 4
Node 3: degree 5
Node 4: degree 4
Node 5: degree 4
Node 6: degree 5
Node 7: degree 5

OUTLINE

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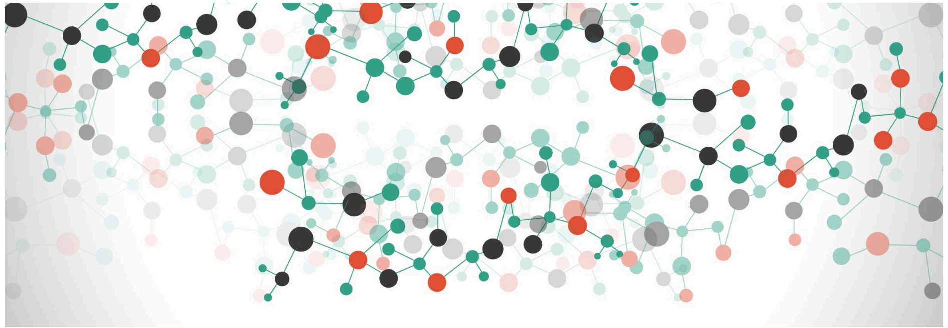
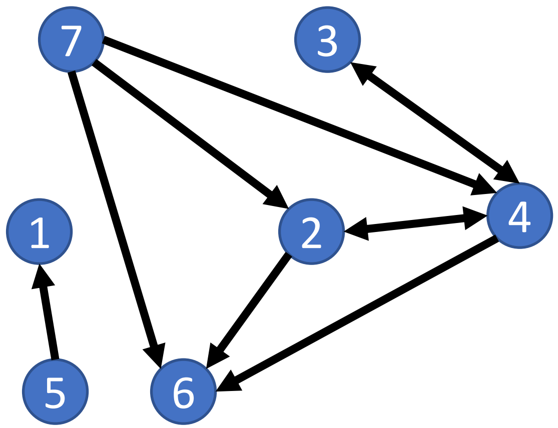
- Preliminaries
- Basic operations



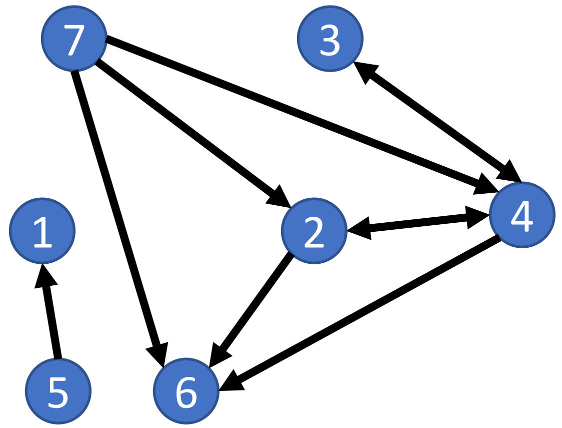
Directed Graphs

- The set E of edges is replaced by a set E of arcs.
- An arc has a direction. An edge runs between two vertices u and v, an arc runs from a vertex u to a vertex v
- Draws the arcs curved allows it to have them between two vertices in both directions. We use double-headed arrows here to indicate that two vertices are connected by arcs in both directions
- This also requires us to extend and amend our previous definitions and terminology (next slide).



Neighborhood

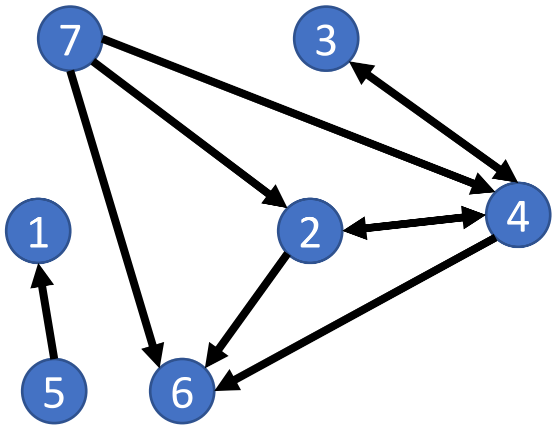
- There are **two sorts of neighbours**
  - A neighbour v of u is an out-neighbour of u if there is an arc from u to v.
  - A neighbour v of u is an in-neighbour of u if there is an arc from v to u.
- There are **two sorts of degrees**
  - The out-degree (number of out-neighbours) and
  - The in-degree (number of in-neighbours).
- Obviously, now $(u, v) \neq (v, u)$.
- Size of a digraph = number of arcs



- 1 is an out-neighbour of 5
- 2 is an in-neighbour of 6
- 2 is an in-neighbour and out-neighbour of 4 (and vice versa)
- 7 has in-degree 0 and out-degree 3
- 4 has in-degree 3 and out-degree 3
- Size is 10

Sources and Sinks

- A vertex with in-degree 0 is called a source (all arcs "flow out" from the vertex).
- A vertex with out-degree 0 is called a sink (all arcs "flow into" the vertex).
- Q: Is it possible to have a node that is both a source as well as sink?



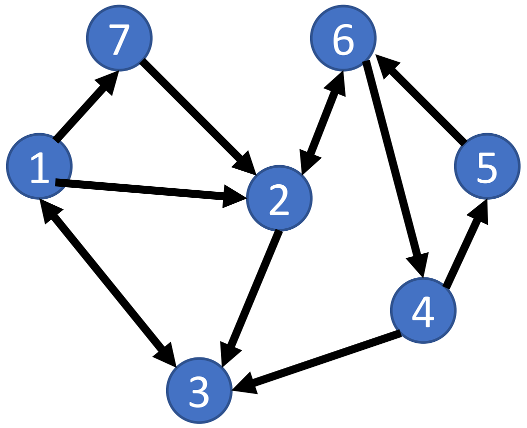
- 1 and 6 are sinks
- 5 and 7 are sources
- The other nodes are neither sinks nor sources

Walk, Path, Cycle

- A walk is a sequence of vertices $v_0, v_1, \ldots, v_n$, such that (v_i, v_{i+1}) is an arc in E for $0 \leq i < n$

- A walk can pass by the same vertex twice, i.e., $v_i = v_j$ is possible even for $i \neq j$

- The length of the walk is n. This is the number of arcs involved.



“4 5 6 4 3 1 2 3 1 7 2” is a walk

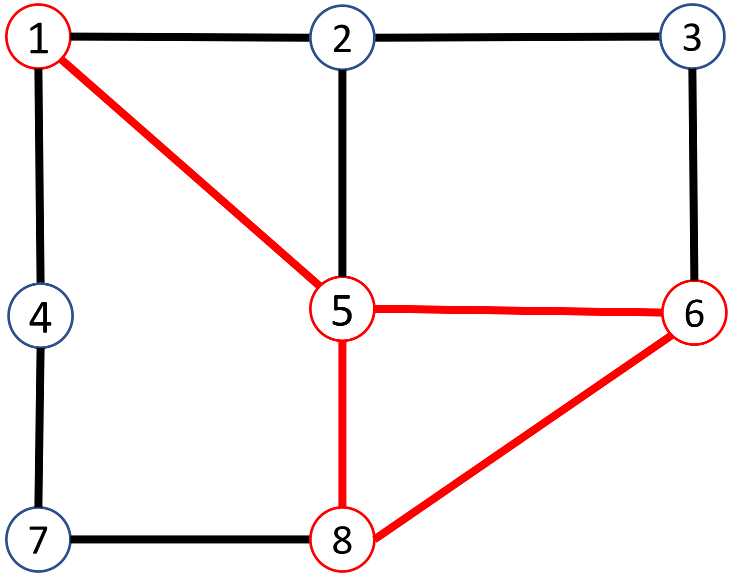
Walk, Path, Cycle (Contd.)

- A path is a walk in which no vertex is repeated

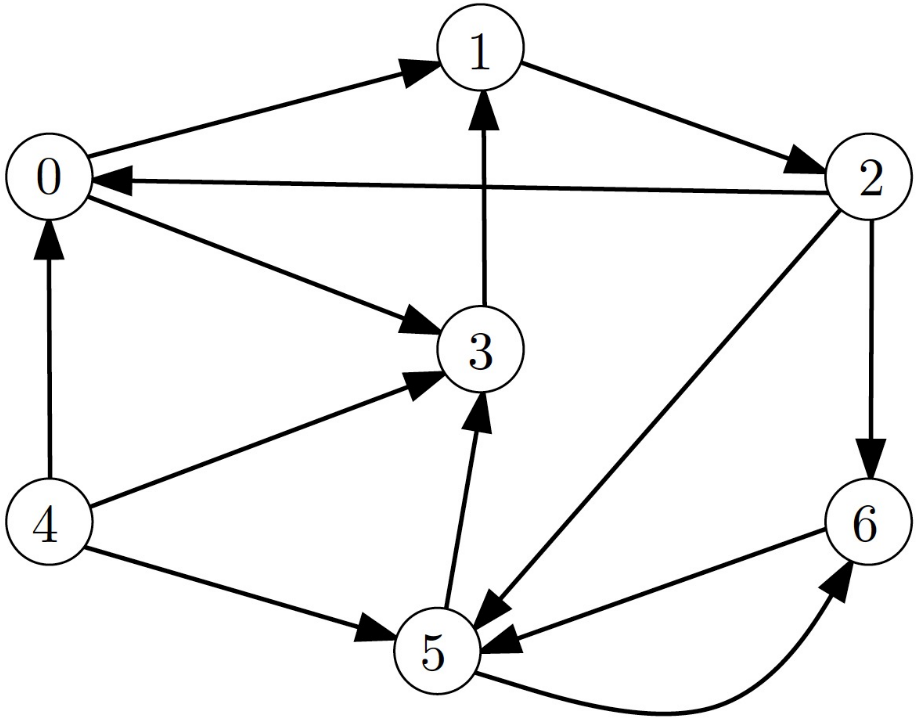
- A cycle is a walk of length 3 or more on a graph or any walk on a digraph where $v_0 = v_n$,
  - A walk that ends in the same vertex that it started in. No vertex is repeated other than the vertex at the start and end.

Exercise: This is NOT a Cycle

(1,5,6,8,5,1)

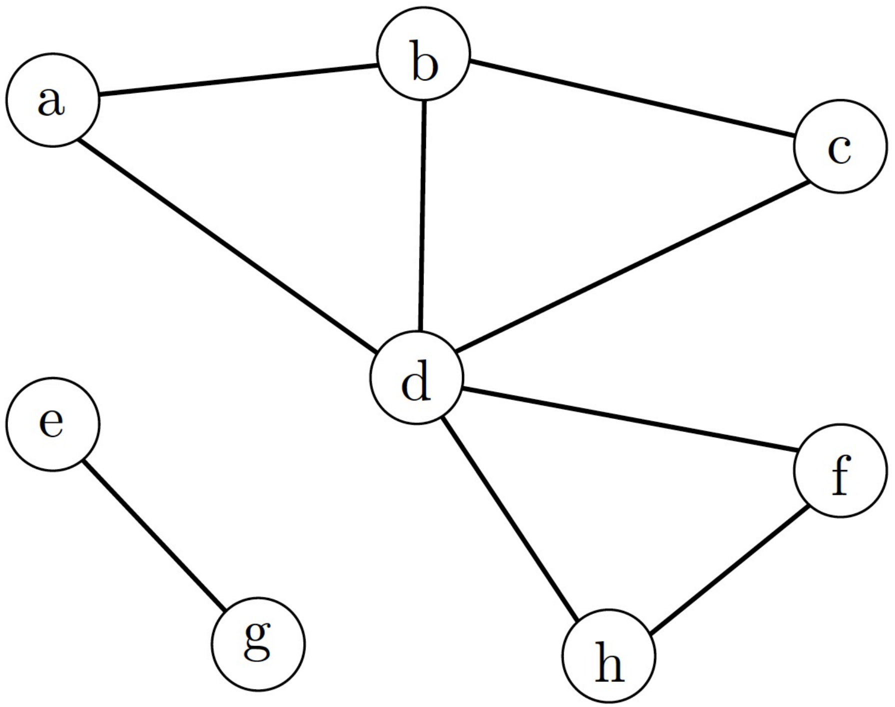


Example: Walks, Paths, and Cycles in a Digraph



Sequence	Walk	Path?	Cycle?
0 2 3	no	no	no
3 1 2	yes	yes	no
1 2 6 5 3 1	yes	no	yes
4 5 6 5	yes	no	no
4 3 5	no	no	no

Example: Walks, Paths, and Cycles in a Digraph

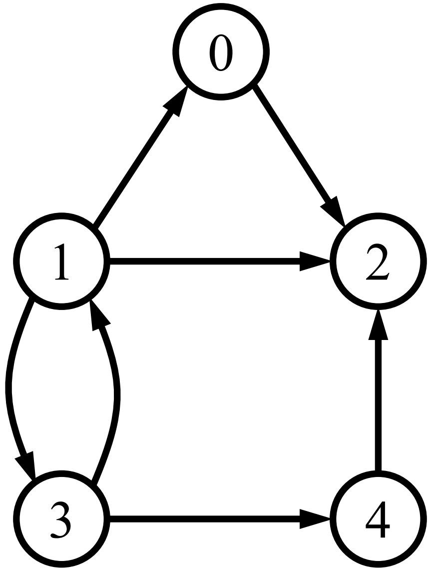


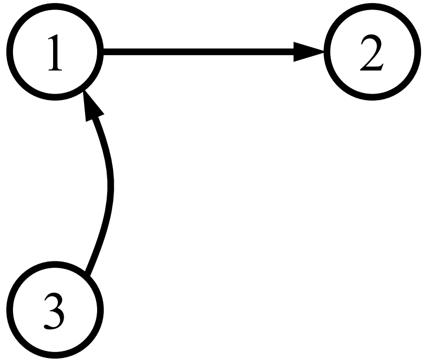
Sequence	Walk	Path?	Cycle?
$a\ b\ c$	yes	yes	no
$e\ g\ e$	yes	no	no
$d\ b\ c\ d$	yes	no	yes
$d\ a\ d\ f$	yes	no	no
$a\ b\ d\ f\ h$	yes	yes	no

Subgraphs and Subdigraphs

- A sub(di)graph $G' = (V', E')$ of a (di)graph $G = (V, E)$ is a (di)graph for which $V' \subseteq V$ and $E' \subseteq E$.

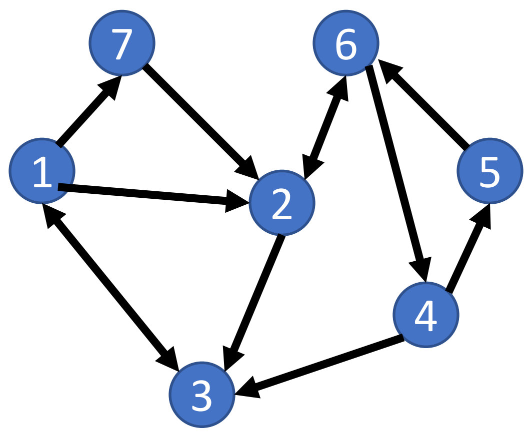
$$G = \left(\begin{array}{c} V = \{0,1,2,3,4\} \\ E = \left\{ \begin{array}{c} (0,2), (1,0), (1,2), \\ (1,3), (3,1), (4,2), \\ (3,4) \end{array} \right\} \end{array} \right)$$



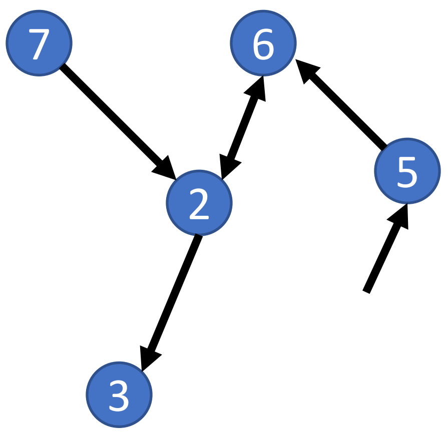


$$G' = \left(\begin{array}{c} V' = \{1,2,3\} \\ E' = \{(1,2), (3,1)\} \end{array} \right)$$

Exercise: Subdigraph or not?

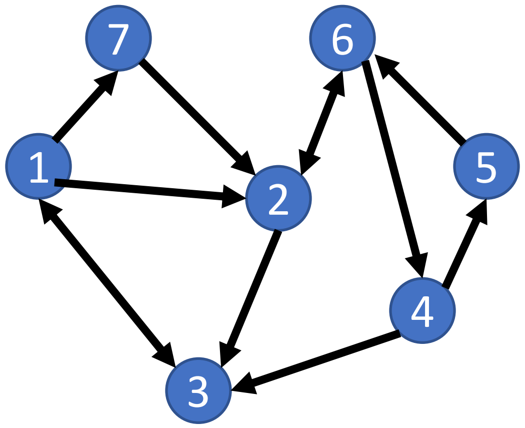


G



G'

Exercise: Subdigraph or not?



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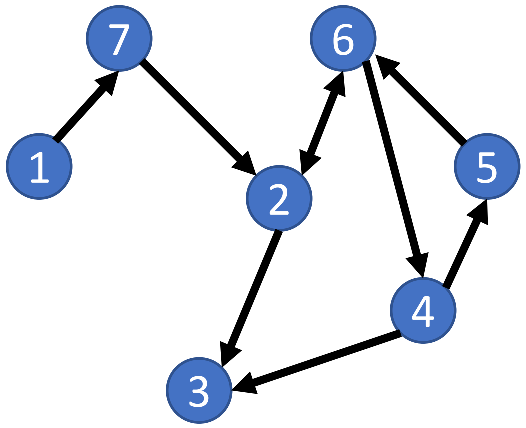
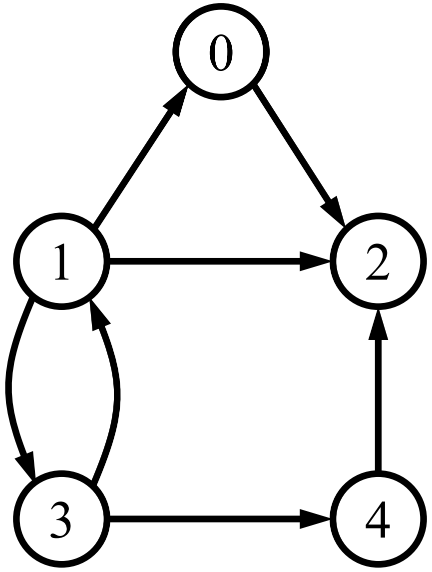


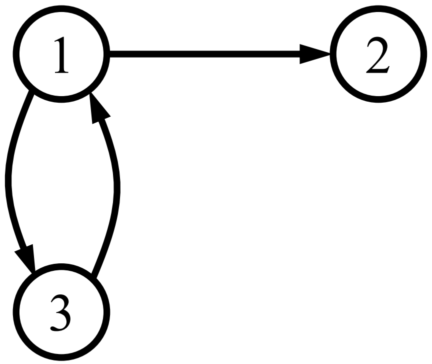
G'

Induced Sub(di)graphs

- A subdigraph induced by a subset V' of V is the digraph $G' = (V', E')$ where $E' = \{(u,v) \in E \,|\, u \in V' \text{and } v \in V'\}$.

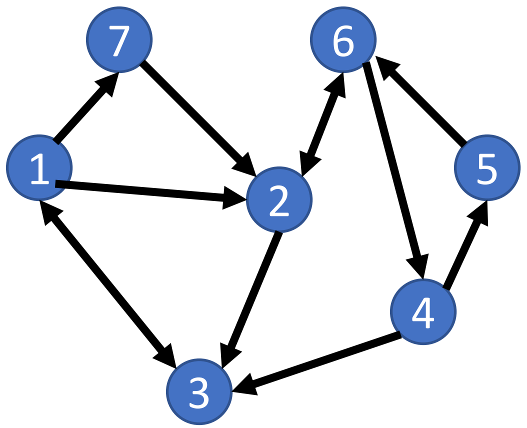
$$G = \begin{pmatrix} V = \{0,1,2,3,4\} \\ E = \begin{Bmatrix} (0,2),(1,0),(1,2), \\ (1,3),(3,1),(4,2), \\ (3,4) \end{Bmatrix} \end{pmatrix}$$





$$G' = \begin{pmatrix} V' = \{1,2,3\} \\ E' = \begin{Bmatrix} (1,2),(1,3), \\ (3,1) \end{Bmatrix} \end{pmatrix}$$

Example: An Induced Subdigraph



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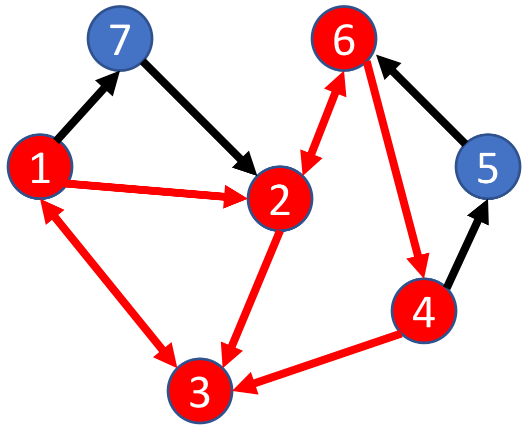


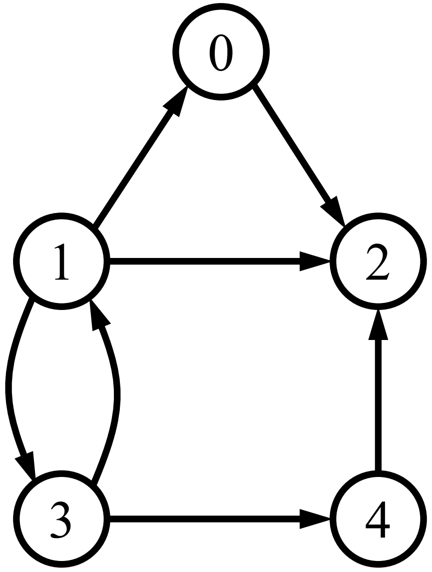
G'

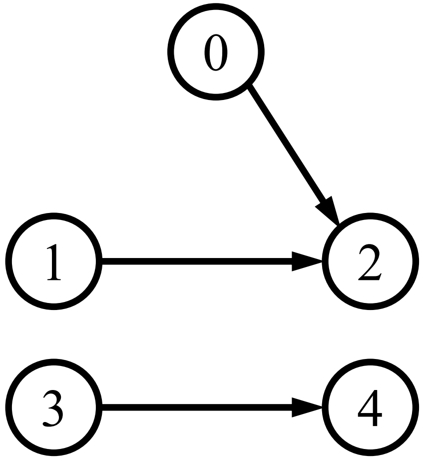
E.g., $V' = \{1,2,3,4,6\}$

Spanning Sub(di)graphs

- A spanning subdigraph contains all nodes, that is, V'=V.

$$G = \left(\begin{array}{c} V = \{0,1,2,3,4\} \\ E = \left\{ \begin{array}{c} (0,2), (1,0), (1,2), \\ (1,3), (3,1), (4,2), \\ (3,4) \end{array} \right\} \end{array} \right)$$

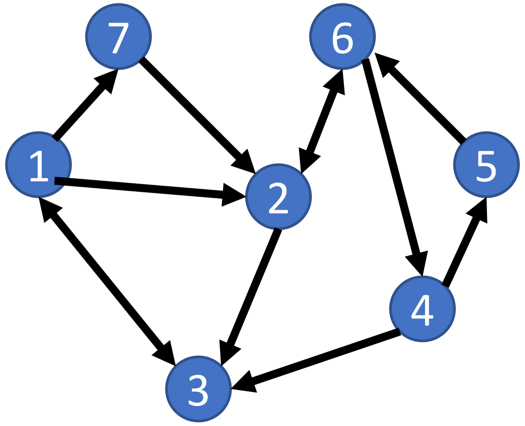




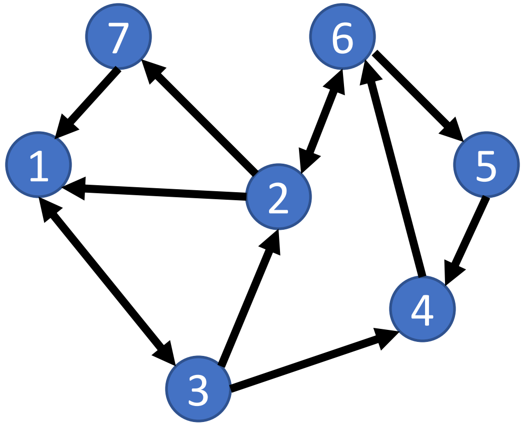
$$G' = \left(\begin{array}{c} V' = \{0,1,2,3,4\} \\ E' = \left\{ \begin{array}{c} (0,2), (1,2), \\ (3,4) \end{array} \right\} \end{array} \right)$$

Reverse Digraph

- The reverse digraph of the digraph $G=(V,E)$, is the digraph $G\_r=(V,E')$ where $(u,\, v) \in E'$ if and only if $(v,\, u) \in E$.



G

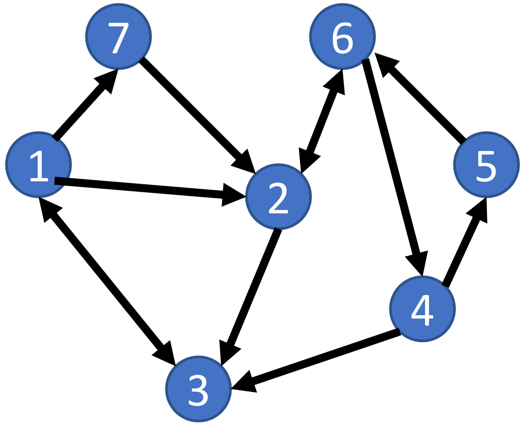


G_r

We simply reverse all the arrows

Underlying Graph of a Digraph

- The underlying graph of a digraph G=(V,E) is the graph G'=(V,E') where E^′$=\{\{u, v\} \mid (u, v) \in E\}$.



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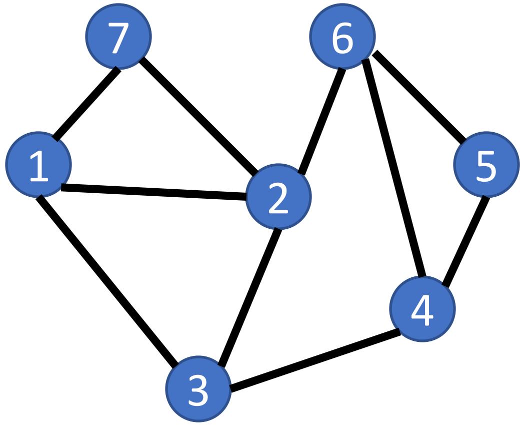

G'

SUMMARY

- Definition of Graphs and Di-Graphs
- Preliminaries on terminology and properties (e.g., size, order, degree)
- Basic operations
  - walking and sub-graphing