Graph Matching

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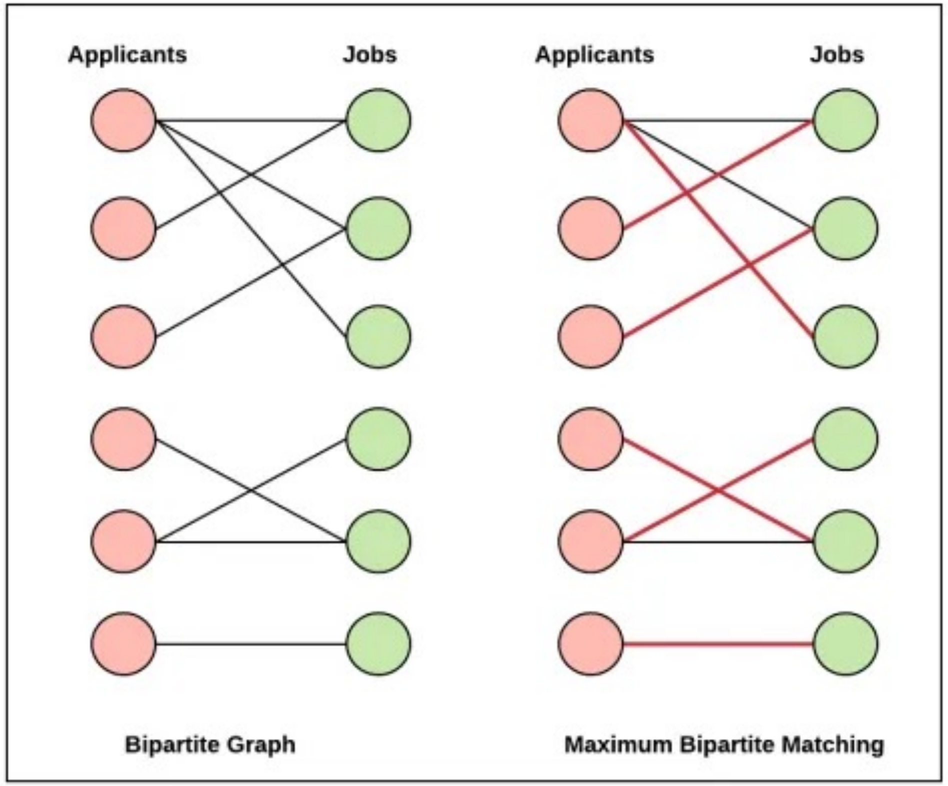
COMPCSI220: WEEK 12

THE UNIVERSITY OF AUCKLAND
Te Whare Wananga o Tamaki Makaurau
NEW ZEALAND

Slides adapted from Mark Wilson, Georgy Gimel'farb, Simone Linz and Tanya Gvozdeva

OUTLINE

- Matching Problem
- Maximal and Maximum Matching
- Alternating Paths
- Augmenting Paths



Matching Problem

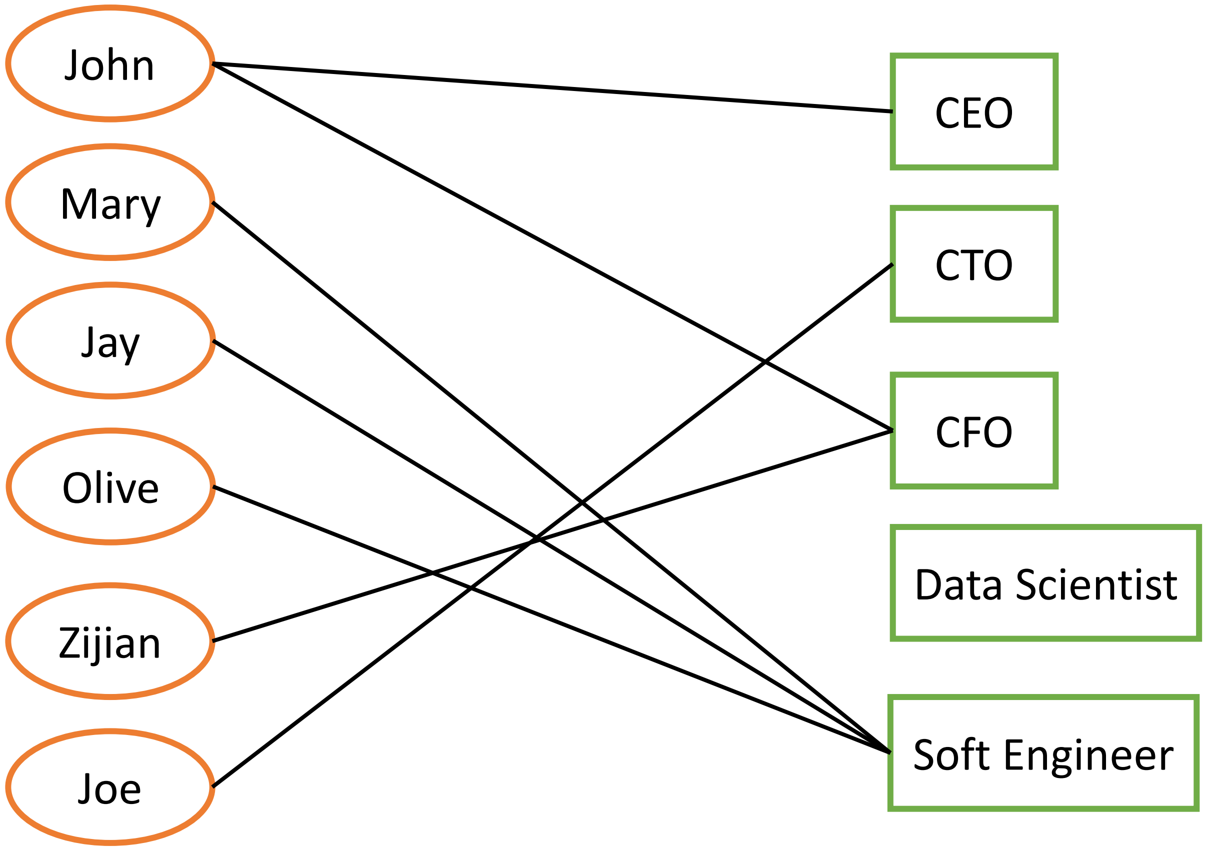
Team

- John
- Mark
- Jay
- Olive
- Zijian
- Joe

Roles

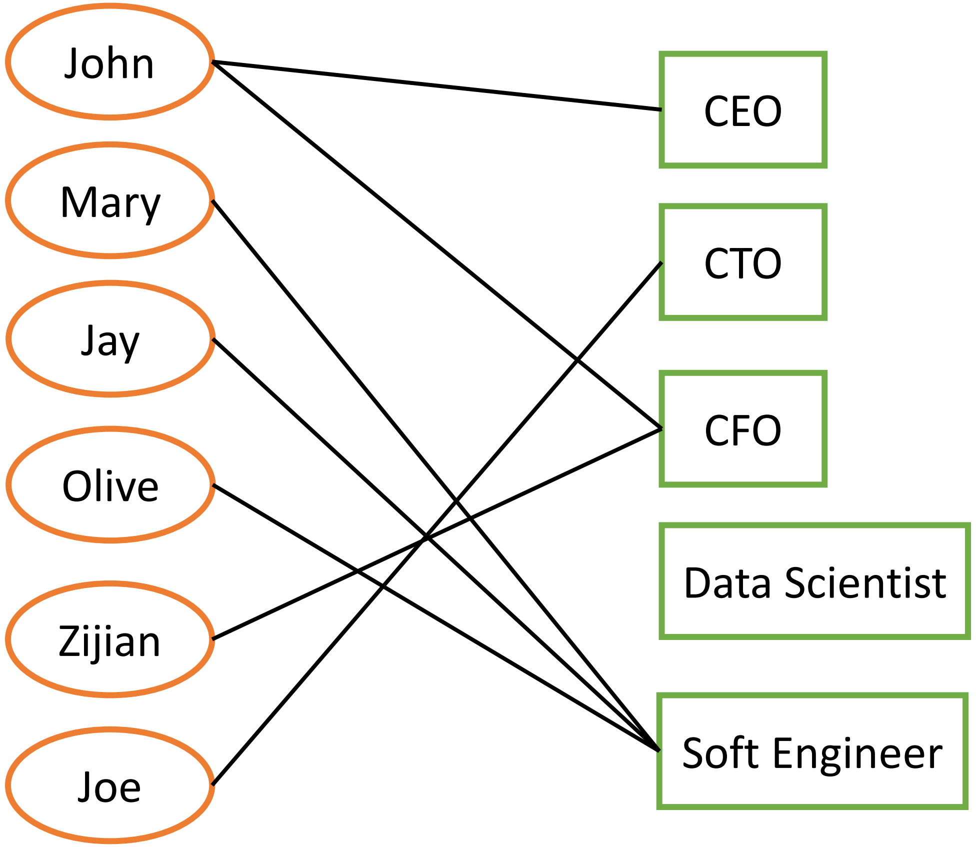
- CEO
- CTO
- CFO
- Data Scientist
- Soft Engineer

Reducing to a Graph Problem



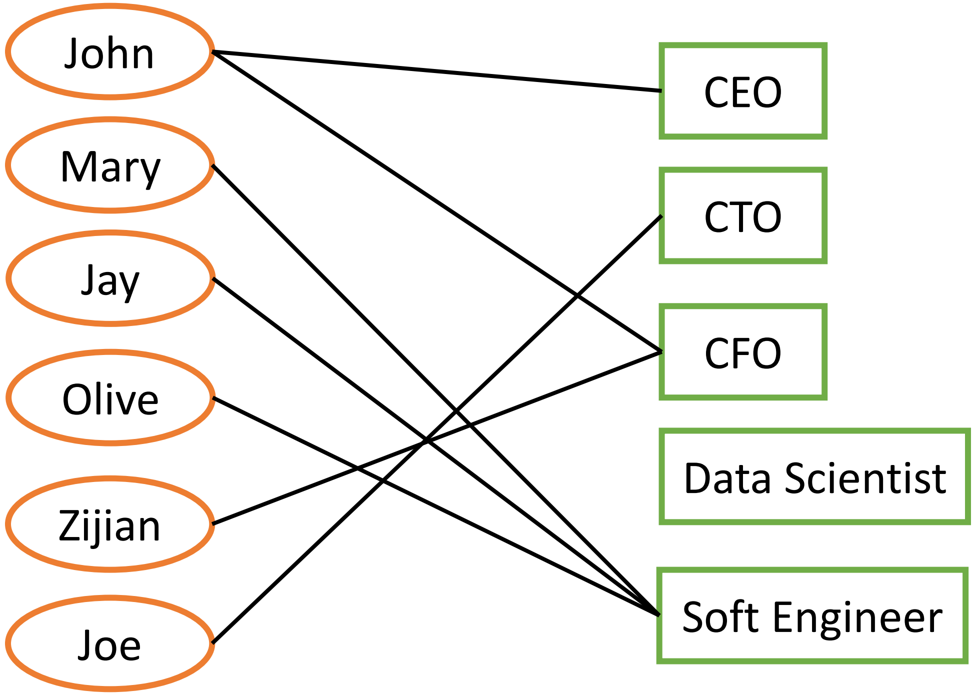
Bi-partite Graphs

- A graph $\mathrm{G}(\mathrm{V}, \mathrm{E})$ is bi-partite if
  - $\mathrm{V}(\mathrm{G})$ can be partitioned in two non-empty disjoint subsets $\{v_0, v_1\}$ s.t.
  - Each edge of $\mathrm{E}(\mathrm{G})$ connects one vertex of v_0 and one vertex of v_1



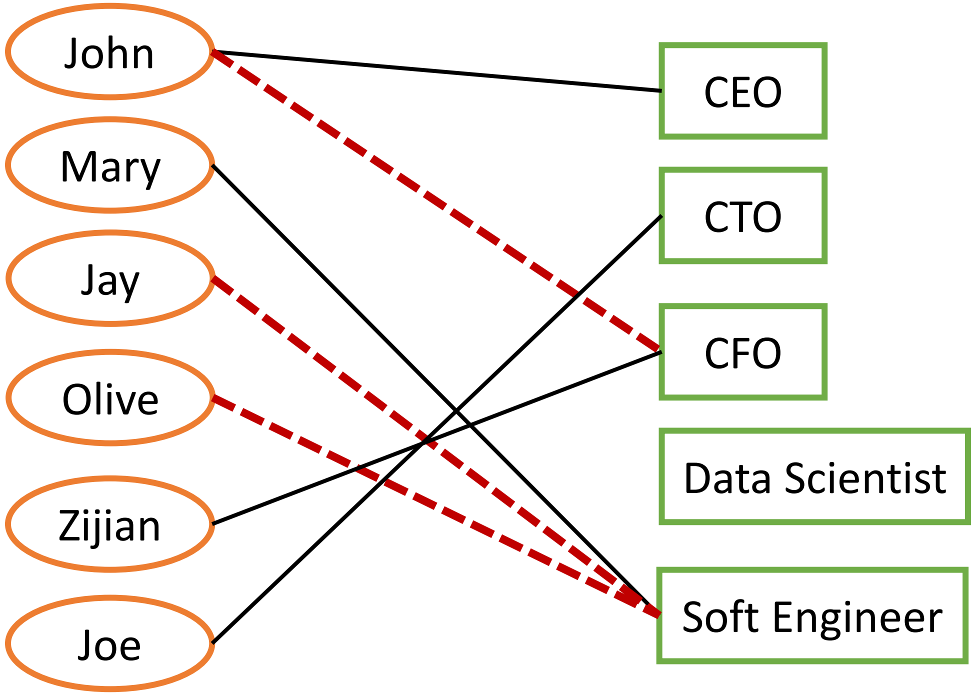
Matching

- We cannot have more than one person for the key roles: CEO, CTO and CFO



Matching (Contd.)

- In other words, we want to find edges that do not start/end on the same vertices

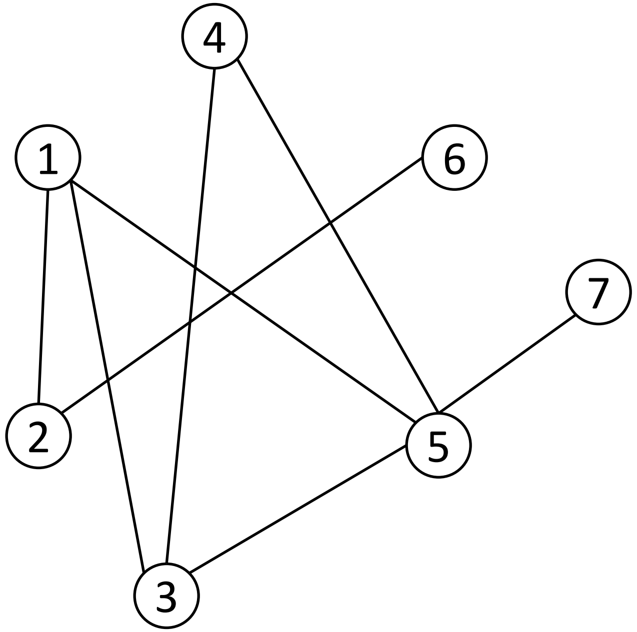


Matchings

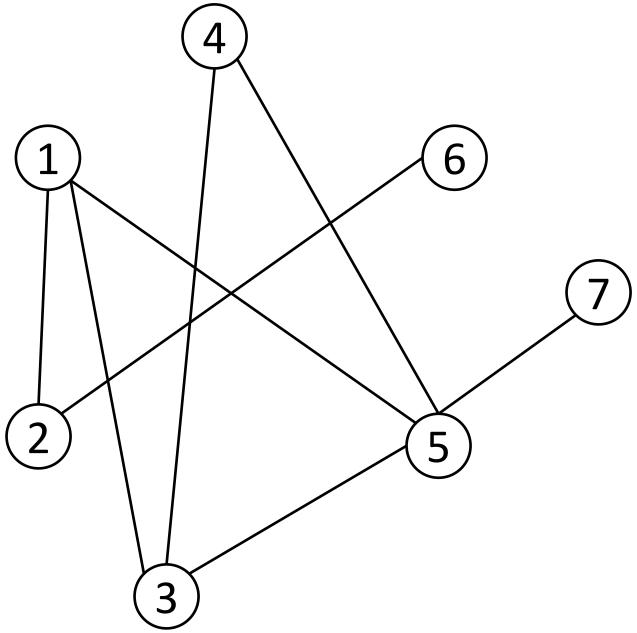
- A matching M is a set of pairwise non-adjacent edges in a graph

Example: Matchings

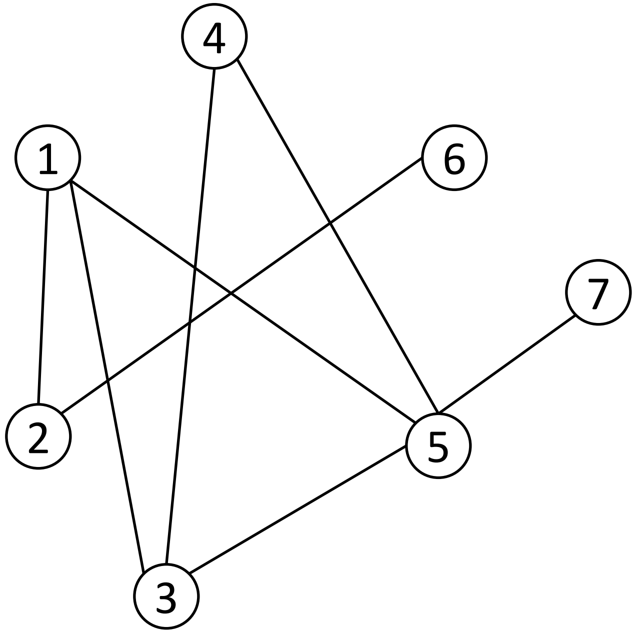
- Is this a matching?



YES



NO



NO

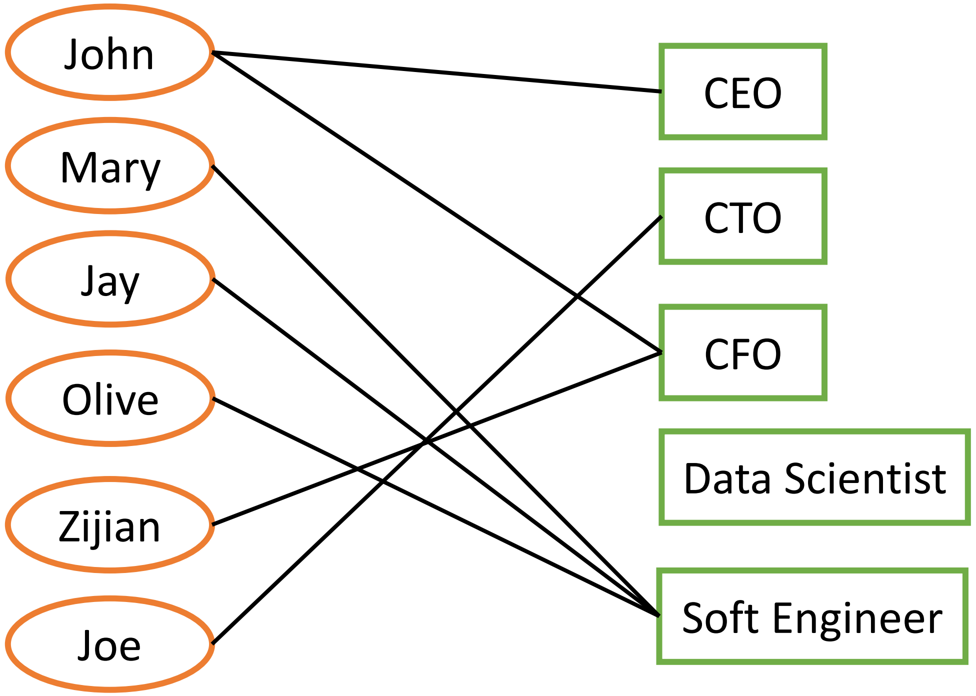
Importance of Matchings

- Matchings are used whenever we need to assign members of a set to each other as exclusive pairs based on suitability criteria

- The members of the set are the vertices, and the edges indicate potentially suitable pairings

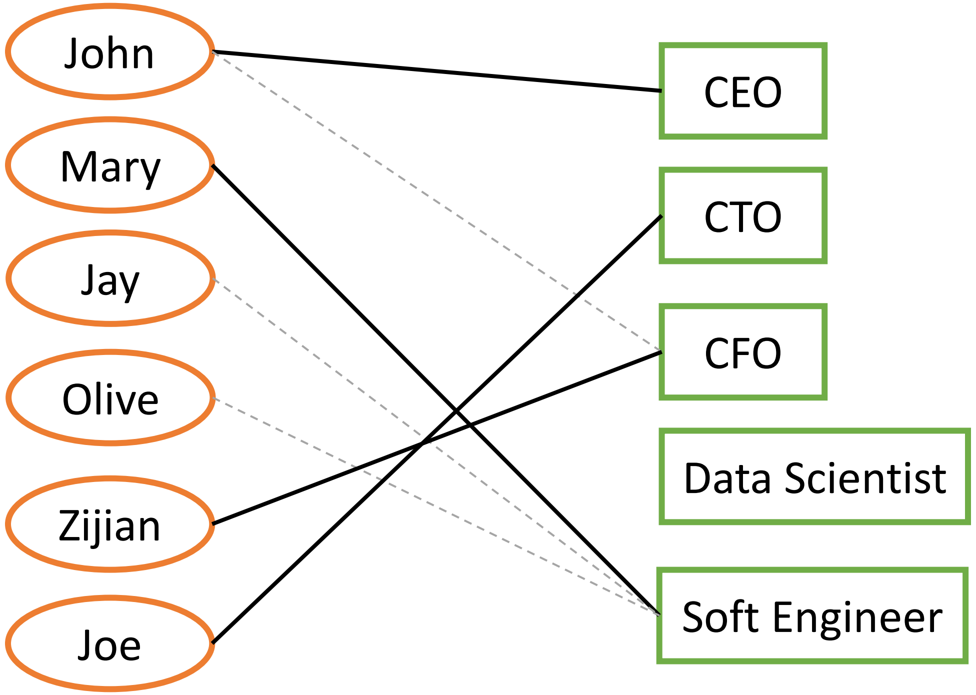
- Matchings are often but not always used in conjunction with bipartite graphs

This is a Possible Matching

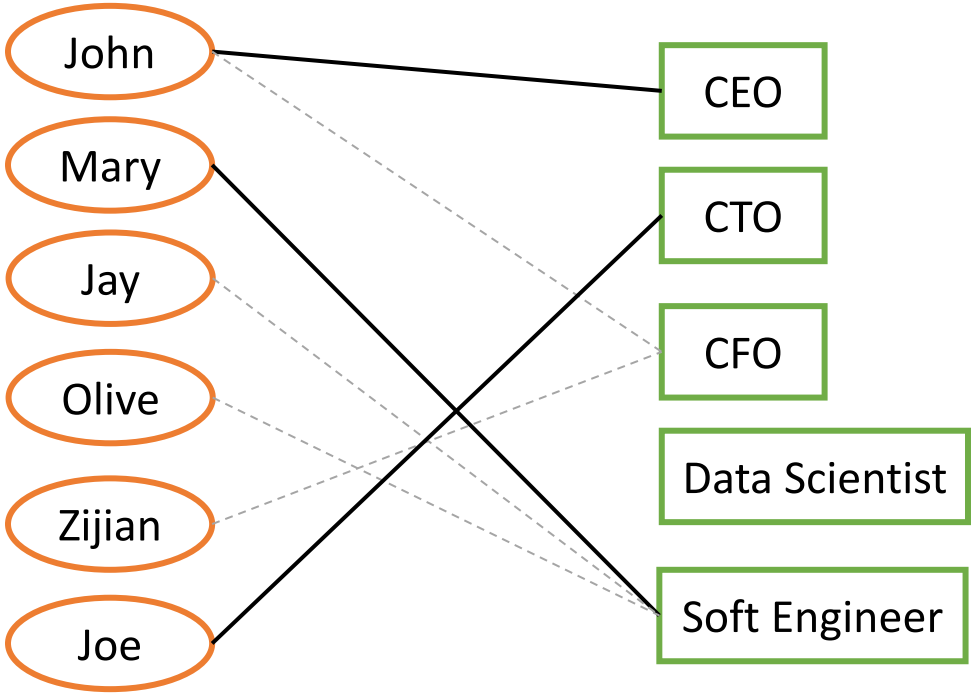


Maximal Matching

- It is not a subset of any other matching



Maximal Matching (Contd.)



Maximal and Maximum Matchings

- A **maximal** matching is a matching that is not a subset of any other matching.
- A **maximum** matching is a matching with the highest number of edges.
- In other words, we cannot find a matching with more edges

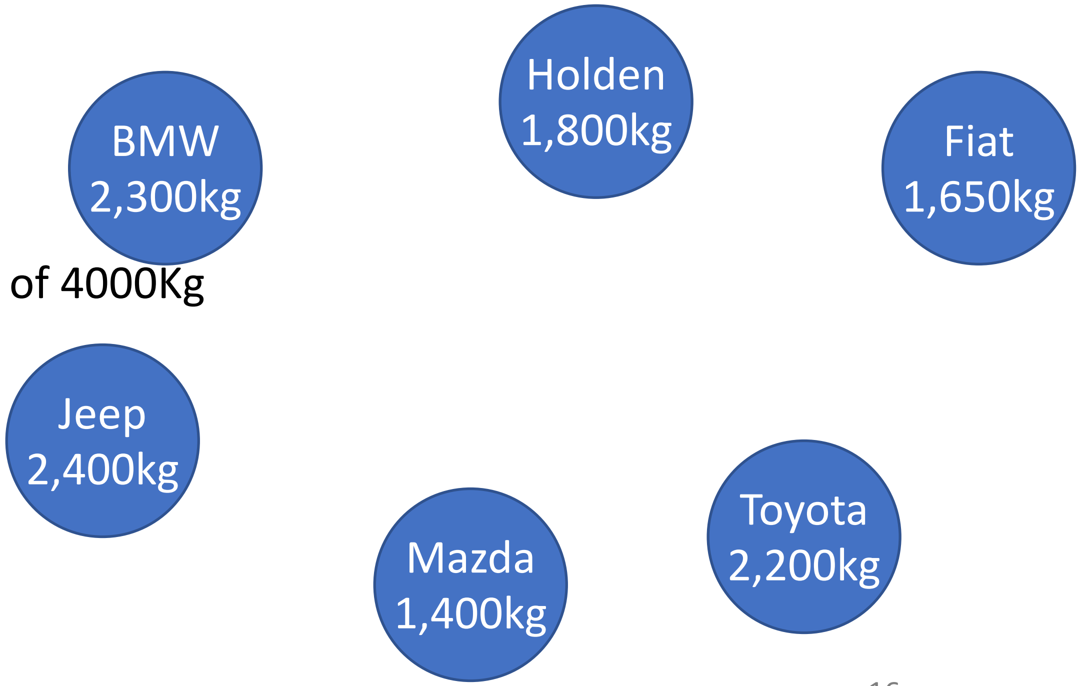
Maximal and Maximum Matchings

- A maximum matching is always a maximal matching, but a maximal matching is not necessarily a maximum matching.
- Note: A maximum matching in an arbitrary graph G can be found in polynomial time. For a polynomial time algorithm see section 5.9 in the textbook.

Exercise: Car Ferry

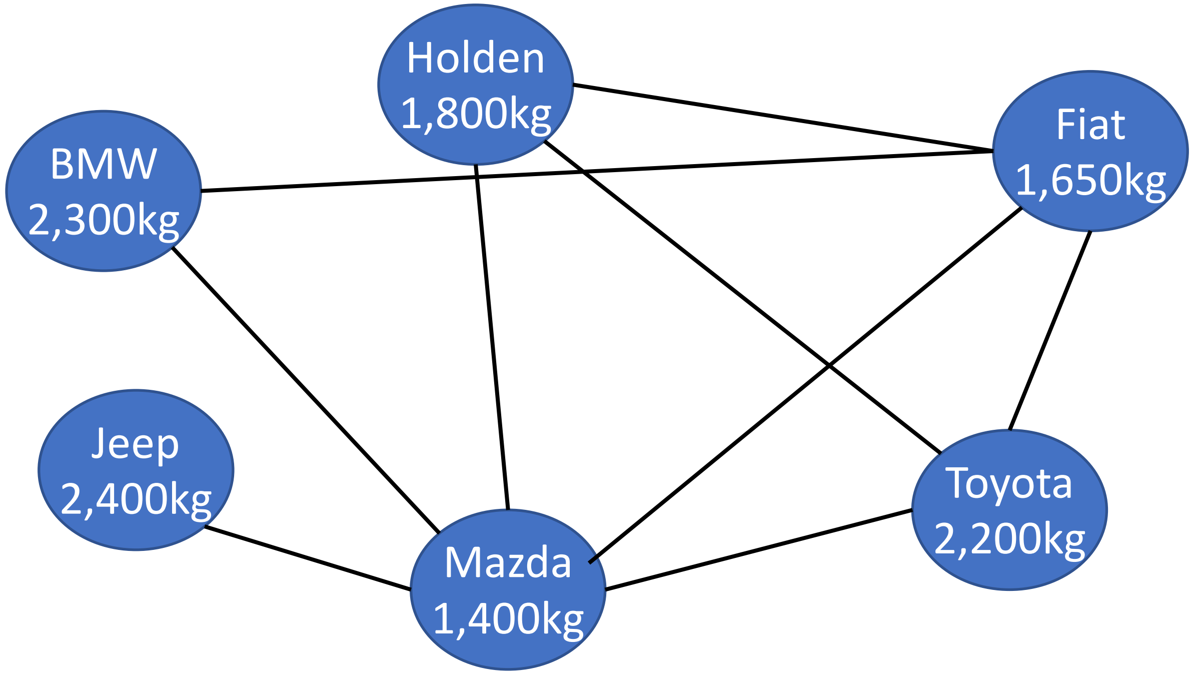
- A small car ferry across a river can take two vehicles at a time with a combined weight of 4,000 kg. On one side, the following cars are waiting for a trip across:

- Please Find
  - All edges between vertices that respect the constraint of 4000Kg
  - An example of maximal matching
  - An example of maximum matching



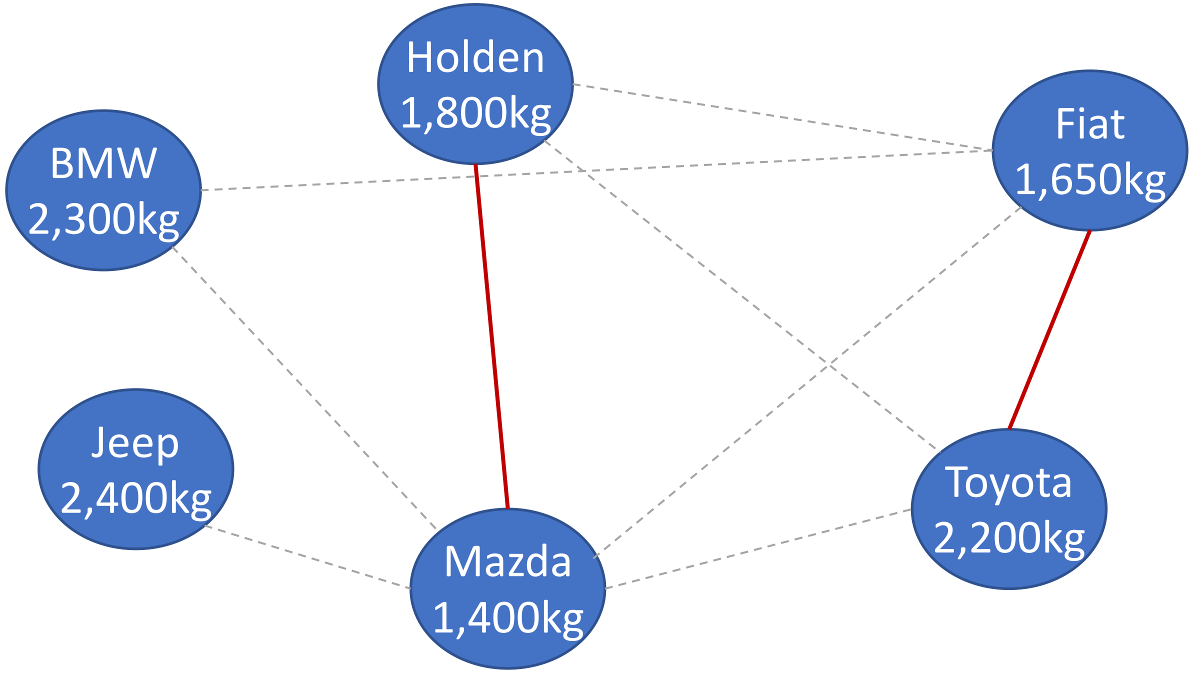
Exercise: Car Ferry

- Edges indicate cars that can be put on the ferry together:



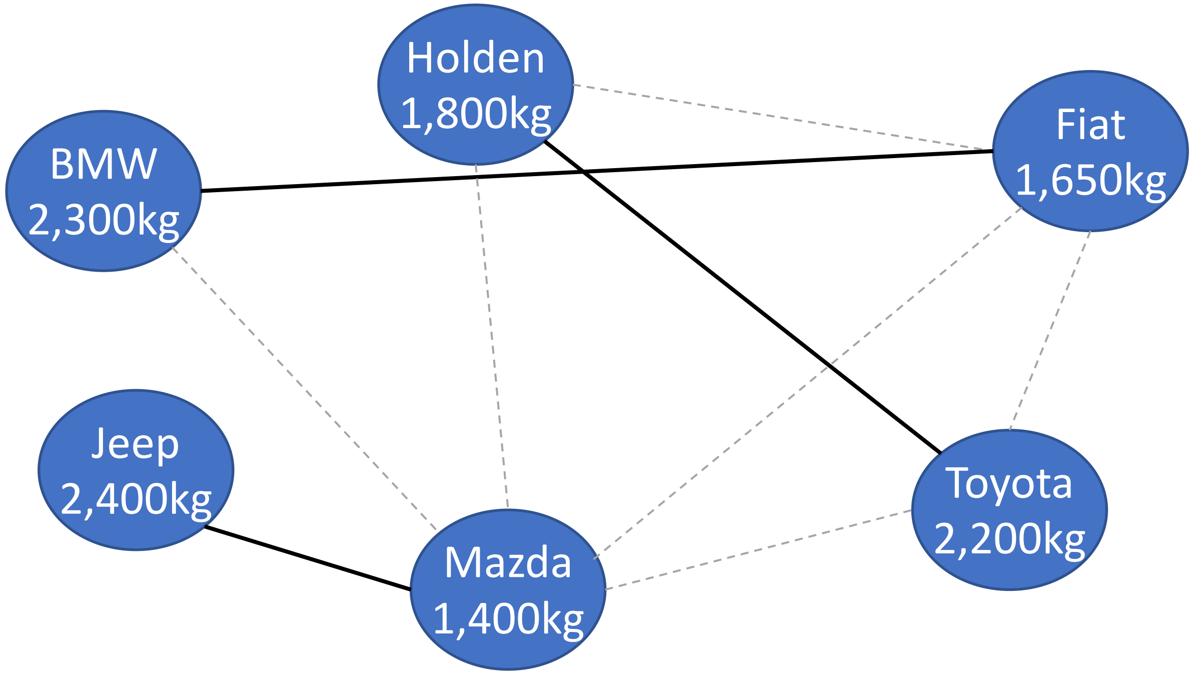
Exercise: Car Ferry

- A matching indicates a possible way in which to get the cars across
- We require four ferry trips: two with a pair of cars and two with a single car each.



Exercise: Car Ferry

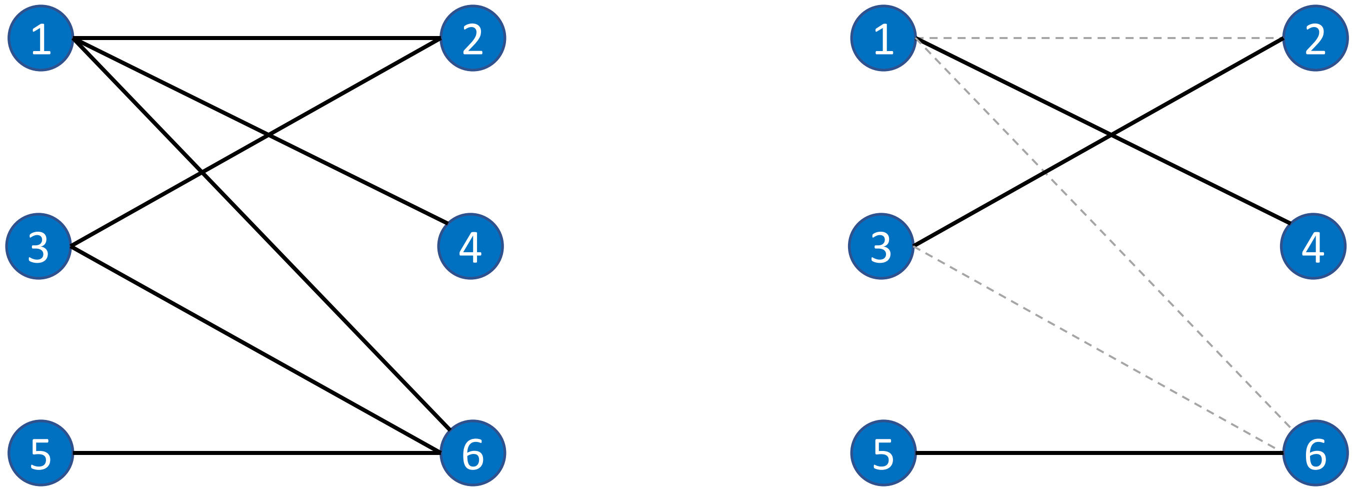
- In this matching, we can get all cars across in three ferry trips:



Finding Maximum Matchings

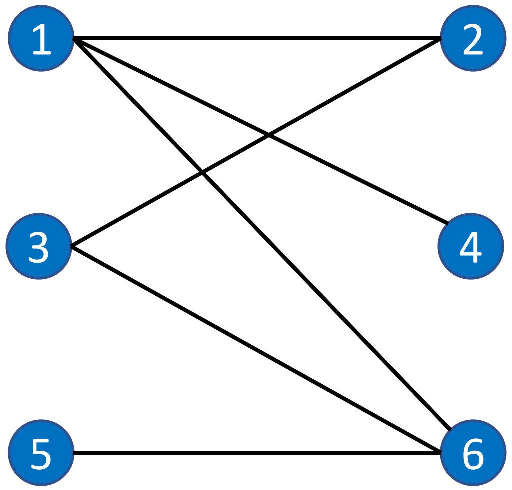
- A maximum matching in an arbitrary graph G can be found in polynomial time

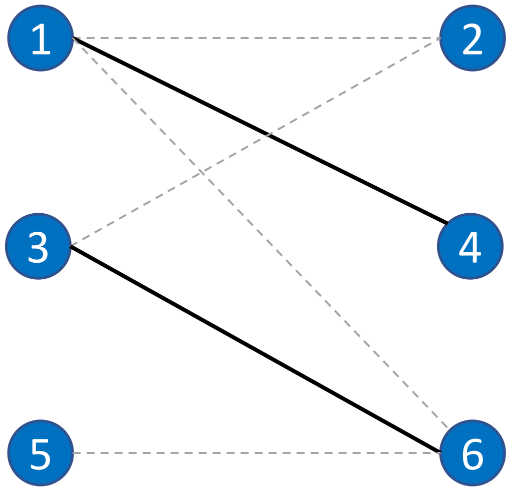
- Given a graph G(V,E), a matching M in G is a set of pairwise non-adjacent edges
  - No two edges share a common vertex



Finding Maximum Matchings (Contd.)

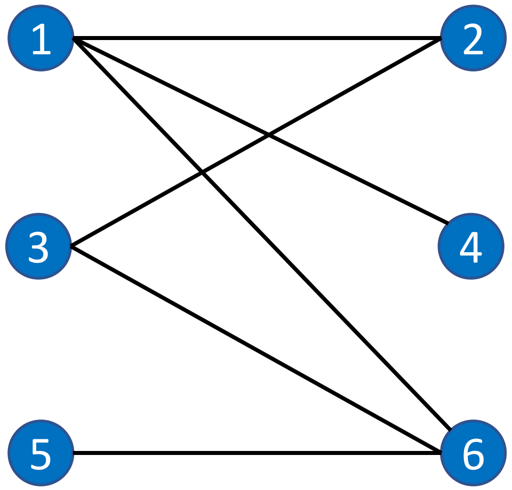
- Given a graph G(V,E), a matching M is a maximal matching if after adding another edge to M, M is not a matching any longer

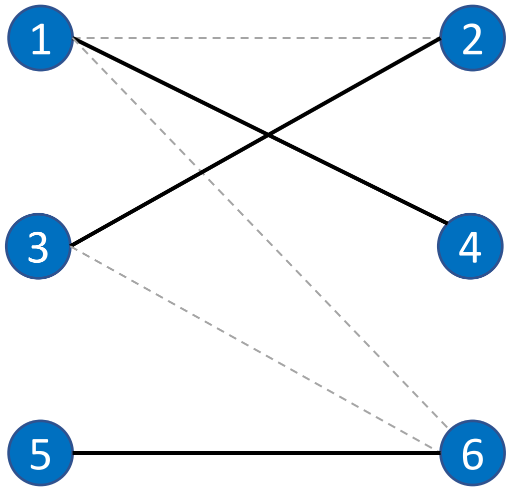




Finding Maximum Matchings (Contd.)

- Given a graph G(V,E), a matching M is a maximum matching if it contains the largest number of edges





Finding Maximal and Maximum Matching

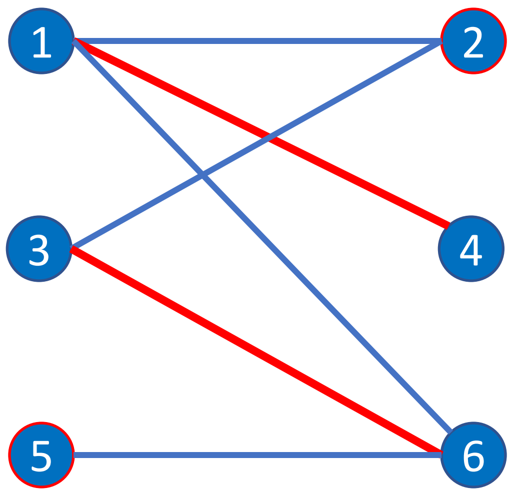
- A simple greedy algorithm that searches maximal matchings:
  - Iterates over all edges in the graph
  - Add an edge to a maximal matching if it is not adjacent to any other edges in the matching

(Un-)Matched Vertex

- A vertex is matched (or saturated) if it is an endpoint of one of the edges in the matching

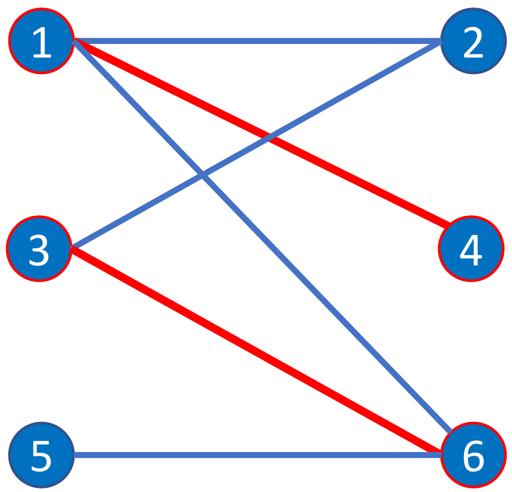
- A vertex is unmatched if no edges of the matching end/start from such vertex

Example: (Un-)Matched Vertex



2 and 5 are unmatched

Example: (Un-)Matched Vertex

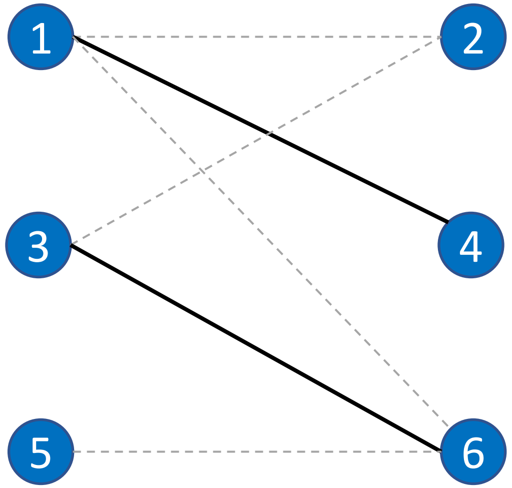


1,4,3,6 are matched

Alternating Paths

- An **alternating path** in a matching M is a path starting from an unmatched vertex in which the edges alternate from being in M and not

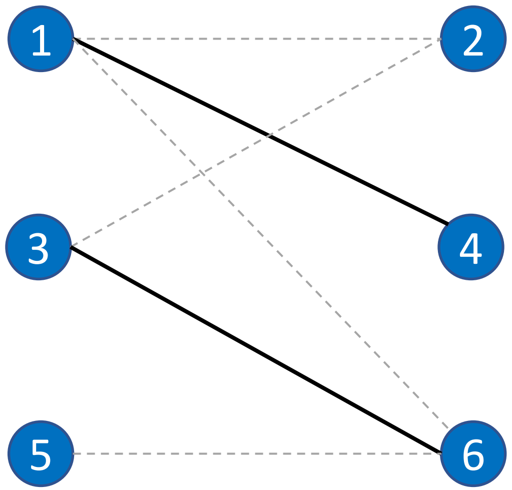
Path: 2,1,4
Path: 5,6,3,2



Augmenting Paths

- An **augmenting path** in a matching M is an alternating path starting and ending in unmatched (free) vertices

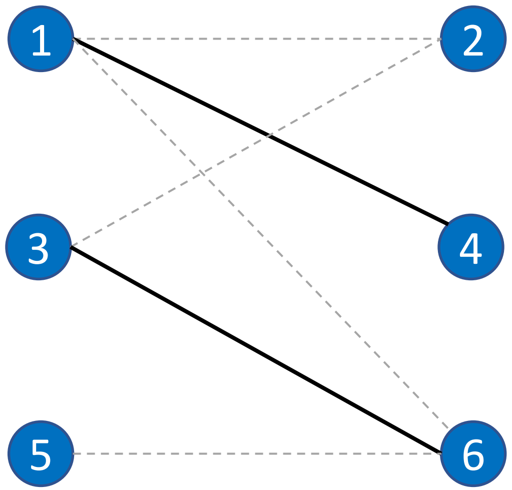
Path: 2,3,6,5



Augmenting Paths Property

- In an augmenting path there is always one more non-matching edge than matching edges:

<u>Path: 2,3,6,5</u>
(2,3) unmatched
(3,6) matched
(6,5) unmatched

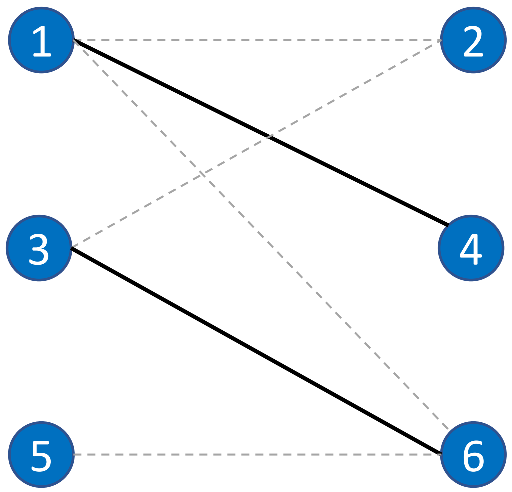


Augmenting Path Property (Contd.)

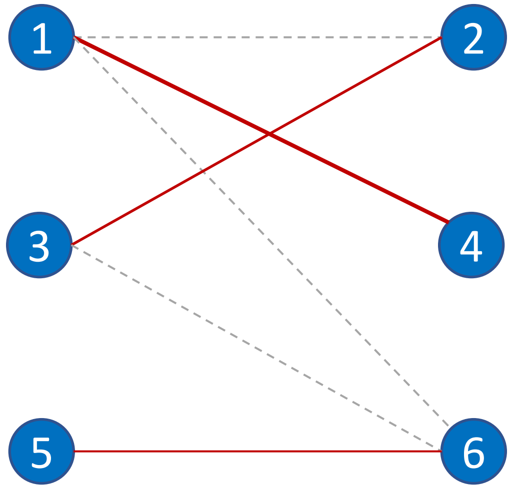
- We can use **augmenting paths** to find maximum matchings: just remove the matched edges from M and add the unmatched ones

We have found our Maximum Matching!!!

Path: 2,3,6,5
(2,3) unmatched
(3,6) matched
(6,5) unmatched



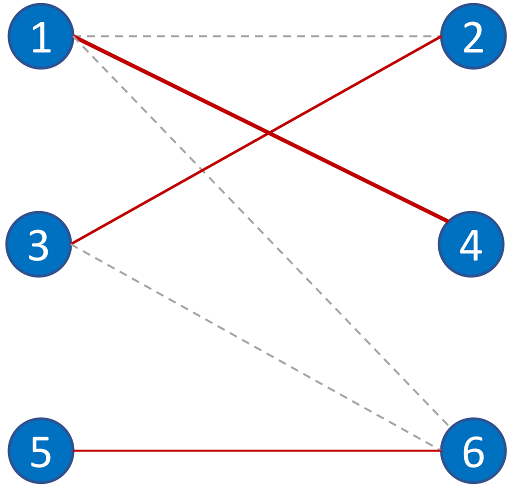
Path: 2,3,6,5
(2,3) unmatched
(3,6) matched
(6,5) unmatched



Augmenting Path Property (Contd.)

- If there is no *augmenting path,* then M is a *maximum matching*

<u>Path: 2,3,6,5</u>
(2,3) unmatched
(3,6) matched
(6,5) unmatched



We have found our Maximum Matching!!!

SUMMARY

- Matching Problem
- Maximal and Maximum Matching
- Alternating Paths
- Augmenting Paths