# Cycles and Girth

Instructor: Meng-Fen Chiang
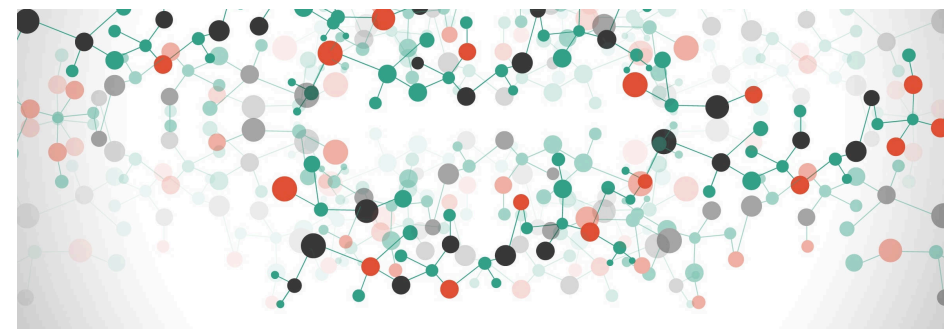
THE UNIVERSITY OF
AUCKLAND
Te Whare Wānanga o Tāmaki Makaurau
NEW ZEALAND

Slides adapted from Mark Wilson, Georgy Gimel'farb, Simone Linz and Tanya Gvozdeva
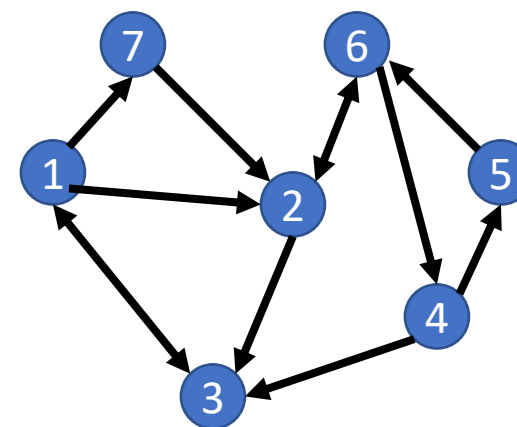
# OUTLINE

- Terminology
  - Walk
  - Cycle
  - Path
  - Girth

- Finding the Smallest Cycle

# RECAP: Walk

- A walk is a sequence of vertices $v_0, v_1, \ldots, v_n$, such that $(v_i, v_{i+1})$ is an arc in $E$ for $0 \leq i < n$

- A walk can pass by the same vertex twice, i.e., $v_i = v_j$ is possible even for $i \neq j$

- The length of the walk is l. This is the number of arcs involved.

E.g., 4 5 6 4 3 1 2 3 1 7 2 is a walk

# Cycle Detection

- Suppose that there is a cycle in $G$ and let $v$ be the node in the cycle visited first by DFS. If $(u, v)$ is an arc in the cycle then it must be a back arc.

- Conversely if there is a back arc, we must have a cycle.

- Suppose that DFS is run on a digraph $G$. Then $G$ is acyclic if and only if $G$ does not contain a back arc.

- A digraph with no cycle is called a directed acyclic graph (DAG).

# Cycles in Graph and Digraph

- A cycle is a walk of length 3 or more on a graph where $v_0 = v_n$

- A cycle on a digraph is a walk of length 2 or more where $v_0 = v_n$
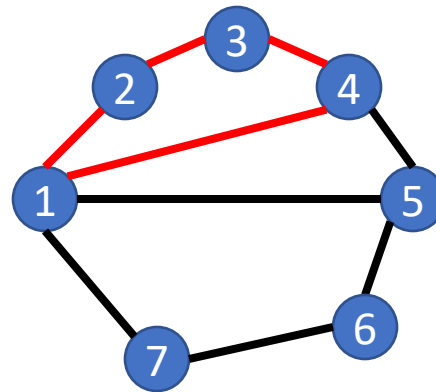
- In a digraph, we talk about directed cycles.

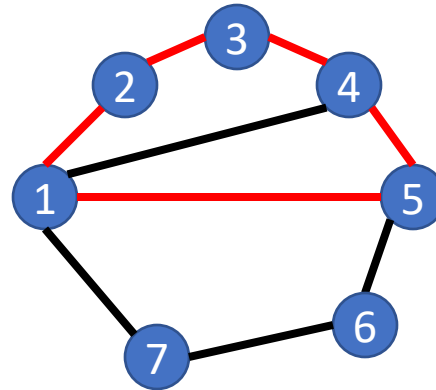# Example: Cycles in Graphs
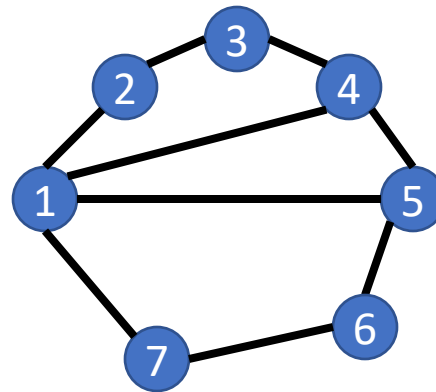
- A graph can have several cycles

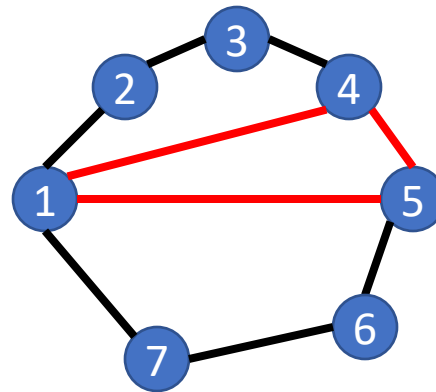# Example: Cycles in Graphs

- A graph can have several cycles

# Example: Cycles in Graphs

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# Example: Cycles in Graphs
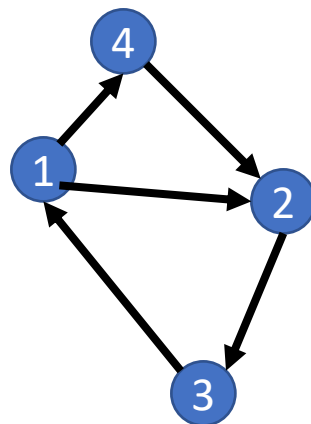
- A graph can have several cycles

# Girth and Digirth

- The length of the shortest cycle in a graph is called the graph's girth.

- NOTE: In the case of a digraph, the girth is determined based on the underlying graph
  - we do not require a walk on the digraph itself.

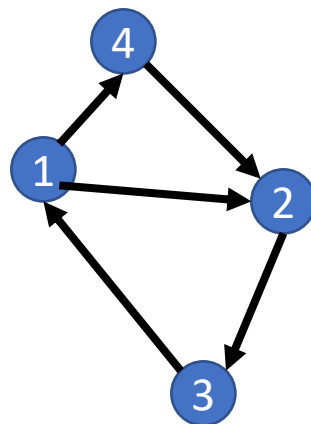- We may refer to the length of the shortest directed cycle of a digraph as its directed girth.

# Examples: graphs and girth
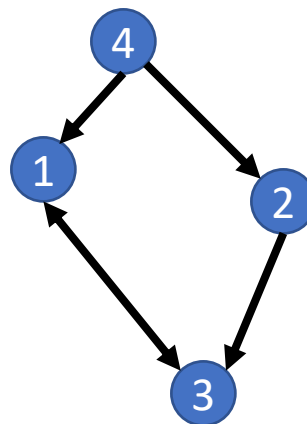


Girth: ?
Directed girth: ?

# Examples: Graphs and Girth
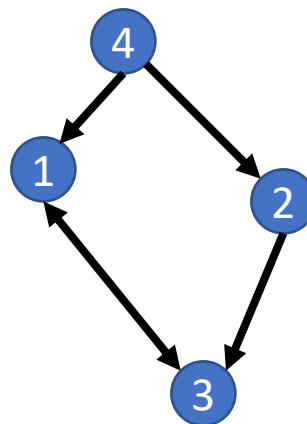


Girth: 3
Directed girth: 3

# Examples: Graphs and Girth
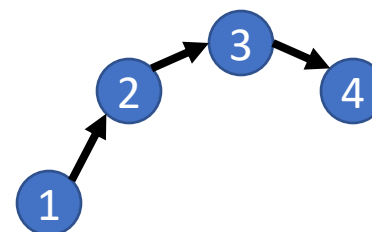


Girth: ?
Directed girth:?

# Examples: Graphs and Girth
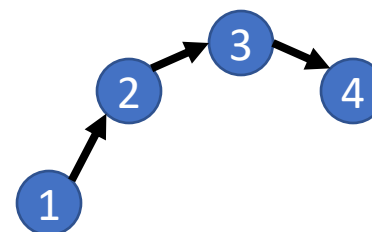


Girth: 4
Directed girth: 2

# Examples: Graphs and Girth
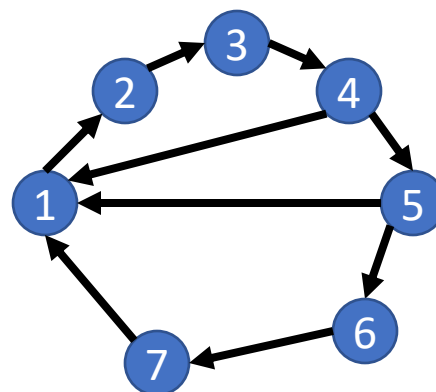


Girth: ?
Directed girth: ?

# Examples: Graphs and Girth



Girth: undefined
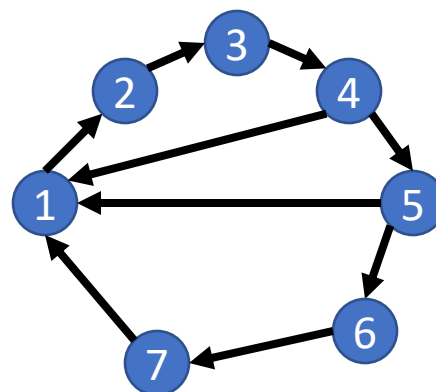Directed girth: undefined

# Examples: Graphs and Girth



Girth: ?
Directed girth: ?

# Examples: Graphs and Girth



Girth: 3
Directed girth: 4

# Finding the Girth of a Graph

- Perform BFSVisit() $|V|$ times, starting at each vertex $v' \in V$ in turn.

- If during a BFSVisit, we encounter a grey neighbour (rather than a white) we have found a cycle
  - The grey node was visited before following another path

- The girth is the smallest cycle in the graph

An important property of BFS is that if it runs from vertex $v$, then every vertex $s$, when it is first reached, then the path that was found from $v$ to $s$ is minimal. Thus, reaching $v$ from $v$ with BFS finds the shortest path from $v$ to itself, namely the shortest cycle that contains $v$.
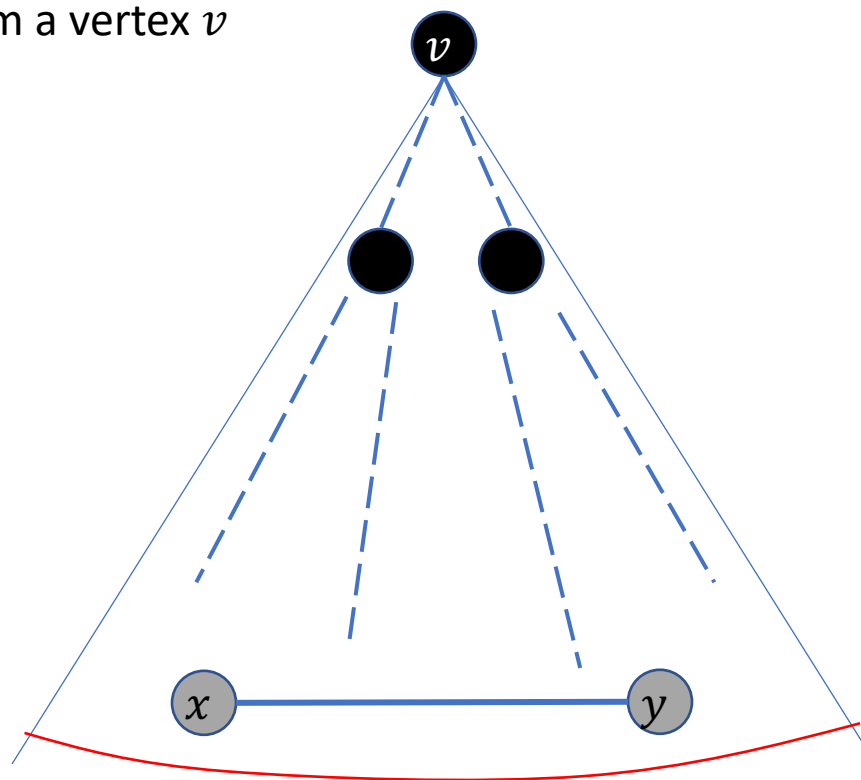
# Finding the Girth of a Graph (Contd.)

Executing BFSVisit from a vertex v

nodes explored level by level

Below, we should only find white nodes
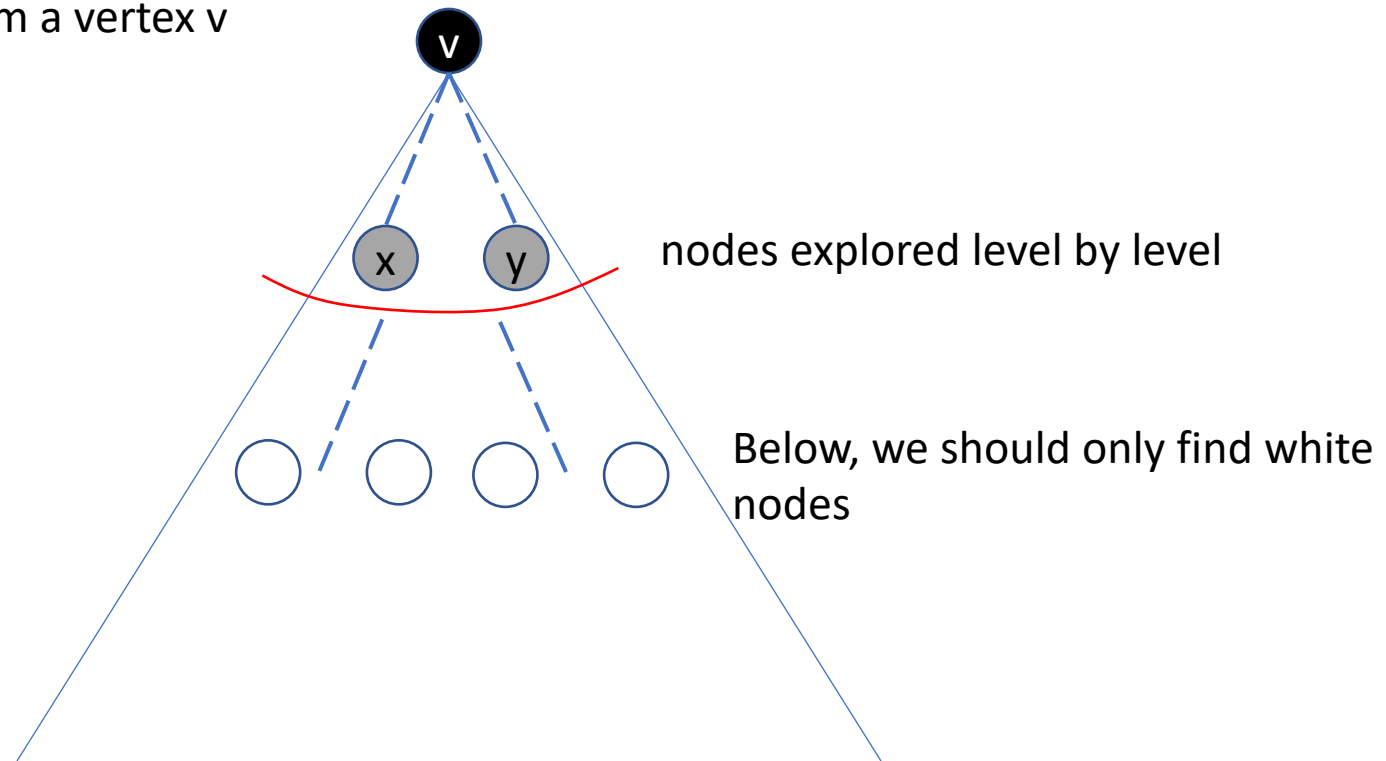
# Finding the Girth of a Graph (Contd.)

Executing BFSVisit from a vertex $v$



But if a cross edge/arc exits then it will connect to a grey vertex
Both $x$ and $y$ are reachable from $v$ therefore there is a cycle through the cross edge

# Finding the Girth of a Graph (Contd.)

- For each vertex $v' \in V$ perform BFSVisit($v$)
  - If a grey neighbour is met, e.g., an edge $(x, y)$ is explored from x where $y$ is grey, continue to the end of the current level and then stop.
  - For each edge $(x, y)$ at this level for which both $x$ and $y$ are grey and for which $v$ is the lowest common ancestor in the search tree, we compute the distances $d(x)$ and $d(y)$ to $v$
  - The length of the cycle involving $x, y$ and $v$ is then $d(x) + d(y) + 1$

- The minimum of these lengths at the level is the smallest cycle that involves $v$

- The smallest cycle among all possible start vertices $v$ is the girth.

# Example: Finding the Smallest Cycle involving Vertex 1
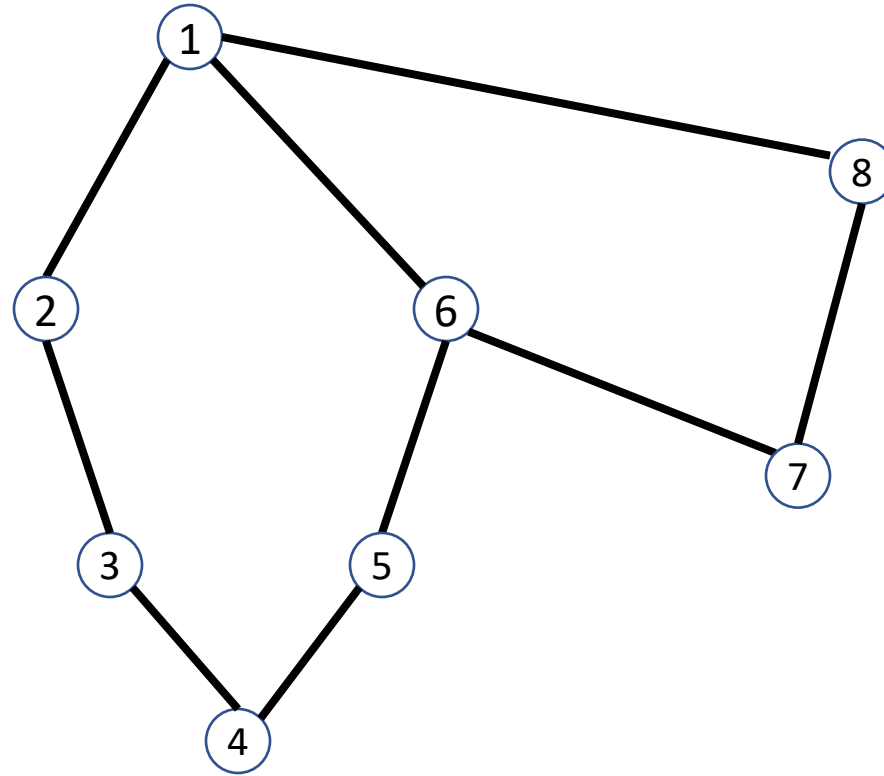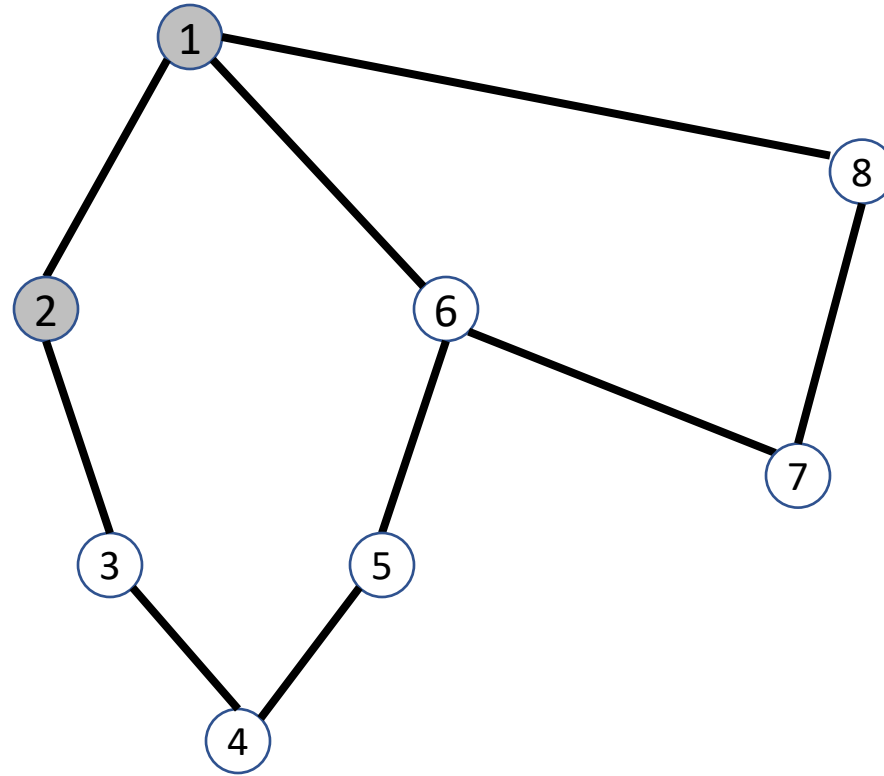
# Example: Finding the Smallest Cycle involving Vertex 1
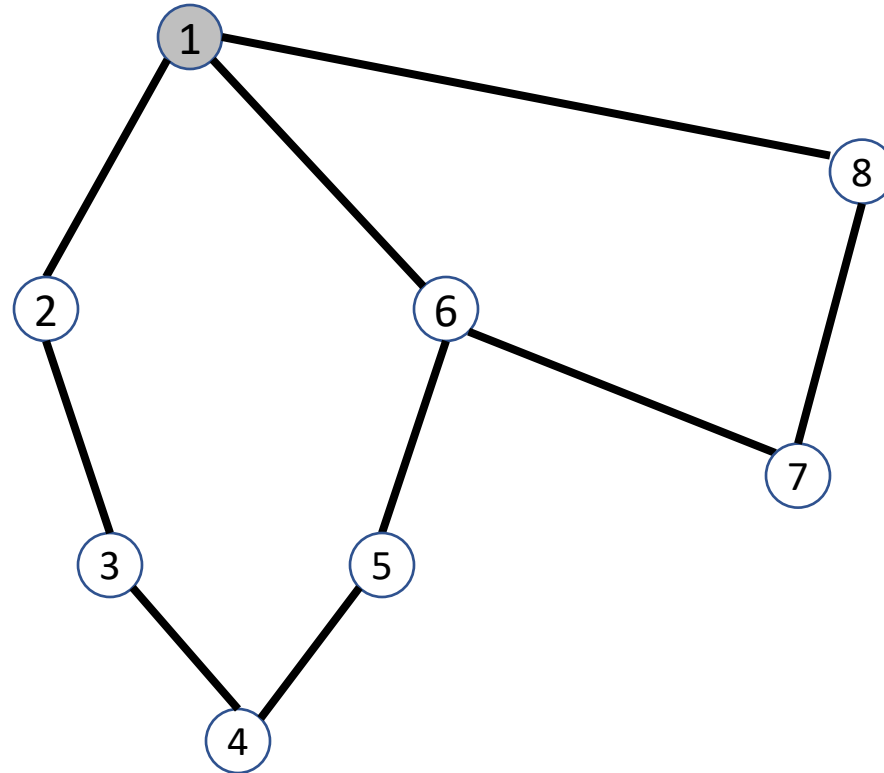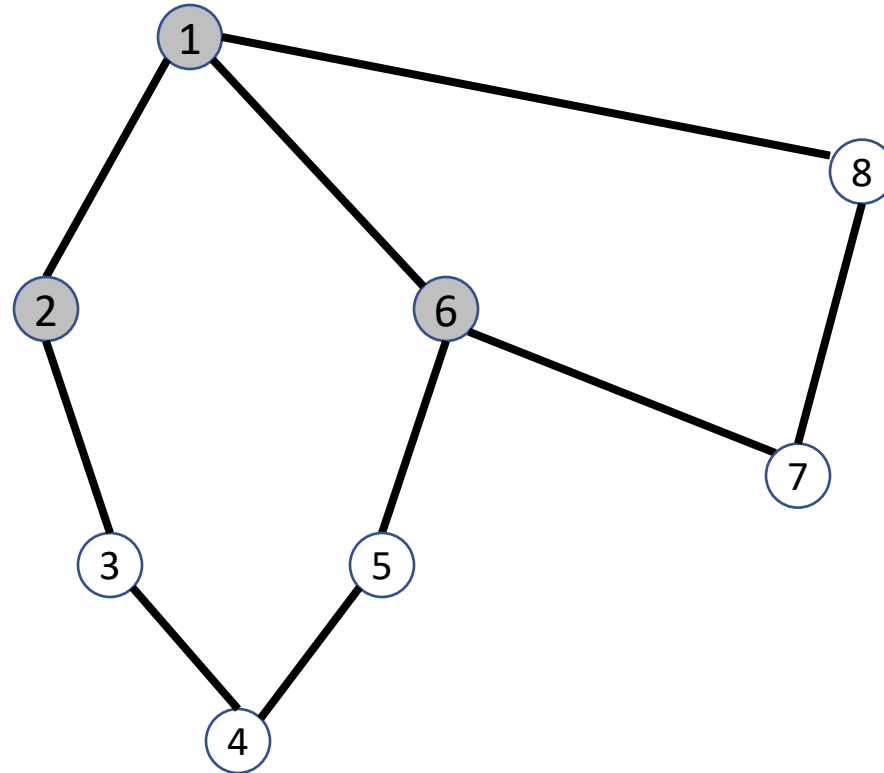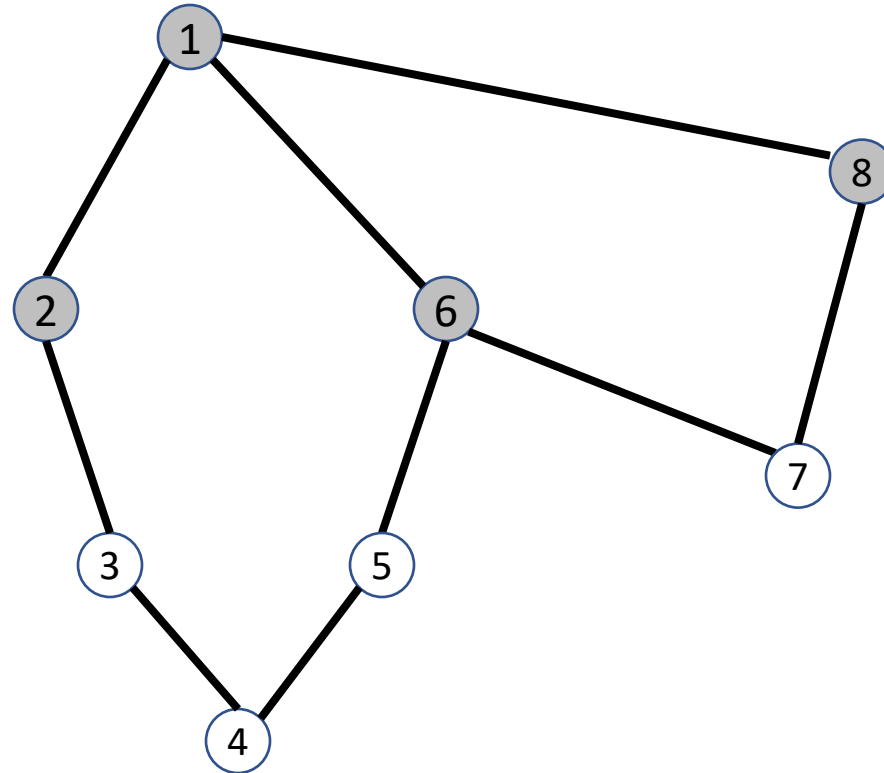


Queue: 1

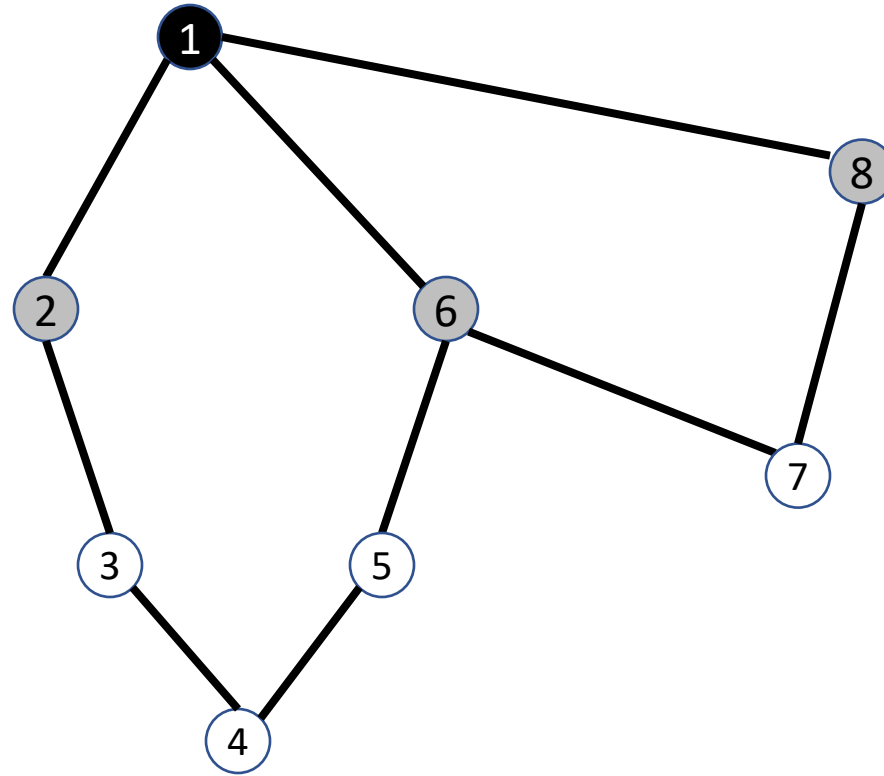# Example: Finding the Smallest Cycle involving Vertex 1



Queue: 1 2

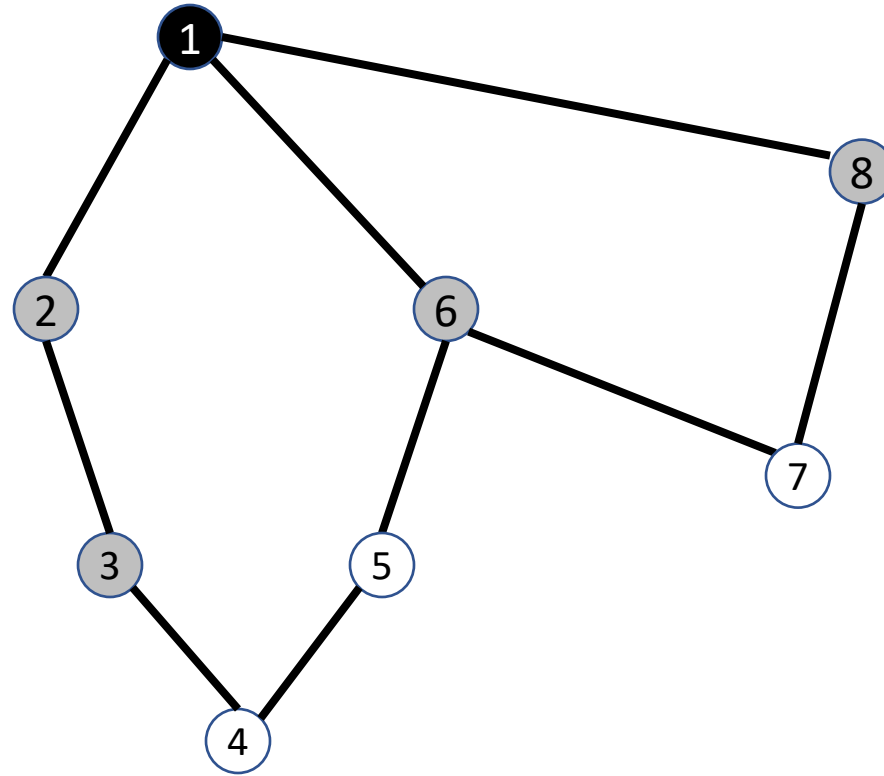# Example: Finding the Smallest Cycle involving Vertex 1



Queue: 1 2 6

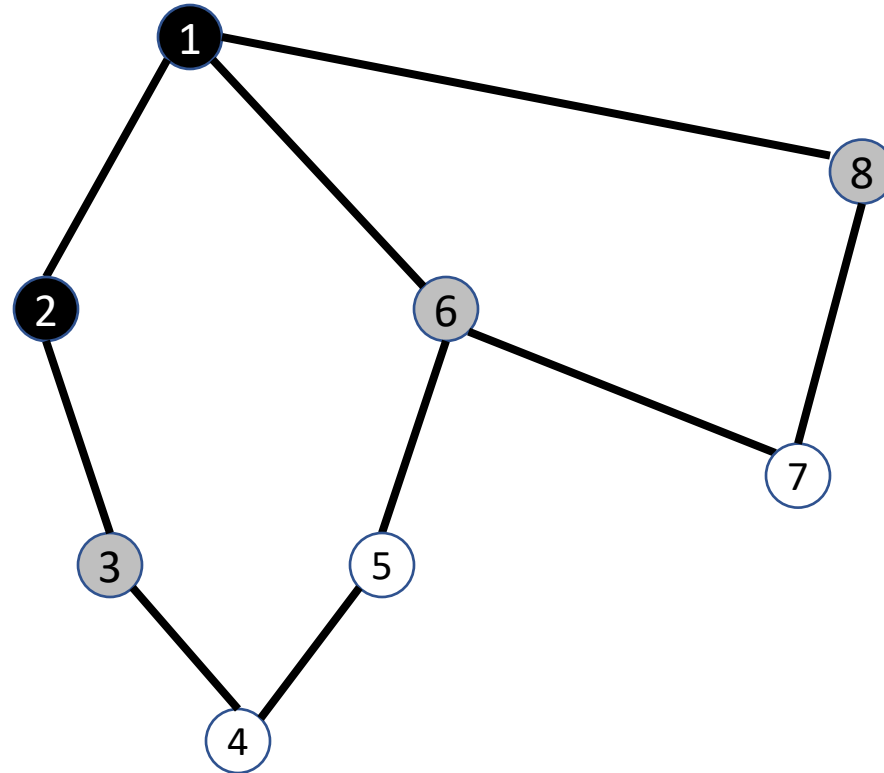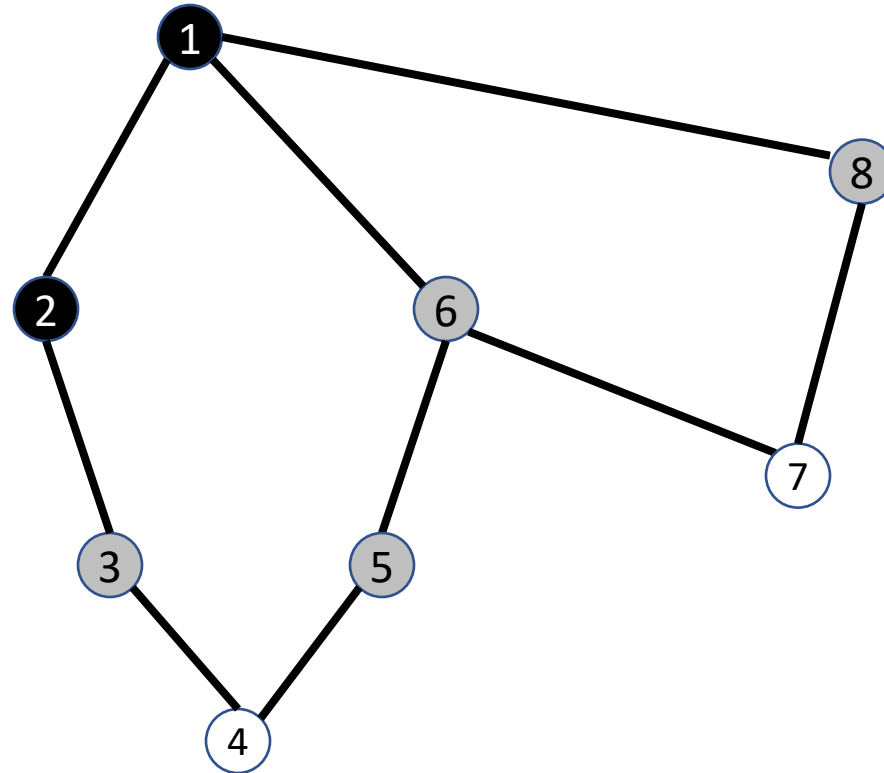# Example: Finding the Smallest Cycle involving Vertex 1



Queue: 1 2 6 8

# Example: Finding the Smallest Cycle involving Vertex 1



Queue: 2 6 8

# Example: Finding the Smallest Cycle involving Vertex 1



Queue: 2 6 8 3

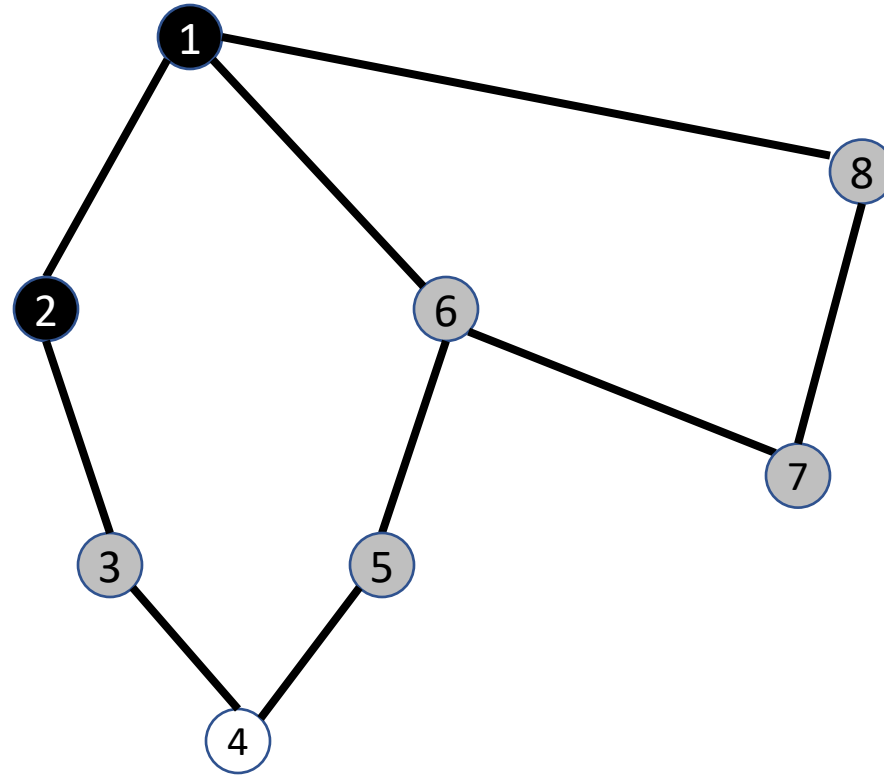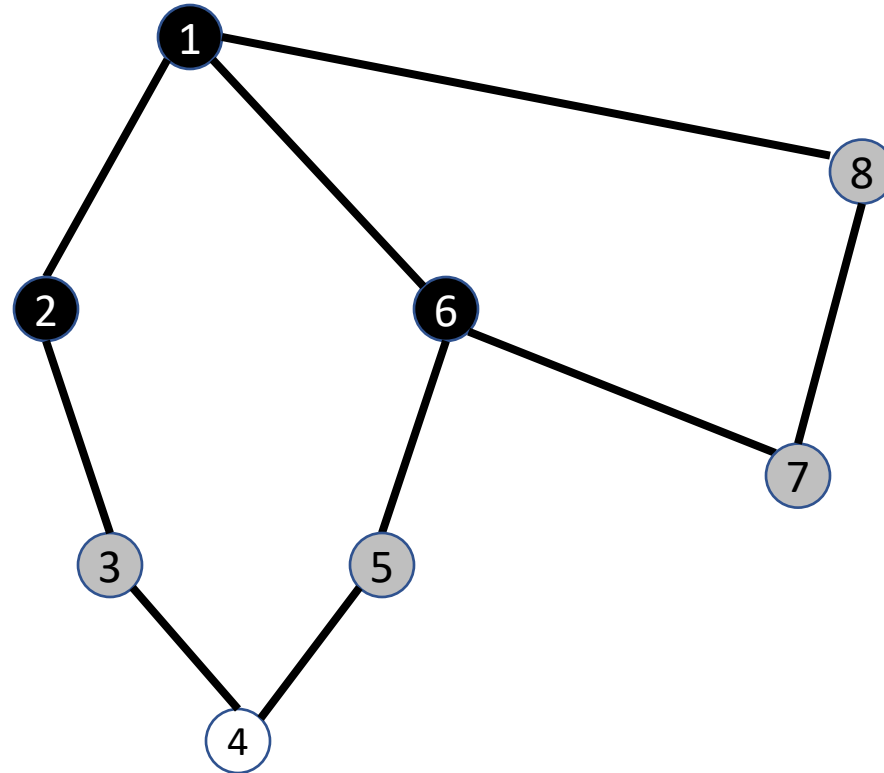# Example: Finding the Smallest Cycle involving Vertex 1



Queue: 6 8 3

# Example: Finding the Smallest Cycle involving Vertex 1



Queue: 6 8 3 5

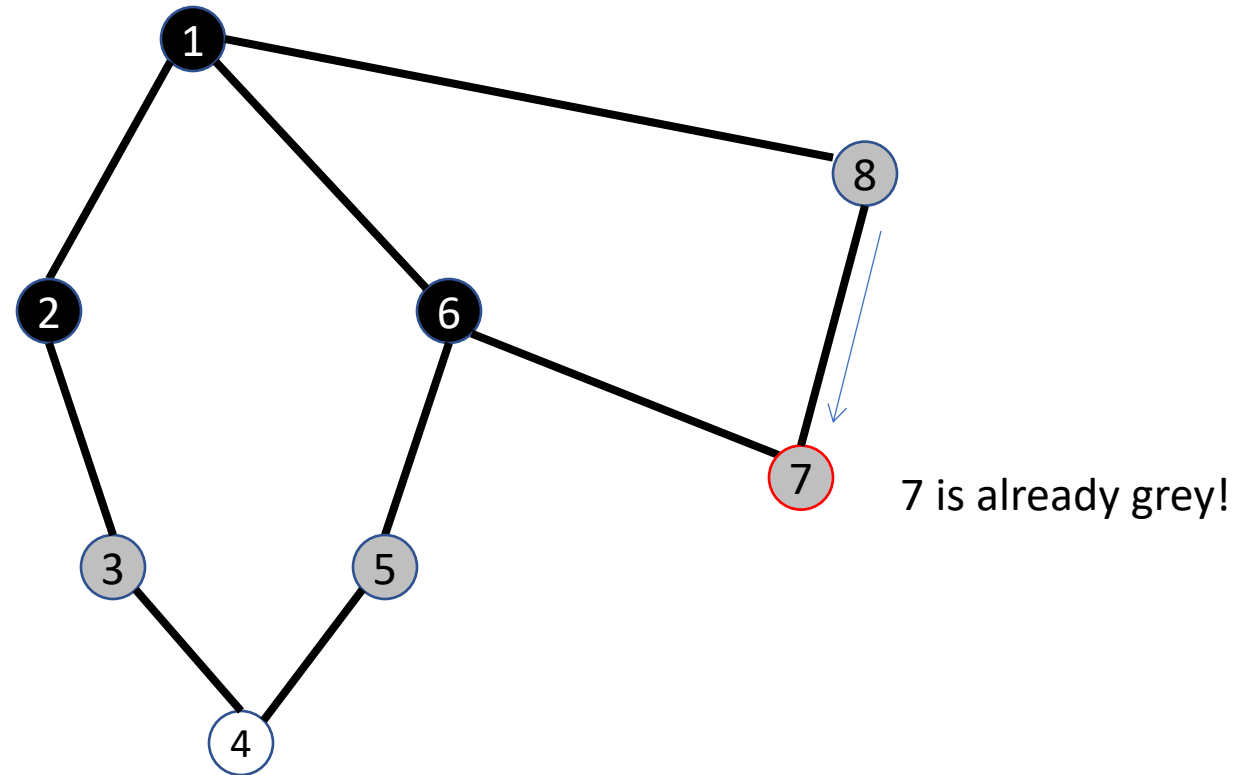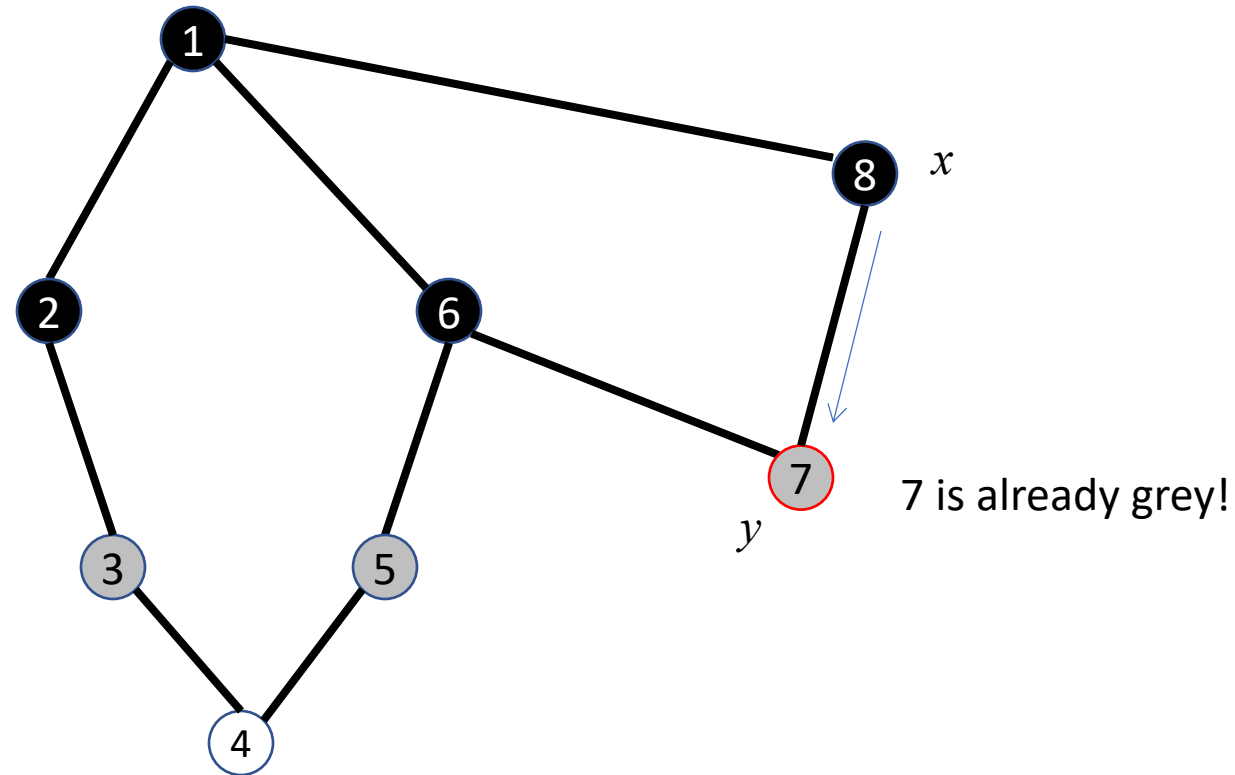# Example: Finding the Smallest Cycle involving Vertex 1



Queue: 6 8 3 5 7

# Example: Finding the Smallest Cycle involving Vertex 1



Queue: 8 3 5 7

# Example: Finding the Smallest Cycle involving Vertex 1
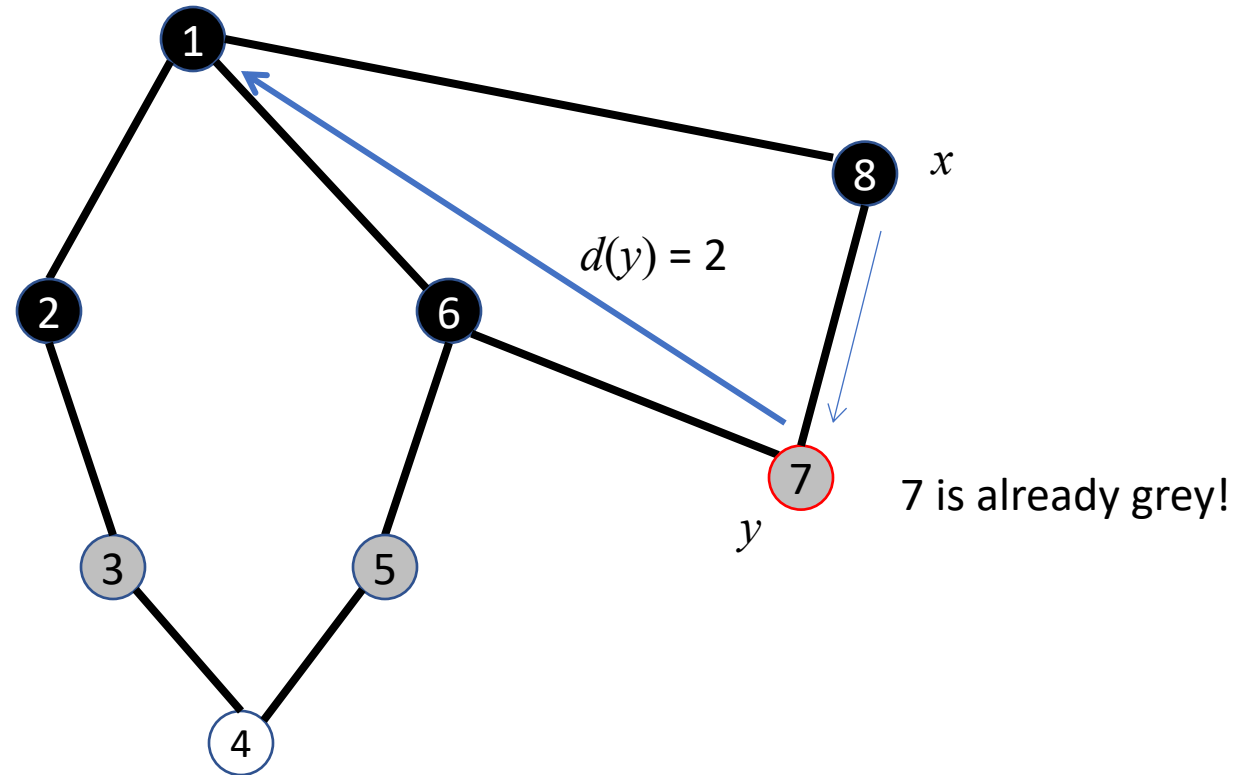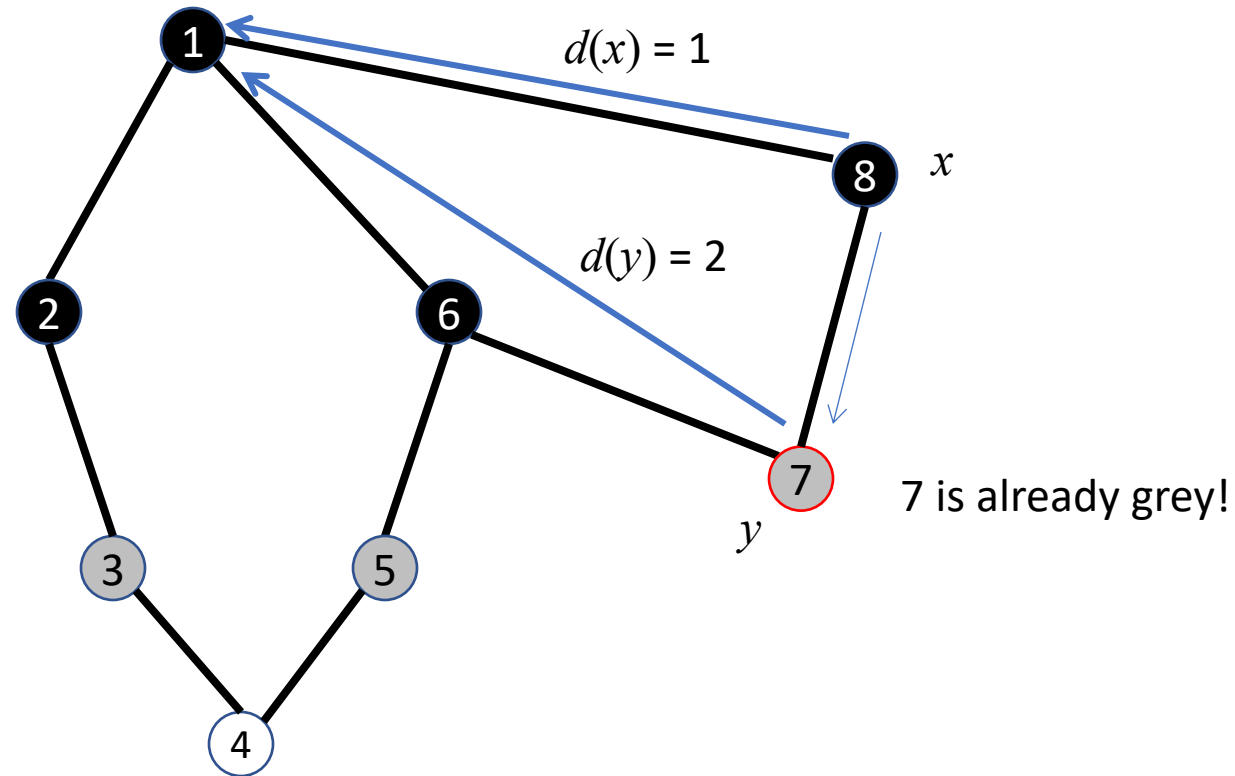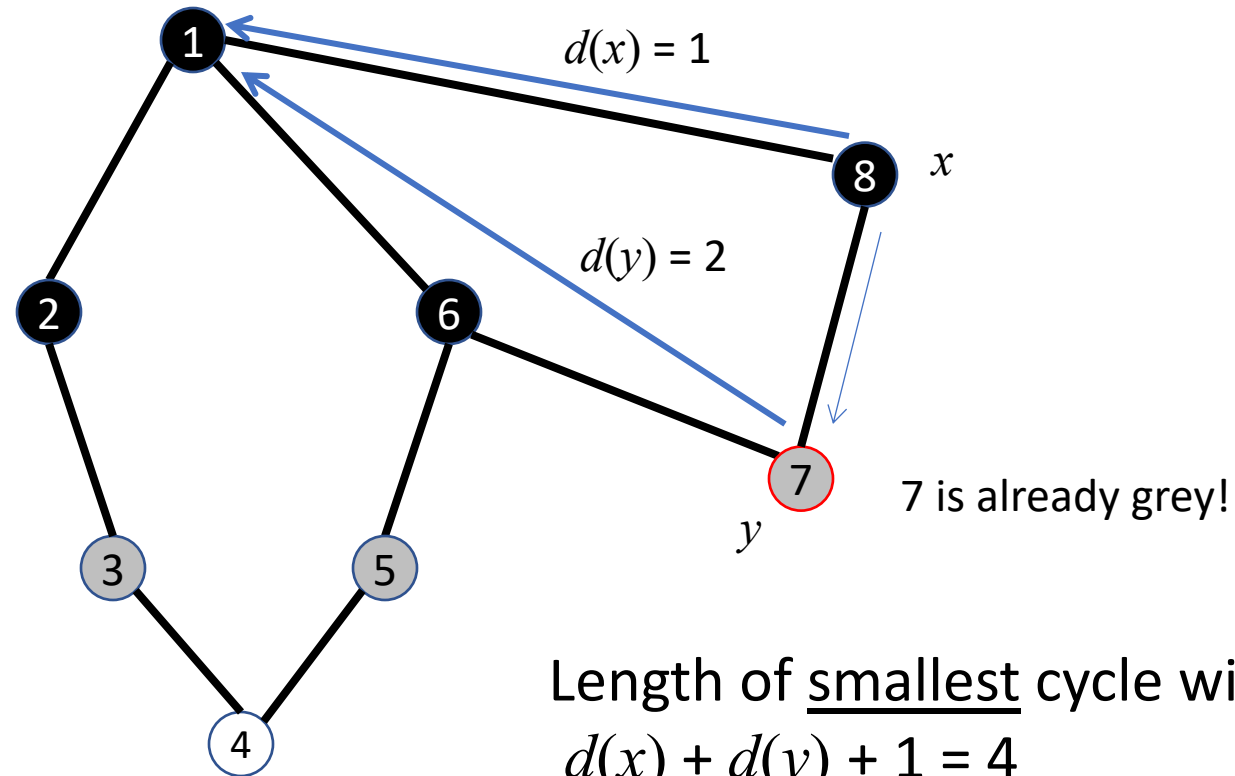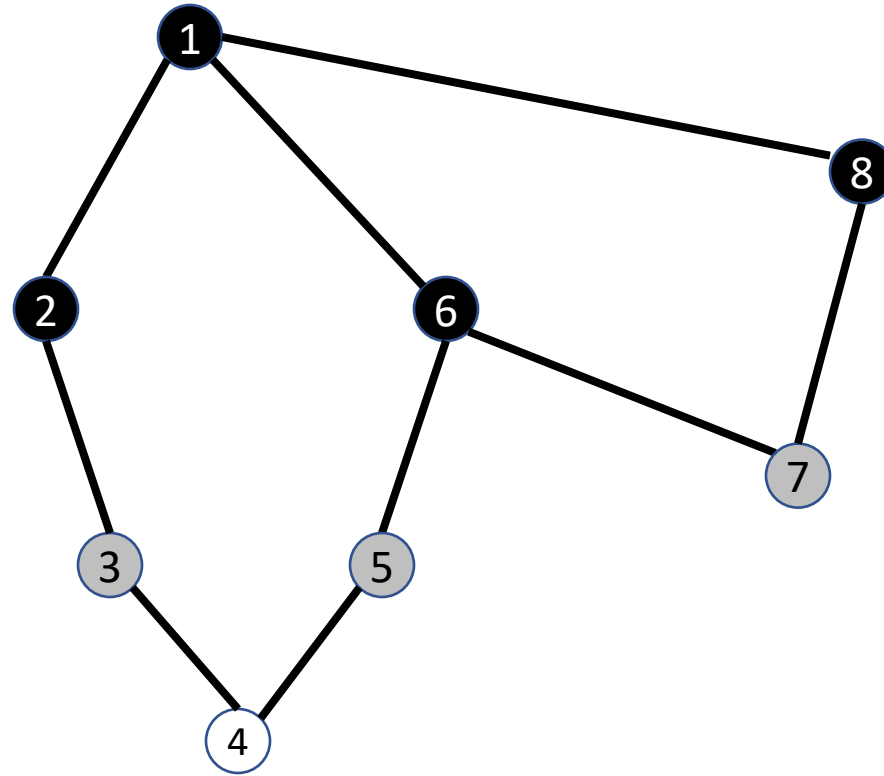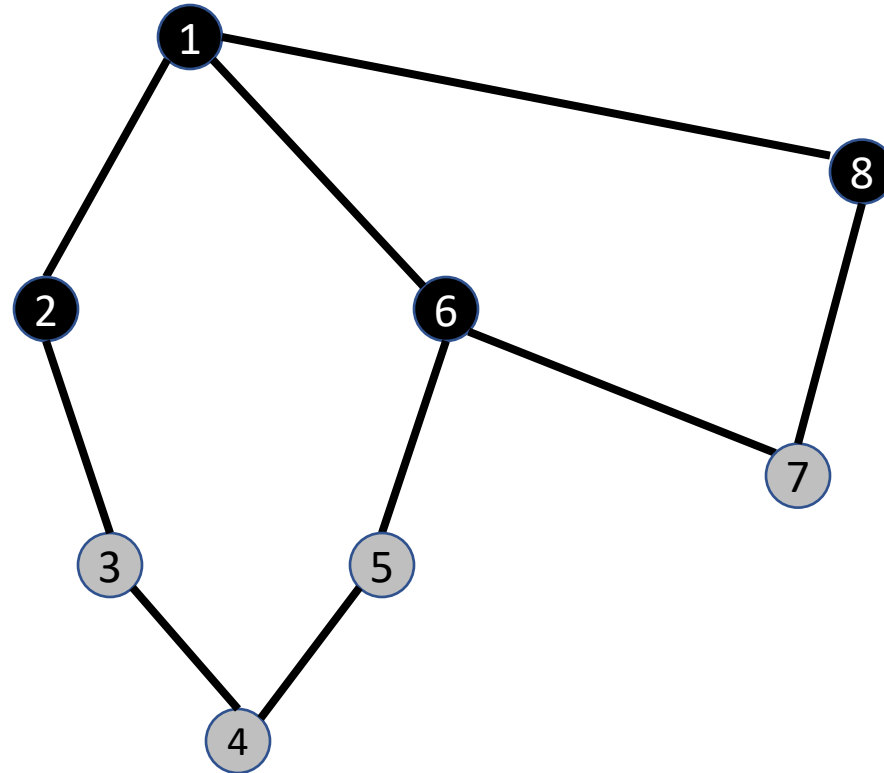


7 is already grey!

Queue: 8 3 5 7

# Example: Finding the Smallest Cycle involving Vertex 1



*x*

*y*

7 is already grey!

Queue: 3 5 7 (stopped)

# Example: Finding the Smallest Cycle involving Vertex 1



$x$

$d(y) = 2$

7 is already grey!

$y$

Queue: 3 5 7 (stopped)

# Example: Finding the Smallest Cycle involving Vertex 1



$d(x) = 1$

$x$

$d(y) = 2$

$y$

7 is already grey!

Queue: 3 5 7 (stopped)

# Example: Finding the Smallest Cycle involving Vertex 1



$d(x) = 1$

$x$

$d(y) = 2$

7 is already grey!

$y$

Length of <u>smallest</u> cycle with vertex 1

$d(x) + d(y) + 1 = 4$

Queue: 3 5 7 (stopped)

# Example: Finding the Smallest Cycle involving Vertex 1
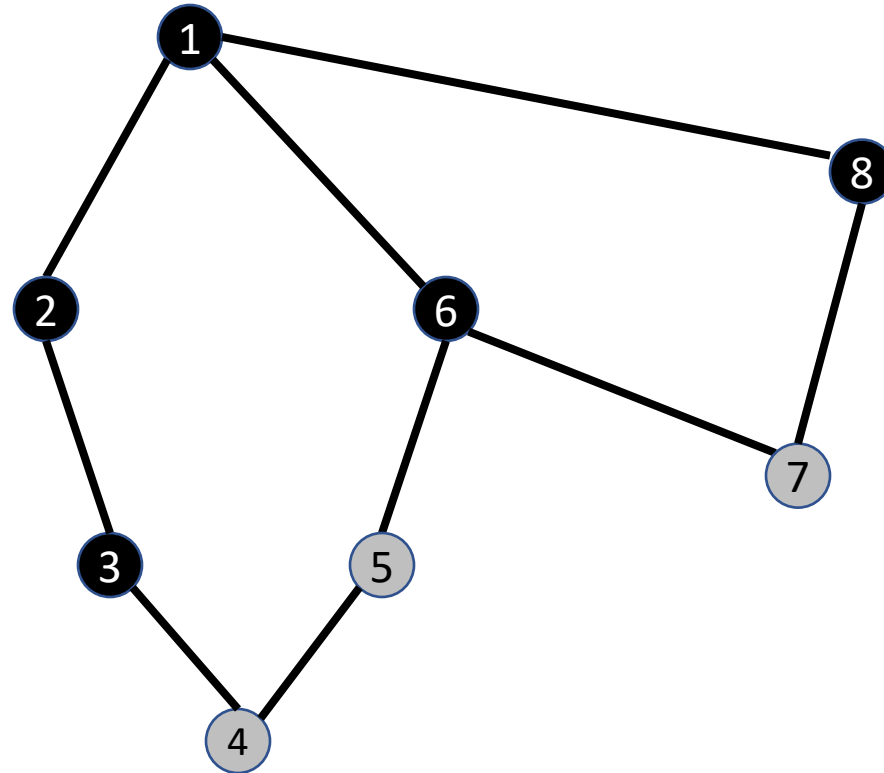


Queue: 3 5 7

# Example: Finding the Smallest Cycle involving Vertex 1
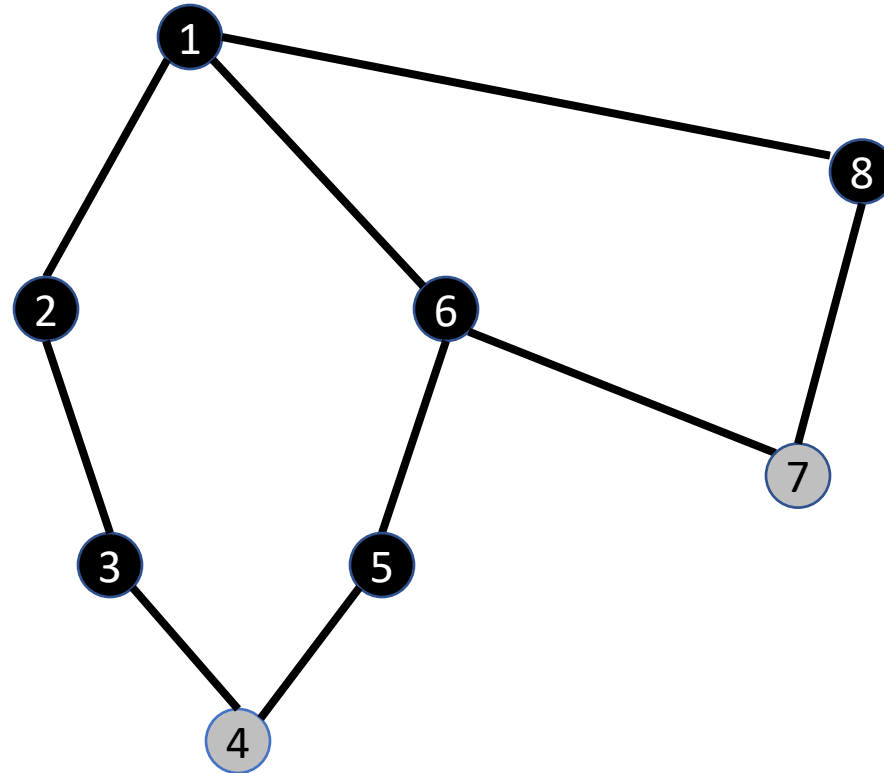


Queue: 3 5 7 4
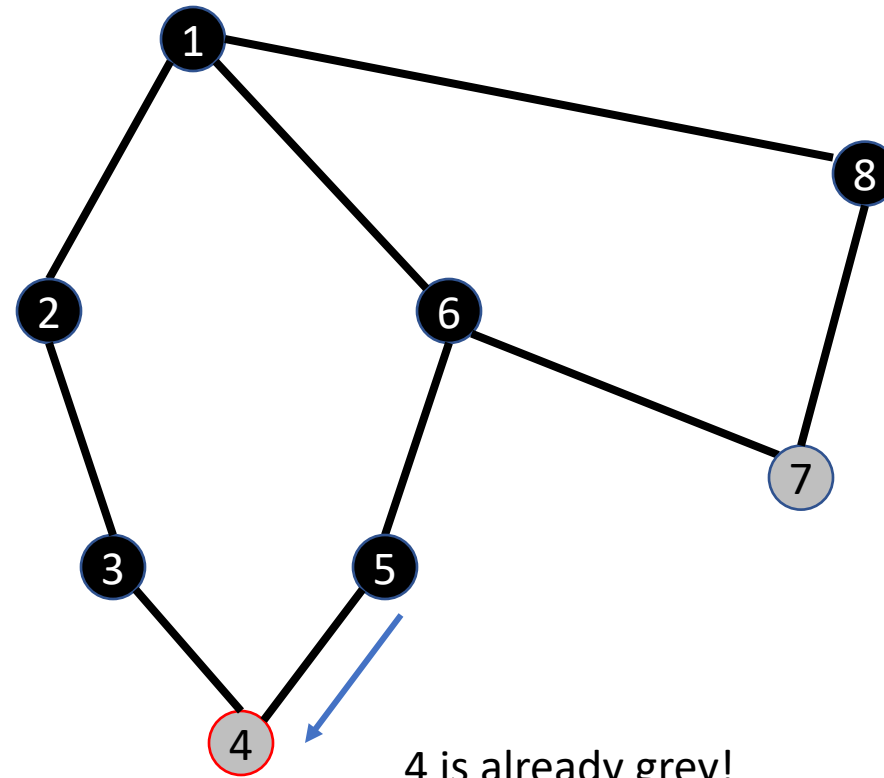
# Example: Finding the Smallest Cycle involving Vertex 1



Queue: 5 7 4

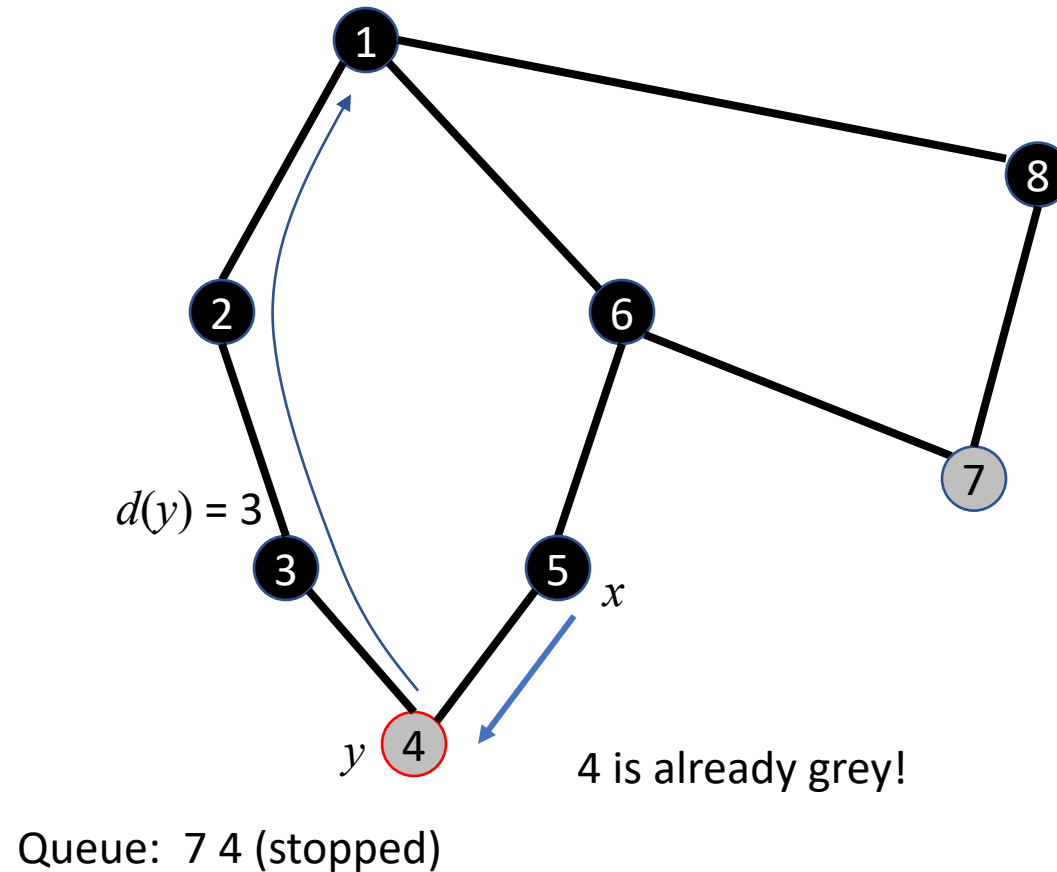# Example: Finding the Smallest Cycle involving Vertex 1



Queue:  7 4
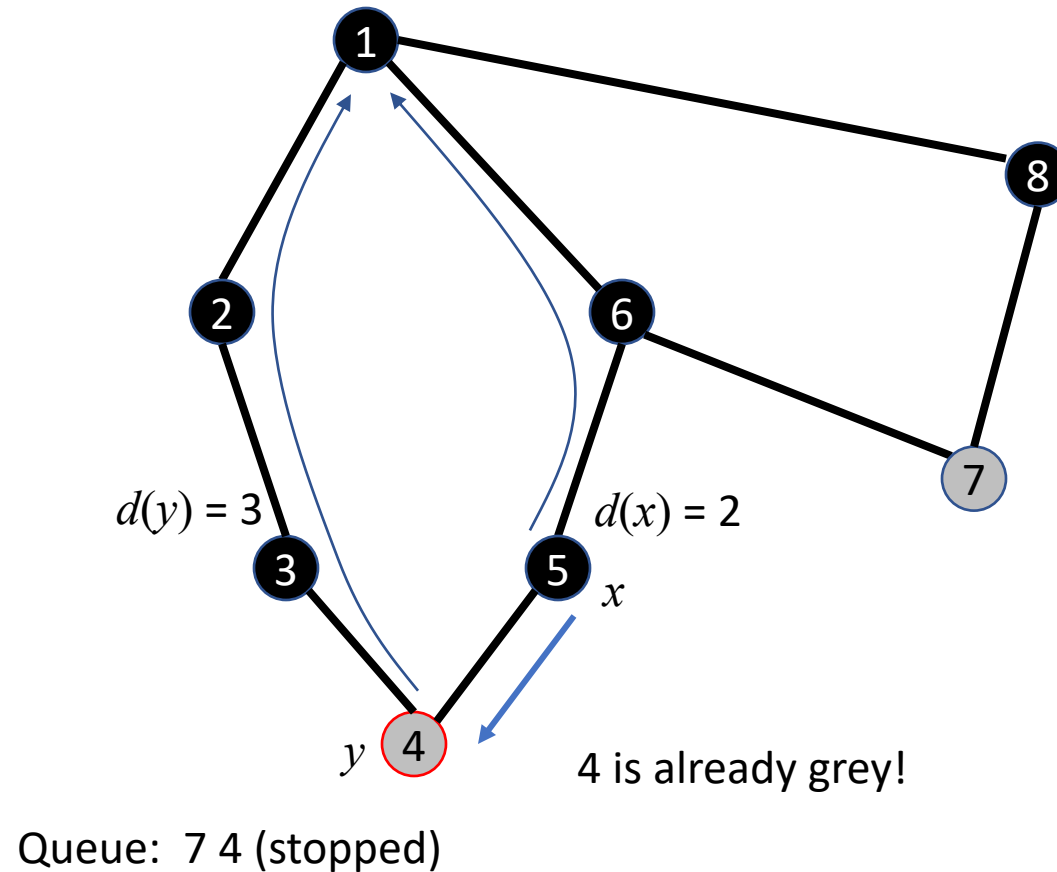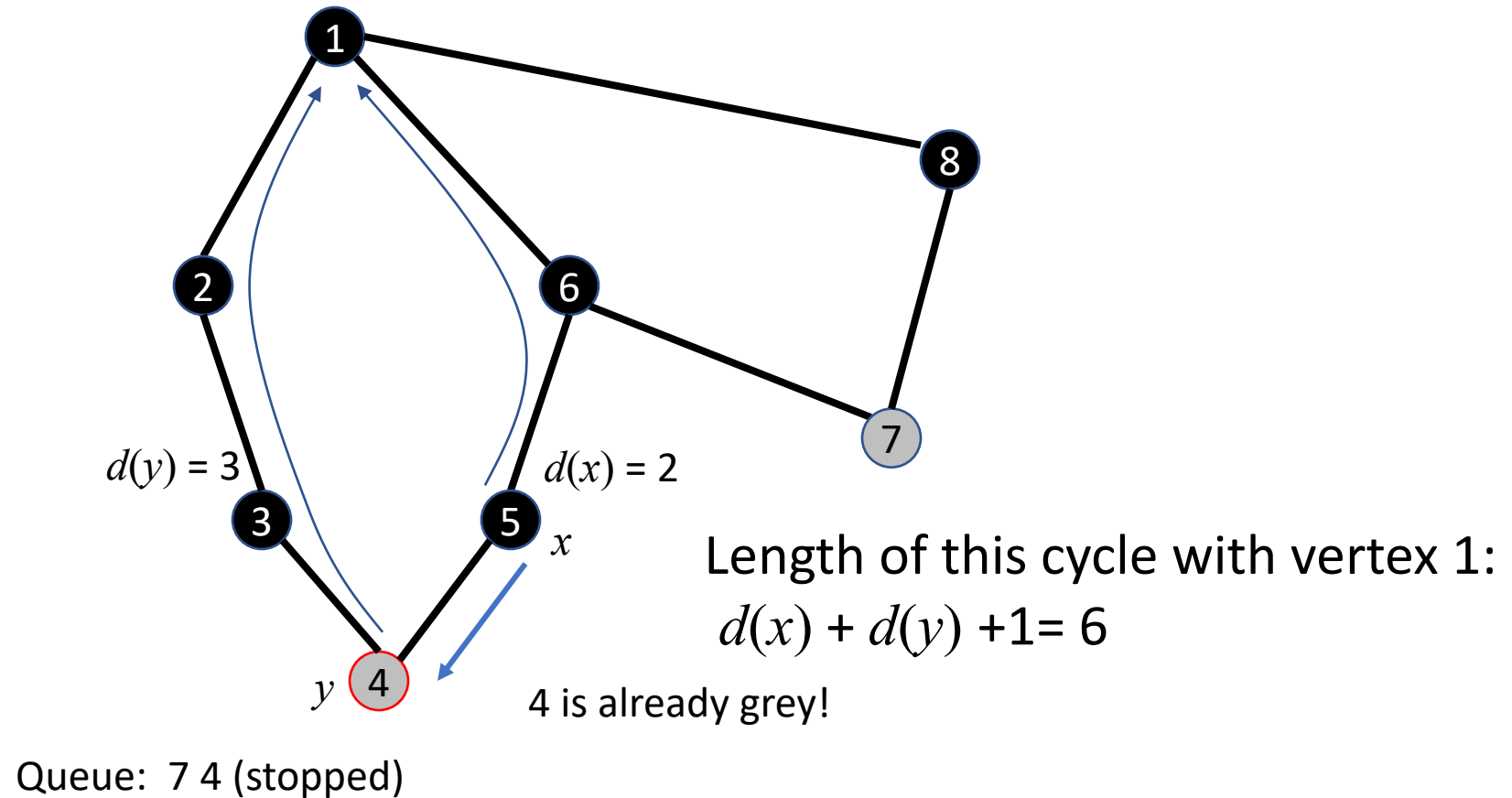
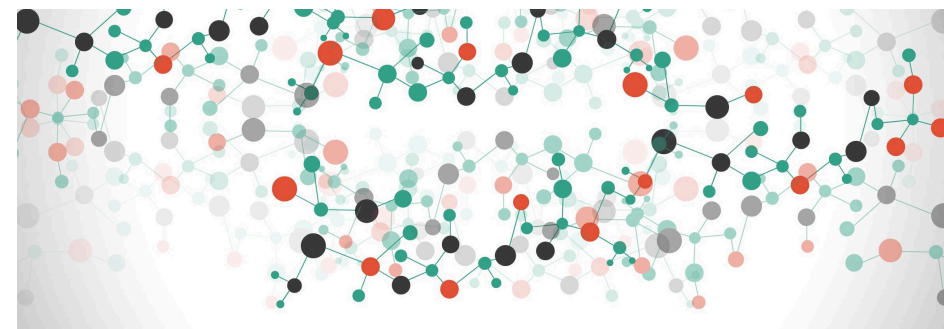# Example: Finding the Smallest Cycle involving Vertex 1



4 is already grey!

Queue:  7 4 (stopped)

# Example: Finding the Smallest Cycle involving Vertex 1



$d(y) = 3$

$y$

$x$

4 is already grey!

Queue:  7 4 (stopped)

# Example: Finding the Smallest Cycle involving Vertex 1



$d(y) = 3$

$d(x) = 2$

$x$

$y$

4 is already grey!

Queue: 7 4 (stopped)

# Example: Finding the Smallest Cycle involving Vertex 1



$d(y) = 3$

$d(x) = 2$

$x$

$y$

Length of this cycle with vertex 1:

$$d(x) + d(y) + 1 = 6$$

4 is already grey!

Queue: 7 4 (stopped)

# SUMMARY

- Terminology
  - Walk
  - Cycle
  - Path
  - Girth
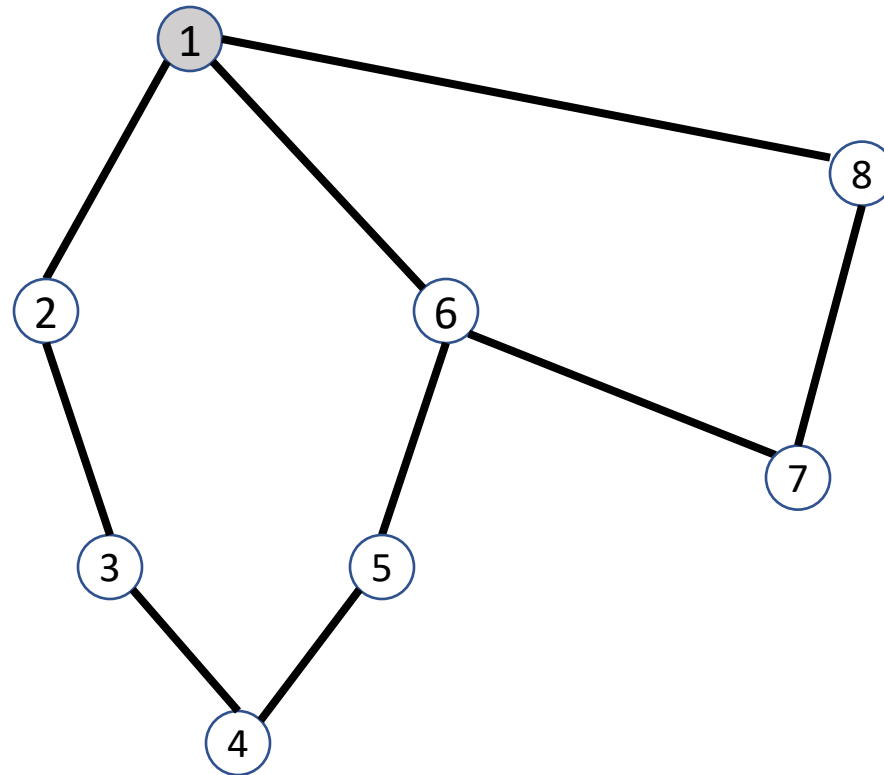
- Finding the Smallest Cycle

# Exercises

- Given a graph $G$, perform a BFSVisit from vertex 1, reporting

- The length of the smallest cycle starting from vertex 1

- The order in which the BFSVisit pops the vertices from the queue

- For each cycle the visit generates, give the cycle length and the vertices involved in clockwise order

- [Assume that we pick the nodes with the smallest label whenever the algorithm has multiple choices.]
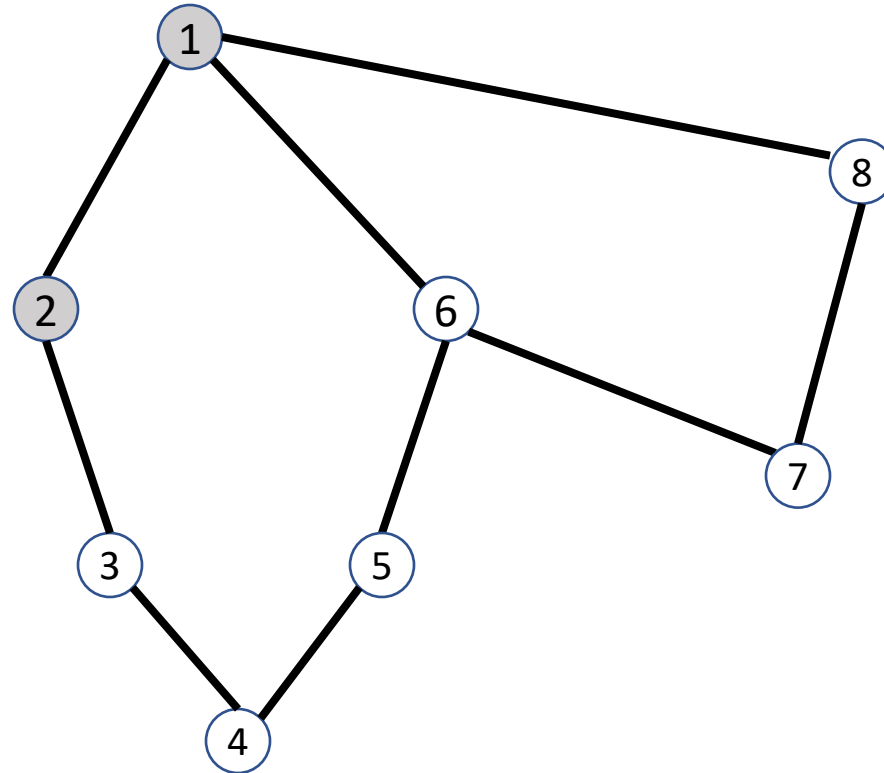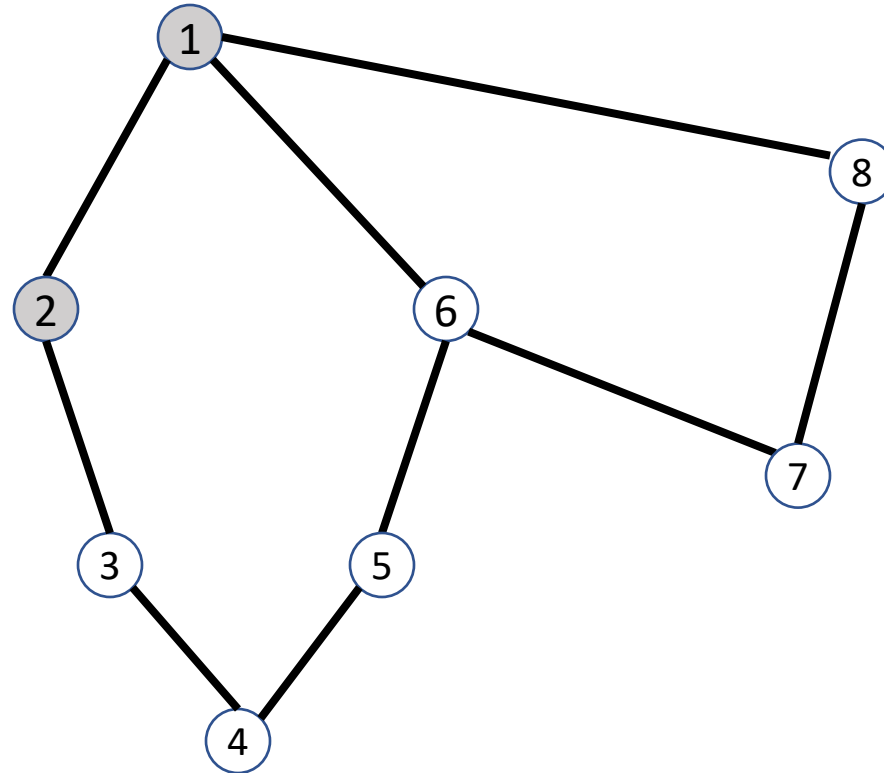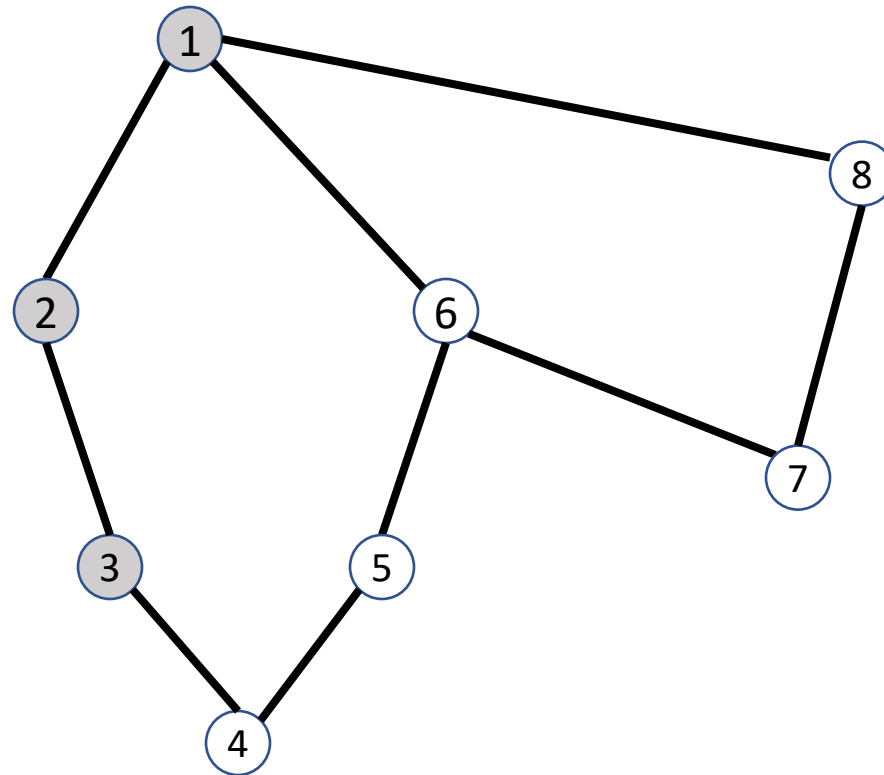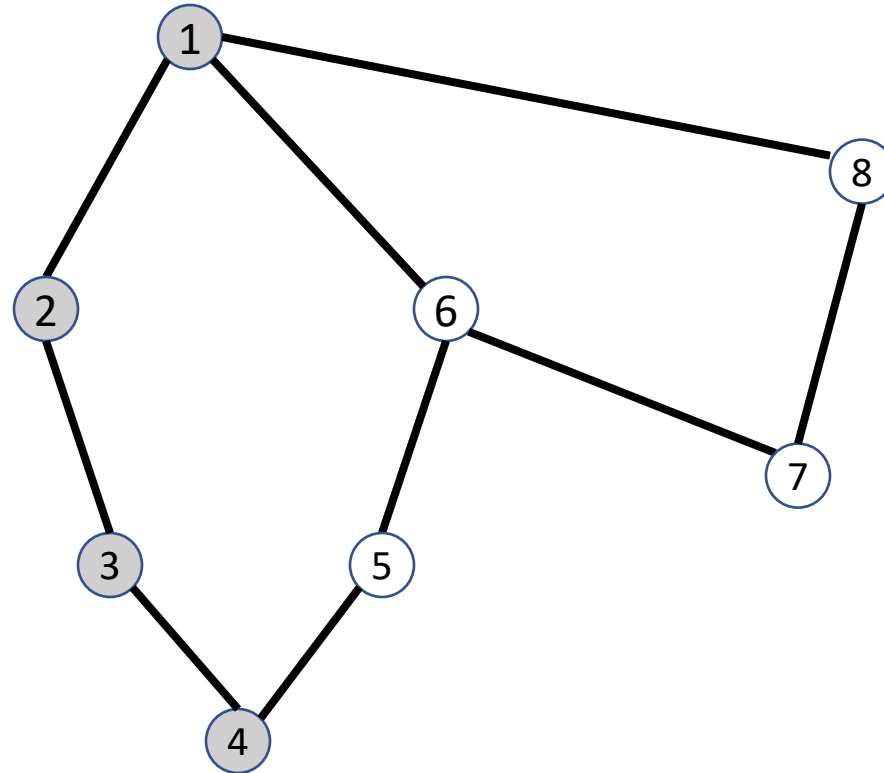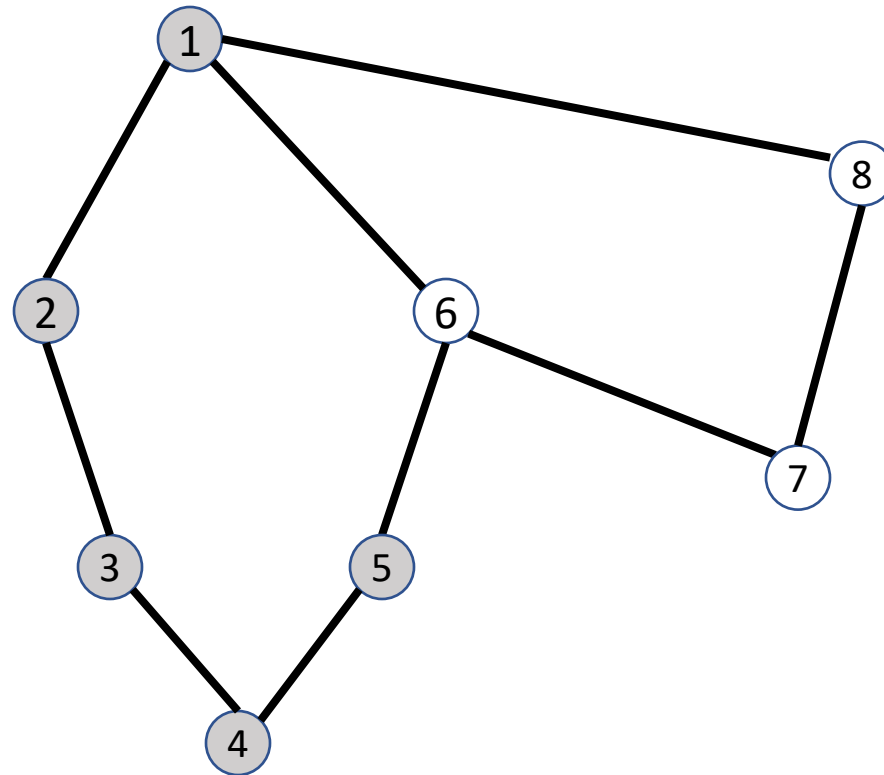
# Exercise: Finding Cycle involving Vertex 1 using DFS
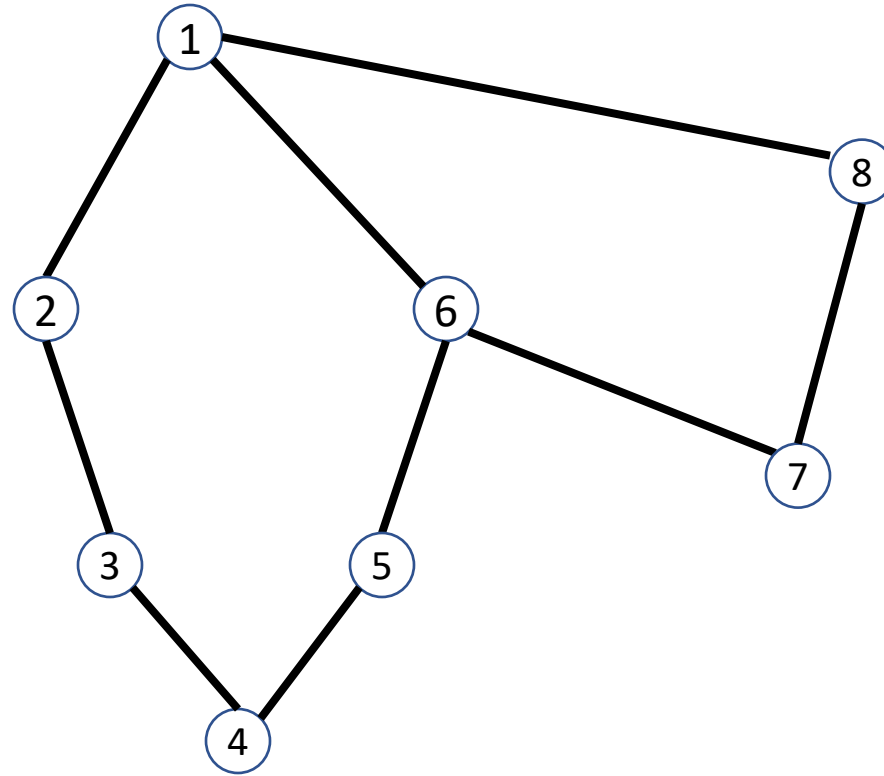
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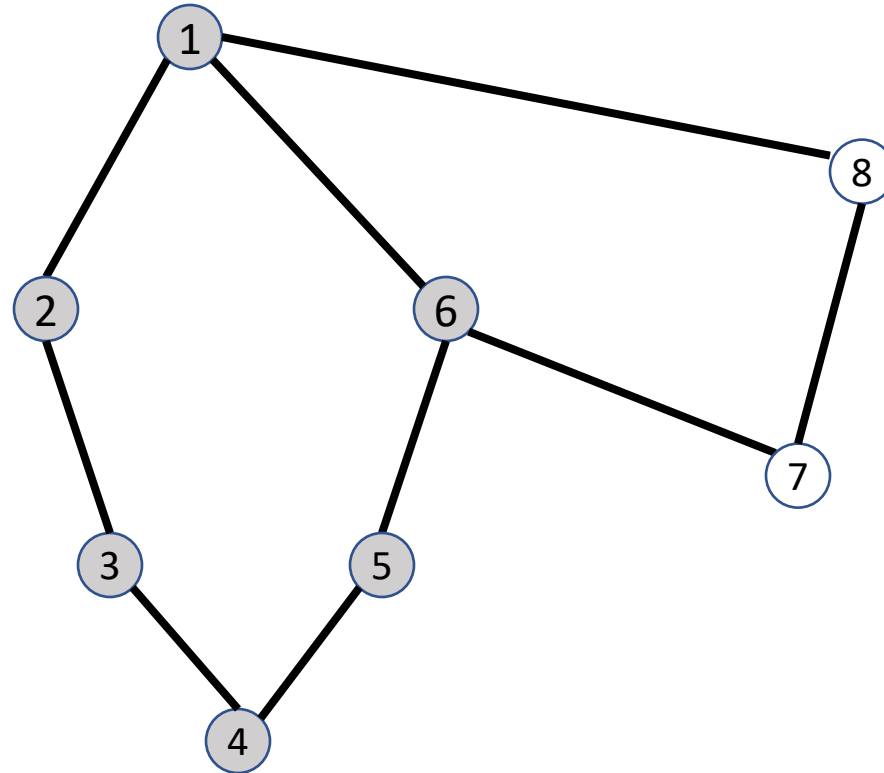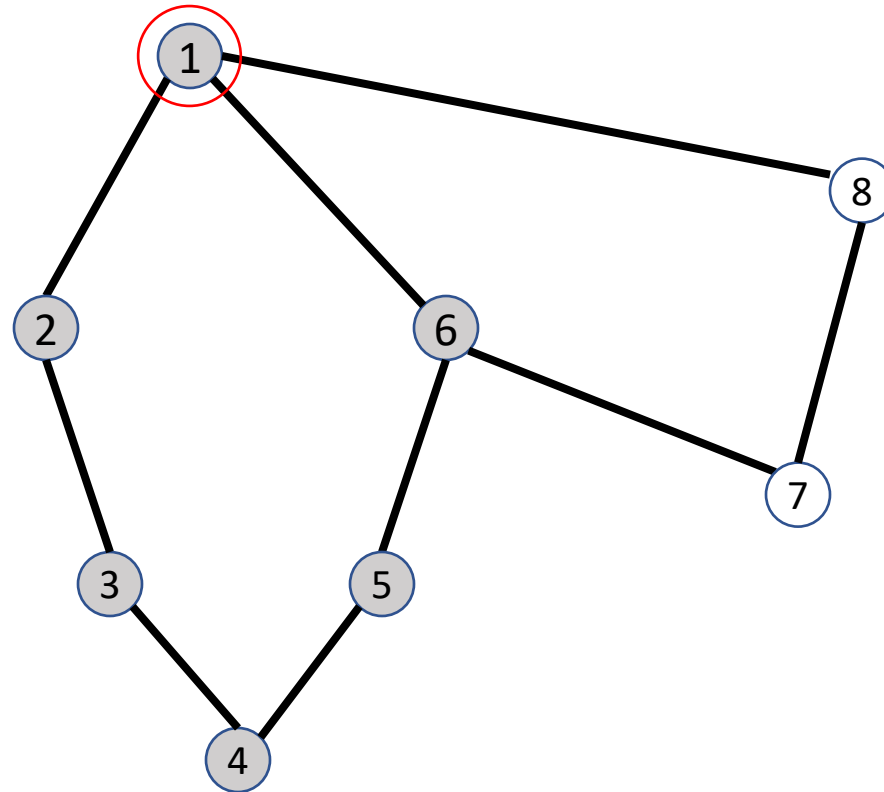
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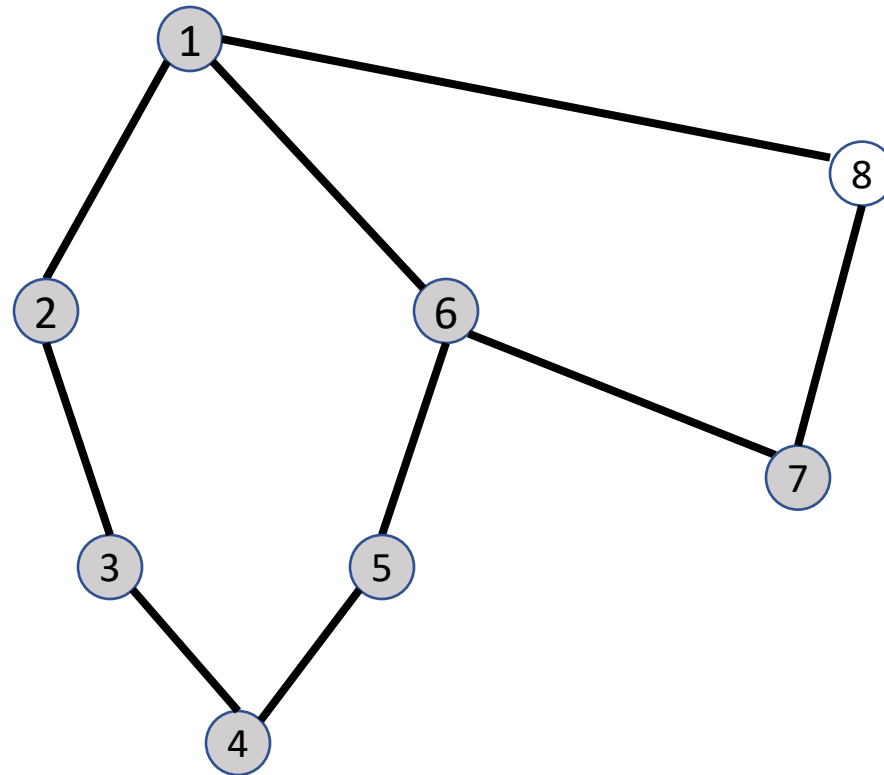
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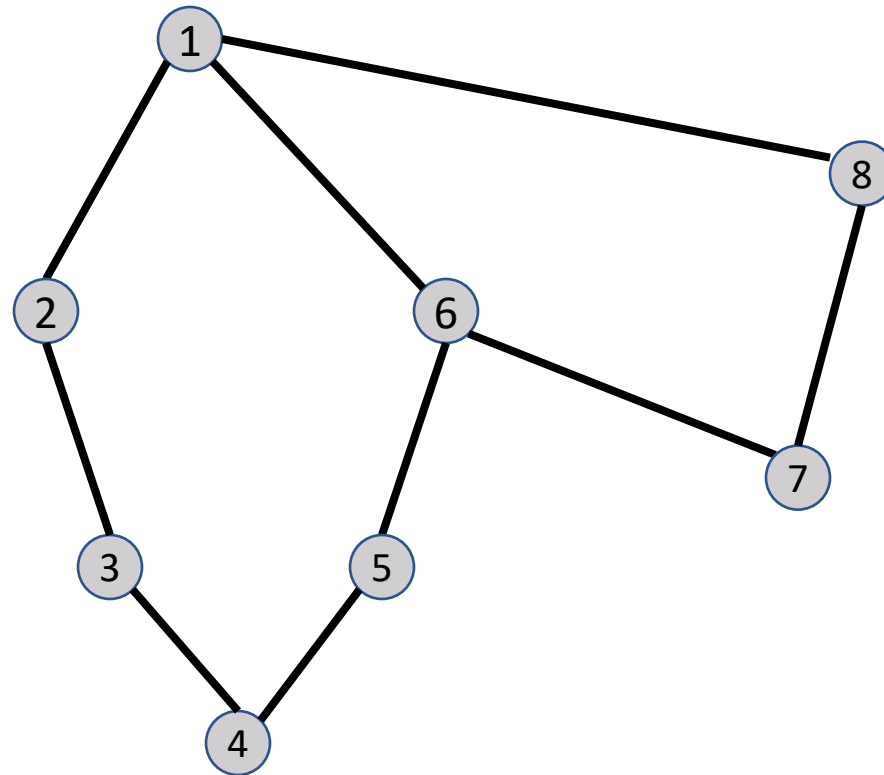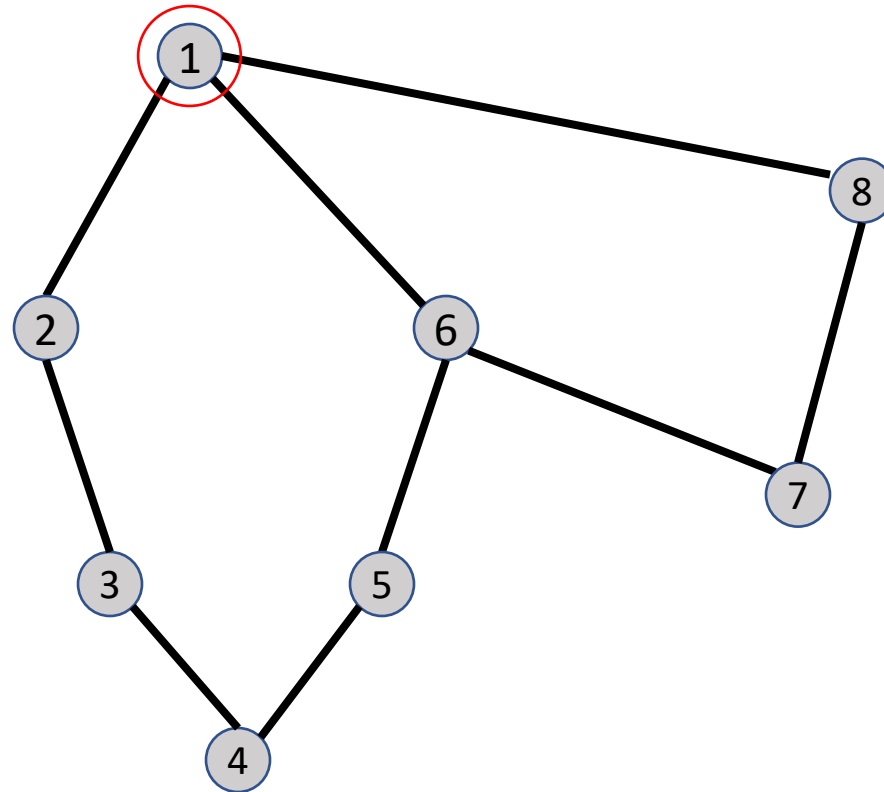# Exercise: Finding Cycle involving Vertex 1 using DFS

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