Binary Search Trees

COMPSCI 220: WEEK 8.6

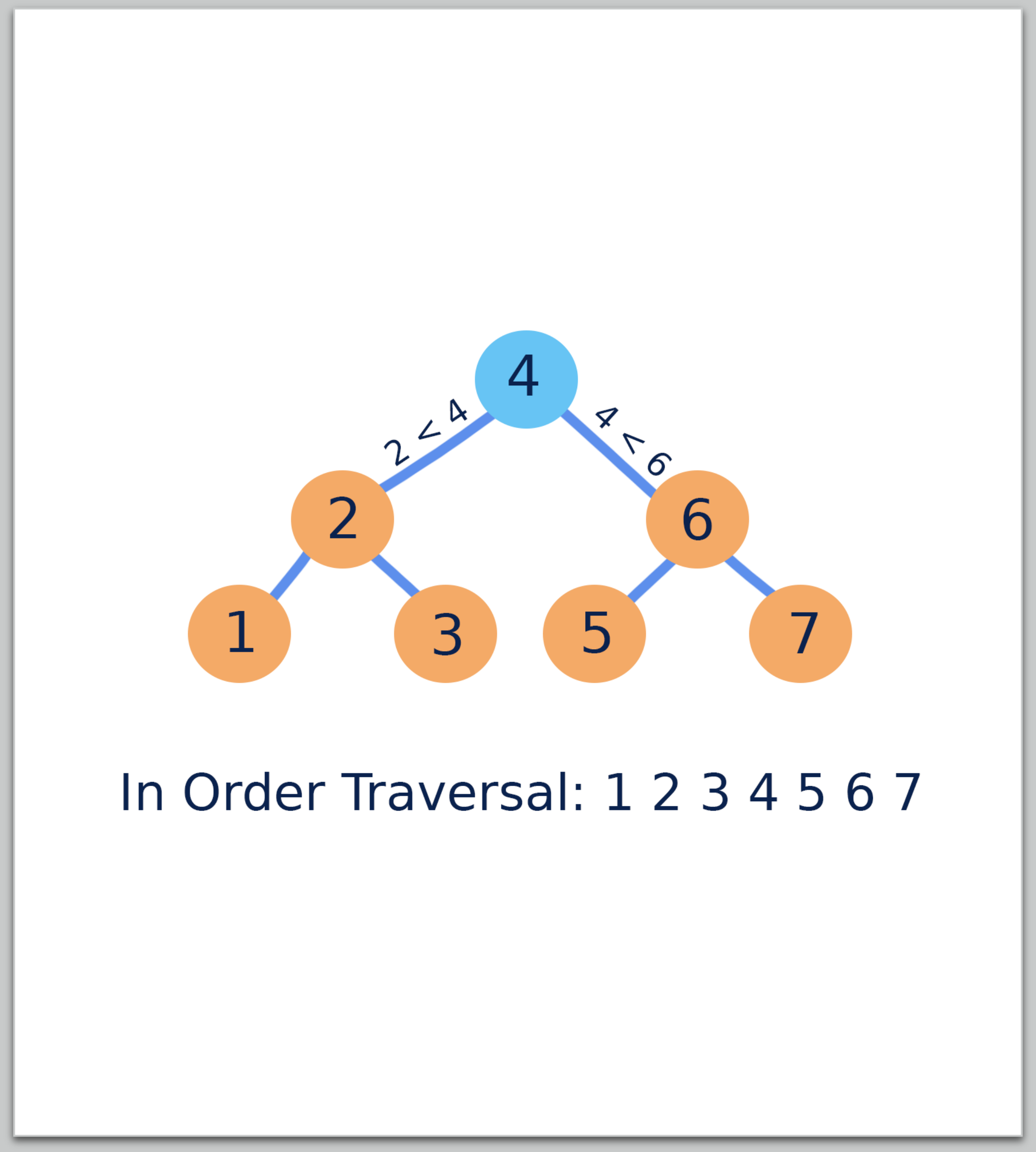
Instructor: Meng-Fen Chiang

THE UNIVERSITY OF AUCKLAND
Te Whare Wananga o Tamaki Makaurau
NEW ZEALAND

Slides adapted from Mark Wilson, Georgy Gimel'farb, Simone Linz and Tanya Gvozdeva

OUTLINE

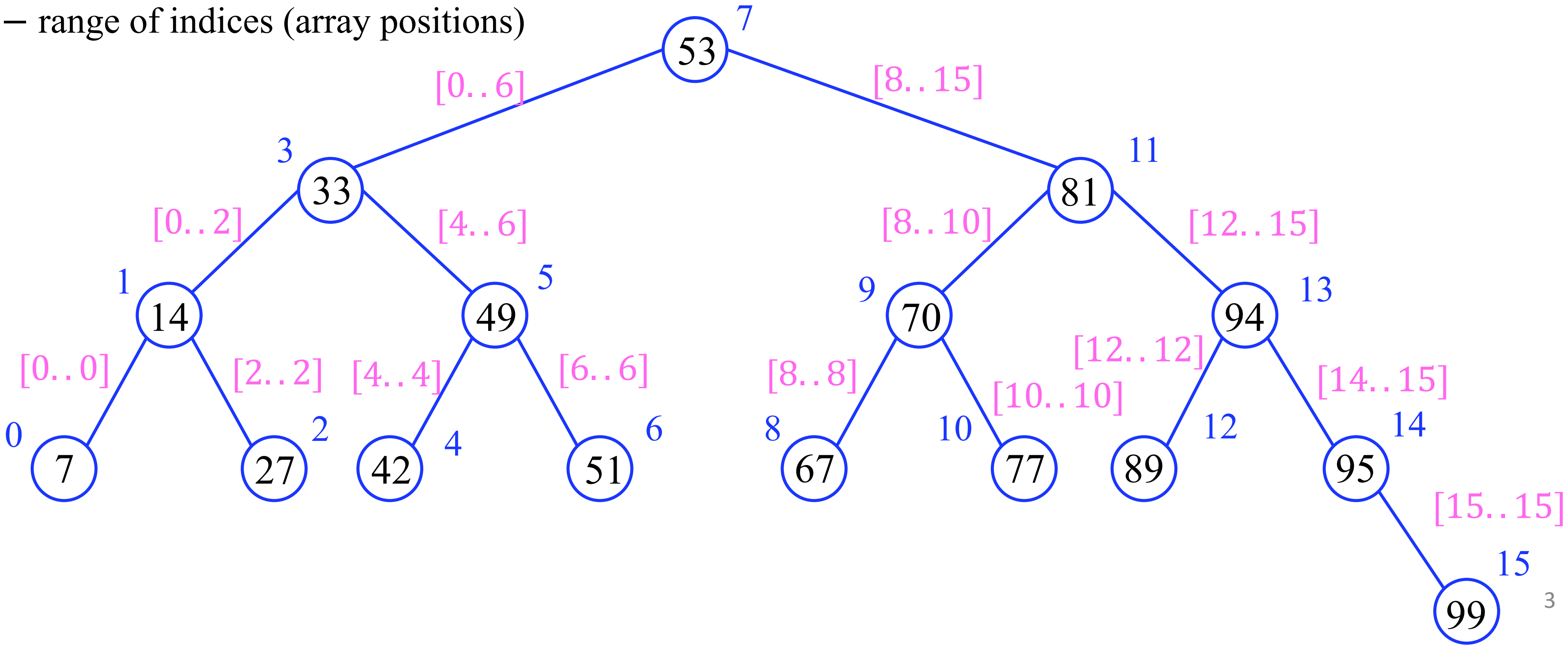
- Tree Data Structure
- Binary Search Tree Operations
- Time Complexity Analysis



Tree Structure of Binary Search

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
7	14	27	33	42	49	51	53	67	70	77	81	89	94	95	99

$[l..r]$ – range of indices (array positions)



Binary Search Tree

- The execution of Binary search (looking for all possible keys) can be described by a decision tree which is called a (static) **binary search tree**.

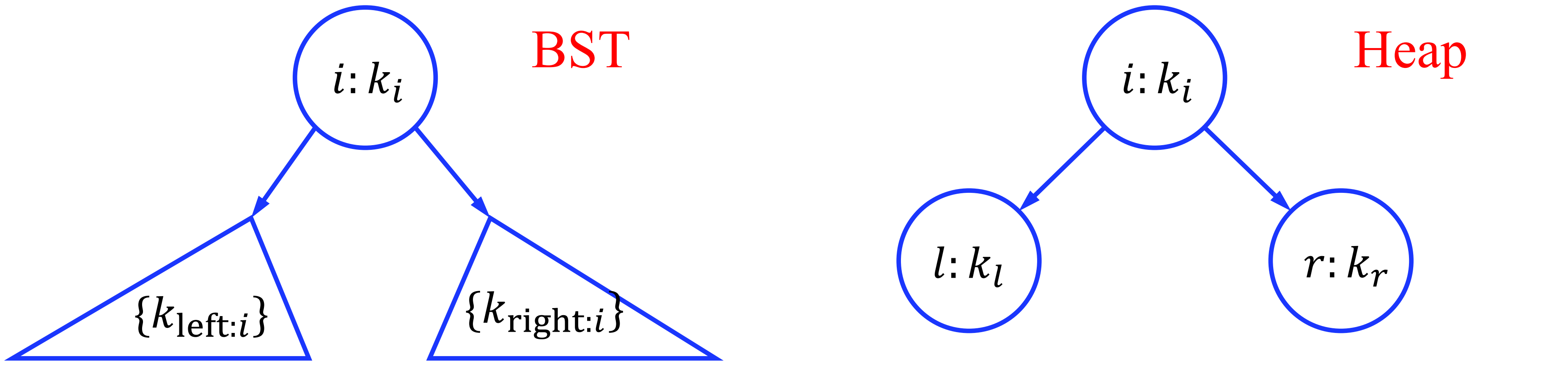
- A binary search tree (**BST**) is a binary tree with the following properties:

1. keys stored in nodes
2. key of each node is $\geq$ the key of every node in the left subtree
3. key of each node is $\leq$ the key of every node in the right subtree

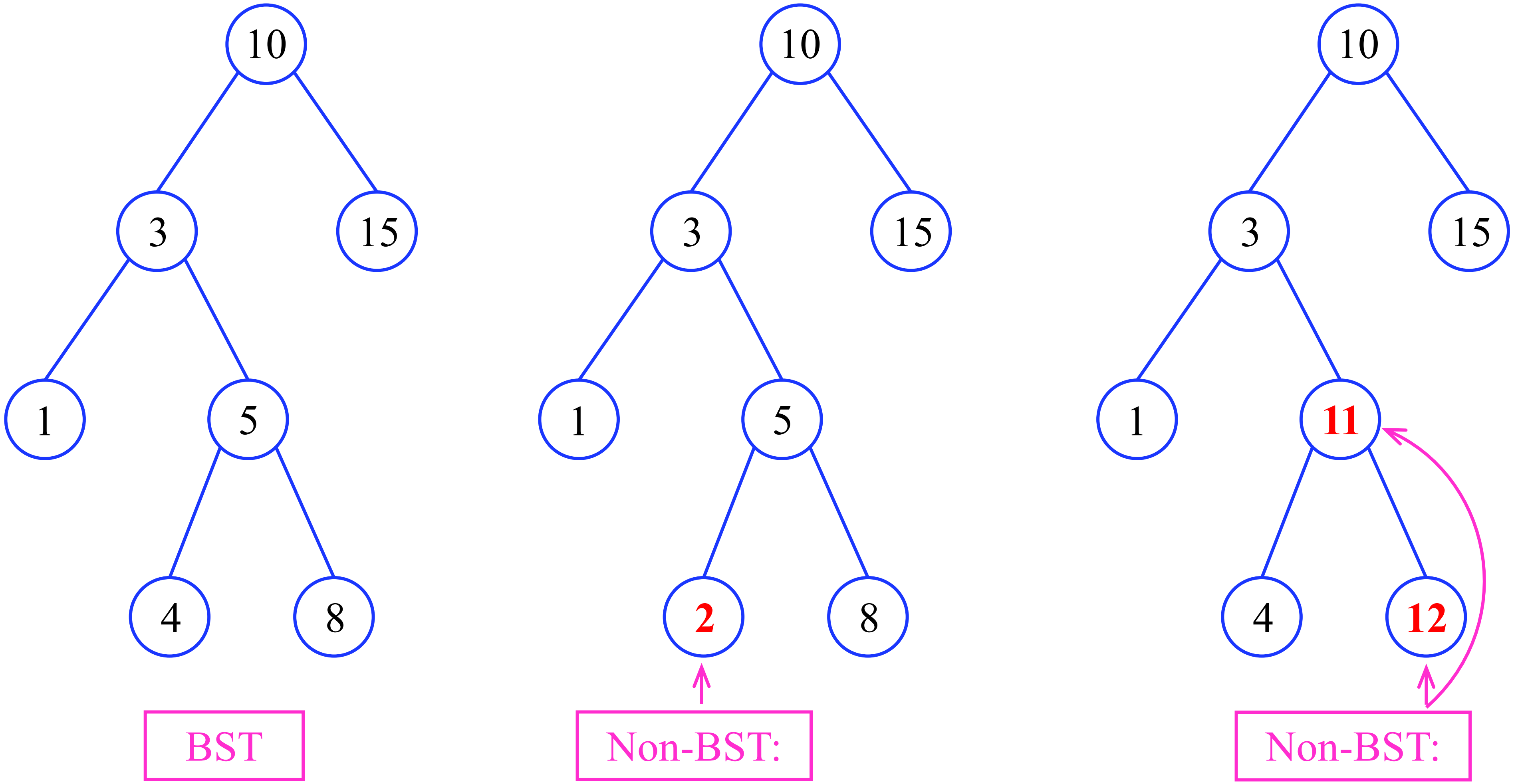
Binary Search Tree: Left-Right Ordering of Keys

- Left-to-right numerical ordering in a BST: for every node i,
  - the values of all the keys $k_{\text{left}:i}$ in the left subtree are smaller than the key k_i in i and
  - the values of all the keys $k_{\text{right}:i}$ in the right subtree are larger than the key k_i in i:

$$\{k_{\text{left}:i}\} \ni l < k_i < r \in \{k_{\text{right}:i}\}$$



Binary Search Tree: Left-Right Ordering of Keys



Key "2" cannot be in the right subtree of key "3".

Key "11” and "12" cannot be in the left subtree of key ”10".

Binary Search Tree Operations

- BST is an explicit data structure implementing the table ADT.
  - BST are more complex than heaps: any node may be removed, not only a root or leaves.
  - The only practical constraint: no duplicate keys (attach them all to a single node).

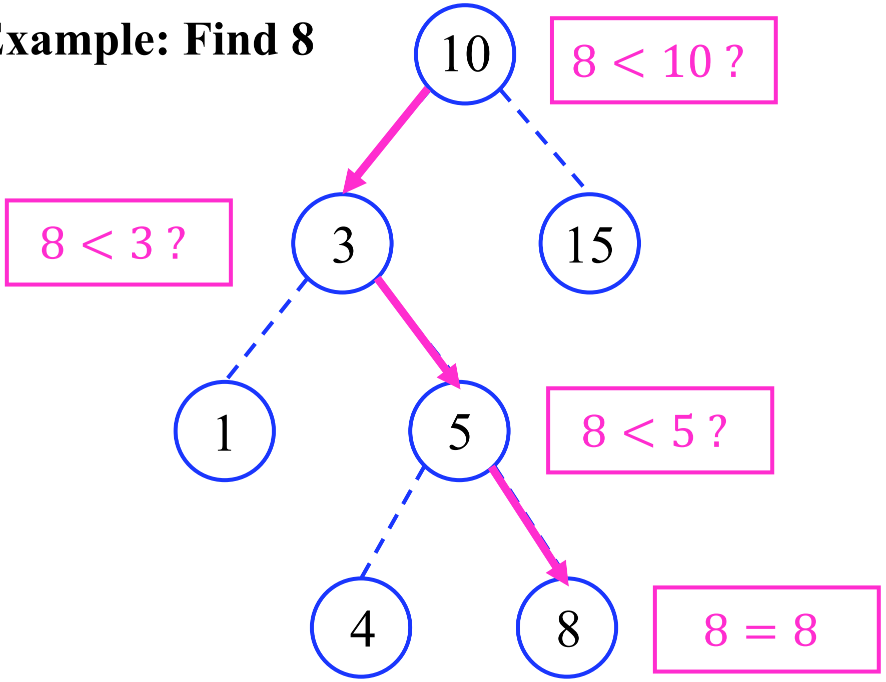
- Basic operations
  - Find a given search key or detect that it is absent in the BST.
  - Insert a node with a given key to the BST if it is not found.
  - FindMin: find the minimum key.
  - FindMax: find the maximum key.
  - Remove a node with a given key and restore the BST if necessary.

BST Operations: Find / Insert a Node

find: a successful binary search

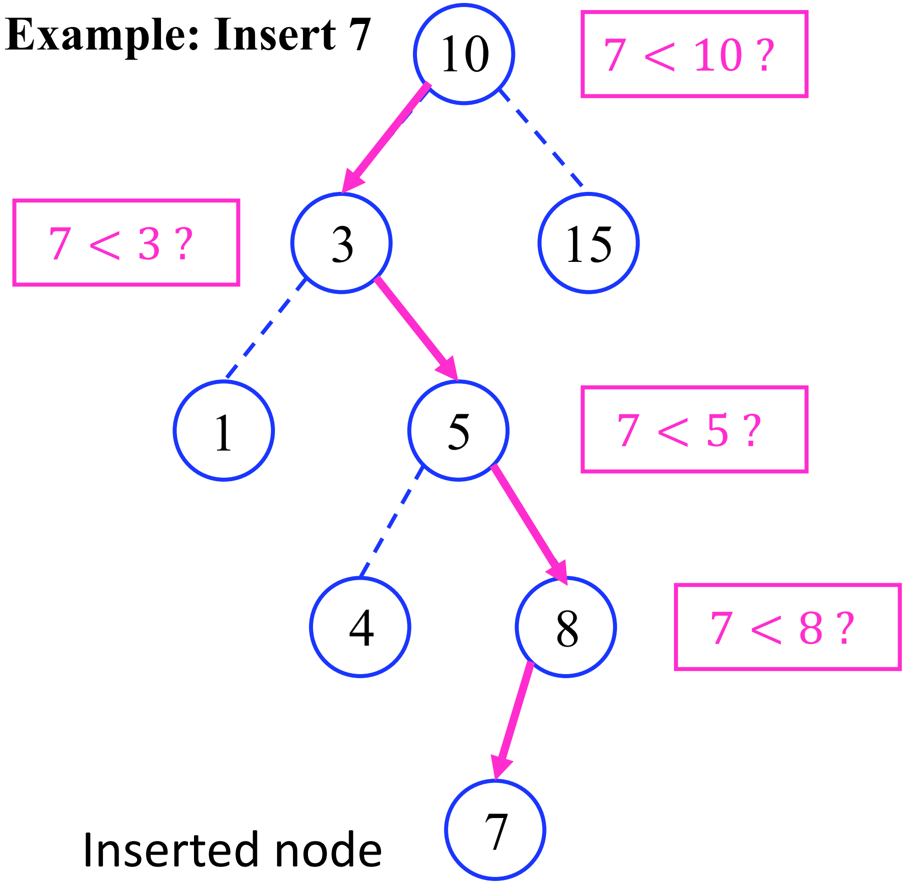
insert: creating a new node at the point where an unsuccessful search stops.

Example: Find 8



Found node

Example: Insert 7



Inserted node

BST Operations: FindMin / FindMax

- Extremely simple: starting at the root, branch repeatedly left (findMin) or right (findMax) if a corresponding child exists.

- The root of the tree plays a role of the pivot in quicksort and quickselect.
- As in quicksort, the in-order traversal of the tree can sort the items:
  - First visit the left subtree;
  - Then visit the root, and
  - Then visit the right subtree.

- $O(\log n)$ average-case and $O(n)$ worst-case running time for find, insert, findMin, and findMax operations, as well as for selecting a single item

BST Operations: Remove a Node

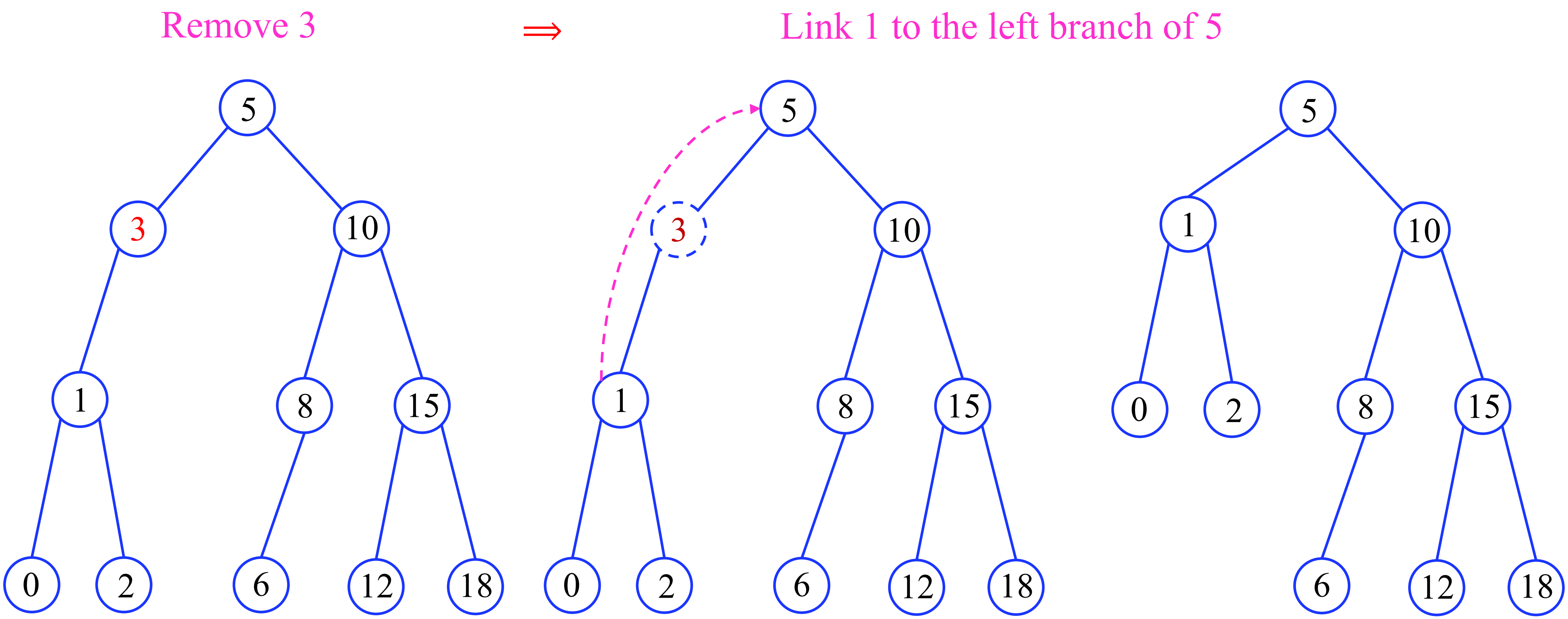
- The most complex because the tree may be disconnected. Need to reconnect some nodes!
  - Reconnection must retain the ordering condition.
  - Reconnection should not needlessly increase the tree height.

BST Operations: Remove a Node

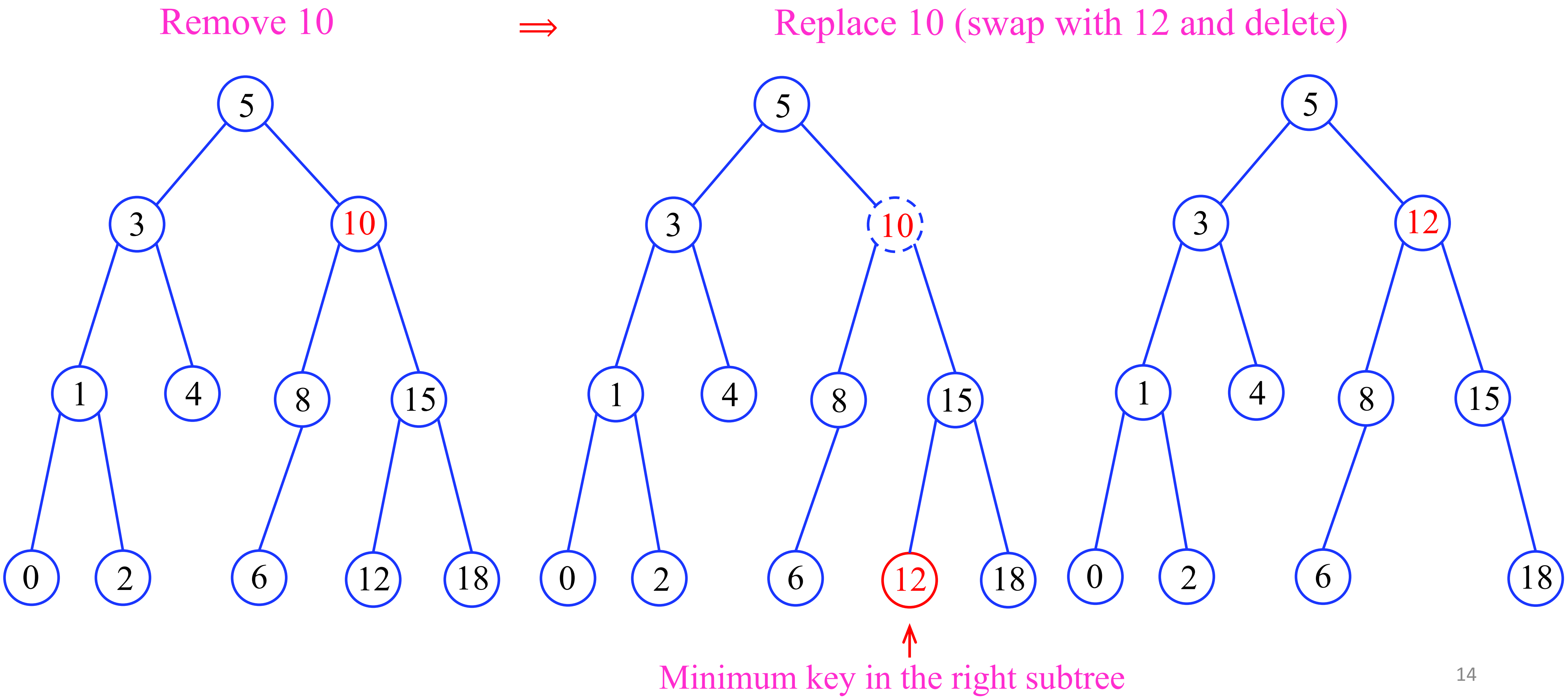
- Standard method of removing a node i with c children:

c	ACTION
0	Simply remove the leaf i.
1	Remove the node i after linking its child to its parent node.
2	Swap the node i with the node j having the smallest key k_j in the right subtree of the node i. After swapping, remove the node i (as now it has at most one right child).

BST Operation: Remove a Node



BST Operation: Remove a Node



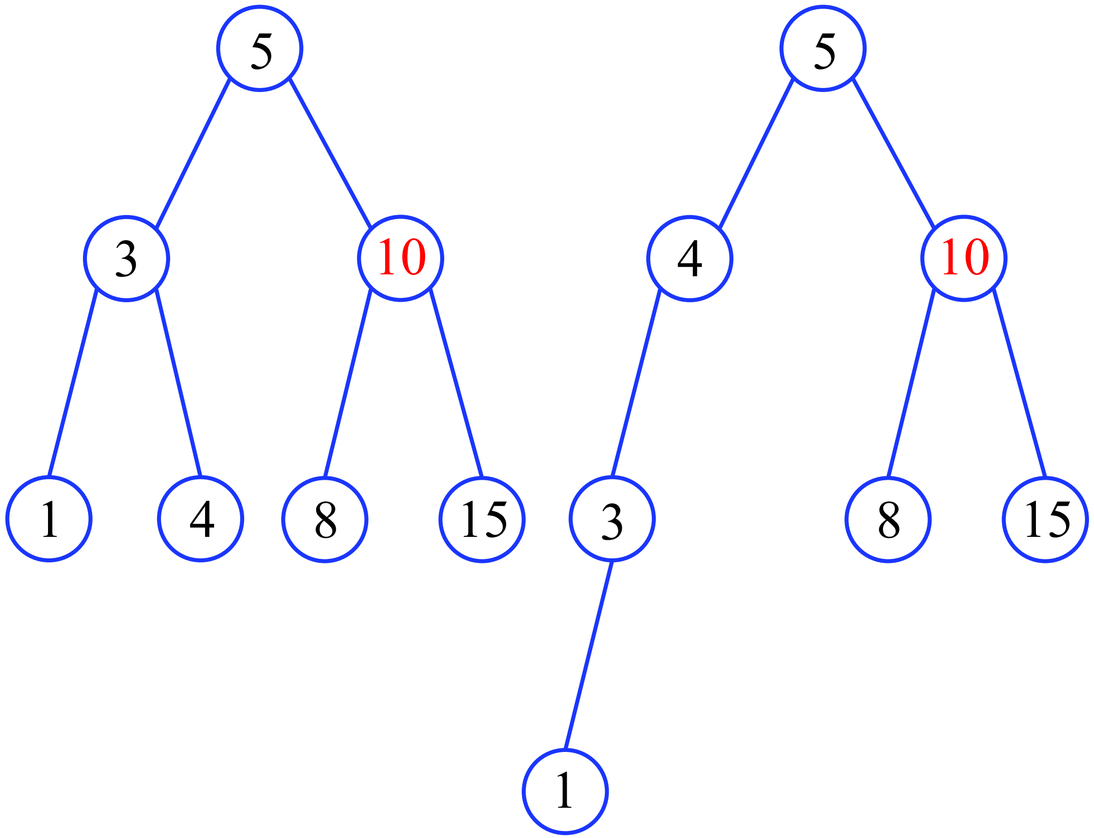
The Worst-Case Time Complexity

- The find, insert, and remove operations in a BST all take time in $O(h)$ in the worst case, where h is the height of the tree

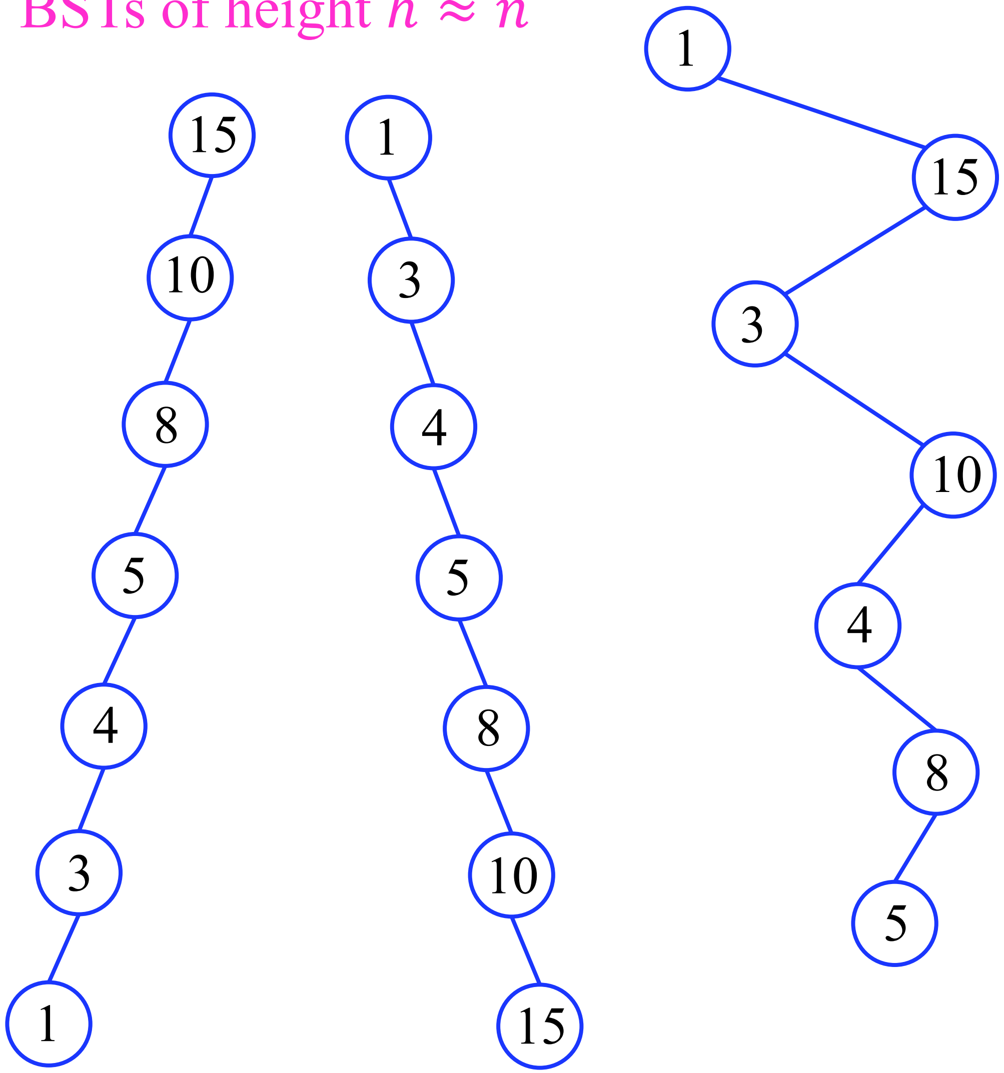
- **Proof**: The running time $T(n)$ of these operations is proportional to the number of nodes visited
  - Find / insert: 1+h
  - Remove: "1 + the depth of the node + the height of its highest subtree" →1+h
  - In each case $T(n) = \Theta(h)$
  - For a well-balanced BST, $T(n) \in O(\log n)$ (logarithmic time)
  - In the worst case $T(n) \in \Theta(n)$ (linear time) because insertions and deletions may heavily destroy the balance

The Worse-Case Time Complexity

BSTs of height $h \approx \log n$

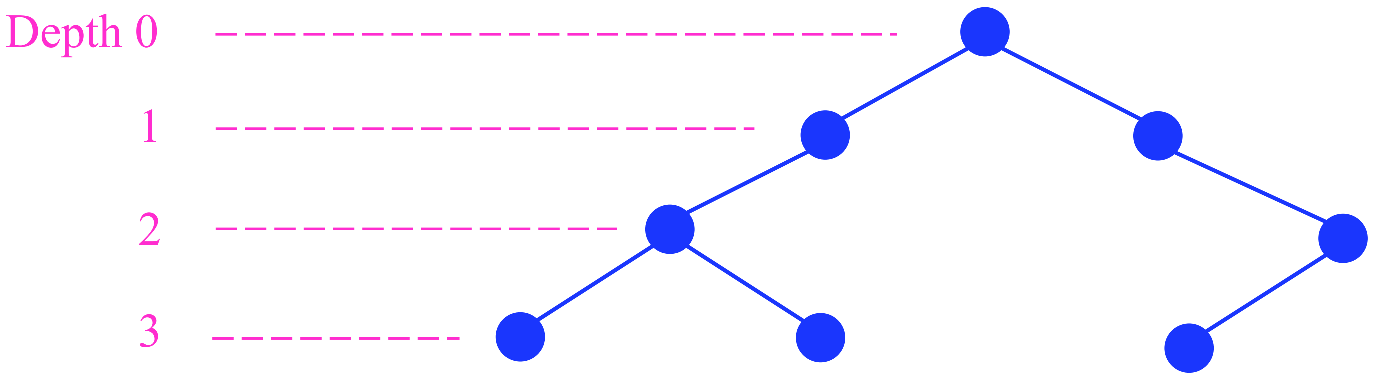


BSTs of height $h \approx n$



The Average-Case Time Complexity

- More balanced trees are more frequent than unbalanced ones.
- **Definition** (**Internal Path Length**): The total internal path length, $S_\tau(n)$, of a binary tree τ is the sum of the depths of all its nodes.

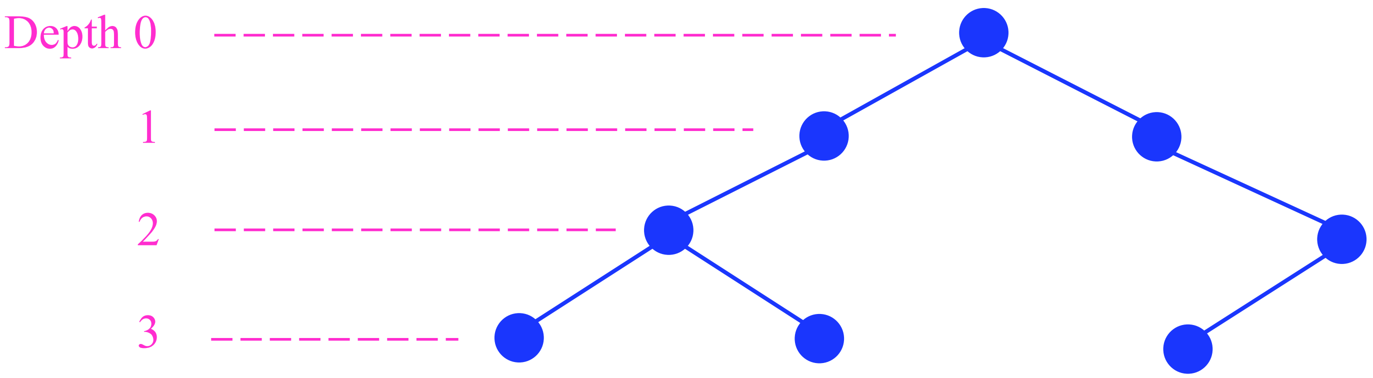


$$S_\tau(8) = 0 + 1 + 1 + 2 + 2 + 3 + 3 + 3 = \mathbf{15}$$

- Average complexity of a successful search in τ: the average node depth, $1/n\ S_\tau(n)$, e.g. $1/8\ S_\tau(8)$ =15/8=1.875 in this example.

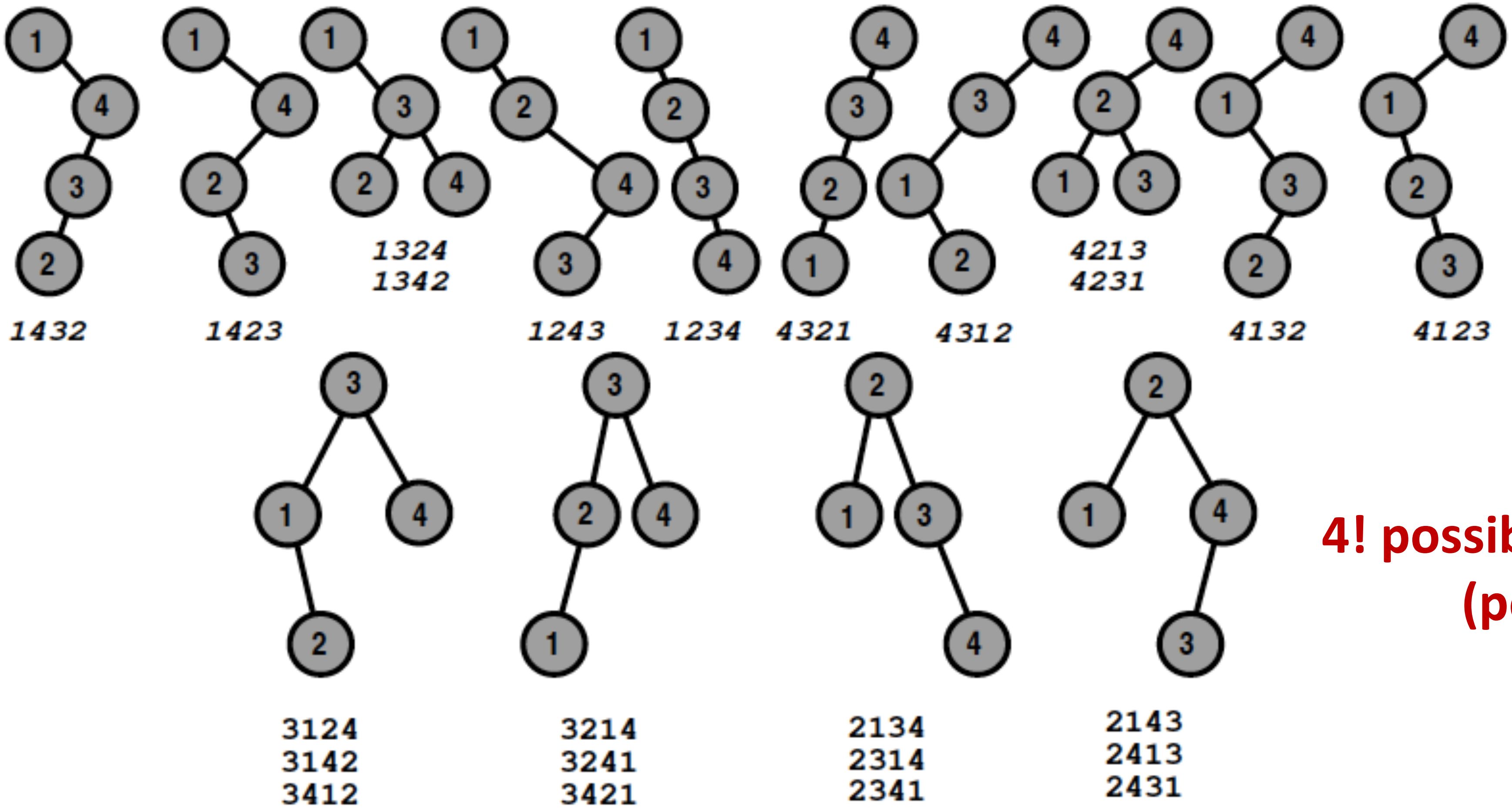
The Average-Case Time Complexity (Contd.)

- Average-case complexity of searching:
  - Averaging $S_\tau(n)$ for all the trees of size n, i.e. for all possible $n!$ insertion orders, occurring with equal probability $\frac{1}{n!}$



$$S_\tau(8) = 0 +$$
$$+1+1$$
$$+2+2$$
$$+3+3+3 = \mathbf{15}$$

Example: All possible BSTs with 4 nodes [1,2,3,4]

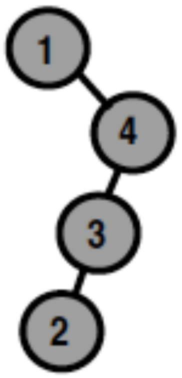


4! possible insertion orders (permutations)

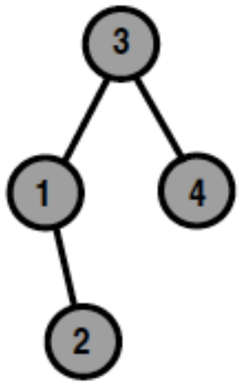
Definition 3.12

- $S_\tau(n)$: the sum of the depths of all its nodes in a binary tree τ

Depth(1)=0
Depth(4)=1
Depth(3)=2
Depth(2)=3



Depth(3)=0
Depth(1)=1
Depth(4)=1
Depth(2)=2



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$\boldsymbol{S_{\tau 1}(n=4)}$**=0+1+2+3 = 6**

$\boldsymbol{S_{\tau 2}(n=4)}$**=0+1+1+2 =4**

- $\frac{1}{n}S_\tau(n)$: the average time complexity of a successful search in a particular tree τ
- $n!$ insertion orders → $n!$ possible trees := $\{\tau_1, \tau_2, \tau_3, \ldots, \tau_{n!}\}$

The $\Theta(\log n)$ Average-case BST Operations

- Let $S(n)$ be the average of the total internal path length, $S_\tau(n)$, over all BST τ created from an empty tree by sequences of n random insertions, each sequence considered as equally possible.

$$S(n) = \frac{1}{n!}[S_{\tau_1}(n) + S_{\tau_2}(n) + S_{\tau_3}(n) + \cdots + S_{\tau_{n!}}(n)]$$

- The expected time for successful and unsuccessful search (insertion and deletion) in such BST is $\Theta(\log n)$
- **Proof**: It should be proven that $S(n) \in \Theta(n \log n)$
  - Obviously, $S(1) = 0$.
  - Any n-node tree, n>1, contains a left subtree with i nodes, a root at level 0, and a right subtree with $n - i - 1$ nodes; $0 \le i \le n - 1$.
  - For a fixed i, $S(n) = (n-1) + S(i) + S(n-i-1)$, as the root adds 1 to the path length of each other node.

$$S(n) = \frac{1}{n}[S(0) + S(n-1) + S(1) + S(n-2) + S(2) + S(n-3) + \cdots + S(n-1) + S(0)] + (n-1)$$

Eq.(1)

$$S(n) = \frac{2}{n} \sum_{0 \le p \le n-1} S(p) + n - 1$$

Eq.(2)

$$nS(n) = 2 \sum_{0 \le p \le n-1} S(p) + n(n-1) = 2S(n-1) + 2 \sum_{0 \le p \le n-2} S(p) + n(n-1)$$

Eq.(3)
Substitution

$$(n-1)S(n-1) = 2 \sum_{0 \le p \le n-2} S(p) + (n-1)(n-2)$$

Eq.(2)-Eq.(3)
Subtraction

$$\begin{aligned} nS(n) &= 2S(n-1) + (n-1)S(n-1) - (n-1)(n-2) + n(n-1) \\ &= (n+1)S(n-1) + 2(n-1) \end{aligned}$$

$$\frac{S(n)}{n+1} = \frac{S(n-1)}{n} + \boxed{\frac{2(n-1)}{n(n+1)}} \longrightarrow \frac{4}{n+1} - \frac{2}{n}$$

"Telescoping" $\frac{S(n)}{n+1} = \frac{S(n-1)}{n} + \frac{4}{n+1} - \frac{2}{n}$ to get the explicit form:

$$\frac{S(n)}{n+1} + \frac{S(n-1)}{n} + \frac{S(n-2)}{n-1} + \cdots + \frac{S(2)}{3} + \frac{S(1)}{2} - \frac{S(n-1)}{n} - \frac{S(n-2)}{n-1} - \cdots - \frac{S(2)}{3} - \frac{S(1)}{2} - \frac{S(0)}{1}$$

$$= \left(\frac{4}{n+1} - \frac{2}{n}\right) + \left(\frac{4}{n} - \frac{2}{n-1}\right) + \left(\frac{4}{n-1} - \frac{2}{n-2}\right) + \cdots + \left(\frac{4}{2} - \frac{2}{1}\right),$$ or

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$$\frac{S(n)}{n+1} = S(0) + 4\left(\frac{1}{n+1} + \cdots + \frac{1}{2}\right) - 2\left(\frac{1}{n} + \cdots + 1\right) = \frac{4}{n+1} + 2\left(\frac{1}{n} + \cdots + \frac{1}{2}\right) - 2$$

$$= \frac{4}{n+1} + 2(H_n - 1) - 2 = 2H_n - 4 + \frac{4}{n+1}$$

Then, the closed-formed formula is
$S(n) = 2(n+1)H_n - 4(n+1) + 4$

This gives $S(n) \in \Theta(n \log n)$

$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is the n^{th} harmonic number and $H_n \in \Theta(\log n)$.

The $\Theta(\log n)$ Average-case BST Operations (Contd.)

- After summing these recurrences for $0 \le i \le n-1$ and averaging, just the same recurrence as for the average-case quicksort analysis is obtained:

$$S(n) = (n-1) + \frac{2}{n} \sum_{i=0}^{n-1} S(i)$$

- Therefore, $S(n) \in \Theta(n \log n)$, and the expected depth of a node $\text{is } \frac{1}{n} S(n) \in \Theta(\log n)$.
- Thus, the average-case search and insertion time is in $\Theta(\log n)$.
- It is possible to prove (but in a more complicate way) that the average-case deletion time is also in $\Theta(\log n)$.
- The BST allow for a special balancing, which prevents the tree height from growing too much, i.e. avoids the worst cases with linear time complexity $\Theta(n)$.

SUMMARY

- Tree Data Structure
- Binary Search Tree Operations
    - find, insert, and remove
- Time Complexity Analysis
    - Worse and average case