Cycles and Girth

Instructor: Meng-Fen Chiang

COMPSCI: WEEK 10.1

THE UNIVERSITY OF AUCKLAND
Te Whare Wānanga o Tāmaki Makaurau
NEW ZEALAND

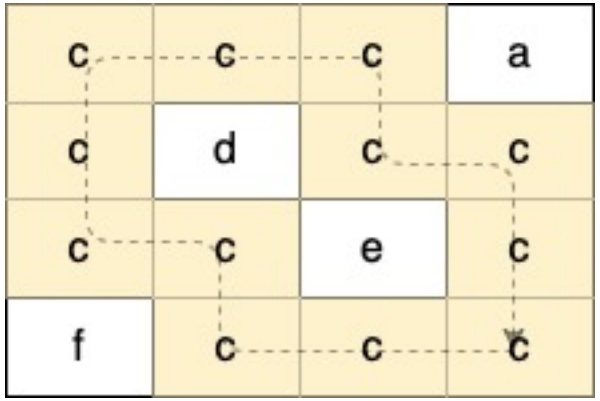
Slides adapted from Mark Wilson, Georgy Gimel'farb, Simone Linz and Tanya Gvozdeva

OUTLINE

- Terminology
  - Cycle
  - Girth

- DFS Application: Cycle Detection
- BFS Application: Girth Computation

- Correctness of Girth Computation

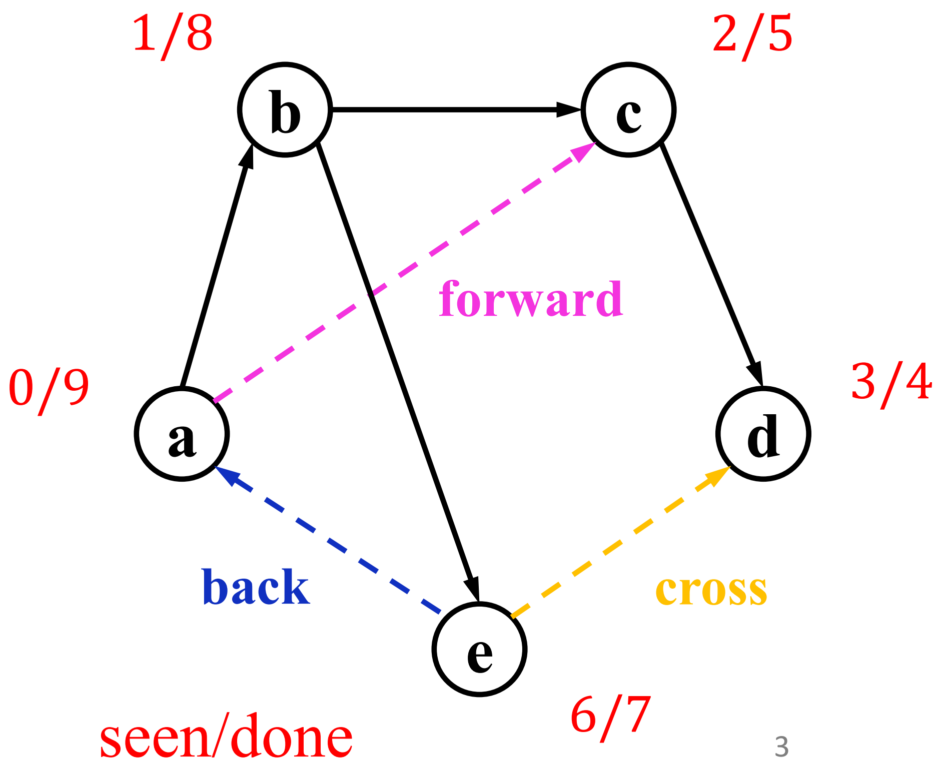


Source: https://leetcode.com/problems/detect-cycles-in-2d-grid/

RECAP: Example 23.7

- Explain how to determine (u,v) in DFS algorithm whether it is a tree-, back-, forward- or cross-arc?

1. If v is white, then (u,v) is a **tree** arc
2. If v is grey, then (u,v) is a **back** arc
3. If v is black, then (u,v) is
   - a **cross** arc($seen[v]<seen[u]$, $done[v]<seen[u]$), or
   - a **forward** arc($seen[u]<seen[v]<done[v]<done[u]$).



Cycle Detection

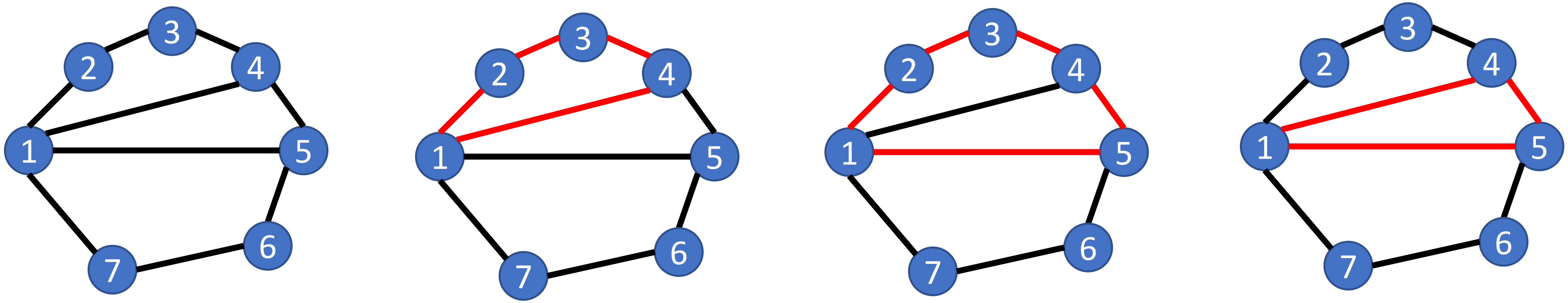
- Suppose that there is a cycle in G and let v be the node in the cycle visited first by DFS. If (u, v) is an arc in the cycle then it must be a back arc.

- Conversely if there is a back arc, we must have a cycle.

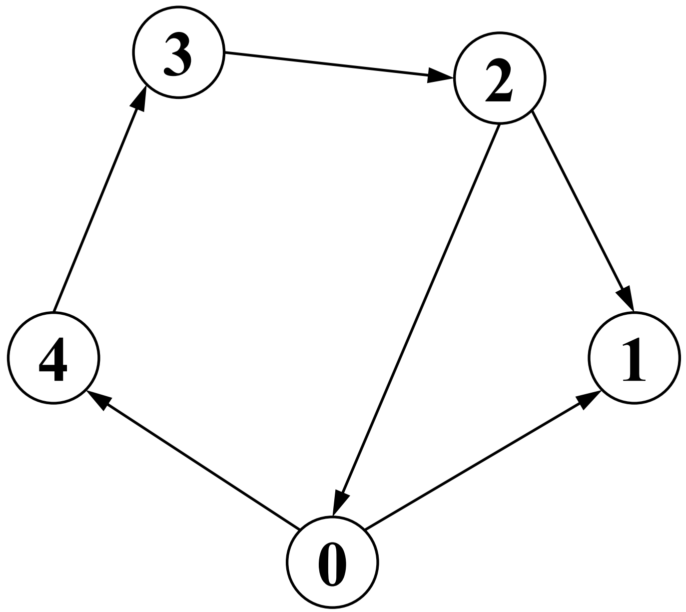
- Suppose that DFS is run on a digraph G. Then G is acyclic if and only if G does not contain a back arc.

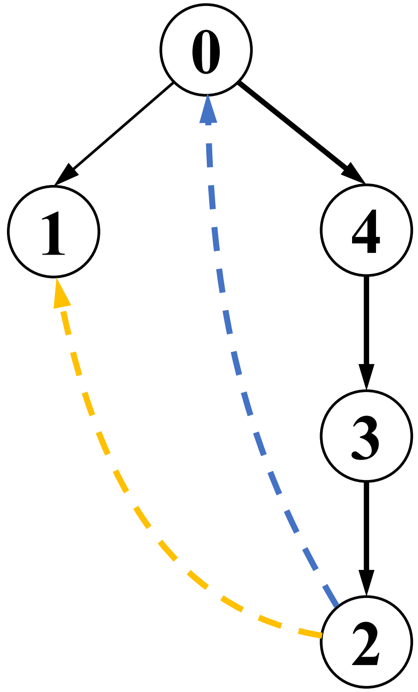
Example: Cycles in Graphs

- A graph can have several cycles



Using DFS to Find Cycles in Digraphs





Back arc detected!

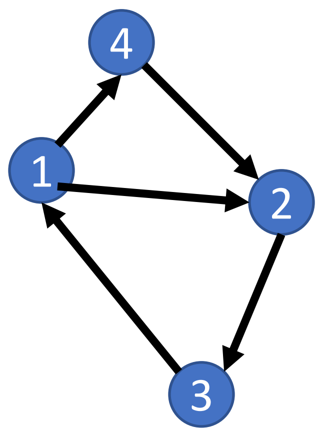
Once DFS finds a cycle, the stack contains the nodes that form the cycle.

Girth and Digirth

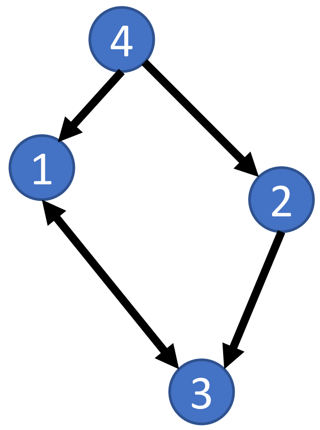
- **Definition.** For a graph (with a cycle), the length of the shortest cycle is called the girth of the graph. If the graph has no cycle then the girth is undefined, can be viewed as $+\infty$.

- **Convention.** For a digraph we use the term girth for its **underlying graph** and the term directed girth for the length of the smallest directed cycle.

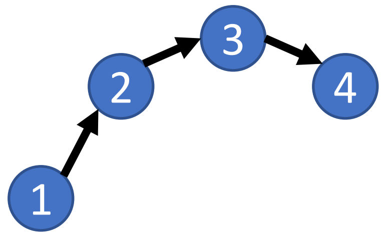
Examples: Girth of a Graph (Digraph)



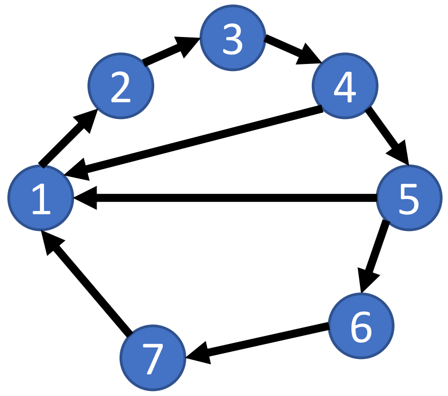
Girth: 3
Directed girth: 3



Girth: 4
Directed girth: 2

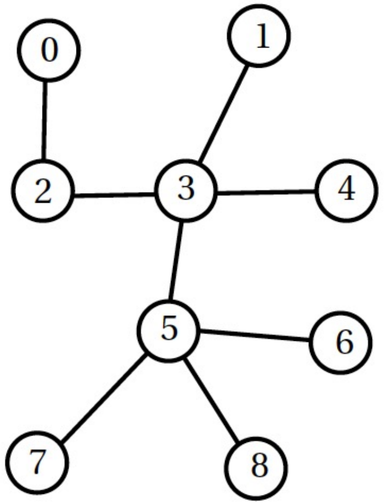


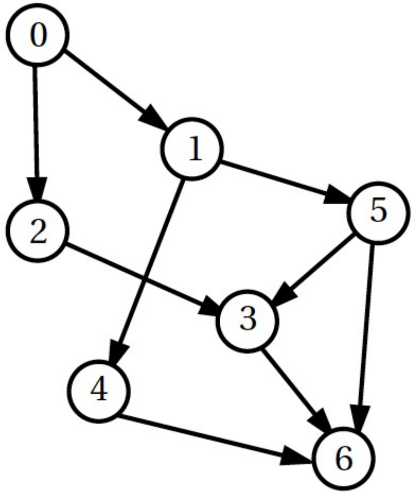
Girth: $+\infty$
Directed girth: $+\infty$

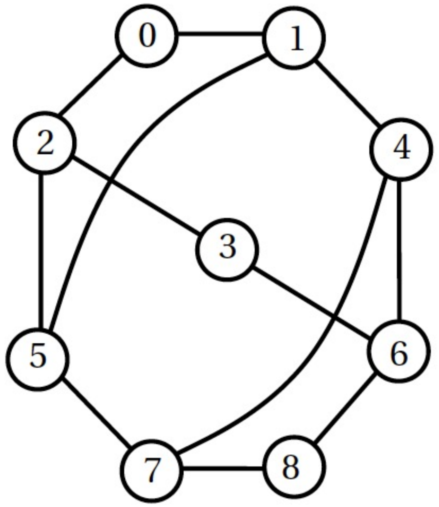


Girth: 3
Directed girth: 4

Examples: Girth of a Graph (Digraph)







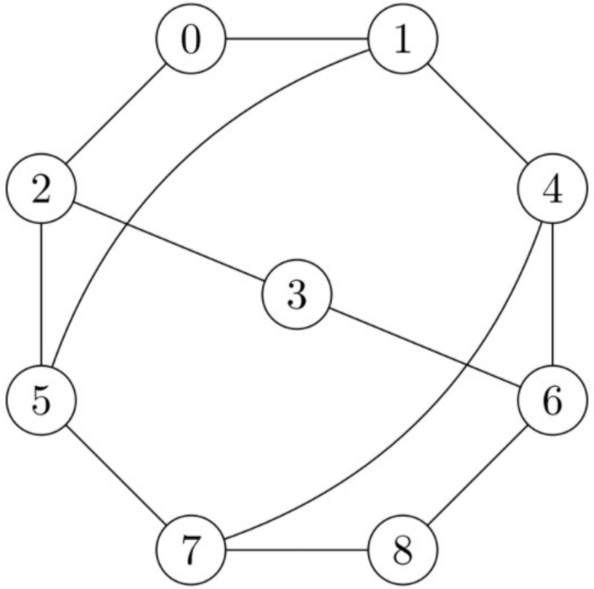
Girth: $+\infty$
Directed girth: undefined

Girth: 3
Directed girth: $+\infty$

Girth: 4
Directed girth: undefined

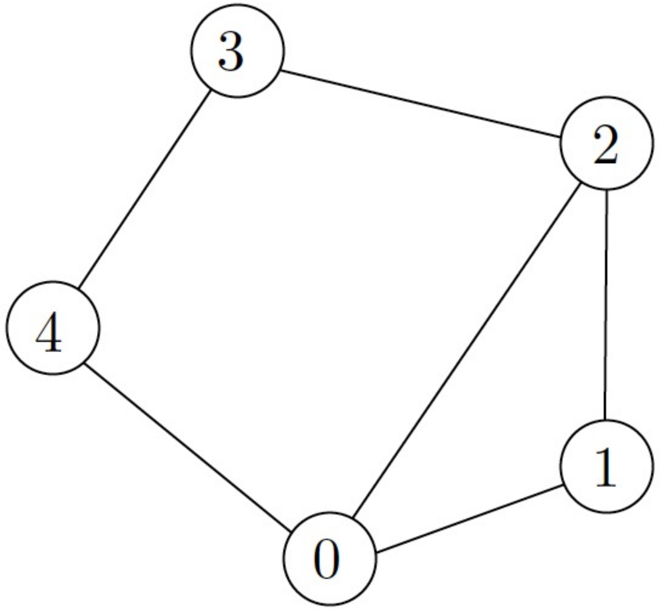
Finding the Girth of a Graph

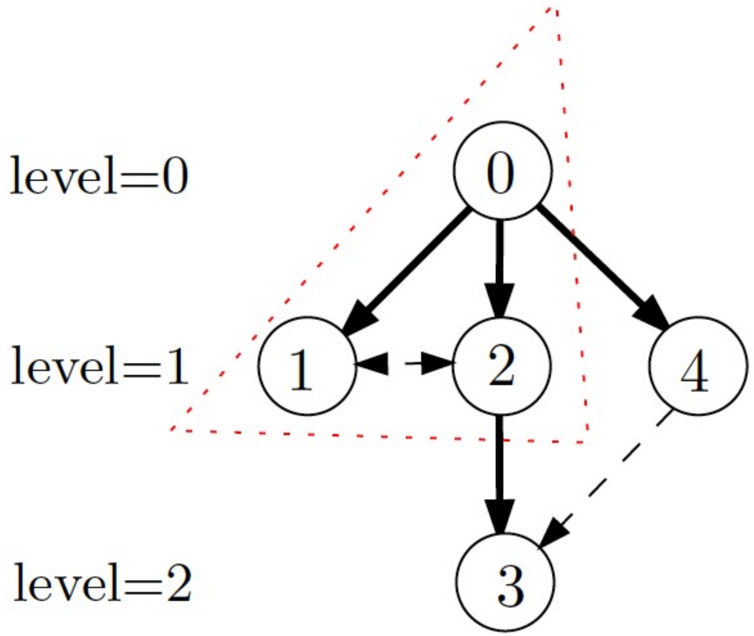
- An easy-to-implement **DFS** idea may not work properly. Consider the **DFS** tree originating from vertex 0 of the graph below. Which is the smallest cycle it finds with the back edges? Which smaller cycle does it miss?



Finding the Girth of a Graph

- **Using BFS to find cycles in graphs**. Cycles can also be easily detected in a graph using BFS. Finding a cycle of minimum length in a graph is not difficult using BFS (better than DFS).





Finding the Girth of a Graph

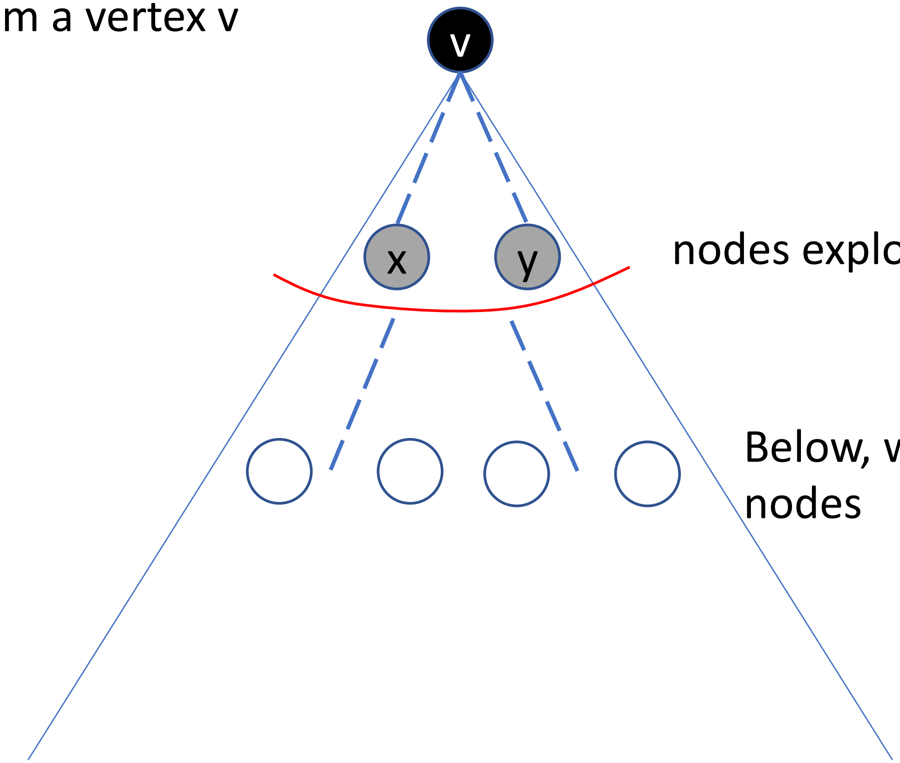
- Perform BFSVisit() $|V|$ times, starting at each vertex $\mathrm{v}' \in V$ in turn.

- If during a BFSVisit, we encounter a grey neighbour (rather than a white) we have found a cycle
  - The grey node was visited before following another path

- An important property of BFS is that if it runs from vertex v, then every vertex s, when it is first reached, then the path that was found from v to s is minimal. Thus, reaching v' from v with BFS finds the shortest path from v to v', namely the shortest cycle that contains v.

Example: Girth of a Graph

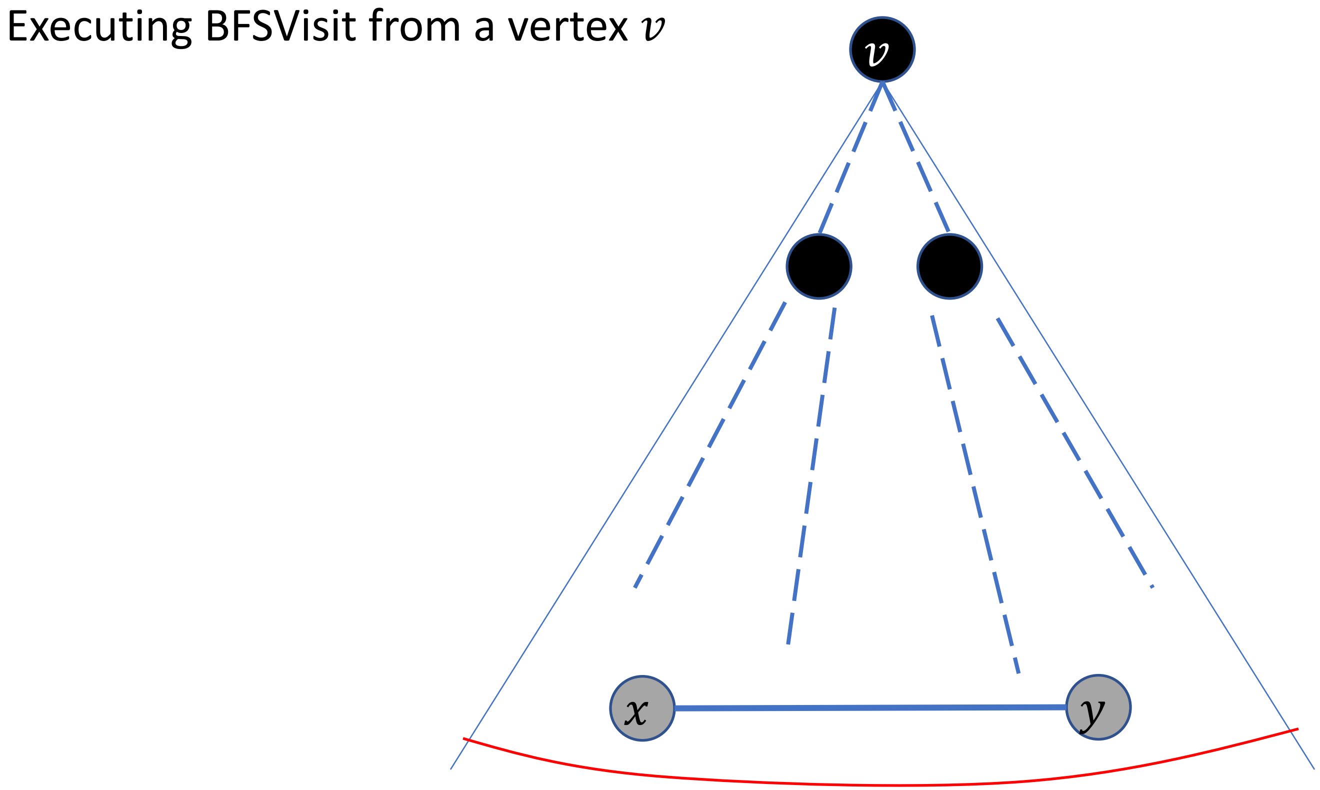
Executing BFSVisit from a vertex v



nodes explored level by level

Below, we should only find white nodes

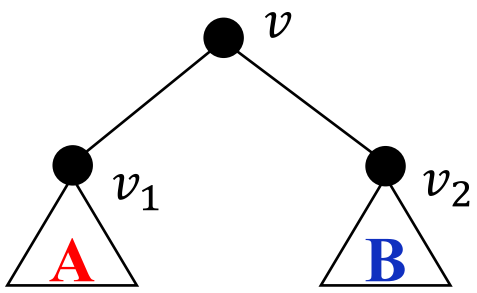
Example: Girth of a Graph

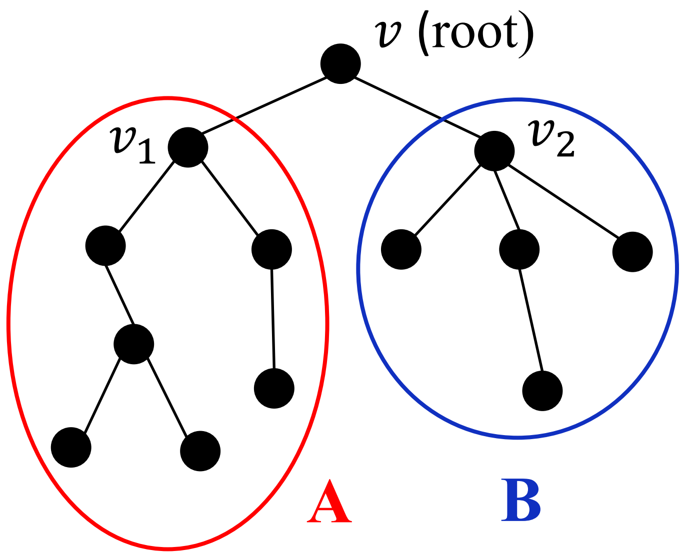


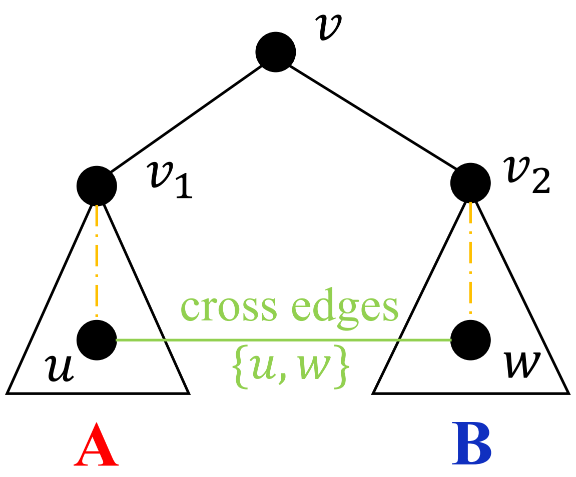
If a cross edge/arc exits then it will connect to a grey vertex

Both x and y are reachable from v therefore there is a cycle through the cross edge

Example: Girth of a Graph



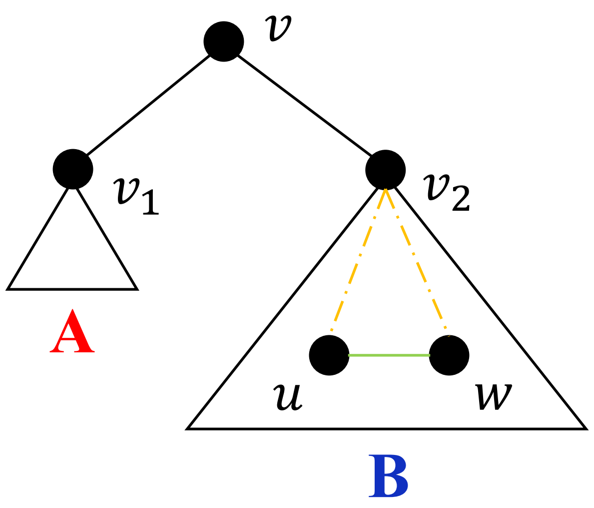




Distance from root v

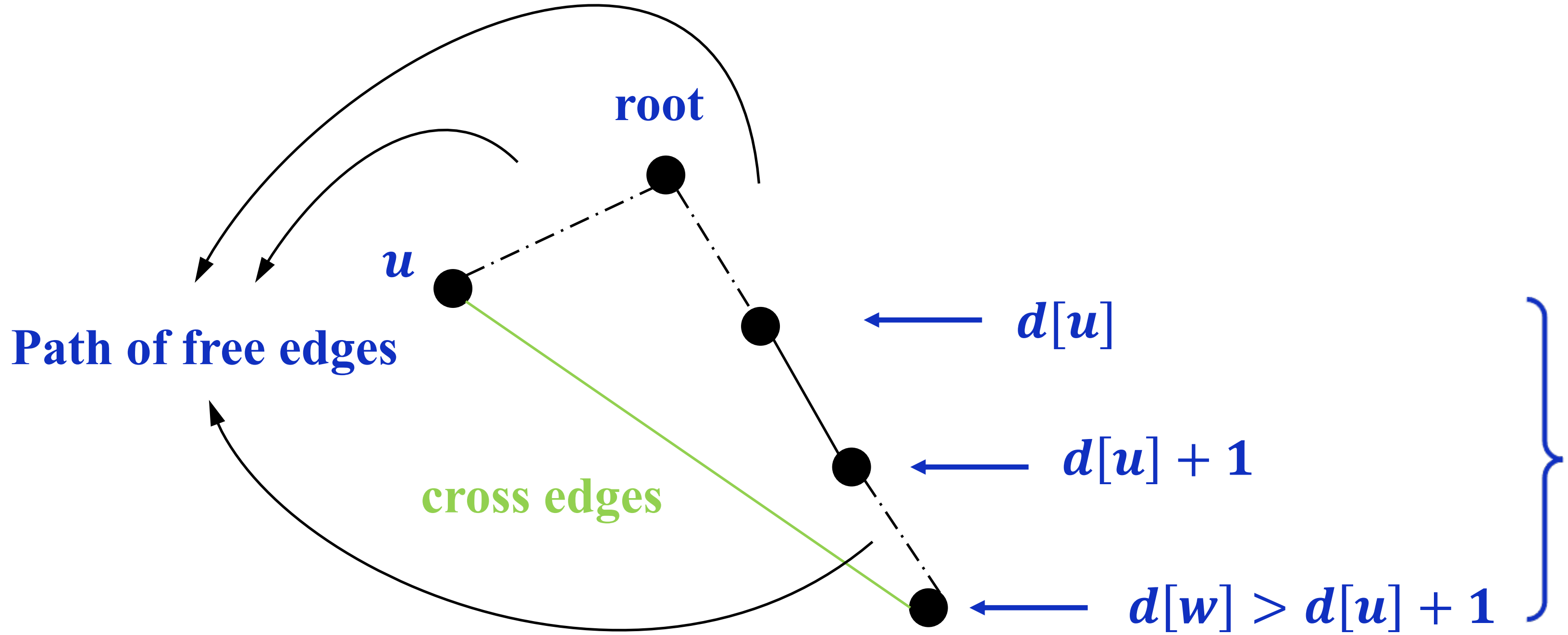
$d[u] + d[w] + 1$

(length of cycle containing v)



BFS: Cross Edges in a Graph

- Let $\{u, w\}$ be a cross edge in a tree. Then either $d[u] = d[w]$ or $|d[u] - d[w]| = 1$
- **Sketch of proof:** Suppose $|d[u] - d[w]|$ >1



$d[w]$ > distance from root to w (shortest path from root to w contains the cross edge)
*contradiction

Facts about Cycle Length

- If vertex v is on at least one cycle then BFS starting at v will find it.
- On detection of a cross edge between descendants u and w, determine whether u and w are in different subtrees below v (the root of the tree).
  - If yes, then a cycle of length $d[u]+d[w]+1$ is found.
  - If no, then a cycle of shorter length is found (but avoids v).
- If $d[u]=d[w]$ then odd length, where v is a common ancestor.
- Otherwise, $d[u]+1=d[w]$ and even length.

Facts about Cycle Length

- cross edge $\{u,w\}$

1. $d[u] = d[w]$

cycle length: $2d[u] + 1 \Longrightarrow$ odd length

cross edge

2. $d[u] + 1 = d[w]$

cycle length: $d[u] + d[w] + 1$

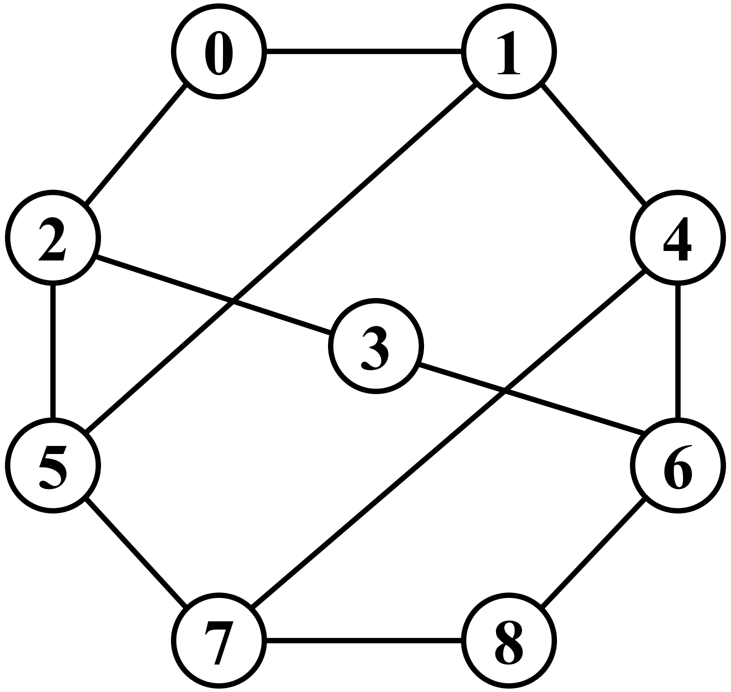
$= d[u] + d[u] + 1 + 1$

$= 2d[u] + 2 \Longrightarrow$ even length

Finding the Girth of a Graph

- To compute girth, we perform BFSVisit(v) procedure once for each $v \in V(G)$ and take minimum.
  - If a grey neighbour is met, e.g., an edge (x, y) is explored from x where y is grey, continue to the end of the current level and then stop.
  - For each edge (x, y) as above on this level, if v is the lowest common ancestor of x and y in the BFS tree, then there is a cycle containing x, y, v of length $l = d(x) + d(y) + 1$.
  - Report the minimum value of l obtained along the current level.
- The minimum of these lengths at the level is the smallest cycle that involves v
- The smallest cycle among all possible start vertices v is the girth.

Example (1): Finding the Girth of a Graph

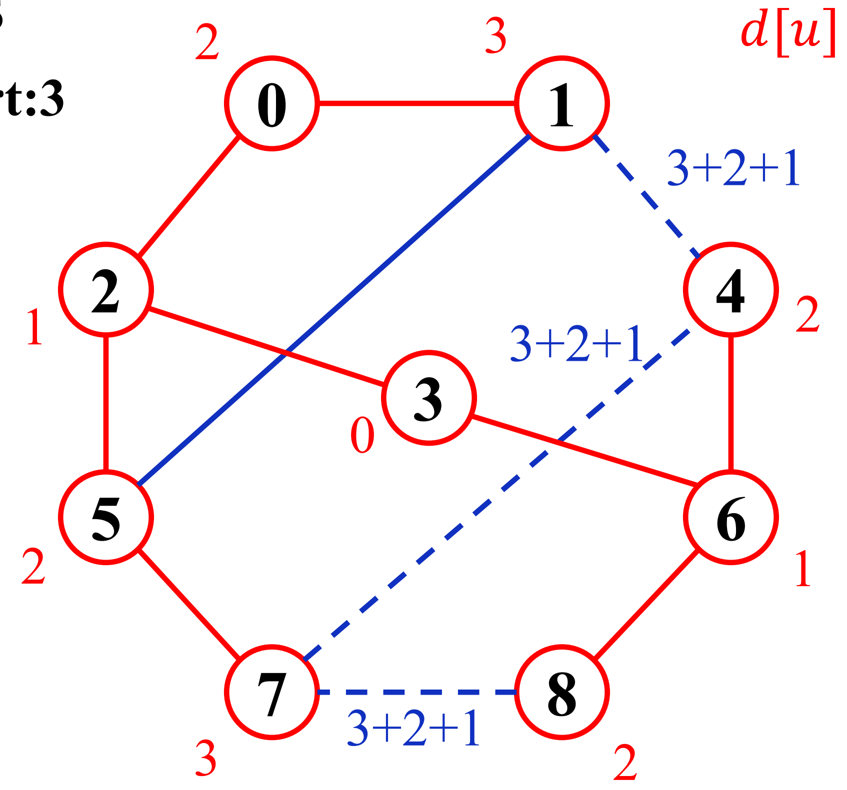


- Run BFS for 3, shortest cycle has length 6.
- Run BFS for any other vertex, shortest cycle has length 4.
- Girth of the graph is 4..

Example (1)

BFS

Start:3



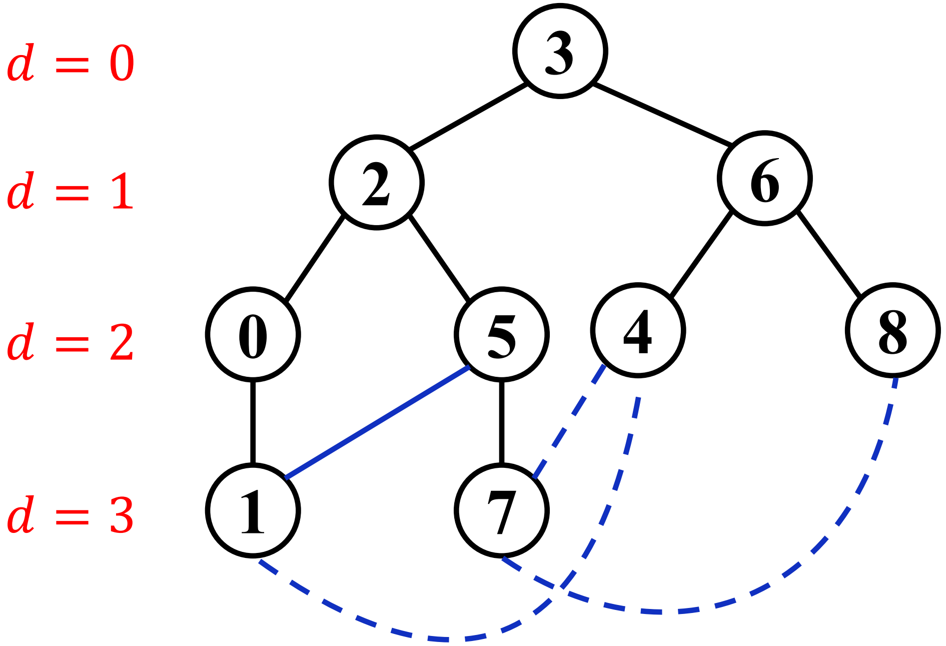
Shortest cycle containing 3 has length 6

tree edges

cross edge same subtree

tree edges different subtree

(relative to root 3)



Example (1)

tree edges

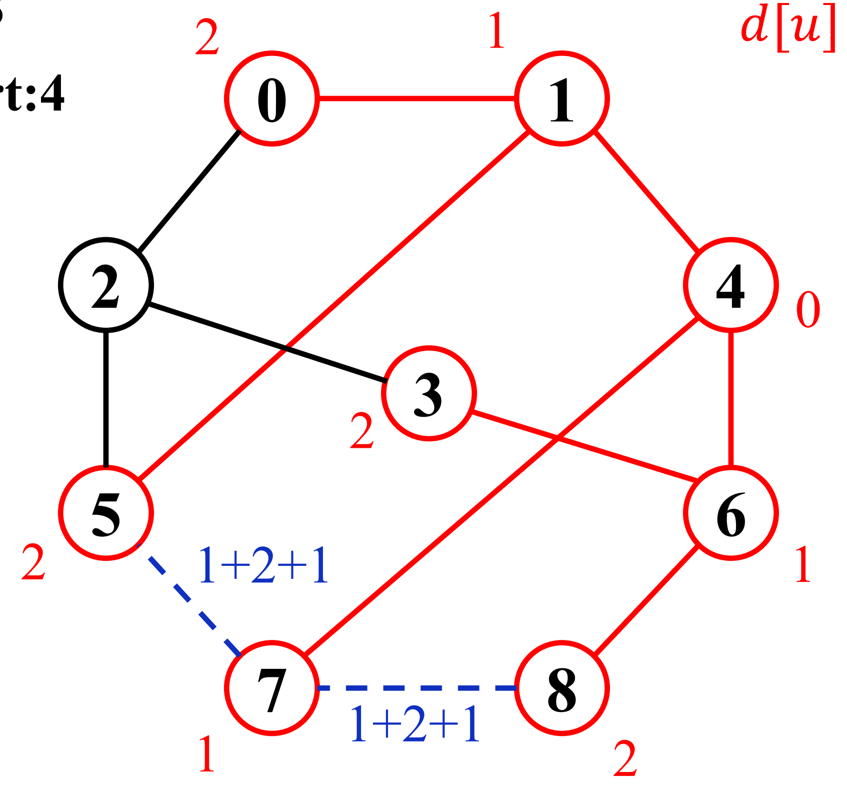
cross edge same subtree

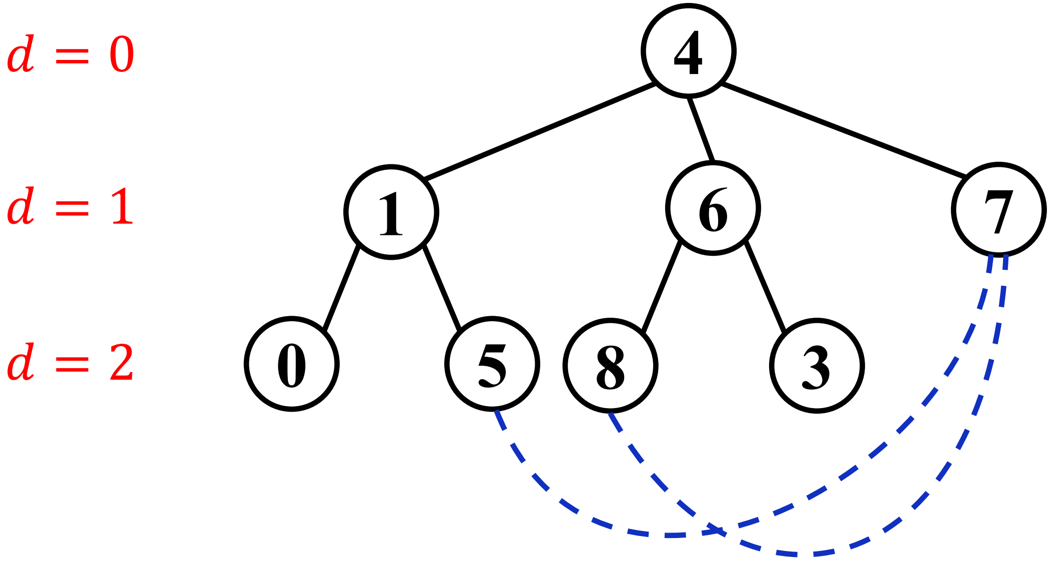
tree edges different subtree

(relative to root 4)

BFS

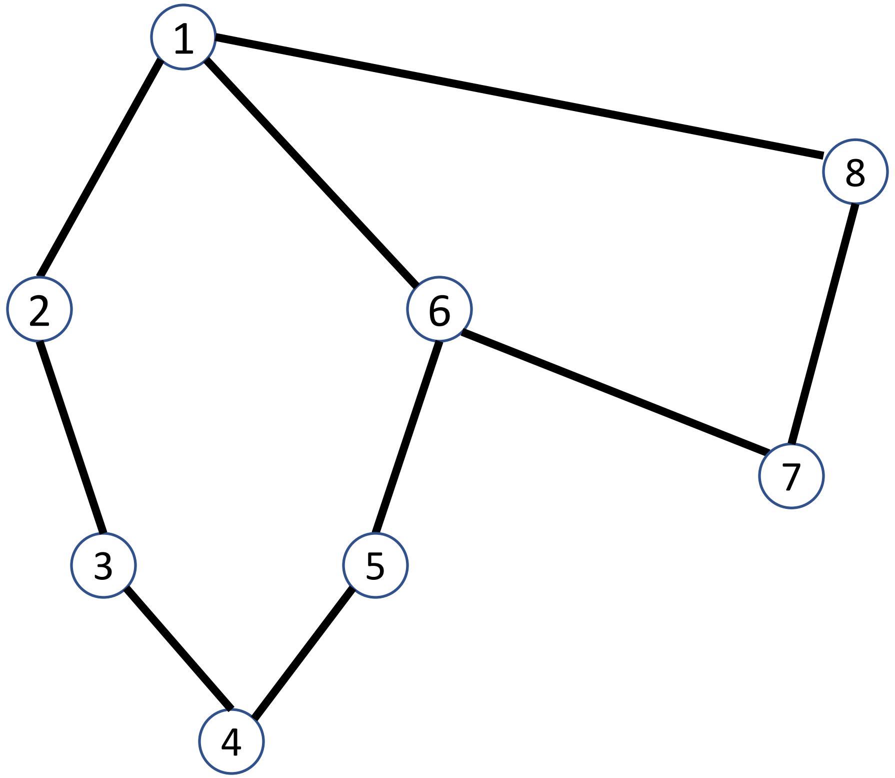
Start:4



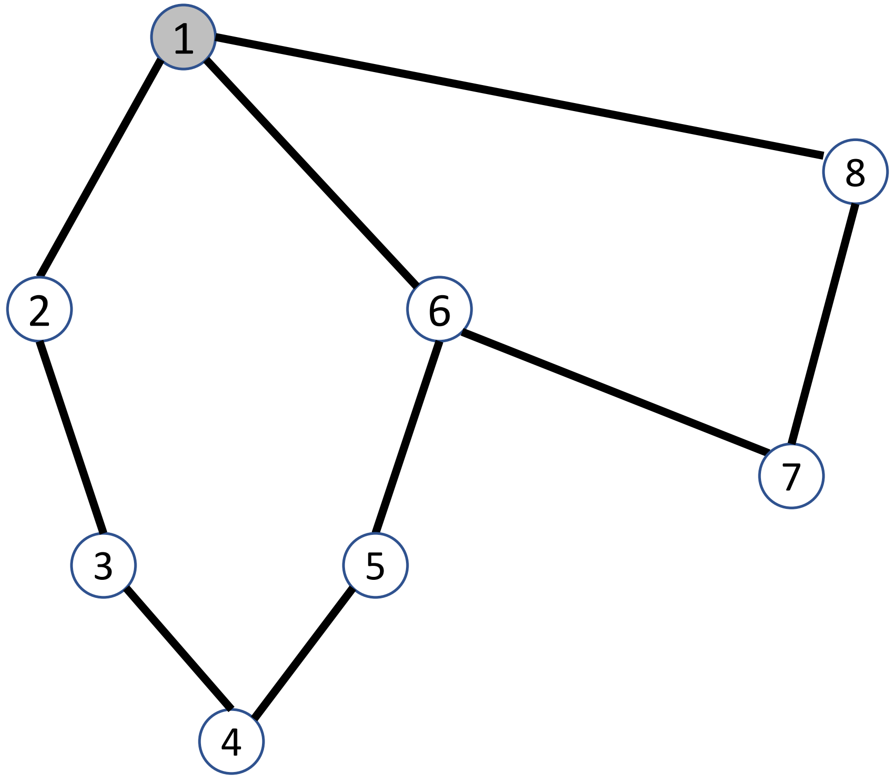


Shortest cycle containing 4 has length 4

Exercise: Finding the Smallest Cycle involving Vertex 1

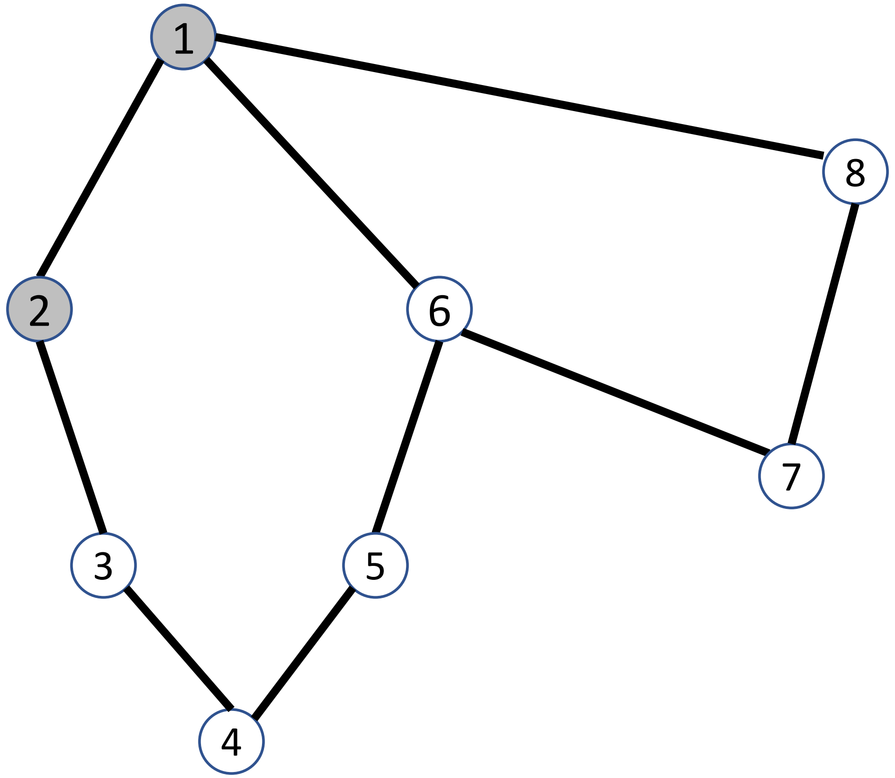


Exercise: Finding the Smallest Cycle involving Vertex 1



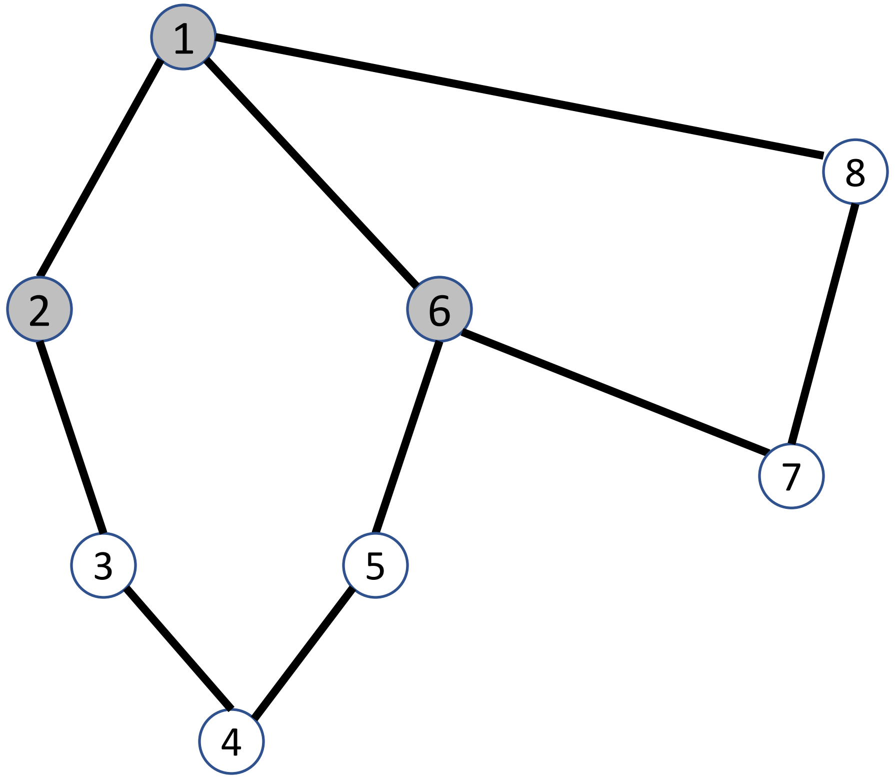
Queue: 1

Exercise : Finding the Smallest Cycle involving Vertex 1



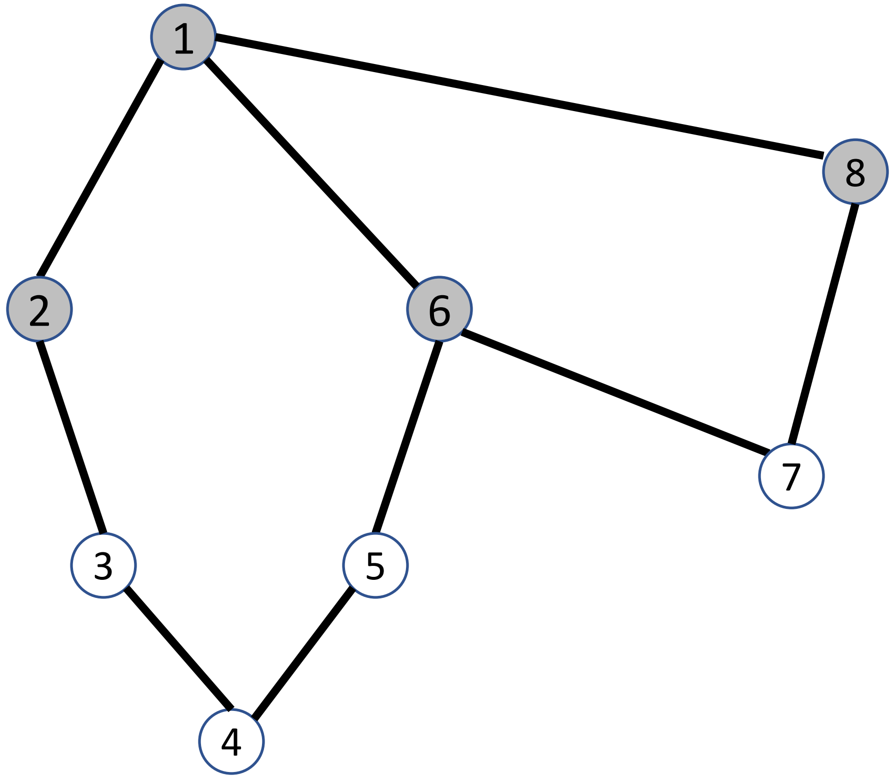
Queue: 1 2

Exercise: Finding the Smallest Cycle involving Vertex 1



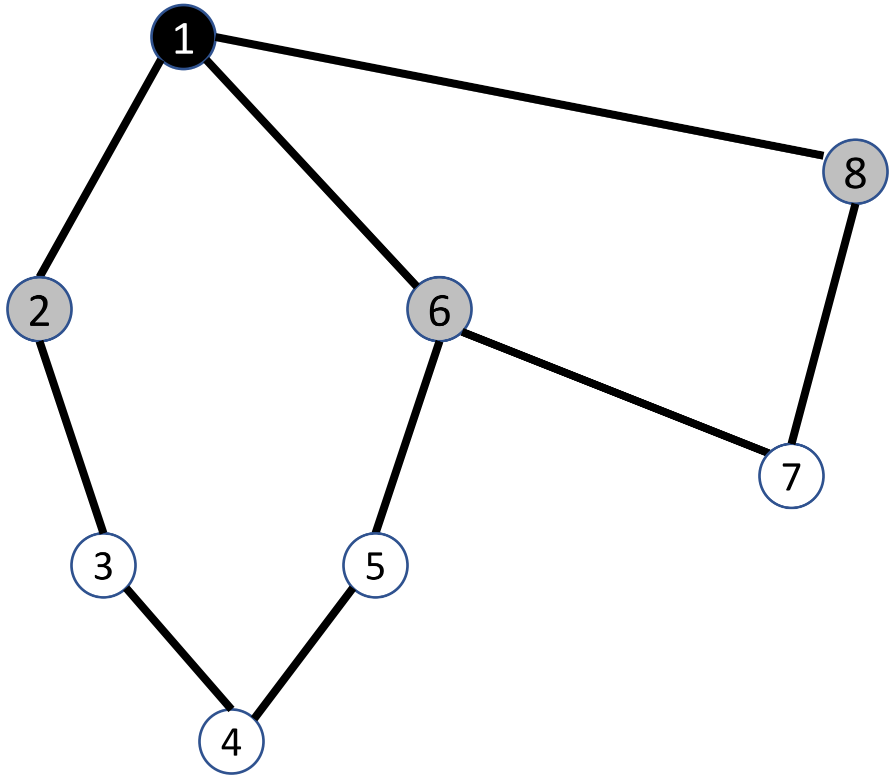
Queue: 1 2 6

Exercise: Finding the Smallest Cycle involving Vertex 1



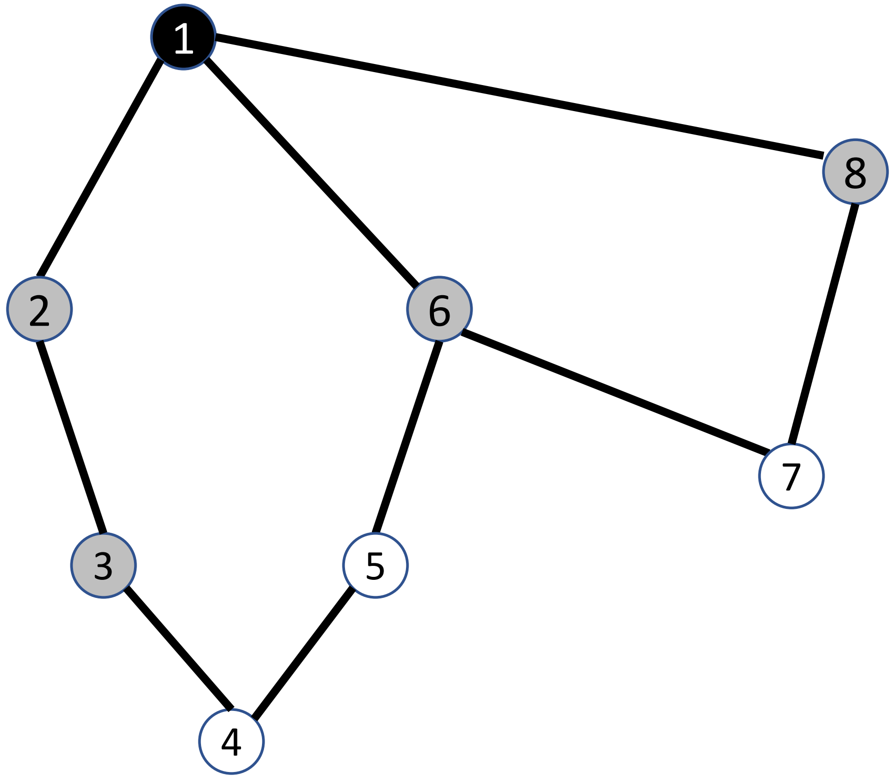
Queue: 1 2 6 8

Exercise: Finding the Smallest Cycle involving Vertex 1



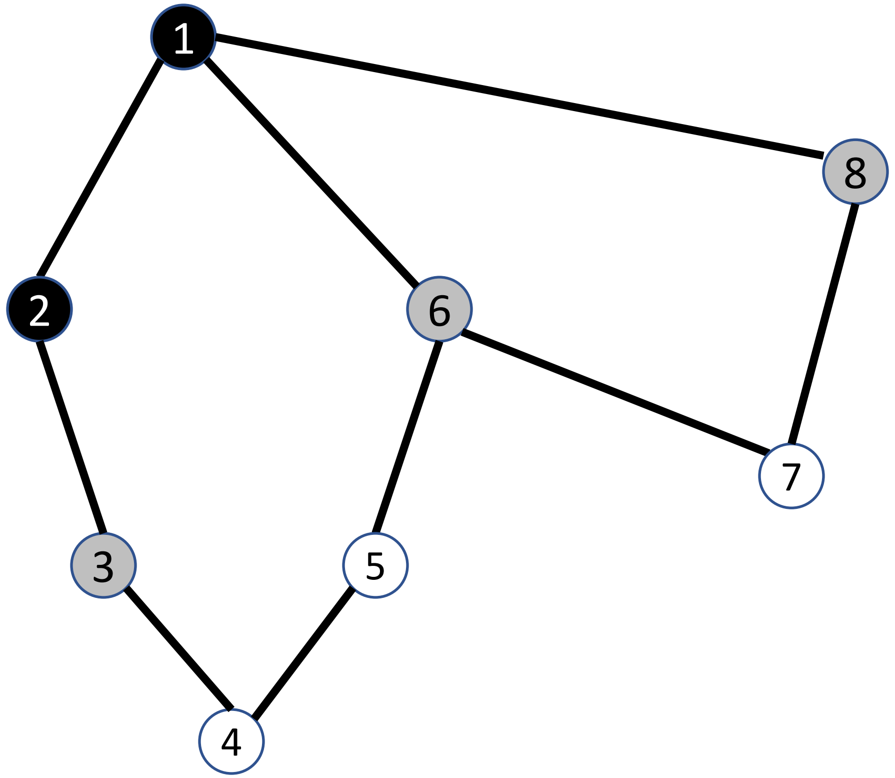
Queue: 2 6 8

Exercise: Finding the Smallest Cycle involving Vertex 1



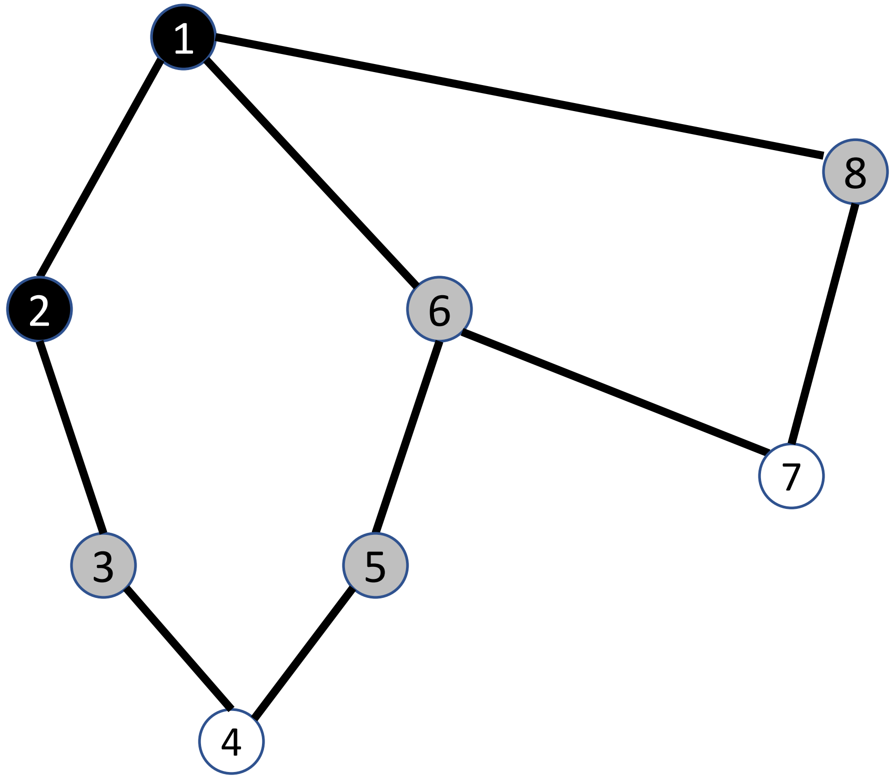
Queue: 2 6 8 3

Exercise: Finding the Smallest Cycle involving Vertex 1



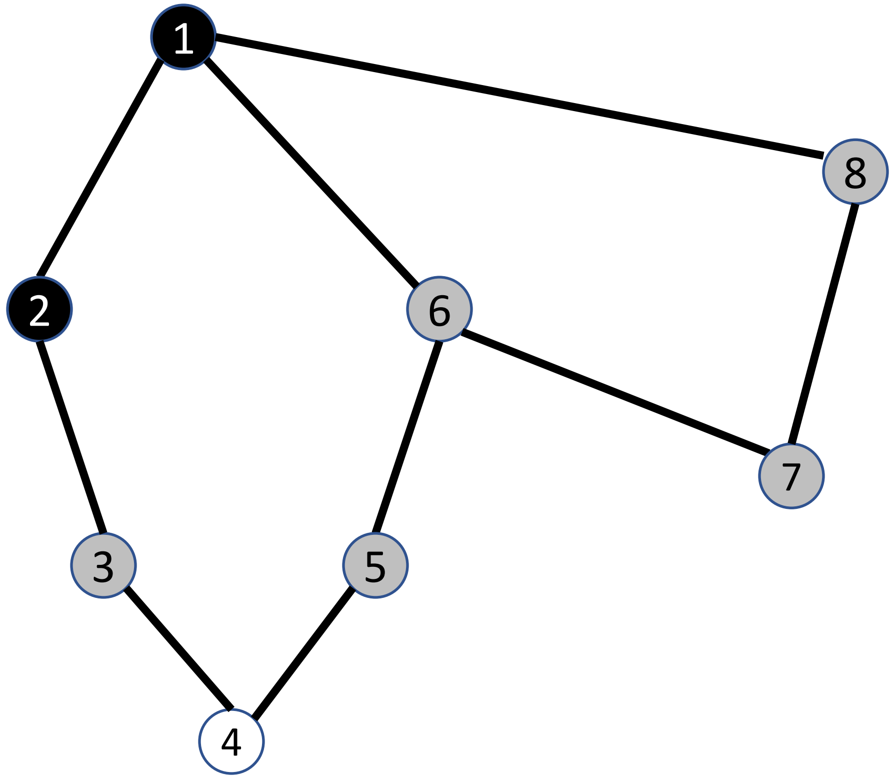
Queue: 6 8 3

Exercise: Finding the Smallest Cycle involving Vertex 1



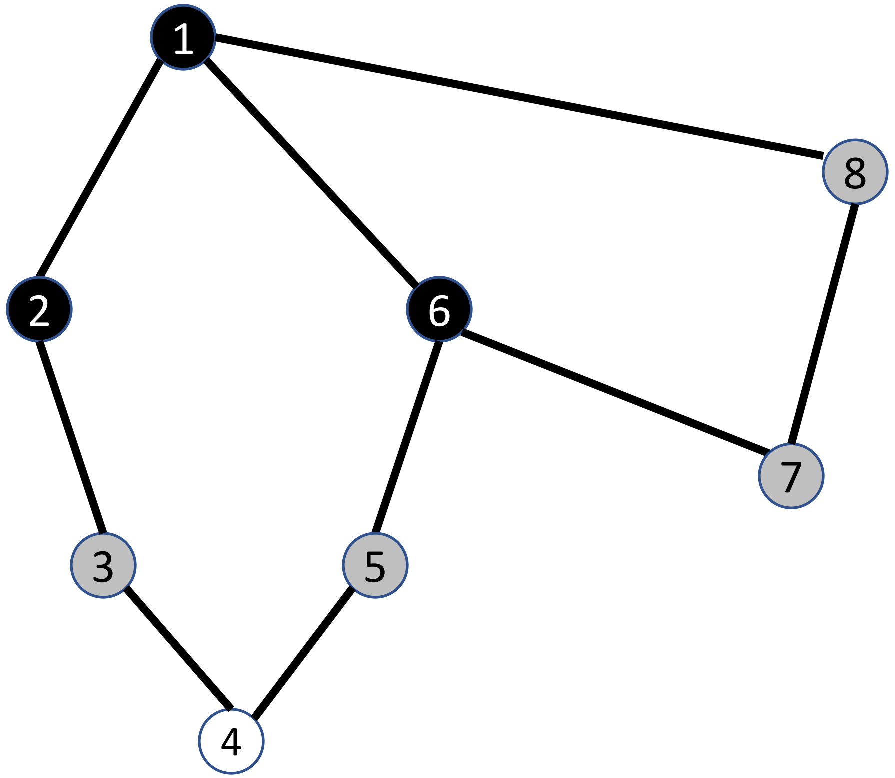
Queue: 6 8 3 5

Exercise: Finding the Smallest Cycle involving Vertex 1



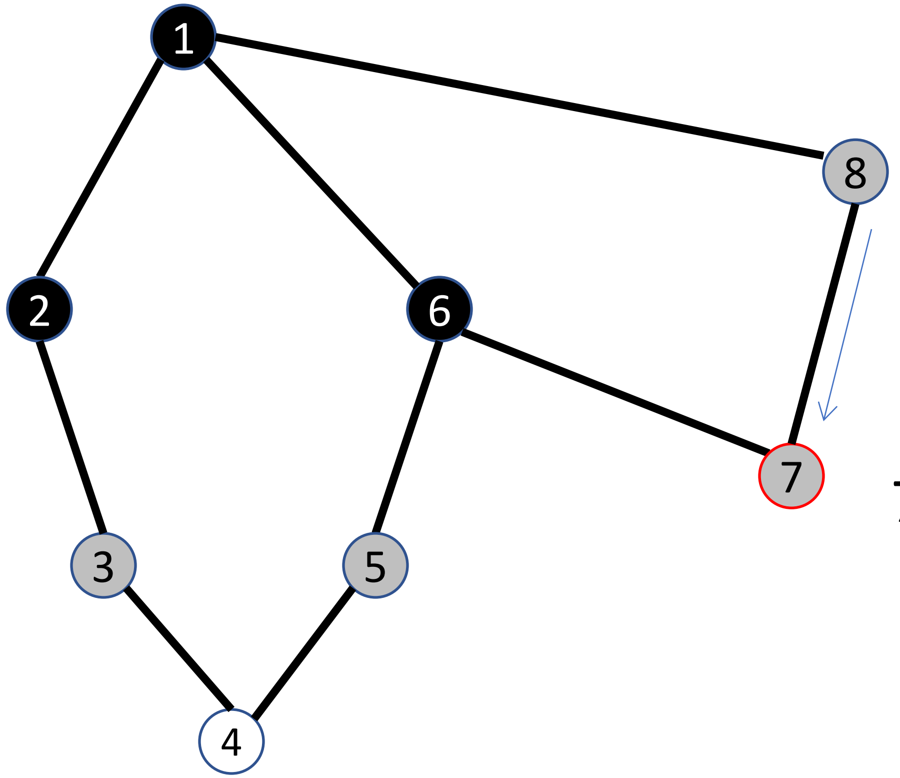
Queue: 6 8 3 5 7

Exercise: Finding the Smallest Cycle involving Vertex 1



Queue: 8 3 5 7

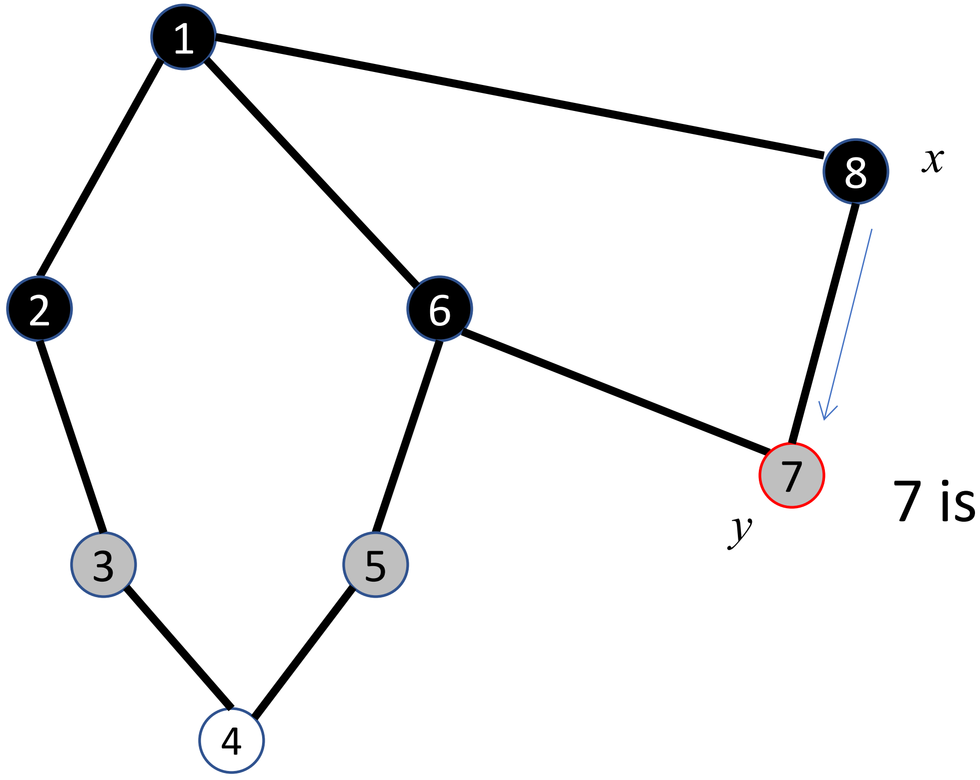
Exercise: Finding the Smallest Cycle involving Vertex 1



7 is already grey!

Queue: 8 3 5 7

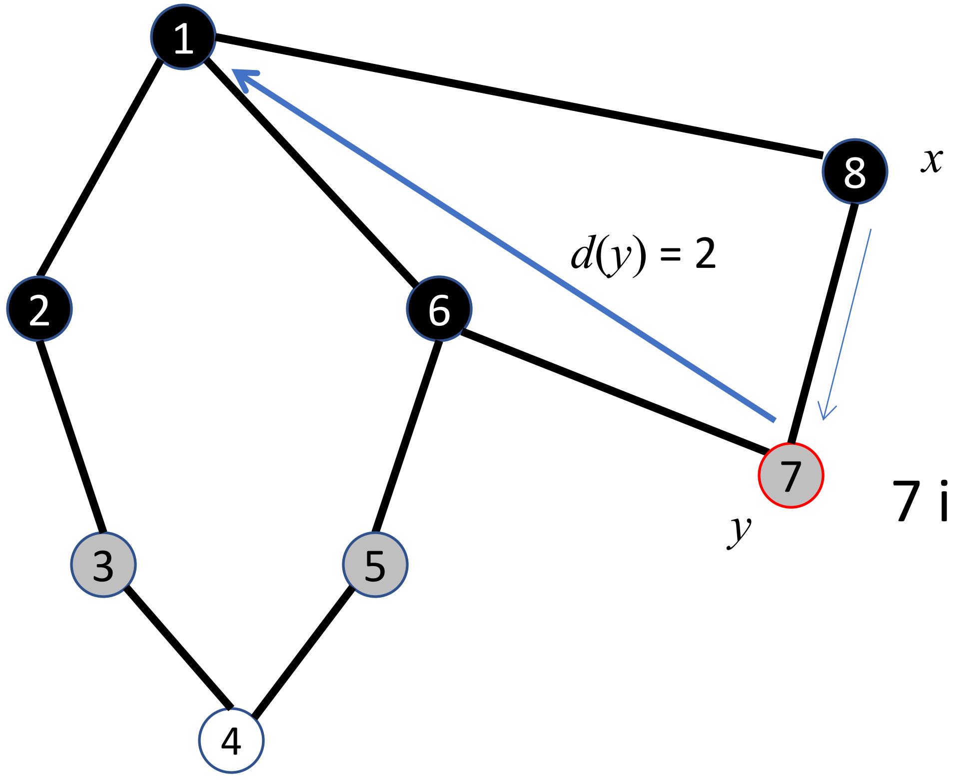
Exercise: Finding the Smallest Cycle involving Vertex 1



7 is already grey!

Queue: 3 5 7 (stopped)

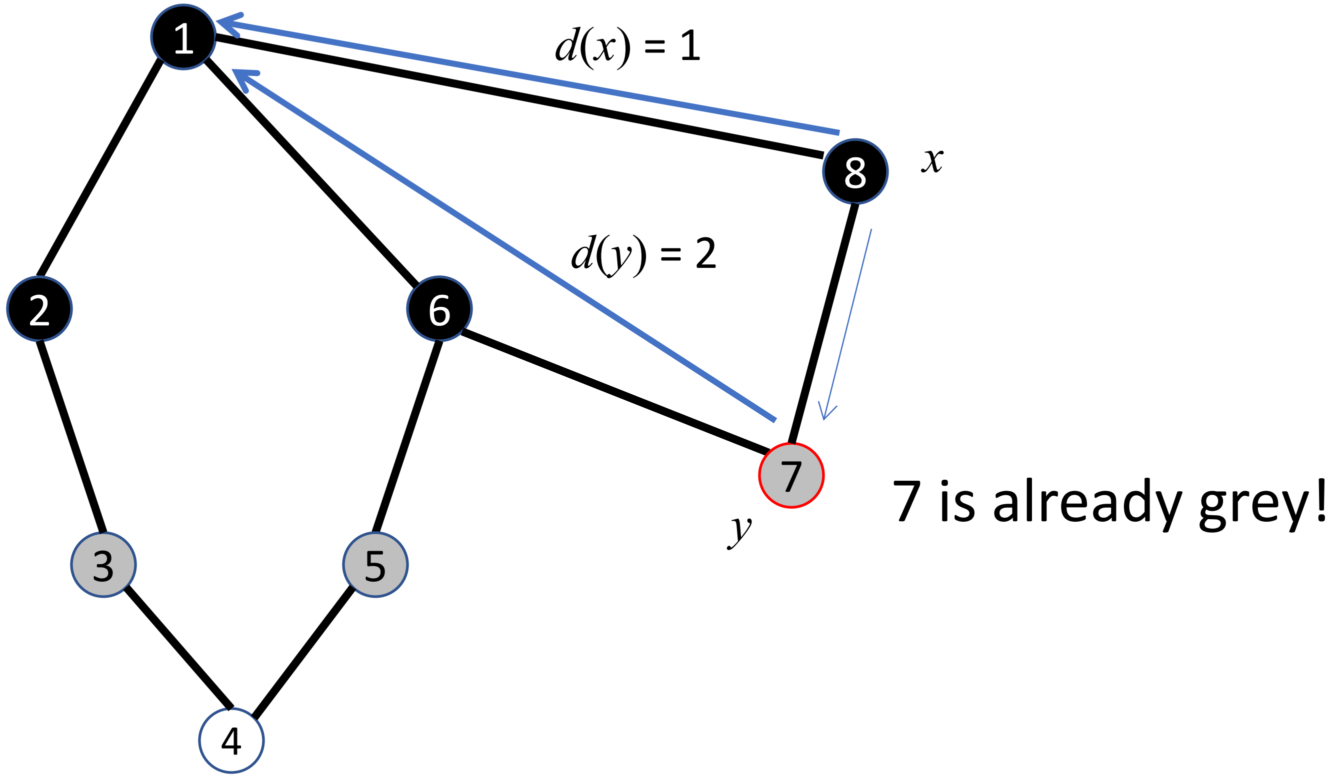
Exercise: Finding the Smallest Cycle involving Vertex 1



7 is already grey!

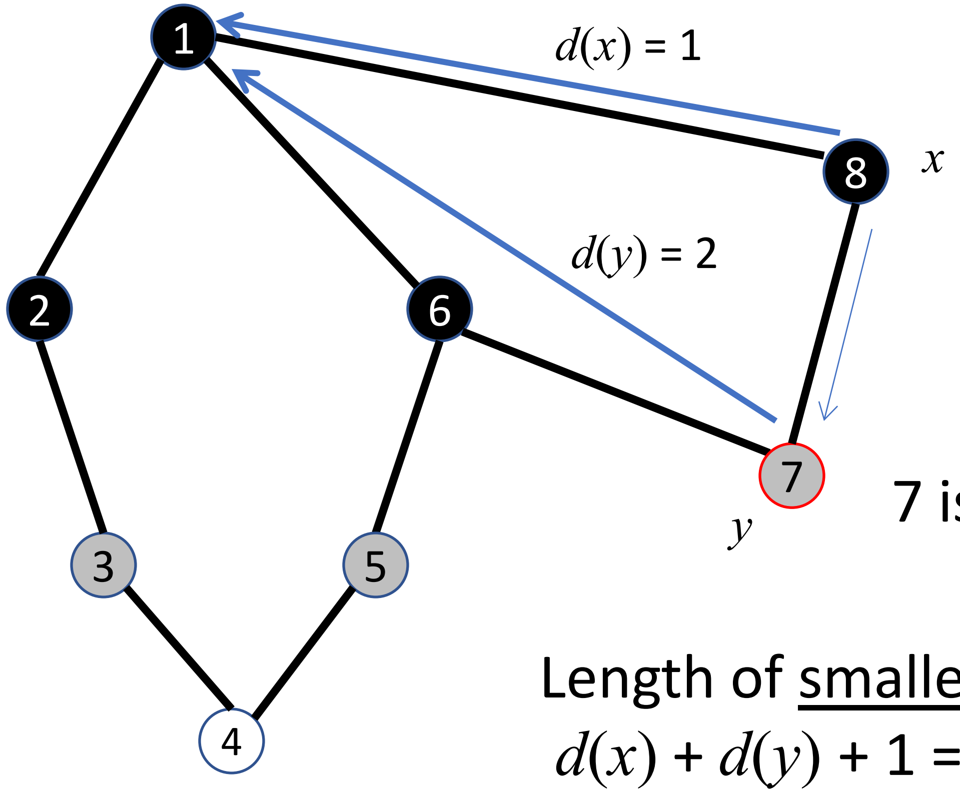
Queue: 3 5 7 (stopped)

Exercise: Finding the Smallest Cycle involving Vertex 1



Queue: 3 5 7 (stopped)

Exercise: Finding the Smallest Cycle involving Vertex 1



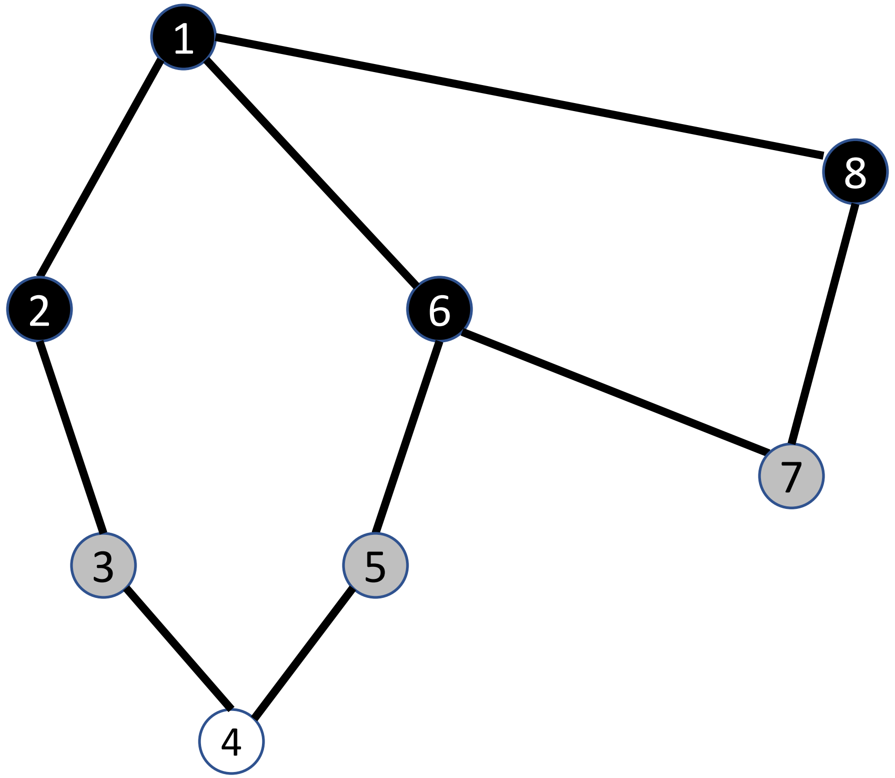
7 is already grey!

Queue: 3 5 7 (stopped)

Length of <u>smallest</u> cycle with vertex 1

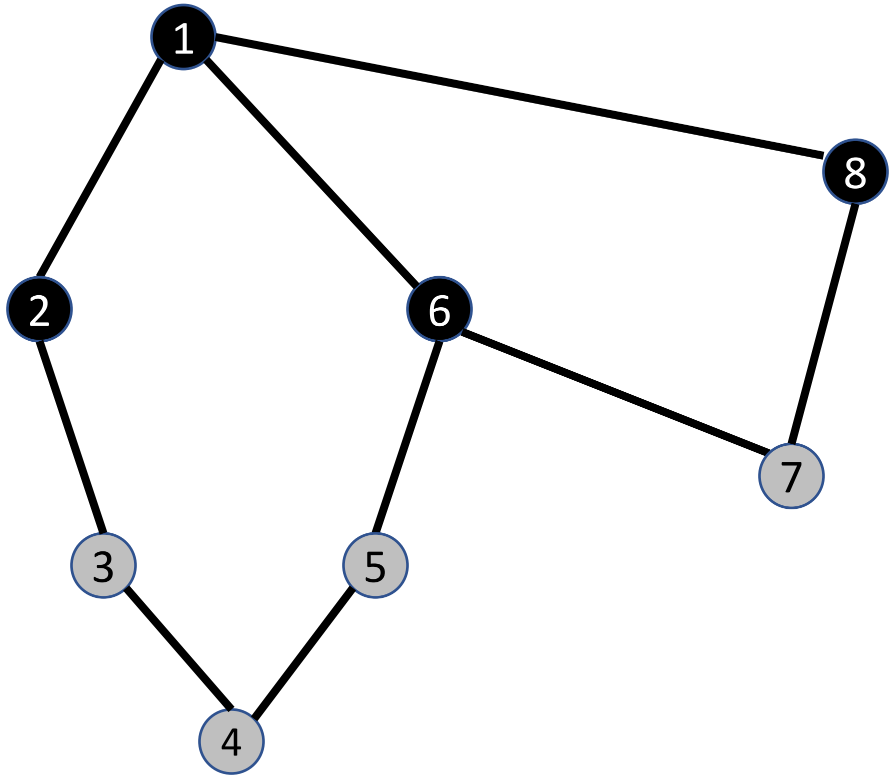
$d(x) + d(y) + 1 = 4$

Exercise: Finding the Smallest Cycle involving Vertex 1



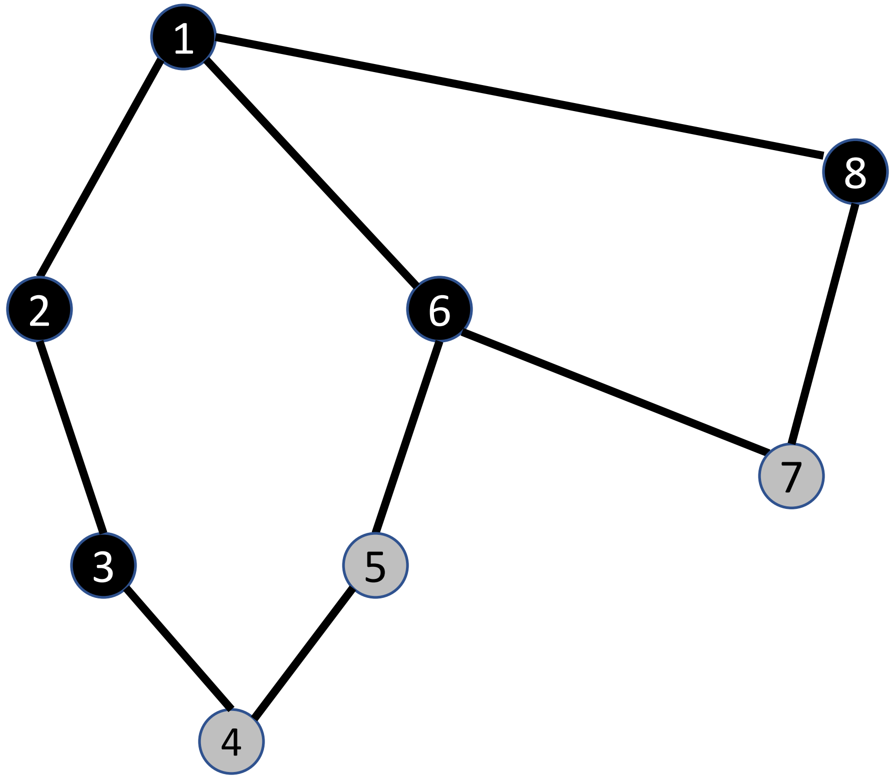
Queue: 3 5 7

Exercise: Finding the Smallest Cycle involving Vertex 1



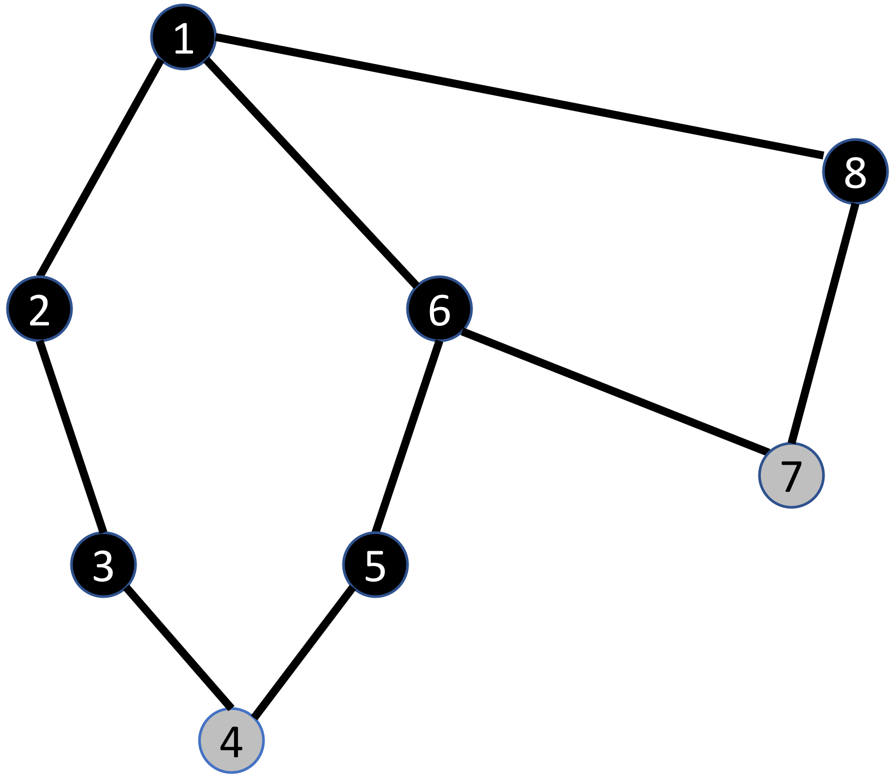
Queue: 3 5 7 4

Exercise: Finding the Smallest Cycle involving Vertex 1



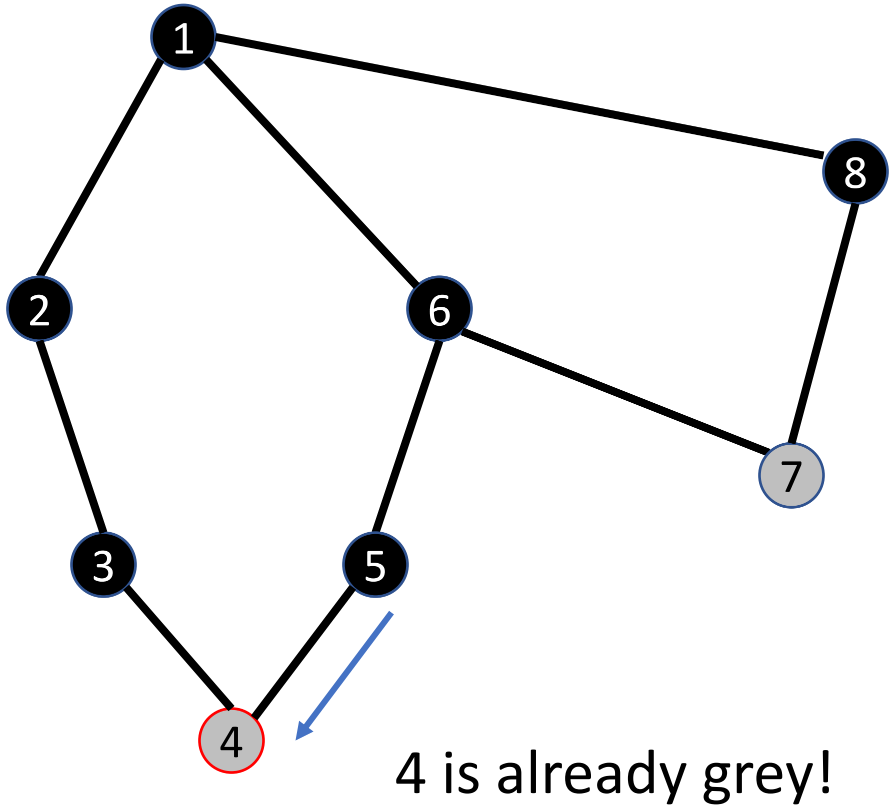
Queue: 5 7 4

Exercise: Finding the Smallest Cycle involving Vertex 1



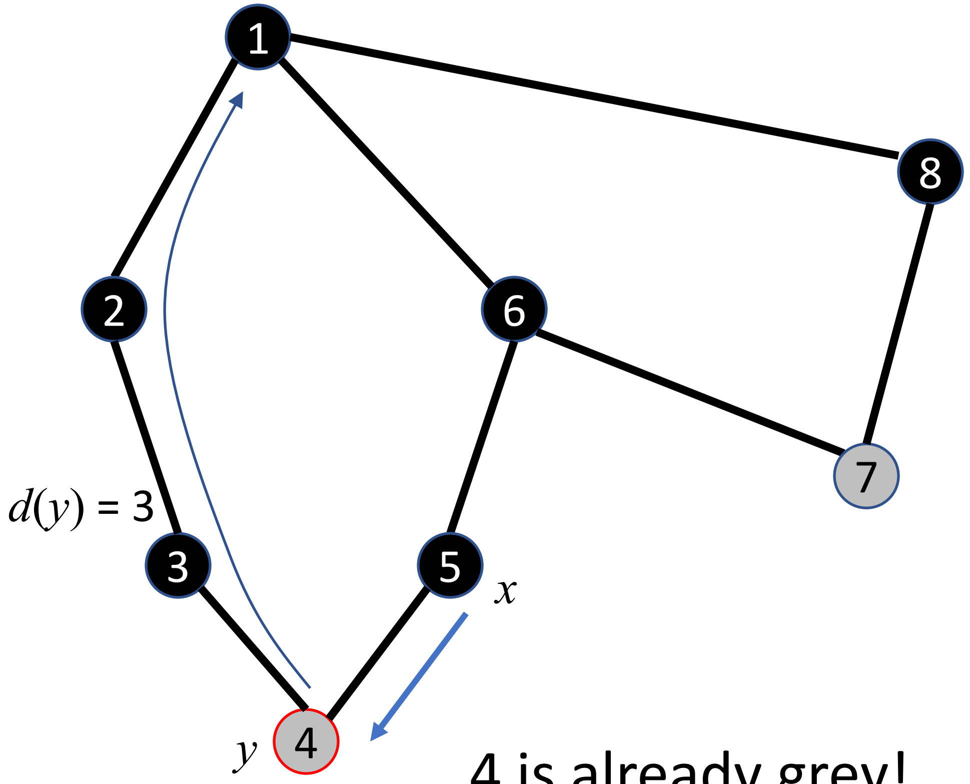
Queue: 7 4

Exercise: Finding the Smallest Cycle involving Vertex 1



Queue: 7 4 (stopped)

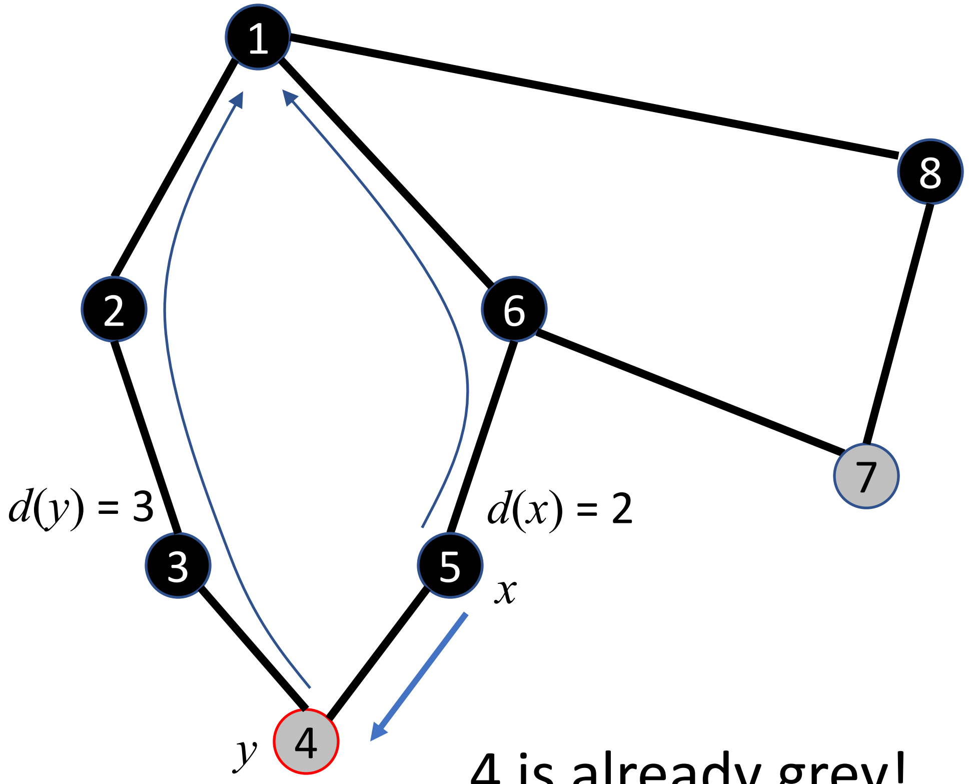
Exercise: Finding the Smallest Cycle involving Vertex 1



4 is already grey!

Queue: 7 4 (stopped)

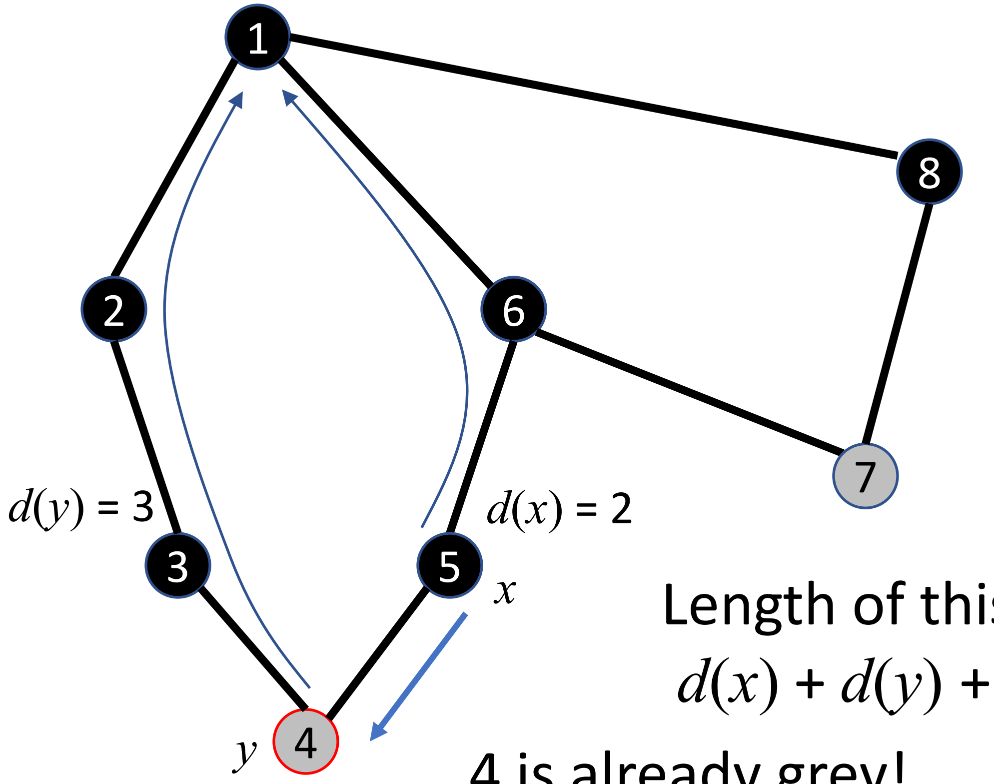
Exercise: Finding the Smallest Cycle involving Vertex 1



4 is already grey!

Queue: 7 4 (stopped)

Exercise: Finding the Smallest Cycle involving Vertex 1



Length of this cycle with vertex 1:

$d(x) + d(y) + 1 = 6$

4 is already grey!

Queue: 7 4 (stopped)

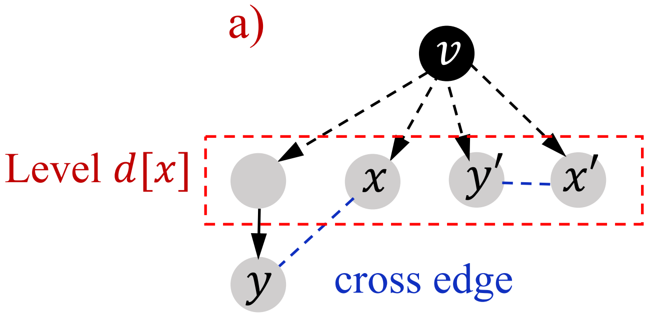
Correctness of the Algorithm

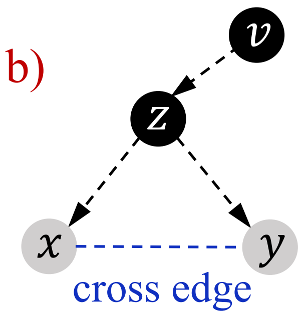
1. Meeting a grey neighbour means that we have found a cycle;
2. The cycle found in this way will be the shortest.

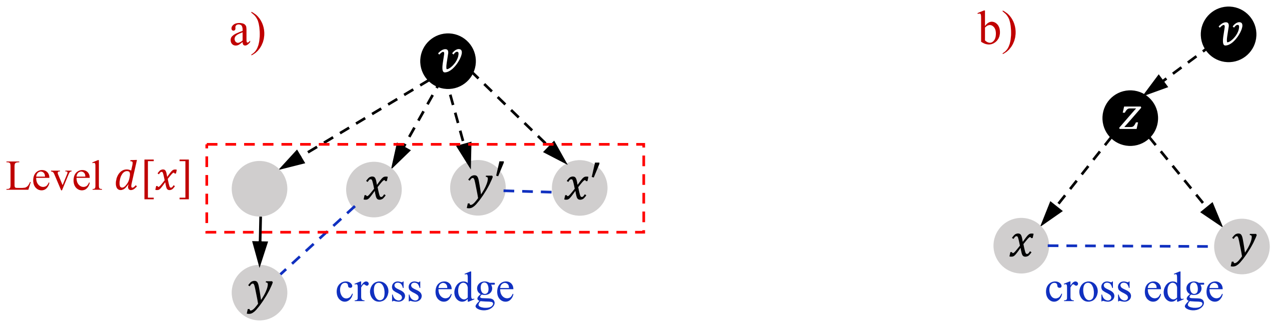
Running BFSVisit(v) above, we have two cases when we first meet a cross edge (x, y) and let $d[x] \leq d[y]$:

a) x and y are in different subtrees of v

b) x and y are in the same subtree of v







Running BFSVisit(v) above gives us two cases when we first meet a cross edge (x,y) and let $d[x] \leq d[y]$:

a) x and y are in different subtrees of v: v,x,y are in the same cycle with length $d[x] + d[y] + 1 \leq 2d[x] + 2$. All cross edges detected at the same level have the same upper bound of cycle length. Then, the algorithm finds the smallest cycle containing v.

b) x and y are in the same subtree of v: a smaller cycle (not including v) exists. Let z be the least common ancestor of x and y. When BFSVisit starts from z, we should find the smallest cycle containing z based on a).

Finding Directed Girth of Digraph

Following procedure finds the (length of the) shortest directed cycle through node v in a digraph.

1. Run BFSVisit(v) starting at node v.
2. The first time a back-arc of the form (x, v) is found, we have found a cycle of length equal to (1 + the depth of x in the search tree).
3. Stop and Return the length found.

- Run the above procedure on every node and pick the length of the shortest cycle

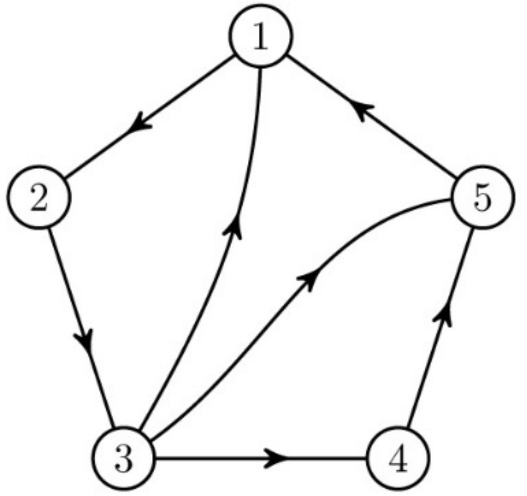
THE UNIVERSITY OF AUCKLAND
Te Whare Wananga o Tamaki Makaurau
NEW ZEALAND

SUMMARY

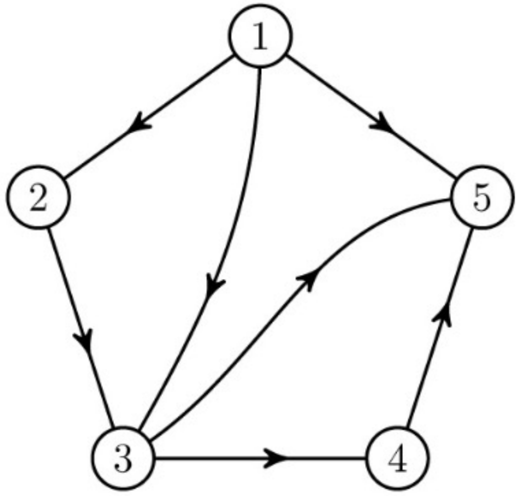
- Terminology
    - Cycle
    - Girth

- DFS Application: Cycle Detection
- BFS Application: Girth Detection

- Correctness of Girth Computation
- Illustrative Examples



(a) A directed cyclic graph.



(b) A directed acyclic graph.