Directed Acyclic Graphs & Topological Order

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COMPCSI220: WEEK 10

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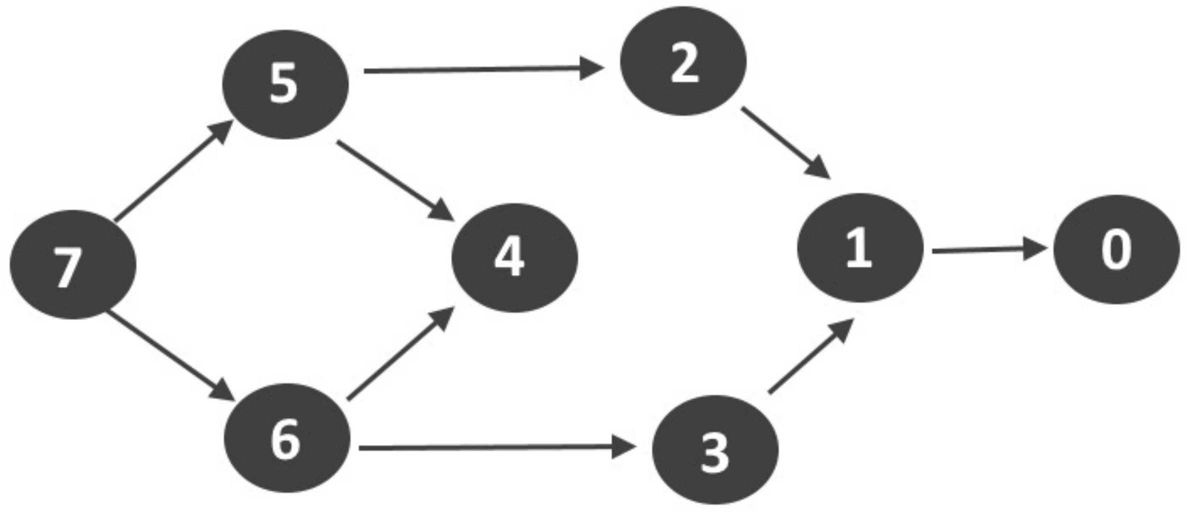
OUTLINE

- Directed Acyclic Graphs (DAGs)
- Topological Orders
  - Definition
  - Illustration
- Topological Sorting
  - Illustrative Examples



Topological Sort : 7 6 5 4 3 2 1 0

Cycle Detection

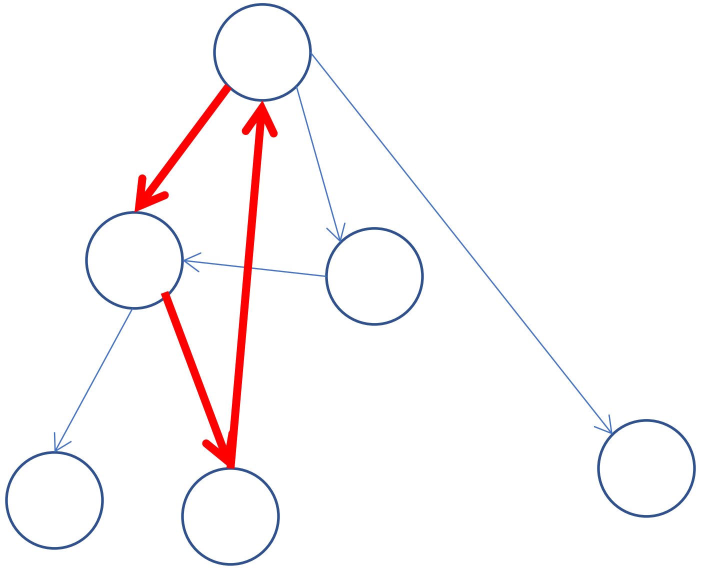
- Suppose that there is a cycle in G and let v be the node in the cycle visited first by DFS. If (u, v) is an arc in the cycle then it must be a back arc.

- Conversely if there is a back arc, we must have a cycle.

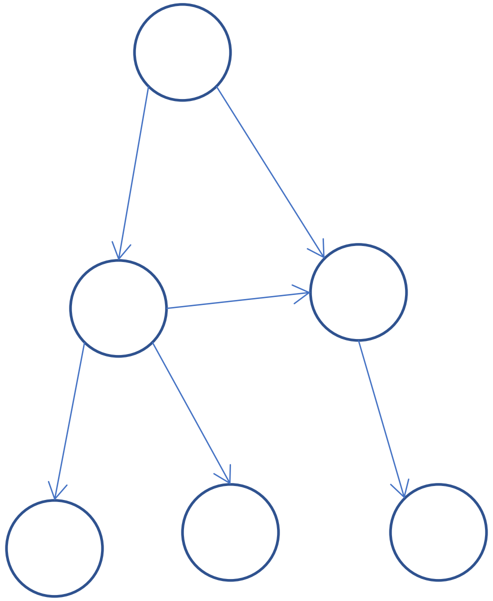
- Suppose that DFS is run on a digraph G. Then G is acyclic if and only if G does not contain a back arc.

- A digraph with no cycle is called a directed acyclic graph (DAG).

Example: Directed Acyclic Graphs (DAGs)



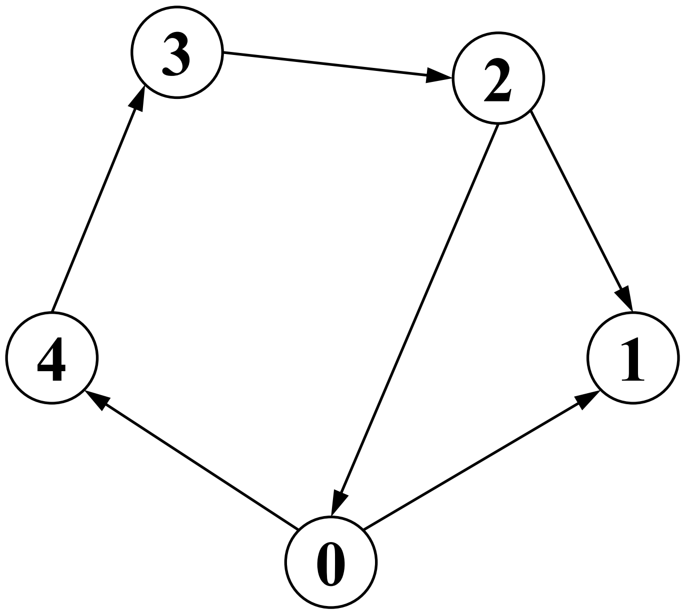
G

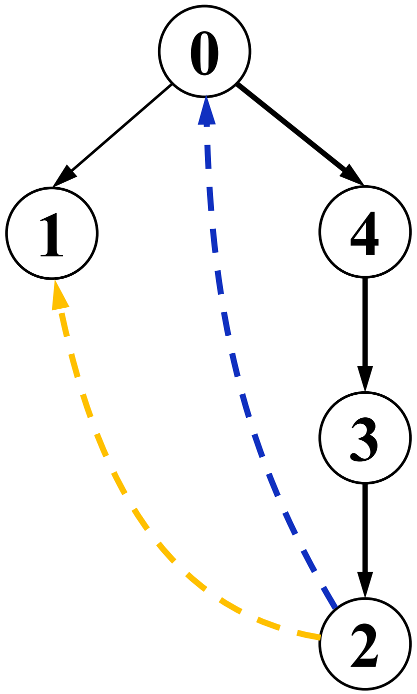


G': Acyclic

Using DFS to Find Cycles in Digraphs

- Once DFS finds a cycle, the stack contains the nodes that form the cycle





Properties: Directed Acyclic Graphs (DAGs)

- Any walk on a DAG G is limited in length

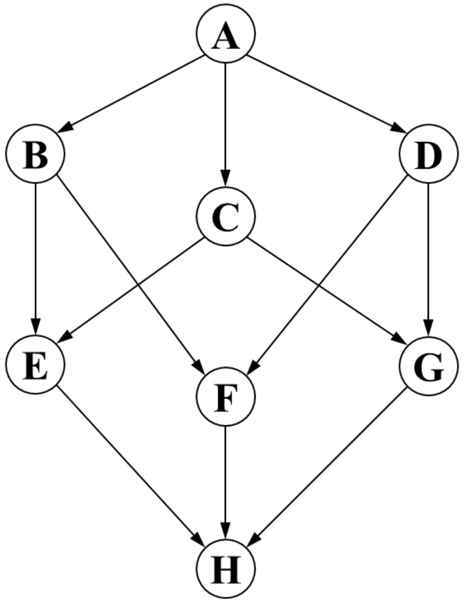
- A DAG must have at least one source and one sink (why?)
  - A graph that has no source or no sink must have a cycle

- Although a DAG has no cycles the underlying graph could have cycles.

- A strongly-connected digraph cannot be a DAG (why?)

Topological Order

- A topological sorting of a digraph G is an ordering on its vertices such that, for each arc (u, v) of G, u appears before v in the ordering.
- To place nodes of a digraph on a line so all arcs go in one direction.
- Main application: scheduling events (arithmetic expressions, university prerequisites, etc).

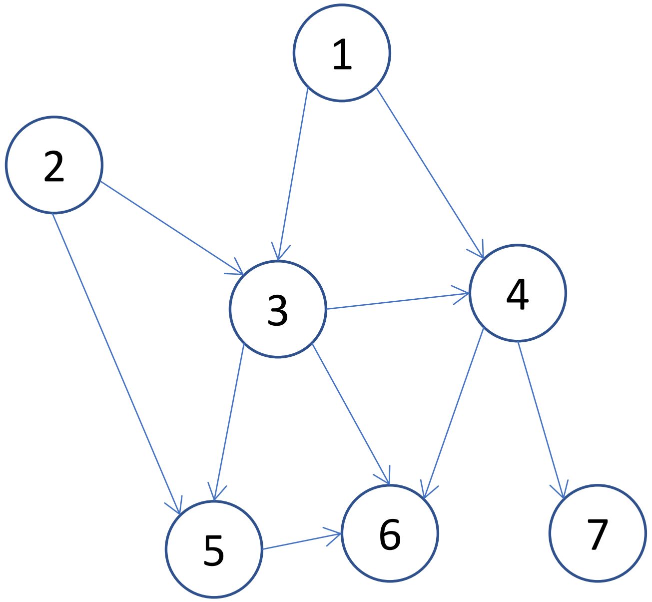


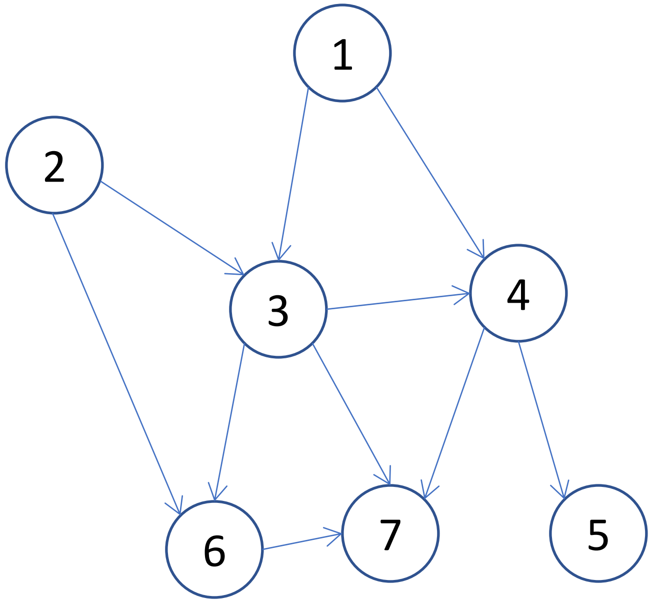
- A D B E C F G H is not a topological sorting, because there is an arc (C,E), but E comes prior to C in the ordering.
- A D B C E F G H is a topological sorting.

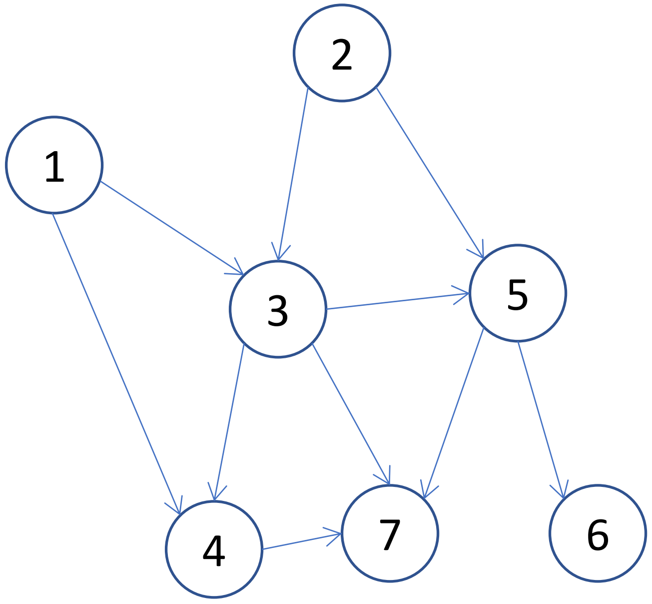
Topological Order (Contd.)

- If G is a DAG, then it is possible to find a topological order of the vertices.
- A topological order is a numbering of the vertices such that an arc (u,v) in the digraph means that u has a smaller number than v.
- Only DAGs have a topological ordering

Example: Topological Order 1



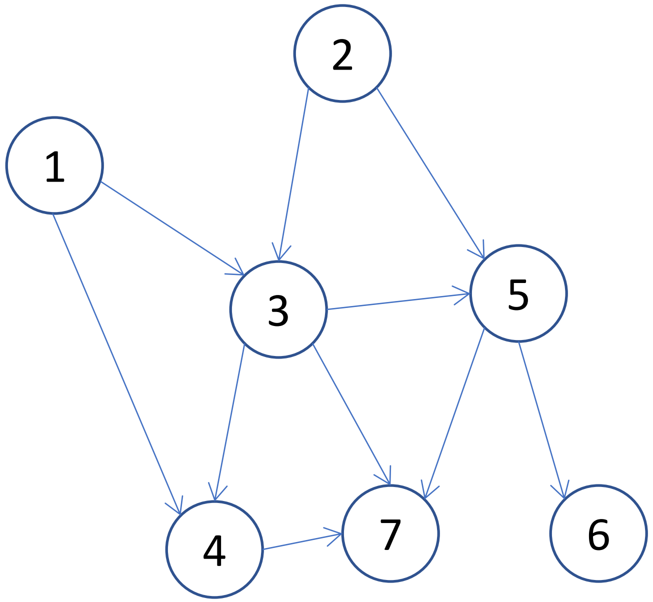




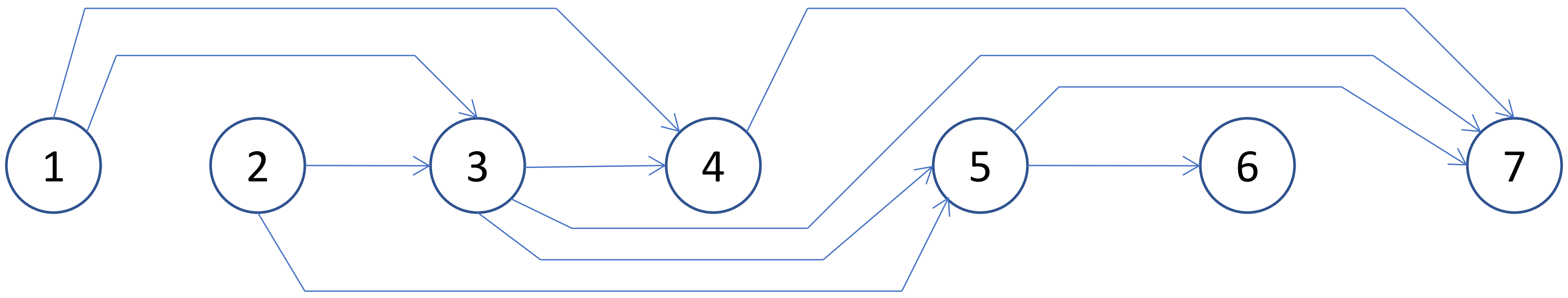
Topological Orders (Contd.)

- There is often more than one topological order for a digraph (see examples on previous slides)
- A topological order is also called a topological sort or a linear order.
- Why "linear" order? Because you can order the vertices in a line and make all arcs point the same way!

Topological Order: to Linear



These two digraphs
are exactly the same!



In linear sort order, all arcs run from left to right!

Topological Sorting

- Topological sorting is possible if and only if digraph is a DAG.
  - If there is a topological order of a digraph, then there is no cycle
    - Suppose there is a cycle $u \rightarrow \cdots \rightarrow v \rightarrow \cdots \rightarrow u$, u and v are both ancestors of each other
  - A DAG always has at least one topological order
    - Every DAG has a source

      Suppose there is no source. Let $u_1, u_2, \ldots, u_n$ be a directed path of maximal length. Then there is no longer directed path that contains $u_1, u_2, \ldots, u_n$. But then u_1 is a source. A contradiction.
    - By removing the source and any out-arcs from the source you still have a DAG.
    - The order we remove the sources forms a topological order

$$\begin{array}{c} \boldsymbol{u_1} \\ \downarrow \\ \boldsymbol{u_2} \\ \downarrow \\ \vdots \\ \downarrow \\ \boldsymbol{u_n} \end{array}$$

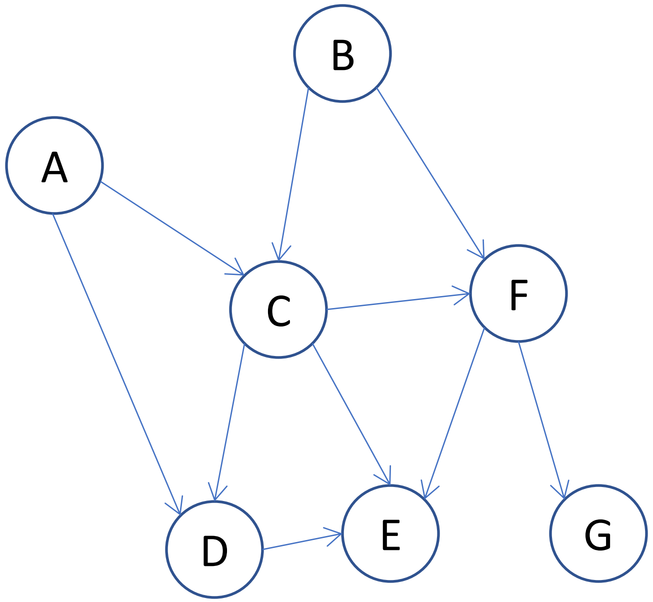
Topological Sorting

- Two solutions:

1. List of finishing times by DFS, in reverse order (since there are no back arcs, each node finishes before anything pointing to it).
2. Zero in-degree sorting – Find a node of in-degree zero, delete it and repeat until all nodes listed.

Example: Topological Order by DFS

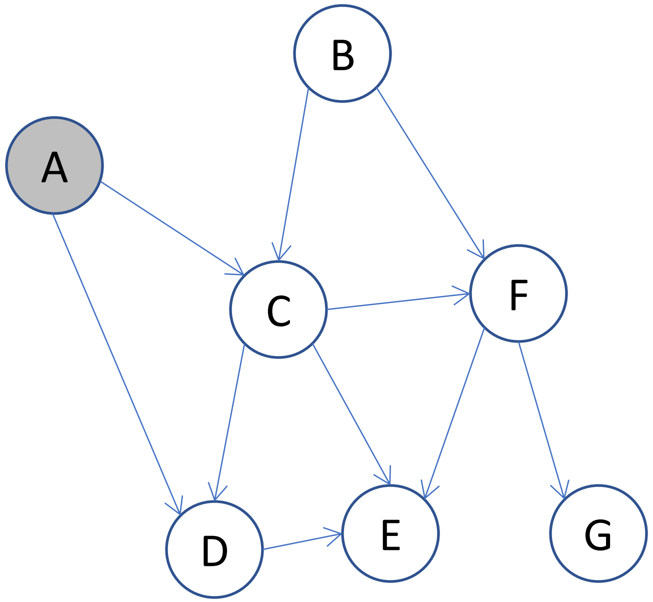
- Starting at the node A:



Example: Topological Order by DFS

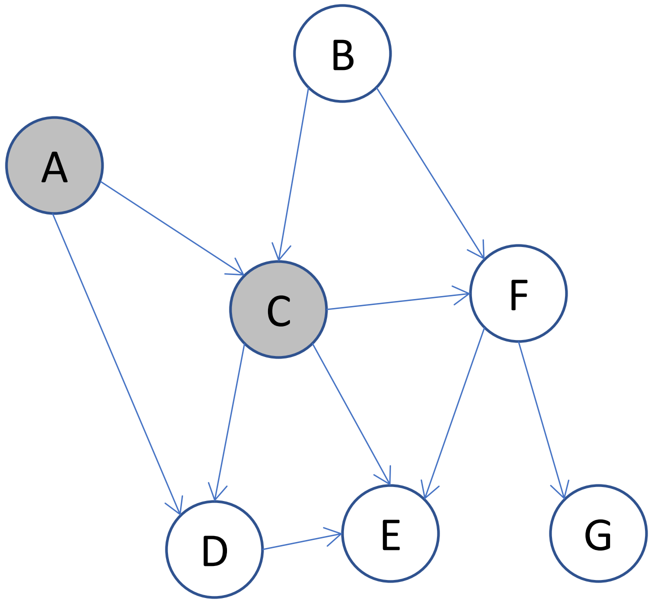
- Starting at the node A:

- push A



Example: Topological Order by DFS

- Starting at the node A:

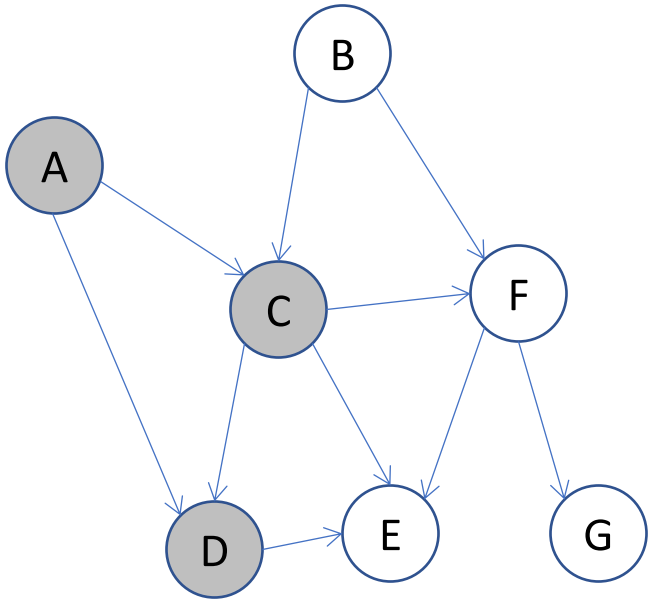


- push A, push C

Example: Topological Order by DFS

- Starting at the node A:

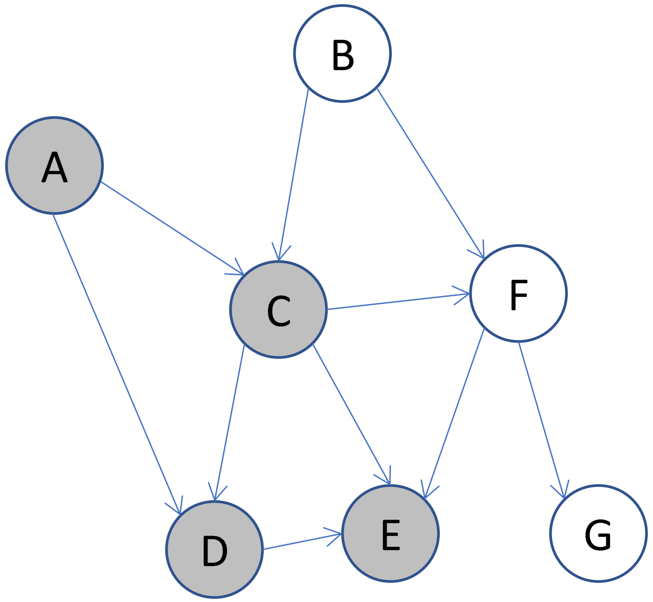
- push A, push C, push D



Example: Topological Order by DFS

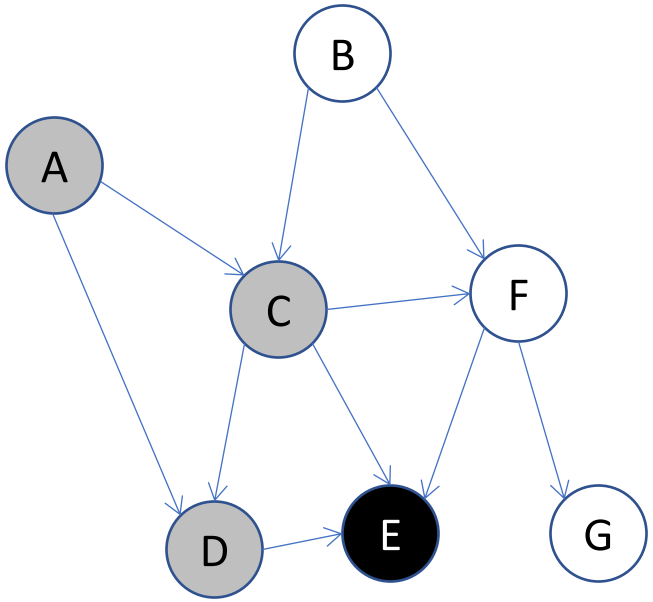
- Starting at the node A:

- push A, push C, push D, push E



Example: Topological Order by DFS

- Starting at the node A:



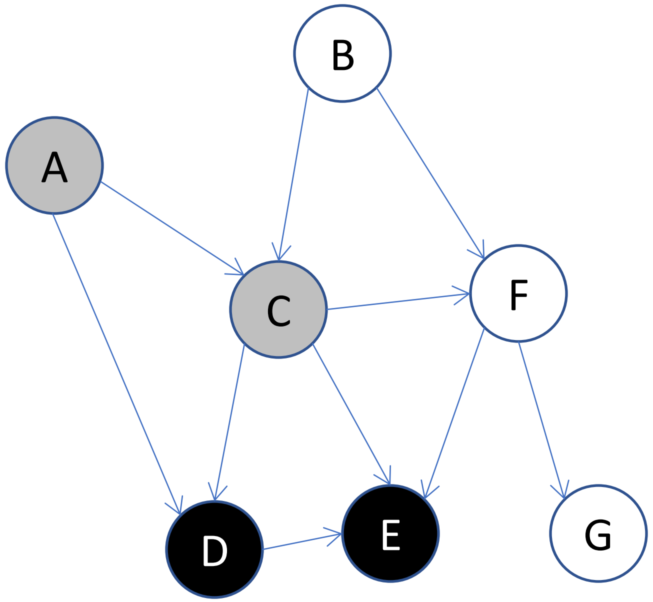
- push A, push C, push D, push E,
- <u>pop E</u>,

- And store E in the last position in the array

						E

Example: Topological Order by DFS

- Starting at the node A:

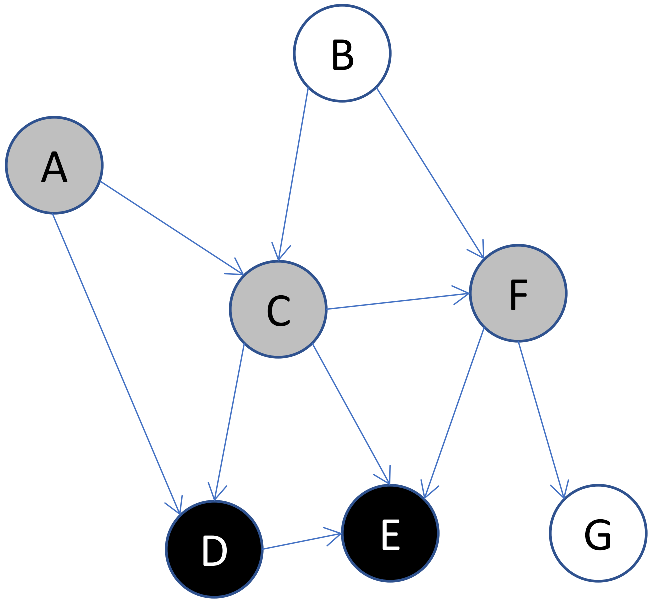


- push A, push C, push D, push E,
- pop E, pop D

- And store D in position array.length()-2

Example: Topological Order by DFS

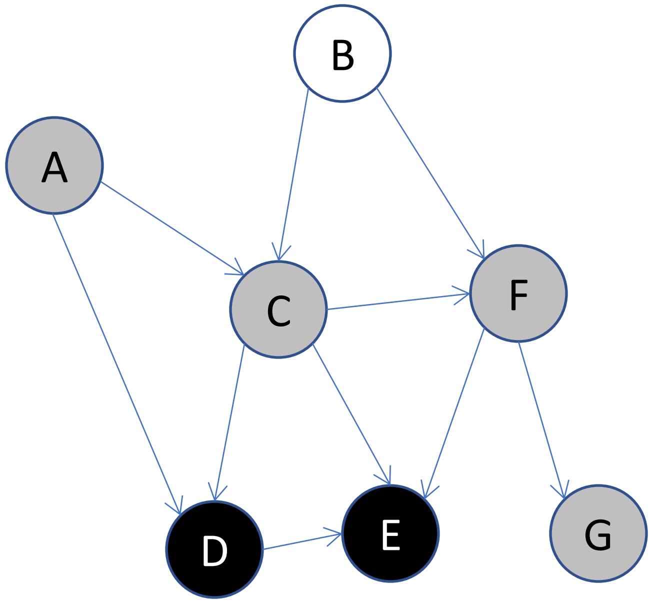
- Starting at the node A:



- push A, push C, push D, push E,
- pop E, pop D, push F

Example: Topological Order by DFS

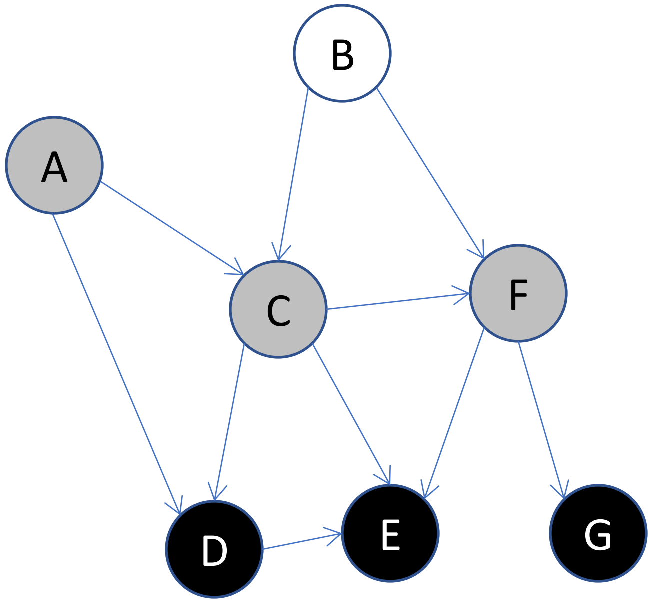
- Starting at the node A:



- push A, push C, push D, push E,
- <u>pop E</u>, <u>pop D</u>, push F, push G

Example: Topological Order by DFS

- Starting at the node A:

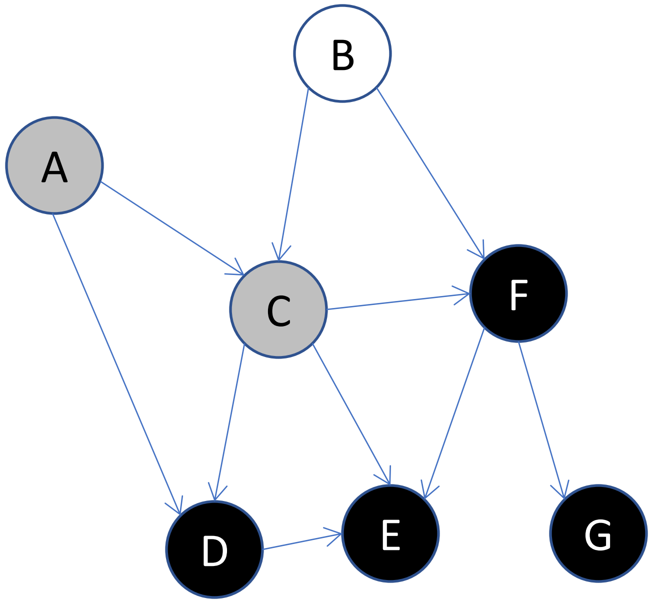


- push A, push C, push D, push E,
- pop E, pop D, push F, push G, pop G

- And store G in position array.length()-3

Example: Topological Order by DFS

- Starting at the node A:



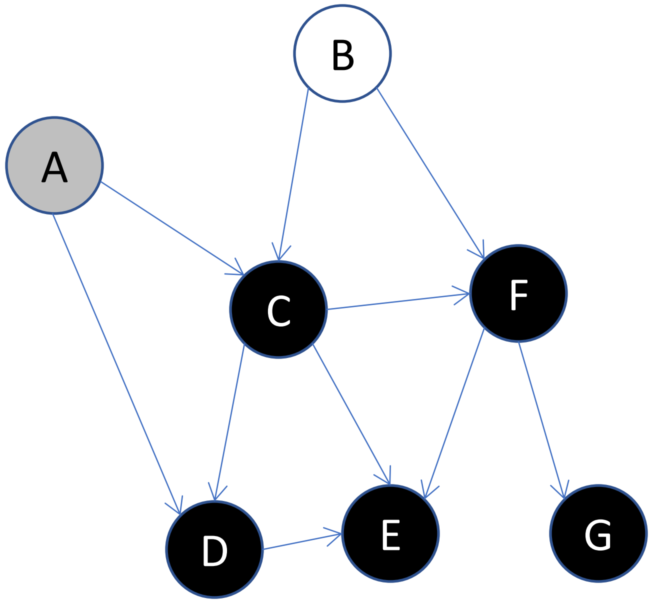
- push A, push C, push D, push E,
- pop E, pop D, push F, push G, pop G
- pop F

- And store F in position array.length()-4

			F	G	D	E

Example: Topological Order by DFS

- Starting at the node A:



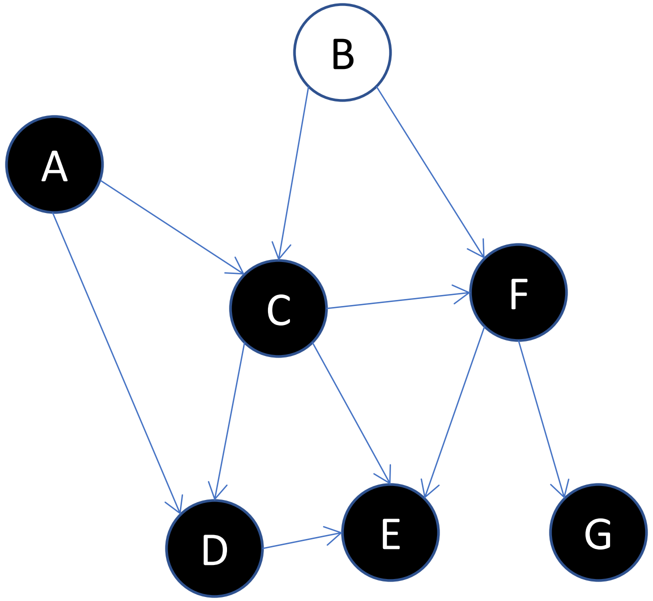
- push A, push C, push D, push E,
- <u>pop E</u>, <u>pop D</u>, push F, push G, <u>pop G</u>
- <u>pop F</u>, <u>pop C</u>

- And store C in position array.length()-5

		C	F	G	D	E

Example: Topological Order by DFS

- Starting at the node A:



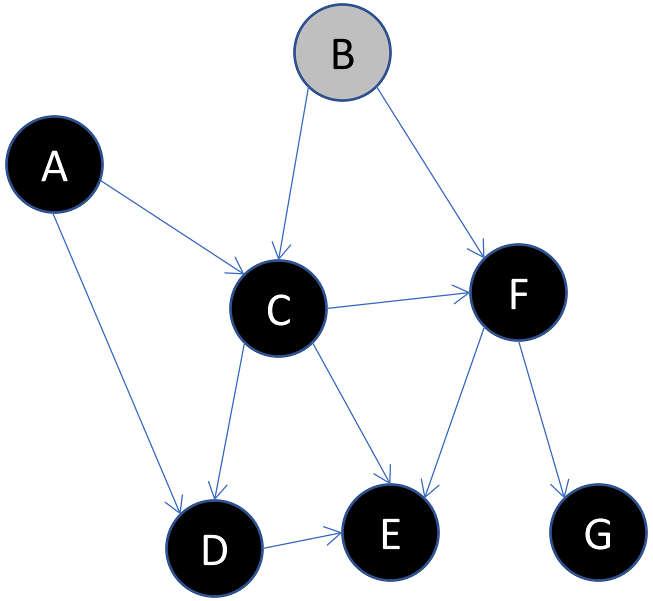
- push A, push C, push D, push E,
- pop E, pop D, push F, push G, pop G
- pop F, pop C, pop A,

- And store A in position array.length()-6

	A	C	F	G	D	E

Example: Topological Order by DFS

- Starting at the node A:

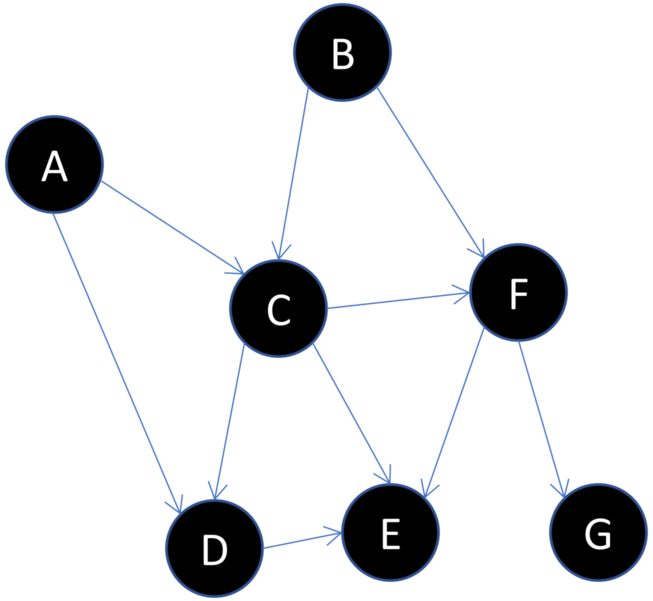


- push A, push C, push D, push E,
- pop E, pop D, push F, push G, pop G
- pop F, pop C, pop A, push B

	A	C	F	G	D	E

Example: Topological Order by DFS

- Starting at the node A:



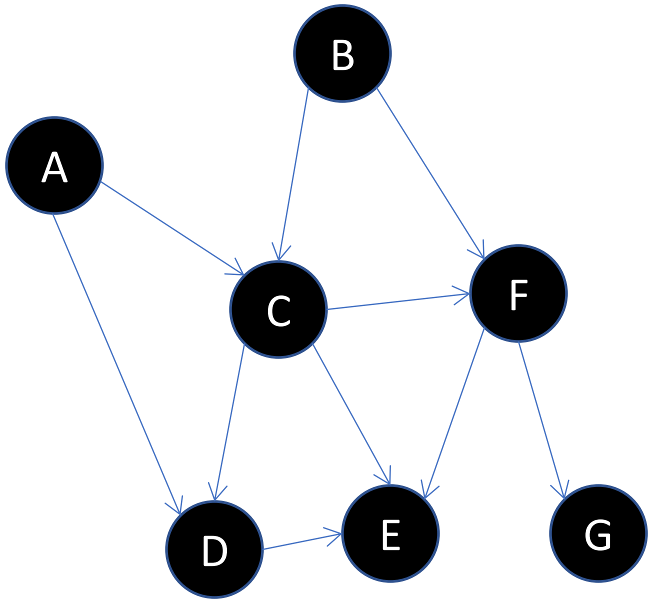
- push A, push C, push D, push E,
- <u>pop E</u>, <u>pop D</u>, push F, push G, <u>pop G</u>
- <u>pop F</u>, <u>pop C</u>, <u>pop A</u>, push B, <u>pop B</u>

- And store B in position array.length()-7

B	A	C	F	G	D	E

Example: Topological Order by DFS

- Starting at the node A:

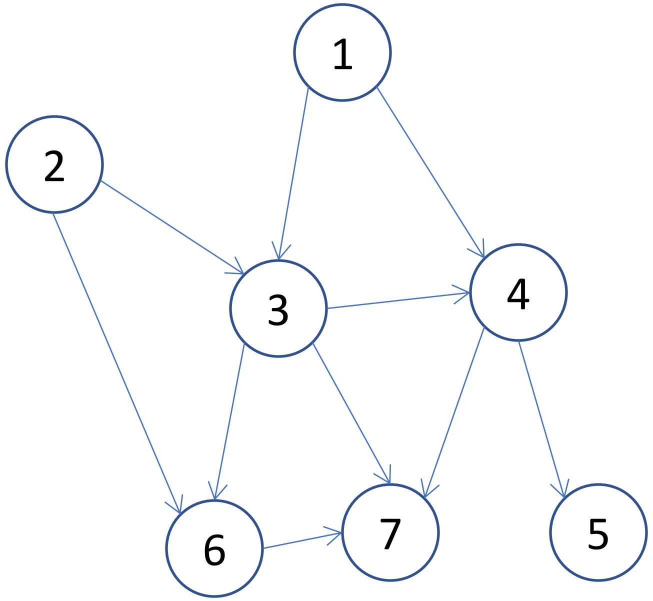


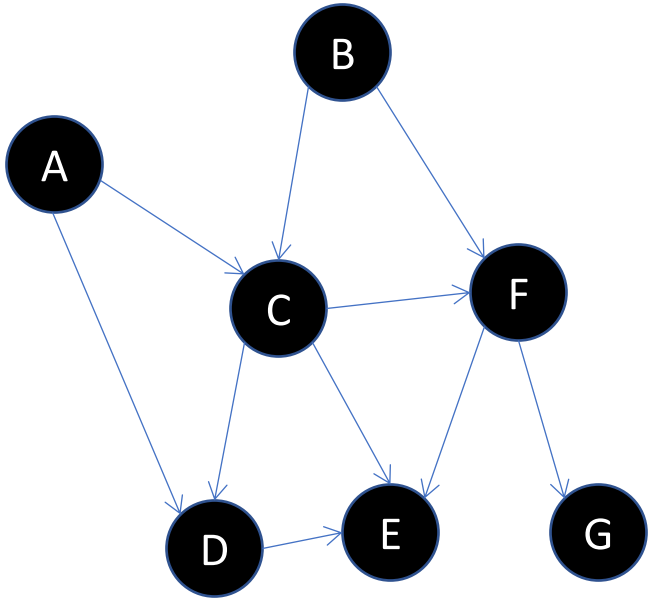
- push A, push C, push D, push E,
- pop E, pop D, push F, push G, pop G
- pop F, pop C, pop A, push B, pop B

- So, we get B=1, A=2, C=3, F=4, G=5, D=6, E=7 (as in example 2)

B	A	C	F	G	D	E

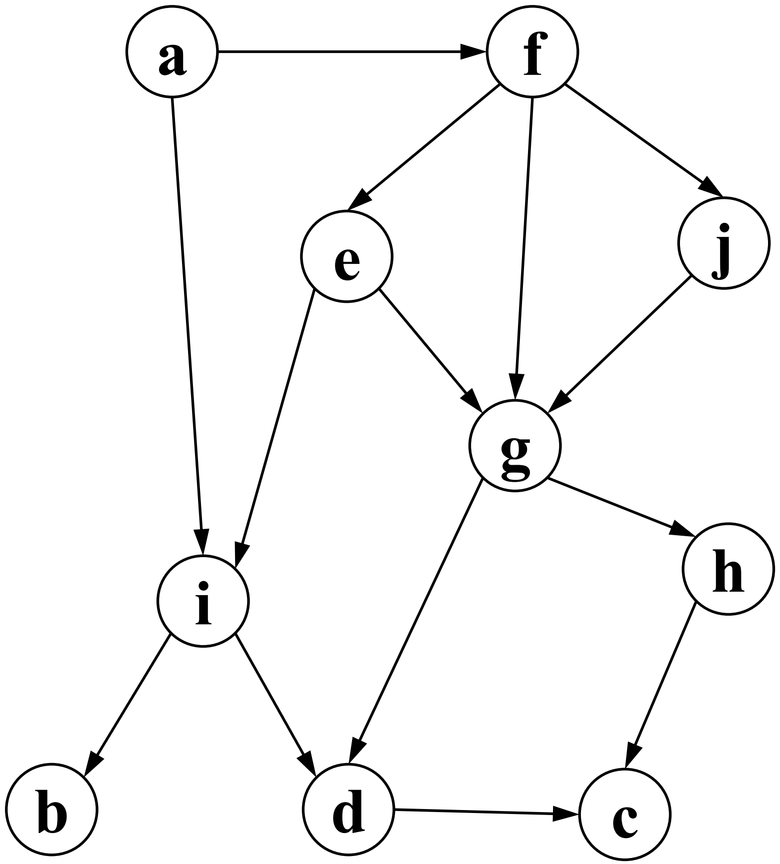
Example: Topological Order 2

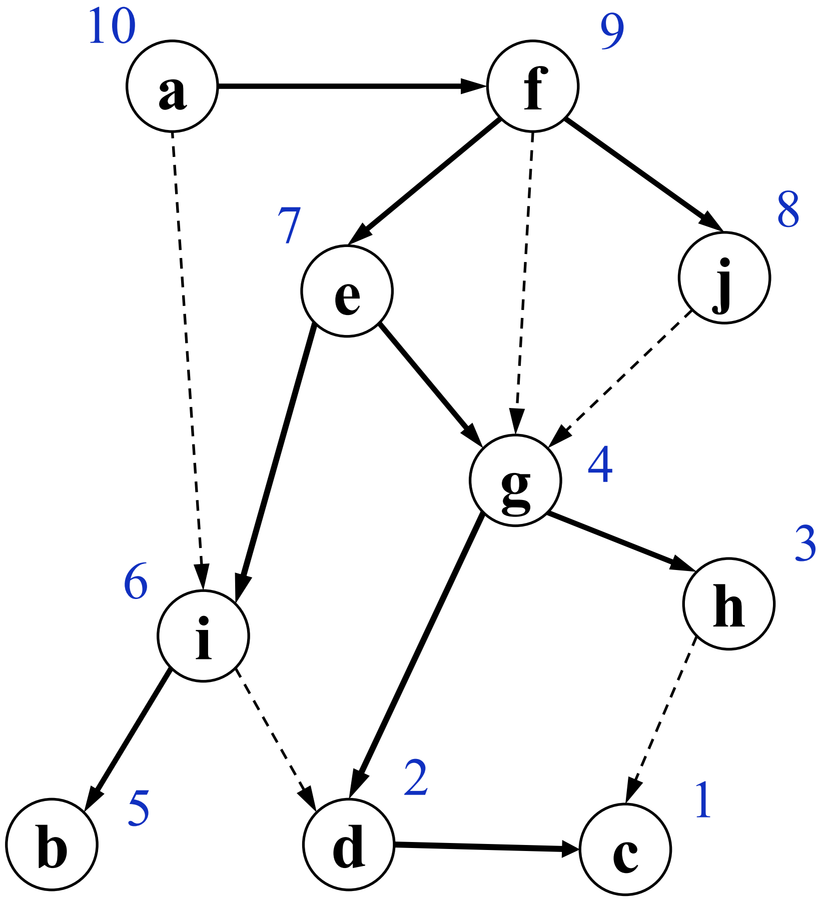




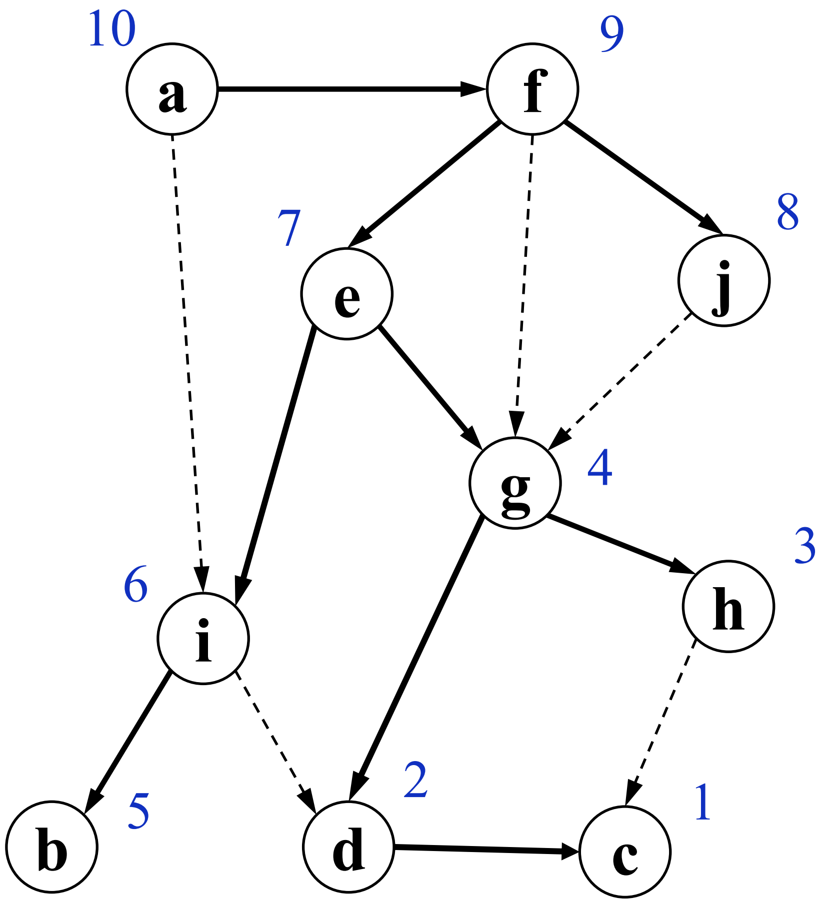
B=1, A=2, C=3, F=4, G=5, D=6, E=7

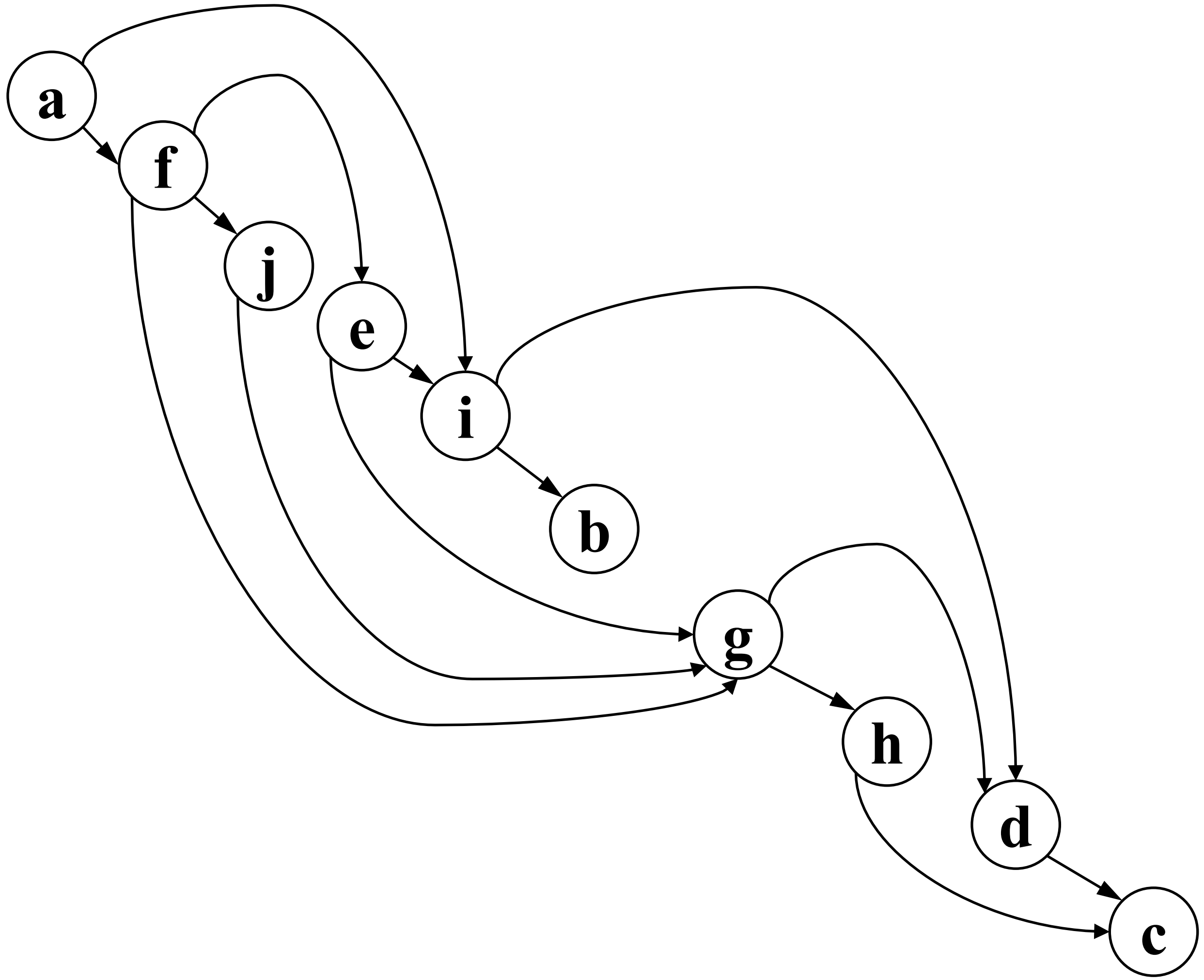
Example: Topological Order





Example: Topological Order





Zero in-degree Sorting

- **Zero indegree sorting**: Start with DAG, say G. Initialize the order list L as empty.

1. Find a source vertex v.

2. Delete v and all its out-going arcs. Append node v to the ordered list L. Since we only delete arcs and vertices, this process does not create a cycle. Hence, the resulting graph is a DAG and has at least a source node.

3. Repeat 1 and 2 under all the vertices have been deleted. Then, we will get a topological order L of G.

Zero in-degree Sorting

- What is the running time of a naive implementation of zero-indegree sorting where a source is found and then removed at each step? How could this idea be made more efficient?
- For each iteration, we need to find zero indegree vertex v, delete v and its out-going arcs. We calculate the in-degree over every node each time, it takes

Adjacency Matrix: $O(n^2)$ Adjacency List: $O(m)$

- For all n iterations:

Adjacency Matrix: $O(n^3)$ Adjacency List: $O(nm)$

- We don't have to calculate the in-degree every time, we can use an array to track the in-degree of every node!

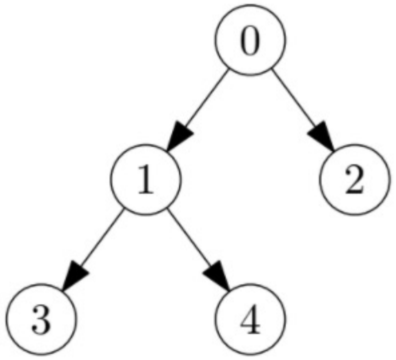
Zero In-degree Sorting: A Faster Algorithm

Algorithm 1 TopSort.

function TopSort(digraph G)
    for $u \in V(G)$ **do**
        $indegree[u] \leftarrow$ indegree of u
    queue Q, $count \leftarrow 0, order = []$
    for $u \in V(G)$ **do**
        if $indegree[u] = 0$ **then** $Q.enqueue(u)$
    while Q is not empty **do**
        $u \leftarrow Q.dequeue(), \quad order.append(u), \quad count \leftarrow count + 1$
        for v as out-neighbor of u **do**
            $indegree[v] \leftarrow indegree[v] - 1$
            if $indegree[v] = 0$ **then** $Q.enqueue(v)$
    if $count$!= $|V(G)|$ **then**
        return $NULL$
    else
        return $order$

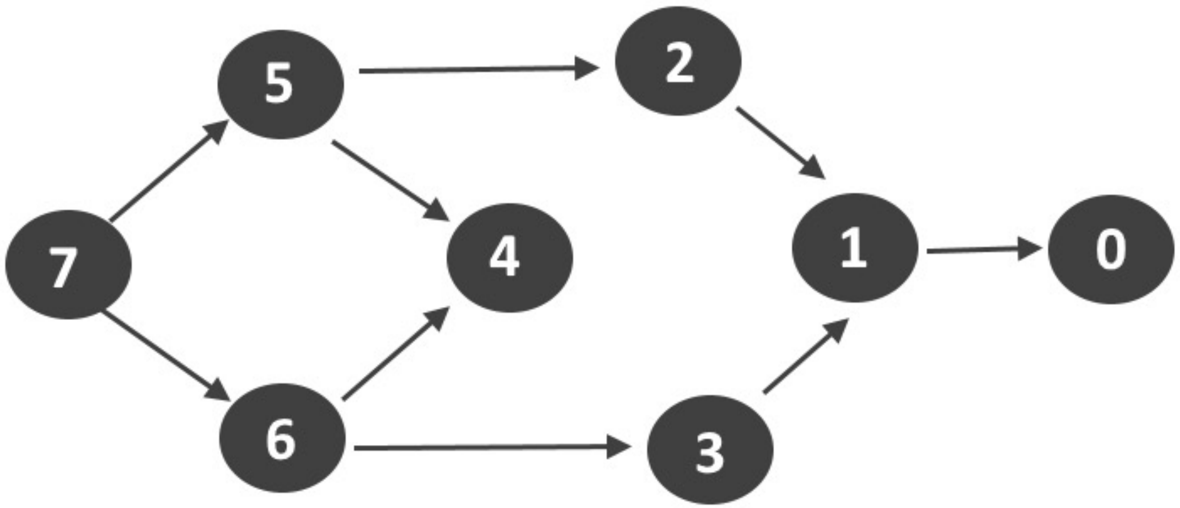
Exercise: Topological Orders

- Topological orders are not unique. List all possible topological orders of the following digraph. We can use zero in-degree sorting.



SUMMARY

- Directed Acyclic Graphs (DAGs)
- Topological Orders
  - Illustrative Examples
- Topological Sorting
  - DFS
  - Zero-indegree Sorting



Topological Sort : 7 6 5 4 3 2 1 0