Priority Queues

COMPSCI 220: WEEK 8.1

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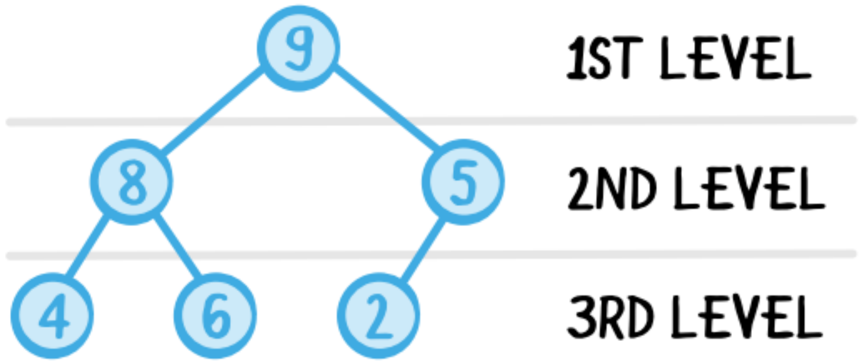
OUTLINE

- Priority Queue
- Heaps
  - Illustrating examples
  - Basic Operations
  - Implementation
  - Complexity Analysis



Definition of Priority Queue

- A priority queue is a container Abstract Data Type (ADT), where each element has a key (from a totally ordered set, as with sorting) called its priority.
  - E.g., Queuing in a bank, VIPs have higher priority
  - E.g., A TODO list highlighting the importance of the items

- Applications: Sorting, Graph algorithms

- Three key operations: insert an element, and to find and delete the element of highest priority
- Implementations: unsorted array, sorted array and binary heap.

Sorted and Unsorted Arrays

- The three key operations: **Insert**, **FindMax** and **DeleteMax**

Unsorted Array

10	7	12	1	4	

→

10	7	1	4		

	FindMax	**DeleteMax**	**Insert**
Unsorted Array	$\Theta(n)$	$\Theta(n)$	$\Theta(1)$

Unsorted array:

- **Find maximum element**: needs $\Theta(n)$ to scan the array
- **Delete maximum element**: needs $\Theta(n)$ to find the max element, and $\Theta(n)$ to move all elements on its right hand side.
- **Insert an element**: Insert at the end in $\Theta(1)$

Sorted and Unsorted Arrays (Contd.)

- The three key operations: **Insert**, **FindMax** and **DeleteMax**

Sorted Array

1	4	7	10	12	

6

1	4	6	7	10	12

	FindMax	**DeleteMax**	**Insert**
Unsorted Array	$\Theta(n)$	$\Theta(n)$	$\Theta(1)$
Sorted Array	$\Theta(1)$	$\Theta(1)$	$\Theta(n)$

Sorted array:

- **Find maximum element**: needs $\Theta(1)$ to retrieve the last element
- **Delete maximum element**: needs $\Theta(1)$ to remove the last element
- **Insert an element**: Find the location to insert in $\Theta(n)$, move at most n elements in $\Theta(n)$

Sorted and Unsorted Arrays (Contd.)

- The three key operations: **Insert**, **FindMax** and **DeleteMax**

Sorted Array

1	4	7	10	12	

6

→

1	4	6	7	10	12

	FindMax	**DeleteMax**	**Insert**
Unsorted Array	Θ(n)	Θ(n)	Θ(1)
Sorted Array	Θ(1)	Θ(1)	Θ(n)
Heap (Binary)	Θ(1)	Θ(logn)	Θ(logn)

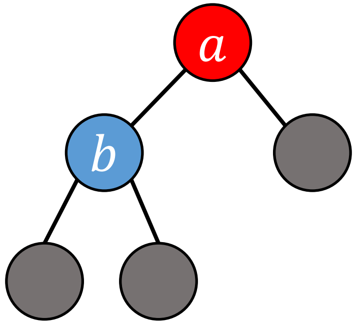
What we want:

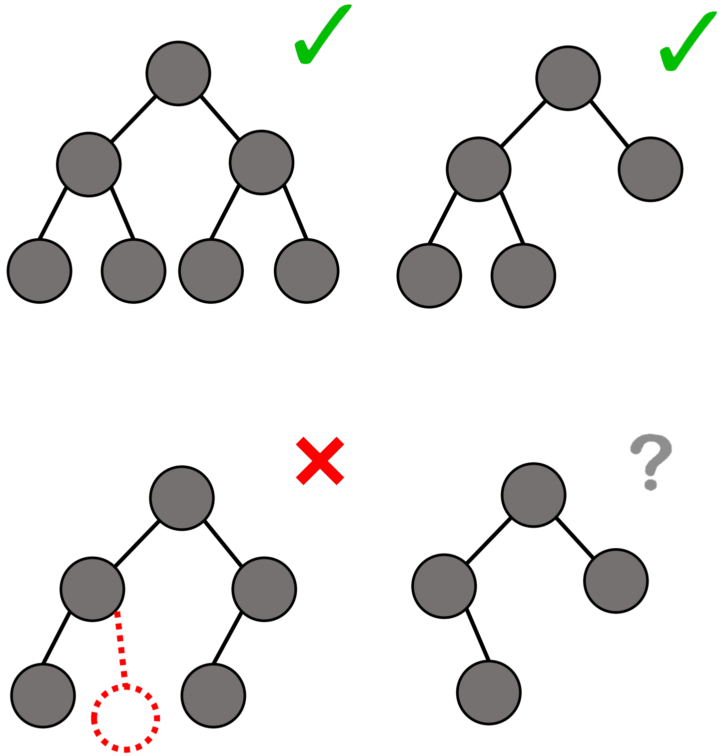
A data structure that can support dynamically organizing the items efficiently:

1. Inserting new items
2. Finding the most important one
3. Deleting the most important one and reorganizing the structure

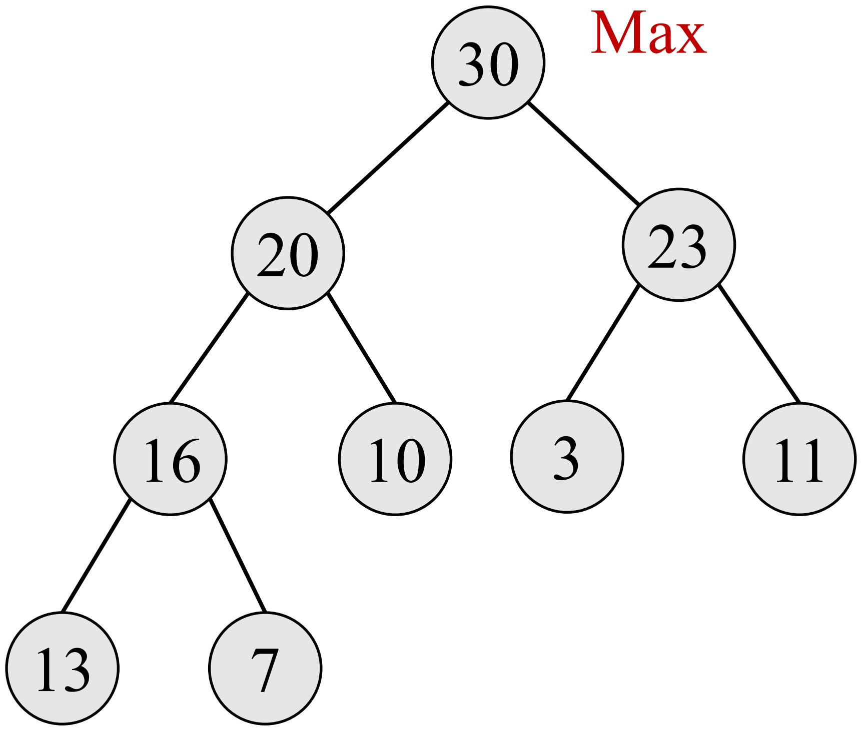
Property of Heaps

- A binary heap: A complete binary tree that satisfies heap ordering property.
  - Complete Binary Tree
    - All levels except the last level are full
    - Nodes in last level are placed left to right
  - Heap ordering property: Suppose a node a has child node b
    - max-heap property: $val(a) \geq val(b)$
    - min-heap property: $val(a) \leq val(b)$

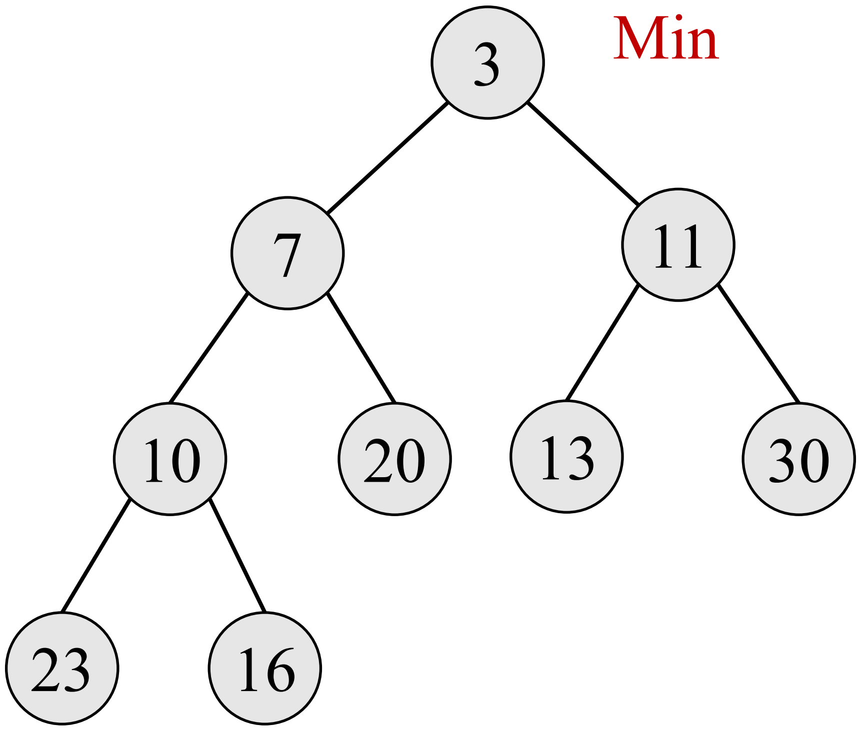




Example: [23, 13, 11, 20, 10, 3, 30, 16, 7] → Heaps



Max Heap



Min Heap

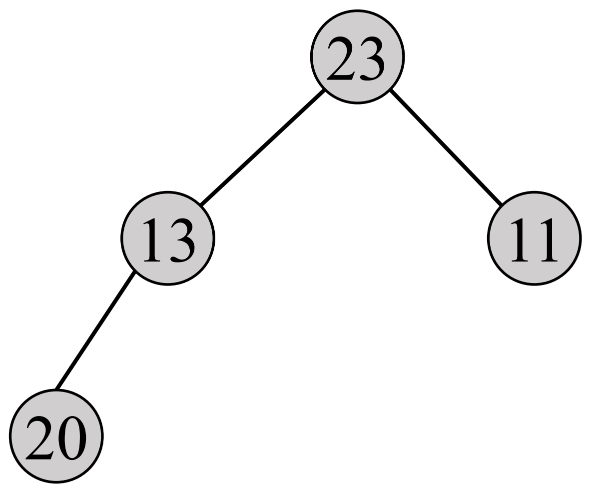
Heap Operations

- FindMax() / FindMin()
- Insert()
- DeleteMax() / DeleteMin()

Insertion

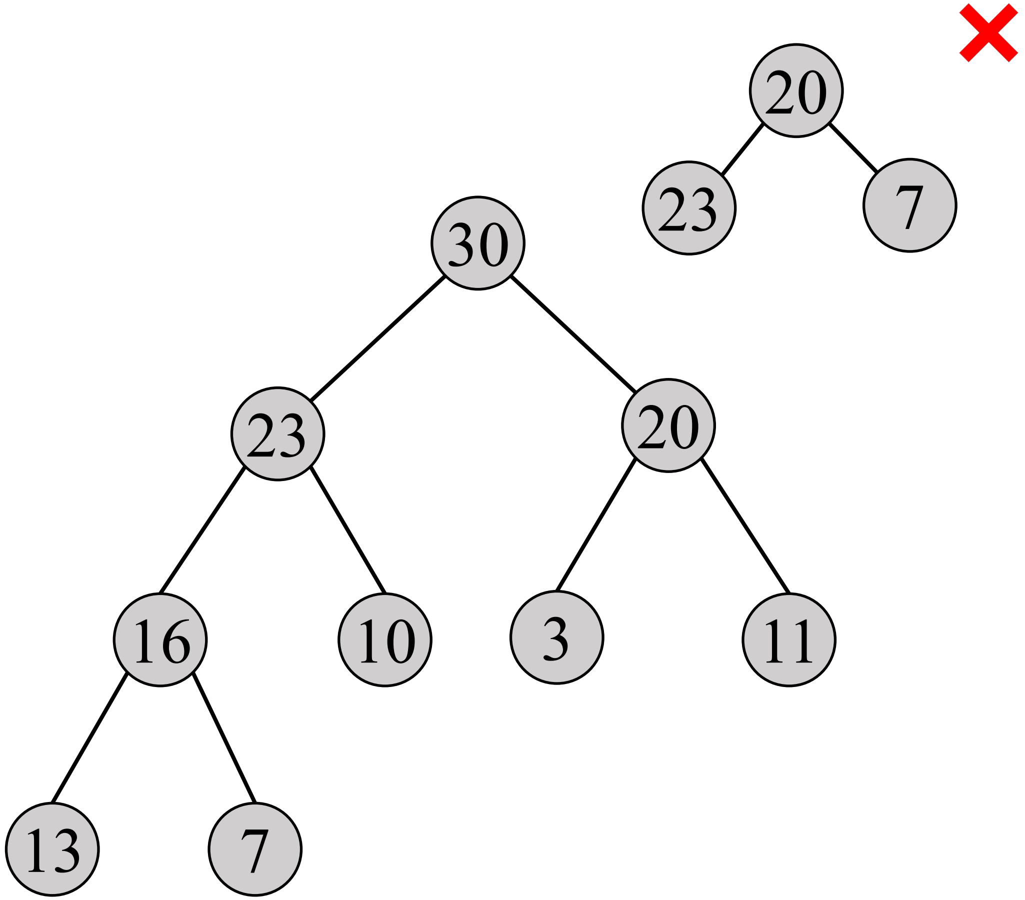
- Step 1 – Insert a new node N at the end of the tree.
- Step 2 – Compare the value of the node n with its parent.
- Step 3 – If the parent is smaller than node N, swap them.
- Step 4 – Repeat step 2 & 3 until heap ordering property holds.

Input → $\boxed{23,}$ 13, 11, 20, 10, 3, 30, 16, 7



Deletion

- Step 1 – Delete the root node and move the last node N to the root.
- Step 2 – Compare the value of node N with its children.
- Step 3 – If node N has smaller value than its children, swap N with the larger child.
- Step 4 – Repeat step 2 & 3 until heap ordering property holds.

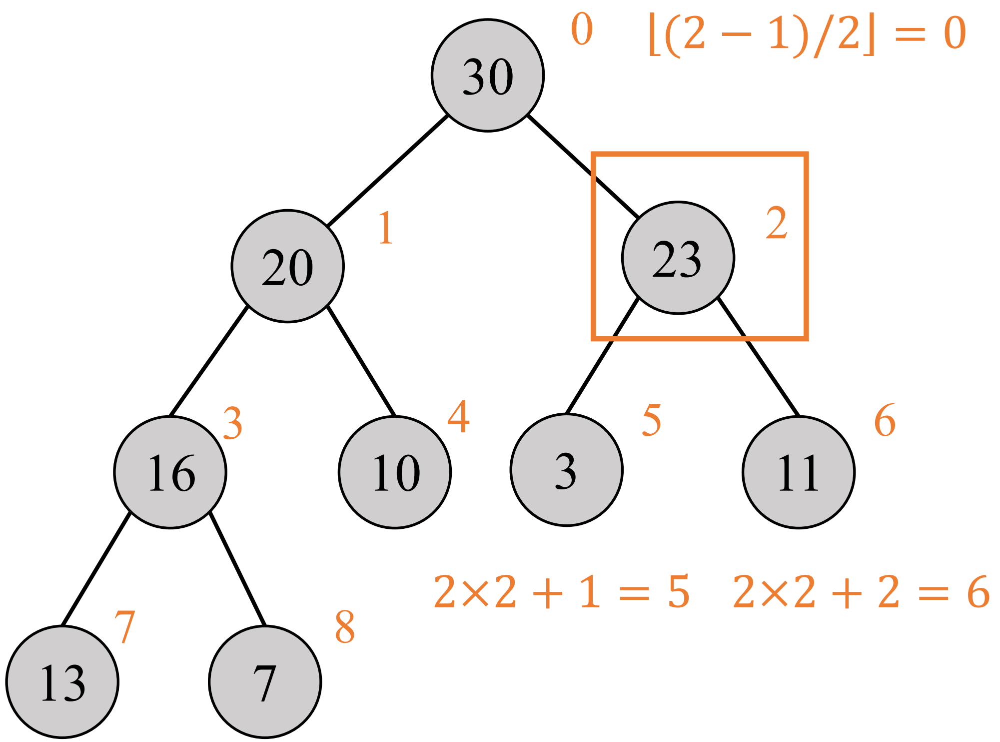


Array Implementation

- Array Implementation for heap
  - Compact representation
  - Easy to swap
- Find parent and children quickly?

position	1	2	3	4	5	6	7	8	9
keys	30	20	23	16	10	3	11	13	7
index	0	1	2	3	4	5	6	7	8

- Index: for the k-th element in the array
  - Left child → $2k$+1
  - Right child → $2k$+2
  - Parent → $\lfloor(k\text{-}1)/2\rfloor$



Implementation - Insertion

- Heap operation on arrays?

Algorithm 1 Insert an element to a heap

function Insert(array $a[0..n-1]$, key x)
  $a \leftarrow append(a[0,..n-1],x)$
  $k \leftarrow n$
  while $k \geq 0$ **do**
    if $a[k] > a[\lfloor (k-1)/2 \rfloor]$ **then**
      $swap(a,k,\lfloor (k-1)/2 \rfloor)$
      $k \leftarrow \lfloor (k-1)/2 \rfloor$
    else
      return

Parent position

Index: for the k-th element in the array
- Left child → $2k$+1
- Right child → $2k+2$
- Parent → $\lfloor (k-1)/2 \rfloor$

Complexity Analysis

- Worst-case analysis
  - FindMax() / FindMin() $\Theta(1)$
  - Insert() $\Theta(h)$
  - DeleteMax() / DeleteMin() $\Theta(h)$

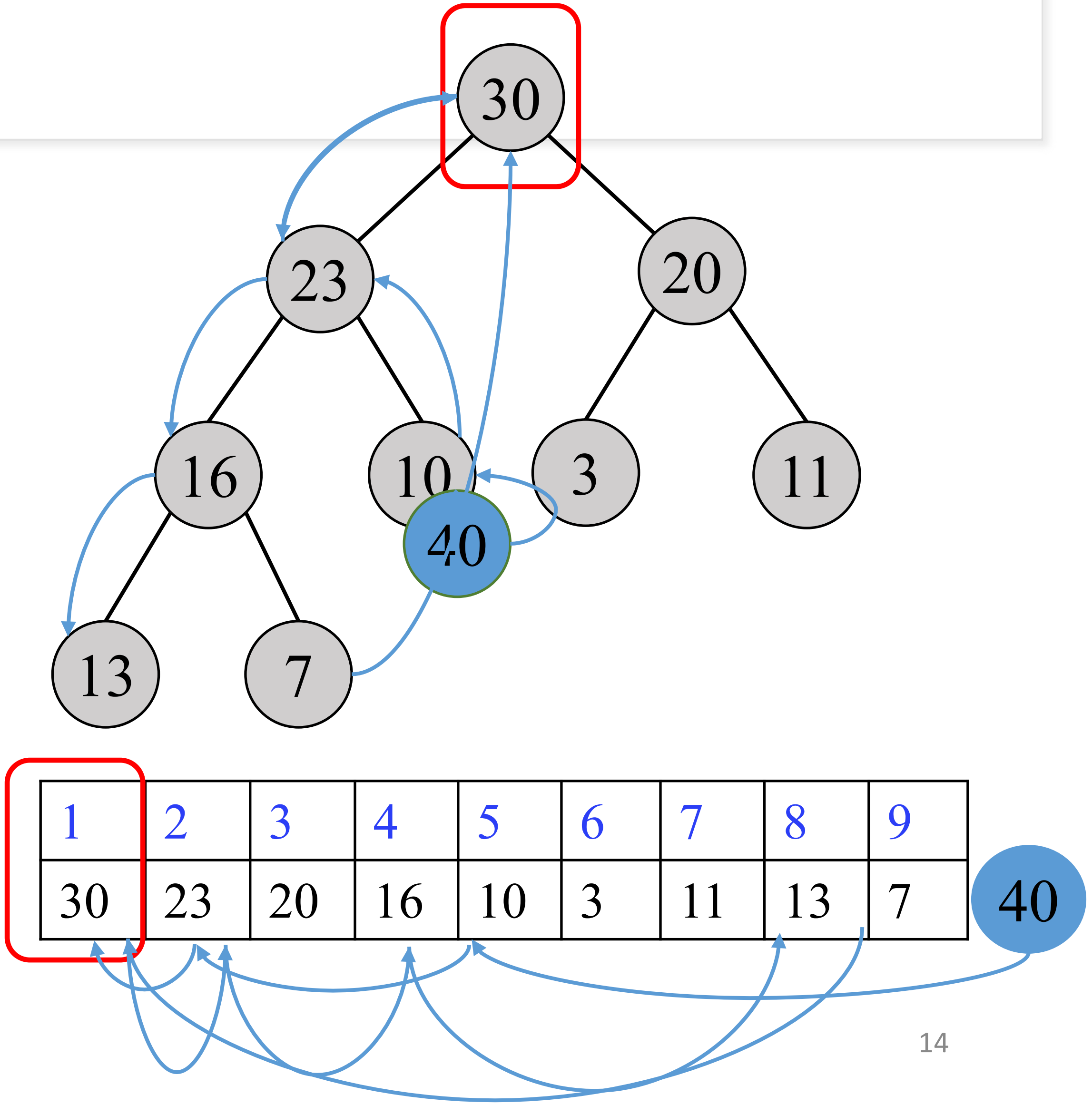
- Let h be the height of the tree. The number of nodes at level $l < h$ is 2^l, then the total number of nodes up to level l is

$$2^0 + 2^1 + \cdots + 2^l = 2^{l+1} - 1$$

- The number of nodes up to level h−1 is then $2^h - 1$, therefore, the number of nodes in a heap of height h always satisfies

$$2^h - 1 + 1 \le n \le 2^{h+1} - 1$$

$$h = \lfloor \log_2 n \rfloor$$



SUMMARY

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