Spring 2015 Statistics 153 (Time Series) : Lecture Five

Aditya Guntuboyina

03 February 2015

1 Last Class

Recall definition of (weak) stationarity. A doubly infinite sequence $\ldots, X_{-2}, X_{-1}, X_0, X_1, X_2, \ldots$ is said to be stationary if

1. $\mathbb{E}X_t$ does not depend on t.
2. The covariance between X_t and X_{t+h} does not depend on t (it depends only on h).

The function $\gamma_X(h) = cov(X_t, X_{t+h})$ is called the autocovariance function of $\{X_t\}$. Check that $\gamma_X(-h) = \gamma_X(h)$.

The function $\rho_X(h) = \gamma_X(h)/\gamma_X(0)$ is called the autocorrelation function of $\{X_t\}$ (this is different from the sample autocorrelation function that we looked at earlier).

2 Examples of Stationary Processes

2.1 White Noise

The random variables $\ldots, X_{-2}, X_{-1}, X_0, X_1, X_2, \ldots$ are said to be *white noise* if they have mean zero and the following covariance:

$$\begin{aligned} \text{cov}(X_{t_1}, X_{t_2}) &= \sigma^2 \qquad \text{if } t_1 = t_2 \\ &= 0 \qquad \text{if } t_1 \neq t_2. \end{aligned} \tag{1}$$

In other words, a the random variables in a white noise series are uncorrelated, have mean zero and a constant variance.

This is clearly a stationary series. What is its acf?

The white noise series is only a very special example of stationarity. Stationarity allows for considerable dependence between successive random variables in the series. The only requirement is that the dependence should be constant over time.

2.2 Moving Average Process of Order 1

Given a white noise series Z_t with variance σ^2 and a number θ, set

$$X_t = Z_t + \theta Z_{t-1}.$$

This is called a *moving average* of order 1. The series is stationary with mean zero and acvf:

$$\begin{aligned} \gamma_X(h) &= \sigma^2(1+\theta^2) \quad \text{if } h=0 \\ &= \theta\sigma^2 \quad \text{if } h=1 \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{2}$$

As a consequence, X_{t_1} and X_{t_2} are uncorrelated whenever t_1 and t_2 are two or more time points apart. This time series has *short memory*.

The autocorrelation function, acf, for $\{X_t\}$ is given by

$$\rho_X(h) = \frac{\theta}{1+\theta^2}$$

for $h = 1$ and 0 for $h > 1$. What is the maximum value that $\rho_X(1)$ can take?

An important motivation for considering the MA(1) process comes from **differencing**. Suppose that we are dealing with time series data for which that model $Y_t = at + b + Z_t$ is appropriate where $\{Z_t\}$ is white noise. The first order differenced series, $X_t = \nabla Y_t$ satisfies $X_t - a = Z_t - Z_{t-1}$. In other words, $X_t - a$ is a moving average process of order 1.

2.3 A Sinusoidal Process

Fix $\lambda \in \mathbb{R}$ and consider two random variables A and B that are uncorrelated, have mean zero and have the same variance σ^2. Consider the process

$$X_t := A\cos(2\pi\lambda t) + B\sin(2\pi\lambda t) \qquad \text{for } t = \dots, -3, -2, -1, 0, 1, 2, \dots$$

This is a stationary process because $\mathbb{E}X_t = 0$ for all t and

$$\text{Cov}(X_t, X_{t+h}) = \sigma^2\left(\cos(2\pi\lambda t)\cos(2\pi\lambda(t+h)) + \sin(2\pi\lambda t)\sin(2\pi\lambda(t+h))\right) = \sigma^2\cos(2\pi\lambda h).$$

The autocovariance function of $\{X_t\}$ is $\gamma_X(h) = \sigma^2\cos(2\pi\lambda h)$ and its autocorrelation function is $\rho_X(h) = \cos(2\pi\lambda h)$. How do realizations from this process look like?