Spring 2015 Statistics 153 (Time Series) : Lecture Six

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05 February 2015

1 Moving Average Process of Order q

More generally, one can consider a Moving Average of order q defined by

$$X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \cdots + \theta_q Z_{t-q} \tag{1}$$

for some numbers $\theta_1, \theta_2, \ldots, \theta_q$. This can be concisely written as $X_t = \sum_{j=0}^{q} \theta_j Z_{t-j}$ where we take $\theta_0 = 1$.

The mean of X_t is clearly 0. For $h \geq 0$, the covariance between X_t and X_{t+h} is given by

$$\text{cov}(X_t, X_{t+h}) = \text{cov}(\sum_{j=0}^{q} \theta_j Z_{t-j}, \sum_{k=0}^{q} \theta_k Z_{t+h-k}) = \sum_{j=0}^{q} \sum_{k=0}^{q} \theta_j \theta_k \text{cov}(Z_{t-j}, Z_{t+h-k}).$$

Because $\{Z_t\}$ is white noise, the covariance between Z_{t-j} and Z_{t+h-k} is non-zero and equal to σ^2 if and only if $t - j = t + h - k$ i.e., if and only if $k = j + h$. But because k has to lie between 0 and q, we must have that j has to lie between 0 and $q - h$. We thus get:

$$\begin{aligned} \gamma_X(h) &= \sigma^2 \sum_{j=0}^{q-h} \theta_j \theta_{j+h} \qquad \text{if } h = 0, 1, \ldots, q \\ &= 0 \qquad \text{if } h > q. \end{aligned} \tag{2}$$

Because $\text{cov}(X_t, X_{t+h})$ does not depend on t, we deduce that $\{X_t\}$ is weakly stationary. The autocovariance and the autocorrelation functions *cut off* after lag q.

2 Backshift Notation

A convenient piece of notation avoids the trouble of writing huge expressions in the sequel. Let B denote the *backshift operator* defined by

$$BX_t = X_{t-1}, B^2 X_t = X_{t-2}, B^3 X_t = X_{t-3}, \ldots$$

and similarly

$$BZ_t = Z_{t-1}, B^2 Z_t = Z_{t-2}, B^3 Z_t = Z_{t-3}, \ldots.$$

Also let I denote the identity operator: $IX_t = X_t$. More generally, we can define polynomial functions of the Backshift operator by, for example,

$$(I + B + 3B^2)X_t = IX_t + BX_t + 3B^2 X_t = X_t + X_{t-1} + 3X_{t-2}.$$

In general, for every polynomial $f(z)$, we can define $f(B)$. One can even extend this notation to negative powers of B which correspond to forward shifts. For example, $B^{-1}X_t = X_{t+1}, B^{-5}X_t = X_{t+5}$ and $(B^3 + 9B^{-2})X_t = X_{t-3} + 9X_{t+2}$ etc.

In this notation, the defining equation $X_t = Z_t + \theta Z_{t-1}$ for the MA(1) process can be written as $X_t = \theta(B)Z_t$ for the polynomial $\theta(z) = 1 + \theta_1 z$.

The defining equation $X_t = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$ for the MA(q) becomes $X_t = \theta(B)Z_t$ for the polynomial $\theta(z) = 1 + \theta_1 z + \ldots \theta_q z^q$.

3 Infinite Order Moving Average

We can extend the definition of moving average processes to even infinite order by taking:

$$X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \cdots + \theta_q Z_{t-q} + \theta_{q+1} Z_{t-q-1} + \ldots. \tag{3}$$

Here, once again, $\{Z_t\}$ represents white noise with mean zero and variance σ_Z^2. In the backshift notation, $X_t = \theta(B)Z_t$ where now $\theta(z)$ is the **power series**:

$$\theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \cdots = \sum_{j=0}^{\infty} \theta_j z^j.$$

The right hand side in (3) is an infinite sum and hence we need to address convergence issues. One particular choice of the weights $\{\theta_i\}$ where one does not encounter problems with convergence is:

$$\theta_i = \phi^i \qquad \text{for some } \phi \text{ with } |\phi| < 1.$$

The resulting process can be written succintly as $X_t = \sum_{i=0}^{\infty} \phi^i Z_{t-i}$. It can be shown, using the fact that $|\phi| < 1$, that the sequence $\sum_{i=0}^{n} \phi^i Z_{t-i}$ converges as $n \to \infty$ to a random variable with finite variance almost surely (and in the mean square or L^2 sense). X_t is defined to be that limit. The fact that the convergence is also in the mean square sense allows us to interchange expectations and sums while calculating means and covariances.

For this choice of weights, it is easy to check that $\{X_t\}$ a stationary process. Its autocovariance function is given by

$$\gamma(h) := \frac{\phi^h \sigma^2}{1 - \phi^2} \qquad \text{for } h \geq 0$$

The autocorrelation function is given by $\rho(h) = \phi^h$ for $h \geq 0$. Note that unlike the case of the Moving Average Model of Order 1, this acf is strictly positive for all lags. But, since $\rho(h)$ drops exponentially as lag increases, this is also taken to be an example of a stationary time series with short range dependence. Note that if ϕ is negative, the acf $\rho(h)$ osillates as h increases.

Here is an important property of this process X_t:

$$\begin{aligned} X_t = Z_t + \phi Z_{t-1} + \phi^2 Z_{t-2} + \ldots &= Z_t + \phi \left(Z_{t-1} + \phi Z_{t-2} + \phi^2 Z_{t-3} + \ldots \right) \\ &= Z_t + \phi X_{t-1} \qquad \text{for every } t = \ldots, -1, 0, 1, \ldots. \end{aligned}$$

Thus X_t satisfies the following first order *difference equation*:

$$X_t = \phi X_{t-1} + Z_t. \tag{4}$$

For this reason, $X_t = \sum_{i=0}^{\infty} \phi^i Z_{t-i}$ is called the *Stationary Autoregressive Process of order one*. Note here that $|\phi| < 1$. In the Backshift notation, the difference equation (4) can be written as $\phi(B)X_t = Z_t$ where $\phi(z) = 1 - \phi z$.

This raises an interesting question. Is $X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}$ the only solution to the difference equation (4) for $|\phi| < 1$?

The answer is no. Indeed, define X_0 to be an arbitrary random variable that is uncorrelated with the white noise series $\{Z_t\}$ and define $X_1, X_2, \dots$ as well as $X_{-1}, X_{-2}, \dots$ using the difference equation (4). The resulting sequence surely satisfies (4). Is it stationary? NO! Because $X_{-1} = X_0/\phi - Z_0/\phi$ and since $|\phi| < 1$ and X_0 and Z_0 are uncorrelated, this would give $\mathrm{var}(X_{-1}) > \mathrm{var}(X_0)$ contradicting stationarity.

Even though the difference equation (4) for $|\phi| < 1$ has many solutions, we shall show below that it has **only one stationary solution** which is given by $X_t = \sum_{i=0}^{\infty} \phi^i Z_{t-i}$. To see this, suppose $\{Y_t\}$ is any other stationary sequence which also satisfies (4) i.e., $Y_t = \phi Y_{t-1} + Z_t$. In that case, by successively using this equation, we obtain

$$
\begin{aligned}
Y_t &= Z_t + \phi Y_{t-1} \\
&= Z_t + \phi Z_{t-1} + \phi^2 Y_{t-2} \\
&= Z_t + \phi Z_{t-1} + \phi^2 Z_{t-2} + \phi^3 Y_{t-3} \\
&= Z_t + \phi Z_{t-1} + \phi^2 Z_{t-2} + \phi^3 Z_{t-3} + \phi^4 Y_{t-4}.
\end{aligned}
$$

In general, one would have $Y_t = \sum_{i=0}^{k} \phi^i Z_{t-i} + \phi^{k+1} Y_{t-k-1}$ for *every* k. The idea is now to let k approach ∞. The first term on the right hand side is $\sum_{i=0}^{k} \phi^i Z_{t-i}$ which we have argued (while defining X_t) converges to $X_t = \sum_{i=0}^{\infty} \phi^i Z_{t-i}$ as k goes to infinity. We need the hypothesis that $\{Y_t\}$ is stationary to deal with the second term. Indeed, because of stationarity, one has

$$
\mathbb{E}\left(\phi^{k+1} Y_{t-k-1}\right)^2 = \phi^{2k+2} \mathbb{E} Y_{t-k-1}^2 = \phi^{2k+2} \mathbb{E} Y_0^2 \to 0 \qquad \text{as } k \to \infty.
$$

It follows therefore that $Y_t = X_t$ almost surely. In other words, only one stationary solution to (4) exists and that equals $\sum_{j=0}^{\infty} \phi^j Z_{t-j}$.

4 Autoregressive Processes of Order One

The autoregressive process $\{X_t\}$ of order one, denoted by AR(1), is defined as a **stationary** process that satisfies the difference equation

$$
X_t - \phi X_{t-1} = Z_t \tag{5}
$$

where $\{Z_t\}$ is white noise.

In the previous section, we have seen that when $|\phi| < 1$, a stationary solution to (5) exists and is given explicitly by the infinite order MA process $X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}$.

What about the case $|\phi| > 1$? Here, $\sum_{j=0}^{\infty} \phi^j Z_{t-j}$ does not obviously make sense. But it can be checked that

$$
X_t = -\frac{Z_{t+1}}{\phi} - \frac{Z_{t+2}}{\phi^2} - \frac{Z_{t+3}}{\phi^3} - \dots \tag{6}
$$

is the unique stationary solution to the difference equation (5) for $|\phi| > 1$. This can be proved in the same way as the $|\phi| < 1$ case. The strange part about (6) is that X_t depends on only future white noise values: $Z_{t+1}, Z_{t+2}, \dots$. As a result, autoregressive processes of order 1 for $|\phi| > 1$ are never used in time series modelling.

What about the case $|\phi| = 1$? Here the difference equation becomes $X_t - X_{t-1} = Z_t$ for $\phi = 1$ and $X_t + X_{t-1} = Z_t$ for $\phi = -1$. These difference equations have **no** stationary solutions. Let us see this for $\phi = 1$ (the $\phi = -1$ case is similar). Note that $X_t - X_0 = Z_1 + \dots + Z_t$ which implies that the variance of $X_t - X_0$ equals $t\sigma^2$ and hence grows with t. This cannot happen if $\{X_t\}$ were stationary.

Here is the AR(1) summary:

1. If $|\phi| < 1$, the difference equation (5) has a unique stationary solution given by $X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}$. The solution clearly only depends on the present and past values of $\{Z_t\}$. It is hence called **causal**.

2. If $|\phi| > 1$, the difference equation (5) has a unique stationary solution given by $X_t = -\sum_{j=1}^{\infty} \phi^{-j} Z_{t+j}$. This is **non-causal**.

3. If $|\phi| = 1$, no stationary solution exists.

Book readings for this lecture: Examples 3.1, 3.2 and 3.3; Definitions 3.3 and 3.4; and the stuff immediately after Definition 3.2.