Spring 2015 Statistics 153 (Time Series) : Lecture Twenty One

Aditya Guntuboyina

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1 DFT Recap

The fact that every vector $(x_0, \dots, x_{n-1})$ can be written as a linear combination of $u^0, u^1, \dots, u^n$ motivates the definition of the Discrete Fourier Transform (DFT).

Given a data vector $(x_0, \dots, x_{n-1})$ of length n, its DFT is given by $b_j, j = 0, 1, \dots, n-1$, where

$$b_j = \sum_{t=0}^{n-1} x_t \exp\left(-\frac{2\pi ijt}{n}\right) \qquad \text{for } j = 0, \dots, n-1.$$

The original data $x_0, \dots, x_{n-1}$ can be recovered from the DFT using:

$$x_t = \frac{1}{n}\sum_{j=0}^{n-1} b_j \exp\left(\frac{2\pi ijt}{n}\right) \qquad \text{for } t = 0, \dots, n-1.$$

The motivation for the definition of the DFT comes from the fact that every vector $(x_0, \dots, x_{n-1})$ can be written as a linear combination of $u^0, \dots, u^{n-1}$ where

$$u^j = (1, \exp(2\pi ij/n), \exp(2\pi i2j/n), \dots, \exp(2\pi i(n-1)j/n)).$$

In other words, u^j is the sinusoid with frequency j/n evaluated at the time points $t = 0, 1, \dots, (n-1)$. For every $x_0, \dots, x_{n-1}$, we have

$$x = \frac{1}{n}\sum_{j=0}^{n-1} b_j u^j. \tag{1}$$

Two basic facts about the DFT are $b_0 = x_0 + \cdots + x_{n-1}$ and $b_{n-j} = \bar{b}_j$ for all $1 \le j \le n-1$. In R, the DFT is calculated by the function `fft()`.

2 Plotting the DFT

Just like we plot our data set $x_0, \dots, x_{n-1}$ for visualization, we would also like to plot the DFT $b_0, \dots, b_{n-1}$. Because b_0 is always just the sum of the data, it is uninteresting and we usually don't want to plot it. Further because $b_{n-j} = \bar{b}_j$, it is enough to look at $b_j, 1 \le j \le n/2$. However the problem is that b_j can in general be complex. We shall therefore plot

$$I(j/n) := \frac{1}{n}|b_j|^2$$

against $j = 1, \dots, n/2$ or against j/n for $1 \le j \le n/2$. This quantity $\{I(j/n) : 1 \le j \le n/2\}$ is known as the periodogram. We define $I(j/n)$ for all $j = 0, \dots, n-1$ but we shall only plot it for $j = 1, \dots, n/2$.

The periodogram values $I(j/n)$ satisfy the following identity:

$$\sum_{t=0}^{n-1} x_t^2 = \sum_{j=0}^{n-1} I(j/n).$$

This is called the sum of squares identity and is a consequence of (1) and the fact that $u^0, \ldots, u^{n-1}$ are orthogonal.

Because $b_{n-j} = \bar{b}_j$, their absolute values are equal and we can re-write the above sum of squares identity in the following way. For $n = 11$,

$$\sum_t x_t^2 = I(0) + 2I(1/n) + 2I(2/n) + 2I(3/n) + 2I(4/n) + 2I(5/n)$$

and for $n = 12$,

$$\sum_t x_t^2 = I(0) + 2I(1/n) + 2I(2/n) + 2I(3/n) + 2I(4/n) + 2I(5/n) + I(6/n).$$

Note that there is no need to put an absolute value on b_6 because it is real.

Because $b_0 = \sum_t x_t = n\bar{x}$, we have $I(0) = n\bar{x}^2$ and hence

$$\sum_t x_t^2 - I(0) = \sum_t x_t^2 - n\bar{x}^2 = \sum_t (x_t - \bar{x})^2.$$

Thus the sum of squares identity can be written for n odd, say $n = 11$, as

$$\sum_t (x_t - \bar{x})^2 = 2I(1/n) + 2I(2/n) + 2I(3/n) + 2I(4/n) + 2I(5/n)$$

and, for n even, say $n = 12$, as

$$\sum_t (x_t - \bar{x})^2 = 2I(1/n) + 2I(2/n) + 2I(3/n) + 2I(4/n) + 2I(5/n) + I(6/n).$$

3 DFT for Sinusoids

To understand the DFT, let us calculate the DFT of the cosine wave $x_t = R\cos(2\pi f_0 t + \phi), t = 0, 1, \ldots, n-1$.

3.1 We can always take $0 \leq f_0 \leq 1/2$

We can, without loss of generality, assume that $0 \leq f_0 \leq 1/2$ because:

1. If $f_0 < 0$, then we can write $\cos(2\pi f_0 t + \phi) = \cos(2\pi(-f_0)t - \phi)$. Clearly, $-f_0 \geq 0$.
2. If $f_0 \geq 1$, then we write
$$\cos(2\pi f_0 t + \phi) = \cos(2\pi[f_0]t + 2\pi(f - [f_0])t + \phi) = \cos(2\pi(f - [f_0])t + \phi),$$
because $\cos(\cdot)$ is periodic with period 2π. Clearly $0 \leq f - [f_0] < 1$.
3. If $f_0 \in [1/2, 1)$, then
$$\cos(2\pi f_0 t + \phi) = \cos(2\pi t - 2\pi(1 - f_0)t + \phi) = \cos(2\pi(1 - f_0)t - \phi)$$
because $\cos(2\pi t - x) = \cos x$ for all integers t. Clearly $0 < 1 - f_0 \leq 1/2$.

Thus given a cosine wave $R\cos(2\pi ft+\phi)$, one can always write it as $R\cos(2\pi f_0 t+\phi')$ with $0 \leq f_0 \leq 1/2$ and a phase ϕ' that is possibly different from ϕ. This frequency f_0 is said to be an **alias** of f. From now on, whenever we speak of the cosine wave $R\cos(2\pi f_0 t + \phi)$, we assume that $0 \leq f_0 \leq 1/2$.

If $\phi = 0$, then we have $x_t = R\cos(2\pi f_0 t)$. When $f_0 = 0$, then $x_t = R$ and so there is no oscillation in the data at all. When $f_0 = 1/2$, then $x_t = R\cos(\pi t) = R(-1)^t$ and so $f_0 = 1/2$ corresponds to the maximum possible oscillation.

The DFT of the sinusoid will be calculated in the next class. The result will be clear from R simulations.

4 Interpreting the DFT

The DFT writes the given data in terms of sinusoids with frequencies of the form k/n. Frequencies of the form k/n are called Fourier frequencies.

Suppose that we are given a dataset $x_0, \dots, x_{n-1}$. We have calculated its DFT: $b_0, b_1, \dots, b_{n-1}$ and we have plotted $|b_j|$ for $j = 1, \dots, (n-1)/2$ for odd n and for $j = 1, \dots, n/2$ for even n.

If we see a single spike in this plot, say at b_k, we are sure that the data is a sinusoid with frequency k/n.

If we get two spikes, say at b_{k_1} and b_{k_2}, then the data is slightly more complicated: it is a linear combination of two sinusoids at frequencies k_1/n and k_2/n with the strengths of these sinusoids depending on the size of the spikes.

Multiple spikes indicate that the data is made up of many sinusoids at Fourier frequencies and, in general, this means that the data is more complicated.

However, sometimes one can see multiple spikes in the DFT even when the structure of the data is not very complicated. A typical example is leakage due to the presence of a sinusoid at a non-Fourier frequency.