# Spring 2015 Statistics 153 (Time Series) : Lecture Seven

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## 1 Moving Average Processes

Recall the Backshift Operator: $B^j X_t = X_{t-j}$. Here $j$ can be positive, negative or zero.

Moving Average Processes can be defined by $X_t = \theta(B) Z_t$ where

1. First Order: $\theta(z) = 1 + \theta z$. This is MA(1).
2. Order $q$: $\theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q$. This is MA($q$).
3. Infinite Order One Way: $\theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \cdots = \sum_{j=0}^{\infty} \theta_j z^j$. One needs a condition on the coefficients $\{\theta_j\}$ for $X_t = \theta(B) Z_t$ to make sense. A sufficient condition is $\sum_{j=0}^{\infty} |\theta_j| < \infty$.
4. Infinite Order Two Way: $\theta(z) = \sum_{j=-\infty}^{\infty} \theta_j z^j$. Once again, one needs a condition such as $\sum_{j=-\infty}^{\infty} |\theta_j| < \infty$ for $X_t = \theta(B) Z_t$ to make sense.

## 2 Autoregressive Processes of Order One

The autoregressive process $\{X_t\}$ of order one, denoted by AR(1), is defined as a **stationary** process that satisfies the difference equation

$$X_t - \phi X_{t-1} = Z_t \tag{1}$$

where $\{Z_t\}$ is white noise.

In the last class, we have seen that when $|\phi| < 1$, a stationary solution to (1) exists and is given explicitly by the infinite order MA process $X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}$. Here is another heuristic way of seeing this. The difference equation (1) can be rewritten as $\phi(B) X_t = Z_t$ where $\phi(B)$ is given by the polynomial $\phi(z) = 1 - \phi z$. Therefore, it is natural that the solution of this equation is

$$X_t = \frac{1}{\phi(B)} Z_t.$$

Now from the formula for the sum of a geometric series (note that $|\phi| < 1$), we have

$$\frac{1}{\phi(z)} = (1 - \phi z)^{-1} = 1 + \phi z + \phi^2 z^2 + \phi^3 z^3 + \dots$$

As a result,

$$X_t = \frac{1}{\phi(B)} Z_t = \left(I + \phi B + \phi^2 B^2 + \dots\right) Z_t = Z_t + \phi Z_{t-1} + \phi^2 Z_{t-2} + \cdots = \sum_{j=0}^{\infty} \phi^j Z_{t-j}.$$

What about the case $|\phi| > 1$? Here, $\sum_{j=0}^{\infty} \phi^j Z_{t-j}$ does not obviously make sense. But we can still write:

$$\frac{1}{\phi(z)} = \frac{1}{1-\phi z} = \frac{-1}{\phi z}\left(1 - \frac{1}{\phi z}\right)^{-1} = -\frac{1}{\phi z} - \frac{1}{\phi^2 z^2} - \frac{1}{\phi^3 z^3} - \dots = -\frac{z^{-1}}{\phi} - \frac{z^{-2}}{\phi^2} - \frac{z^{-3}}{\phi^3} - \dots$$

Therefore, it makes sense to conjecture

$$X_t = \left(-\frac{z^{-1}}{\phi} - \frac{z^{-2}}{\phi^2} - \frac{z^{-3}}{\phi^3} - \dots\right) Z_t = -\frac{Z_{t+1}}{\phi} - \frac{Z_{t+2}}{\phi^2} - \frac{Z_{t+3}}{\phi^3} - \dots \tag{2}$$

as the unique stationary solution to the difference equation (1) for $|\phi| > 1$. This is indeed true and can be proved in the same way as the $|\phi| < 1$ case. The strange part about (2) is that $X_t$ depends on only future white noise values: $Z_{t+1}, Z_{t+2}, \dots$. As a result, autoregressive processes of order 1 for $|\phi| > 1$ are never used in time series modelling.

What about the case $|\phi| = 1$? Here the difference equation becomes $X_t - X_{t-1} = Z_t$ for $\phi = 1$ and $X_t + X_{t-1} = Z_t$ for $\phi = -1$. These difference equations have **no** stationary solutions. Let us see this for $\phi = 1$ (the $\phi = -1$ case is similar). Note that $X_t - X_0 = Z_1 + \dots + Z_t$ which implies that the variance of $X_t - X_0$ equals $t\sigma^2$ and hence grows with $t$. This cannot happen if $\{X_t\}$ were stationary.

Here is the AR(1) summary:

1. If $|\phi| < 1$, the difference equation (1) has a unique stationary solution given by $X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}$. The solution clearly only depends on the present and past values of $\{Z_t\}$. It is hence called **causal**.
2. If $|\phi| > 1$, the difference equation (1) has a unique stationary solution given by $X_t = -\sum_{j=1}^{\infty} \phi^{-j} Z_{t+j}$. This is **non-causal**.
3. If $|\phi| = 1$, no stationary solution exists.

This summary can be reinterpreted in terms of the polynomial $\phi(z) = 1 - \phi z$. The root of this polynomial is $1/\phi$.

1. If the magnitude of the root of $\phi(z)$ is strictly larger than 1, then $\phi(B)X_t = Z_t$ has a unique causal stationary solution.
2. If the magnitude of the root of $\phi(z)$ is strictly smaller than 1, then $\phi(B)X_t = Z_t$ has a unique stationary solution which is non-causal.
3. If the magnitude of the root of $\phi(z)$ is exactly equal to one, then $\phi(B)X_t = Z_t$ has no stationary solution.

These conclusions hold for difference equations $\phi(B)X_t = Z_t$ for more general polynomials $\phi(z)$ as well. We will study these in the next class.

**Book Readings:** The initial part (up to page 98) of Chapter 3 of the book.