Spring 2015 Statistics 153 (Time Series) : Lecture Eight

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1 ARMA Processes

The combination of ideas behind the Autoregressive and Moving Average processes give us ARMA processes, which provide a large class of stationary models for modelling residuals after trend and seasonality are removed.

Definition 1.1 (ARMA(p, q))**.** *The doubly infinite sequence $\{X_t\}$ is said to be an ARMA(p, q) process if*

1. *$\{X_t\}$ is stationary.*
2. *$\{X_t\}$ satisfies the following difference equation*

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q} \tag{1}$$

   for every t.

Here $\{Z_t\}$ is white noise.

If we let $\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p$ and $\theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q$, the difference equation (1) becomes $\phi(B)X_t = \theta(B)Z_t$. **Whenever we talk of an ARMA process, we assume that the polynomials $\theta(z)$ and $\phi(z)$ have no common roots**.

This class of models includes the examples of stationary models that we have so far studied:

1. **White noise**: Corresponds to $\phi(z) = 1$ and $\theta(z) = 1$.
2. **Moving Average**: Corresponds to $\phi(z) = 1$ and $\theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q$.
3. **Autoregressive Process**: Corresponds to $\theta(z) = 1$. $\phi(z) = 1 - \phi z$ is autoregressive process of order one. $\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p$ is called an autoregressive process of order p.

Definition 1.2 (Causal ARMA process)**.** *An ARMA(p, q) process X_t is said to be causal if it can be written as $X_t = Z_t + \psi_1 Z_{t-1} + \psi_2 Z_{t-2} + \cdots = \sum_{j==0}^{\infty} \psi_j Z_{t-j}$ for some sequence ψ_j with $\sum_{j=0}^{\infty} |\psi_j| < \infty$.*

We ask the same questions as in the AR(1) case:

1. For what values of $\phi_1, \dots, \phi_p$ and $\theta_1, \dots, \theta_q$ does there exist a stationary solution to the general difference equation (1).
2. For what values of $\phi, \dots, \phi_p$ and $\theta_1, \dots, \theta_q$ does there exist a **causal** stationary solution to (1).

3. How does one calculate the acvf and acf of this process?

The answer to Question 1 is: **The difference equation $\phi(B)X_t = \theta(B)Z_t$ has a unique stationary solution if and only if the polynomial $\phi(z)$ has no roots having magnitude exactly equal to one**. It should be kept in mind that polynomials in general can have complex roots.

The answer to Question 2 is: **The difference equation has a unique stationary solution that is causal if and only if $\phi(z)$ has no roots having magnitude less than or equal to one**.

In the case of the AR(1) model, the polynomial ϕ equals $\phi(z) = 1 - \phi z$ whose root is given by $1/\phi$ and its magnitude is $1/|\phi|$. In this case, the above answers give the same conclusions as in the last section.

Heuristic explanation of the answers:

1. **Stationarity**: The stationary solution to $\phi(B)X_t = \theta(B)Z_t$ is $X_t = \theta(B)/\phi(B)Z_t$. Let $\psi(z) = \theta(z)/\phi(z)$ so that the solution becomes $X_t = \psi(B)Z_t$. We have seen that we can make sense of $\psi(B)Z_t$ provided $\psi(z)$ is of the form $\psi(z) = \sum_{j=-\infty}^{\infty} \psi_j z^j$ with $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$. So a sufficient condition for the existence of a stationary solution to $\phi(B)X_t = \theta(B)Z_t$ is that the ratio $\psi(z) = \theta(z)/\phi(z)$ can be expanded into a series $\sum_{j=-\infty}^{\infty} \psi_j z^j$ for which $\sum_j |\psi_j| < \infty$. The condition $\sum_j |\psi_j| < \infty$ is equivalent to saying that the series $\sum_{j=-\infty}^{\infty} \psi_j z^j$ is absolutely convergent at z with magnitude $= 1$. This is only possible if $\psi(z)$ does not equal zero at every z with magnitude 1. In other words, all roots of $\phi(z)$ have magnitude not equal to 1.

2. **Causality**: For causality we need $\psi(z) = \theta(z)/\phi(z)$ to be of the form $\sum_{j=0}^{\infty} \psi_j z^j$ (note that the sum is only over $j \geq 0$) where $\sum_{j=0}^{\infty} |\psi_j| < \infty$. In other words $\psi(z)$ is a power series satisfying $\sum_{j \geq 0} |\psi_j| < \infty$. As a consequence, $\psi(z)$ is well-defined for all z with $|z| \leq 1$. This can only happen if $\phi(z)$ is non-zero for all z with $|z| \leq 1$. In other words, all roots of $\phi(z)$ have magnitude strictly larger than one.

Suppose that a causal solution exists i.e., that all roots of $\phi(z)$ have magnitude strictly larger than one. How do we find the solution explicitly? The solution is given by $X_t = \psi(B)Z_t$ where $\psi(z) = \theta(z)/\phi(z)$. Let $\psi(z) = \psi_0 + \psi_1 z + \psi_2 z^2 + \dots$ (note that this is a power series because of causality). Because $\psi(z) = \theta(z)/\phi(z)$, we have

$$(1 - \phi_1 z - \cdots - \phi_p z^p)(\psi_0 + \psi_1 z + \dots) = 1 + \theta_1 z + \cdots + \theta_q z^q.$$

Equate the coefficients of z^j on both sides for $j = 0, 1, 2, \dots$ to get

$$1 = \psi_0, \quad \theta_1 = \psi_1 - \psi_0\phi_1, \quad \theta_2 = \psi_2 - \psi_1\phi_1 - \psi_0\phi_2, \quad \theta_3 = \psi_3 - \phi_1\psi_2 - \phi_2\psi_1 - \phi_3\psi_0, \quad \dots$$

Another way is to write $\phi(z) = (1 - a_1 z)(1 - a_2 z)\dots(1 - a_p z)$ where $1/a_1, \dots, 1/a_p$ are the (possibly complex) roots of the $a_1, \dots, a_p$ each satisfying $|a_i| < 1$ so that

$$\begin{aligned}
\psi(z) &= \frac{\theta(z)}{\phi(z)} \\
&= \frac{\theta(z)}{(1 - a_1 z)\dots(1 - a_p z)} \\
&= \theta(z)(1 - a_1 z)^{-1}\dots(1 - a_p z)^{-1} \\
&= \theta(z)(1 + a_1 z + a_1^2 z^2 + \dots)(1 + a_2 z + a_2^2 z^2 + \dots)\dots(1 + a_p z + a_p^2 z^2 + \dots).
\end{aligned}$$

The product above can, in principle, be multiplied out to get the expression $\psi_0 + \psi_1 z + \psi_2 z^2 + \dots$.

Both these techniques for determining $\psi_1, \psi_2, \dots$ can be very tedious in some cases.

Example 1.3. *Consider the following ARMA(1, 1) difference equation:*

$$X_t - 0.5X_{t-1} = Z_t + 0.4Z_{t-1}$$

where $\{Z_t\}$ is white noise. Does this have a unique stationary solution? Is it causal? Find the solution.

The autoregressive polynomial is $\phi(z) = 1 - 0.5z$. The moving average polynomial is $\theta(z) = 1 + 0.4z$. ϕ has only one root: $z = 2$. This root has magnitude $\neq 1$; hence there exists a unique stationary solution. Moreover, the root also has magnitude > 1; hence the unique stationary solution is causal. To find the solution, we need to find $\psi(z) = \theta(z)/\phi(z)$. In other words:

$$(1 - 0.5z)(\psi_0 + \psi_1 z + \psi_2 z^2 + \dots) = (1 + 0.4z).$$

Equate coefficients of z^j on both sides.

Book Readings: Pages 93, 94, 95, Example 3.10