Spring 2015 Statistics 153 (Time Series) : Lecture Nine

Aditya Guntuboyina

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1 Calculation of Autocovariance of ARMA Processes

From now on, we shall consider only causal, stationary ARMA processes: $\phi(B)X_t = \theta(B)Z_t$.

1.1 First method

A causal, stationary ARMA process can be explicitly written as

$$X_t = \psi(B)Z_t = \psi_0 Z_t + \psi_1 Z_{t-1} + \psi_2 Z_{t-2} + \dots \tag{1}$$

where $\psi(z) = \theta(z)/\phi(z)$. Note that ψ_0 will always equal one. This explicit representation can be used to calculate the acvf:

$$\gamma_X(h) = \mathrm{cov}(X_t, X_{t+h}) = \sigma_Z^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h} \qquad \text{for } h \geq 0, \tag{2}$$

where σ_Z^2 is the variance of the white noise process Z_t. The acf (autocorrelation function) can be calculated from this via $\rho_X(h) = \gamma_X(h)/\gamma_X(0)$.

This method requires the explicit evaluation of $\psi_0, \psi_1, \dots$ which can be quite cumbersome sometimes.

As an example consider the first order moving average process: $X_t = Z_t + \theta Z_{t-1}$. This is of the form (1) with $\psi_0 = 1, \psi_1 = \theta$ and $\psi_j = 0$ for $j \geq 2$. Thus (2) with these ψ_js gives

$$\gamma_X(0) = \sigma_Z^2(1+\theta^2) \qquad \gamma_X(1) = \sigma_Z^2\theta \qquad \gamma_X(h) = 0 \ \text{ for } h \geq 2.$$

The corresponding autocorrelations are

$$\rho_X(1) = \frac{\theta}{1+\theta^2} \qquad \rho_X(h) = 0 \ \text{ for } h \geq 2.$$

Thus for MA(1), autocorrelations drop after the first lag. The autocorrelation at lag one has the same sign as θ and its maximum and minimum values are 1/2 (for $\theta = 1$) and -1/2 (for $\theta = -1$) respectively.

For the second order MA process: $X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}$, we have $\psi_0 = 1, \psi_1 = \theta_1, \psi_2 = \theta_2$ and $\psi_j = 0$ for $j \geq 3$. The identity (2) with these ψ_js gives

$$\gamma_X(0) = \sigma_Z^2(1+\theta_1^2+\theta_2^2) \quad \gamma_X(1) = \sigma_Z^2\theta_1(1+\theta_2) \quad \gamma_X(2) = \sigma_Z^2\theta_2 \quad \gamma_X(h) = 0 \text{ for } h \geq 3.$$

The corresponding autocorrelations are given by

$$\rho_X(1) = \frac{\theta_1(1+\theta_2)}{1+\theta_1^2+\theta_2^2} \qquad \rho_X(2) = \frac{\theta_2}{1+\theta_1^2+\theta_2^2} \qquad \rho_X(h) = 0 \quad \text{for } h \geq 3.$$

Let us now look at AR(1) given by $X_t - \phi X_{t-1} = Z_t$. For this to be stationary and causal, we have seen that the condition $|\phi| < 1$ is necessary and sufficient. In this case, $X_t = Z_t + \phi Z_{t-1} + \phi^2 Z_{t-2} + \dots$. This is same as (1) if we take $\psi_j = \phi^j$ for $j = 0, 1, 2, \dots$. Thus formula (2) for the autocovariance gives:

$$\gamma_X(h) = \sigma_Z^2 \sum_{j=0}^{\infty} \phi^j \phi^{j+h} = \sigma_Z^2 \phi^h \sum_{j=0}^{\infty} \phi^{2j} = \sigma_Z^2 \frac{\phi^h}{1-\phi^2} \qquad \text{for } h > 0,$$

which results in the simple expression for the autocorrelation: $\rho_X(h) = \phi^h$ for $h > 0$.

1.2 Second Method

The second method works particularly well for AR processes (i.e., the moving average polynomial $\theta(z) = 1$) of order $p > 1$ where it might be hard to explicitly find the coefficients ψ_j in (2).

Consider a causal, stationary ARMA satisfying the difference equation:

$$X_t - \phi_1 X_{t-1} - \dots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \dots + \theta_q Z_{t-q}.$$

Now fix $k \geq 0$ and take the covariance of both sides of the above equation with X_{t-k}. On the left hand side, we would get:

$$\operatorname{cov}(\phi(B)X_t, X_{t-k}) = \gamma_X(k) - \phi_1\gamma_X(k-1) - \dots - \phi_p\gamma_X(k-p). \tag{3}$$

To evaluate the right hand side, we use the expression $X_t = \psi_0 Z_t + \psi_1 Z_{t-1} + \dots$ (valid only under causality) to get

$$\operatorname{cov}(\theta(B)Z_t, X_{t-k}) = \operatorname{cov}(Z_t + \theta_1 Z_{t-1} + \dots + \theta_q Z_{t-q}, \psi_0 Z_{t-k} + \psi_1 Z_{t-k-1} + \psi_2 Z_{t-k-2} + \dots)$$

If $k > q$, the above covariance is zero. And if $k \leq q$, then

$$\operatorname{cov}(\theta(B)Z_t, X_{t-k}) = (\psi_0\theta_k + \psi_1\theta_{k+1} + \dots + \psi_{q-k}\theta_q)\,\sigma_Z^2. \tag{4}$$

Equating (3) and (4), we get

$$\gamma_X(k) - \phi_1\gamma_X(k-1) - \dots - \phi_p\gamma_X(k-p) = (\psi_0\theta_k + \psi_1\theta_{k+1} + \dots + \psi_{q-k}\theta_q)\,\sigma_Z^2 \qquad \text{for } 0 \leq k \leq q \tag{5}$$

and

$$\gamma_X(k) - \phi_1\gamma_X(k-1) - \dots - \phi_p\gamma_X(k-p) = 0 \qquad \text{for } k > q. \tag{6}$$

The autocovariance function γ_X is obtained by solving these equations. The advantage to this method is that we only need to find out $\psi_0 = 1, \psi_1, \dots, \psi_{q-k}$ instead of the whole sequence $\{\psi_j\}$.

Example 1.1 (AR(1) Process). *For AR(1), $p = 1$ so that left hand side in* (5) *and* (6) *becomes $\gamma_X(k) - \gamma_X(k-1)$. Also $q = 0$ and thus* (5) *which only holds for $k \leq q = 0$ gives only one equation (recall $\gamma_X(-h) = \gamma_X(h)$):*

$$\gamma_X(0) - \phi\gamma_X(1) = \sigma_Z^2\psi_0\theta_0. \tag{7}$$

On the other hand, (6) *holds for all $k > q = 0$ which implies*

$$\gamma_X(k) - \phi\gamma_X(k-1) = 0 \qquad \textit{for all } k \geq 1. \tag{8}$$

This equation is known as the ***Yule-Walker equation for order 1****.*

Note that (8) *allows us to deduce $\gamma_X(1) = \phi\gamma_X(0)$, $\gamma_X(2) = \phi\gamma_X(1) = \phi^2\gamma_X(0)$ and, in general, $\gamma_X(k) = \phi^k\gamma_X(0)$ for $k \geq 1$. The equations* (7) *and* (8) *for $k = 1$ together imply $\gamma_X(k) = \sigma_Z^2\psi_0\theta_0/(1-\phi^2)$. Finally, observe that $\theta_0 = 1$ and $\psi_0 = 1$.*

Example 1.2 (ARMA(1,1)). *The difference equation is $X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}$. Just as in the AR(1) case, we need $|\phi| < 1$ to ensure stationarity and causality. We have $p = 1, \phi_0 = 1, \phi_1 = \phi$ and $q = 1, \theta_0 = 1, \theta_1 = \theta$.*

The equations (5) *and* (6) *give*

$$\gamma_X(0) - \phi\gamma_X(-1) = (1 + \psi_1\theta_1)\sigma_Z^2 \qquad \textit{for } k = 0,$$

$$\gamma_X(1) - \phi\gamma_X(0) = \theta\sigma_Z^2 \qquad \textit{for } k = 1,$$

and

$$\gamma_X(k) = \phi\gamma_X(k-1) \qquad \textit{for } k \geq 2.$$

The number ψ_1 is the coefficient of z in $(1 + \theta z)(1 + \phi z + \phi^2 z^2 + \dots)$ and equals $\theta + \phi$. Solving these equations, we get

$$\gamma_X(0) = \sigma_Z^2 \frac{1 + \theta^2 + 2\phi\theta}{1 - \phi^2} \quad \gamma_X(k) = \phi^{k-1}\sigma_Z^2 \frac{(\theta + \phi)(1 + \theta\phi)}{1 - \phi^2}.$$

This results in the autocorrelations:

$$\rho_X(k) = \frac{(\theta + \phi)(1 + \theta\phi)}{1 + \theta^2 + 2\phi\theta}\phi^{k-1} \qquad \textit{for } k \geq 1.$$

The autocorrelation at lag one is not equal to ϕ. After lag one, subsequent autocorrelations decay exponentially with factor ϕ. When $\theta = 0$, we get back the autocorrelations for AR(1).

Book Readings: Section 3.4 (until Example 3.11).