Spring 2015 Statistics 153 (Time Series) : Lecture Ten

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19 February 2015

1 Autocovariance and Autocorrelation Functions of ARMA Processes

Consider the ARMA(p, q) process: $\phi(B)X_t = \theta(B)Z_t$ with $\phi(z) = 1-\phi_1 z-\cdots-\phi_p z^p$ and $\theta(z) = 1+\theta_1 z+\cdots+\theta_q z^q$. Assume that it is causal and stationary i.e., all roots of $\phi(z)$ have magnitude strictly larger than one. There are two methods for calculating $\gamma_X(h) = \mathrm{cov}(X_t, X_{t+h})$ and $\rho_X(h) = \gamma_X(h)/\gamma_X(0)$:

1. Use the explicit representation: $X_t = \sum_{j\geq 0} \psi_j Z_{t-j}$ which gives $\gamma_X(h) = \sigma_Z^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h}$ for $h \geq 0$.
2. Solve the following set of equations

$$\gamma_X(k)-\phi_1\gamma_X(k-1)-\cdots-\phi_p\gamma_X(k-p) = (\psi_0\theta_k + \psi_1\theta_{k+1} + \cdots + \psi_{q-k}\theta_q)\,\sigma_Z^2 \qquad \text{for } 0 \leq k \leq q \quad (1)$$

and

$$\gamma_X(k) - \phi_1\gamma_X(k-1) - \cdots - \phi_p\gamma_X(k-p) = 0 \qquad \text{for } k > q. \quad (2)$$

In the last class, we have used the second technique to find the autocorrelation function for the ARMA(1, 1) process $X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}$:

$$\rho_X(k) = \frac{(\theta+\phi)(1+\theta\phi)}{1+\theta^2+2\phi\theta}\phi^{k-1} \qquad \text{for } k \geq 1.$$

The equation (2) is an example of a difference equation.

2 Difference Equations

2.1 First Order

Consider the equations:

$$u_k - \alpha u_{k-1} = 0 \qquad \text{for } k = 1, 2, \dots \quad (3)$$

and $u_0 = b_0$. Here $\alpha \neq 0$ is a constant.

This is a first order difference equation. Also known as the order one linear homogeneous recurrence relation with constant coefficients. This equation is very easy to solve and the solution is given by $u_k = \alpha^k u_0 = b_0\alpha^k$. One can also write the equation (3) using the Backshift operator as: $f(B)u_k = 0$ where $f(z) = 1 - \alpha z$. The root of this polynomial f is $z_0 = 1/\alpha$. So the solution can also be written as $u_k = b_0 z_0^{-k}$.

2.2 Second Order

The defining equations are:

$$u_k - \alpha_1 u_{k-1} - \alpha_2 u_{k-2} = 0 \qquad \text{for } k = 2, 3, \ldots \tag{4}$$

with two initial conditions $u_0 = b_0$ and $u_1 = b_1$. Assume that α_1 and α_2 are real numbers with $\alpha_2 \neq 0$ because otherwise we will be back to the first order case. In backshift notation, this becomes $f(B)u_k = 0$ where $f(z) = 1 - \alpha_1 z - \alpha_2 z^2$. Let the roots of f be z_1 and z_2. Remember that the roots can be in general complex, in which case, we would have that $z_2 = \bar{z}_1$.

If z_1 and z_2 are real and if $z_1 \neq z_2$, then the solution to (4) is given by

$$u_k = c_1 z_1^{-k} + c_2 z_2^{-k}$$

where c_1 and c_2 are real numbers determined by the initial conditions:

$$u_0 = c_1 + c_2 = b_0 \quad \text{and} \quad u_1 = c_1 z_1^{-1} + c_2 z_2^{-1} = b_1.$$

If $z_1 = z_2$ and the common root (which is necessarily real) is denoted by z_0 (say), then the solution to (4) is given by

$$u_k = z_0^{-k}(c_1 + c_2 k)$$

where c_1 and c_2 are determined by the initial conditions:

$$u_0 = c_1 = b_0 \quad \text{and} \quad u_1 = (c_1 + c_2) z_0^{-1} = b_1.$$

If z_1 and z_2 are non-real, then $z_2 = \bar{z}_1$ and the solution is given by

$$u_k = c_1 z_1^{-k} + \bar{c}_1 \bar{z}_1^{-k}$$

where the complex number c_1 (which has a real part and an imaginary part) is determined by

$$u_0 = c_1 + \bar{c}_1 = b_0 \quad \text{and} \quad u_1 = c_1 z_1^{-1} + \bar{c}_1 \bar{z}_1^{-1} = b_1.$$

3 Autocorrelation Function for AR(2)

For the special case of AR(2), the equation (2), upon dividing both sides by $\gamma_X(0)$ and noting that $q = 0$, gives

$$\rho_X(k) - \phi_1 \rho_X(k-1) - \phi_2 \rho_X(k-2) = 0 \qquad \text{for } k = 1, 2, \ldots$$

This is a set of order two difference equations which we can solve by the method we have just learned. Note that these equations start from $k = 1$ so the initial conditions need to be in terms of $\rho_X(0)$ and $\rho_X(-1)$. We know that $\rho_X(0) = 1$ and that $\rho_X(-1) = \rho_X(1)$. These two can be used as initial conditions. Alternately, we can take the equation for $k = 1$ above and use the constraints $\rho_X(0) = 1$ and $\rho_X(-1) = \rho_X(1)$ to obtain $\rho_X(1) = \phi_1/(1 - \phi_2)$. We could then use $\rho_X(0) = 1$ and $\rho_X(-1) = \phi_1/(1 - \phi_2)$ as the initial values.

The solution depends on the roots of the autoregressive polynomial $\phi(z) = 1 - \phi_1 z - \phi_2 z^2$. Let the roots be z_1 and z_2. Because we assumed causality, we have $|z_1| > 1$ and $|z_2| > 1$. Also, $z_1 + z_2 = -\phi_1/\phi_2$ and $z_1 z_2 = -1/\phi_2$.

1. Suppose z_1 and z_2 are both real and distinct. Then the autocorrelation function is given by

$$\rho_X(h) = c_1 z_1^{-h} + c_2 z_2^{-h}$$

2. Suppose $z_1 = z_2$ and denote the common root (which is real) by z_0. Then

$$\rho_X(h) = z_0^{-h}(c_1 + c_2 h).$$

3. Suppose $z_2 = \bar{z}_1$. Then

$$\rho_X(h) = c_1 z_1^{-h} + \bar{c}_1 \bar{z}_1^{-h}.$$

   Write the complex number z_1 as $|z_1|e^{i\theta}$ so that

$$\rho_X(h) = |z_1|^{-h}\left(c_1 e^{-i\theta h} + \bar{c}_1 e^{i\theta h}\right).$$

   Suppose $\bar{c}_1 = ae^{ib}$, then we get that $\rho_X(h) = 2a|z_1|^{-h}\cos(h\theta + b)$.

The constants c_1 and c_2 in the first two cases and the constants a and b in the third case are chosen so as to satisfy the initial conditions $\rho_X(0) = 1$ and $\rho_X(1) = \phi_1/(1 - \phi_2)$.

Book Readings: Section 3.3.