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Cumulative prospect theory and imprecise risk

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Abstract

Tversky and Kahneman have worked out an appealing model of decision making under uncertainty, involving rank- and sign-dependent utilities. This model, cumulative prospect theory (CPT), as well as related models proposed by other authors, has received wide acclaim. Available information and psychological attitude facing ambiguity jointly determine the subjective likelihood values the decision maker attributes to events, expressed by either one of two capacities depending on the prospect of either gains or losses; unfortunately, neither interpretation of these capacities nor prevision of their links are straightforward. An insight into these issues is given by studying consistency of CPT with certain generalized expected utility models, when faced with objective data described by lower–upper probability intervals. Means of testing the existence of subjectively lower–upper probabilized events are obtained, as well as means of evaluating ambiguity aversion. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Savage's (1954) subjective expected utility theory (SEU) is well known and widely admitted for a normative model. Ellsberg's (1961) experiments, however, show that ambiguous situations often lead a decision maker (d.m.) to violate Savage's sure-thing principle (a broad review about uncertainty and ambiguity from the experimental point of view can be found in Camerer and Weber (1992) and Camerer (1995)); for example, recall the famous Ellsberg urn which contains 90 red, black, or yellow balls, with exactly

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30 red ones, and from which one ball is drawn. With the prospect of a payoff of either 0\$ or 1000\$, many subjects prefer a bet on red to a bet on black, even though they prefer a bet on black or yellow to a bet on red or yellow. Such a preference is not consistent with SEU: the subjective probability P should satisfy the contradictory inequalities $P(R) > P(B)$ and $P(R \cup Y) < P(B \cup Y)$.

Choquet expected utility (CEU) theory, essentially developed by Schmeidler (1989), Gilboa (1987), and Sarin and Wakker (1992), gives an appealing solution to Ellsberg's paradox, by relaxing the additivity of the subjective probability and justifying an expected utility criterion related to a capacity. Nevertheless, Ellsberg's paradox is by far not the only systematic bias (as regards the SEU theory) observed by experimenters; in particular, subjects seem to respond, rather than to the absolute gains and losses, to their relative values with respect to the status quo, as pointed out by Tversky and Kahneman (1986) (see also Cohen et al., 1987). So as to cope with this further challenge, cumulative prospect theory (CPT), a development of prospect theory (Kahneman and Tversky, 1979), makes use of two capacities — one on the loss side, one on the gain side (Tversky and Kahneman, 1992; Sarin and Wakker, 1994; Wakker and Tversky, 1993; see also Luce and Fishburn, 1991, 1995). CPT generalizes the CEU model, while allowing a much greater flexibility.

A different point of view can be adopted: Ellsberg's paradox can also be solved by taking in account the whole available (objective) information, namely a set of probabilities which may be characterized by its lower (or upper) envelope. Such a position is called *imprecise risk*; it must be noted that it is an instance of a fairly common situation since, in an often natural way, information under uncertainty is quantified by means of probability intervals. Imprecise risk may be handled by generalizing the expected utility (EU) theory (Von Neumann and Morgenstern, 1944; Herstein and Milnor, 1953) to convex sets of lower envelopes defined on the set of consequences. Related models have been elaborated (Jaffray, 1989a,b; Hendon et al., 1994; Philippe et al., 1999), which are referred to as EUIR (expected utility under imprecise risk) models in the sequel. An additional, very simple, EUIR model is presented in Appendix A; assuming no particular properties of the lower envelope, it is designed to meet exactly the general assumptions that are made in the sequel. It must be noted that, in the EUIR models, the vNM utility function is defined on the *powerset* of the set of consequences, so that the values of this function themselves express the d.m.'s psychological attitude with respect to gains and losses.

The goal of this paper is twofold: to study the compatibility of CPT and EUIR, and to provide possible interpretations of the capacities involved in CPT. Firstly, both CPT and EUIR models solve simultaneously the Ellsberg paradox and the gains and losses issue, but they arise from quite different theories. Previous rank- or sign-dependent models were designed to accommodate violations of EU under risk, making use of transformations of the objective probability; nevertheless, the recent rank- and sign-dependent models we consider in this paper cope with very general situations of uncertainty, without necessary reference to any objectively probabilized information. Accordingly, there is no prior theoretical contradiction for preference of a d.m. to be described by CPT in a general uncertainty context and, simultaneously, by EU or EUIR when faced with objective data. Comparing the two corresponding decision criteria together will

shed light on the d.m.'s attitude with respect to ambiguity; indeed, this attitude is likely to be expressed through the capacities in CPT, and through the vNM utility function in EUIR models.

Secondly, whatever the axiomatization, neither are the relations between the two capacities specified in the CPT model, nor their interpretation clearly founded. On the one hand, intuition suggests that these capacities need support the main consequences of the d.m.'s ambiguity attitude when facing gains and when facing losses; on the other hand, it seems unlikely that these capacities be completely independent since they should depend in part on the d.m.'s psychological attitude, and possibly other factors, and in part on the d.m.'s available information about events, which is certainly a common factor.

Connection between CEU theory (within Savage's settings) and Jaffray's particular EUIR model is studied in a previous paper (Jaffray and Philippe, 1997). The present paper partially resumes the devices already used in the latter. It is moreover shown that many results still hold for a quite general EUIR model, so that they cannot be thought to rely on the particular form of Jaffray's criterion. The following line of attack is proposed: consider a d.m. who usually agrees with EU theory under risk, makes use of a general EUIR criterion under imprecise risk, and adopts CPT only in an ambiguous, non-objectively probabilized situation; further consider particular situations, in which objective information about events exists and is known to the d.m., and assume that her behavior in such situations conforms to both the specific EUIR model and the universal CPT model.

As a first result, we show that CPT and EU are reconcilable if, and only if, both capacities of the CPT model coincide with the objective probability when restricted to risky events. As a second result, consistency of the two models under imprecise risk is shown to impose necessary links between each of the capacities of CPT and the objective data, and, in turn, links between the two capacities themselves. A representation of these capacities is provided, in which the areas of information and psychology are clearly distinguished. For ambiguous decisions, the consistency of the two criteria forces their reduction to particular cases; this suggests that, in actual situations, only few d.m.s will simultaneously agree with the two models. Nevertheless, both particular expressions of the criteria isolate a common pair of indices (α^+ , α^-) which reveal the attitude of the d.m. facing ambiguity. We further establish both necessary and sufficient conditions for the CPT criterion to reduce to the above-mentioned particular case. The latter conditions focus on the capacities of CPT, and may be tested in view of their potential interpretations. We finally consider two such interpretations. On the one hand, we study the possible extension of the algebra of imprecisely probabilized events, and moreover discuss the revelation of subjective imprecise risk when no prior objective information is available. On the other hand, we link the values of the pair (α^+ , α^-) with the tendency towards ambiguity aversion; accordingly, we are led to reconsider former characterizations of ambiguity aversion and to suggest the use of a weaker property.

The two decision criteria considered are introduced in Section 3, and the consequences of their coexistence are studied in Section 4, while Section 5 deals with the converse problem of the detection of subjective lower and upper probabilities in a decision frame such as CPT, and Section 6 with possible interpretations of the capacities

in CPT. Finally, a new EUIR model is axiomatized in Appendix A. Some definitions are first recalled in Section 2.

2. Definitions

Let \mathbf{E} be a set and $\mathbf{A} \subseteq 2^{\mathbf{E}}$ a boolean algebra. A function $w: \mathbf{A} \rightarrow [0, 1]$ is *monotone* when $w(B) \leq w(A)$ as soon as $B \subseteq A$, *convex* when $w(A \cup B) + w(A \cap B) \geq w(A) + w(B)$ for all $A, B \in \mathbf{A}$. A *capacity* v defined on (\mathbf{E}, \mathbf{A}) is a monotone function: $\mathbf{A} \rightarrow [0, 1]$ which satisfies the conditions

$$v(\emptyset) = 0 \text{ and } v(\mathbf{E}) = 1.$$

If v is a capacity on a measurable space (\mathbf{E}, \mathbf{A}) and ξ an \mathbf{A} -measurable function from \mathbf{E} to the set \mathbb{R} of real numbers, the *Choquet integral* (Choquet, 1953) of ξ w.r.t. v is defined by

$$\int_{\mathbf{E}} \xi \, dv = - \int_{-\infty}^0 (1 - v(\xi > x)) \, dx + \int_0^{+\infty} v(\xi > x) \, dx. \quad (1)$$

The *dual capacity* V of v is defined by

$$V(A) = 1 - v(A^c) \text{ for all } A \in \mathbf{A}.$$

Throughout this paper, dual capacities are denoted with a same, capital and small, letter. Given two capacities v, v' and a real number λ , the straightforward property

$$v' = \lambda v + (1 - \lambda)V \Leftrightarrow V' = \lambda V + (1 - \lambda)v$$

will be frequently used later on. If λ differs from $1/2$, this property results in

$$v' = \lambda v + (1 - \lambda)V \Leftrightarrow v = \frac{\lambda}{2\lambda - 1} v' + \frac{\lambda - 1}{2\lambda - 1} V'. \quad (2)$$

A capacity is called a *symmetry* when it equals its dual capacity, and probabilities are precisely the convex symmetries. In the sequel, the subset $\{v = V\}$ of \mathbf{A} is called the class of the *unambiguous* events (with reference to v). This class contains the empty set and is closed with respect to complementation, so that our definition meets the minimal requirements concerning unambiguity. Of course, each event is unambiguous with reference to a probability.

Let \mathbf{P} be a set of probabilities defined on \mathbf{A} , the lower probability f of \mathbf{P} is defined on the latter algebra by

$$f(A) = \inf_{Q \in \mathbf{P}} Q(A) \text{ (for short, } f = \inf \mathbf{P}).$$

It is easily seen that f is a capacity, and that its dual capacity F is the upper probability of \mathbf{P} . The set \mathbf{P} is *m-closed* (closed w.r.t. majorization) when it is identical to the set $\text{core}(f)$ of all the probabilities which (pointwise) dominate its lower probability f on \mathbf{A} ; therefore m-closeness property makes \mathbf{P} characterizable by its lower probability.

Remark 1. Given a capacity v defined on \mathbf{A} and a chain $\mathbf{C} \subseteq \mathbf{A}$, let the symbol $|_{\mathbf{C}}$ stand for restriction to \mathbf{C} . The two following statements are equivalent (Kindler, 1986).

- (i) There is Q additive on \mathbf{A} such that $Q(\emptyset) = 0$, $Q \geq v$, and $v|_{\mathbf{C}} = Q|_{\mathbf{C}}$;
- (ii) For all $A_i \in \mathbf{A}$ and all $C_j \in \mathbf{C}$,

$$\sum_{i=1 \dots n} \mathbf{1}_{A_i} = \sum_{j=1 \dots m} \mathbf{1}_{C_j} \Rightarrow \sum_{i=1 \dots n} v(A_i) \leq \sum_{j=1 \dots m} v(C_j).$$

Furthermore, the particular case of $\mathbf{C} = \{A, \mathbf{E}\}$ for each $A \in \mathbf{A}$ establishes a both necessary and sufficient condition for v to be a lower probability (Schmeidler, 1972; Wolf, 1977) — the two following statements are equivalent:

- (i) $\text{core}(v) \neq \emptyset$ and $v = \inf \text{core}(v)$;
- (ii) for all $A \in \mathbf{A}$, all $A_i \in \mathbf{A}$, and all nonnegative integers p and q ,

$$\sum_{i=1 \dots n} \mathbf{1}_{A_i} = p\mathbf{1}_A + q\mathbf{1}_{\mathbf{E}} \Rightarrow \sum_{i=1 \dots n} v(A_i) \leq pv(A) + q.$$

Let now \mathbf{E} be a *finite* set. Given a function $w: 2^{\mathbf{E}} \rightarrow [0, 1]$, its *Möbius inverse* (Rota, 1964) ω is defined on $2^{\mathbf{E}}$ by

$$\omega(A) = \sum_{B \in 2^{\mathbf{E}}, B \subseteq A} (-1)^{|A| - |B|} w(B)$$

and is conversely related to w by the formula

$$w(A) = \sum_{B \in 2^{\mathbf{E}}, B \subseteq A} \omega(B). \quad (3)$$

Considering (3), the Möbius inverse ν of a capacity v may be seen as the distribution of a signed measure μ on $(2^{\mathbf{E}}, 2^{2^{\mathbf{E}}})$, associated with v by the relation $v(A) = \mu(2^A)$ (see e.g. Gilboa and Schmeidler, 1995). When v is non-negative, μ is of course a probability measure and v is often called a belief function (Shafer, 1976). A point mass μ_B on $(2^{\mathbf{E}}, 2^{2^{\mathbf{E}}})$ is associated with a particular belief function e_B , called an *elementary belief function*, defined by:

$$\text{if } A \supseteq B \text{ then } e_B(A) = 1 \text{ else } e_B(A) = 0.$$

Elementary belief functions allow formula (3) to be also written, in a unique way,

$$w = \sum_{B \in 2^{\mathbf{E}}} \omega(B) e_B. \quad (4)$$

Recall that, ν denoting a capacity defined on a finite space $(\mathbf{E}, 2^{\mathbf{E}})$, and v denoting its Möbius inverse, the Choquet integral w.r.t. ν of a function ξ from \mathbf{E} to \mathbb{R} can be written (see e.g. Gilboa and Schmeidler, 1995)

$$\int_{\mathbf{E}} \xi d\nu = \sum_{X \subseteq \mathbf{E}} \nu(X) \min_{x \in X} \xi(x). \quad (5)$$

Additionally, since clearly

$$\int_{\mathbf{E}} \xi \, dV = - \int_{\mathbf{E}} -\xi \, dv, \quad (6)$$

the dual capacity V of the capacity v satisfies

$$\int_{\mathbf{E}} \xi \, dV = \sum_{X \subseteq \mathbf{E}} v(X) \max_{x \in X} \xi(x).$$

If v is moreover convex, the set $\text{core}(v)$ of all probabilities on $(\mathbf{E}, 2^{\mathbf{E}})$ which dominate v is not empty (Shapley, 1971), and the above Choquet integral satisfies (Choquet, 1953, 54.2)

$$\int_{\mathbf{E}} \xi \, dv = \inf_{Q \in \text{core}(v)} \int_{\mathbf{E}} \xi \, dQ. \quad (7)$$

3. The decision criteria

3.1. Cumulative prospect theory

Let \mathbf{S} be the set of the states of nature, and $\mathbf{A} \subseteq 2^{\mathbf{S}}$ an algebra of events. Let \mathbf{X} be the set of consequences, endowed with its powerset algebra; the status quo consequence is denoted by γ . A decision δ is an \mathbf{A} -measurable finite-ranged mapping: $\mathbf{S} \rightarrow \mathbf{X}$, and we will write

$$\delta = (A_1, c_1; \dots; A_n, c_n)$$

when $A_i = \delta^{-1}(\{c_i\})$ for $1 \leq i \leq n$ and $\{A_1, \dots, A_n\}$ is a partition of \mathbf{S} ; a decision δ is n -ranged if n is the cardinality of $\delta(\mathbf{S})$. The set \mathbf{D} of the decisions to be considered is assumed to contain each constant (i.e. one-ranged) decision (\mathbf{S}, c) , so that preference on \mathbf{D} induces preference on \mathbf{X} . Both preference relations are denoted by the symbol \succeq (\sim and $>$ for their respective symmetric and asymmetric parts). The further symbols \mathbf{X}^+ and \mathbf{X}^- stand for the subsets $\{c \in \mathbf{X}: c \succeq \gamma\}$ and $\{c \in \mathbf{X}: \gamma \succeq c\}$ respectively, namely gains and losses in the CPT model; for the sake of interest, we assume in the rest of the paper that neither \mathbf{X}^+ nor \mathbf{X}^- reduces to $\{\gamma\}$. The consequences of a decision $(A_1, c_1; \dots; A_n, c_n)$ are agreed to be indexed in increasing order:

$$c_n \succeq \dots \succeq c_k \succeq \dots \succeq c_1.$$

Under uncertainty, cumulative prospect theory is based on the following frame: consequences being described as deviations with regard to the status quo, the d.m. expresses simultaneously her information on ambiguous events and its impact on preference by means of two capacities v^+ and v^- defined on the algebra of events \mathbf{A} . Given an event A in \mathbf{A} , the value $v^+(A)$ could, e.g., be obtained by searching a probabilized event B for which the d.m. shows indifference between a bet on A and a bet on B with the prospect of only gains; likewise, $v^-(A)$ could be obtained via bets against

A and B , with the prospect of only losses (Sarin and Wakker, 1994). Criterion (8) presented hereafter is derived from several axiomatic systems.

We assume that there exist two capacities v^+ and v^- defined on \mathbf{A} , and a bounded function $u: \mathbf{X} \rightarrow \mathbb{R}$ with $u(\gamma) = 0$, such that preference of the d.m. on the set \mathbf{D} of decisions is representable by the following CPT criterion:

for each $\delta = (A_1, c_1; \dots; A_n, c_n)$ in \mathbf{D} with $c_n \succeq \dots \succeq c_k \sim \gamma \succeq \dots \succeq c_1$,

$$U(\delta) = \sum_{i=1}^{k-1} \left[v^- \left(\bigcup_{j=1}^i A_j \right) - v^- \left(\bigcup_{j=1}^{i-1} A_j \right) \right] u(c_i) \\ + \sum_{i=k+1}^n \left[v^+ \left(\bigcup_{j=i}^n A_j \right) - v^+ \left(\bigcup_{j=i+1}^n A_j \right) \right] u(c_i). \quad (8)$$

Using the Choquet integral (1), another expression of (8) is

$$U(\delta) = \int_{\mathbf{s}} u^+ \circ \delta \, dv^+ - \int_{\mathbf{s}} u^- \circ \delta \, dv^-, \quad (9)$$

where $u^+(x) = \max(0, u(x))$ and $u^- = (-u)^+$.

Notice that, if $v^+ = V^- = v$ (i.e. $v^+(A) + v^-(A^c) = 1$ on \mathbf{A}), then by (6)

$$\int_{\mathbf{s}} u^+ \circ \delta \, dv^+ - \int_{\mathbf{s}} u^- \circ \delta \, dv^- = \int_{\mathbf{s}} u^+ \circ \delta \, dv + \int_{\mathbf{s}} -u^- \circ \delta \, dv,$$

so that criterion (8) reads

$$U(\delta) = \int_{\mathbf{s}} u \circ \delta \, dv, \quad (10)$$

which is precisely the CEU criterion with respect to v and the utility function u . In particular, if v is a probability measure then criterion (10) is the SEU criterion with respect to v and u . Therefore, the CPT criterion may be thought of as a generalization of the CEU criterion, this last criterion in turn generalizing the SEU criterion.

Conversely, Proposition 1 shows that, faced with probabilized information, the consistency of criterion (8) with the EU criterion with respect to a probability measure P forces the identities $v^+ = v^- = P$. Observe that this result is not self-evident since, in the CPT model, capacities are not necessarily additive on probabilized events. A priori, they only have to be ordinaly equivalent with the involved probability.

3.2. Linear utility under imprecise risk

Under imprecise risk, the d.m. is able to express her information with the help of an m -closed set \mathbf{P} of probabilities (instead of only one under risk) defined on the considered algebra of events. In this section and the following one, we assume that this kind of information is available for certain events of \mathbf{A} , which constitute a subalgebra of \mathbf{A} that is denoted by \mathbf{A}_{ULP} in the sequel. The additional symbol \mathbf{D}_{ULP} will stand for the subset of \mathbf{D} containing the \mathbf{A}_{ULP} -measurable decisions.

The subset of \mathbf{A}_{ULP} on which lower and upper probabilities f and F coincide (i.e. the class of unambiguous events with reference to f) is denoted by \mathbf{A}_p ; it is closed with respect to complementation and disjoint union (a class of sets with such properties is sometimes called a λ -system), because $-f$ and F are subadditive. The common restriction to \mathbf{A}_p of f and F is denoted by P , and P is additive on \mathbf{A}_p . When f is convex, it is known (Jaffray and Philippe, 1997) that \mathbf{A}_p is an algebra, so that P is a probability on \mathbf{A}_p (put in other words, \mathbf{A}_p is an algebra iff the restriction of f to the algebra generated by \mathbf{A}_p is convex), but \mathbf{A}_p is not closed under intersection (nor union, of course) in general; nevertheless, each finite partition of \mathbf{S} composed of elements of \mathbf{A}_p generates an algebra that is contained in \mathbf{A}_p , so that the expression ‘ \mathbf{A}_p -measurable decision’ makes sense. The subset of \mathbf{D}_{ULP} containing all such decisions is denoted by \mathbf{D}_p . The case of $\mathbf{A}_{\text{ULP}} = \mathbf{A}_p$ is not considered in this paper, since imprecise risk reduces then to risk; that is why \mathbf{A}_p is now assumed to be strictly contained in \mathbf{A}_{ULP} .

Several instances of the EUIR model have been axiomatized, all of which requiring that f be at least convex. This requirement is relaxed in the further instance that is presented in Appendix A, which not only illustrates the concepts and methods used but also fits the assumptions made in Section 4. Each decision $\delta \in \mathbf{D}_{\text{ULP}}$ induces a capacity $f \circ \delta^{-1}$ on \mathbf{X} , which is denoted, for short, by f_δ in the sequel. Since δ is finite-ranged, we may identify f_δ with its restriction to $\delta(\mathbf{S})$, so that we can consider the Möbius inverse of f_δ (with respect to $\delta(\mathbf{S})$). The assumption is made that, under imprecise risk, choice between decisions may be identified with choice between the induced capacities on the set \mathbf{X} of consequences. Accordingly, preference, denoted by \succeq again, is given on a convex set Γ of capacities containing the set $\{f_\delta: \delta \in \mathbf{D}_{\text{ULP}}\}$, and is consistent with the restriction to \mathbf{D}_{ULP} of the preference relation \succeq on decisions:

$$\delta \succeq \delta' \Leftrightarrow f_\delta \succeq f_{\delta'}.$$

Application of EU theory to the convex set Γ is supported by representation (4); it leads to the decision criterion H defined on \mathbf{D}_{ULP} in the following way. Given a decision $\delta \in \mathbf{D}_{\text{ULP}}$, denote by ϕ_δ the Möbius inverse of the induced capacity f_δ defined on $(\delta(\mathbf{S}), 2^{\delta(\mathbf{S})})$. If Γ contains both sets $\{f_\delta: \delta \in \mathbf{D}_{\text{ULP}}\}$ and $\{e_C: C \subseteq \delta(\mathbf{S}), \delta \in \mathbf{D}_{\text{ULP}}\}$, and if there exists a linear utility Λ representing \succeq on Γ , set $h(C) = \Lambda(e_C)$ and $H(\delta) = \Lambda(f_\delta)$ to obtain

$$H(\delta) = \sum_{C \subseteq \delta(\mathbf{S})} h(C) \phi_\delta(C). \quad (11)$$

Notice that the vNM utility function h is defined on a set of finite subsets of consequences. Ultimately, *insofar as* H and h derive from the linear utility function Λ , both are unique up to a (common) positive affine transformation.

In particular, for each $\delta \in \mathbf{D}_p$ the Möbius inverse ϕ_δ vanishes everywhere but at the singletons, and $\phi_\delta(\{c\}) = P(\delta^{-1}(c))$ for all $c \in \delta(\mathbf{S})$. Accordingly, criterion (11) turns into

$$H(\delta) = \sum_{C \subseteq \delta(\mathbf{S})} h(\{c\}) P(\delta^{-1}(c)), \quad (12)$$

which is the EU criterion with respect to the probability P , with a von Neumann utility function under risk given on \mathbf{X} by $c \mapsto h(\{c\})$.

It must be noted that, commonly, the d.m.'s information is directly expressed with the help of a probability distribution ϕ on $2^{\mathbf{S}}$ (\mathbf{S} finite), and that the capacity f is a by-product. As an unsophisticated example, let us consider once again the Ellsberg urn described in the Introduction. The d.m. only knows that the ball may be red with probability $1/3$ and black or yellow with probability $2/3$. Denote by $\mathbf{S} = R \cup B \cup Y$ the set of the 90 colored balls and by ϕ the probability distribution on $(2^{\mathbf{S}}, 2^{2^{\mathbf{S}}})$ given by

$$\phi(R) = \frac{1}{3} \text{ and } \phi(B \cup Y) = \frac{2}{3}.$$

It can be shown that the true probability on \mathbf{S} is only known to belong to the set $\text{core}(f)$, where f , defined on $(\mathbf{S}, 2^{\mathbf{S}})$ by

$$f(A) = \sum_{B \in 2^{\mathbf{S}}, B \subseteq A} \phi(B),$$

is a belief function (see e.g. Nguyen and Walker (1994), or Jaffray and Wakker (1994) for a more general set-up); thus criterion (11) may be seen as the EU criterion with respect to the probability on $2^{\mathbf{S}}$ whose distribution is ϕ .

4. Consistency

4.1. The general case

Let us summarize the hypotheses that have been made about the d.m.'s beliefs and information: the algebra of events \mathbf{A} is endowed with two capacities v^+ and v^- ; it includes a subalgebra \mathbf{A}_{ULP} endowed with an m-closed set of probabilities and its lower and upper envelopes f and F ; \mathbf{A}_{ULP} strictly includes the class \mathbf{A}_p on which $f = F$, endowed with the additive set function P which is the common restriction of f and F .

On the other hand, preference on \mathbf{D} and its restriction to \mathbf{D}_{ULP} are representable by the criteria U and H respectively, defined in (8) and (11). As demonstrated in Proposition 1, the consistency of the two criteria U and H implies several properties, in particular their identity on \mathbf{D}_{ULP} (up to scale and location). The key of this result is that the events of \mathbf{A}_p , in other words those which are probabilized on account of f , are also unambiguous events with reference to v^+ and v^- : these two capacities are additive on \mathbf{A}_p , and take the same values as P does. Obtaining this last property requires that the class \mathbf{A}_p be rich enough: for each $(\lambda_1, \lambda_2, \lambda_3)$ in the 3-simplex there should exist in \mathbf{A}_p a partition (A_1, A_2, A_3) of \mathbf{S} such that $P(A_i) = \lambda_i$, $i = 1 \dots 3$; for short, \mathbf{A}_p should allow each three-valued lottery.

Proposition 1. *Assume that \mathbf{A}_p allows any three-valued lottery and that \mathbf{D}_p contains each two- and three-ranged decision. Then, when H , given by (11), and the restriction to \mathbf{D}_{ULP} of U , given by (8), represent the same preference,*

$$(i) v^+ = v^- = P \text{ on } \mathbf{A}_p,$$

and, after calibration of h ,

$$(ii) h(\{c\}) = u(c) \text{ for all } c \in \mathbf{X},$$

$$(iii) H = U \text{ on } \mathbf{D}_{\text{ULP}}.$$

Proof. Let \mathbf{D}^+ be the subset of \mathbf{D} containing the decisions with all values in \mathbf{X}^+ . The restriction of U to \mathbf{D}^+ is none other than the CEU criterion w.r.t. v^+ . As a result, using the assumptions above, the consistency of CEU and EU forces

$$v^+ = P \text{ on } \mathbf{A}_p \quad (13)$$

as soon as $u(\mathbf{X}^+)$ contains at least two non-zero values (Jaffray and Philippe, 1997; (ii) of Proposition 1). It is not difficult to check that (13) still holds if \mathbf{X}^+ contains only one strict gain. We get in the same way $V^- = P$ on \mathbf{A}_p , where V^- is the dual capacity of v^- , by considering the decisions with all values in \mathbf{X}^- . Therefore $P = V^- = v^-$ on \mathbf{A}_p , and (i) is proved.

Now, since both U and H represent \succeq on \mathbf{D}_{ULP} , there exists an increasing function $\kappa: \mathbb{R} \rightarrow \mathbb{R}$ such that $U = \kappa \circ H$ on \mathbf{D}_{ULP} . Considering the constant decisions, we get $u(c) = \kappa(h(\{c\}))$ for all $c \in \mathbf{X}$. If necessary, one may choose two consequences c and c' such that $u(c) \neq u(c')$, and calibrate h so that

$$h(\{c\}) = u(c) \text{ and } h(\{c'\}) = u(c').$$

According to (i), for any $\delta = (A_1, c_1; \dots; A_n, c_n)$ in \mathbf{D}_p criterion U writes

$$U(\delta) = \sum_{i=1}^n P(A_i)u(c_i).$$

If $\delta = (A, c_1; A^c, c_2)$ with $A \in \mathbf{A}_p$, $P(A) = p$, and $c_2 > c_1$, then

$$\begin{aligned} U(\delta) &= pu(c_1) + (1-p)u(c_2) = p\kappa(h(\{c_1\})) + (1-p)\kappa(h(\{c_2\})) \\ &= \kappa(H(\delta)) = \kappa(ph(\{c_1\}) + (1-p)h(\{c_2\})). \end{aligned}$$

Because of the richness assumption about \mathbf{A}_p , $H(\mathbf{D}_p)$ is the convex hull of $u(\mathbf{X})$. So, $H(\mathbf{D}_p)$ is an interval and the restriction $\kappa|_{H(\mathbf{D}_p)}$ is affine, thus, using the above calibration, $\kappa|_{H(\mathbf{D}_p)}$ is the identity function; therefore $h(\{c\}) = u(c)$ on \mathbf{X} , which proves (ii), and $H = U$ on \mathbf{D}_p . Moreover,

$$H(\mathbf{D}_p) = \kappa(H(\mathbf{D}_p)) \subseteq \kappa(H(\mathbf{D}_{\text{ULP}})) = U(\mathbf{D}_{\text{ULP}}). \quad (14)$$

Finally, $H(\mathbf{D}_p)$ contains $U(\mathbf{D}_{\text{ULP}})$ since $U(\mathbf{D}_{\text{ULP}})$ is contained in the convex hull of $u(\mathbf{X})$. So, together with (14) we get $H(\mathbf{D}_p) = H(\mathbf{D}_{\text{ULP}})$ for κ is increasing. Thus $H = U$ on \mathbf{D}_{ULP} . \square

This first proposition describes first consequences of the consistency of the two criteria under risk, and under imprecise risk. As proved in the following two

propositions, the identity on \mathbf{D}_{ULP} of the two criteria U and H implies particular properties for each one. On the one hand, the restrictions to \mathbf{A}_{ULP} of v^+ and v^- have to be linear combinations of f and F ; as could be expected, this property forces a convexity relation between the two capacities v^+ and v^- . On the other hand, the utility function h is ‘equivalent’ (in a sense which is made precise later) with a utility function g that only depends on extremal consequences.

Proposition 2. *Using the assumptions of Proposition 1, if \mathbf{D}_{ULP} moreover contains each two-ranged decision then there exists a unique pair $(\alpha^+, \alpha^-) \in \mathbb{R}^2$ such that, on \mathbf{A}_{ULP} :*

$$\begin{cases} v^+ = \alpha^+ f + (1 - \alpha^+) F \\ v^- = \alpha^- F + (1 - \alpha^-) f \end{cases} \quad (15)$$

As a particular consequence: once restricted to \mathbf{A}_{ULP} , either v^+ or v^- is a convex linear combination of the other one and its dual capacity.

Proof. By Proposition 1, equalities (15) are straightforward on \mathbf{A}_p ; besides, such a pair (α^+, α^-) is necessarily unique since, according to the assumption $\mathbf{A}_{\text{ULP}} \neq \mathbf{A}_p$, f and F are not identical.

Choose a consequence $M > \gamma$ and an event $A \in \mathbf{A}_{\text{ULP}}$ such that $F(A) > f(A)$, and let $\delta = (A^c, \gamma; A, M)$. By Proposition 1, we have

$$f(A)u(M) + (F(A) - f(A))h(\{\gamma, M\}) = H(\delta) = U(\delta) = v^+(A)u(M).$$

Therefore, we may define a number α^+ by

$$1 - \alpha^+ = \frac{v^+(A) - f(A)}{F(A) - f(A)} = \frac{h(\{\gamma, M\})}{u(M)}, \quad (16)$$

and α^+ clearly depends on the choice of neither the event A nor the gain M . Considering a decision $\delta' = (A, m; A^c, \gamma)$ with $\gamma > m$, we get similarly a number

$$\alpha^- = \frac{V^-(A) - f(A)}{F(A) - f(A)} = \frac{h(\{m, \gamma\})}{u(m)}, \quad (17)$$

which depends neither on the event A nor on the loss m . For this pair (α^+, α^-) , (15) holds.

Now, assume that both α^+ and α^- differ from $1/2$. After few calculations, equalities (15) yield the relations

$$\begin{aligned} v^+ &= \frac{(\alpha^+ - \alpha^-)v^- + (1 - \alpha^+ - \alpha^-)V^-}{1 - 2\alpha^-}, \\ v^- &= \frac{(\alpha^- - \alpha^+)v^+ + (1 - \alpha^+ - \alpha^-)V^+}{1 - 2\alpha^+}; \end{aligned}$$

if α^- lies between α^+ and $1 - \alpha^+$ then the first combination is convex, else the second one is.

Finally, if $\alpha^+ = 1/2$ then

$$\frac{1}{2}(V^- + v^-) = \frac{1}{2}(\alpha^- f + (1 - \alpha^-)F + \alpha^- F + (1 - \alpha^-)f) = v^+;$$

likewise, $\alpha^- = 1/2$ yields $v^- = 1/2(v^+ + V^+)$. \square

Proposition 2 shows that several results in Jaffray and Philippe (1997) still hold for a very general EUIR model, without the prior hypothesis that the vNM utility function h only depends on extremal consequences — in the sequel, the (indifference class of) minimal and maximal consequence(s) in the finite subset C of \mathbf{X} are denoted by m_C and M_C . A similar property, however, is implied by the forthcoming proposition: the values of criterion H would be left unchanged if the preferences on (finite elements of) $2^{\mathbf{X}}$ of the d.m. were represented by the function g linked to the vNM utility function u on \mathbf{X} by the relation

$$g(C) = \alpha(m_C, M_C)u(m_C) + (1 - \alpha(m_C, M_C))u(M_C), \quad (18)$$

where α is a constant as long as m_C and M_C are either both gains or both losses, but remains local as soon as m_C is a loss and M_C a gain.

Because this result is of use later on, we state it in a slightly more general form: no special assumption is made about the current algebra.

Proposition 3. *Let three capacities v^+ , v^- , and f be given on an algebra \mathbf{A} of \mathbf{S} and linked by relations (15). Then, for each \mathbf{A} -measurable decision δ , ϕ_δ standing for the Möbius inverse of f_δ , and for each utility u representing \succeq on \mathbf{X} ,*

$$\int_{\mathbf{S}} u^+ \circ \delta \, dv^+ - \int_{\mathbf{S}} u^- \circ \delta \, dv^- = \sum_{C \subseteq \delta(\mathbf{S})} g(C) \phi_\delta(C),$$

where g is defined on finite subsets of \mathbf{X} by (18), with the pessimism index α given by

$$\alpha(m_C, M_C) = \begin{cases} \alpha^+ & \text{if } M_C \succeq m_C \succeq \gamma, \\ \alpha^- & \text{if } \gamma \succeq M_C \succeq m_C, \\ \frac{\alpha^+ u(M_C) - \alpha^- u(m_C)}{u(M_C) - u(m_C)} & \text{if } M_C \succ \gamma \succ m_C. \end{cases} \quad (19)$$

Proof. Let $\delta = (A_i, c_i)$ with $c_n \succeq \dots \succeq c_k \sim \gamma \succeq \dots \succeq c_1$ be an \mathbf{A} -measurable decision. By substituting (15) and using the Möbius inverse ϕ_δ of f_δ together with (5), we get

$$\begin{aligned} & \int_{\mathbf{S}} u^+ \circ \delta \, dv^+ - \int_{\mathbf{S}} u^- \circ \delta \, dv^- \\ &= \alpha^+ \int_{\mathbf{S}} u^+ \circ \delta \, df + (1 - \alpha^+) \int_{\mathbf{S}} u^+ \circ \delta \, dF - \alpha^- \int_{\mathbf{S}} u^- \circ \delta \, dF - (1 - \alpha^-) \int_{\mathbf{S}} u^- \circ \delta \, df \\ &= \alpha^+ \int_{\delta(\mathbf{S})} u^+ \, df_\delta + (1 - \alpha^+) \int_{\delta(\mathbf{S})} u^+ \, dF_\delta - \alpha^- \int_{\delta(\mathbf{S})} u^- \, dF_\delta - (1 - \alpha^-) \int_{\delta(\mathbf{S})} u^- \, df_\delta \\ &= \sum_{C \subseteq \delta(\mathbf{S})} g(C) \phi_\delta(C), \end{aligned}$$

where $g(C)$ is the following expression:

$$\alpha^+ \min_C u^+ + (1 - \alpha^+) \max_C u^+ - \alpha^- \max_C u^- - (1 - \alpha^-) \min_C u^-.$$

Given a subset C of $\delta(S)$, three cases occur: if $\gamma \succeq M_C$ then $\min_C u^+ = \max_C u^+ = 0$, whence

$$g(C) = \alpha^- u(m_C) + (1 - \alpha^-) u(M_C);$$

if $m_C \succeq \gamma$ then $g(C) = \alpha^+ u(m_C) + (1 - \alpha^+) u(M_C)$ follows in the same way, and if $M_C \succeq \gamma \succeq m_C$ then $g(C) = \alpha^- u(m_C) + (1 - \alpha^+) u(M_C)$ holds finally. The third equality in (19) is a direct consequence. \square

Note that $\alpha(m, M)$ in Proposition 3 is a constant if and only if $\alpha^+ = \alpha^-$, which is the case of dual capacities v^+ and v^- ; in fact, it has been seen that the CPT criterion and the CEU criterion are then identical.

Propositions 1, 2, and 3 bring to the fore the pair (α^+, α^-) as the essential mark of the attitude of the d.m. facing ambiguity. Values to be taken by this pair are discussed hereafter, while some first elements are given that are intended to shed light on its interpretation.

4.2. Discussion of particular cases

Proposition 2 provides one with a pair (α^+, α^-) of real numbers without special properties, but the identity $H = U$ on \mathbf{D}_{ULP} constrains the range of the potential values of this pair. Let $(m, M) \in \mathbf{X}^2$ with $u(M) > u(m)$, let $A \in \mathbf{A}_{\text{ULP}}$ with $f(A) < F(A)$, and let $\delta = (A, m; A^\circ, M)$. Since $U(\delta)$ lies in the interval $[u(m), u(M)]$, $H(\delta)$ does too, and this writes

$$F(A^\circ)u(m) - f(A^\circ)u(M) \leq (F(A) - f(A))h(\{m, M\}) \leq F(A)u(M) - f(A)u(m).$$

It is tedious but not difficult to verify that the latter inequalities are equivalent with

$$\max(|2\alpha^+ - 1|, |2\alpha^- - 1|) \leq \inf_{\mathbf{A}_{\text{ULP}} \setminus \mathbf{A}_P} \frac{F + f}{F - f} = \frac{1 + \rho_0}{1 - \rho_0}, \text{ where } \rho_0 = \inf_{F(A) \neq 0} \frac{f}{F}(A).$$

Consequently, if there exists $A \in \mathbf{A}_{\text{ULP}}$ such that $f(A) = 0$ and $F(A) > 0$ then $(\alpha^+, \alpha^-) \in [0, 1]^2$, so that each linear combination in both Propositions 2 and 3 is a convex one. Such a situation is required in Jaffray (1989a) and Philippe et al. (1999), where a dominance axiom yields $u(m_C) \leq h(C) \leq u(M_C)$. If two-ranged decisions δ are considered, the latter property of h is equivalent with the following property of H (see the proof of Proposition 4 hereafter and Appendix A):

$$\inf_{Q \in \text{core}(f)} \int_S u \circ \delta \, dQ \leq H(\delta) \leq \sup_{Q \in \text{core}(f)} \int_S u \circ \delta \, dQ.$$

In a context of imprecise risk, this property is quite appealing; nevertheless, in view of possible interpretations of the capacities of the CPT model, the general EUIR model

considered here does not force $(\alpha^+, \alpha^-) \in [0, 1]^2$. For example, consider an Ellsberg urn containing 100 either red or yellow balls, with at least $n < 50$ balls of each color; the possible events are R (a red ball is drawn from the urn) and Y . Further consider an algebra \mathbf{B} endowed with a probability P the range of which is $[0, 1]$ (such an algebra can be generated by iterative tosses of a fair coin), and assume that each event in \mathbf{B} is independent with both events R and Y . Now, define \mathbf{A}_{ULP} as the algebra generated by \mathbf{B} and R ; it can easily be verified that the available information about \mathbf{A}_{ULP} is characterized by the lower probability f given for A, A' in \mathbf{B} by

$$f(A \cap R \cup A' \cap Y) = \frac{n}{100} \max(P(A), P(A')) + \left(1 - \frac{n}{100}\right) \min(P(A), P(A')).$$

Moreover, \mathbf{A}_p is the set $\{A \cap R \cup A' \cap Y : (A, A') \in \mathbf{B}^2, P(A) = P(A')\}$, and we get, for $P(A) \neq P(A')$,

$$\frac{F+f}{F-f}(A \cap R \cup A' \cap Y) = \frac{100}{100-2n} \frac{P(A) + P(A')}{|P(A) - P(A')|},$$

so that $\max(|2\alpha^+ - 1|, |2\alpha^- - 1|) \leq (1 - n/50)^{-1}$. Therefore, large values of n (i.e. precise information about the urn) enable large gaps between α^+ and $1/2$ or α^- and $1/2$.

Let us now focus attention on Proposition 3: at first sight, the mapping g described in the latter does not necessarily represent the preference of the d.m. on the finite subsets of \mathbf{X} . As a matter of fact, uniqueness of h refers to a preference defined on the convex hull Γ of the union

$$\{f_\delta : \delta \in \mathbf{D}_{\text{ULP}}\} \cup \{e_C : C \subseteq \delta(\mathbf{S}), \delta \in \mathbf{D}_{\text{ULP}}\},$$

but U only represents the preference relation on the set $\{f_\delta : \delta \in \mathbf{D}_{\text{ULP}}\}$. The equality $U = H$ holds, of course, but it does not necessarily entail the equality $h = g$; actually, no decision δ is guaranteed such that $f_\delta = e_C$ for a finite $C \subseteq \mathbf{X}$, unless *totally uncertain* events (i.e. events on which $F - f$ equals 1) are available. Nevertheless, preference of the d.m. on the set $\{e_C : C \subseteq \delta(\mathbf{S}), \delta \in \mathbf{D}_{\text{ULP}}\}$ could be represented by g without modifying the values of the criterion on \mathbf{D}_{ULP} ; in fact, $h = g$ seems to hold frequently, but let us consider first a negative example. Let Q be a probability on \mathbf{A}_{ULP} and let $f = Q^2$; it can easily be verified that f is convex, that $1/2(f + F) = Q$, and that, for each three-ranged decision δ , the Möbius inverse ϕ_δ of f_δ vanishes at $\delta(\mathbf{X})$. If \mathbf{D}_{ULP} only contains k -ranged decisions with $k \leq 3$, we cannot conclude that $h = g$ (in this case, however, values of h on sets C with $|C| \geq 3$ are vain).

Now, let us present a favourable example. Assume for the sake of simplicity that \mathbf{D}_{ULP} contains every \mathbf{A}_{ULP} -measurable decision, and let $\delta = (A_i, c_i) \in \mathbf{D}_{\text{ULP}}$. Proposition 3 yields the identity

$$\sum_{C \subseteq \delta(\mathbf{S})} (h(C) - g(C)) \phi_\delta(C) = 0.$$

If, for each $n \geq 2$, \mathbf{A}_{ULP} contains a subalgebra \mathbf{A}_n generated by n atoms such that the Möbius inverse ϕ_n of f w.r.t. \mathbf{A}_n satisfies $\phi_n(\mathbf{S}) \neq 0$, then $h(C) = g(C)$ holds for each finite subset C of \mathbf{X} , as shown by induction on the cardinality of C — use the

above-mentioned identity and remark that h and g coincide at the singletons by (19) and Proposition 1.

Note that the latter example establishes that h and g always coincide at the pairs, because each event in $\mathbf{D}_{\text{ULP}}|\mathbf{D}_p$ generates a fitting algebra \mathbf{A}_2 . Also remark that the richness assumption about \mathbf{A}_{ULP} is conceivable. For instance, algebras \mathbf{A}_n for $n \geq 3$ may be obtained in the following way: consider an Ellsberg urn that contains n colored balls, n different colors c_1, \dots, c_n being possible but their distribution unknown; then take the algebra generated by the n events ‘one c_i -colored ball is drawn from the urn’. The Möbius inverse of f w.r.t. each \mathbf{A}_n takes the value 1 at \mathbf{S} and vanishes elsewhere, just like f itself. Further notice that these n events form a totally uncertain partition of \mathbf{S} .

Let us finally examine the case of a convex lower probability f . The next Proposition states that both consistent criteria coincide on \mathbf{D}_{ULP} with a noticeable Hurwicz-like criterion. Denote by H_α the Hurwicz criterion with (constant) index α :

$$H_\alpha(\delta) = \alpha \inf_{Q \in \text{core}(f)} \int_{\mathbf{S}} u \circ \delta \, dQ + (1 - \alpha) \sup_{Q \in \text{core}(f)} \int_{\mathbf{S}} u \circ \delta \, dQ. \quad (20)$$

Let $\delta = (A_i, c_i) \in \mathbf{D}$ with $c_n \succeq \dots \succeq c_k \sim \gamma \succeq \dots \succeq c_1$; we define the decision δ^+ by

$$\delta^+(x) = \begin{cases} \delta(x) & \text{if } x \in \bigcup_{i=k+1}^n A_i, \\ \gamma & \text{otherwise} \end{cases};$$

likewise, we define the decision δ^- with $\bigcup_{i=1}^{k-1} A_i$.

Proposition 4. *Using the assumptions of Proposition 2: if f is moreover convex, then, for each δ in \mathbf{D}_{ULP} ,*

$$U(\delta) = H(\delta) = H_{\alpha^+}(\delta^+) + H_{\alpha^-}(\delta^-).$$

Proof. Referring to the proof of Proposition 3 together with (7), we get

$$\begin{aligned} U(\delta) &= \alpha^+ \inf_{Q \in \text{core}(f_\delta)} \int_{\delta(\mathbf{S})} u^+ \, dQ + (1 - \alpha^+) \sup_{Q \in \text{core}(f_\delta)} \int_{\delta(\mathbf{S})} u^+ \, dQ \\ &\quad - \alpha^- \sup_{Q \in \text{core}(f_\delta)} \int_{\delta(\mathbf{S})} u^- \, dQ - (1 - \alpha^-) \inf_{Q \in \text{core}(f_\delta)} \int_{\delta(\mathbf{S})} u^- \, dQ. \end{aligned}$$

Since f is convex and δ is simple, it is known (Jaffray and Philippe, 1997) that $\text{core}(f_\delta)$ is exactly the set of all induced probabilities $Q_\delta = Q \circ \delta^{-1}$ with $Q \in \text{core}(f)$. Then the claimed formula is easily verified. \square

The above proof shows that the convexity of f is not a necessary condition for the result to hold, since a convex f_δ satisfying the property $\text{core}(f_\delta) = \delta(\text{core}(f))$ suffice. Furthermore, if $h(C) = u(M_C)$ for all finite C then

$$H(\delta) = \int_{\mathbf{S}} u \circ \delta \, dF = \sup_{Q \in \text{core}(f)} \int_{\mathbf{S}} u \circ \delta \, dQ;$$

likewise, $h(C) = u(m_C)$ entails

$$H(\delta) = \int_S u \circ \delta \, df = \inf_{Q \in \text{core}(f)} \int_S u \circ \delta \, dQ.$$

Gilboa and Schmeidler (1989) and Chateauneuf (1991) independently justify a model which gives rise to a situation of purely subjective imprecise risk, described by a lower expectation; the resulting maxmin criterion has the latter form.

5. The reciprocal problem

We have shown that the consistency of the CPT and EUIR criteria implies special links between the capacities involved, these links conversely leading to a special form of the CPT criterion. In view of possible interpretations, discussed in the next section, it is important to know both necessary and sufficient conditions for such links to hold; that is why we study here the two following issues. Given two capacities v and v' defined on an algebra \mathbf{A} , like, for instance, in the CPT model, does there exist a capacity f defined on an algebra $\mathbf{A}' \subseteq \mathbf{A}$ and a pair (α, α') in \mathbb{R}^2 such that the restrictions to \mathbf{A}' of v and v' satisfy

$$\begin{cases} v = \alpha f + (1 - \alpha)F \\ v' = \alpha' f + (1 - \alpha')F \end{cases} \quad (21)$$

Moreover, in which cases could either f or F be a lower probability?

First consider the following result. Proposition 2 shows that the existence of a solution implies that either v or v' is a convex linear combination of the other and its dual. This situation can easily be tested:

Lemma 1. *There exists $\beta \in \mathbb{R}$ such that $v = \beta v' + (1 - \beta)V'$ on \mathbf{A}' if and only if the two following conditions hold for all A in \mathbf{A}' :*

- (i) *the differences $v(A) - v(A^c)$ and $v'(A) - v'(A^c)$ are equal.*
- (ii) *ratios between the probability gaps $1 - v(A) - v(A^c)$ and $1 - v'(A) - v'(A^c)$ are constant.*

Proof. First,

$$v = \beta v' + (1 - \beta)V' \Leftrightarrow \begin{cases} v = \beta v' + (1 - \beta)V' \\ V = \beta V' + (1 - \beta)v \end{cases} \Leftrightarrow \begin{cases} v + V = v' + V' \\ V - v = (2\beta - 1)(V' - v') \end{cases}$$

Recall that $V(A) = 1 - v(A^c)$, $V'(A) = 1 - v'(A^c)$, and the result is straightforward. \square

Turning now to our first problem, we begin by settling the case of both symmetries v and v' . According to the preceding lemma, their restrictions to \mathbf{A}' must coincide; therefore, the CPT criterion turns into the CEU criterion with respect to the symmetry v . Conditions (21) result in

$$(2\alpha - 1)(F - f) = (2\alpha' - 1)(F - f) = 0.$$

If $F = f$ then $F = f = v$, so that the non-trivial solutions obtain for the values $\alpha = \alpha' = 1/2$. This particular instance of our problem is solved in Jaffray and Philippe (1997; Proposition 4). Therefore, we assume hereafter that (at least) one of the two capacities, say v , is not a symmetry. Two simple but useful consequences must be mentioned: in (21), α must differ from $1/2$ and f cannot be a symmetry. From this remark, together with Proposition 2 and (2), the following lemma is easily derived:

Lemma 2. *If $v = \alpha f + (1 - \alpha)F$, then there exists α' such that (21) holds if, and only if, there exists $\beta \in \mathbb{R}$ such that $v' = \beta v + (1 - \beta)V$. In the latter case, β is unique and $\alpha' = \alpha(1 - \beta) + \beta(1 - \alpha)$ is the only value for which (21) holds.*

Consequently, our problem is tantamount to finding $\alpha \in \mathbb{R}$ and a corresponding capacity f_α such that the following equality (E_α) holds on \mathbf{A}' :

$$(E_\alpha) \quad v = \alpha f_\alpha + (1 - \alpha)F_\alpha;$$

by (2) indeed, such a capacity is unique given α .

With this end in view, let us define, for any capacity v defined on \mathbf{A}' , the coefficient

$$\rho(v) = \inf_{A \subset B, v(A^\complement) \neq v(B^\complement)} \frac{v(B) - v(A)}{v(A^\complement) - v(B^\complement)}.$$

Clearly, $\rho(v) \geq 0$; besides, if there exists a pair $A \subset B$ such that $v(B) = v(A)$ and $v(A^\complement) \neq v(B^\complement)$, then $\rho(v) = \rho(V) = 0$, else the sets

$$\{(A, B): A \subset B, v(A^\complement) \neq v(B^\complement)\} \text{ and } \{(A, B): B^\complement \subset A^\complement, v(B) \neq v(A)\}$$

are identical. Therefore, $\rho(v) = \rho(V)$ and $0 \leq \rho(v) \leq 1$; moreover, the equality $\rho(v) = 1$ holds if and only if v is a symmetry.

The number $\rho(v)$ enables us to characterize the set of all values α in \mathbb{R} for which (E_α) has a solution f_α .

Proposition 5. *Let v be a capacity defined on an algebra \mathbf{A}' , v not being a symmetry. Let $\alpha \in \mathbb{R}$. There exists a capacity f_α such that $v = \alpha f_\alpha + (1 - \alpha)F_\alpha$ if, and only if,*

$$|2\alpha - 1| \geq \frac{1 - \rho(v)}{1 + \rho(v)}, \quad (22)$$

in which case f_α is moreover unique.

Proof. According to (2), (E_α) is equivalent with $(2\alpha - 1)f_\alpha = \alpha v - (1 - \alpha)V$. The mapping f_α (uniquely) defined by the latter relation is a capacity if, and only if, it is monotone, that is

$$\text{for all pairs } A \subset B \text{ in } \mathbf{A}', \frac{\alpha}{2\alpha - 1} (v(B) - v(A)) \geq \frac{1 - \alpha}{2\alpha - 1} (V(B) - V(A)),$$

which, in turn, is equivalent to

$$\text{for all pairs } A \subset B \text{ in } \mathbf{A}', v(B) - v(A) \geq \frac{1 - |2\alpha - 1|}{1 + |2\alpha - 1|} (v(A^c) - v(B^c)).$$

The latter condition holds if and only if

$$\rho(v) \geq \frac{1 - |2\alpha - 1|}{1 + |2\alpha - 1|}, \text{ i.e., } |2\alpha - 1| \geq \frac{1 - \rho(v)}{1 + \rho(v)}. \quad \square$$

Some remarks about the preceding result. It must be noted that $\rho(v) = 0$ forces values of α to keep out of the interval $(0, 1)$. Further notice that inequality (22) always holds for such values of α , because $\rho(v) \in [0, 1]$; nevertheless, as demonstrated later, we cannot hope for the corresponding f_α to be a lower probability unless either v or V is itself a lower probability. Alain Chateauneuf pointed out to the author that, if the algebra \mathbf{A}' is finite, the coefficient $\rho(v)$ does not vanish if and only if the following very simple condition holds:

$$\text{for all } A \subseteq B, v(A) = v(B) \Rightarrow v(A^c) = v(B^c).$$

Accordingly, values of α in $[0, 1]$ are always permitted under this condition. Ultimately, if we want both α and α' to be in $[0, 1]$, then (21) holds if, and only if, α satisfies

$$\frac{1 - \rho(v)}{1 + \rho(v)} \leq |2\alpha - 1| \leq \min\left(1, \frac{1}{|2\beta - 1|}\right),$$

where β comes from the necessary relation $v' = \beta v + (1 - \beta)V$.

Let us turn, finally, to our second problem: among the solutions of (E_α) , does there exist either a lower or an upper probability? A necessary and sufficient condition is established in the next proposition, where the coefficient $\rho(v)$ is involved again. Denote by v_ρ the mapping

$$v_\rho = \frac{1}{1 - \rho(v)}(v - \rho(v)V). \quad (23)$$

Note that $\rho(v) < 1$ since v is not a symmetry, and that, according to Proposition 5, v_ρ is a capacity: v_ρ and V_ρ are the solutions of (E_α) for the minimal possible value of $|2\alpha - 1|$.

Proposition 6. *Let v be a capacity defined on \mathbf{A}' , v being not a symmetry. There exists $\alpha \in \mathbb{R}$ and a lower probability (resp. a convex capacity) f_α for which (E_α) holds if, and only if, either v_ρ or V_ρ is a lower probability (resp. a convex capacity). In addition, $\alpha > 1/2$ in the first case, and $\alpha < 1/2$ in the second case.*

Proof. As noted above, v_ρ is a capacity, and v moreover writes

$$v = \frac{1}{1 + \rho(v)}(v_\rho + \rho(v)V_\rho).$$

If v_ρ is a lower probability, (E_α) holds for $\alpha = (1 + \rho(v))^{-1} > 1/2$; likewise, if V_ρ is a lower probability, then (E_α) holds for $\alpha = \rho(v)(1 + \rho(v))^{-1} < 1/2$.

Conversely, assume that there exist α and a lower probability f_α such that (E_α) holds. After substitution, v_ρ writes

$$v_\rho = \lambda f_\alpha + (1 - \lambda)F_\alpha, \text{ where } \lambda = \frac{1}{1 - \rho(v)}(\alpha - \rho(v)(1 - \alpha)).$$

Since $(1 - \rho(v))|2\lambda - 1| = (1 + \rho(v))|2\alpha - 1| \geq (1 - \rho(v))$ by Proposition 5, either λ or $(1 - \lambda)$ is nonpositive.

Now, making use of Remark 1, take $A \in \mathbf{A}$, $A_i \in \mathbf{A}$, and nonnegative integers p and q , such that $\sum_{i=1 \dots n} \mathbf{1}_{A_i} = p\mathbf{1}_A + q\mathbf{1}_S$. Then the assumption about f_α yields

$$\sum_{i=1 \dots n} f_\alpha(A_i) \leq pf_\alpha(A) + q, \text{ and } \sum_{i=1 \dots n} F_\alpha(A_i) \geq pF_\alpha(A) + q.$$

Therefore, if $\lambda \leq 0$ then $V_\rho = \lambda F_\alpha + (1 - \lambda)f_\alpha$ is also a lower probability, else v_ρ is. The convex case is obtained in a similar way. \square

The same argument shows that any relation $v = \alpha f_\alpha + (1 - \alpha)F_\alpha$ with f_α a lower probability and α out of $(0, 1)$ implies that either v or V is itself a lower probability. Besides, if two lower probabilities f_α and f_β are solutions of (E_α) and (E_β) , $2\alpha - 1$ and $2\beta - 1$ have the same sign; moreover, $|2\alpha - 1| \geq |2\beta - 1|$ implies $\text{core}(f_\alpha) \subseteq \text{core}(f_\beta)$. The latter remarks are specially useful in the next section, which deals with possible interpretations of the capacities in CPT.

6. Interpretations

In this section, we consider a d.m. who complies with a general model of decision under uncertainty similar to CPT, and whose preference on \mathbf{D} may be represented by the criterion U given in (8). First studying in Section 6.1 the revelation of subjective imprecise risk, we show in Section 6.2 how pessimism, a natural notion in EUIR models, helps to elucidate the concept of ambiguity aversion in the CEU (or CPT) context.

6.1. Subjective imprecise risk

At first, assume that the d.m.'s information is partially described by an imprecise risk model and that her preference on \mathbf{D}_{ULP} is representable by the criterion H given in (11). Using the assumptions of Section 3, there exists by Proposition 2 a unique pair (α^+, α^-) such that relations (15) hold on \mathbf{A}_{ULP} . Using the results of Section 5, it is possible to know if f can be extended to a capacity f' defined on a subalgebra \mathbf{A}' of \mathbf{A} strictly containing \mathbf{A}_{ULP} (of course, \mathbf{A}' might be \mathbf{A} itself). For this purpose, it suffices to check on \mathbf{A}' equality of differences and proportionality of probability gaps as indicated by Lemma 1, and condition (22) with whichever of v^+ and v^- is not a symmetry. If these conditions are fulfilled, there exists a unique capacity f' defined on \mathbf{A}' and satisfying the conditions

$$\begin{cases} v^+ = \alpha^+ f' + (1 - \alpha^+) F' \\ v^- = \alpha^- F' + (1 - \alpha^-) f' \end{cases} \quad (24)$$

In particular, the restriction of f' to \mathbf{A}_{ULP} is f by uniqueness (Lemma 2 and Proposition 5). Should f' be a lower probability, one would naturally consider \mathbf{A}' as a set of events about which the d.m. reveals subjective imprecise risk. On the other hand, by uniqueness again, conditions (24) cannot hold with any different pair (β^+, β^-) instead of (α^+, α^-) .

Secondly, assume that no objective information is available — in other words, the algebra \mathbf{A}_{ULP} reduces to $\{\emptyset, \mathbf{S}\}$. Consider any subalgebra \mathbf{A}' of \mathbf{A} , and take for granted that there exist numbers α^+ and α^- in \mathbb{R} and a capacity f such that (15) holds on \mathbf{A}' . Let H be defined on \mathbf{D}' , the set of all \mathbf{A}' -measurable decisions of \mathbf{D} , by

$$H(\delta) = \sum_{C \subseteq \delta(\mathbf{S})} \phi_\delta(C) g(C), \quad (25)$$

where ϕ_δ is the Möbius inverse of the capacity f_δ defined on $\delta(\mathbf{S})$, and g the utility function defined in (18) and (19) via the mappings u and α . Then, Proposition 3 states that criteria H and U are identical. If f is a lower probability, the previous interpretation of v^+ and v^- as revealing subjective imprecise risk may be resumed. In the latter situation, however, uniqueness of the pair (α^+, α^-) (thus uniqueness of the corresponding f) is not theoretically guaranteed; the alternative is discussed hereafter.

On the one hand, if the d.m. agrees with EUIR when faced with objective imprecise risk, then the mapping α is one of her intrinsic characteristics. Indeed, the vNM utility function h on the powerset of outcomes is unique up to scale and location, so that α is uniquely determined by the primitive preference on a set of elementary belief functions with finite support. In the EUIR model, the mapping α is interpreted as a local pessimism index (see Jaffray, 1989b) as regards the potential consequences of a decision (when only extremal ones are taken in account among a set of outcomes). Using relations (16) and (17), values α^+ and α^- could be tested in two ways. Inspired by Jaffray (1989b), the following first device may be used: fix an arbitrary gain c and, denoting by $[\gamma, c]$ the preference interval $\{c' \in \mathbf{X}^+ : c \succeq c'\}$, search an objectively probabilized event A for which the d.m. shows indifference between

- (i) to bet against A with the prospect of c , status quo otherwise,
- (ii) to receive certainly an outcome taken in $[\gamma, c]$, without further information.

Then $P(A) = \alpha^+$. In the same way, α^- is constructed by considering a loss c , and bets on A ; again, $P(A) = \alpha^-$.

A second approach is possible, which requires that a totally uncertain event A exists in \mathbf{A}_{ULP} . Intuition suggests that, once eliminated the share of objective data, only remains the part of the individual's state of mind facing ambiguity. In actual fact, by (16) and (17), $1 - \alpha^+ = v^+(A)$ and $\alpha^- = v^-(A)$.

Since α^+ and α^- are unique, the capacity f , if it exists, is unique too. Existence of f for the given pair (α^+, α^-) can be tested by calculating the coefficient $\rho(v)$ on \mathbf{A}' . If f exists and is a lower probability, it may be interpreted as describing subjective imprecise risk, expressed by a unique set $\text{core}(f)$ of subjective probabilities on \mathbf{A}' . Then f and

(α^+, α^-) enable the distinction, in v^+ and v^- , between the part of subjective information and the part of psychological attitude towards ambiguity.

On the other hand, one could think that uniqueness of (α^+, α^-) only makes sense if the d.m. agrees with an EUIR model when faced with objective data. Evaluating the pair (α^+, α^-) requires the existence of (differently) lower and upper probabilized events, that she may deny. Nevertheless, the lack of uniqueness does not prevent from an interpretation in terms of subjective imprecise risk. Suppose for instance that v^+ is not a symmetry, then by Proposition 5 the possible α^+ s form two symmetrical intervals, and by Lemma 2 each possible α^+ is associated with a unique α^- and a unique capacity f_{α^+} . Moreover, if α^+ runs through an interval I , then α^- continuously runs through an interval, too, and the family $(\text{core}(f_{\alpha^+}))_{\alpha^+ \in I}$ is monotone with respect to inclusion. Further notice that, if f_{α^+} and f_{β^+} are two fitting lower probabilities, we have

$$V^+ - v^+ = (2\alpha^+ - 1)(F_{\alpha^+} - f_{\alpha^+}) = (2\beta^+ - 1)(F_{\beta^+} - f_{\beta^+}), \quad (26)$$

so that α^+ and β^+ are either both less (optimism) or both more (pessimism) than $1/2$. Finally, according to Proposition 6, if f_{α^+} is a fitting lower probability for a given α^+ then f_{β^+} is still a lower probability for each fitting β^+ which is nearer to $1/2$. These robustness properties allow the following interpretation: the capacities v^+ and v^- may reveal subjective imprecise risk on certain events, but then the subjective information cannot be uniquely described; nevertheless, whatever this subjective information, the d.m. is either systematically pessimistic or systematically optimistic when she considers the possible outcomes of an ambiguous decision. Such a situation may be uncovered by using Proposition 6. When v_ρ is an envelope, its core provides one with the maximal set of subjective probabilities that enables the interpretation of v^+ and v^- in terms of subjective imprecise risk, and the pessimism index which achieves the minimal deviation from $1/2$ is calculated from $v(\rho)$. It must be noted that testing v_ρ only requires the values of v^+ or v^- on \mathbf{A}' ; in any situation, it can therefore be carried out prior to an eventual further interview of the decision maker.

Be that as it may, revealing a subalgebra \mathbf{A}' with subjective imprecise risk makes it possible to display the d.m.'s pessimism; in turn, it can bring to the fore her ambiguity aversion, as discussed hereafter.

6.2. Ambiguity aversion

Under imprecise risk with a utility function $h(C) = \alpha(C)u(m_C) + (1 - \alpha(C))u(M_C)$, pessimism is characterized by the property $\alpha(C) > 1/2$. In accordance with the above discussion, let us call pessimistic a d.m. who uses a CEU criterion with a capacity v satisfying (E_α) for a lower probability f_α and an index $\alpha > 1/2$. Intuitively, pessimism should be an instance of ambiguity aversion in the CEU (or CPT) context; since there is no final consensus yet about defining the latter notion, a comparison requires more than one point of view: we first consider the original one, turn then to a very recent study, and finally propose a further definition.

On the basis of the seminal paper of Schmeidler (1989), ambiguity aversion in the CEU context has often been characterized as involving a convex capacity. In Schmeidler

(1989) (as well as in Gilboa and Schmeidler (1989)), the Anscombe–Aumann framework is adopted and the following axiom is shown to characterize convexity of the capacity. For all (lottery) acts h_1 and h_2 , and for any $\lambda \in [0, 1]$:

$$h_1 \sim h_2 \Rightarrow \lambda h_1 + (1 - \lambda)h_2 \succeq h_2.$$

Schmeidler proposes an intuitive interpretation of the latter: ‘Averaging *utility* distributions makes the decision maker better off’. Within Savage’s settings, Chateauneuf (1991) gives a similar interpretation of convexity as revealing ambiguity aversion (Axioms A6, A7, and Remark 9). According to (5), a CEU maximizer with respect to a convex v exhibits the same criterion, therefore the same preference on \mathbf{D} , as a purely pessimistic ($\alpha = 1$) d.m. under EUIR with an information described by v . Therefore, pure pessimism about consequences and ambiguity aversion (in Schmeidler’s sense) about events are identical notions; nevertheless, Proposition 6 shows that, if $\rho(v) > 0$, then the d.m.’s preference may also be represented by a CEU criterion with respect to the capacity v_ρ , which is also convex, of course, but is associated with a pessimism index $1/2 < \alpha < 1$. This remark supports the more and more prevalent idea that Schmeidler’s definition captures more than ambiguity aversion.

In a very appealing work, Ghirardato and Marinacci (1999) propose both comparative and absolute definitions of ambiguity aversion; let us first examine the comparative one. In the CEU context (for convenience, we assume that both preferences, hereafter, have essential events), their result is: a preference \succeq_2 is more ambiguity averse than a preference \succeq_1 if and only if the two corresponding capacities satisfy $v_1 \geq v_2$ and the vNM indices are identical up to scale and location. Consider now two d.m.s whose criterions under imprecise risk have the form (25) with the same information described by a lower probability f and the same vNM index u , so that only their pessimism indices α_1 and α_2 may differ. According to (5), both are CEU maximizers for the respective capacities $\alpha_1 f + (1 - \alpha_1)F$ and $\alpha_2 f + (1 - \alpha_2)F$. Since

$$(\alpha_1 f + (1 - \alpha_1)F) - (\alpha_2 f + (1 - \alpha_2)F) = (\alpha_2 - \alpha_1)(F - f),$$

one of the two d.m.s is more ambiguity averse than the other if and only if she is more pessimistic than the other is; this result complies with intuition, provided the relation ‘more pessimistic than’ is fittingly defined. Turning back to CPT, it is easy to verify, from identities (15), that

$$\alpha^+ \geq \alpha^- \Leftrightarrow v^+ \leq v^- \text{ on } \mathbf{A}',$$

a property which may now be interpreted as: the d.m. is more ambiguity averse in the domain of gains than in the domain of losses. Notice this comparative property does not require uniqueness of the pair (α^+, α^-) — in Lemma 2, the sign of $\alpha' - \alpha = \beta(2\alpha - 1)$ is fixed.

Next examine their absolute definition: in the CEU context, ambiguity aversion is characterized by a *dominated* capacity (i.e. its core is not empty). This is consistent with Schmeidler’s point of view, since a convex capacity is always dominated. Unfortunately, the forthcoming example shows that this is not entirely consistent with the absolute

definition of pessimism ($\alpha > 1/2$). Let \mathbf{A}' be an algebra generated by three atoms A , B , C , and let v and f be defined on \mathbf{A}' by

	\emptyset	A	B	C	$A \cup B$	$A \cup C$	$B \cup C$	S
v	0	0.202	0.102	0.402	0.502	0.802	0.702	1
f	0	0.1	0	0.3	0.4	0.7	0.6	1

The following points are easy to verify: $\rho(v) = 17/33$, $v_\rho = f$, which is convex, $v = 0.66f + 0.34F$, but v is not dominated. That is to say, a CEU maximizer for v is pessimistic without being ambiguity averse (as defined above). Therefore, if we wish both notions to agree, we have to find an *absolute* definition of ambiguity aversion, which should be weaker than the former but still agreeing with the comparative one.

Under imprecise risk, pessimism is characterized by $\alpha \geq 1/2$, and a constant $\alpha = 1/2$ means neutrality for this notion. By (5), preferences agreeing with an EUIR model and neutral for pessimism may be represented by a CEU criterion with respect to a symmetry — the half-sum of the upper and lower probabilities involved. Diverging from the theory in Ghirardato and Marinacci (1999), which uses the more restrictive SEU preferences as benchmarks for ambiguity neutrality, we propose to call *ambiguity neutral* any preference that is representable by CEU with a symmetry (of course, as long as only bets are involved, SEU preferences cannot be distinguished from symmetrical CEU preferences). It must be noted that the latter property is perfectly agreeing with the definition of unambiguous events given in Section 2, because symmetries are precisely the capacities for which each event is unambiguous.

Since we aim to concur with the comparative definition of ambiguity aversion given by Ghirardato and Marinacci, we naturally propose to call *ambiguity averse*, in the CEU context, a preference that is more ambiguity averse than an ambiguity neutral preference. Since $1/2(v + V)$ is a symmetry, it is very easy to see that a CEU preference is now ambiguity averse if, and only if, the corresponding capacity v satisfies the property $v \leq V$ on \mathbf{A}' . Since a dominated v always satisfies $v \leq V$, our definition is still consistent with, but weaker than, the definition in Ghirardato and Marinacci (1999). The following axiom presents a behavioral characterization of this property, provided that a certain richness of the class of the unambiguous events is assumed.

Axiom AA (ambiguity aversion). For any event $A \in \mathbf{A}'$, there exists an unambiguous event B and consequences $c' > c$ such that

$$(B, c; B^c, c') \succeq (A, c; A^c, c') \text{ and } (B^c, c; B, c') \succeq (A^c, c; A, c').$$

In other words, both events B and B^c are considered by the subject as more likely than A and A^c are; in particular, if she thinks the ambiguous event A and the unambiguous event B are equally likely, the d.m. prefers a bet against B to a bet against A .

To conclude, we have seen that the d.m.'s attitude when faced with ambiguous situations of choice is expressed by values she attributes to sets of either states of the

nature or outcomes, depending on the current model — CPT or EUIR. We have put in a prominent position a pair of numbers which provide with a basis for respective interpretations of these values. This is obtained at the expense of a reduction of both criteria to very particular cases. Nevertheless, CPT and the general EUIR model appear to be closer to one another, in the sense of a less specific common reduction, than the CEU model and Jaffray's EUIR model are. An overall criterion is still to be discovered and justified, which would contain each of the precedent ones as a particular instance. The results in this paper, once compared with the previous results in Jaffray and Philippe (1997), could supply some keys. This is the matter of future work.

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Appendix A. Linear utility for bets under imprecise risk

We present in this appendix a particular EUIR model. Because we restrict our attention to either risky prospects or ambiguous bets (i.e. two-ranged decisions), we get rid of special hypotheses about the lower probability on events: in the former models, the latter is at least assumed to be convex.

We primarily assume that the (objective) information of the decision maker is described by a set \mathbf{P} of probabilities, defined on an algebra \mathbf{A} of events of the set \mathbf{S} of the states of the nature. We further assume that \mathbf{P} is characterizable by its lower probability f , i.e. $f = \inf \mathbf{P}$ and $\mathbf{P} = \text{core}(f)$. The set $\{f = F\}$ is denoted by \mathbf{A}_p , and called the set of the risky events; this set is only a λ -system in general, not necessarily an algebra. The common restriction to \mathbf{A}_p of f and F is additive, and it is denoted by P .

A set \mathbf{X} of consequences is given, endowed with its powerset algebra. A decision is defined as an \mathbf{A} -measurable mapping from \mathbf{S} to \mathbf{X} , and risky decisions are those which are \mathbf{A}_p -measurable. Let δ be a decision, symbols Q_δ and f_δ stand respectively for the probability $Q \circ \delta^{-1}$ and the capacity $f \circ \delta^{-1}$ induced on \mathbf{X} by δ ; the set of all the probabilities on \mathbf{X} that are induced by δ from elements of \mathbf{P} is denoted by \mathbf{P}_δ .

Preference of the decision maker is given on a set $\mathbf{D} = \mathbf{D}_b \cup \mathbf{D}_r$ which contains either bets or risky finite-ranged decisions.

Proposition 7. *For each δ in \mathbf{D} , $f_\delta = \min \mathbf{P}_\delta$ and $\text{core}(f_\delta) = \mathbf{P}_\delta$.*

Proof. Let $A \in \mathbf{A}$, and apply Kindler's result (see Remark 1) to the chain (A, \mathbf{S}) : there exists $Q \in \text{core}(f)$ such that $Q(A) = f(A)$. Since $\text{core}(f) = \mathbf{P}$, the property $f = \min \mathbf{P}$ is established. Let $\delta \in \mathbf{D}$ and $Q \in \text{core}(f)$, then $Q_\delta \geq f_\delta$ on $2^{\mathbf{X}}$. Since $f = \min \mathbf{P}$, we get $f_\delta = \min \mathbf{P}_\delta$.

In particular, $\mathbf{P}_\delta \subseteq \text{core}(f_\delta)$. Finally, let $Q' \in \text{core}(f_\delta)$. If $\delta \in \mathbf{D}_r$, then each Q in

$\text{core}(f)$ clearly satisfies $Q_\delta = Q'$, so that $\text{core}(f_\delta) = \mathbf{P}_\delta$. If $\delta = (A, c; A^c, c') \in \mathbf{D}_b$, then $\delta^{-1}(2^{\mathbf{X}}) = \{\emptyset, A, A^c, \mathbf{S}\}$. Let $\lambda \in [0, 1]$ be such that

$$Q'(\{c\}) = \lambda f(A) + (1 - \lambda)F(A),$$

let $(Q_l, Q_u) \in \text{core}(f)^2$ be s.t. $Q_l(A) = f(A)$ and $Q_u(A) = F(A)$, and let $Q = \lambda Q_l + (1 - \lambda)Q_u$. Then $Q \in \text{core}(f)$, and the identity $Q_\delta = Q'$ is easily verified. Thus $\text{core}(f_\delta) = \mathbf{P}_\delta$ again. \square

This result permits the justification of the following reduction axiom (we paraphrase Fishburn (1988, p. 27)):

Axiom A0. For comparative purposes of preference and choice in decision of \mathbf{D} , it suffices to characterize each alternative in terms of its induced lower probability over potential consequences.

Therefore, we consider as a primitive a preference relation \succeq defined on a convex set Γ of capacities on \mathbf{X} , that contains f_δ for each δ in \mathbf{D} , and which is made precise soon. First notice that, for each δ in \mathbf{D} , f_δ is a monotone capacity of infinite order on $(\delta(\mathbf{S}), 2^{\delta(\mathbf{S})})$: the Möbius inverse of f_δ with respect to $2^{\delta(\mathbf{S})}$ is clearly nonnegative. Now, we define Γ as the convex hull of the set

$$\mathbf{E} = \{e_{\{c, c'\}} : \exists \delta \in \mathbf{D}_b, \{c, c'\} = \delta(\mathbf{S})\} \cup \{e_{\{c\}} : \exists \delta \in \mathbf{D}, c \in \delta(\mathbf{S})\}.$$

Note that \mathbf{E} only contains extremal elements of Γ . From (4) and the previous remarks, it follows easily that, for each δ in \mathbf{D} , Γ contains the induced capacity f_δ . The d.m.'s preference on Γ is first assumed to be transitive and complete. Put in other words:

Axiom A1. \succeq on Γ is a weak order.

Jaffray (1989a) proves the following statement:

$$\begin{aligned} \text{let } (g, g') \in \Gamma^2, \lambda \in [0, 1], \text{ then } \text{core}(\lambda g + (1 - \lambda)g') \\ = \lambda \text{core}(g) + (1 - \lambda) \text{core}(g'). \end{aligned}$$

Basing one's argument on the latter result, he further justifies the validity, in the context of imprecise risk, of the following independence and continuity axioms (Jensen, 1967).

Axiom A2. For all g_1, g_2, g in Γ , and all λ in $(0, 1)$,

$$g_1 \succ g_2 \Rightarrow \lambda g_1 + (1 - \lambda)g \succ \lambda g_2 + (1 - \lambda)g.$$

Axiom A3. For all g_1, g_2, g in Γ , there are some α, β in $(0, 1)$ such that

$$g_1 \succ g \succ g_2 \Rightarrow \begin{cases} \alpha g_1 + (1 - \alpha)g_2 \succ g \\ g \succ \beta g_1 + (1 - \beta)g_2 \end{cases}.$$

If a capacity g is convex, then $g(C) = g(C') = 1$ implies $g(C \cap C') = 1$. Let $g \in \Gamma$. On

the one hand, g is convex. On the other hand, there exists a finite $C \in 2^X$ such that $g(C) = 1$. Consequently, there exists a unique finite $C_g \in 2^X$ such that

$$g(C_g) = 1 \text{ and } (g(C) = 1 \Rightarrow C \supseteq C_g).$$

We call C_g the support of g , and denote it by $\text{supp}(g)$.

Now, applying the celebrated vNM's Theorem together with (4) yields

Theorem 1 (von Neumann and Morgenstern). *Using the above assumptions, if \succeq on Γ satisfies Axioms A1, A2, and A3, then it may be represented by the cardinal utility function Λ defined on Γ by*

$$\Lambda(g) = \sum_{C \subseteq \text{supp}(g), \varphi(C) \neq 0} \Lambda(e_C) \varphi(C),$$

where φ stands for the Möbius inverse of g with respect to $2^{\text{supp}(g)}$.

Finally, define $H(\delta) = \Lambda(f_\delta)$ for each $\delta \in \mathbf{D}$, and $h(C) = \Lambda(e_C)$ for each $e_C \in \mathbf{E}$.

Corollary 1. *Using the assumptions of Theorem 1, \succeq on \mathbf{D} may be represented by the utility function H defined on \mathbf{D} by*

$$H(\delta) = \sum_{C \subseteq \delta(\mathbf{S})} h(C) \phi_\delta(C),$$

where ϕ_δ is the Möbius inverse of f_δ with respect to $2^{\delta(\mathbf{S})}$.

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