Advanced Multivariable Differential Calculus

Joseph Breen

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Department of Mathematics
University of California, Los Angeles

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Preface

These notes are based on lectures from Math 32AH, an honors multivariable differential calculus course at UCLA I taught in the fall of 2020. Briefly, the goal of these notes is to develop the theory of differentiation in arbitrary dimensions with more mathematical maturity than a typical calculus class, with an eye towards more advanced math. I wouldn't go so far as to call this a *multivariable analysis* text, but the level of rigor is fairly high. These notes borrow a fair amount in terms of overlying structure from *Calculus and Analysis in Euclidean Space* by Jerry Shurman, which was the official recommended text for the course. There are, however, a number of differences, ranging from notation to omission, inclusion, and presentation of most topics. The heart of the notes is Chapter 6, which discusses the theory of differentiation and all of its applications; the first five chapters essentially lay the necessary mathematical foundation of analysis and linear algebra.

As far as prerequisites are concerned, I only assume that you are comfortable with all of the usual topics in single variable calculus (limits, continuity, derivatives, optimization, integrals, sequences, series, Taylor polynomials, etc.). In particular, I *do not assume any prior knowledge of linear algebra*. Linear algebra naturally permeates the entire set of notes, but all of the necessary theory is introduced and explained. In general, I cover only the minimal amount of linear algebra needed, so it would be a bad idea to use these notes as a linear algebra reference.

Exercises at the end of each section correspond to homework and exam problems I assigned during the course. There are many topics that are standard in multivariable calculus courses (like the notion of projecting one vector onto another) that are introduced and studied in the exercises, so keep this in mind. Challenging (optional) exercises are marked with a $(*)$. I've also included some appendices that delve into more advanced (optional) topics.

Any comments, corrections, or suggestions are welcome!

Chapter 1

Preliminaries

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Chapter 2

Euclidean Space

Single variable calculus is the study of functions of one variable. In slightly fancier language, single variable calculus is the study of functions $f : \mathbb{R} \to \mathbb{R}$. In order to study functions of many variables — which is the goal of multivariable calculus — we first need to understand the underlying universe which hosts all of the forthcoming math. This universe is *Euclidean space*, a generalization of the set of real numbers $\mathbb{R}$ to *higher dimensions* (whatever that means!). This chapter introduces the basic algebra and geometry of Euclidean space.

2.1 The elements of Euclidean space

We begin with the definition of Euclidean space.

Definition 2.1. Define n**-dimensional Euclidean space**, read as "R-n", as follows:

$$\mathbb{R}^n := \{ (x_1, \ldots, x_n) \; : \; x_j \in \mathbb{R} \} .$$

Elements of $\mathbb{R}^n$ are called **vectors**, and elements of $\mathbb{R}$ are called **scalars**.

In words, n-dimensional Euclidean space is the set of tuples of n real numbers, and by definition its elements are vectors. You may have preconceived notions of a vector as being an arrow, but you shouldn't really think of them this way — at least, not initially. A vector is just an element of $\mathbb{R}^n$.

There are a number of common notations used to represent vectors. For example,

$$(x_1, \ldots, x_n) \quad \text{and} \quad \langle x_1, \ldots, x_n \rangle \quad \text{and} \quad \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

are all commonly used to denote an element of $\mathbb{R}^n$. I will likely use all three notations throughout these notes. You may object: *Joe, this is confusing! Why don't you just pick one and stick with it?* To that, I have three answers:

(i) I am too lazy to consistently stick with one notation.

(ii) Sometimes, one notation is more convenient than another, depending on the context of the problem.

(iii) An important skill as a mathematician is to be able to recognize and adapt to unfamiliar notation. There is almost never a universal notation for any mathematical object, and this will become apparent as you read more books, take more classes, and talk to more mathematicians. Learning to quickly recognize what one notation means in a certain context is extremely valuable![1]

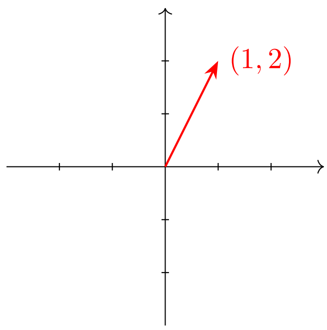
In any case, here are some down to earth examples of vectors.

Example 2.2.

$$(1,2) \in \mathbb{R}^2 \qquad \begin{pmatrix} 0 \\ \pi \\ 1000 \end{pmatrix} \in \mathbb{R}^3 \qquad \langle 500, 0, 0, -e \rangle \in \mathbb{R}^4$$

One thing I *will* consistently do in these notes is use boldface letters to indicate elements of $\mathbb{R}^n$. For example, I might say: "Let $\mathbf{x} \in \mathbb{R}^n$ be a vector." Letters that are not in boldface will usually denote scalars.

Although I said that you shouldn't think of vectors as arrows, you actually can (and should, sometimes) think of them that way. In particular, a vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ has a geometric interpretation as *the arrow emanating from the origin* $(0, \ldots, 0)$ *and terminating at the coordinate* $(x_1, \ldots, x_n)$. For example, in $\mathbb{R}^2$ we could draw the vector $(1,2)$ like this:



Sometimes it can be helpful to think of the vector $(x_1, \ldots, x_n)$ as an arrow with terminal point given by its coordinates, and other times it is better to think of it as simply its terminal point. However, I'll reiterate what I said above: a vector is simply an element of $\mathbb{R}^n$. An arrow is just a tool for visualization.

2.2 The algebra of Euclidean space

The set of real numbers $\mathbb{R}$ comes equipped with a number of algebraic operations like addition and multiplication, together with a host of rules like the distributive law that govern their interactions. Much of this algebraic structure extends naturally to $\mathbb{R}^n$, though some of it is more subtle (like multiplication). Our first task is to establish some basic algebraic definitions and rules in Euclidean space.

I'll begin by defining the notion of *vector addition* and *scalar multiplication*.

Definition 2.3. Let $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ be vectors, and let $\lambda \in R$ be a scalar.

(i) Vector addition is defined as follows:

$$\mathbf{x} + \mathbf{y} := (x_1 + y_1, \ldots, x_n + y_n) \in \mathbb{R}^n.$$

[1] One day, you may even have to interact with a physicist. If this ever happens, you should be prepared to encounter some highly unusual notation.

(ii) Scalar multiplication is defined as follows:

$$\lambda\mathbf{x} := (\lambda x_1, \dots, \lambda x_n) \in \mathbb{R}^n.$$

In words, to add two vectors (of the same dimension) you just add the corresponding components, and to multiply a vector by a scalar you just multiply each component by that scalar. Note that we have not yet defined how to multiply two vectors together; we'll talk about this later.

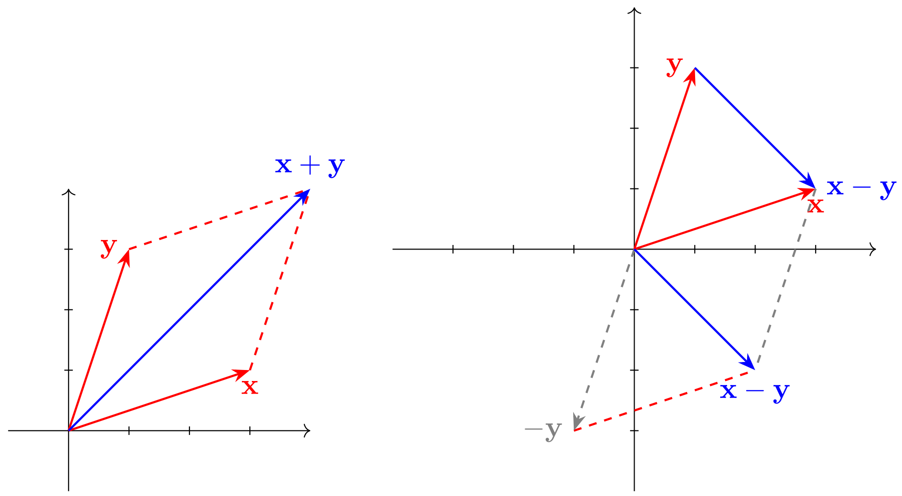
Example 2.4.

$$2\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}2\\4\\6\end{pmatrix} \qquad \begin{pmatrix}1\\0\\-1\end{pmatrix} + \begin{pmatrix}2\\2\\3\end{pmatrix} = \begin{pmatrix}3\\2\\2\end{pmatrix}$$

If we interpret vectors as arrows emanating from the origin, vector addition and scalar multiplication have nice geometric interpretations. In particular,

- the vector $\mathbf{x}+\mathbf{y}$ is the main diagonal of the parallelogram generated by $\mathbf{x}$ and $\mathbf{y}$, and
- the vector $\lambda\mathbf{x}$ is the arrow $\mathbf{x}$, stretched by a factor of λ.

See the figure below. One important consequence of the above two statements is that the vector $\mathbf{x}-\mathbf{y}$ is the off diagonal of the parallelogram generated by $\mathbf{x}$ and $\mathbf{y}$, travelling from $\mathbf{y}$ to $\mathbf{x}$, shifted appropriately.



The following definition should be clear.

Definition 2.5. Two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are **parallel** if there is a scalar $\lambda \in \mathbb{R}$ such that $\mathbf{x} = \lambda\mathbf{y}$ or $\lambda\mathbf{x} = \mathbf{y}$.

Remark 2.6. The above discussion should make one thing clear: sometimes, it can be helpful to visualize vectors as arrows emanating from points other than the origin. For example, it is natural to think of $\mathbf{x}-\mathbf{y}$ as the off diagonal arrow, beginning at the terminal point of $\mathbf{y}$. But I'll reiterate once more what I said above: a vector is simply an element of $\mathbb{R}^n$. Drawing arrows is just a way to visualize such elements. In particular, you should really think of *all vectors as emanating from the origin*.

We conclude this section with a summary of all of the algebraic rules governing vector addition and scalar multiplication. All of these should feel like natural extensions of corresponding rules in $\mathbb{R}$.

Proposition 2.7. *(Vector space axioms)*

(i) *For all* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$,
$$(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z}).$$

(ii) *Let* $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^n$. *Then* $\mathbf{0} + \mathbf{x} = \mathbf{x}$ *for all* $\mathbf{x} \in \mathbb{R}^n$.

(iii) *For all* $\mathbf{x} \in \mathbb{R}^n$, *there is a* $\mathbf{y} \in \mathbb{R}^n$ *(namely,* $-1\mathbf{x}$*) such that* $\mathbf{x} + \mathbf{y} = \mathbf{0}$.

(iv) *For all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,
$$\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}.$$

(v) *For all* $\lambda, \mu \in \mathbb{R}$ *and all* $\mathbf{x} \in \mathbb{R}^n$,
$$\lambda(\mu\mathbf{x}) = (\lambda\mu)\mathbf{x}.$$

(vi) *For all* $\mathbf{x} \in \mathbb{R}^n$, $1\mathbf{x} = \mathbf{x}$.

(vii) *For all* $\lambda, \mu \in \mathbb{R}$ *and all* $\mathbf{x} \in \mathbb{R}^n$,
$$(\lambda + \mu)\mathbf{x} = \lambda\mathbf{x} + \mu\mathbf{x}.$$

(viii) *For all* $\lambda \in \mathbb{R}$ *and all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,
$$\lambda(\mathbf{x} + \mathbf{y}) = \lambda\mathbf{x} + \lambda\mathbf{y}.$$

Proof. I'll prove one of these to show you how these arguments go, and I'll make you prove the rest of the properties as an exercise. They are all straightforward and follow from the well-known properties of $\mathbb{R}$.

Let's prove (viii). Fix $\lambda \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Write $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$. Then
$$\begin{aligned}
\lambda(\mathbf{x} + \mathbf{y}) &= \lambda \left[\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right] \\
&= \lambda \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix} \\
&= \begin{pmatrix} \lambda(x_1 + y_1) \\ \vdots \\ \lambda(x_n + y_n) \end{pmatrix} \\
&= \begin{pmatrix} \lambda x_1 + \lambda y_1 \\ \vdots \\ \lambda x_n + \lambda y_n \end{pmatrix} \\
&= \begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix} + \begin{pmatrix} \lambda y_1 \\ \vdots \\ \lambda y_n \end{pmatrix} \\
&= \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \lambda \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \\
&= \lambda\mathbf{x} + \lambda\mathbf{y}.
\end{aligned}$$

It may seem silly to write all of this out when this statement seems obvious, but this is a nontrivial fact that requires proof based on our definitions. In particular, in the second equality I used the definition of vector addition. In the third equality, I used the definition of scalar multiplication. In the fourth equality, I used the the distributive law of the real numbers, then in the fifth and sixth equalities I used the definitions of vector addition and scalar multiplication again.

The proofs of (i)-(vii) are left as an exercise. □

Remark 2.8. The reason I named this proposition the *vector space axioms* is because there is a more general notion of something called a *vector space*. Briefly and imprecisely, a vector space is any abstract set that satisfies (i)-(viii). For example, the set of functions $f : \mathbb{R} \to \mathbb{R}$ satisfies all of the above properties, and thus is a "vector space." I won't discuss abstract vector spaces in these notes — the only one that we care about is $\mathbb{R}^n$. If you're interested, you can look at Appendix A which discusses some more advanced linear algebra. Broadly speaking, linear algebra is the study of abstract vector spaces.

2.3 The geometry of Euclidean space

Now that we have mastered the basic algebra of Euclidean space, it's time to start discussing its *geometry*. By this, I mean things like *length* and *angle*.

2.3.1 Inner products

The object that gives rise to geometry in Euclidean space is something called an *inner product*. Roughly, an inner product is a way to "multiply" two vectors $\mathbf{x}$ and $\mathbf{y}$ to produce a scalar.

Definition 2.9. An **inner product** is a function $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfying:

(i) For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,
$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle .$$

(ii) For all $\mathbf{x} \in \mathbb{R}^n$, $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ and $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

(iii) For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$,
$$\begin{aligned} \langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle &= \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle \\ \langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle &= \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle \\ \langle \lambda \mathbf{x}, \mathbf{y} \rangle &= \lambda \langle \mathbf{x}, \mathbf{y} \rangle \\ \langle \mathbf{x}, \lambda \mathbf{y} \rangle &= \lambda \langle \mathbf{x}, \mathbf{y} \rangle . \end{aligned}$$

Remark 2.10. Property (i) is referred to as *symmetry*, property (ii) is *positive definitness*, and property (iii) is *bilinearity*. Note that, by symmetry, some of the conditions in (iii) are superfluous: bilinearity in the second component (equalities 2 and 4) follows from bilinearity in the first component (equalities 1 and 3).

The point of this definition is that an inner product is an operation that should obey the same rules as numerical multiplication. For example, numerical multiplication is symmetric: for any $x, y \in \mathbb{R}$, $xy = yx$.

It is important to note that in the definition above, I used the word "an." That suggests there are multiple different inner products, and this is indeed the case. However, there is a standard one defined as follows.

Definition 2.11. The **standard inner product**, also known as the **dot product**, is defined as follows: for $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} := x_1 y_1 + \cdots x_n y_n.$$

Typically I will write $\langle \mathbf{x}, \mathbf{y} \rangle$ to refer to the standard inner product. I will give you some examples of other inner products in the exercises, but throughout the notes we will only use the standard one. The above definition also introduces the alternative notation $\mathbf{x} \cdot \mathbf{y}$, which may feel more natural.

Example 2.12.

$$\left\langle \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\rangle = 1 \cdot 3 + 2 \cdot 4 = 11.$$

Even though I named the object above the *standard inner product*, we have to actually verify that it is an inner product. In other words, we have to show that it satisfies (i)-(iii) in Definition 2.9.

Proposition 2.13. *The standard inner product* $\langle \cdot, \cdot \rangle$, *, i.e., the dot product, is an inner product.*

Proof. I'll verify one of the properties here (property (ii)) and I will leave the others as an exercise.

Let $\mathbf{x} = (x_1, \ldots, x_n)$. Then

$$\langle \mathbf{x}, \mathbf{x} \rangle = x_1^2 + \cdots + x_n^2.$$

Since $x_j^2 \geq 0$ for all j, it follows that $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$.

Next, we prove the "if and only if" statement. Note $\langle \mathbf{0}, \mathbf{0} \rangle = 0^2 + \cdots 0^2 = 0$. Now, suppose that $\langle \mathbf{x}, \mathbf{x} \rangle = 0$. Then

$$x_1^2 + \cdots + x_n^2 = 0.$$

Since $x_j^2 \geq 0$, it necessarily follows that each $x_j = 0$ and thus $\mathbf{x} = \mathbf{0}$ as desired.

Properties (i) and (iii) are left as exercises. □

The reason that an inner product gives rise to geometric structure is because it allows to define the notion of *length* and *angle*.

Definition 2.14. The **length** or **magnitude** or **norm** of a vector $\mathbf{x} \in \mathbb{R}^n$ is

$$\|\mathbf{x}\| := \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}.$$

This definition makes mathematical sense by (ii) in Definition 2.9, and it should make intuitive sense because it coincides with the notion of the absolute value of a number: if $x \in \mathbb{R}$, then $|x| = \sqrt{x^2}$. In other words, for vectors in $\mathbb{R}$, the length/magnitude/norm is the same as the absolute value. Using the standard inner product, the magnitude of $(x_1, \ldots, x_n)$ is just the distance from the origin to the terminal point of the arrow using the usual distance formula.

Example 2.15.

$$\left\| \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\| = \sqrt{\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}} = \sqrt{1^2 + 2^2} = \sqrt{5}.$$

The following properties of the norm follow immediately from the definition of an inner product.

Proposition 2.16.

(i) For all $\mathbf{x} \in \mathbb{R}^n$, $\|\mathbf{x}\| \geq 0$ *and* $\|\mathbf{x}\| = 0$ *if and only if* $\mathbf{x} = \mathbf{0}$.

(ii) For all $\mathbf{x} \in \mathbb{R}^n$ *and* $\lambda \in \mathbb{R}$, $\|\lambda \mathbf{x}\| = |\lambda| \, \|\mathbf{x}\|$.

2.3.2 Some important inequalities

In order to talk about the angle between two vectors, we need to take a short but important detour to talk about the *Cauchy-Schwarz inequality*. Using this inequality, we will define the notion of angle and also prove another important geometric inequality: the *triangle inequality*.

The Cauchy-Schwarz inequality

Theorem 2.17 (Cauchy-Schwarz inequality)**.** *For all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

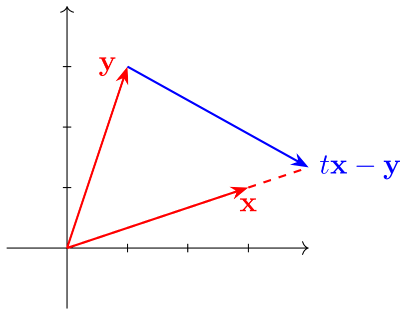
$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \, \|\mathbf{y}\| \, .$$

Remark 2.18. There are many different well-known proofs of the Cauchy-Schwarz inequality, including the one I'll present here. I've provided two other proofs in the form of exercises at the end of the chapter.

Proof. The following proof is one of the cutest proofs, but it is perhaps not obvious. Fix $\mathbf{x}$ and $\mathbf{y}$ and define a function $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$f(t) = \langle t\mathbf{x} - \mathbf{y}, t\mathbf{x} - \mathbf{y} \rangle \, .$$

Note that $f(t) = \|t\mathbf{x} - \mathbf{y}\|^2 \geq 0$.



On the other hand, we can expand f algebraically using the bilinearity properties of the inner product. Namely,

$$\begin{aligned}
f(t) &= \langle t\mathbf{x} - \mathbf{y}, t\mathbf{x} - \mathbf{y} \rangle \\
&= \langle t\mathbf{x}, t\mathbf{x} - \mathbf{y} \rangle - \langle \mathbf{y}, t\mathbf{x} - \mathbf{y} \rangle \\
&= \langle t\mathbf{x}, t\mathbf{x} \rangle - \langle t\mathbf{x}, \mathbf{y} \rangle - \langle \mathbf{y}, t\mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle \\
&= t^2 \langle \mathbf{x}, \mathbf{x} \rangle - t \langle \mathbf{x}, \mathbf{y} \rangle - t \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle \, .
\end{aligned}$$

In the last equality we treated t as a scalar, pulling it out of each argument of the inner product. Next, note that by symmetry the two middle terms are the same. Thus, simplifying the above expression gives

$$f(t) = \|\mathbf{x}\|^2 t^2 - 2 \langle \mathbf{x}, \mathbf{y} \rangle t + \|\mathbf{y}\|^2 \, .$$

This is a quadratic polynomial in the variable t! It looks funny because the constants are given by norms and inner products of vectors, but it's a quadratic polynomial nonetheless.

Since $f(t) \geq 0$, this quadratic polynomial cannot have two distinct real roots. It follows[2] that the discriminant of this polynomial is nonpositive: $b^2 - 4ac \leq 0$. This means that

$$(2\langle \mathbf{x}, \mathbf{y} \rangle)^2 - 4\|\mathbf{x}\|^2 \|\mathbf{y}\|^2 \leq 0.$$

After some simple algebraic manipulation, this becomes

$$(\langle \mathbf{x}, \mathbf{y} \rangle)^2 \leq \|\mathbf{x}\|^2 \|\mathbf{y}\|^2.$$

Taking square roots gives the desired inequality:

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \|\mathbf{y}\|.$$

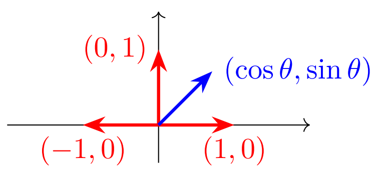
□

In words, the Cauchy-Schwarz inequality says that the inner product of two vectors is less than (in magnitude) the product of their norms. Typically, the statement of Cauchy-Schwarz comes with the following addendum: *equality occurs if and only if the vectors are parallel*. I encourage you to think about why this is true, but for the purpose of these notes I'm not too concerned about that part of the statement.

Angles

The innocent looking Cauchy-Schwarz inequality has a number of important applications, the first of which being the ability to define the angle between two vectors. Before giving the definition, I want to convince you that the standard inner product tells us something about our usual notion of angle. Let's compute the dot product of the vector $(1, 0) \in \mathbb{R}^2$ (a vector pointing in the positive x direction) with various other vectors:

$$\begin{pmatrix}1\\0\end{pmatrix} \cdot \begin{pmatrix}1\\0\end{pmatrix} = 1$$
$$\begin{pmatrix}1\\0\end{pmatrix} \cdot \begin{pmatrix}0\\1\end{pmatrix} = 0$$
$$\begin{pmatrix}1\\0\end{pmatrix} \cdot \begin{pmatrix}-1\\0\end{pmatrix} = -1$$
$$\begin{pmatrix}1\\0\end{pmatrix} \cdot \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} = \cos\theta.$$



The first computation shows that when we take the dot product of two vectors going in the same direction, we get a positive number. The second computations shows that the dot product of two perpendicular vectors is 0. The third computations shows that the dot product of two vectors going in opposite directions gives a negative number. Finally, the fourth computation generalizes all of these and suggests that the dot product somehow detects the cosine of the angle between the vectors. This should motivate the following definition.

[2] You may have to dig deep into the depths of your memory to remember the quadratic formula: if $at^2 + bt + c = 0$, then $t = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$.

Definition 2.19. Fix $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. The **angle between x and y** is the number

$$\theta = \arccos\left(\frac{\langle \mathbf{x}, \mathbf{y}\rangle}{\|\mathbf{x}\|\,\|\mathbf{y}\|}\right).$$

Remark 2.20. Recall that the domain for $\arccos(t)$ is $|t| \leq 1$. By Cauchy-Schwarz, $\left|\frac{\langle \mathbf{x},\mathbf{y}\rangle}{\|\mathbf{x}\|\|\mathbf{y}\|}\right| \leq 1$, and so this definition makes sense. In other words, we are allowed to define the angle between two vectors this way because of Cauchy-Schwarz.

This definition gives an alternative geometric way of computing the standard inner product of two vectors.

Proposition 2.21. *Fix $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and let θ be the angle between $\mathbf{x}$ and $\mathbf{y}$. Then*

$$\langle \mathbf{x}, \mathbf{y}\rangle = \|\mathbf{x}\|\,\|\mathbf{y}\| \cos\theta.$$

Proof. Rearrange the expression in the above definition. □

The next definition should also be clearly motivated by the above discussion.

Definition 2.22. Two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are **orthogonal** or **perpendicular** if $\langle \mathbf{x}, \mathbf{y}\rangle = 0$.

Remark 2.23. Rant incoming. Pay close attention to the fact that the above statement about orthogonality is a *definition*, and not a *proposition*. In other words, I have *defined* what it means to be orthogonal, using the inner product. I did not *conclude* that if two vectors are orthogonal, then their inner product is 0. This might seem weird, or it might seem like a trivial thing for me to start ranting about, but this is actually very important. You likely have a vast amount of geometric prejudice built up in two and three dimensions from your mathematical career so far. In particular, you had already studied things like distance and angles in a geometry class. But in this chapter, you should rid all of that from your mind. At the beginning of the chapter I defined $\mathbb{R}^n$ as a set, just by describing its elements. A priori, it has *no other structure*, algebraic or geometric. A priori, there is no notion of angle or length, even in the seemingly familiar $\mathbb{R}^2$. The point of this section is that *a choice of inner product gives rise to geometry*, and not the other way around. We define the notion of length and angle using an inner product, and if we picked a different inner product then we would get a different notion of length and angle. Anyway, this isn't really a big deal but it's an important mindset to have as you delve further into math. Carry on.

The triangle inequality

We conclude this section with an inequality of comparable importance to the Cauchy-Schwarz inequality.

Theorem 2.24 (Triangle inequality)**.** *For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,*

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|.$$

Proof. The moral of the following proof is that it is often easier to deal with squared norms as opposed to norms, the reason being that squared norms are naturally translated into inner products.

In particular, note that

$$\begin{aligned}\|\mathbf{x} + \mathbf{y}\|^2 &= \langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y}\rangle \\ &= \langle \mathbf{x}, \mathbf{x}\rangle + \langle \mathbf{y}, \mathbf{x}\rangle + \langle \mathbf{x}, \mathbf{y}\rangle + \langle \mathbf{y}, \mathbf{y}\rangle \\ &= \|\mathbf{x}\|^2 + 2\langle \mathbf{x}, \mathbf{y}\rangle + \|\mathbf{y}\|^2.\end{aligned}$$

Here we have used the bilinearity and symmetry properties of the inner product. By Cauchy-Schwarz, $\langle \mathbf{x}, \mathbf{y} \rangle \leq \|\mathbf{x}\| \|\mathbf{y}\|$. Applying this to the middle term above gives

$$\|\mathbf{x} + \mathbf{y}\|^2 \leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\| \|\mathbf{y}\| + \|\mathbf{y}\|^2 .$$

We can now cleverly factor the right hand side!

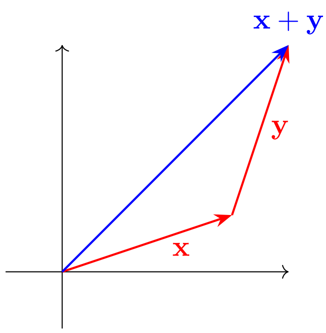
$$\|\mathbf{x} + \mathbf{y}\|^2 \leq (\|\mathbf{x}\| + \|\mathbf{y}\|)^2.$$

Taking square roots gives the desired inequality:

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\| .$$

□

This proof is a nice application of Cauchy-Schwarz, but it hides the geometric intuition of the statement. Consider the following picture of vectors $\mathbf{x}, \mathbf{y}$, and $\mathbf{x} + \mathbf{y}$. The triangle inequality says that traveling in a straight line (along $\mathbf{x} + \mathbf{y}$) covers a shorter amount of distance than taking a detour (along $\mathbf{x}$, then along $\mathbf{y}$).



2.4 Exercises

1. Let $\mathbf{a} = \langle 1, 2, 3 \rangle$ and $\mathbf{b} = \langle 1, 0, 1 \rangle$.

   (a) Compute $\mathbf{a} \cdot \mathbf{b}$. Is the angle between the vectors acute or obtuse?

   (b) Compute the angle between $\mathbf{a}$ and $\mathbf{b}$.

   (c) Compute a unit vector pointing in the same direction as $\mathbf{a}$.

2. Show that the points $(1, 3, 0)$, $(3, 2, 1)$, and $(4, 4, 1)$ form a right triangle in $\mathbb{R}^3$.

3. Describe all the vectors $\langle x, y, z \rangle$ which are orthogonal to the vector $\langle 1, 1, 1 \rangle$.

4. Prove that $\mathbb{R}^n$ satisfies the rest of the vector space axioms.

5. Prove that the standard inner product satisfies the rest of the inner product properties.

6. The goal of this exercise is to get you to think about the geometry of vector addition and how it interacts with inner products and magnitudes.

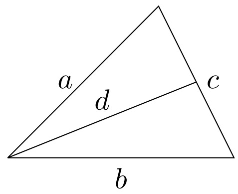
   (a) Suppose that $\mathbf{u}$ and $\mathbf{v}$ are unit vectors in $\mathbb{R}^n$, i.e., $\|\mathbf{u}\| = \|\mathbf{v}\| = 1$. Compute $\langle \mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle$. What does this result tell you about a rhombus?

   (b) Prove the *parallelogram identity*: for any vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$,

   $$\|\mathbf{a} + \mathbf{b}\|^2 + \|\mathbf{a} - \mathbf{b}\|^2 = 2\|\mathbf{a}\|^2 + 2\|\mathbf{b}\|^2 .$$

   Draw a picture and explain why this is named the parallelogram identity.

(c) Let a, b, c denote the side lengths of an arbitrary triangle. Let d be the length of the line segment from the midpoint of the c-side to the opposite vertex.



Show that
$$a^2 + b^2 = \frac{1}{2}c^2 + 2d^2.$$

7. Prove that
$$\big|\, \|\mathbf{a}\| - \|\mathbf{b}\| \,\big| \leq \|\mathbf{a} + \mathbf{b}\|$$
for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. This is the *reverse triangle inequality*.

[Hint: The normal triangle inequality may be helpful.]

8. Given two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, the **projection of $\mathbf{a}$ onto $\mathbf{b}$** is the vector
$$\operatorname{proj}_{\mathbf{b}} \mathbf{a} := \frac{\langle \mathbf{a}, \mathbf{b} \rangle}{\|\mathbf{b}\|^2} \mathbf{b}.$$

(a) Show that $\mathbf{a} - \operatorname{proj}_{\mathbf{b}} \mathbf{a}$ is orthogonal to $\mathbf{b}$.

(b) Draw a (generic) picture of $\mathbf{a}, \mathbf{b}, \operatorname{proj}_{\mathbf{b}} \mathbf{a}$, and $\mathbf{a} - \operatorname{proj}_{\mathbf{b}} \mathbf{a}$.

(c) Show that
$$\|\mathbf{a}\|^2 = \|\operatorname{proj}_{\mathbf{b}} \mathbf{a}\|^2 + \|\mathbf{a} - \operatorname{proj}_{\mathbf{b}} \mathbf{a}\|^2 .$$
What famous theorem have you just proven?

(d) Show that $\|\operatorname{proj}_{\mathbf{b}} \mathbf{a}\| \leq \|\mathbf{a}\|$ and then use this to give a proof of the Cauchy-Schwarz inequality:
$$|\langle \mathbf{a}, \mathbf{b} \rangle| \leq \|\mathbf{a}\| \, \|\mathbf{b}\| .$$

(e) Compute the distance from the point $(2, 3)$ to the line $y = \frac{1}{2}x$.

9. Fix $\theta \in [0, 2\pi)$ and define a map $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ by
$$T_\theta(x, y) = (x \cos\theta + y \sin\theta, -x \sin\theta + y \cos\theta) .$$

(a) Compute $T_{\frac{\pi}{2}}(1, 0)$ and $T_{\frac{\pi}{2}}(0, 1)$.

(b) Show that $\|T_\theta(x, y)\| = \|(x, y)\|$.

(c) Show that $\langle T_\theta(x_1, y_1), T_\theta(x_2, y_2) \rangle = \langle (x_1, y_1), (x_2, y_2) \rangle$.

(d) Give a geometric interpretation of (b) and (c), and then give a geometric description of T_θ as a whole.

10. A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subset \mathbb{R}^n$ is called **linearly independent** if
$$\lambda_1 \mathbf{v}_1 + \cdots + \lambda_k \mathbf{v}_k = \mathbf{0}$$
implies $\lambda_1 = \lambda_2 = \cdots = \lambda_k = 0$. In words, the set of vectors is linearly independent if each vector "points in a new direction." The goal of this exercise is to get you thinking about linear independence and to convince you that this verbal slogan is true.

(a) Show that a set of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\} \subset \mathbb{R}^n$ is linearly independent if and only if $\mathbf{v}_1$ and $\mathbf{v}_2$ are not parallel.

(b) Show that if $\mathbf{v}_j = \mathbf{0}$ for some j, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subset \mathbb{R}^n$ is not linearly independent.

(c) Give an example of three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ such that $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is linearly independent.

(d) Give an example of three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ such that $\{\mathbf{a}, \mathbf{b}\}$ is linearly independent, but $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is not linearly independent.

(e) Show that any set of 3 vectors in $\mathbb{R}^2$ is not linearly independent.

11. ($*$) Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ be linearly independent vectors (here $n \geq 3$). Write down an expression involving $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ which gives a nonzero vector orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

   [Hint: It may seem odd that I gave you a third vector $\mathbf{c}$*, seemingly unrelated to* $\mathbf{a}$ *and* $\mathbf{b}$*, and I only want a vector orthogonal to* $\mathbf{a}$ *and* $\mathbf{b}$*. Treat the vector* $\mathbf{c}$ *as a starting point from which to produce the desired vector.]*

12. ($*$) Show that for any positive numbers x, y, z, w,

$$(x+y+z+w)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{w}\right) \geq 16.$$

13. ($*$) In this problem, I'll walk you through the proof of a generalization of the Cauchy-Schwarz inequality called *Hölder's inequality*. It says that if $p, q > 1$ such that $\frac{1}{p}+\frac{1}{q} = 1$, then

$$|\langle \mathbf{a}, \mathbf{b} \rangle| \leq \left(\sum_{j=1}^{n} |a_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} |b_j|^q\right)^{\frac{1}{q}}.$$

   Cauchy-Schwarz is the special case $p = q = 2$.

   (a) Show that the above inequality is true if either $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$.

   (b) Show that

$$|\langle \mathbf{a}, \mathbf{b} \rangle| \leq \sum_{j=1}^{n} |a_j b_j|.$$

   (c) Use concavity of $\ln x$ to show that if $x, y > 0$, then

$$\ln(xy) \leq \ln\left(\frac{1}{p}x^p + \frac{1}{q}y^q\right).$$

   [Hint: A function is ***concave*** *if for all* $t \in [0,1]$, $tf(x)+(1-t)f(y) \leq f(tx+(1-t)y)$*.]*

   (d) Prove that for $x, y > 0$,

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}.$$

   (e) Let

$$x = \frac{|a_j|}{\left(\sum_{j=1}^{n} |a_j|^p\right)^{\frac{1}{p}}} \quad \text{and} \quad y = \frac{|b_j|}{\left(\sum_{j=1}^{n} |b_j|^q\right)^{\frac{1}{q}}}$$

   and apply the inequality in the previous part to prove Hölder's inequality.

Chapter 3

Some Analysis

For the rest of these notes, we will primarily be interested in studying functions of several variables:

$$f : \mathbb{R}^n \to \mathbb{R}^m.$$

For example, $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y, z) = xy + e^z$ and $g : \mathbb{R}^2 \to \mathbb{R}^3$ given by $g(x, y) = (xy, y + z, 3)$ are such functions. In particular, our goal is to extend all of the familiar techniques and tools from single variable calculus to these multivariable functions.

Here is a list of single variable techniques we can use to study functions $f : \mathbb{R} \to \mathbb{R}$:

- We can take limits of functions.
- We can investigate continuity of functions.
- We can take derivatives of functions and compute tangent lines.
- We can find local minima and maxima to solve optimization problems.
- We can use the derivative to investigate local invertibility.

By the end of these notes, we will be able to do all of the above (and hopefully more!) for functions $f : \mathbb{R}^n \to \mathbb{R}^m$. To do so semi-rigorously requires a small amount of analysis. The purpose of this chapter is to discuss the necessary theory of limits and continuity of multivariable functions.

3.1 Limits of sequences

As a reader, you are likely already familiar with sequences $\{a_n\} = \{a_1, a_2, \dots\}$ of real numbers, at least at the level of calculus. Given an infinite sequence, we care about its limiting behavior as n approaches infinity. For example, given the sequence $a_n = \frac{1}{n^3+1}$ we can observe that as $n \to \infty$ the denominator grows very large while the numerator remains constant. Thus,

$$\lim_{n\to\infty} \frac{1}{n^3 + 1} = 0.$$

That's all well and good, but we want to approach the study of sequential convergence with more mathematical rigor. That is the goal of this section.

Now that we're comfortable with $\mathbb{R}^n$, the first formal definition I will give is what it means for a sequence of *vectors* to converge.

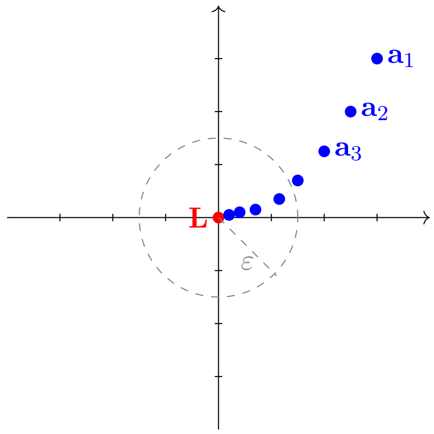
Definition 3.1. Let $\{\mathbf{a}_n\} \subset \mathbb{R}^m$ be a sequence of vectors in $\mathbb{R}^m$. We say that **the sequence converges to the vector $\mathbf{L} \in \mathbb{R}^m$**, written $\mathbf{a}_n \to \mathbf{L}$ or $\lim_{n\to\infty} \mathbf{a}_n = \mathbf{L}$, if: for any $\varepsilon > 0$, there is an N such that $n > N$ implies

$$\|\mathbf{a}_n - \mathbf{L}\| < \varepsilon.$$

A definition like this takes some time to parse if it is your first time seeing it. In words, this definition is a game: I give you a small number ε, and I tell you that I want your sequence to be ε-close to the vector $\mathbf{L}$. If you can always find a point in the sequence where this is permanently achieved, then you win. Indeed, note that the set

$$\{\,\mathbf{x} \in \mathbb{R}^m \;:\; \|\mathbf{x} - \mathbf{L}\| < \varepsilon\,\}$$

is a ball of radius ε centered around $\mathbf{L}$.



Example 3.2. Let's prove using the formal definition above that

$$\lim_{n\to\infty} \frac{1}{n^3+1} = 0.$$

Here $a_n = \frac{1}{n^3+1}$ and $L = 0$. These are numbers instead of vectors, but numbers are just vectors in $\mathbb{R}^1$ after all!

Let $\varepsilon > 0$ be a small, fixed number. According to the above definition, we want to show that there is some choice of N such that

$$\left|\frac{1}{n^3+1} - 0\right| < \varepsilon$$

if $n > N$. Here the absolute value bars have replaced the norm, since we're dealing with numbers. Note that $\left|\frac{1}{n^3+1} - 0\right| < \varepsilon$ is equivalent to

$$\frac{1}{n^3+1} < \varepsilon$$

since we assume that $n > 0$. This is further equivalent to $\frac{1}{\varepsilon} < n^3 + 1$ and thus

$$n > \left(\frac{1}{\varepsilon} - 1\right)^{\frac{1}{3}}.$$

This is precisely the kind of condition we seek! Let N be any integer larger than $\left(\frac{1}{\varepsilon} - 1\right)^{\frac{1}{3}}$. Then if $n > N$, the above work implies that

$$\left|\frac{1}{n^3+1} - 0\right| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we have shown that $\lim_{n\to\infty} \frac{1}{n^3+1} = 0$.

To get some more practice with this definition, let's prove the following well-known properties of limits.

Proposition 3.3. *If* $\mathbf{a}_n \to \mathbf{L} \in \mathbb{R}^m$ *and* $\mathbf{b}_n \to \mathbf{K} \in \mathbb{R}^m$*, then* $\mathbf{a}_n + \mathbf{b}_n \to \mathbf{L} + \mathbf{K} \in \mathbb{R}^m$*. In other words, the limit of a sum is the sum of the limits, provided both limits exist.*

Proof. This proposition is a nice application of the triangle inequality from the previous chapter. Let $\varepsilon > 0$. We want to show that for some choice N,

$$\|(\mathbf{a}_n + \mathbf{b}_n) - (\mathbf{L} + \mathbf{K})\| < \varepsilon$$

when $n > N$. Note that

$$\begin{aligned} \|(\mathbf{a}_n + \mathbf{b}_n) - (\mathbf{L} + \mathbf{K})\| &= \|(\mathbf{a}_n - \mathbf{L}) + (\mathbf{b}_n - \mathbf{K})\| \\ &\leq \|\mathbf{a}_n - \mathbf{L}\| + \|\mathbf{b}_n - \mathbf{K}\|. \end{aligned}$$

Here, we used the triangle inequality in the last line. The point of this rearrangement is that we know $\mathbf{a}_n \to \mathbf{L}$ and $\mathbf{b}_n \to \mathbf{K}$, so heuristically we can make each of the above terms small. Indeed, pick N_1 so that $n > N_1$ implies $\|\mathbf{a}_n - \mathbf{L}\| < \frac{\varepsilon}{2}$ and pick N_2 such that $n > N_2$ implies $\|\mathbf{b}_n - \mathbf{K}\| < \frac{\varepsilon}{2}$. Each of these choices are possible because $\mathbf{a}_n \to \mathbf{L}$ and $\mathbf{b}_n \to \mathbf{K}$.

Then if $n > \max(N_1, N_2)$, we have

$$\|(\mathbf{a}_n + \mathbf{b}_n) - (\mathbf{L} + \mathbf{K})\| \leq \|\mathbf{a}_n - \mathbf{L}\| + \|\mathbf{b}_n - \mathbf{K}\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we have shown $\mathbf{a}_n + \mathbf{b}_n \to \mathbf{L} + \mathbf{K}$. □

Other well known limit properties are left as exercises to the reader.

3.2 Limits of functions

The next limit generalization we are interested in is that of a limit of a function. The following definition is similar in spirit to that of the limit of a sequence.

Definition 3.4. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and let $\mathbf{x}_0 \in \mathbb{R}^n$ and $\mathbf{L} \in \mathbb{R}^m$. We say that

$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = \mathbf{L}$$

if for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$$

implies

$$\|f(\mathbf{x}) - \mathbf{L}\| < \varepsilon.$$

The intuition here is essentially the same as before. In words, this definition is a game: I give you a small number ε, and you want the values of f to be ε-close to $\mathbf{L}$ to win the game. If you can find a $\delta > 0$ such that f is ε-close in a δ-neighborhood of $\mathbf{x}_0$, then you win.

Fortunately, in these notes we typically will not use this definition directly. Instead, there is a convenient characterization of function limits in terms of sequences. In words, the following proposition says: to check if a limit of a function exists, approach that point along all possible sequences.

Proposition 3.5. *Let* $f : \mathbb{R}^n \to \mathbb{R}^m$*,* $\mathbf{x}_0 \in \mathbb{R}^n$*, and* $\mathbf{L} \in \mathbb{R}^m$*. Then*

$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = \mathbf{L}$$

if and only if for all sequences $\mathbf{a}_n \to \mathbf{x}_0$*,*

$$\lim_{n \to \infty} f(\mathbf{a}_n) = \mathbf{L}.$$

Proof. This is left as a challenge exercise to the reader. For now, hopefully the statement seems believable! $\square$

An important consequence of this result is the following observation.

Corollary 3.6. *If* $\mathbf{a}_n, \mathbf{b}_n \to \mathbf{x}_0$ *and*

$$\lim_{n\to\infty} f(\mathbf{a}_n) \neq \lim_{n\to\infty} f(\mathbf{b}_n)$$

then $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})$ *does not exist.*

In other words, if you can find two different sequential approaches to the point $\mathbf{x}_0$ that give you different limits, you have shown that the limit cannot exist. The following examples will demonstrate the utility of this fact.

Example 3.7. Consider the following limit:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}.$$

Here we are considering the function $f : \mathbb{R}^2 \setminus \{\mathbf{0}\} \to \mathbb{R}$ defined by $f(x,y) = \frac{xy}{x^2+y^2}$ and $\mathbf{x}_0 = (0,0)$. I claim that the limit does not exist, and we will show this by finding two different sequences of vectors that approach the origin which yield different limits.

First, consider $\mathbf{a}_n = (\frac{1}{n}, 0)$. Then $\mathbf{a}_n \to \mathbf{0}$. Note that

$$f(\mathbf{a}_n) = \frac{\frac{1}{n}\cdot 0}{(\frac{1}{n})^2 + 0^2} = 0.$$

Thus, $\lim_{n\to\infty} f(\mathbf{a}_n) = 0$. In words, we have approached the origin along the x-axis, and we got 0. The next natural choice of approach to take is along the y-axis, so let $\mathbf{b}_n = (0, \frac{1}{n})$. Then $\mathbf{b}_n \to \mathbf{0}$, and

$$f(\mathbf{b}_n) = \frac{0\cdot\frac{1}{n}}{0^2 + (\frac{1}{n})^2} = 0.$$

So $\lim_{n\to\infty} f(\mathbf{b}_n) = 0$. This is an important point: *even though we approach the origin along both the x and y axes and 0, this does not mean the limit is 0!*

Indeed, let $\mathbf{c}_n = (\frac{1}{n}, \frac{1}{n})$, which approaches the origin along the path $y = x$. Then $\mathbf{c}_n \to \mathbf{0}$, and

$$f(\mathbf{c}_n) = \frac{(\frac{1}{n})^2}{(\frac{1}{n})^2 + (\frac{1}{n})^2} = \frac{1}{2}.$$

Thus, $\lim_{n\to\infty} f(\mathbf{c}_n) = 0$. Since $\frac{1}{2} \neq 0$, the overall limit does not exist!

I'll emphasize one more time that in order for a multivariable limit to exist, it must exist yield the same value *along all possible approaches*. The next example we will see shows that even approaching along all lines is not enough! Before that, I want to remark that oftentimes it is convenient to supersede the sequential approach with an equivalent smooth path approach, rather than with a sequence. For example, approaching $(0,0)$ along the sequence $(\frac{1}{n}, 0)$ is the same as sending $x \to 0$ along the line $y = 0$, or approaching $(1,0)$ via the sequence $(1+\frac{1}{n}, \frac{1}{n})$ is the same as sending $x \to 1$ along the line $y = x - 1$, etc. We will demonstrate this approach in the next example.

Example 3.8. Consider the following limit:

$$\lim_{(x,y)\to(0,0)} \frac{x^4y}{x^8+y^2}.$$

I claim that approaching the origin along *any* line with give a value of 0. We could check a handful of lines individually ($y=0$, $x=0$, $y=x$, etc.) but to make my point I'll deal with a whole family of lines in one go. In particular, let's approach $(0,0)$ along the line $y=mx$, where $m\neq 0$ is any fixed number. This handles all linear approaches, with the exception of the x-axis and y-axis.

Approaching along $y=mx$ gives

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^4(mx)}{x^8+(mx)^2} &= \lim_{x\to 0} \frac{mx^5}{x^8+m^2x^2} \\ &= \lim_{x\to 0} \frac{mx^3}{x^6+m^2} \\ &= \frac{0}{0+m^2} \\ &= 0.\end{aligned}$$

Thus, approaching along any such line $y=mx$ gives 0. If we approach along $y=0$, we get

$$\lim_{(x,y)\to(0,0)} \frac{x^4\cdot 0}{x^8+(0)^2} \lim_{x\to 0} 0 = 0$$

and along $x=0$ we similarly get

$$\lim_{(x,y)\to(0,0)} \frac{0^4\cdot y}{0^8+y^2} = \lim_{y\to 0} 0 = 0.$$

Thus approaching along all lines gives 0. Unfortunately, this does not imply that the overall limit is 0! When we say "all possible approaches" we really mean *all*, including nonlinear approaches.

In particular, if we approach the origin along $y=x^4$, we get

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^4\cdot x^4}{x^8+(x^4)^2} &= \lim_{x\to 0} \frac{x^8}{x^8+x^8} \\ &= \lim_{x\to 0} \frac{x^8}{2x^8} \\ &= \lim_{x\to 0} \frac{1}{2} \\ &= \frac{1}{2}.\end{aligned}$$

Since $\frac{1}{2}\neq 0$, we have found at least two different paths that yield different values and therefore the limit does not exist!

Let's look at a few more examples to see some more techniques involved in computing multivariable limits.

Example 3.9. Consider the following limit:

$$\lim_{(x,y,z)\to(0,0,0)} \frac{\sin(x^2+y^2+z^2)}{x^2+y^2+z^2}.$$

The trick here is turn this into a single variable limit. Indeed, note that

$$x^2+y^2+z^2 = \|(x,y,z)\|^2.$$

Thus, we may write

$$\lim_{(x,y,z)\to(0,0,0)} \frac{\sin(x^2+y^2+z^2)}{x^2+y^2+z^2} = \lim_{\mathbf{x}\to\mathbf{0}} \frac{\sin(\|\mathbf{x}\|^2)}{\|\mathbf{x}\|^2}.$$

Let $t = \|\mathbf{x}\|^2$. Then as $\mathbf{x} \to \mathbf{0}$, $t \to 0^+$. Thus, we may further write

$$\lim_{\mathbf{x}\to\mathbf{0}} \frac{\sin(\|\mathbf{x}\|^2)}{\|\mathbf{x}\|^2} = \lim_{t\to 0^+} \frac{\sin t}{t}.$$

With that, we have converted the multivariable limit in a single variable limit with a variable substitution! Now we have single variable techniques, like L'Hopital's rule, at our disposable. Indeed, $\lim_{t\to 0^+} \frac{\sin t}{t}$ is indeterminate of type $\frac{0}{0}$ and therefore by L'Hopital's rule

$$\lim_{t\to 0^+} \frac{\sin t}{t} = \lim_{t\to 0^+} \frac{\cos t}{1} = 1.$$

Example 3.10. Consider the following limit:

$$\lim_{(x,y)\to(0,0)} \frac{x^4y^2}{x^4+3y^6}.$$

You can check that approaching along all lines yields 0. You can also try to find some nonlinear paths to break the limit like in the above example, but you'll have a much more difficult time. It turns out that this limit does exist and equals 0. Instead of a variable substitution, we will prove this by a squeeze theorem argument (which is the only real way to prove that a multivariable limit exists).

Note that

$$0 \le \frac{x^4y^2}{x^4+3y^6} \le \frac{x^4y^2}{x^4} \le y^2.$$

These inequalities are valid since all powers are even and thus all quantities in sight are positive. Since $\lim_{(x,y)\to(0,0)} y^2 = \lim_{y\to 0} y^2 = 0$, we have

$$0 \le \lim_{(x,y)\to(0,0)} \frac{x^4y^2}{x^4+3y^6} \le \lim_{(x,y)\to(0,0)} y^2 = 0$$

and therefore

$$\lim_{(x,y)\to(0,0)} \frac{x^4y^2}{x^4+3y^6} = 0.$$

Sometimes a squeeze theorem argument can be a bit more delicate, especially if you have odd powers; some care needs to be taken in dealing with positive and negative cases.

Typically, you can handle this easily with absolute values.

Example 3.11. Consider the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}.$$

As before, you can investigate various paths to build a hunch that the limit exists and equals 0. Mimicking the previous example, you may be tempted to write down the following inequality:

$$0 \le \frac{x^3}{x^2+y^2} \le \frac{x^3}{x^2} = x.$$

This string of equalities is *entirely wrong*, since x and x^3 can be negative. The fix is to just use absolute values. Here is a true inequality:

$$0 \le \left|\frac{x^3}{x^2+y^2}\right| \le \left|\frac{x^3}{x^2}\right| = |x|.$$

Thus,

$$0 \le \lim_{(x,y)\to(0,0)} \left|\frac{x^3}{x^2+y^2}\right| \le \lim_{(x,y)\to(0,0)} |x| = 0.$$

Formally, what we have shown is

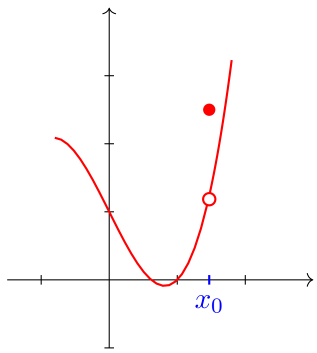
$$\lim_{(x,y)\to(0,0)} \left|\frac{x^3}{x^2+y^2}\right| = 0.$$

Essentially by definition of convergence of a limit (from a different perspective, continuity of the absolute value function; see the next section) this implies

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = 0.$$

3.3 Continuity

Recall from single variable calculus that a function $f : \mathbb{R} \to \mathbb{R}$ is *continuous* at a point $x \in \mathbb{R}$ if $\lim_{x\to x_0} f(x) = f(x_0)$. This statement implies two things: the limit on the left side of the equality *exists*, *and* it equals the value of the function at that point. For example, one could have a function where $\lim_{x\to x_0} f(x)$ exists but does not equal $f(x_0)$:



Such a function would not be continuous at x_0, even though the limit of the function exists at that point.

The definition of continuity for multivariable limits is essentially the same.

Definition 3.12. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is **continuous at** $\mathbf{x}_0 \in \mathbb{R}^n$ if

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = f(\mathbf{x}_0).$$

We say f **is continuous** (on $\mathbb{R}^n$) if f is continuous at all $\mathbf{x}_0 \in \mathbb{R}^n$.

Using the sequential characterization of multivariable limits from the previous section, we immediately have the following very convenient characterization of continuity.

Proposition 3.13 (Sequential continuity). *A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $\mathbf{x}_0$ if and only if, for every sequence $\mathbf{x}_n \to \mathbf{x}_0$,*

$$\lim_{n\to\infty} f(\mathbf{x}_n) = f(\mathbf{x}_0).$$

This may seem like a silly thing to explicitly say, but it is surprisingly helpful in proving statements about continuity. For example, we can easily prove the familiar property that the sum of two continuous functions is also continuous.

Proposition 3.14. *Suppose that $f, g : \mathbb{R}^n \to \mathbb{R}^m$ are both continuous at $\mathbf{x}_0 \in \mathbb{R}^n$. Then $f + g$ is continuous at $\mathbf{x}_0$.*

Proof. Using the sequential characterization of continuity, we don't have to mess around with ε's or δ's or anything like that. We simply choose an arbitrary sequence $\mathbf{x}_n$ such that $\mathbf{x}_n \to \mathbf{x}_0$. Then

$$\begin{aligned}
\lim_{\mathbf{x}_n\to\mathbf{x}_0} (f+g)(\mathbf{x}) &= \lim_{\mathbf{x}_n\to\mathbf{x}_0} (f(\mathbf{x}) + g(\mathbf{x})) \\
&= \lim_{\mathbf{x}_n\to\mathbf{x}_0} f(\mathbf{x}) + \lim_{\mathbf{x}_n\to\mathbf{x}_0} g(\mathbf{x}) \\
&= f(\mathbf{x}_0) + g(\mathbf{x}_0) \\
&= (f+g)(\mathbf{x}_0).
\end{aligned}$$

Here we used the additivity property of limits from earlier and the assumption that f and g are continuous at $\mathbf{x}_0$. Since this computation holds for all sequences $\mathbf{x}_n \to \mathbf{x}_0$, it follows that $f + g$ is continuous at $\mathbf{x}_0$. $\square$

Proposition 3.15. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function. Write*

$$f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$$

where each $f_j : \mathbb{R}^n \to \mathbb{R}$. If each f_j is continuous, then f is continuous.

Solution. Suppose that each f_j is continuous. Fix $\mathbf{x}_0 \in \mathbb{R}^n$ and let $\mathbf{x}_n$ be any sequence in $\mathbb{R}^n$ such that $\mathbf{x}_n \to \mathbf{x}_0$. Then we know from lecture that $\lim_{n\to\infty} f_j(\mathbf{x}_n) = f(\mathbf{x}_0)$ for all j. Thus,

$$\begin{aligned}
\lim_{n\to\infty} f(\mathbf{x}_n) &= \lim_{n\to\infty} (f_1(\mathbf{x}_n), \ldots, f_m(\mathbf{x}_n)) \\
&= \Big(\lim_{n\to\infty} f_1(\mathbf{x}_n), \ldots, \lim_{n\to\infty} f_m(\mathbf{x}_n)\Big) \\
&= (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})) \\
&= f(\mathbf{x}_0).
\end{aligned}$$

Since $\lim_{n\to\infty} f(\mathbf{a}_n) = f(\mathbf{x}_0)$ for all sequences $\mathbf{x}_n \to \mathbf{x}_0$, it follows that f is continuous at $\mathbf{x}_0$. Since $\mathbf{x}_0$ was arbitrary, it follows that f is continuous. $\square$

Other similar properties of continuity (e.g. the product of continuous functions is continuous) are left to be stated and proved in the exercises.

Example 3.16. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{y^5}{x^4+y^4} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0) \end{cases}.$$

We claim that f is not continuous at $(0,0)$. Indeed, we will show that

$$\lim_{(x,y)\to(0,0)} \frac{y^5}{x^4+y^4} \neq 1.$$

Note that if we approach the origin along $y = 0$, we get

$$\lim_{(x,y)\to(0,0)} \frac{0^5}{x^4+0^4} = \lim_{y\to 0} 0 = 0.$$

Since we have found one approach to the origin that gives us a number other than 1, it follows that

$$\lim_{(x,y)\to(0,0)} \frac{y^5}{x^4+y^4} \neq 1.$$

Therefore, f is not continuous at $(0,0)$.

Observe the subtlety here: it is not necessary to actually show that the limit exists at the origin to show the function is not continuous. The limit does exist and equals 0, but all we had to do is *show that the limit cannot be* 1. That is all that is necessary to conclude that the function is *not* continuous.

3.4 Exercises

1. Use the formal definition of a limit to compute the following limits:

   (a)
   $$\lim_{n\to\infty} \frac{n}{n+1}.$$

   (b)
   $$\lim_{n\to\infty} n^2.$$

   (c)
   $$\lim_{n\to\infty} \frac{n^2-n+1}{n^2+1}.$$

2. Evaluate the following limits, or show that they don't exist.

   (a)
   $$\lim_{(x,y)\to(0,0)} \frac{y^2}{x^2+y^2}.$$

   (b)
   $$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{xy^2}.$$

   (c)
   $$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$$

(d)
$$\lim_{(x,y)\to(0,0)} \frac{y\sqrt{|x|}}{|x|+y^2}$$

(e)
$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^4}{x+y^2}.$$

(f)
$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{xyzw}{x^4+2y^4+3z^4+4w^4}.$$

3. Let $f:\mathbb{R}^2\to\mathbb{R}$ be a function defined by
$$f(x,y)=\begin{cases}\cos x & \text{if } \sqrt{x^2+y^2}=\frac{1}{n} \text{ for some } n=1,2,3,\dots\\ 0 & \text{otherwise}\end{cases}.$$
Compute $\lim_{(x,y)\to(0,0)} f(x,y)$ or show that it does not exist.

4. For $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$,
$$\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{0},\mathbf{0})} \frac{\langle\mathbf{x},\mathbf{y}\rangle\,\mathbf{x}-\|\mathbf{y}\|\,\mathbf{y}}{\|\mathbf{x}\|+\|\mathbf{y}\|}$$
Note that $(\mathbf{x},\mathbf{y})$ and $(\mathbf{0},\mathbf{0})$ are elements of $\mathbb{R}^n\times\mathbb{R}^n$.

5. Compute the following limit or show that it does not exist.
$$\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{x\sin y}.$$

6. Let $h:\mathbb{R}^2\to\mathbb{R}$ be defined by
$$h(x,y)=\begin{cases}1 & (x,y)\neq(0,0) \text{ with } x+y \text{ rational}\\ 0 & (x,y)\neq(0,0) \text{ with } x+y \text{ irrational}\end{cases}.$$
Compute $\lim_{(x,y)\to(0,0)} h(x,y)$ or show that it doesn't exist.

7. Suppose that $f:\mathbb{R}^n\to\mathbb{R}^m$ and $g:\mathbb{R}^m\to\mathbb{R}^k$ are continuous functions. Prove that the composition $g\circ f:\mathbb{R}^n\to\mathbb{R}^k$ is continuous.

8. Show that if $f,g:\mathbb{R}^n\to R^m$ are continuous functions, then $\langle f,g\rangle:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ is continuous.

9. Show that if $f:\mathbb{R}^n\to\mathbb{R}$ and $g:\mathbb{R}^n\to\mathbb{R}$ are continuous and $g(\mathbf{x})\neq 0$ for all $\mathbf{x}$, then $\frac{f}{g}$ is continuous.

10. For each of the following functions, either prove that there is a choice of $c\in\mathbb{R}$ such that the function is continuous, or show that for any choice of c, the function is not continuous.

(a) Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by
$$f(x,y)=\begin{cases}\frac{xy^2}{(x-1)^2+y^2} & (x,y)\neq(1,0)\\ c & (x,y)=(1,0)\end{cases}.$$

(b) Let $g:\mathbb{R}^4\to\mathbb{R}$ be defined by
$$g(x,y,z,w)=\begin{cases}\frac{zx^2+zy^4}{x^2+y^2+z^4+w^4} & (x,y,z,w)\neq(0,0,0,0)\\ c & (x,y,z,w)=(0,0,0,0)\end{cases}.$$

11. For each of the following statements, say whether they are *true* or *false*. If a statement is true, prove it. If a statement is false, provide a counter example.

    (a) If the image of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is path connected,[1] then f is continuous.

    (b) If the graph[2] of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is path connected, then f is continuous.

12. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is a function such that the limit as $(x, y) \to (0, 0)$ along all paths of the form $y = x^n$ and $x = y^m$ is 0. Here, $n, m > 0$ are any whole numbers. In other words,

$$\lim_{x\to 0} f(x, x^n) = \lim_{y\to 0} f(y^m, y) = 0$$

    for all choices of $n, m > 0$. In even more words, this means that the limit of f as you approach the origin along all "polynomial curves" is 0.

    Is it true that $\lim_{(x,y)\to(0,0)} f(x, y) = 0$? If it is, prove it. If it's false, give a counter example.

13. ($*$) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is a function such that the limit as $(x, y) \to (0, 0)$ along all paths of the form $|y|^\beta = |x|^\alpha$ and along the axes $x = 0$ and $y = 0$. Here, $\alpha, \beta > 0$ are any positive real numbers. In other words,

$$\lim_{x\to 0} f(x, |x|^\gamma) = \lim_{y\to 0} f(|y|^\gamma, y) = \lim_{x\to 0} f(x, 0) = \lim_{x\to 0} f(0, y)$$

    for all real numbers $\gamma > 0$.

    Is it true that $\lim_{(x,y)\to(0,0)} f(x, y) = 0$? If it is, prove it. If it's false, give a counter example.

[1] Here, "path connected" just means that any two points are connected by a path. Don't worry about anything doing anything super formal.

[2] The **graph** of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is the set

$$\text{graph}(f) = \{\, (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^m \;:\; \mathbf{y} = f(\mathbf{x}) \,\}\,.$$

Chapter 4

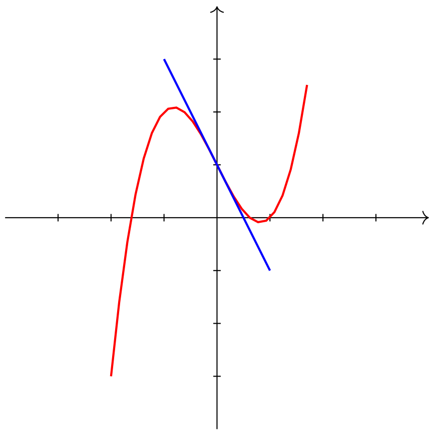
Some Linear Algebra

In this chapter, we will develop a small amount of linear algebra — just enough for our multivariable calculus goals. The theory of linear algebra is much deeper than what is presented in this chapter, and if you find any of this material interesting I encourage you to check out Appendix A or a text like *Linear Algebra Done Right* by Sheldon Axler.

Before diving into the theory, I want to give some motivation as to what linear algebra *is* and why we need it to study multivariable calculus. After all, multivariable calculus often comes *before* linear algebra in a standard engineering-level math curriculum. I want to convince you that this is nonsense.

Here is the motto of single variable calculus:

Curvy things are hard to understand, so let's approximate them with straight things.



This is the idea behind derivatives, tangent lines, optimization, and so on. This approach is effective because straight things — in other words, linear things — in one dimension are easy to understand. Linear functions are studied in a high school algebra class! To reiterate: single variable calculus is the approximation of (one-variable) curvy things with (one-variable) straight things, and because we understand (one-variable) straight things, this is effective.

In multivariable calculus, we want to understand functions of many variables. Precisely, we are interested in functions $f : \mathbb{R}^n \to \mathbb{R}^m$. Our motto will be the same: curvy things are hard, so let's approximate them with straight things. But in order for this to be effective, *we need to understand straight things in higher dimensions*. Linear algebra is exactly the study of straight things in higher dimensions, so obviously we should learn some linear algebra before we do multivariable calculus!

4.1 Linear maps

If you've studied some linear algebra, there is a chance that you may have a prejudice along the lines of "linear algebra = matrice." If so, I want you to rid that from your mind, at least for now. We are going to approach the subject from a slightly more sophisticated and abstract mindset.

The following definition is the main definition of this chapter, and is one of the most important definitions in all of math.

Definition 4.1. A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is **linear** if it satisfies the following two properties:

(i) For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,
$$T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y}).$$

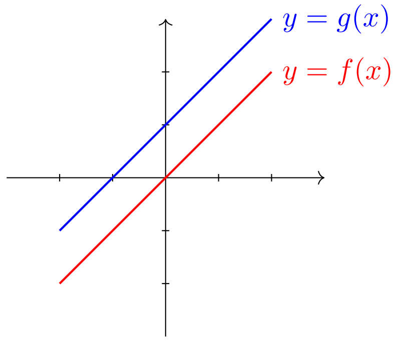
(ii) For all $\mathbf{x} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$,
$$T(\lambda \mathbf{x}) = \lambda T(\mathbf{x}).$$

Remark 4.2. Before we look at some examples, allow me to preach a bit more. First, we have already begun studying functions $f : \mathbb{R}^n \to \mathbb{R}^m$ in the previous chapter. This definition simply identifies a subclass of the functions we've already been studying. In that sense, linear maps aren't a *new* object. Finally, note that I haven't said anything about matrices yet! That will come later. For now, linearity is a completely abstract property, independent of matrices.

Example 4.3. In this example, let's reconcile this abstract notion of linearity with familiar one variable functions.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = mx$ for some $m \in \mathbb{R}$. Then
$$f(x + y) = m(x + y) = mx + my = f(x) + f(y)$$
and
$$f(\lambda x) = m(\lambda x) = \lambda(mx) = \lambda f(x).$$
Thus, f is linear. This is probably unsurprising, as the graph of f is a line.

(b) However, let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = mx + 1$ for some $m \in \mathbb{R}$. The graph of g is also a line, but I claim that g is *not* linear. Indeed,
$$\begin{aligned} g(0 \cdot 1) &= g(0) = 0m + 1 = 1 \\ 0 \cdot g(1) &= 0(m + 1) = 0. \end{aligned}$$
So $g(0 \cdot 1) \neq 0 \cdot g(1)$, and thus g cannot be linear!



This example suggests that being *linear* is more than just being *straight*. In particular, the graph of g in the example above is certainly a line, yet g is not linear. In order for a function to be linear, the graph must pass through the origin. Said more precisely and more generally:

Proposition 4.4. *If $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then $T(\mathbf{0}) = \mathbf{0}$.*

Proof. Note that

$$T(\mathbf{0}) = T(0\mathbf{0}) = 0T(\mathbf{0}) = \mathbf{0}.$$

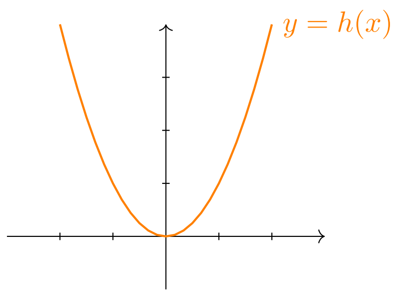
In the first equality, we used the observation that if we multiply the zero vector by the *scalar* 0, we still get the zero vector. In the second equality we used linearity of T to pull the scalar 0 out, and in the third equality we used the fact that 0 times any vector is the zero vector. □

Linearity can fail for other reasons as well.

Example 4.5. Let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(x) = x^2$. Then h is not linear. Indeed,

$$h(2 \cdot 1) = 2^2 = 4$$
$$2 \cdot h(1) = 2(1)^2 = 2.$$

Since $h(2 \cdot 1) \neq 2 \cdot h(1)$, h is not linear (even though $h(0) = 0$). This result should not be surprising, because the graph of h is not *straight*.



Using only the abstract definition of linearity, we can easily characterize all linear functions $\mathbb{R} \to \mathbb{R}$. The following proposition shows that linear functions of one variable have to look like the function f from the first example.

Proposition 4.6. *If $T : \mathbb{R} \to \mathbb{R}$ is linear, then $T(x) = mx$ for some $m \in \mathbb{R}$.*

Proof. Note that

$$T(x) = T(x \cdot 1) = xT(1).$$

Let $m = T(1)$. Then we have shown that $T(x) = mx$.

This argument is special to the one dimensional case, because we can treat $x \in \mathbb{R}$ as both a scalar and a vector. □

We have reconciled and completely classified our understanding of linearity for functions $\mathbb{R} \to \mathbb{R}$, and the rest the chapter is dedicated to understanding linearity of functions $\mathbb{R}^n \to \mathbb{R}^m$. Before moving on, I want to give one more example of a linear function.

Example 4.7. Above, I defined linearity for functions of from $\mathbb{R}^n$ to $\mathbb{R}^m$. In truth, the definition of linearity can be extended to a much more general setting (see Appendix A). For the moment, don't worry about the details and consider the following function:

$$\frac{d}{dx} : \{\text{smooth functions } \mathbb{R} \to \mathbb{R}\} \to \{\text{smooth functions } \mathbb{R} \to \mathbb{R}\}$$

defined by $\frac{d}{dx}(f) := f'(x)$. In words, I'm thinking about the familiar derivative operator $\frac{d}{dx}$ as a function from the world of smooth[a] functions back to itself.

Ignoring the fact that this is not a function $\mathbb{R}^n \to \mathbb{R}^m$, I claim that $\frac{d}{dx}$ is linear! Indeed, using familiar properties of the derivative, we have

$$\frac{d}{dx}(f+g) = \frac{d}{dx}(f) + \frac{d}{dx}(g)$$

and

$$\frac{d}{dx}(\lambda f) = \lambda \frac{d}{dx}(f).$$

The idea of the derivative being a linear map will resurface a little while later.

[a]Here, "smooth" just means you can take as many derivatives as you like. For example, a polynomial would be a smooth function.

4.2 Matrices

Despite my insistence in the previous section of introducing linear algebra without matrices, matrices do play a fundamental role in linear algebra and they are intimitely connected to the notion of linearity. Before discussing this relationship, though, we will simply define them and discuss their basic operations.

Definition 4.8. An $m \times n$ **(real) matrix** A is an array of real numbers with m rows and n columns:

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{m,1} & \cdots & \cdots & a_{m,n} \end{pmatrix} = (a_{i,j})_{1 \le i \le m, 1 \le j \le n}.$$

The set of $m \times n$ real matrices is denoted $M_{m\times n}(\mathbb{R})$, so we write $A \in M_{m\times n}(\mathbb{R})$.

Remark 4.9. In the above definition, the order of m and n in $m \times n$ is important; they are not interchangeable. That is to say, the number of rows always comes first when referencing the size of a matrix. Likewise, in the above array the convention is to let $a_{i,j}$ denote the entry in row i and column j. The second notation $(a_{i,j})$ introduced above is a compact representation of the entire matrix.

Example 4.10.

$$\begin{pmatrix} 1 & 0 & \pi \\ 0 & 2 & 1000 \end{pmatrix} \in M_{2\times 3}(\mathbb{R}) \qquad \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \in M_{3\times 1}(\mathbb{R}) \qquad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in M_{3\times 3}(\mathbb{R}).$$

The last matrix is called the 3×3 *identity matrix*.

The second matrix in the above example should look familiar — it looks like a vector! Indeed, the space of $m\times 1$ matrices is "equivalent" to the set of m-dimensional row vectors. Formally,

$$M_{m\times 1}(\mathbb{R}) \cong \mathbb{R}^m.$$

When I say "equivalent," I actually mean "isomorphic as vector spaces," but don't worry about that phrase for now.[1] This just means that you can treat a matrix with a single column as a vector, and vice versa.

Now, let's discuss the basic algebraic operations on matrices: addition and scalar multiplication.

Definition 4.11. Define matrix addition and scalar multiplication as follows.

(i) If $\lambda \in \mathbb{R}$ and $A = (a_{i,j}) \in M_{m\times n}(\mathbb{R})$, then

$$\lambda A := (\lambda a_{i,j}) = \begin{pmatrix} \lambda a_{1,1} & \lambda a_{1,2} & \cdots & \lambda a_{1,n} \\ \lambda a_{2,1} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \lambda a_{m,1} & \dots & \dots & \lambda a_{m,n} \end{pmatrix}$$

(ii) If $A = (a_{i,j}), B = (b_{i,j}) \in M_{m\times n}(\mathbb{R})$, then

$$A + B := (a_{i,j} + b_{i,j}) = \begin{pmatrix} a_{1,1}+b_{1,1} & a_{1,2}+b_{1,2} & \cdots & a_{1,n}+b_{1,n} \\ a_{2,1}+b_{2,1} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{m,1}+b_{m,1} & \dots & \dots & a_{m,n}+b_{m,n} \end{pmatrix}$$

In words, this definition says: to multiply a matrix by a number, just multiply every entry by that number. Likewise, to add two matrices together, just add the corresponding components together. Note that you can only add matrices of the same dimensions! Here are some examples.

Example 4.12.

$$2\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 1 \\ 4 & 4 & 1 \end{pmatrix}$$

4.2.1 Matrix-vector multiplication

Matrix addition and scalar multiplication is straightforward and hopefully unsurprising. Matrix algebra gets a little more interesting when we start discussing multiplication. Before defining how to multiply two matrices in general, we need to define how to multiply a matrix and vector. In particular, given $A \in M_{m\times n}(\mathbb{R})$ and a vector $\mathbf{x} \in \mathbb{R}^n$, we want to define a new vector $A\mathbf{x} \in \mathbb{R}^m$. Take note of the dimensions here: an n-vector will be multiplied on the left by a matrix with m rows and n columns to produce an n-vector. Ultimately, we want to think about multiplication by $A \in M_{m\times n}(\mathbb{R})$ as a function $\mathbb{R}^n \to \mathbb{R}^m$.

[1]If you want to worry about it, check out Appendix A

To define this matrix-vector product, I'll use yet another notation to depict a matrix. Namely, for $A \in M_{m\times n}(\mathbb{R})$ I'll write

$$A = \begin{pmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \\ | & | & & | \end{pmatrix}$$

where each $\mathbf{a}_j \in \mathbb{R}^m$. In words, the vector $\mathbf{a}_j \in \mathbb{R}^m$ is the jth column of A.

Definition 4.13. Let $A \in M_{m\times n}(\mathbb{R})$ with columns $\mathbf{a}_j \in \mathbb{R}^m$. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Define

$$A\mathbf{x} = \begin{pmatrix} | & & | \\ \mathbf{a}_1 & \cdots & \mathbf{a}_n \\ | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} := x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n \in \mathbb{R}^n.$$

In words, this definition says: scale the jth column of A with x_j, and add all of the resulting vectors together. Notably, the number of columns of A has to match the number of entries of $\mathbf{x}$.

Example 4.14.

$$\begin{aligned} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} &= 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-1)\begin{pmatrix} 2 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 3 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 1 \end{pmatrix}. \end{aligned}$$

There are a number of reasons why matrix-vector multiplication is defined in this odd way. Historically, the purpose is to reformulate the data in a linear system of equations. For us, the reason is one that I mentioned earlier: we want to think about mutliplication by A as a linear function from $\mathbb{R}^n$ to $\mathbb{R}^m$. This is made precise with the following proposition.

Proposition 4.15. *Let $A \in M_{m\times n}(\mathbb{R})$. The map $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is linear.*

Proof. Let's prove one of the properties of linearity here, and I'll leave the other property as an exercise. In particular, fix $\lambda \in \mathbb{R}$ and let's show that $T(\lambda\mathbf{x}) = \lambda T(\mathbf{x})$. Write $\mathbf{x} = (x_1, \ldots, x_n)$ and let $\mathbf{a}_j$ for $j = 1, \ldots, n$ be the columns of A. Then

$$\begin{aligned} T(\lambda\mathbf{x}) = A(\lambda\mathbf{x}) &= \begin{pmatrix} | & & | \\ \mathbf{a}_1 & \cdots & \mathbf{a}_n \\ | & & | \end{pmatrix} \begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix} \\ &= (\lambda x_1)\mathbf{a}_1 + \cdots (\lambda x_n)\mathbf{a}_n \\ &= \lambda\left(x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n\right) \\ &= \lambda A\mathbf{x} \\ &= \lambda T(\mathbf{x}). \end{aligned}$$

In words, we used the definition of matrix-vector multiplication to invoke the distributive properties of vector addition and scalar multiplication from Chapter 2. $\square$

4.3 The standard matrix of a linear map

One way to conceptualize the previous proposition is that given an $m \times n$ matrix, we can produce a linear map from $\mathbb{R}^n \to \mathbb{R}^m$:

$$A \in M_{m\times n}(\mathbb{R}) \qquad \rightsquigarrow \qquad T : \mathbb{R}^n \to \mathbb{R}^m \text{ linear}$$

In this section, we want to go the other way: given an abstract linear map, we will produce a matrix:

$$T : \mathbb{R}^n \to \mathbb{R}^m \text{ linear} \qquad \rightsquigarrow \qquad A \in M_{m\times n}(\mathbb{R}).$$

To do so, I need to introduce a standard set of vectors.

Definition 4.16. The **standard basis of** $\mathbb{R}^n$ is $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, where

$$\mathbf{e}_1 := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \cdots \quad \mathbf{e}_1 := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

In words, the vector $\mathbf{e}_j$ is the vector with 0's in all components except the jth component, which has a 1. This set of vectors is called the standard basis because every vector $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ admits a natural decomposition into these vectors:

$$\begin{aligned}\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} &= \begin{pmatrix} x_1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 \\ \vdots \\ x_n \end{pmatrix} \\ &= x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n.\end{aligned}$$

Moreover, this decomposition is unique. In general a *basis* of $\mathbb{R}^n$ is any set of vectors for every which every other vector admits a similar unique decomposition, but this is a concept from linear algebra that we will not worry about. You only need to care about the standard basis.

Definition 4.17. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear. The **standard matrix of** T is

$$\begin{pmatrix} | & | & & | \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \\ | & | & & | \end{pmatrix} \in M_{m\times n}(\mathbb{R}).$$

In words, this definition says: to compute the standard matrix of a linear map, feed all of the standard basis vectors to T and make the resulting vectors the columns of the matrix.

Before we make mathematical sense of this definition, it's important to make grammatical sense of it. The map T is from $\mathbb{R}^n$ to $\mathbb{R}^m$, so $T(\mathbf{e}_j)$ is an m-vector. Indeed, the above matrix has m rows, and n columns.

The following proposition explains why we call such a matrix the *standard matrix* of T.

Proposition 4.18. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear, and let $A \in M_{m\times n}(\mathbb{R})$ be its standard matrix. Then $T(\mathbf{x}) = A\mathbf{x}$.*

Proof. Let $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$. Then

$$\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n.$$

Thus,

$$\begin{aligned}
T(\mathbf{x}) &= T(x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n) \\
&= T(x_1\mathbf{e}_1) + \cdots + T(x_n\mathbf{e}_n) \\
&= x_1T(\mathbf{e}_1) + \cdots + x_nT(\mathbf{e}_n) \\
&= \begin{pmatrix} | & & | \\ T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_n) \\ | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \\
&= A\mathbf{x}.
\end{aligned}$$

In the first few equalities we made heavy use of linearity of T, and then we used the definition of matrix-vector multiplication. □

Remark 4.19. This result may seem unremarkable, but to me its of utmost importance. We started with an abstract linear map from $\mathbb{R}^n \to \mathbb{R}^m$, knowing absolutely nothing about it aside from the defining properties of linearity. Using only these properties, we have concluded that the abstract map must be realized by matrix multiplication. That is a very powerful fact! In other words, we have essentially characterized linearity of maps $\mathbb{R}^n \to \mathbb{R}^m$ as those given by matrix multiplication. To summarize what we've shown in comparison to our earlier understanding of one-dimensional linearity:

- A function $T : \mathbb{R} \to \mathbb{R}$ is linear if and only if there is some number a such that $T(x) = ax$.
- A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if and only if there is some matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

The latter statement is a natural generalization of the former.

Example 4.20.

(a) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the identity map: $T(\mathbf{x}) = \mathbf{x}$. Then the standard matrix of T is the $n \times n$ *identity* matrix:

$$I_n := \begin{pmatrix} | & & | \\ T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_n) \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ \mathbf{e}_1 & \cdots & \mathbf{e}_n \\ | & & | \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

In words, this example is pointing out that the "do nothing" transformation should correspond to multiplication by the identity matrix. Indeed, you can easily check that $I_n\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

(b) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be be given by $T(\mathbf{x}) = A\mathbf{x}$ for some $A \in M_{m\times n}(\mathbb{R})$. Then the standard matrix of T is A. Indeed, if the columns of A are $\mathbf{a}_j$ for $j = 1, \ldots, n$, then

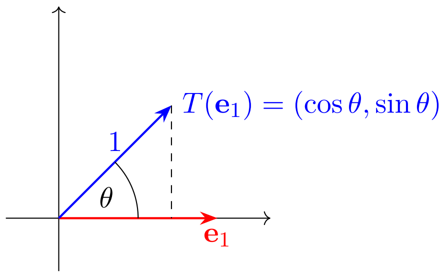
$$T(\mathbf{e_j}) = A\mathbf{e}_j = \mathbf{a}_j.$$

This claim should not be surprising at all. One way to think about this example is that it shows that T is represented *uniquely* by its standard matrix.

To contrast with the above somewhat trivial examples, here is a more interesting one. It may be insightful to look at Exercise 9 from Chapter 2 after reading the following example.

Example 4.21. Fix $\theta \in [0, 2\pi)$ and let $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be map which rotates the input vector counter-clockwise by θ radians. For example, $T_{\frac{\pi}{2}}(1,0) = (0,1)$. I claim that this map is linear, but I won't rigorously justify this. I encourage you to think about why T_θ is linear geometrically (for instance, if you rotate a vector and then scale it, this is the same thing as scaling it first and then rotating it).

Let's compute the standard matrix of T_θ. To do this, we need to compute $T_\theta(\mathbf{e}_1)$ and $T_\theta(\mathbf{e}_2)$. This is essentially a trigonometry computation:



The length of $\mathbf{e}_1$ is 1, so the length of $T(\mathbf{e}_1)$ is also 1. Basic trigonometry then implies

$$T(\mathbf{e}_1) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$

A similar computation shows that

$$T(\mathbf{e}_2) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}.$$

Thus, the standard matrix of T_θ is

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

An example like this demonstrates the utility of the standard matrix: we now have a straightforward algebraic way to rotate *any* vector, and all we had to do is explicitly compute how to rotate *two* special vectors.

4.4 Matrix multiplication

The goal of this section is to define matrix multiplication in full generality and to connect it with the more abstract notion of linear maps.

Definition 4.22. Let $A \in M_{m\times n}(\mathbb{R})$ and $B \in M_{n\times k}(\mathbb{R})$. Let $\mathbf{b}_1, \dots, \mathbf{b}_k \in \mathbb{R}^n$ be the columns of B. Define $AB \in M_{m\times k}(\mathbb{R})$ by

$$AB := \begin{pmatrix} | & | & & | \\ A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_k \\ | & | & & | \end{pmatrix} \in M_{m\times k}(\mathbb{R}).$$

Remark 4.23. The dimensions are important here are not arbitrary. In particular, an $m \times n$ matrix can be multiplied on the right by an $n \times k$ matrix; that is, the "inner" dimensions have to agree. Furthermore, the result is an $m \times k$ matrix, given by the "outer" dimensions. Thus, matrix multiplication is not commutative ($AB \neq BA$) in general! If AB is defined, it may be the case that BA is not even defined, let alone equal to AB.

In words, this says: to compute AB, multiply each column of B by A. Once again, it is helpful to simply make grammatical sense of this definition. Since B is $n \times k$, each column $\mathbf{b}_j$ is an element of $\mathbb{R}^n$. Since A is $m \times n$, the product $A\mathbf{b}_j$ is well defined and produces an m vector. Thus, the matrix AB will have k columns and m rows.

Example 4.24. Let

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 2 & 1 \end{pmatrix}.$$

Note that A is 2×2 and B is 2×3, so the product AB is well-defined: $(2 \times 2)(2 \times 3)$. Moreover, the result should be a 2×3 matrix. Indeed,

$$\begin{aligned} AB &= \begin{pmatrix} | & | & | \\ A\begin{pmatrix} 1 \\ 0 \end{pmatrix} & A\begin{pmatrix} 2 \\ 2 \end{pmatrix} & A\begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ | & | & | \end{pmatrix} \\ &= \begin{pmatrix} | & | & | \\ 1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 1 \end{pmatrix} & 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} & -1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ | & | & | \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & -1 \\ 2 & 6 & -1 \end{pmatrix}. \end{aligned}$$

The product BA is not defined.

The following proposition explains why matrix multiplication is defined in this way. In particular, matrix multiplication corresponds to composition of linear maps.

Proposition 4.25. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^k \to \mathbb{R}^n$ be linear, and let $A \in M_{m\times n}(\mathbb{R})$ and $B \in M_{n\times k}(\mathbb{R})$ be their standard matrices, respectively. The standard matrix for the composition $T \circ S : \mathbb{R}^k \to \mathbb{R}^m$ is AB.*

Proof. Fix $\mathbf{x} \in \mathbb{R}^k$. Note that

$$\begin{aligned} (T \circ S)(\mathbf{x}) = T(S(\mathbf{x})) &= T(B\mathbf{x}) \\ &= A(B\mathbf{x}). \end{aligned}$$

Here we have used the fact that A is the standard matrix for T and B is the standard matrix for S. A straightforward claim (left as an exercise) is that matrix multiplication is associative. In particular, $A(B\mathbf{x}) = (AB)\mathbf{x}$. Thus, we have shown that $(T \circ S)(\mathbf{x}) = (AB)\mathbf{x}$. Thus, the standard matrix for $T \circ S$ is AB. □

Remark 4.26. You should think of the following diagram when reading this proposition:

$$\mathbb{R}^k \xrightarrow{S} \mathbb{R}^n \xrightarrow{T} \mathbb{R}^m \qquad \rightsquigarrow \qquad \mathbb{R}^k \xrightarrow{B} \mathbb{R}^n \xrightarrow{A} \mathbb{R}^m.$$

In words, applying S first and then T corresponds to multiplying by B first, and then by A.

4.5 Invertibility

In this section, we briefly discuss the notion of *invertibility*. In a standard linear algebra class, invertibility is extremely important and thus is studied to death. Here, I'm only going to give you the bare-bones definition and a little bit of intuition. Invertibility will not be as important for us as in a linear algebra class, at least not for now.

Definition 4.27. A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is **invertible** if there is a $g : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$(f \circ g) : \mathbb{R}^n \to \mathbb{R}^n \quad \text{and} \quad (g \circ f) : \mathbb{R}^n \to \mathbb{R}^n$$

are both the identity maps, i.e., $(f \circ g)(\mathbf{x}) = \mathbf{x}$ and $(g \circ f)(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. If such a g exists, we write $f^{-1} := g$ and call f^{-1} the *inverse of* f.

Remark 4.28. This definition can be stated in much more generality, for example for a function $f : \mathbb{R}^n \to \mathbb{R}^m$ or even a function $f : X \to Y$ between arbitrary sets. This definition is simply describing *bijectivity*, in case that word means anything to you. The reason why I am only defining invertibility for functions $f : \mathbb{R}^n \to \mathbb{R}^n$ is because we are primarily interested in invertibility of linear functions. Because the goal of this chapter is to develop as little linear algebra as necessary, some simplifications have to be made. This really isn't consequential for our purposes!

Remark 4.29. The intuition behind this definition is that f^{-1} should "undo" the action f. In other words, if you apply f and then apply f^{-1}, it should look like you did nothing (the identity map).

Example 4.30.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3$. Then f is invertible and $f^{-1}(x) = x^{\frac{1}{3}}$. Indeed, note that

$$f^{-1}(f(x)) = \left(x^3\right)^{\frac{1}{3}} = x \quad \text{and} \quad f(f^{-1}(x)) = \left(x^{\frac{1}{3}}\right)^3 = x.$$

In more familiar language, this inverse function arises from solving the equation $y = x^3$ for x: $x = y^{\frac{1}{3}}$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Then f is *not* invertible. Indeed, suppose there was a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(g(x)) = x$ for all $x \in \mathbb{R}$. Then $[g(x)]^2 = x$ for all $x \in \mathbb{R}$, but this is impossible if x is negative. Thus, no g can exist.

The intuition here should be straightforward: suppose $f(x) = x^2$ and I told you that $f(?) = 4$. Can you tell me what the mystery number ? is? You can't, because it could be either -2 or 2. Thus, you cannot "undo" the action of f.

As I mentioned above, we are primarily interested in invertibility of linear functions. Towards that goal, we have the following important fact.

Proposition 4.31. *Suppose that $T : \mathbb{R}^n \to \mathbb{R}^n$ is linear and invertible. Then $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is linear.*

Proof. Fix $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. We wish to show that $T^{-1}(\mathbf{x} + \mathbf{y}) = T^{-1}(\mathbf{x}) + T^{-1}(\mathbf{y})$. Because we don't know much about invertibility aside from its defining properties, we have to use the following clever trick:

$$T^{-1}(\mathbf{x} + \mathbf{y}) = T^{-1}(T(T^{-1}(\mathbf{x})) + T(T^{-1}(\mathbf{y}))).$$

We have used the fact that $T(T^{-1}(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. The reason why this is useful is because we can invoke the known linearity (in particular, additivity) of T:

$$T^{-1}(\mathbf{x} + \mathbf{y}) = T^{-1}(T(T^{-1}(\mathbf{x})) + T(T^{-1}(\mathbf{y}))) = T^{-1}(T(T^{-1}(\mathbf{x}) + T^{-1}(\mathbf{y}))).$$

Now, the T^{-1} and T in the outer layers compose to the identity by definition of invertibility, and we are left with

$$T^{-1}(\mathbf{x}+\mathbf{y}) = T^{-1}(\mathbf{x}) + T^{-1}(\mathbf{y})$$

as desired. The calculation to show $T^{-1}(\lambda\mathbf{x}) = \lambda T^{-1}(\mathbf{x})$ is similar.

□

This is an important fact, because it allows us to discuss invertibility of matrices. In particular, if T is linear (and thus has a standard matrix) and invertible, then T^{-1} is linear as well and also admits a standard matrix. Using this fact, we can define what it means for a matrix to be invertible.

Definition 4.32. A matrix $A \in M_{n\times n}(\mathbb{R})$ is **invertible** if the function $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is invertible. In this case, the standard matrix of T^{-1} is denoted A^{-1} and is called the **inverse of** A**.**

Remark 4.33. The definition of invertibility combined with Proposition 4.25 implies that if A is invertible, then

$$AA^{-1} = I_n \quad \text{and} \quad A^{-1}A = I_n.$$

Currently, I've told you almost nothing about *how* to find the inverse of a matrix. We will not have to invert many matrices in these notes, so this isn't a problem; if you're interested, a class in linear algebra would develop the tools to invert a matrix. The few examples I'll give here will be based on the intuition of "undoing the transformation."

Example 4.34.

(a) Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ be the 2×2 identity matrix. The corresponding transformatiion $T(\mathbf{x}) = A\mathbf{x} = \mathbf{x}$ is the "do nothing" transformation. To undo this transformation, we don't have to do anything. So A should be invertible, and the inverse matrix should be A itself! Indeed:

$$AA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 + 0\cdot 0 & 0\cdot 1 + 1\cdot 0 \\ 1\cdot 0 + 0\cdot 1 & 0\cdot 0 + 1\cdot 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In other words, if I_n denotes the $n \times n$ identity matrix, then I_n is invertible and $I_n^{-1} = I_n$.

(b) Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then A is *not* invertible. I'll give you a rigorous proof of this fact first, and then I'll explain the intuition.

Suppose that $B \in M_{2\times 2}(\mathbb{R})$ is a matrix such $BA = I_2$. Then

$$I_2 = BA = \begin{pmatrix} B\begin{pmatrix} 1 \\ 0 \end{pmatrix} & B\begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{pmatrix}.$$

The second column of this matrix is $B\mathbf{0} = \mathbf{0}$, which is a contradiction. Thus, no such B exists, and in particular A is not invertible.

Intuitively, the map $T(\mathbf{x}) = A\mathbf{x}$ with this choice of A is the map which projects the input vector onto the $\mathbf{e}_1$ direction (the x-axis). I encourage to you think about why this is the case. Such a transformation cannot be undone, for a couple reasons. One of them is that many different vectors get projected onto the same vector. For example, $T(1, c) = (1, 0)$ for all $c \in \mathbb{R}$.

Example 4.35. We'll conclude this section with a slightly more interesting example of an invertible matrix. Fix $\theta \in [0, 2\pi)$ and let

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Recall from Example 4.21 that A is the standard matrix for the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ which is "rotation by θ radians." To investigate the invertibility of A, we wish to undo the action of rotating by θ radians. It should be unsurprising that rotating by $-\theta$ radians will undo the action of rotating by θ radians. Thus, we hypothesize that the inverse matrix of A is

$$B = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Let's verify that B is in fact the inverse of A. Note that

$$\begin{aligned} AB &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\theta + \sin^2\theta & \sin\theta\cos\theta - \sin\theta\cos\theta \\ \cos\theta\sin\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

A very similar computatiion show that $BA = I_2$ as well. Thus, since $AB = BA = I_2$, it follows that A is invertible and $A^{-1} = B$.

4.6 Exercises

1. Define the following matrices:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & -1 \end{pmatrix} \quad C = \begin{pmatrix} 2 & -1 & -2 \\ 1 & 0 & 0 \\ 3 & 3 & 1 \end{pmatrix}.$$

   Compute AB and BC. Explain why the products BA and CB do not make sense.

2. For each of the following statements, say whether they are *true* or *false*. If a statement is true, prove it. If a statement is false, provide a counter example.

   (a) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subseteq \mathbb{R}^n$ is a set of linearly independent vectors and A is an $n \times n$ matrix, then
   $$\{A\mathbf{v}_1, \ldots, A\mathbf{v}_k\}$$
   is a set of linearly independent vectors.

   (b) If A and B are real matrices such that $AB = I_n$, where I_n is the $n \times n$ identity matrix, then A and B are invertible.

3. Let $A = (a_{i,j})$ be an $n \times n$ real matrix. The **transpose** of a A is defined to be $A^T := (a_{j,i})$. In words, the transpose of A is the matrix obtained by "flipping A across the diagonal." For example,
$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}.$$

A matrix A is **symmetric** if $A^T = A$.

Let A be a symmetric $n \times n$ real matrix and define a function $\langle \cdot, \cdot \rangle_A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$\langle \mathbf{x}, \mathbf{y} \rangle_A := \langle A\mathbf{x}, \mathbf{y} \rangle$$

where the right hand side is the standard inner product on $\mathbb{R}^n$, i.e., the dot product.

(a) If $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ and

$$A_0 = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$$

then compute $\langle \mathbf{x}, \mathbf{y} \rangle_{A_0}$.

(b) Show that $\langle \cdot, \cdot \rangle_A$ is *symmetric*, i.e., $\langle \mathbf{x}, \mathbf{y} \rangle_A = \langle \mathbf{y}, \mathbf{x} \rangle_A$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. (Here A is no longer the matrix from part (a), this statement is for a general symmetric matrix A.)

(c) Show that $\langle \cdot, \cdot \rangle_A$ is bilinear, i.e.,

$$\langle \lambda \mathbf{x}, \mathbf{y} \rangle_A = \lambda \langle \mathbf{x}, \mathbf{y} \rangle_A \qquad \text{for all } \lambda \in \mathbb{R}$$

and

$$\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle_A = \langle \mathbf{x}, \mathbf{z} \rangle_A + \langle \mathbf{y}, \mathbf{z} \rangle_A .$$

(Note that, by symmetry, you only have to show bilinearity in the first component.)

(d) Let $A_0 = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$ be the matrix from (a). Show that $\langle \cdot, \cdot \rangle_{A_0}$ is *positive definite*, i.e.,

$$\langle \mathbf{x}, \mathbf{x} \rangle_{A_0} \geq 0 \quad \text{and} \quad \langle \mathbf{x}, \mathbf{x} \rangle_{A_0} = 0 \text{ iff } \mathbf{x} = \mathbf{0}.$$

Conclude that for this choice of matrix, $\langle \cdot, \cdot \rangle_{A_0}$ is an inner product on $\mathbb{R}^2$.

(e) For a general symmetric matrix A, is it true that $\langle \cdot, \cdot \rangle_A$ is always positive definite? What about if A is invertible? If these statements are true, prove them. If they are false, provide counter examples.

4. Fix $\mathbf{a} \in \mathbb{R}^n$.

(a) Define $T : \mathbb{R}^n \to \mathbb{R}$ by $T(\mathbf{x}) = \langle \mathbf{a}, \mathbf{x} \rangle$. Show that T is linear and compute the standard matrix for T.

(b) Define $P : \mathbb{R}^n \to \mathbb{R}^n$ by $P(\mathbf{x}) = \operatorname{proj}_{\mathbf{a}} \mathbf{x}$. Show that P is linear. Is P invertible? Why or why not?

(c) Define $S : \mathbb{R}^n \to \mathbb{R}^n$ by $T(\mathbf{x}) = \mathbf{x} + \mathbf{a}$. Is S linear? Is S invertible? Why or why not?

5. The **kernel** of a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ is

$$\ker T := \{ \mathbf{x} \in \mathbb{R}^n \ : \ T(\mathbf{x}) = \mathbf{0} \} .$$

In words, the kernel of T is the set of things that T sends to $\mathbf{0} \in \mathbb{R}^m$.

Let A be an $m \times n$ matrix. Define $T : \mathbb{R}^n \to \mathbb{R}^m$ by $T(\mathbf{x}) = A\mathbf{x}$.

(a) Show that the columns of A are linearly independent vectors if and only if $\ker T = \{\mathbf{0}\}$.

(b) Suppose now that A is $n \times n$. Show that if A is invertible, then the $\ker T = \{0\}$. Conclude that if A is invertible, then the columns of A are linearly independent.

6. The **exponential** of an $n \times n$ matrix is defined to be the matrix $e^A := I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$. Here, convergence of the right hand side is defined by convergence of each component of the matrix. Let

$$A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

for some $\lambda \in \mathbb{R}$. Compute e^A.

7. ($*$) In this problem, I'll tell you about a norm that you can put on the set of all square matrices. More generally, this will also describe a norm on the set of all linear maps from a vector space to itself. Let A be an $n \times n$ real matrix. Define

$$\|A\|_{op} := \max_{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|=1} \|A\mathbf{x}\| .$$

On the right hand side, all of the norms are the usual vector norm. In words, this norm says: take the length of $A\mathbf{x}$, where $\mathbf{x}$ is any unit vector; the biggest possible length you can produce is by definition the norm of the matrix A.

(a) Show that $\|A\|_{op} = 0$ if and only if A is the 0 matrix.

(b) Show that $\|\lambda A\|_{op} = |\lambda| A$ for any $\lambda \in \mathbb{R}$.

(c) Show that the triangle inequality holds:

$$\|A + B\|_{op} \leq \|A\|_{op} + \|B\|_{op} .$$

(d) Use this norm to show that the function $T(\mathbf{x}) = A\mathbf{x}$ is continuous.

(e) Show that if A is invertible,

$$\left\|A^{-1}\right\|_{op} = \frac{1}{\|A\|_{op}}.$$

8. ($*$) Let $\mathbb{C} = \left\{ a + bi \ : \ a, b \in \mathbb{R},\ i^2 = -1 \right\}$ be the set of complex numbers. Let $\phi : \mathbb{C} \to \mathbb{R}^2$ be given by $\phi(a + bi) := (a, b)$, let $m_{a+bi} : \mathbb{C} \to \mathbb{C}$ be given by $m_{a+bi}(z) = (a + bi)z$, and let $M_A : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $M_A(\mathbf{x}) = A\mathbf{x}$ for any 2×2 matrix A.

(a) Let $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Define $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = (\phi^{-1} \circ M_J \circ \phi)(z).$$

Show that $f = m_i$.

(b) More generally, let $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Define $g : \mathbb{C} \to \mathbb{C}$ by

$$g(z) = (\phi^{-1} \circ M_A \circ \phi)(z).$$

Show that $f = m_{a+bi}$.

(c) Give some vague explanation as to what you've shown with these two statements.

Chapter 5

Curves, Lines, and Planes

In this short chapter, we take a detour of sorts to discuss parametric curves, the cross product, and plane geometry in $\mathbb{R}^3$. There are some standard topics (tangent vectors, curvature) that I have left as exercises, so I encourage you as a reader to work through these (in particular, Exercises ? and ?). In general, I have provided a large number of exercises to make up for the brevity of the chapter.

5.1 Parametric curves

Definition 5.1. A **parametric curve** is a continuous function $\mathbf{r} : \mathbb{R} \to \mathbb{R}^n$. The domain may also be a subinterval $[a, b] \subset \mathbb{R}$. The **trace** of the parameteric curve is the image of $\mathbf{r}$ in $\mathbb{R}^n$. The domain variable, typically t, is called the **parameter**.

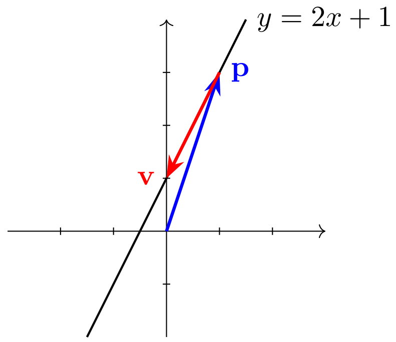
Example 5.2. Fix $\mathbf{p}, \mathbf{v} \in \mathbb{R}^n$ and let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^n$ be defined by

$$\mathbf{r}(t) := \mathbf{p} + t\mathbf{v}.$$

As long as $\mathbf{v} \neq \mathbf{0}$, the trace of this parametric curve will be a line. The vector $\mathbf{p}$ is an arbitrary point on the line and $\mathbf{v}$ is any vector parallel to the line, typically called a *direction vector*.

Concretely, consider the line $y = 2x + 1$ in $\mathbb{R}^2$. One possible parametrization of this line is given by $\mathbf{p} = (1, 3)$ and $\mathbf{v} = (-1, -2)$:

$$\mathbf{r}_1(t) = (1, 3) + t(-1, -2) = (1 - t, 3 - 2t).$$



Another parametrization can be generated by picking $x = t$ and then choosing y to satisfy $y = 2x + 1$. That is, $y = 2t + 1$. Then $\mathbf{r}_2(t) = (t, 2t + 1) = (0, 1) + t(1, 2)$.

This example demonstrates an important fact about parametric curves: *two different parametrizations can trace out the same curve.* It also gives a general principle for how to parametrize any line: *find any point on the line, and then find any direction vector, i.e., a vector parallel to the line.*

Very generally, the intuition behind parametric curves is that the parameter t represents time, and the trace of the curve is the path of a particle wandering around in $\mathbb{R}^n$. Different parametrizations of the same space curve correspond to different journeys along that path, where the differences can be speed, starting point, direction, etc. The next example demonstrates this point of view.

Example 5.3. Consider the following two parametric curves $\mathbf{r}_j : \mathbb{R} \to \mathbb{R}^2$:

$$\mathbf{r}_1(t) = (\cos t, \sin t)$$
$$\mathbf{r}_2(t) = (\sin(2t), \cos(2t))$$

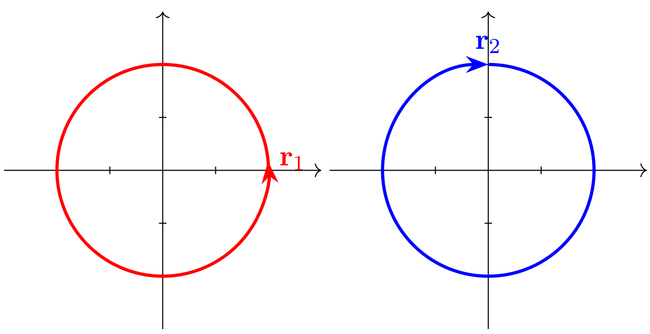
I claim that both of these parametric curves parametrize the unit circle. Indeed, the unit circle is defined by the cartesian equation $x^2 + y^2 = 1$ and

$$(\cos t)^2 + (\sin t)^2 = 1 = (\sin(2t))^2 + (\cos(2t))^2.$$

Even though $\mathbf{r}_1$ and $\mathbf{r}_2$ trace out the same curve, they are distinctly different as parametric curves. For example, when $t = 0$, $\mathbf{r}_1(0) = (1, 0)$ but $\mathbf{r}_2(0) = (0, 1)$. This means that the journeys of particles 1 and 2 start at different points. Furthermore, $\mathbf{r}_1$ traverses the circle counter clockwise, and $\mathbf{r}_2$ traverses the circle clockwise, twice as fast! Indeed, at time $\frac{\pi}{4}$,

$$\mathbf{r}_1\left(\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$
$$\mathbf{r}_2\left(\frac{\pi}{4}\right) = (0, -1).$$

The first curve traverses a quarter of the circle in $\frac{\pi}{4}$ time units, while the second curve traverses half of the circle in the same amount of time because of the factors of 2 in the trig functions.

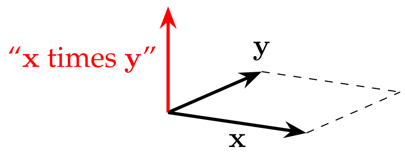


There is much more to be said about parametric curves in general. There are a number of exercises at the end of the chapter to guide you through some of these interesting topics.

5.2 The cross product

In Chapter 2, we introduced the notion of an inner product as a way to multiply two vectors together to get a scalar. This product not only behaves like our usual sense of multiplication, but it encodes a great deal of geometric information. A natural question is whether it is possible to multiply two vectors to get another *vector* in a geometrically motivated way.

In general, this isn't possible. But in $\mathbb{R}^3$, there is a very nice geometric way to motivate such a product. Given two nonparallel vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, consider the plane that they generate. In three dimensions, there is a unique orthogonal direction to this plane (in contrast to higher dimensions, where there are *many* orthogonal directions). We could seek a product "$\mathbf{x}$ times $\mathbf{y}$" which is a vector pointing in this orthogonal direction.



This is the geometric motivation for the following definition.

Definition 5.4. The **cross product** of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ is

$$\mathbf{x} \times \mathbf{y} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} := \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix} \in \mathbb{R}^3.$$

Remark 5.5. In my opinion, this is kind of a lame definition. Even though I gave some geometric motivation for the definition above, it is not clear at all why this definition captures that intuition. In truth, there are other ways to develop and introduce the notion of a cross product with more elegance and sophistication, but these have no time or place in a this set of notes. If you're interested in this, see Appendix A, or Google some buzzwords like *wedge product* and *exterior algebra*. The approach I will take in this section is to give you the lame definition and claim that it satisfies some nice desirable geometric properties. In the exercises, you will verify this for yourself.

Example 5.6. We begin with two quick computational examples.

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \cdot 0 - 0 \cdot 1 \\ 0 \cdot 0 - 1 \cdot 0 \\ 1 \cdot 1 - 0 \cdot 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \cdot 0 - 0 \cdot 0 \\ 0 \cdot 1 - 0 \cdot 0 \\ 0 \cdot 0 - 1 \cdot 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

More succinctly, $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$ and $\mathbf{e}_2 \times \mathbf{e}_1 = -\mathbf{e}_3$. This suggests a number of important properties of the cross product already. First, it seems that $\mathbf{e}_1 \times \mathbf{e}_2$ is in fact orthogonal to both $\mathbf{e}_1$ and $\mathbf{e}_2$, so this checks out in terms of our original geometric motivation. Second, evidently $\mathbf{e}_1 \times \mathbf{e}_2$ is not the same as $\mathbf{e}_2 \times \mathbf{e}_1$! We have shown that

$$\mathbf{e}_1 \times \mathbf{e}_2 = -(\mathbf{e}_2 \times \mathbf{e}_1).$$

In other words, the cross product is *not* commutative! The direction of the cross product obeys something called the *right hand rule*, which will be discussed more in depth below.

Before stating the interesting geometric properties of the cross product, it is important to record some basic algebraic properties.

Proposition 5.7. *For all* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3$ *and* $\lambda \in \mathbb{R}$*,*

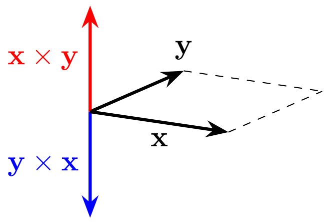
(i)
$$\mathbf{x} \times (\mathbf{y} + \mathbf{z}) = \mathbf{x} \times \mathbf{y} + \mathbf{x} \times \mathbf{z} \quad \text{and} \quad (\mathbf{x} + \mathbf{y}) \times \mathbf{z} = \mathbf{x} \times \mathbf{z} + \mathbf{y} \times \mathbf{z}.$$

(i)
$$(\lambda \mathbf{x}) \times \mathbf{y} = \lambda(\mathbf{x} \times \mathbf{y}) = \mathbf{x} \times (\lambda \mathbf{y}).$$

(iii)
$$\mathbf{x} \times \mathbf{y} = -(\mathbf{y} \times \mathbf{x}).$$

Proof. These are all straightforward (but annoying) algebraic computations using the definition; they are left as exercises to the reader. □

Remark 5.8. The first two properties roughly say that the cross product defines a bilinear map (linear in each entry), just like the inner product (and just like the usual numerical multiplication). The third property has already been briefly discussed: the cross product is not commutative, and reversing the order of multiplication reverses the direction of the resulting vector. In particular, the direction of the cross product obeys the *right hand rule*: using your right hand, if $\mathbf{x}$ is your index finger and $\mathbf{y}$ your middle finger, then $\mathbf{x} \times \mathbf{y}$ points in the direction of your thumb. See the figure below.



Example 5.9. Another interesting property of the cross product is that it is not associative! Indeed, note that
$$(\mathbf{e}_1 \times \mathbf{e}_2) \times \mathbf{e}_2 = \mathbf{e}_3 \times \mathbf{e}_2.$$
A similar computation to the previous example shows that $\mathbf{e}_3 \times \mathbf{e}_2 = -\mathbf{e}_1$.

On the other hand,
$$\mathbf{e}_1 \times (\mathbf{e}_2 \times \mathbf{e}_2) = \mathbf{e}_1 \times \mathbf{0} = \mathbf{0}.$$
This follows from another similar computation to show that $\mathbf{e}_2 \times \mathbf{e}_2 = \mathbf{0}$. So we have shown that

$$\begin{aligned} (\mathbf{e}_1 \times \mathbf{e}_2) \times \mathbf{e}_2 &= -\mathbf{e}_1 \\ \mathbf{e}_1 \times (\mathbf{e}_2 \times \mathbf{e}_2) &= \mathbf{0} \end{aligned}$$

and thus the cross product is not associative.

Now we state the more interesting geometric properties of the cross product. The following proposition shows that the cross product does in fact capture the intuition given by the preliminary motivation.

Proposition 5.10. *For any* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$*,*

(i) The vector $\mathbf{x} \times \mathbf{y}$ *is orthogonal to both* $\mathbf{x}$ *and* $\mathbf{y}$*.*

(ii) The number $\|\mathbf{x} \times \mathbf{y}\|$ *is the area of the parallelogram generated by* $\mathbf{x}$ *and* $\mathbf{y}$*.*

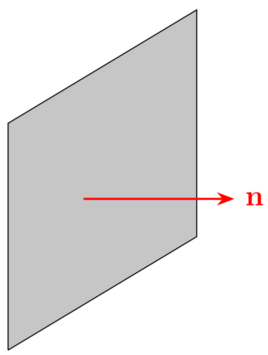
(iii) If θ is the angle between $\mathbf{x}$ and $\mathbf{y}$, then $\|\mathbf{x} \times \mathbf{y}\| = \|\mathbf{x}\| \|\mathbf{y}\| \sin\theta$.

Remark 5.11. Observe that (ii) and (iii) are essentially the same statement, and that they imply indirectly that $\mathbf{x} \times \mathbf{x} = \mathbf{0}$ for any vector $\mathbf{x}$. Furthermore, (iii) suggests that the cross product is somehow dual to the inner product; the inner product measures how *similar* two vectors are, while the cross product measures how *different* they are.

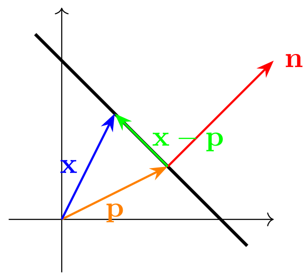
Proof. The proof of these properties is left as an (important!) exercise to the reader. $\square$

5.3 Planes

The goal of this section is to describe planes in $\mathbb{R}^3$, and one immediate application of the cross product is its utility in this endeavor. In particular, to describe a plane in $\mathbb{R}^3$ we will adopt the following strategy: *find a vector orthogonal to the plane, and describe the plane as all points orthogonal to that vector.* We will call such a vector a *normal vector*, and usually denote it $\mathbf{n}$.



To make this a bit more precise, we have to consider what happens with a plane that doesn't go through the origin. Consider the following figure, which represents an appropriate side view of a plane in $\mathbb{R}^3$.



Given a plane (in black), a normal vector $\mathbf{n}$, and any point $\mathbf{p}$ on the plane, we can describe *all* points on the plane as the points $\mathbf{x}$ such that $\mathbf{x} - \mathbf{p}$ is orthogonal to $\mathbf{n}$. This motivates the following definition.

Definition 5.12. A **plane** in $\mathbb{R}^3$ is a set of the form

$$\{\, \mathbf{x} \in \mathbb{R}^3 \;:\; \langle \mathbf{x} - \mathbf{p}, \mathbf{n} \rangle = 0 \,\}$$

where $\mathbf{p} \in \mathbb{R}^3$ is a fixed vector and $\mathbf{n} \neq \mathbf{0}$ is a **normal vector**.

Remark 5.13. In $\mathbb{R}^n$, we can define a *hyperplane* in exactly the same way: for a fixed $\mathbf{p}, \mathbf{n} \in \mathbb{R}^n$, a **hyperplane** is a set of the form

$$\{\, \mathbf{x} \in \mathbb{R}^n \;:\; \langle \mathbf{x} - \mathbf{p}, \mathbf{n} \rangle = 0 \,\}\,.$$

Remark 5.14. Note that normal vectors are not unique. If $\mathbf{n}$ is a normal vector to a plane, then $\lambda\mathbf{n}$ is also a normal vector for any scalar $\lambda \neq 0$.

We can turn the above definition into something more down to earth by writing out all of the vectors in components. Let $\mathbf{x} = (x, y, z)$, $\mathbf{n} = (a, b, c)$, and $\mathbf{p} = (x_0, y_0, z_0)$. Then

$$\langle \mathbf{x} - \mathbf{p}, \mathbf{n} \rangle = 0$$

is equivalent to

$$\begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \qquad \Rightarrow \qquad a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

This is further equivalent to

$$ax + by + cz = d$$

where $d = ax_0 + by_0 + cz_0$. Thus, we have a number of standard ways to describe a plane with a single equation.

To summarize: to find an equation of a plane, you need a normal vector and a point on a plane. Typically, to get a normal vector, a good strategy is to take the cross product of two vectors parallel to the plane.

Example 5.15. Let $\mathbf{r}_1(t) = \langle t + 3, 1 - t, 3t + 3 \rangle$ and $\mathbf{r}_2(t) = \langle t + 5, -t, 3t \rangle$ be two parallel lines (that do not intersect). Let's find the equation of the plane containing both lines.

Observe that the two lines are in fact parallel because they have parallel (the same) direction vectors. A direction vector for both lines is $\langle 1, -1, 3 \rangle$. This vector lies "in" the plane we seek.

To find the desired equation, we need a point and a normal vector. Because the plane contains both lines, to get a point we can use any point on either line. For example, $\mathbf{r}_2(0) = (5, 0, 0)$ is a point on the line. To get a normal vector, we will take the cross product of two vectors parallel to the plane. One of those vectors is the direction vector $\mathbf{a} := \langle 1, -1, 3 \rangle$ from above. To get another vector, we connect any two points on the lines. For example, since $\mathbf{r}_1(0) = (3, 1, 3)$ and $\mathbf{r}_2(0) = (5, 0, 0)$, we may take

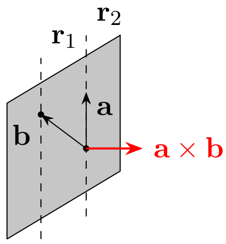
$$\mathbf{b} = \langle 3, 1, 3 \rangle - \langle 5, 0, 0 \rangle = \langle -2, 1, 3 \rangle.$$

Then a normal vector for our plane is

$$\mathbf{a} \times \mathbf{b} = \langle 1, -1, 3 \rangle \times \langle -2, 1, 3 \rangle = \langle -6, -9, -1 \rangle.$$

Thus, the equation of the plane we seek is

$$-6(x - 5) - 9(y - 0) - 1(z - 0) = 0 \qquad \Rightarrow \qquad 6x + 9y + z = 30.$$



Example 5.16. Let $\mathbf{r}_1(t) = \langle t, t+2, 2t\rangle$ and $\mathbf{r}_2(t) = \langle t+1, 2t+1, t+1\rangle$ be two skew lines. Let's find the equation of the plane which lies exactly halfway between the two lines and intersects neither.

Because we want our plane to not intersect either of the given lines, we need the direction vectors of both $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ to be parallel to the plane. Thus, we can get a normal vector for the plane by taking the cross product of direction vectors for the given lines:

$$\langle 1,1,2\rangle \times \langle 1,2,1\rangle = \langle -3,1,1\rangle.$$

The slightly more difficult part is finding a point on the plane. The relevant condition is the fact that the plane lies exactly halfway between the skew lines. We can locate any point on this plane by connecting a point on one line to a point on the other (any such points will work — prove this as an extra exercise!) and travelling halfway along this vector; see the figure below. To make this precise, let

$$\mathbf{p} := \mathbf{r}_1(0) = \langle 0,2,0\rangle$$
$$\mathbf{q} := \mathbf{r}_2(0) = \langle 1,1,1\rangle$$

be points on the first and second line, respectively. Let

$$\mathbf{a} := \mathbf{q} - \mathbf{p} = \langle 1,1,1\rangle - \langle 0,2,0\rangle = \langle 1,-1,1\rangle.$$

Then a point on the plane we seek is

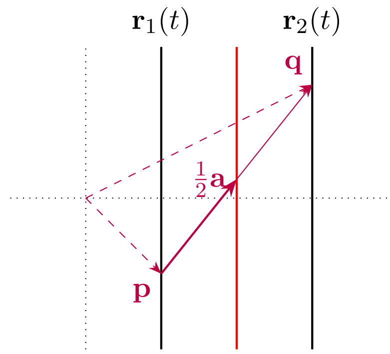
$$\mathbf{p} + \frac{1}{2}\mathbf{a} = \langle 0,2,0\rangle + \frac{1}{2}\langle 1,-1,1\rangle = \langle \frac{1}{2}, \frac{3}{2}, \frac{1}{2}\rangle.$$

Using the normal vector we computed above, the plane we seek is

$$-3\left(x - \frac{1}{2}\right) + 1\left(y - \frac{3}{2}\right) + 1\left(z - \frac{1}{2}\right) = 0.$$

Simplifying gives

$$-3x + y + z = \frac{1}{2}.$$



5.4 Exercises

1. The goal of this problem is for you to investigate and prove some of the properties of the cross product.

   (a) Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$. Prove that $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

(b) Prove that
$$\|\mathbf{a} \times \mathbf{b}\|^2 + \langle \mathbf{a}, \mathbf{b} \rangle^2 = \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 .$$

(c) Use 1(b) to prove that $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

(d) Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ be three vectors, and let X be the *parallelepiped generated by* $\mathbf{a}, \mathbf{b}, \mathbf{c}$, i.e., the set
$$X := \{ \lambda_1 \mathbf{a} + \lambda_2 \mathbf{b} + \lambda_3 \mathbf{c} \;:\; \lambda_j \in [0,1] \} .$$
Prove that the volume of X is $| \langle \mathbf{a}, \mathbf{b} \times \mathbf{c} \rangle |$.
[Hint: Note that we have not discussed or defined the determinant! Recall that the volume of a prism is the area of its base times its vertical height and use 1(c).]

2. For each of the following statements, determine whether it is *true* or *false*. If it is true, prove it. If it is false, give a counter example.

   (a) Suppose that $\mathbf{v}$ and $\mathbf{w}$ are parallel vectors and that $\mathbf{p} \neq \mathbf{q}$ are different vectors. Then $\mathbf{r}_1(t) = \mathbf{p} + t\mathbf{v}$ and $\mathbf{r}_2(t) = \mathbf{q} + t\mathbf{w}$ are parameterizations for different lines.

   (b) A parameterization for the unit circle is $\mathbf{r}(t) = \langle \sin(t^2), \cos(t^2) \rangle$, $t \in \mathbb{R}$.

   (c) If $\operatorname{proj}_{\mathbf{b}} \mathbf{a} = \mathbf{0}$, then $\|\mathbf{a} \times \mathbf{b}\|$ is a positive number.

   (d) If $\mathbf{a} \cdot \mathbf{b} > 0$, then $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

   (e) The area of the triangle spanned by vectors $\mathbf{a}$ and $\mathbf{b}$ is the same as the area of the triangle spanned by vectors $\mathbf{a}$ and $\mathbf{b} - \mathbf{a}$.

3. Compute $(0, 1, 2) \times (3, 2, 1)$.

4. Compute the area of the triangle with vertices $(0, 1, 0), (1, 2, 2), (-1, -1, 0)$.

5. Write down parametrizations of the following curves:

   (a) A clockwise circle of radius 4 centered at the $(1, 1)$ where the position at $t = 0$ is $(-3, 1)$.

   (b) The line passing through the points $(1, 2, 3, 4)$ and $(-1, 0, 4, 1)$ in $\mathbb{R}^4$.

   (c) The curve of intersection between the surfaces $x^2 + y^2 = 1$ and $x + y + z = 1$ in $\mathbb{R}^3$.

6. Find the equation of the plane with normal vector $\langle 1, 2, 5 \rangle$ which passes through the point $(-1, 3, 4)$.

7. Find the equation of the plane containing the points $(1, 2, 3)$, $(3, 2, 1)$, and $(-1, 0, 2)$.

8. Find the equation of the plane containing the line $\mathbf{r}(t) = \langle t + 1, 3t - 1, t \rangle$ and the point $(10, 10, 10)$.

9. Find the equation of the plane which is parallel to the plane $x - y + 9z = 0$ and contains the point $(1, 1, 1)$.

10. Let $\mathbf{r}_1(t) = \langle 4 - 2t, 5t - 2, 2t \rangle$ and $\mathbf{r}_2(t) = \langle t + 1, 2t + 1, t + 1 \rangle$ be two lines. Find their point of intersection and then find the equation of the plane containing both lines.

11. Find the equation of a plane perpendicular to the planes $x + y - 3z = 0$ and $-x + 2y + 2z = 1$.

12. Let $P_1 = \{ \langle \mathbf{x}, \mathbf{n}_1 \rangle = 0 \}$ and $P_2 = \{ \langle \mathbf{x}, \mathbf{n}_2 \rangle = 0 \}$ be two planes in $\mathbb{R}^3$, where $\mathbf{n}_1, \mathbf{n}_2$ are not parallel. Let $\mathbf{v} \neq \mathbf{0} \in \mathbb{R}^3$ be a third vector and let L be the line parameterized by $\mathbf{r}(t) = t\mathbf{v}$ for $t \in \mathbb{R}$.

(a) Write down a condition[1] on $\mathbf{v}$ so that there exists a unique plane P_3 satisfying

$$P_3 \supset (P_1 \cap P_2) \cup L.$$

Assuming this condition, find the equation for this plane P_3.

(b) Write down a condition on $\mathbf{v}$ such that there is a unique plane P_3 containing the three terminal points of $\mathbf{n}_1, \mathbf{n}_2$, and $\mathbf{v}$. Assuming this condition, find the equation for this plane P_3.

(c) Find the equation of a plane which, together with P_1 and P_2, encloses a region which is an (infinitely long) *isosceles triangular prism* such that the *area of any (isosceles) triangular cross section of the prism* is 1. (Note that there are four correct answers.)

13. Let $\mathbf{r} = (r_1, \ldots, r_n) : \mathbb{R} \to \mathbb{R}^n$ be a differentiable[2] parametric curve. Define its **tangent vector**, also called its **velocity**, as

$$\mathbf{r}'(t) = \begin{pmatrix} r_1'(t) \\ \vdots \\ r_n'(t)) \end{pmatrix}.$$

Unsurprisingly, the tangent vector is a vector "tangent to the curve." We can similarly define $\mathbf{r}''(t)$ and all higher order derivatives. The **speed** of a parametric curve is $\|\mathbf{r}'(t)\|$ and the **acceleration** is $\|\mathbf{r}''(t)\|$.

(a) Let $\mathbf{r}_1(t) = (\cos t, \sin t)$. Compute $\mathbf{r}'(0)$ and $\mathbf{r}'(\pi/2)$. Draw a picture and make sense of this. Do the same with $\mathbf{r}_2(t) = (\cos(2t), \sin(2t))$.

(b) Prove that if $\|\mathbf{r}(t)\|$ is constant, then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$. Draw a picture and explain why this makes sense geometrically.

(c) Let $\mathbf{r}(t) = (\cos t, \sin t, t)$ be a parametric helix. Compute the tangent line (in parametric form) to the curve at the point $(-1, 0, \pi)$.

14. Let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^n$ be a differentiable parametric curve such that $\|\mathbf{r}'(t)\| = 1$ for all $t \in \mathbb{R}$. The **curvature** of the curve at time t is defined to be $\kappa(t) := \|\mathbf{r}''(t)\|$.

(a) Explain why this definition of curvature seems to actually coincide with our intuitive understanding of curvature.

(b) Explain why the condition $\|\mathbf{r}'(t)\| = 1$ is necessary in order for this definition to coincide with our intuitive definition of curvature. (For example, can you provide an example of a parametric curve $\mathbf{r} : \mathbb{R} \to \mathbb{R}^n$ such that the trace of the curve is "straight," but $\|\mathbf{r}''(t)\| \neq 0$?)

(c) In $\mathbb{R}^3$, we can define the curvature of any parametric curve $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$ with $\|\mathbf{r}'(t)\| > 0$ as:

$$\kappa(t) := \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$

Prove that this definition agrees with the definition in part (a) if $\|\mathbf{r}'(t)\| \equiv 1$.

(d) Compute the curvature of a circle of radius $R > 0$. What happens as $R \to 0$ and $R \to \infty$? Explain why this makes sense geometrically.

[1] Don't worry about *proving* that your condition is correct. I just want you to think about it and give some intuitive justification.

[2] For now, this just means that each component is differentiable.

(e) ($*$) Suppose that $\mathbf{r} : \mathbb{R} \to \mathbb{R}^n$ is differentiable and satisfies $\|\mathbf{r}'(t)\| > 0$. Prove that there exists an invertible function $f : \mathbb{R}_s \to \mathbb{R}_t$ such that[3]

$$\left\|\tilde{\mathbf{r}}'(s)\right\| = 1$$

for all $s \in \mathbb{R}$, where $\tilde{r} : \mathbb{R}_s \to \mathbb{R}^n$ is defined as $\tilde{\mathbf{r}} := \mathbf{r} \circ f$.

[3] The point of this problem is that if you have a parametric curve $\mathbf{r}$ and you want to compute the curvature but $\|\mathbf{r}'(t)\|$ isn't necessarily identically 1, you have a problem. The fix is to *reparametrize* the curve via the invertible function $f : \mathbb{R}_s \to \mathbb{R}_t$. This creates a new parameter s over which the reparametrized curve $\tilde{\mathbf{r}}$ has unit speed. Furthermore, the trace of $\mathbf{r}$ and $\tilde{\mathbf{r}}$ are the same, so we can compute the curvature of the original curve by computing $\|\tilde{\mathbf{r}}''(s)\|$.

Chapter 6

Differentiation

After many pages of preparation, we are finally ready to discuss differentiation of multivariable functions. In the same spirit as the rest of the notes, we want to discuss the derivative with more formality than in a typical multivariable calculus class.

6.1 Defining the derivative

To motivate the formal definition of the derivative of a function $f : \mathbb{R}^n \to \mathbb{R}^m$, it is important to revisit the more familiar definition from single variable calculus. After all — however we decide to define the multivariable derivative, it should be a direct generalization of the usual derivative.

6.1.1 Revisiting the derivative in one variable

Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a point $x_0 \in \mathbb{R}$ if the following limit exists:

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

If the limit exists, we denote the limit $f'(x_0)$ and name it the *derivative of f at x_0*. There is an obvious problem when we try to naively generalize this to a function $f : \mathbb{R}^n \to \mathbb{R}^m$. Namely, if $\mathbf{x}_0, \mathbf{h} \in \mathbb{R}^n$, the expression

$$\frac{f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0)}{\mathbf{h}}$$

is ill-defined, because there is no notion of *division* by a vector. A possible fix is to consider the following limit:

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0)}{\|\mathbf{h}\|}.$$

By dividing by the *norm* of the vector $\mathbf{h}$, we now have a sensible mathematical expression. The problem is that this would force the usual one-variable derivative to be

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{|h|}$$

and this doesn't make sense. Indeed, consider $f(x) = x$. We all know that $f'(0) = 1$, and anything else would be absurd. But

$$\lim_{h \to 0} \frac{f(0 + h) - f(0)}{|h|} = \lim_{h \to 0} \frac{h}{|h|}$$

and this limit does not exist! Thus, we have to do something else.

The fix is to slightly change our perspective on what we mean by *the derivative*. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $x_0 \in \mathbb{R}$. Then

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}.$$

Recalling the formal definition of a limit from Chapter 3, this means that

$$\lim_{h\to 0} \left| \frac{f(x_0+h) - f(x_0)}{h} - f'(x_0) \right| = 0,$$

which further means that

$$\lim_{h\to 0} \frac{|f(x_0+h) - f(x_0) - f'(x_0)h|}{|h|} = 0.$$

The next step may seem a little odd. Define a linear map $T : \mathbb{R}_h \to \mathbb{R}$ by $T(h) = f'(x_0)h$. Note that the *variable* is h, and $f'(x_0)$ is a fixed number. Thus, a function f is differentiable at x_0 if there is a linear transformation $T : \mathbb{R}_h \to \mathbb{R}$ such that

$$\lim_{h\to 0} \frac{|f(x_0+h) - f(x_0) - T(h)|}{|h|} = 0.$$

We call the linear map $T(h)$ *the derivative*. By the characterization of one-variable linear maps from Chapter 4, there is some number $m \in \mathbb{R}$ such that $T(h) = mh$. We denote this number m by $f'(x_0)$. Observe the change in perspective: the derivative of f at x_0 is *not* the number $f'(x_0)$; it is the linear map $T(h) = f'(x_0)h$.

6.1.2 The multivariable derivative

The point of the above change in perspective is that it does generalize appropriately to any dimension.

Definition 6.1. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and fix $\mathbf{x}_0 \in \mathbb{R}^n$. We say that f **is differentiable at** $\mathbf{x}_0$ if there exists a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{\mathbf{h}\to \mathbf{0}} \frac{\|f(\mathbf{x}_0+\mathbf{h}) - f(\mathbf{x}_0) - T(\mathbf{h})\|}{\|\mathbf{h}\|} = 0.$$

If such a T exists, we call the linear map $Df(\mathbf{x}_0) := T$ the **derivative of** f **at** $\mathbf{x}_0$. If f is differentiable at all points in its domain, we say that f **is differentiable.**

This definition is important enough to deserve some verbal emphasis. The derivative of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ at a point $\mathbf{x}_0$ is another *linear* function $Df(\mathbf{x}_0) : \mathbb{R}^n \to \mathbb{R}^m$. The notation is a bit awkward: the whole linear map is named $Df(\mathbf{x}_0)$, so that the input of the function is $\mathbf{h} \in \mathbb{R}^n$ and the output is $Df(\mathbf{x}_0)(\mathbf{h}) \in \mathbb{R}^m$.

For a fixed $\mathbf{x}_0$, note that the function $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^m$ defined by $\tilde{f}(\mathbf{h}) := f(\mathbf{x}_0+\mathbf{h}) - f(\mathbf{x}_0)$ satisfies $\tilde{f}(\mathbf{0}) = \mathbf{0}$. In words, $\tilde{f}$ is a shifted version of f based at $\mathbf{x}_0$ so that the graph of f passes through the origin. The derivative $Df(\mathbf{x}_0)$ is the the linear map which best approximates $\tilde{f}$; this witnesses the philosophy that calculus is the process of approximating non-linear objects by linear ones.

6.2 Properties of the derivative

The definition of the multivariable derivative is theoretically elegant, but it doesn't really tell us anything about how to compute a derivative. We will discuss that soon, but first I want to record some basic properties of the derivative.

The first observation to make it that the derivative is unique. This is implied by the language of the definition above, but it is a nontrivial fact and is important to say! The proof of the following proposition may be skipped on a first reading.

Proposition 6.2. *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\mathbf{x}_0 \in \mathbb{R}^n$. Then the derivative $Df(\mathbf{x}_0)$ is unique.*

Proof. This is stated a bit informally, so here is what I really mean: suppose that $T_1, T_2 : \mathbb{R}^n \to \mathbb{R}^m$ are two linear maps such that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|f(\mathbf{x}_0+\mathbf{h}) - f(\mathbf{x}_0) - T_j(\mathbf{h})\|}{\|\mathbf{h}\|} = 0$$

for $j = 1, 2$. We want to show that, in fact, $T_1 = T_2$.

Note that

$$\begin{aligned}
\frac{\|T_1(\mathbf{h}) - T_2(\mathbf{h})\|}{\|\mathbf{h}\|} &= \frac{\|T_1(\mathbf{h}) - f(\mathbf{x}_0+\mathbf{h}) + f(\mathbf{x}_0) + f(\mathbf{x}_0+\mathbf{h}) - f(\mathbf{x}_0) - T_2(\mathbf{h})\|}{\|\mathbf{h}\|} \\
&\leq \frac{\|T_1(\mathbf{h}) - f(\mathbf{x}_0+\mathbf{h}) + f(\mathbf{x}_0)\| + \|f(\mathbf{x}_0+\mathbf{h}) - f(\mathbf{x}_0) - T_2(\mathbf{h})\|}{\|\mathbf{h}\|} \\
&= \frac{\|f(\mathbf{x}_0+\mathbf{h}) - f(\mathbf{x}_0) - T_1(\mathbf{h})\|}{\|\mathbf{h}\|} + \frac{\|f(\mathbf{x}_0+\mathbf{h}) - f(\mathbf{x}_0) - T_2(\mathbf{h})\|}{\|\mathbf{h}\|}
\end{aligned}$$

Here we used the triangle inequality in the numerator to get the desired inequality. By assumption, it follows that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|T_1(\mathbf{h}) - T_2(\mathbf{h})\|}{\|\mathbf{h}\|} \leq 0 + 0 = 0.$$

We will use this fact to conclude that $T_1 = T_2$. Since $T_j(\mathbf{0}) = \mathbf{0}$ by linearity, fix $\mathbf{h}_0 \neq \mathbf{0} \in \mathbb{R}^n$. Then as $t \to 0$, $t\mathbf{h}_0 \to \mathbf{0}$, and thus the above limit implies

$$\begin{aligned}
0 &= \lim_{t\to 0} \frac{\|T_1(t\mathbf{h}_0) - T_2(t\mathbf{h}_0)\|}{\|t\mathbf{h}_0\|} \\
&= \lim_{t\to 0} \frac{\|t(T_1(\mathbf{h}_0) - T_2(\mathbf{h}_0))\|}{\|t\mathbf{h}_0\|} \\
&= \lim_{t\to 0} \frac{|t|\,\|T_1(\mathbf{h}_0) - T_2(\mathbf{h}_0)\|}{|t|\,\|\mathbf{h}_0\|} \\
&= \lim_{t\to 0} \frac{\|T_1(\mathbf{h}_0) - T_2(\mathbf{h}_0)\|}{\|\mathbf{h}_0\|} \\
&= \frac{\|T_1(\mathbf{h}_0) - T_2(\mathbf{h}_0)\|}{\|\mathbf{h}_0\|}.
\end{aligned}$$

Here we used linearity of T_j to pull the scalar t out and ultimately cancel it out with the t in the denominator. This implies that $\|T_1(\mathbf{h}_0) - T_2(\mathbf{h}_0)\| = 0$ and thus $T_1(\mathbf{h}_0) - T_2(\mathbf{h}_0) = \mathbf{0}$, from which it follows that $T_1(\mathbf{h}_0) = T_2(\mathbf{h}_0)$. Since this holds for all $\mathbf{h}_0 \in \mathbb{R}^n$, we have shown that $T_1 = T_2$. □

Another important and familiar fact about the derivative is that differentiability implies continuity of one variable functions. Indeed, this holds for multivariable functions. Again, the following proof may be skipped on a first reading.

Proposition 6.3. *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\mathbf{x}_0 \in \mathbb{R}^n$. Then f is continuous at $\mathbf{x}_0$.*

Proof. We want to show that

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{x}_0.$$

This is equivalent to showing that

$$\lim_{\mathbf{h}\to\mathbf{0}} f(\mathbf{x}_0+\mathbf{h}) = f(\mathbf{x}_0)$$

which is further equivalent to

$$\lim_{\mathbf{h}\to\mathbf{0}} [f(\mathbf{x}_0+\mathbf{h}) - f(\mathbf{x}_0)] = \mathbf{0}.$$

Since $Df(\mathbf{x}_0) : \mathbb{R}^n \to \mathbb{R}^m$ is linear, $Df(\mathbf{x}_0)(\mathbf{h}) = \mathbf{0}$. Thus, the above limit is further equivalent to

$$\lim_{\mathbf{h}\to\mathbf{0}} [f(\mathbf{x}_0+\mathbf{h}) - f(\mathbf{x}_0) - Df(\mathbf{x}_0)(\mathbf{h})] = \mathbf{0}.$$

To show this, it suffices to show

$$\lim_{\mathbf{h}\to\mathbf{0}} \|f(\mathbf{x}_0+\mathbf{h}) - f(\mathbf{x}_0) - Df(\mathbf{x}_0)(\mathbf{h})\| = 0.$$

Note that, if $\|\mathbf{h}\| \leq 1$,

$$\begin{aligned}\|f(\mathbf{x}_0+\mathbf{h}) - f(\mathbf{x}_0) - Df(\mathbf{x}_0)(\mathbf{h})\| &= \frac{\|\mathbf{h}\|\,\|f(\mathbf{x}_0+\mathbf{h}) - f(\mathbf{x}_0) - Df(\mathbf{x}_0)(\mathbf{h})\|}{\|\mathbf{h}\|} \\ &\leq \frac{\|f(\mathbf{x}_0+\mathbf{h}) - f(\mathbf{x}_0) - Df(\mathbf{x}_0)(\mathbf{h})\|}{\|\mathbf{h}\|}.\end{aligned}$$

By assumption, the larger expression limits to 0 as $\mathbf{h} \to \mathbf{0}$, and thus by the squeeze theorem we have shown

$$\lim_{\mathbf{h}\to\mathbf{0}} \|f(\mathbf{x}_0+\mathbf{h}) - f(\mathbf{x}_0) - Df(\mathbf{x}_0)(\mathbf{h})\| = 0$$

as desired! □

Another well known property of the one-variable derivative is that $\frac{d}{dx}(c) = 0$. In words, the derivative of a constant is 0. This is indeed the case in higher dimensions, interpreted appropriately.

Proposition 6.4. *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ is the constant map defined by $f(\mathbf{x}) = \mathbf{c}$ for some $\mathbf{c} \in \mathbb{R}^m$. Then f is differentiable at all points $\mathbf{x}_0$ and $Df(\mathbf{x}_0) = 0$, where $0 : \mathbb{R}^n \to \mathbb{R}^m$ is the 0 map.*

Proof. Fix $\mathbf{x}_0$. We need to show that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|f(\mathbf{x}_0+\mathbf{h}) - f(\mathbf{x}_0) - 0(\mathbf{h})\|}{\|\mathbf{h}\|} = 0.$$

But $f(\mathbf{x}_0+\mathbf{h}) = \mathbf{c}$, $f(\mathbf{x}_0) = \mathbf{c}$, and $0(\mathbf{h}) = \mathbf{0}$, so

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|f(\mathbf{x}_0+\mathbf{h}) - f(\mathbf{x}_0) - 0(\mathbf{h})\|}{\|\mathbf{h}\|} = \lim_{\mathbf{h}\to\mathbf{0}} \frac{\|\mathbf{c}-\mathbf{c}-\mathbf{0}\|}{\|\mathbf{h}\|} = \lim_{\mathbf{h}\to\mathbf{0}} \frac{\|\mathbf{0}\|}{\|\mathbf{h}\|} = 0$$

as desired. □

The next proposition should feel a little uncomfortable. This discomfort is due to our change in perspective. In particular, consider the function $f(x) = 2x$. In calculus, we would say that the derivative of f at any point is the number 2: $f'(x_0) = 2$. But with our new perspective, the derivative of f at x_0 is actually a function of h:

$$Df(x_0)(h) = f'(x_0)h = 2h.$$

Note that we began with the function $2x$, and the derivative at x_0 is the function $2h$. The variables are different, but they are the same function! In other words, $Df(x_0) = f$. Indeed, this generalizes to any linear map: the derivative of a linear transformation is itself!

Proposition 6.5. *Suppose that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map. Then T is differentiable at any $\mathbf{x}_0 \in \mathbb{R}^n$ and $DT(\mathbf{x}_0) = T$.*

Proof. Fix $\mathbf{x}_0 \in \mathbb{R}^n$. We need to show that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|T(\mathbf{x}_0+\mathbf{h}) - T(\mathbf{x}_0) - T(\mathbf{h})\|}{\|\mathbf{h}\|} = 0.$$

We can use linearity of T to great effect. Indeed, note that

$$\begin{aligned}
\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|T(\mathbf{x}_0+\mathbf{h}) - T(\mathbf{x}_0) - T(\mathbf{h})\|}{\|\mathbf{h}\|} &= \lim_{\mathbf{h}\to\mathbf{0}} \frac{\|T(\mathbf{x}_0) + T(\mathbf{h}) - T(\mathbf{x}_0) - T(\mathbf{h})\|}{\|\mathbf{h}\|} \\
&= \lim_{\mathbf{h}\to\mathbf{0}} \frac{\|\mathbf{0}\|}{\|\mathbf{h}\|} \\
&= 0
\end{aligned}$$

as desired. □

The next property is another familiar one: linearity of the derivative operator.

Proposition 6.6. *Suppose that $f, g : \mathbb{R}^n \to \mathbb{R}^m$ are differentiable at $\mathbf{x}_0 \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. Then $f + g$ and λf are differentiable at $\mathbf{x}_0$,*

$$D(f+g)(\mathbf{x}_0) = Df(\mathbf{x}_0) + Dg(\mathbf{x}_0)$$

and

$$D(\lambda f)(\mathbf{x}_0) = \lambda Df(\mathbf{x}_0).$$

Proof. We show the latter statement first. As usual, we need to show that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|(\lambda f)(\mathbf{x}_0+\mathbf{h}) - (\lambda f)(\mathbf{x}_0) - \lambda Df(\mathbf{x}_0)(\mathbf{h})\|}{\|\mathbf{h}\|} = 0.$$

But we can simply factor out all the λ's; note that

$$\begin{aligned}
\frac{\|(\lambda f)(\mathbf{x}_0+\mathbf{h}) - (\lambda f)(\mathbf{x}_0) - \lambda Df(\mathbf{x}_0)(\mathbf{h})\|}{\|\mathbf{h}\|} &= \frac{\|\lambda f(\mathbf{x}_0+\mathbf{h}) - \lambda f(\mathbf{x}_0) - \lambda Df(\mathbf{x}_0)(\mathbf{h})\|}{\|\mathbf{h}\|} \\
&= |\lambda| \frac{\|f(\mathbf{x}_0+\mathbf{h}) - f(\mathbf{x}_0) - Df(\mathbf{x}_0)(\mathbf{h})\|}{\|\mathbf{h}\|}
\end{aligned}$$

Since f is differentiable at $\mathbf{x}_0$,

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|f(\mathbf{x}_0+\mathbf{h}) - f(\mathbf{x}_0) - Df(\mathbf{x}_0)(\mathbf{h})\|}{\|\mathbf{h}\|} = 0.$$

Thus,

$$\lim_{\mathbf{h}\to\mathbf{0}} |\lambda| \frac{\|f(\mathbf{x}_0+\mathbf{h}) - f(\mathbf{x}_0) - Df(\mathbf{x}_0)(\mathbf{h})\|}{\|\mathbf{h}\|} = 0$$

as desired.

The proof of additivity is similar and is left as an exercise. □

The last important property I'm going to state is the chain rule. We will revisit the chain rule later to bring it down to earth, but for now I'll leave you with the abstract statement. The proof is quite technical and you should probably skip it for now.

Theorem 6.7 (Chain rule)**.** *Suppose that $g : \mathbb{R}^k \to \mathbb{R}^n$ is differentiable at $\mathbf{x}_0 \in \mathbb{R}^k$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $g(\mathbf{x}_0) \in \mathbb{R}^n$. Then $f \circ g : \mathbb{R}^k \to \mathbb{R}^m$ is differentiable at $\mathbf{x}_0 \in \mathbb{R}^k$ and*

$$D(f \circ g)(\mathbf{x}_0) = Df(g(\mathbf{x}_0)) \circ Dg(\mathbf{x}_0).$$

Proof. As usual, we want to show that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|(f\circ g)(\mathbf{x}_0+\mathbf{h})-(f\circ g)(\mathbf{x}_0)-(Df(g(\mathbf{x}_0))\circ Dg(\mathbf{x}_0))(\mathbf{h})\|}{\|\mathbf{h}\|}=0.$$

Note that

$$\begin{aligned}
(f\circ g)(\mathbf{x}_0{+}\mathbf{h}) &- (f\circ g)(\mathbf{x}_0)-(Df(g(\mathbf{x}_0))\circ Dg(\mathbf{x}_0))(\mathbf{h})\\
&= f(g(\mathbf{x}_0+\mathbf{h}))-f(g(\mathbf{x}_0))-Df(g(\mathbf{x}_0))\,[Dg(\mathbf{x}_0))(\mathbf{h})]\\
&= f(g(\mathbf{x}_0+\mathbf{h}))-f(g(\mathbf{x}_0))\\
&\qquad - Df(g(\mathbf{x}_0))\,[g(\mathbf{x}_0+\mathbf{h})-g(\mathbf{x}_0)-g(\mathbf{x}_0+\mathbf{h})+g(\mathbf{x}_0)+Dg(\mathbf{x}_0))(\mathbf{h})]\\
&= f(g(\mathbf{x}_0+\mathbf{h}))-f(g(\mathbf{x}_0))-Df(g(\mathbf{x}_0))\,[g(\mathbf{x}_0+\mathbf{h})-g(\mathbf{x}_0)]\\
&\qquad + Df(g(\mathbf{x}_0))\,[-g(\mathbf{x}_0+\mathbf{h})+g(\mathbf{x}_0)+Dg(\mathbf{x}_0))(\mathbf{h})]\\
&= f(g(\mathbf{x}_0+\mathbf{h}))-f(g(\mathbf{x}_0))-Df(g(\mathbf{x}_0))\,[g(\mathbf{x}_0+\mathbf{h})-g(\mathbf{x}_0)]\\
&\qquad - Df(g(\mathbf{x}_0))\,[g(\mathbf{x}_0+\mathbf{h})-g(\mathbf{x}_0)-Dg(\mathbf{x}_0))(\mathbf{h})]\,.
\end{aligned}$$

Here we have cleverly inserted a number of terms (highlighted in red) into a forest of parentheses and made use of linearity of the operator $Df(g(\mathbf{x}_0)):\mathbb{R}^n\to\mathbb{R}^m$. Thus, by the triangle inequality,

$$\begin{aligned}
&\frac{\|(f\circ g)(\mathbf{x}_0+\mathbf{h})-(f\circ g)(\mathbf{x}_0)-(Df(g(\mathbf{x}_0))\circ Dg(\mathbf{x}_0))(\mathbf{h})\|}{\|\mathbf{h}\|}\\
&\qquad\le \frac{\|f(g(\mathbf{x}_0+\mathbf{h}))-f(g(\mathbf{x}_0))-Df(g(\mathbf{x}_0))\,[g(\mathbf{x}_0+\mathbf{h})-g(\mathbf{x}_0)]\|}{\|\mathbf{h}\|}\\
&\qquad\qquad + \frac{\|Df(g(\mathbf{x}_0))\,[g(\mathbf{x}_0+\mathbf{h})-g(\mathbf{x}_0)-Dg(\mathbf{x}_0))(\mathbf{h})]\|}{\|\mathbf{h}\|}
\end{aligned}\tag{6.1}$$

We consider each of the two terms on the right hand side of the inequality separately.

<u>The blue term.</u>

Consider the blue expression first. Let $\tilde{\mathbf{h}}:=g(\mathbf{x}_0+\mathbf{h})-g(\mathbf{x}_0)$. Since g is differentiable at $\mathbf{x}_0$, it is continuous at $\mathbf{x}_0$. Thus, as $\mathbf{h}\to\mathbf{0}$ we have $\tilde{\mathbf{h}}\to\mathbf{0}$. Therefore,

$$\begin{aligned}
&\lim_{\mathbf{h}\to\mathbf{0}}\frac{\|f(g(\mathbf{x}_0+\mathbf{h}))-f(g(\mathbf{x}_0))-Df(g(\mathbf{x}_0))\,[g(\mathbf{x}_0+\mathbf{h})-g(\mathbf{x}_0)]\|}{\|\mathbf{h}\|}\\
&\qquad=\lim_{\tilde{\mathbf{h}}\to\mathbf{0}}\frac{\left\|f(g(\mathbf{x}_0)+\tilde{\mathbf{h}})-f(g(\mathbf{x}_0))-Df(g(\mathbf{x}_0))[\tilde{\mathbf{h}}]\right\|}{\|\mathbf{h}\|}\\
&\qquad\qquad=\lim_{\tilde{\mathbf{h}}\to\mathbf{0}}\frac{\left\|f(g(\mathbf{x}_0)+\tilde{\mathbf{h}})-f(g(\mathbf{x}_0))-Df(g(\mathbf{x}_0))[\tilde{\mathbf{h}}]\right\|}{\left\|\tilde{\mathbf{h}}\right\|}\cdot\frac{\left\|\tilde{\mathbf{h}}\right\|}{\|\mathbf{h}\|}.
\end{aligned}$$

Note that, by differentiability of f at $g(\mathbf{x}_0)$,

$$\lim_{\tilde{\mathbf{h}}\to\mathbf{0}}\frac{\left\|f(g(\mathbf{x}_0)+\tilde{\mathbf{h}})-f(g(\mathbf{x}_0))-Df(g(\mathbf{x}_0))[\tilde{\mathbf{h}}]\right\|}{\left\|\tilde{\mathbf{h}}\right\|}=0.$$

This is exactly the point of the variable substitution I made. However, to conclude that the entire limit above is 0 we need to consider the $\frac{\|\tilde{\mathbf{h}}\|}{\|\mathbf{h}\|}$ term as well. In particular, I will show that this term is bounded as $\mathbf{h} \to \mathbf{0}$. Note that

$$\begin{aligned}
\frac{\left\|\tilde{\mathbf{h}}\right\|}{\|\mathbf{h}\|} &= \frac{\|g(\mathbf{x}_0+\mathbf{h}) - g(\mathbf{x}_0)\|}{\|\mathbf{h}\|} \\
&= \frac{\|g(\mathbf{x}_0+\mathbf{h}) - g(\mathbf{x}_0) - Dg(\mathbf{x}_0)(\mathbf{h}) + Dg(\mathbf{x}_0)(\mathbf{h})\|}{\|\mathbf{h}\|} \\
&\leq \frac{\|g(\mathbf{x}_0+\mathbf{h}) - g(\mathbf{x}_0) - Dg(\mathbf{x}_0)(\mathbf{h})\|}{\|\mathbf{h}\|} + \frac{\|Dg(\mathbf{x}_0)(\mathbf{h})\|}{\|\mathbf{h}\|}.
\end{aligned}$$

Here we have again inserted some terms and used the triangle inequality. By differentiability of g at $\mathbf{x}_0$, the first term limits to 0 as $\mathbf{h} \to \mathbf{0}$, so we may assume that the first term is bounded above by 1. It remains to bound the second term. That this term is bounded is a general fact about linear maps (in finite dimensions!) that I could have stated separately much earlier, but alas — I will prove it here. Write $\mathbf{h} = h_1\mathbf{e}_1 + \cdots + h_k\mathbf{e}_k$. Then, using linearity of $Dg(\mathbf{x}_0)$, we have

$$Dg(\mathbf{x}_0)(\mathbf{h}) = Dg(\mathbf{x}_0)(h_1\mathbf{e}_1 + \cdots + h_k\mathbf{e}_k) = h_1 Dg(\mathbf{x}_0)(\mathbf{e}_1) + \cdots + h_k Dg(\mathbf{x}_0)(\mathbf{e}_k).$$

By the triangle inequality,

$$\begin{aligned}
\|Dg(\mathbf{x}_0)(\mathbf{h})\| &\leq |h_1|\,\|Dg(\mathbf{x}_0)(\mathbf{e}_1)\| + \cdots + |h_k|\,\|Dg(\mathbf{x}_0)(\mathbf{e}_k)\| \\
&\leq \|\mathbf{h}\|\,\|Dg(\mathbf{x}_0)(\mathbf{e}_1)\| + \cdots + \|\mathbf{h}\|\,\|Dg(\mathbf{x}_0)(\mathbf{e}_k)\| \\
&= \|\mathbf{h}\|\,(\|Dg(\mathbf{x}_0)(\mathbf{e}_1)\| + \cdots + \|Dg(\mathbf{x}_0)(\mathbf{e}_k)\|)
\end{aligned}$$

Here we used the observation that $|h_k| \leq \|\mathbf{h}\|$ to get the second inequality. Let

$$M := \|Dg(\mathbf{x}_0)(\mathbf{e}_1)\| + \cdots + \|Dg(\mathbf{x}_0)(\mathbf{e}_k)\| < \infty.$$

Then we have shown that $\|Dg(\mathbf{x}_0)(\mathbf{h})\| \leq M\,\|\mathbf{h}\|$ and thus $\frac{\|Dg(\mathbf{x}_0)(\mathbf{h})\|}{\|\mathbf{h}\|} \leq M$. At last, this implies that $\frac{\|\tilde{\mathbf{h}}\|}{\|\mathbf{h}\|} \leq 1+M$ as $\mathbf{h} \to \mathbf{0}$. Using this and the earlier remark about differentiability of f at $g(\mathbf{x}_0)$, we can finally show that the blue term limits to 0. We have

$$\begin{aligned}
&\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|f(g(\mathbf{x}_0+\mathbf{h})) - f(g(\mathbf{x}_0)) - Df(g(\mathbf{x}_0))\,[g(\mathbf{x}_0+\mathbf{h}) - g(\mathbf{x}_0)]\|}{\|\mathbf{h}\|} \\
&\qquad\qquad \leq \lim_{\tilde{\mathbf{h}}\to\mathbf{0}} \frac{\left\|f(g(\mathbf{x}_0)+\tilde{\mathbf{h}}) - f(g(\mathbf{x}_0)) - Df(g(\mathbf{x}_0))[\tilde{\mathbf{h}}]\right\|}{\left\|\tilde{\mathbf{h}}\right\|} \cdot (1+M) = 0.
\end{aligned}$$

Thus,

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|f(g(\mathbf{x}_0+\mathbf{h})) - f(g(\mathbf{x}_0)) - Df(g(\mathbf{x}_0))\,[g(\mathbf{x}_0+\mathbf{h}) - g(\mathbf{x}_0)]\|}{\|\mathbf{h}\|} = 0. \tag{6.2}$$

<u>The brown term.</u>

Next we will show that the brown term limits to 0 as well. By the same argument as above that showed that $\|Dg(\mathbf{x}_0)(\mathbf{h})\| \leq M\,\|\mathbf{h}\|$ for some number M, it follows that

$$\|Df(g(\mathbf{x}_0))\,[g(\mathbf{x}_0+\mathbf{h}) - g(\mathbf{x}_0) - Dg(\mathbf{x}_0))(\mathbf{h})]\| \leq M_2\,\|g(\mathbf{x}_0+\mathbf{h}) - g(\mathbf{x}_0) - Dg(\mathbf{x}_0))(\mathbf{h})\|$$

for some number M_2. Thus,

$$\begin{aligned}
&\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|Df(g(\mathbf{x}_0))\,[g(\mathbf{x}_0+\mathbf{h}) - g(\mathbf{x}_0) - Dg(\mathbf{x}_0))(\mathbf{h})]\|}{\|\mathbf{h}\|} \\
&\qquad\qquad \leq \lim_{\mathbf{h}\to\mathbf{0}} M_2 \frac{\|g(\mathbf{x}_0+\mathbf{h}) - g(\mathbf{x}_0) - Dg(\mathbf{x}_0))(\mathbf{h})\|}{\|\mathbf{h}\|}.
\end{aligned}$$

By differentiability of g at $\mathbf{x}_0$, this right hand side limits to 0. Thus,

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|Df(g(\mathbf{x}_0))\left[g(\mathbf{x}_0+\mathbf{h})-g(\mathbf{x}_0)-Dg(\mathbf{x}_0))(\mathbf{h})\right]\|}{\|\mathbf{h}\|} = 0. \tag{6.3}$$

Finally, applying (6.2) and (6.3) to the main estimate (6.1), we have

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|(f\circ g)(\mathbf{x}_0+\mathbf{h})-(f\circ g)(\mathbf{x}_0)-(Df(g(\mathbf{x}_0))\circ Dg(\mathbf{x}_0))(\mathbf{h})\|}{\|\mathbf{h}\|} \leq 0+0$$

and therefore

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|(f\circ g)(\mathbf{x}_0+\mathbf{h})-(f\circ g)(\mathbf{x}_0)-(Df(g(\mathbf{x}_0))\circ Dg(\mathbf{x}_0))(\mathbf{h})\|}{\|\mathbf{h}\|} = 0$$

as desired.

□

6.3 Partial derivatives and the Jacobian

All of the above theory of the derivative as an abstract linear transformation is great, and generalizes the one variable theory in a pleasing way. Unfortunately, it tells us almost nothing about how to actually compute derivatives. We computed the derivative of any linear function $T : \mathbb{R}^n \to \mathbb{R}^m$, but that's about it. The goal of this section is to bring the abstract theory down to earth, and study differentiation from a more computational perspective.

We begin by introducing a separate notion of multivariable differentiation which is more geometrically motivated. A priori, this may or may not have anything to do with our notion of the derivative above. (It actually will, but we'll have to prove that.) Anyway, for the moment, consider a function $f : \mathbb{R}^2 \to \mathbb{R}$. The graph of this function lives in $\mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$, and in particular is some kind of surface.

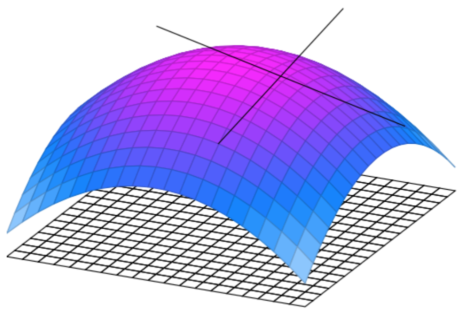


Figure 6.1: The graph of a function $f(x, y)$ with two different tangent lines at a given point.

In single variable calculus, given the graph of a function we wanted to find the equation of a line tangent to the graph. The key to doing so was determining the instantaneous slope of the graph at that point. As mentioned in the introduction of this chapter, this was done by taking successively more refined difference quotients based at the corresponding point in the domain:

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}.$$

Given the graph of a function $f(x,y)$, we could also seek tangent lines at a point. However, at a point on a surface there are many different tangent lines; not just one. We can produce different tangent lines by restricting our movement in the domain at the desired point. In particular, at $\mathbf{x}_0 \in \mathbb{R}^2$ we could study how the function changes if we only vary the x coordinate, keeping the y coordinate fixed. Using the above different quotient, this would give us the slope of the graph "in the x direction." Likewise, we could restrict our movement in the domain to be in the y direction to compute the slope of the graph "in the y direction." We could actually move in *any* fixed direction, but for now we will restrict our movement to fixed-coordinate direction.

This discussion motivates the following definition.

Definition 6.8. Let $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{x}_0 \in \mathbb{R}^n$. The j**th partial derivative of** f **at** $\mathbf{x}_0$ is

$$\frac{\partial f}{\partial x_j}(\mathbf{x}_0) = f_{x_j}(\mathbf{x}_0) := \lim_{t\to 0} \frac{f(\mathbf{x}_0 + t\mathbf{e}_j) - f(\mathbf{x}_0)}{t}.$$

Here $\mathbf{e}_j \in \mathbb{R}^n$ is the standard jth basis vector.

Remark 6.9. We will also write e.g. $\frac{\partial f}{\partial y}$ or f_y when using different variable names.

As described by the motivation above, you can think of the number $\frac{\partial f}{\partial x_j}(\mathbf{x}_0)$ as the instantaneous slope of the tangent line to the graph of f at $(\mathbf{x}_0, f(\mathbf{x}_0)$ in the x_j direction.

The nice thing about partial derivatives is that they more naturally generalize the computational techniques of single variable calculus. For example, given a function $f(x,y)$ of two variables, you compute the partial with respect to x as follows: pretend that y is a constant, and take the derivative as you normally would. Likewise, the partial with respect to y is computed by simply pretending that x is constant. I won't prove this claim, as the proof is the same as that of the usual single variable rules.

Example 6.10. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = xe^{xy}$. Then

$$\begin{aligned}\frac{\partial f}{\partial x}(x,y) &= \frac{\partial}{\partial x}\left(xe^{xy}\right) \\ &= \frac{\partial}{\partial x}(x)\cdot e^{xy} + x\cdot\frac{\partial}{\partial x}(e^{xy}) \\ &= e^{xy} + xye^{xy}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial f}{\partial y}(x,y) &= \frac{\partial}{\partial y}(xe^{xy}) \\ &= x\frac{\partial}{\partial y}(e^{xy}) \\ &= x^2e^{xy}.\end{aligned}$$

Now we return to considering functions $f : \mathbb{R}^n \to \mathbb{R}^m$. Because f outputs vectors in $\mathbb{R}^n$, we can always write

$$f = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}$$

where $f_j : \mathbb{R}^n \to \mathbb{R}$ for $j = 1, \ldots, m$. Each of these functions f_j are scalar-valued, so we can compute their partial derivatives. In the next definition, we collect all of the partial derivatives into a matrix.

Definition 6.11. Let $f = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{x}_0 \in \mathbb{R}^n$. The **Jacobian matrix of f at $\mathbf{x}_0$**, denoted $f'(\mathbf{x}_0)$, is

$$f'(\mathbf{x}_0) := \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}_0) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}_0) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{x}_0) \end{pmatrix} = \left(\frac{\partial f_i}{\partial x_j}(\mathbf{x}_0) \right) \in M_{m \times n}(\mathbb{R})$$

provided that all of the above partial derivatives exist.

Example 6.12. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$f(x, y, z) = \begin{pmatrix} xyz \\ e^y + x^2 \end{pmatrix}.$$

Let's compute the Jacobian matrix $f'(x, y, z)$. Note that $f'(x, y, z)$ will be a 2×3 matrix. Here, $f_1(x, y, z) = xyz$ and $f_2(x, y, z) = e^y + x^2$. Thus,

$$\begin{aligned} f'(x, y, z) &= \begin{pmatrix} (f_1)_x & (f_1)_y & (f_1)_z \\ (f_2)_x & (f_2)_y & (f_2)_z \end{pmatrix} \\ &= \begin{pmatrix} yz & xz & xy \\ 2x & e^y & 0 \end{pmatrix}. \end{aligned}$$

The above expression gives the Jacobian matrix as a *function* of the input variables. More in tune with the preceding discussion would be to compute the Jacobian at a single *point*. For example,

$$f'(1, 2, 3) = \begin{pmatrix} 6 & 3 & 2 \\ 2 & e^2 & 0 \end{pmatrix}.$$

The connection between partial derivatives and the abstract derivative from the previous subsection is given by the following theorem. Briefly, the theorem says that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\mathbf{x}_0 \in \mathbb{R}^n$, then the standard matrix of $Df(\mathbf{x}_0) : \mathbb{R}^n \to \mathbb{R}^m$ is the Jacobian matrix!

Theorem 6.13. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{x}_0 \in \mathbb{R}^n$. If f is differentiable at $\mathbf{x}_0$, then all of the partial derivatives of the components of f exist at $\mathbf{x}_0$, and the standard matrix of $Df(\mathbf{x}_0) : \mathbb{R}^n \to \mathbb{R}^m$ is $f'(\mathbf{x}_0) \in M_{m \times n}(\mathbb{R})$. In other words,*

$$Df(\mathbf{x}_0)(\mathbf{h}) = f'(\mathbf{x}_0)\mathbf{h}.$$

Remark 6.14. It is important to be able to grammatically parse the above equality! On the left side, $Df(\mathbf{x}_0)$ is an abstract linear map with variable $\mathbf{h} \in \mathbb{R}^n$. On the right, $f'(\mathbf{x}_0)$ is an $m \times n$ matrix and thus $f'(\mathbf{x}_0)\mathbf{h}$ is a matrix-vector product.

Proof. We will consider the quantity

$$\left\| \frac{f(\mathbf{x}_0 + t\mathbf{e}_j) - f(\mathbf{x}_0)}{t} - Df(\mathbf{x}_0)(\mathbf{e}_j) \right\|.$$

Here $t > 0$. Note that

$$\begin{aligned}\left\| \frac{f(\mathbf{x}_0 + t\mathbf{e}_j) - f(\mathbf{x}_0)}{t} - Df(\mathbf{x}_0)(\mathbf{e}_j) \right\| &= \left\| \frac{f(\mathbf{x}_0 + t\mathbf{e}_j) - f(\mathbf{x}_0) - tDf(\mathbf{x}_0)(\mathbf{e}_j)}{t} \right\| \\ &= \left\| \frac{f(\mathbf{x}_0 + t\mathbf{e}_j) - f(\mathbf{x}_0) - Df(\mathbf{x}_0)(t\mathbf{e}_j)}{t} \right\| \\ &= \frac{\|f(\mathbf{x}_0 + t\mathbf{e}_j) - f(\mathbf{x}_0) - Df(\mathbf{x}_0)(t\mathbf{e}_j)\|}{\|t\mathbf{e}_j\|}.\end{aligned}$$

In the last equality, we used the fact that $\|\mathbf{e}_j\| = 1$ to rewrite $|t| = \|t\mathbf{e}_j\|$. By definition of differentiability of f at $\mathbf{x}_0$,

$$\lim_{t\to 0} \frac{\|f(\mathbf{x}_0 + t\mathbf{e}_j) - f(\mathbf{x}_0) - Df(\mathbf{x}_0)(t\mathbf{e}_j)\|}{\|t\mathbf{e}_j\|} = 0$$

since $\mathbf{h} := t\mathbf{e}_j \to \mathbf{0}$ as $t \to 0$. Thus, we have shown that

$$\lim_{t\to 0} \left\| \frac{f(\mathbf{x}_0 + t\mathbf{e}_j) - f(\mathbf{x}_0)}{t} - Df(\mathbf{x}_0)(\mathbf{e}_j) \right\| = 0.$$

By definition of convergence of a sequence of vectors, this implies that

$$\lim_{t\to 0} \frac{f(\mathbf{x}_0 + t\mathbf{e}_j) - f(\mathbf{x}_0)}{t} = Df(\mathbf{x}_0)(\mathbf{e}_j).$$

It follows that the jth column of the standard matrix of $Df(\mathbf{x}_0)$ is given by the m-vector

$$\lim_{t\to 0} \frac{f(\mathbf{x}_0 + t\mathbf{e}_j) - f(\mathbf{x}_0)}{t}.$$

Write $f = (f_1, \ldots, f_m)$. The ith component of the above limit is then the number

$$\lim_{t\to 0} \frac{f_j(\mathbf{x}_0 + t\mathbf{e}_j) - f_j(\mathbf{x}_0)}{t} = \frac{\partial f_i}{\partial x_j}(\mathbf{x}_0).$$

Thus, we have shown that all partial derivatives of the components of f exist, and that the (i, j)-th entry of the standard matrix of $Df(\mathbf{x}_0)$ is $\frac{\partial f_i}{\partial x_j}(\mathbf{x}_0)$. Hence the standard matrix of $Df(\mathbf{x}_0)$ is $f'(\mathbf{x}_0)$. $\square$

While this theorem is fantastic and nicely ties together the abstract notion of the derivative with the more down to earth notion of the Jacobian matrix, it is not without its subtleties and complications. For example, note that in the definition of the Jacobian matrix I did *not* require the function f to be differentiable. Indeed, all of the partial derivatives of a function can exist (and hence, the Jacobian $f'(\mathbf{x}_0)$ can exist) without the function being differentiable at that point! The above theorem says that *if* f is differentiable at that point, then all of the partial derivatives exist at that point. The converse is *not* true. Even if the partial derivatives exist at some point, it does not necessarily mean the function is differentiable.

Let's see two examples of this phenomenon.

Example 6.15. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}.$$

I claim that both partial derivatives $f_x(0,0)$ and $f_y(0,0)$ exist, but that f is not differentiable at $(0,0)$.

Let's first compute the partial derivatives at the origin. Because f is a rational function everywhere *except* the origin, we cannot use the usual power and product and quotient rules to compute the partials. We need to use the original limit definition. For example,

$$\begin{aligned} f_x(0,0) &= \lim_{t\to 0} \frac{f(\mathbf{0}+t\mathbf{e}_1) - f(\mathbf{0})}{t} \\ &= \lim_{t\to 0} \frac{f(t,0) - f(0,0)}{t} \\ &= \lim_{t\to 0} \frac{\frac{t^3}{t^2+0^2} - 0}{t} \\ &= \lim_{t\to 0} \frac{t^2}{t^2} \\ &= 1. \end{aligned}$$

Thus, $f_x(0,0)$ exists and equals 1. Similarly,

$$\begin{aligned} f_y(0,0) = \lim_{t\to 0} \frac{f(\mathbf{0}+t\mathbf{e}_2) - f(\mathbf{0})}{t} = \lim_{t\to 0} \frac{f(0,t) - f(0,0)}{t} = \lim_{t\to 0} \frac{\frac{0^3}{0^2+t^2} - 0}{t} &= \lim_{t\to 0} 0 \\ &= 0. \end{aligned}$$

Thus, $f_y(0,0)$ exists and equals 0.

Next, we claim that f is not differentiable at $(0,0)$. If it were, by Theorem 6.13, then $Df(\mathbf{0})(\mathbf{h}) = f'(\mathbf{0})\mathbf{h}$. By the above computations, $f'(\mathbf{0}) = \begin{pmatrix} 1 & 0 \end{pmatrix}$. Thus, writing $\mathbf{h} = (h_1, h_2)$, we have

$$\begin{aligned} \lim_{\mathbf{h}\to\mathbf{0}} \frac{\|f(\mathbf{0}+\mathbf{h}) - f(\mathbf{0}) - f'(\mathbf{0})\mathbf{h}\|}{\|\mathbf{h}\|} &= \lim_{(h_1,h_2)\to(0,0)} \frac{\frac{h_1^3}{h_1^2+h_2^2} - 0 - h_1}{\sqrt{h_1^2+h_2^2}} \\ &= \lim_{(h_1,h_2)\to(0,0)} \frac{h_1^3 - h_1(h_1^2+h_2^2)}{(h_1^2+h_2^2)^{\frac{3}{2}}}. \end{aligned}$$

Approaching the origin along $h_1 = h_2$ gives

$$\lim_{h_1\to 0} \frac{h_1^3 - 2h_1^3}{(2h_1^2)^{\frac{3}{2}}} = \lim_{h_1\to 0} \frac{-1}{2^{\frac{3}{2}}} = -\frac{1}{2^{\frac{3}{2}}}.$$

Thus, the above limit cannot be 0, and therefore f is not differentiable at the origin.

Here is an even simpler example.

Example 6.16. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}.$$

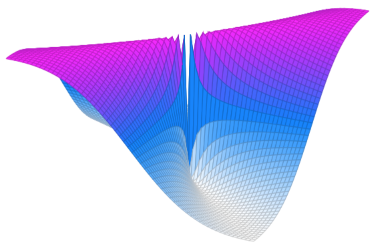
As before, I claim that both partial derivatives of f at $(0,0)$ exist, but that f is not differentiable at $(0,0)$. In fact, f is not even continuous at $(0,0)$!

Let's compute $f_x(0,0)$. Like the previous example, we have to use the limit definition. Thus,

$$f_x(0,0) = \lim_{t\to 0} \frac{f(\mathbf{0}+t\mathbf{e}_1) - f(\mathbf{0})}{t} = \lim_{t\to 0} \frac{f(t,0) - f(0,0)}{t} = \lim_{t\to 0} \frac{\frac{t\cdot 0}{t^2+0^2} - 0}{t} = \lim_{t\to 0} 0$$
$$= 0.$$

One can similarly show that $f_y(0,0) = 0$. Thus, both partials exist at the origin.

On the other hand, f is not continuous at $(0,0)$ because the limit $\lim_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x})$ doesn't even exist. Indeed, approaching along $x = 0$ gives a limit of 0 and approaching along $y = x$ gives a limit of $\frac{1}{2}$. Since f is not continuous at $(0,0)$, it cannot be differentiable at $(0,0)$ since differentiability implies continuity.



Fortunately, a partial converse to Theorem 6.13 exists, if we strengthen the hypothesis a bit. Briefly, the following theorem says that if the partial derivatives of a function exist *and are continuous* at some point, then the function is differentiable at that point. The proof is omitted.

Theorem 6.17. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{x}_0 \in \mathbb{R}^n$. Write $f = (f_1, \ldots, f_m)$. If the function $\frac{\partial f_i}{\partial x_j} : \mathbb{R}^n \to \mathbb{R}$ exists in a neighborhood (i.e., any open ball) around $\mathbf{x}_0$ and is continuous on that neighborhood, then f is differentiable at $\mathbf{x}_0$.*

6.3.1 Revisiting the chain rule

Recall from earlier the multivariable chain rule. Briefly, if $g : \mathbb{R}^k \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ are differentiable, then $g \circ f : \mathbb{R}^k \to \mathbb{R}^m$ is differentiable and

$$D(f \circ g)(\mathbf{x}_0) = Df(g(\mathbf{x}_0)) \circ Dg(\mathbf{x}_0).$$

The quantities $D(f\circ g)(\mathbf{x}_0)$, $Df(g(\mathbf{x}_0))$, and $Dg(\mathbf{x}_0)$ above are abstract linear maps. Now that we've studied the Jacobian, we can rephrase the chain rule in a more down to earth manner. In particular, since matrix multiplication corresponds to composition of linear transformations, and since the Jacobian matrix is the standard matrix of the derivative, we immediately have the following proposition.

Proposition 6.18 (Chain rule in coordinates)**.** *Suppose that $g : \mathbb{R}^k \to \mathbb{R}^n$ is differentiable at $\mathbf{x}_0 \in \mathbb{R}^k$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $g(\mathbf{x}_0) \in \mathbb{R}^n$. Then $f \circ g : \mathbb{R}^k \to \mathbb{R}^m$ is differentiable at $\mathbf{x}_0$ and*

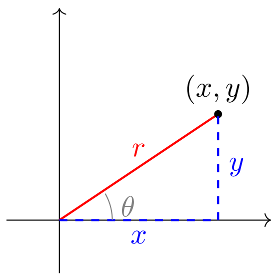
$$(f \circ g)'(\mathbf{x}_0) = f'(g(\mathbf{x}_0))g'(\mathbf{x}_0).$$

This looks just like the chain rule from single variable calculus! Indeed, the familiar number $f'(x_0)$ associated to a function $f : \mathbb{R} \to \mathbb{R}$ is the Jacobian of the abstract derivative $Df(x_0) : \mathbb{R} \to \mathbb{R}$. As usual, it is important to parse the grammar of the above statement. The matrix $(f \circ g)'(\mathbf{x}_0)$ is $m \times k$, the matrix $f'(g(\mathbf{x}_0))$ is $m \times n$, and the matrix $g'(\mathbf{x}_0)$ is $n \times k$.

I want to pick about this version of the chain rule even more. To do so, I will briefly discuss a different coordinate system called *polar coordinates*.

Polar coordinates

Consider a point $(x, y) \in \mathbb{R}^2$ which is not the origin. The two numbers x and y describe the horizontal and vertical position of the point, respectively. Sometimes it may be helpful to describe the point using different numbers: its *radius* $r > 0$, i.e., its distance from the origin, and the angle θ the point makes with the positive x-axis:



Some basic trigonometry suggests the following relations between x, y, r, θ:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ r^2 &= x^2 + y^2 \\ \tan\theta &= \frac{y}{x}. \end{aligned}$$

A chain rule computation

Now let's return to the chain rule. Suppose we have a function $f : \mathbb{R}^2_{x,y} \to R$ and we want to do some calculus with it. Perhaps with this particular function f it is more interesting to ask how f changes with respect to r and θ, rather than with respect to x and y. Let's define the function $\phi : (0, \infty)_r \times [0, 2\pi) \subset \mathbb{R}^2 \to \mathbb{R}^2_{x,y}$ by

$$\phi(r, \theta) = (r\cos\theta, r\sin\theta).$$

This function ϕ is basically the function which describes the correspondence between polar coordinates and cartesian coordinates. Indeed, the components of ϕ read $x = r\cos\theta$ and $y = r\sin\theta$. Finally let $\tilde{f} : (0, \infty)_r \times [0, 2\pi) \subset \mathbb{R}^2 \to \mathbb{R}$ be defined by $\tilde{f} = f \circ \phi$. In words, $\tilde{f}$ is essentially "f in polar coordinates."

So, to study how f changes with respect to a radial and angular change, we can compute $\frac{\partial \tilde{f}}{\partial r}$ and $\frac{\partial \tilde{f}}{\partial \theta}$. Applying the Jacobian version of the chain rule above gives

$$\begin{aligned} \tilde{f}'(r, \theta) &= f'(\phi(r, \theta))\phi'(r, \theta) \\ &= f'(x, y)\phi'(r, \theta). \end{aligned}$$

Note that

$$f'(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} \in M_{1\times 2}(\mathbb{R}).$$

Since $\phi(r,\theta) = (x,y)$, we can write the Jacobian $\phi'(r,\theta)$

$$\phi'(r,\theta) = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} \in M_{2\times 2}(\mathbb{R}).$$

We can compute this explicitly, but for the moment I'll leave it as above. The chain rule then tells us that

$$\begin{aligned} \begin{pmatrix} \frac{\partial \tilde{f}}{\partial r} & \frac{\partial \tilde{f}}{\partial \theta} \end{pmatrix} &= \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} & \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} \end{pmatrix}. \end{aligned}$$

Equating each component gives the following equations:

$$\begin{aligned} \frac{\partial \tilde{f}}{\partial r} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} \\ \frac{\partial \tilde{f}}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}. \end{aligned}$$

Let me explain the significance of these equations. We begun with f, which depends on x and y. Via polar coordinates, each of x and y further depend on r and θ. This gives us a diagram of variable dependencies:

$$\begin{tikzcd} & & f \arrow[dl] \arrow[dr] & & \\ & x \arrow[dl] \arrow[d] & & y \arrow[d] \arrow[dr] & \\ r & \theta & & r & \theta \end{tikzcd}$$

If we want to understand how f changes with respect to, say, θ, there are two possible avenues of dependence on θ: via x, and via y.

$$\begin{tikzcd} & & f \arrow[dl] \arrow[dr] & & \\ & x \arrow[dl] \arrow[d] & & y \arrow[d] \arrow[dr] & \\ r & \theta & & r & \theta \end{tikzcd}$$

The total change of f (really, $\tilde{f}s$) as θ changes is then computed by considering both avenues of change. The equation

$$\frac{\partial \tilde{f}}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}$$

says exactly this.

This diagram of variable dependence yielding partial derivative expression is typically how the multivariable chain rule is presented.

6.4 Higher order derivatives

Given a function $f : \mathbb{R}^n \to \mathbb{R}^m$, we defined the derivative of f at a point $\mathbf{x}_0 \in \mathbb{R}^n$, which was a linear map $Df(\mathbf{x}_0) : \mathbb{R}^n \to \mathbb{R}^m$. A natural question is how to define higher order derivatives, like the second derivative. It may be tempting to say: well, $Df(\mathbf{x}_0)$ is a function from $\mathbb{R}^n \to \mathbb{R}^m$ and I know how to compute the derivative of such a thing, so therefore the second derivative at $\mathbf{x}_0$ should be $Df(\mathbf{x}_0)$. Unfortunately, this is a bit naive. Consider the following single variable non-example: let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = x^2$. Then $f'(x) = 2x$ and so $f'(1) = 2$. If we want to compute the second derivative at 1, i.e., the number $f''(1)$, we simply differentiate the number $f'(1)$ to get 0. Therefore, $f''(1) = 0$. This is obviously wrong! Similarly, computing the derivative of $Df(\mathbf{x}_0)$ is akin to differentiating the value of the derivative at a single point. What we really want to do is differentiate the function Df as a function of $\mathbf{x}$, rather than $Df(\mathbf{x}_0)$ as a function of $\mathbf{h}$. After all, $Df(\mathbf{x}_0)$ is linear and hence the derivative of $Df(\mathbf{x}_0)$ is itself! This would not be very interesting as a second derivative.

A descent into abstract oblivion

Motivated by the above discussion, the correct notion of the second derivative should involve computing the derivative of Df, not $Df(\mathbf{x}_0)$. However, this introduces some new complications. What is the codomain of Df? Well, if we feed Df a point $\mathbf{x}$, we get a linear map $Df(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^m$. Thus, the output of the function Df is a linear map. If we let $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ denote the set of linear maps $T : \mathbb{R}^n \to \mathbb{R}^m$, then we have argued that Df is a function of the following form:

$$Df : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m).$$

This is interesting, but poses a problem in terms of differentiation. We know how to differentiate functions $\mathbb{R}^n \to \mathbb{R}^m$, but we have no clue how to differentiate functions $\mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$.

It turns out that this is possible, but it requires some more sophisticated tools. In particular, one needs to equip the (vector) space $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ with a *norm*. Just for fun, if we did this then presumably the second derivative $D^2 f$ would be a map from $\mathbb{R}^n$ to the space of linear maps from $\mathbb{R}^n$ to the space of linear maps from $\mathbb{R}^n \to \mathbb{R}^m$:

$$D^2 f : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)).$$

Yikes! This quickly becomes intimidating. Like I mentioned, it is actually possible to make coherent sense of this, but we will not do so here.

Some fake math

In the next subsection, I'll state the definition of the second derivative as we'll use it in these notes. We need to do a bit of fake math (slightly handwavy is probably a better term) to motivate this definition.

First, one simplification we will make is to consider functions of the form $f : \mathbb{R}^n \to \mathbb{R}$, rather than functions $f : \mathbb{R}^n \to \mathbb{R}^m$. In this case, the above discussion implies that Df is a map of the following form:

$$Df : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}).$$

Via the standard matrix, we can effectively equate the space of linear maps $\mathbb{R}^n \to \mathbb{R}$ with the set of $1 \times n$ matrices, i.e., with the set of row-vectors. More precisely, $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$ is *isomorphic as a vector space* to $M_{1\times n}(\mathbb{R})$, but don't worry if that means nothing to you. Thus, we have

$\mathcal{L}(\mathbb{R}^n, \mathbb{R}) \cong M_{1\times n}(\mathbb{R})$. Similarly, we can take the transpose of a $1 \times n$ row vector to get an $n \times 1$ column vector, i.e., an element of $\mathbb{R}^n$. Thus,

$$\mathcal{L}(\mathbb{R}^n, \mathbb{R}) \cong M_{1\times n}(\mathbb{R}) \cong M_{n\times 1}(\mathbb{R}) \cong \mathbb{R}^n.$$

Using this chain of equivalences, we can thus view the derivative function Df of a function $f : \mathbb{R}^n \to \mathbb{R}$ in the following way:

$$Df : \mathbb{R}^n \to \mathbb{R}^n.$$

This is now a function that we *can* differentiate!

The second derivative

The above discussion motivates the following definition.

Definition 6.19. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable. The **gradient vector** $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ is

$$\nabla f(\mathbf{x}) := (f'(\mathbf{x}))^T = \begin{pmatrix} f_{x_1}(\mathbf{x}) \\ f_{x_2}(\mathbf{x}) \\ \vdots \\ f_{x_n}(\mathbf{x}) \end{pmatrix}.$$

Example 6.20. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(x, y, z) = xy^2 + z$. Then the Jacobian matrix of f is

$$\begin{pmatrix} y^2 & 2xy & 1 \end{pmatrix}$$

and the gradient vector of f is

$$\begin{pmatrix} y^2 \\ 2xy \\ 1 \end{pmatrix}.$$

The gradient is extremely important, and we will return to it in a later section. For now, it is simply a tool for us to formally define the second derivative.

Definition 6.21. A function $f : \mathbb{R}^n \to \mathbb{R}$ is **twice-differentiable** if $f : \mathbb{R}^n \to \mathbb{R}$ and $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ are both differentiable. If f is twice-differentiable, the **second derivative of f at $\mathbf{x}$** is

$$D^2 f(\mathbf{x}) := D(\nabla f)(\mathbf{x}).$$

In words, we define the second derivative of a function to be the usual derivative of the gradient vector. Note that since $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$, $D(\nabla f)(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map. Thus, we may view $D^2 f(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^n$ as a linear map and consider its standard matrix, which will be a square matrix.

Definition 6.22. Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice-differentiable. The **Hessian matrix**, denoted $f''(\mathbf{x}) \in M_{n\times n}(\mathbb{R})$, is the standard matrix of $D^2 f(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^n$. In other words,

$$f''(\mathbf{x}) = (\nabla f)'(\mathbf{x}) = ((f')^T)'(\mathbf{x}).$$

We are finally about to bring second derivatives completely down to earth. Let's compute the Hessian matrix of a function $f : \mathbb{R}^n \to \mathbb{R}$ explicitly. Since

$$(\nabla f)(\mathbf{x}) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(\mathbf{x}) \\ \vdots \\ \frac{\partial f}{\partial x_n}(\mathbf{x}) \end{pmatrix}$$

it follows from the usual Jacobian computation that

$$f''(\mathbf{x}) = \begin{pmatrix} \frac{\partial}{\partial x_1}\frac{\partial f}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial}{\partial x_n}\frac{\partial f}{\partial x_1}(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_1}\frac{\partial f}{\partial x_n}(\mathbf{x}) & \cdots & \frac{\partial}{\partial x_n}\frac{\partial f}{\partial x_n}(\mathbf{x}) \end{pmatrix}.$$

In words, the Hessian matrix is the $n \times n$ matrix of all *second-order partial derivatives* of f!

Definition 6.23. Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice-differentiable. The **second-order partial derivatives** are

$$f_{x_j x_i} = \frac{\partial^2 f}{\partial x_i \, \partial x_j} := \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right).$$

To clarify the variable order in the above definition, if $f(x, y)$ then

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).$$

We will discuss more of the geometric implications of the second derivative soon, but for now we will close this subsection with a simple computational example that collects all of the notions of "derivative" that we have discussed so far.

Example 6.24. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x^2 y$. We will compute:

$$f_x, f_y, f_{xx}, f_{xy}, f_{yx}, f_{yy}, f'(x,y), Df(x,y)(h_1,h_2), \nabla f(x,y), f''(x,y).$$

First, we have all of the partial derivatives:

$$\begin{aligned} f_x(x,y) &= 2xy \\ f_y(x,y) &= x^2 \\ f_{xx}(x,y) &= (f_x)_x(x,y) = 2y \\ f_{xy}(x,y) &= (f_x)_y(x,y) = 2x \\ f_{yx}(x,y) &= (f_y)_x(x,y) = 2x \\ f_{yy}(x,y) &= (f_y)_y(x,y) = 0. \end{aligned}$$

Next, the Jacobian is the following 1×2 matrix:

$$f'(x,y) = \begin{pmatrix} 2xy & x^2 \end{pmatrix}.$$

The Jacobian is the standard matrix of the derivative $Df(x, y)$, thus

$$Df(x,y)(h_1,h_2) = f'(x,y)\mathbf{h} = \begin{pmatrix} 2xy & x^2 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = 2xyh_1 + x^2 h_2.$$

The gradient vector is the transpose of the Jacobian matrix:

$$\nabla f(x,y) = \begin{pmatrix} 2xy \\ x^2 \end{pmatrix}$$

and the Hessian is the Jacobian of the gradient:

$$f''(x,y) = \begin{pmatrix} f_x x(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{pmatrix} = \begin{pmatrix} 2y & 2x \\ 2x & 0 \end{pmatrix}.$$

An important observation from the previous example is that the mixed partial derivatives f_{xy} and f_{yx} were the same. Usually, this is not a coincidence! The following result shows that *with some assumptions*, you can take partial derivatives in any order you like.

Proposition 6.25 (Clairaut's theorem). *Let $f : \mathbb{R}^n \to \mathbb{R}$. Suppose that*

$$\frac{\partial^2 f}{\partial x_i \, \partial x_j} : \mathbb{R}^n \to \mathbb{R}$$

exists and is continuous for all $1 \leq i, j \leq n$. Then

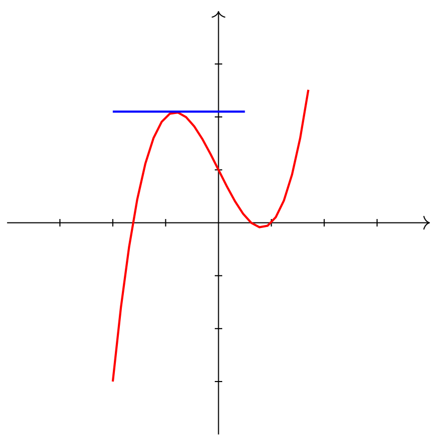
$$\frac{\partial^2 f}{\partial x_i \, \partial x_j} = \frac{\partial^2 f}{\partial x_j \, \partial x_i}$$

for all $1 \leq i, j \leq n$.

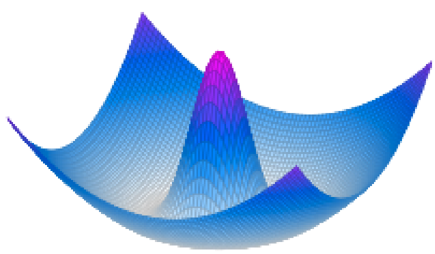
To reiterate, this result (whose proof is deferred) says that under nice circumstances, the order of partial derivatives does not matter. A quick corollary is that if $f : \mathbb{R}^n \to \mathbb{R}$ satisfies the hypotheses of Clairaut's theorem, then the Hessian matrix $f''(\mathbf{x})$ is symmetric! The exercises provide an example of a function that has continuous first order partial derivatives, but that the mixed second order partial derivatives disagree at a point.

6.5 Extreme values and optimization

One of the most useful applications of single variable calculus is the ability to optimize functions and locate local extrema by seeking *critical points*, i.e., points where the derivative is 0.



It doesn't really make sense to optimize a function $f : \mathbb{R}^n \to \mathbb{R}^m$, but we do want to optimize functions $f : \mathbb{R}^n \to \mathbb{R}$. For example, given a function $f : \mathbb{R}^2 \to \mathbb{R}$ we would seek hills and valleys on the graph:



Definition 6.26. Let $f : \mathbb{R}^n \to \mathbb{R}$. A point $\mathbf{x}_0 \in \mathbb{R}^n$ is a

(i) **local maximum** if there is a neighborhood of $\mathbf{x}_0$ such that $f(\mathbf{x}) \leq f(\mathbf{x}_0)$ for all $\mathbf{x}$ in that neighborhood, and is a

(ii) **local minimum** if there is a neighborhood of $\mathbf{x}_0$ such that $f(\mathbf{x}) \geq f(\mathbf{x}_0)$ for all $\mathbf{x}$ in that neighborhood.

In either case, $\mathbf{x}_0$ is a **local extremum** and $f(\mathbf{x}_0)$ is a **local extreme (maximum / minimum) value**.

Definition 6.27. Let $f : \mathbb{R}^n \to \mathbb{R}$. A point $\mathbf{x}_0 \in \mathbb{R}^n$ is a **critical point** if $Df(\mathbf{x}_0) : \mathbb{R}^n \to \mathbb{R}$ is the zero transformation, or if f is not differentiable at $\mathbf{x}_0$.

We have the following direct generalization of the critical point theorem from single variable calculus.

Theorem 6.28 (Critical point theorem)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$. If $\mathbf{x}_0 \in \mathbb{R}^n$ is a local extremum, then $\mathbf{x}_0$ is a critical point.*

Proof. The idea of the proof is very simple, and uses the single variable theorem. Suppose that f is differentiable at $\mathbf{x}_0$. We will show that $f'(\mathbf{x}_0)$ is the 0-matrix by showing $f_{x_j}(\mathbf{x}_0) = 0$ for all j. Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(t) := f(\mathbf{x}_0 + t\mathbf{e}_j).$$

In words, the function g is the restriction of f along the x_j axis. Because $\mathbf{x}_0$ is a local extremum of f, $t = 0$ must be a local extremum of g. Since f is differentiable, g is differentiable. Thus, by the single variable critical point theorem, $g'(0) = 0$. By the Jacobian form of the multivariable chain rule,

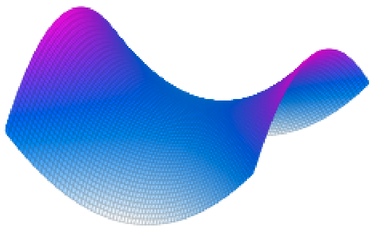
$$g'(t) = f'(\mathbf{x}_0 + t\mathbf{e}_j) \cdot \mathbf{e}_j$$

and so $0 = g'(0) = f'(\mathbf{x}_0) \cdot \mathbf{e}_j$. Since $f'(\mathbf{x}_0) \cdot \mathbf{e}_j$ is the jth entry of the matrix $f'(\mathbf{x}_0)$, it follows that $f'(\mathbf{x}_0)$ is the 0 matrix, as desired. $\square$

Note that the theorem does *not* say that if $\mathbf{x}_0$ is a critical point, than it is a local extremum. This is not true! The logic is important: if $\mathbf{x}_0$ is a local extremum, then it is a critical point.

Example 6.29. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = x^2 + y^2$. Note that $f'(x, y) = \begin{pmatrix} 2x & 2y \end{pmatrix}$ and so $f'(x, y) = \mathbf{0}$ when $(x, y) = (0, 0)$. Note that this *does not* imply (yet) that $(0, 0)$ is a local extreme value; we have simply identified $(0, 0)$ as a critical point. However, note that $f(x, y) \geq 0$ for all $(x, y) \in \mathbb{R}^2$ and $f(0, 0) = 0$. Thus, $f(x, y) \geq f(0, 0)$ for all $(x, y) \in \mathbb{R}^2$ and so $(0, 0)$ is local minimum.

So far, the theory of local extreme values generalizes quite nicely to higher dimensions. However, there is a new phenomenon that occurs in higher dimensions that does not occur in one dimension! This phenomenon is that of a *saddle point*. The classic example is given by the function $f(x, y) = x^2 - y^2$ at the point $(0, 0)$. One can verify that $(0, 0)$ is a critical point of f, but that $(0, 0)$ is *not* a local minumum or a local maximum. When restricted to the line $\mathbf{r}_1(t) = (t, 0)$, the function is $f(\mathbf{r}_1(t)) = t^2$, which suggests the origin could be a local *minumum*, but when restricted to the line $\mathbf{r}_2(t) = (0, t)$, the function is $f(\mathbf{r}_2(t)) = -t^2$, which suggests the origin could be a local *maximum*! Indeed, it is neither.



Definition 6.30. Let $f : \mathbb{R}^n \to \mathbb{R}$. A point $\mathbf{x}_0 \in \mathbb{R}^n$ is a **saddle point** if $\mathbf{x}_0$ is a critical point of f, but is not a local extremum.

Example 6.31. The critical point $(0,0)$ of the function $f(x,y) = x^2 - y^2$ described above is a saddle point.

Ultimately, we want a generalization of the second derivative test to more easily classify ciritical points as either local extrema or saddle points. Recall from single variable calculus that if $x_0 \in \mathbb{R}$ is a critical point of $f : \mathbb{R} \to \mathbb{R}$, then if

(i) $f''(x_0) > 0$, the point x_0 is a local minimum, and

(ii) $f''(x_0) < 0$, then point x_0 is a local maximum, and

(iii) $f''(x_0) = 0$ then no conclusion can be made.

To generalize this, we need to study the Hessian matrix $f''(\mathbf{x}_0)$.

Definition 6.32. Suppose that $\mathbf{x}_0 \in \mathbb{R}^n$ is a critical point of a twice-differentiable function $f : \mathbb{R}^n \to \mathbb{R}$. The critical point $\mathbf{x}_0$ is **nondegenerate** if the Hessian matrix $f''(\mathbf{x}_0)$ is invertible. If $f''(\mathbf{x}_0)$ is not invertible, then $\mathbf{x}_0$ is **degenerate**.

Next, I will remind you of some notation and language from a previous exercise. Namely, let $A \in M_{n\times n}(\mathbb{R})$ be a symmetric square matrix and define a function $\langle \cdot, \cdot \rangle_A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$\langle \mathbf{x}, \mathbf{y} \rangle_A := \langle A\mathbf{x}, \mathbf{y} \rangle$$

where the inner product on the right side is the standard inner product. The previous exercise studied in part *positive definiteness* of such a function. I will remind you what this means, along with some other terms.

Definition 6.33. A symmetric matrix A is

(i) **positive definite** if for all $\mathbf{x} \neq \mathbf{0}$, then $\langle \mathbf{x}, \mathbf{x} \rangle_A > 0$,

(ii) **negative definite** if for all $\mathbf{x} \neq \mathbf{0}$, then $\langle \mathbf{x}, \mathbf{x} \rangle_A < 0$,

(iii) **indefinite** if there is an $\mathbf{x}_1$ such that $\langle \mathbf{x}_1, \mathbf{x}_1 \rangle_A > 0$ and an $\mathbf{x}_2$ such that $\langle \mathbf{x}_2, \mathbf{x}_2 \rangle_A < 0$.

If you happen to know some more linear algebra than what we've discussed, we can equivalently describe the above terms in the following way:

(i) The symmetric matrix A is positive definite if all of the eigenvalues of A are positive.

(ii) The symmetric matrix A is negative definite if all of the eigenvalues of A are negative.

(iii) The symmetric matrix A is indefinite if A has both positive and negative eigenvalues.

If this doesn't mean anything to you, don't worry.

Example 6.34. Define

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then A is positive definite, B is indefinite, and C is negative definite. Indeed, if $\mathbf{x} \neq \mathbf{0}$ then

$$\langle \mathbf{x}, \mathbf{x} \rangle_A = \|\mathbf{x}\|^2 > 0 \quad \text{and} \quad \langle \mathbf{x}, \mathbf{x} \rangle_C = -\|\mathbf{x}\|^2 < 0$$

which proves positive definiteness and negative definitness of A and C, respectively. Finally, note that

$$\langle \mathbf{e}_1, \mathbf{e}_1 \rangle_B = \|\mathbf{e}_1\|^2 = 1 > 0 \quad \text{and} \quad \langle \mathbf{e}_2, \mathbf{e}_2 \rangle_B = \langle -\mathbf{e}_2, \mathbf{e}_2 \rangle = -\|\mathbf{e}_2\|^2 = -1 < 0$$

which proves indefiniteness of B.

Using this language, we can state the second derivative test in all of its glory. The first version I will state is relatively abstract and not very helpful in practice, especially since we have not discussed about positive definiteness and negative definiteness. After I state the full abstract version, I will state a more practical down-to-earth version for functions $f : \mathbb{R}^2 \to \mathbb{R}$.

Theorem 6.35 (The second derivative test). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice-differentiable with continuous second partials and let $\mathbf{x}_0 \in \mathbb{R}^n$ be a nondegenerate critical point.*

(i) If $f''(\mathbf{x}_0)$ is positive definite, then $\mathbf{x}_0$ is a local minimum.

(ii) If $f''(\mathbf{x}_0)$ is negative definite, then $\mathbf{x}_0$ is a local maximum.

(iii) If $f''(\mathbf{x}_0)$ is indefinite, then $\mathbf{x}_0$ is a saddle point.

Remark 6.36. Note that this theorem *assumes that $\mathbf{x}_0$ is nondegenerate*, that is, it assumes that $f''(\mathbf{x}_0)$ is invertible. This is important! Similarly, the assumption that f has continuous second partials is to ensure that $f''(\mathbf{x}_0)$ is a symmetric matrix by Clairaut's theorem.

Proof idea. I will omit a formal proof of the abstract second derivative test, but I will explain the idea. Recall from single variable calculus that we may improve the theory of tangent lines by computing higher order Taylor polynomials. In particular, the tangent line to $f : \mathbb{R} \to \mathbb{R}$ at $x = x_0$ has equation

$$T_1(h) = f(x_0) + f'(x_0)h$$

where we have used the variable $h := x - x_0$ to mimic our multivariable setup. This tangent line function is in fact the degree 1 Taylor polynomial of f at x_0. The degree 2 Taylor polynomial gives a *better* approximation near x_0:

$$T_2(h) = f(x_0) + f'(x_0)h + \frac{1}{2}f''(x_0)h^2.$$

In other words,

$$f(x_0 + h) \approx f(x_0) + f'(x_0)h + \frac{1}{2}f''(x_0)h^2$$

when h is small.

Given a function $f : \mathbb{R}^n \to \mathbb{R}$, we can do the exact same thing. A first degree Taylor approximation to f at $\mathbf{x}_0 \in \mathbb{R}^n$ is given by

$$f(\mathbf{x}_0 + \mathbf{h}) \approx f(\mathbf{x}_0) + f'(\mathbf{x}_0)\mathbf{h}.$$

This is essentially just the definition of the derivative $Df(\mathbf{x}_0)(\mathbf{h})(= f'(\mathbf{x}_0)\mathbf{h})$. It turns out that we get a degree 2 approximation as follows:

$$f(\mathbf{x}_0 + \mathbf{h}) \approx f(\mathbf{x}_0) + f'(\mathbf{x}_0)\mathbf{h} + \frac{1}{2}\left\langle f''(\mathbf{x}_0)\mathbf{h}, \mathbf{h}\right\rangle$$

for $\mathbf{h}$ small. The term $\frac{1}{2}\langle f''(\mathbf{x}_0)\mathbf{h}, \mathbf{h}\rangle$ is the analogue of the term $\frac{1}{2}f''(x_0)h^2$ above; observe that this inner product is "quadratic" in $\mathbf{h}$. If $\mathbf{x}_0$ is a critical point of f, $f'(\mathbf{x}_0)$ is the 0 matrix, and thus

$$f(\mathbf{x}_0 + \mathbf{h}) \approx f(\mathbf{x}_0) + \frac{1}{2}\left\langle f''(\mathbf{x}_0)\mathbf{h}, \mathbf{h}\right\rangle$$

for $\mathbf{h}$ small. Thus, to classify $\mathbf{x}_0$ as a local extremum, it suffices to understand the behavior of the term $\frac{1}{2}\langle f''(\mathbf{x}_0)\mathbf{h}, \mathbf{h}\rangle$ near $\mathbf{h} = \mathbf{0}$. In particular, if $\frac{1}{2}\langle f''(\mathbf{x}_0)\mathbf{h}, \mathbf{h}\rangle$ behaves like a local minumum with respect to $\mathbf{h}$ at $\mathbf{0}$, then by the above approximation it follows that f will behave like a local minumum near $\mathbf{x}_0$, and so on.

Suppose first that $f''(\mathbf{x}_0)$ is positive definite. Then by definition, $\frac{1}{2}\langle f''(\mathbf{x}_0)\mathbf{h}, \mathbf{h}\rangle > 0$ for all $\mathbf{h} \neq 0$. Fix any small direction vector $\mathbf{h}_0$. By the above remark,

$$\frac{1}{2}\left\langle f''(\mathbf{x}_0)t\mathbf{h}_0, t\mathbf{h}_0\right\rangle = t^2 \cdot \frac{1}{2}\left\langle f''(\mathbf{x}_0)\mathbf{h}_0, \mathbf{h}_0\right\rangle = Ct^2$$

for some constant $C > 0$ and all t near 0. In words, what I've shown here is that along the $\mathbf{h}_0$ direction, this term behaves like a positive multiple of t^2. This describes a local minimum in the $\mathbf{h}_0$ direction. Since $\mathbf{h}_0$ was arbitrary, it follows that $\frac{1}{2}\langle f''(\mathbf{x}_0)\mathbf{h}, \mathbf{h}\rangle$ has a local minimum at $\mathbf{h} = \mathbf{0}$, and therefore by the above approximation f has a local minimum at $\mathbf{x}_0$, as desired. The argument for when $f''(\mathbf{x}_0)$ is negative definite is nearly identical, except that the constant C is negative and thus every direction behaves like a local maximum.

Finally, suppose that $f''(\mathbf{x}_0)$ is indefinite. Then there is some $\mathbf{h}_1 \neq 0$ such that

$$\frac{1}{2}\left\langle f''(\mathbf{x}_0)\mathbf{h}_1, \mathbf{h}_1\right\rangle > 0.$$

Repeating the trick above, it follows that for all t near 0,

$$\frac{1}{2}\left\langle f''(\mathbf{x}_0)t\mathbf{h}_1, t\mathbf{h}_1\right\rangle = C_1t^2$$

for some $C_1 > 0$, and thus f behaves like a local minimum in the $\mathbf{h}_1$ direction. On the other hand, there is also some $\mathbf{h}_2 \neq \mathbf{0}$ such that $\frac{1}{2}\langle f''(\mathbf{x}_0)\mathbf{h}_2, \mathbf{h}_2\rangle < 0$. Then

$$\frac{1}{2}\left\langle f''(\mathbf{x}_0)t\mathbf{h}_2, t\mathbf{h}_2\right\rangle = C_2t^2$$

for some $C_2 < 0$, and thus f behaves like a local maximum in the $\mathbf{h}_2$ direction. It follows that f is neither a local minimum or a local maximum and thus $\mathbf{x}_0$ is a saddle point. This is the idea of the proof. $\square$

As I mentioned above, in practice this version of the theorem isn't so helpful. For a function $f : \mathbb{R}^2 \to \mathbb{R}$, we can unravel the meaning of definiteness to get a very practical version of the second derivative test.

Theorem 6.37 (The second derivative test on $\mathbb{R}^2$)**.** *Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is twice-differentiable with continuous second partials. Suppose that $\mathbf{x}_0 \in \mathbb{R}^2$ is a critical point of f.*

(i) If $f_{xx}(\mathbf{x}_0) > 0$ and $f_{xx}(\mathbf{x}_0)f_{yy}(\mathbf{x}_0) - f_{xy}(\mathbf{x}_0)^2 > 0$, then $\mathbf{x}_0$ is a local minimum.

(ii) If $f_{xx}(\mathbf{x}_0) < 0$ and $f_{xx}(\mathbf{x}_0)f_{yy}(\mathbf{x}_0) - f_{xy}(\mathbf{x}_0)^2 > 0$, then $\mathbf{x}_0$ is a local maximum.

(iii) If $f_{xx}(\mathbf{x}_0)f_{yy}(\mathbf{x}_0) - f_{xy}(\mathbf{x}_0)^2 < 0$, then $\mathbf{x}_0$ is a saddle point.

(iv) If $f_{xx}(\mathbf{x}_0)f_{yy}(\mathbf{x}_0) - f_{xy}(\mathbf{x}_0)^2 = 0$, then $\mathbf{x}_0$ is a degenerate critical point and no conclusion can be made.

Remark 6.38. Statements (i)-(iii) above correspond to statements (i)-(iii) in the abstract second derivative test. Statement (iv) is included for clarity; in this case the Hessian matrix is not invertible. To make a conclusion about a degenerate critical point, you should mimic what I did above with the function $f(x, y) = x^2 - y^2$. Namely, directly analyze the behavior of the function along parametric curves, for example of the form $\mathbf{r}_1(t) = (t, 0)$ and $\mathbf{r}_2(t) = (0, t)$. Finally, the quantity

$$f_{xx}(\mathbf{x}_0)f_{yy}(\mathbf{x}_0) - f_{xy}(\mathbf{x}_0)^2$$

is called the *determinant* of the Hessian matrix:

$$\det \begin{pmatrix} f_{xx}(\mathbf{x}_0) & f_{xy}(\mathbf{x}_0) \\ f_{xy}(\mathbf{x}_0) & f_{yy}(\mathbf{x}_0) \end{pmatrix} := f_{xx}(\mathbf{x}_0)f_{yy}(\mathbf{x}_0) - f_{xy}(\mathbf{x}_0)^2.$$

We have not discussed the determinant in these notes, so don't worry abou this — but if you happen to have more linear algebra experience it is insightful to think about the relation between the abstract second derivative test, the practical second derivative test, and the eigenvalues of the symmetric matrix $f''(\mathbf{x}_0)$.

Proof. The following proof can be simplified greatly with the theory of determinants and eigenvalues. Thus, it can safely be skipped for now, but it is interesting to see how one can study definiteness of 2×2 matrices in a very down and dirty manner, without having to appeal to determinants and eigenvalues.

First, we prove (i). Assume that $f_{xx}(\mathbf{x}_0) > 0$ and $f_{xx}(\mathbf{x}_0)f_{yy}(\mathbf{x}_0) - f_{xy}(\mathbf{x}_0)^2 > 0$. By the abstract second derivative test, we need to show that $f''(\mathbf{x}_0)$ is positive definite. That is, we want to show that if $\mathbf{h} \neq \mathbf{0}$ then $\langle \mathbf{h}, \mathbf{h} \rangle_{f''(\mathbf{x}_0)} > 0$. We do this by considering two cases of such vectors. First, note that

$$\langle \mathbf{e}_1, \mathbf{e}_1 \rangle_{f''(\mathbf{x}_0)} = \left\langle f''(\mathbf{x}_0)\mathbf{e}_1, \mathbf{e}_1 \right\rangle = f_{xx}(\mathbf{x}_0) > 0.$$

It follows that if $\mathbf{h} = (h_1, 0)$ for any $h_1 \neq 0$, then

$$\langle \mathbf{h}, \mathbf{h} \rangle_{f''(\mathbf{x}_0)} = h_1^2 \langle \mathbf{e}_1, \mathbf{e}_1 \rangle_{f''(\mathbf{x}_0)} > 0.$$

That was the first case. Next, suppose that $\mathbf{h} = (h_1, h_2)$ with $h_2 \neq 0$; here h_1 could possibly be 0. Then

$$\begin{aligned} \langle \mathbf{h}, \mathbf{h} \rangle_{f''(\mathbf{x}_0)} &= \begin{pmatrix} f_{xx}(\mathbf{x}_0) & f_{xy}(\mathbf{x}_0) \\ f_{xy}(\mathbf{x}_0) & f_{yy}(\mathbf{x}_0) \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \cdot \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \\ &= \begin{pmatrix} h_1 f_{xx}(\mathbf{x}_0) + h_2 f_{xy}(\mathbf{x}_0) \\ h_1 f_{xy}(\mathbf{x}_0) + h_2 f_{yy}(\mathbf{x}_0) \end{pmatrix} \cdot \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \\ &= h_1^2 f_{xx}(\mathbf{x}_0) + 2h_1h_2 f_{xy}(\mathbf{x}_0) + h_2^2 f_{yy}(\mathbf{x}_0) \\ &= [f_{xx}(\mathbf{x}_0)]\, h_1^2 + [2h_2 f_{xy}(\mathbf{x}_0)]\, h_1 + \left[h_2^2 f_{yy}(\mathbf{x}_0)\right]. \end{aligned}$$

We view this as a quadratic polynomial in h_1. Note that the discriminant of this h_1 polynomial is

$$(2h_2 f_{xy}(\mathbf{x}_0))^2 - 4h_2^2 f_{xx}(\mathbf{x}_0) f_{yy}(\mathbf{x}_0) = 4h_2^2 \left(f_{xy}(\mathbf{x}_0)^2 - f_{xx}(\mathbf{x}_0) f_{yy}(\mathbf{x}_0)\right) < 0.$$

The last inequality comes from the assumption of (i). Since the discriminant of this quadratic h_1 polynomial is negative, it has no real roots. Thus, it is either the case that

$$[f_{xx}(\mathbf{x}_0)]\, h_1^2 + [2h_2 f_{xy}(\mathbf{x}_0)]\, h_1 + \left[h_2^2 f_{yy}(\mathbf{x}_0)\right] > 0$$

for all h_1 or

$$[f_{xx}(\mathbf{x}_0)]\, h_1^2 + [2h_2 f_{xy}(\mathbf{x}_0)]\, h_1 + \left[h_2^2 f_{yy}(\mathbf{x}_0)\right] < 0$$

for all h_1. By plugging in $h_1 = 0$, it suffices to determine the sign of $h_2^2 f_{yy}(\mathbf{x}_0)$ and thus to determine the sign of $f_{yy}(\mathbf{x}_0)$. Note that if $f_{yy}(\mathbf{x}_0) < 0$, then since $f_{xx}(\mathbf{x}_0) > 0$ it would follow that

$$f_{xx}(\mathbf{x}_0) f_{yy}(\mathbf{x}_0) - f_{xy}(\mathbf{x}_0)^2 \leq f_{xx}(\mathbf{x}_0) f_{yy}(\mathbf{x}_0) < 0$$

which is a contradiction. Thus, $f_{yy}(\mathbf{x}_0) > 0$, and therefore

$$\langle \mathbf{h}, \mathbf{h} \rangle_{f''(\mathbf{x}_0)} > 0.$$

With these two cases, we have shown that $\langle \mathbf{h}, \mathbf{h} \rangle_{f''(\mathbf{x}_0)} > 0$ for all nonzero $\mathbf{h}$, and thus $f''(\mathbf{x}_0)$ is positive definite. The proof of (ii) is very similar and is omitted.

Next, we prove (iii). Suppose that $f_{xx}(\mathbf{x}_0) f_{yy}(\mathbf{x}_0) - f_{xy}(\mathbf{x}_0)^2 < 0$. For any $\mathbf{h} = (h_1, h_2)$ with $h_2 \neq 0$, the above computation gives

$$\langle \mathbf{h}, \mathbf{h} \rangle_{f''(\mathbf{x}_0)} = [f_{xx}(\mathbf{x}_0)]\, h_1^2 + [2h_2 f_{xy}(\mathbf{x}_0)]\, h_1 + \left[h_2^2 f_{yy}(\mathbf{x}_0)\right]$$

as before. Also as before, the discriminant of this h_1 polynomial is

$$4h_2^2 \left(f_{xy}(\mathbf{x}_0)^2 - f_{xx}(\mathbf{x}_0) f_{yy}(\mathbf{x}_0)\right) > 0.$$

Since the discriminant is positive, the h_1 polynomial has two real roots and thus $\langle \mathbf{h}, \mathbf{h} \rangle_{f''(\mathbf{x}_0)}$ achieves both negative and positive values. It follows that $f''(\mathbf{x}_0)$ is indefinite, and then (iii) follows from (iii) in the abstract second derivative test.

Finally we prove (iv). Suppose that $f_{xx}(\mathbf{x}_0) f_{yy}(\mathbf{x}_0) - f_{xy}(\mathbf{x}_0)^2 = 0$. First, suppose that at least one of $f_{xy}(\mathbf{x}_0)$ and $f_{yy}(\mathbf{x}_0)$ is nonzero. Note that

$$\begin{aligned} \begin{pmatrix} f_{xx}(\mathbf{x}_0) & f_{xy}(\mathbf{x}_0) \\ f_{xy}(\mathbf{x}_0) & f_{yy}(\mathbf{x}_0) \end{pmatrix} \begin{pmatrix} f_{yy}(\mathbf{x}_0) \\ -f_{xy}(\mathbf{x}_0) \end{pmatrix} &= \begin{pmatrix} f_{xx}(\mathbf{x}_0) f_{yy}(\mathbf{x}_0) - f_{xy}(\mathbf{x}_0)^2 \\ f_{yy}(\mathbf{x}_0) f_{xy}(\mathbf{x}_0) - f_{yy}(\mathbf{x}_0) f_{xy}(\mathbf{x}_0) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

Then by a previous exercise, since there is a nonzero vector in the kernel of the Hessian $f''(\mathbf{x}_0)$, the matrix $f''(\mathbf{x}_0)$ is not invertible and thus $\mathbf{x}_0$ is degenerate. If $f_{xy}(\mathbf{x}_0) = f_{yy}(\mathbf{x}_0) = 0$, then one can easily verify that $f''(\mathbf{x}_0)\mathbf{e}_2 = \mathbf{0}$ and thus $f''(\mathbf{x}_0)$ is not invertible in this case as well.

□

All of this has been very abstract. In the following example we will show how to find critical points and classify them using the second derivative test in practice.

Example 6.39. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x^2y + xy^2 - 3xy$. We will identify and classify all of the critical points of f. First, note that

$$f_x(x, y) = 2xy + y^2 - 3y$$
$$f_y(x, y) = x^2 + 2xy - 3x.$$

To find critical points, we want $f'(x, y)$ to be the 0 matrix. Thus, we wish to solve the system of equations

$$\begin{cases} f_x(x, y) = 0 \\ f_y(x, y) = 0 \end{cases} \quad \rightsquigarrow \quad \begin{cases} 2xy + y^2 - 3y = 0 \\ x^2 + 2xy - 3x = 0 \end{cases}.$$

Factoring out common terms in each equation gives

$$\begin{cases} y(2x + y - 3) = 0 \\ x(x + 2y - 3) = 0 \end{cases}.$$

Thus, $f_x(x, y) = 0$ when $y = 0$ or when $2x + y + 3 = 0$, and $f_y(x, y) = 0$ when $x = 0$ or when $x + 2y - 3 = 0$.

(i) If $x = 0$, then $f_x(x, y) = 0$ when $y = 0$ or when $y = 3$. Thus, two critical points are
$$(0, 0) \quad \text{and} \quad (0, 3).$$

(ii) If $y = 0$, then $f_y(x, y) = 0$ when $x = 0$ or when $x = 3$. Since we have already recorded $(0, 0)$ as a critical point, this gives us a new critical point at $(3, 0)$.

(iii) The remaining cases to consider is when x and y are both nonzero. Then we need to solve the system of equations
$$\begin{cases} 2x + y - 3 = 0 \\ x + 2y - 3 = 0 \end{cases}.$$
Adding the two equations gives $3x + 3y = 6$, and hence $x = 2 - y$. Plugging this into the second equation gives $y = 1$ and thus $x = 1$. Therefore, the last critical point is $(1, 1)$.

Thus, the function f has four critical points:

$$(0, 0), (0, 3), (3, 0), (1, 1).$$

Now we classify each critical point with the second derivative test. Note that

$$f_{xx}(x, y) = 2y, \quad f_{yy}(x, y) = 2x, \quad f_{xy}(x, y) = 2x + 2y - 3.$$

Thus,

$$f''(0, 0) = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix} \quad f''(0, 3) = \begin{pmatrix} 6 & 3 \\ 3 & 0 \end{pmatrix} \quad f''(3, 0) = \begin{pmatrix} 0 & 3 \\ 3 & 6 \end{pmatrix} \quad f''(1, 1) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Since $0^2 - (-3)^2 = -9 < 0$, $(0, 0)$ is a saddle point. Since $6 \cdot 0 - 3^2 = -9 < 0$, $(3, 0)$ and $(0, 3)$ are both saddle points. Since $2^2 - 1^2 = 3 > 0$ and $2 > 0$, $(1, 1)$ is a local minimum.

6.5.1 Global extrema on closed and bounded domains

So far, we have only discussed *local* extrema of a function $f : \mathbb{R}^n \to \mathbb{R}$. In this subsection, we want to understand how to find *global* extreme values on some domain, if they exist.

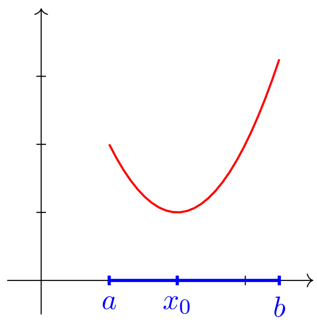
Definition 6.40. Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$. A point $\mathbf{x}_0 \in D$ is a **global minimum** if $f(\mathbf{x}) \geq f(\mathbf{x}_0)$ for all $\mathbf{x} \in D$. Likewise, $\mathbf{x}_0 \in D$ is a **global maximum** if $f(\mathbf{x}) \leq f(\mathbf{x}_0)$ for all $\mathbf{x} \in D$.

Recall from single variable calculus that a continuous function $f : [a, b] \to \mathbb{R}$ is guaranteed to have a global minimum and global maximum value (and that if the domain is an open interval, the function may not have global extrema). Furthermore, the general recipe for locating global extrema is as follows:

(1) First, search for any local extrema by identifying critical points.

(2) Second, check the boundary points $x = a$ and $x = b$.

The logic here is that a local extremum *could* be a global extremum, but the global minimum and maximum may occur at the boundary points. For example, consider the following graph:

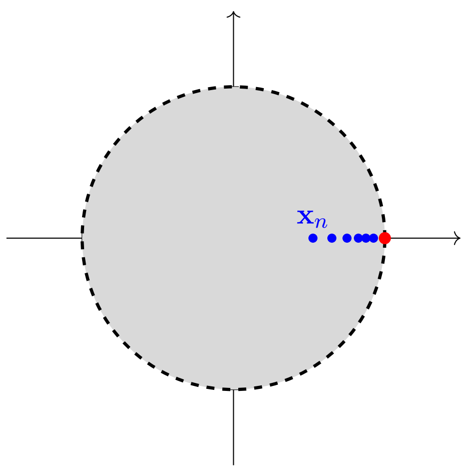


The global minimum coincides with the local minimum $x = x_0$, but the global maximum value occurs at the endpoint $x = b$.

To generalize this phenomenon, we begin by describing carefully what it means for a set in $\mathbb{R}^n$ to be *closed* and *bounded*.

Definition 6.41. A set $D \subset \mathbb{R}^n$ is **closed** if $\{\mathbf{x}_n\} \subset D$ and $\mathbf{x}_n \to \mathbf{x}_0 \in \mathbb{R}^n$ implies $\mathbf{x}_0 \in D$.

In words, with a bit more exposition: a set is closed if every convergent sequence *in the set* converges to a point *also in the set*. It is instructive to consider a non-example. Let $U \subset \mathbb{R}^2$ be the unit disc $\{x^2 + y^2 < 1\}$:



The strict inequality sign $<$ is important; this indicates that the standard unit circle is *not* a part of the set. Consider the sequence of vectors

$$\mathbf{x}_n = \left(1 - \frac{1}{n}, 0\right).$$

Note that each $\mathbf{x}_n \in U$, since $(1 - 1/n)^2 + 0^2 < 1$. Also note that $\mathbf{x}_n \to (1, 0)$, but that the vector $(1, 0)$ is *not* in the set U! This means that U is *not* closed. On the other hand, the unit disc $D = \{x^2 + y^2 \leq 1\}$ cut out by an inclusive inequality $\leq$ *is* a closed set. In practice, *closed sets are described by inclusive inequalities ($\leq$, $\geq$).* If there is a set described (even partially) with a strict inequality sign, it is probably not closed.

Definition 6.42. A set $D \subset \mathbb{R}^n$ is **bounded** if there is an $R > 0$ such that

$$D \subset \{\|\mathbf{x}\| \leq R\}.$$

In words, a set is bounded if it is contained in some large enough ball; that is, the set doesn't wander off to infinity. As a non-example, the set $\{x > 0, y > 0\}$ is not bounded (it is also not closed). In any case, with the notion of closed and bounded, we can generalize the familiar extreme value theorem from single variable calculus. We defer the proof.

Theorem 6.43 (Extreme value theorem)**.** *Let $D \subset \mathbb{R}^n$ be a closed and bounded set. If $f : D \to \mathbb{R}$ is continuous, then f has a global minimum and a global maximum.*

Remark 6.44. If D is not closed or bounded, a continuous function may not have a global minimum or maximum. For example, the set $D = \{0 \leq x \leq 1\}$ is closed, but not bounded (D is an infinite vertical strip) and the function $f(x, y) = y$ has no global minimum or maximum on D. Likewise, if $D = \{x^2 + y^2 < 1\}$ then D is bounded but not closed, and the function $f(x, y) = y$ again has no global minimum or global maximum.

In practice, here is how you go about searching for global extrema of a continuous function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ on a closed and bounded domain D:

(1) Search for local extrema in the domain. Note that it is not necessary to classify the local extrema, unless you want to just for fun.

(2) Search for *boundary* critical points. By this I mean, let $\mathbf{r} : [a, b] \to D$ be a parameteric curve that traces out the boundary of D (practically, the curve(s) obtained by setting all inequalities $\leq$ to equalities $=$). Then perform the usual one variable optimization process on the function $g(t) := f(\mathbf{r}(t))$. Note that you need to check the endpoints (the boundary) of your parameter interval $[a, b]$.

(3) From the above two steps, you have gathered a list of candidate extrema $\mathbf{x}_1, \ldots, \mathbf{x}_k \in D$. Feed each of these points to f to get a list of numbers

$$f(\mathbf{x}_1), \ldots, f(\mathbf{x}_k).$$

This list may not actually be a list (you may have infinitely many points to check, depending on the behavior of the function). In any case, the biggest number in the list is the global maximum value, and a corresponding $\mathbf{x}_j$ (there may be multiple!) is a global maximum. Likewise with the global minimum.

The above recipe is a bit vague, so let's do a couple examples to see this in action.

Example 6.45. Let $D = \{ (x, y) \in \mathbb{R}^2 \,:\, x^2 - 1 \le y \le 1 \}$. Let's find the global extrema of $f : D \to \mathbb{R}$ defined by $f(x, y) = y^2$.

First, note that D is both closed and bounded and that f is continuous, so global extrema are guaranteed to exist by the extreme value theorem. We begin by searching for critical points of f inside D. Note that

$$f'(x, y) = \begin{pmatrix} 0 & 2y \end{pmatrix}.$$

Thus, f has a critical point at any point with $y = 0$. In other words, $(x, 0)$ is a critical point of f for any $x^2 - 1 \le 0$ and thus $-1 \le x \le 1$ (in this case, these critical points are degenerate).

Now we check the boundary. Note that there are two components of the boundary $y = 1$ and $y = x^2 - 1$. These components intersect when $x^2 - 1 = 1$ and thus when $x = \pm\sqrt{2}$. The boundary curve $y = 1$ can then be parametrized by $\mathbf{r}_1(t) = (t, 1)$ for $-\sqrt{2} \le t \le \sqrt{2}$. The function f restricted to this boundary segment is then $g_1(t) = f(\mathbf{r}_1(t)) = 1^2 = 1$ on $[-\sqrt{2}, \sqrt{2}]$. Every value of t is a critical point of g_1, so we must further consider points of the form $(t, 1)$ for $-\sqrt{2} \le t \le \sqrt{2}$. Note that this collection of points includes the boundary points of the domain interval.

The lower boundary curve can be parametrized by $\mathbf{r}_2(t) = (t, t^2 - 1)$ for $-\sqrt{2} \le t \le \sqrt{2}$. The corresponding boundary function is $g_2(t) = (t^2 - 1)^2$ on $[-\sqrt{2}, \sqrt{2}]$. Note that

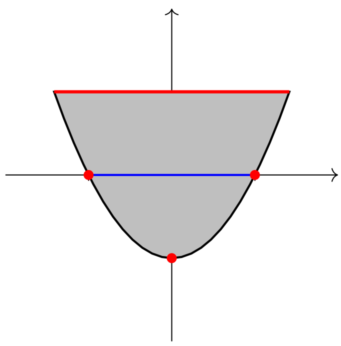
$$g_2'(t) = 2(t^2 - 1) \cdot t^2$$

and thus g_2 has critical points at $t = -1, 0, 1$. The value $t = 0$ yields the point $(0, -1)$. The values $t = \pm 1$ yield $(\pm 1, 0)$, which are already being considered by earlier analysis. We must also consider the boundary points $t = \pm\sqrt{2}$ which yield $(\pm\sqrt{2}, 1)$, which are also being considered from the analysis of g_1.

To summarize, we have the following class of candidate points to check:

$$\begin{aligned}
&(x, 0) \text{ for } -1 \le x \le 1, \\
&(t, 1) \text{ for } -\sqrt{2} \le t \le \sqrt{2}, \\
&(0, -1).
\end{aligned}$$

Note that $f(x, 0) = 0$, $f(t, 1) = 1^2 = 1$, and $f(0, -1) = (-1)^2$. It follows that the global minimum value of f on D is 0, occurring at points $(x, 0)$ for $-1 \le x \le 1$, and the global maximum value of f is 1, occurring at $(0, -1)$ and $(t, 1)$ for $-\sqrt{2} \le t \le \sqrt{2}$.



The above figure indicates the domain D is gray, the interior line of critical points in blue, and the boundary critical points in red.

Example 6.46. Let $D = \{x^2 + y^2 \leq 4\}$. Let's find the global extrema of the function $f : D \to \mathbb{R}$ defined by $f(x, y) = x + y$.

As in the previous example, D is both closed and bounded and f is continuous, so global extrema are guaranteed to exist. Note that

$$f'(x, y) = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

which never vanishes, and thus f has no interior critical points.

Next, we search for boundary critical points. There is one boundary curve cut out by $x^2 + y^2 = 4$. We can parametrize this curve by $\mathbf{r}(t) = (2\cos t, 2\sin t)$ for $0 \leq t \leq 2\pi$. The restriction of f to this curve is

$$g(t) = f(\mathbf{r}(t)) = 2\cos t + 2\sin t$$

on $[0, 2\pi]$. Note that

$$g'(t) = -2\sin t + 2\cos t.$$

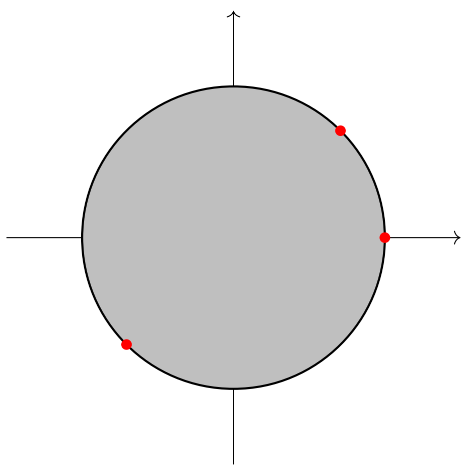
Thus, critical points of g occur when $\sin t = \cos t$. On $[0, 2\pi]$ this happens when $t = \frac{\pi}{4}$ and $t = \frac{5\pi}{4}$. We also need to check the boundary points $t = 0$ and $t = 2\pi$. These four values of t yield the following points:

$$\begin{aligned} \mathbf{r}\left(\frac{\pi}{4}\right) &= (\sqrt{2}, \sqrt{2}) \\ \mathbf{r}\left(\frac{5\pi}{4}\right) &= (-\sqrt{2}, -\sqrt{2}) \\ \mathbf{r}(0) &= (2, 0) \\ \mathbf{r}(2\pi) &= (2, 0). \end{aligned}$$

Thus, there are three candidate points to check: $(\sqrt{2}, \sqrt{2})$, $(-\sqrt{2}, -\sqrt{2})$, and $(2, 0)$. Note that

$$\begin{aligned} f(\sqrt{2}, \sqrt{2}) &= \sqrt{2} + \sqrt{2} = 2\sqrt{2} \\ f(-\sqrt{2}, -\sqrt{2}) &= -\sqrt{2} + -\sqrt{2} = -2\sqrt{2} \\ f(2, 0) &= 2 + 0 = 2. \end{aligned}$$

The global minimum value of f is clearly $-2\sqrt{2}$, which occurs at $(-\sqrt{2}, -\sqrt{2})$. Since $2\sqrt{2} > 2$ it follows that the global maximum value of f is $2\sqrt{2}$, which occurs at $(\sqrt{2}, \sqrt{2})$.



6.6 Applications of the gradient

Earlier, we introduced the *gradient vector* of a function $f : \mathbb{R}^n \to \mathbb{R}$, by definition the transpose of the Jacobian matrix:

$$\nabla f(\mathbf{x}) = \begin{pmatrix} f_{x_1}(\mathbf{x}) \\ \vdots \\ f_{x_n}(\mathbf{x}) \end{pmatrix}.$$

At the time, it was only introduced for the purpose of defining the second derivative. In this section, we study the geometric significance of the gradient vector in its own right.

6.6.1 Directional derivatives

We begin by defining a more general notion of partial derivative: that of a *directional derivative*. Recall that if $f : \mathbb{R}^n \to \mathbb{R}$, then $\frac{\partial f}{\partial x_j}$ is a derivative which informally measures how much the function f is changing in the x_j direction. Recall that the actual definition of a partial derivative is

$$\frac{\partial f}{\partial x_j}(\mathbf{x}_0) = \lim_{t\to 0} \frac{f(\mathbf{x}_0 + t\mathbf{e}_j) - f(\mathbf{x}_0)}{t}$$

provided that this limit exists. Using these observations, we can generalize how to take a derivative in *any* direction; not just an axial direction.

Definition 6.47. Let $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{x}_0 \in \mathbb{R}^n$. Let $\mathbf{v} \in \mathbb{R}^n$ be a unit vector, i.e., $\|\mathbf{v}\| = 1$. The **directional derivative of f in the $\mathbf{v}$ direction at** $\mathbf{x}_0$ is

$$\nabla_{\mathbf{v}} f(\mathbf{x}_0) := \lim_{t\to 0} \frac{f(\mathbf{x}_0 + t\mathbf{v}) - f(\mathbf{x}_0)}{t}$$

provided the limit exists.

Remark 6.48. Observe that we require $\mathbf{v}$ to be a *unit* vector in this definition. Also, since each $\mathbf{e}_j$ is a unit vector is follows that partial derivatives are directional derivatives. In particular,

$$\frac{\partial f}{\partial x_j}(\mathbf{x}_0) = \nabla_{\mathbf{e}_j} f(\mathbf{x}_0).$$

Next, recall that

$$\frac{\partial f}{\partial x_j}(\mathbf{x}_0) = Df(\mathbf{x}_0)(\mathbf{e}_j) = f'(\mathbf{x}_0)\mathbf{e}_j = \langle \nabla f(\mathbf{x}_0), \mathbf{e}_j \rangle$$

since the $(1, j)$ entry of the standard matrix $f'(\mathbf{x}_0)$ of $Df(\mathbf{x}_0)$ is exactly $\frac{\partial f}{\partial x_j}(\mathbf{x}_0)$. The following result should be unsurprising.

Theorem 6.49. *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{x}_0 \in \mathbb{R}^n$. Let $\mathbf{v} \in \mathbb{R}^n$ be a unit vector. If f is differentiable at $\mathbf{x}_0 \in \mathbb{R}^n$, then $\nabla_{\mathbf{v}} f(\mathbf{x}_0)$ exists and*

$$\nabla_{\mathbf{v}} f(\mathbf{x}_0) = Df(\mathbf{x}_0)(\mathbf{v}).$$

Equivalently,

$$\nabla_{\mathbf{v}} f(\mathbf{x}_0) = f'(\mathbf{x}_0)\mathbf{v} = \langle \nabla f(\mathbf{x}_0), \mathbf{v} \rangle.$$

Proof. The proof is almost identical to the proof of Theorem 6.13. In particular, note that

$$\begin{aligned}\left\| \frac{f(\mathbf{x}_0+t\mathbf{v})-f(\mathbf{x}_0)}{t} - Df(\mathbf{x}_0)(\mathbf{v}) \right\| &= \left\| \frac{f(\mathbf{x}_0+t\mathbf{v})-f(\mathbf{x}_0)-tDf(\mathbf{x}_0)(\mathbf{v})}{t} \right\| \\ &= \frac{\|f(\mathbf{x}_0+t\mathbf{v})-f(\mathbf{x}_0)-Df(\mathbf{x}_0)(t\mathbf{v})\|}{\|t\mathbf{v}\|}.\end{aligned}$$

Since f is differentiable at $\mathbf{x}_0$,

$$\lim_{t\to 0} \frac{\|f(\mathbf{x}_0+t\mathbf{v})-f(\mathbf{x}_0)-Df(\mathbf{x}_0)(t\mathbf{v})\|}{\|t\mathbf{v}\|} = 0$$

and thus we have shown that

$$\lim_{t\to 0} \left\| \frac{f(\mathbf{x}_0+t\mathbf{v})-f(\mathbf{x}_0)}{t} - Df(\mathbf{x}_0)(\mathbf{v}) \right\| = 0.$$

By definition of limit convergence, this implies

$$\lim_{t\to 0} \frac{f(\mathbf{x}_0+t\mathbf{v})-f(\mathbf{x}_0)}{t} = Df(\mathbf{x}_0)(\mathbf{v}).$$

Thus, $\nabla_{\mathbf{v}} f(\mathbf{x}_0)$ exists and equals $Df(\mathbf{x}_0)(\mathbf{v})$, as desired.

The equivalent statements follow from the observation that $\nabla f'(\mathbf{x}_0) = f'(\mathbf{x}_0)^T$, and the fact that if $\mathbf{a}$ is a column vector (an $n \times 1$ matrix), then

$$\langle \mathbf{a}, \mathbf{v} \rangle = \mathbf{a}^T \mathbf{v}$$

where the quantity on the right is a $(1 \times n) \cdot (n \times 1)$ matrix product.

□

In words, the above theorem gives a simple way to compute directional derivatives without using the limit definition (provided the function is differentiable): you just compute the Jacobian (or gradient) and multiply (or inner product) by the vector $\mathbf{v}$.

Example 6.50. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(x,y,z) = x^2 z + e^{yz}$. Let $\mathbf{v} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{4}}, \frac{1}{\sqrt{4}})$. Let's compute $\nabla_{\mathbf{v}} f(1,2,1)$.

First, observe that f is differentiable at all points, and that $\|\mathbf{v}\| = 1$ so that we are allowed to take a directional derivative with respect to this vector. Note that

$$\nabla f(x,y,z) = \begin{pmatrix} f_x(x,y,z) \\ f_y(x,y,z) \\ f_z(x,y,z) \end{pmatrix} = \begin{pmatrix} 2xz \\ ze^{yz} \\ x^2 + ye^{yz} \end{pmatrix}$$

so that

$$\nabla f(1,2,1) = \begin{pmatrix} 2 \\ e^2 \\ 1+2e^2 \end{pmatrix}.$$

Thus, by the preceding theorem we have

$$\begin{aligned}\nabla_{\mathbf{v}} f(1,2,1) = \langle \nabla f(1,2,1), \mathbf{v} \rangle &= \begin{pmatrix} 2 \\ e^2 \\ 1+2e^2 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{4}} \\ \frac{1}{\sqrt{4}} \end{pmatrix} \\ &= 2 \cdot \frac{1}{\sqrt{2}} + e^2 \cdot \frac{1}{\sqrt{4}} + (1+2e^2) \cdot \frac{1}{\sqrt{4}}.\end{aligned}$$

The geometric interpretation of directional derivatives is identical to that of partial derivatives, since a directional derivative is just a slight generalization of a partial derivative. Namely, $\nabla_{\mathbf{v}} f(\mathbf{x}_0)$ represents the instantaneous rate of change of the function f in the $\mathbf{v}$ direction. Said differently, if $\mathbf{r}(t) = \mathbf{x}_0 + t\mathbf{v}$ is the parametric curve in $\mathbb{R}^n$ which passes through $\mathbf{x}_0$ at $t = 0$ in the direction $\mathbf{v}$ and $g(t) := f(\mathbf{r}(t))$ is the restriction of f to this line, then $\nabla_{\mathbf{v}} f(\mathbf{x}_0)$ is the slope of the function g at $t = 0$. This follows essentially from the definition of the directional derivative, but we can alternatively see this with a chain rule computation now that we have the above theorem:

$$g'(0) = f'(\mathbf{r}(0))\mathbf{r}'(0) = f'(\mathbf{x}_0)\mathbf{v} = \nabla_{\mathbf{v}} f(\mathbf{x}_0).$$

The gradient as the direction of maximal change

Finally, we can give a nice geometric interpretation to the gradient vector $\nabla f(\mathbf{x}_0)$. Because $f : \mathbb{R}^n \to \mathbb{R}$ and $\nabla f(\mathbf{x}_0) \in \mathbb{R}^n$, we can view the gradient vector as a vector in the domain of f. In particular, we view $\nabla f(\mathbf{x}_0)$ as a vector emanating from the point $\mathbf{x}_0$ to indicate a direction of travel. It turns out that, informally, *the gradient vector points in the direction of maximal change of the function.* In other words, if you're at a point $\mathbf{x}_0$ in the domain of f and you want f to increase as fast as possible, you should walk in the $\nabla f(\mathbf{x}_0)$ direction. More formally,

Proposition 6.51. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable and suppose that $\mathbf{x}_0 \in \mathbb{R}^n$ is not a critical point of f. Let $\mathbf{v}_0 := \frac{\nabla f(\mathbf{x}_0)}{\|\nabla f(\mathbf{x}_0)\|}$ be the unit vector in the gradient direction. Then for all unit vectors $\mathbf{v} \in \mathbb{R}^n$,*

$$\nabla_{-\mathbf{v}_0} f(\mathbf{x}_0) \leq \nabla_{\mathbf{v}} f(\mathbf{x}_0) \leq \nabla_{\mathbf{v}_0} f(\mathbf{x}_0).$$

Moreover,

$$\nabla_{\mathbf{v}_0} f(\mathbf{x}_0) = \|\nabla f(\mathbf{x}_0)\| \quad \textit{and} \quad \nabla_{-\mathbf{v}_0} f(\mathbf{x}_0) = -\|\nabla f(\mathbf{x}_0)\|$$

Remark 6.52. One of the assumptions above is that $\mathbf{x}_0$ is not a critical point. This is necessary to define the unit vector $\mathbf{v}_0 = \frac{\nabla f(\mathbf{x}_0)}{\|\nabla f(\mathbf{x}_0)\|}$, since $\nabla f(\mathbf{x}_0)$ is the zero vector if $\mathbf{x}_0$ is a critical point. Finally, the "moreover" statement says in words that the value of the directional derivative in the $\mathbf{v}_0$ direction (the gradient direction) is the *length* of the gradient vector.

Remark 6.53. We will sometimes abuse the language of this proposition and say *the gradient points in the direction of maximal change*, even though by the convention in these notes we can only take the directional derivative with respect to a unit vector, and $\nabla f(\mathbf{x}_0)$ may not be a unit vector. This is mostly inconsequential, as long as you (the reader) understand the difference!

Proof. We begin by computing $\nabla_{\mathbf{v}_0} f(\mathbf{x}_0)$ and $\nabla_{-\mathbf{v}_0} f(\mathbf{x}_0)$, proving the "moreover" statement. By Theorem 6.49,

$$\nabla_{\mathbf{v}_0} f(\mathbf{x}_0) = \langle \nabla f(\mathbf{x}_0), \mathbf{v}_0 \rangle = \left\langle \nabla f(\mathbf{x}_0), \frac{\nabla f(\mathbf{x}_0)}{\|\nabla f(\mathbf{x}_0)\|} \right\rangle = \frac{1}{\|\nabla f(\mathbf{x}_0)\|} \|\nabla f(\mathbf{x}_0)\|^2 = \|\nabla f(\mathbf{x}_0)\|.$$

The fact that $\nabla_{-\mathbf{v}_0} f(\mathbf{x}_0) = -\|\nabla f(\mathbf{x}_0)\|$ follows from bilinearity of the inner product.

Next, we prove the inequality. By Theorem 6.49, for any unit vector $\mathbf{v}$ we have

$$\nabla_{\mathbf{v}} f(\mathbf{x}_0) = \langle \nabla f(\mathbf{x}_0), \mathbf{v} \rangle.$$

By the Cauchy-Schwarz inequality,

$$|\nabla_{\mathbf{v}} f(\mathbf{x}_0)| \leq \|\nabla f(\mathbf{x}_0)\| \|\mathbf{v}\|.$$

Since $\|\mathbf{v}\| = 1$, it follows that

$$-\|\nabla f(\mathbf{x}_0)\| \leq \nabla_{\mathbf{v}} f(\mathbf{x}_0) \leq \|\nabla f(\mathbf{x}_0)\|.$$

Thus,

$$\nabla_{-\mathbf{v}_0} f(\mathbf{x}_0) \leq \nabla_{\mathbf{v}} f(\mathbf{x}_0) \leq \nabla_{\mathbf{v}_0} f(\mathbf{x}_0)$$

as desired. □

The fact that the gradient points in the direction of greatest ascent is a theoretically nice result, but it also has actual practical applications. For example, there is a class of algorithms called *gradient descent* that are designed to locate local minimum values of a multivariable function. Doing so efficiently is important for many applied math problems. Briefly, start at some point in your domain, compute the negative gradient, and move a little bit in that direction. Repeat this process at the new point. Repeat again, and so on. If all goes well, this process will eventually bring to to a local minimum of the function.

6.6.2 Level curves and contour plots

We now discuss a visual tool for studying functions $f : \mathbb{R}^2 \to \mathbb{R}$. Given such a function, the graph will be a surface in $\mathbb{R}^3$, which oftentimes can be unwieldy to draw or even visualize. Instead, we will consider only the domain $\mathbb{R}^2$ and draw curves that contain information about the function.

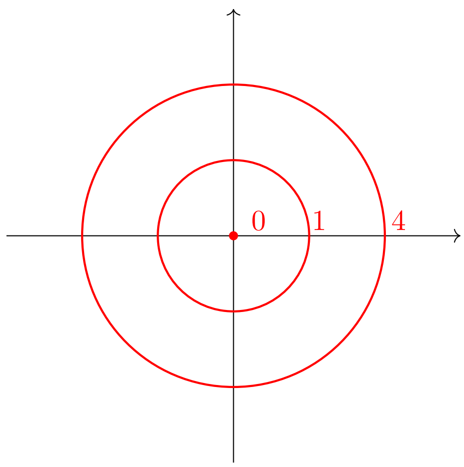
Definition 6.54. Let $f : \mathbb{R}^2 \to \mathbb{R}$. A **level curve** or **level set** of f is a curve in $\mathbb{R}^2$ defined by $f(x, y) = c$ for some constant c.

In words, a level curve corresponding to the value c is the set of all points in the domain for which f has value c.

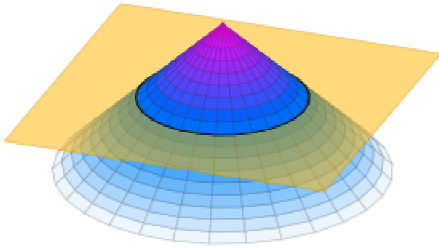
Example 6.55. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x^2 + y^2$. We can consider various level curves:

$$\begin{aligned} x^2 + y^2 &= 4 \\ x^2 + y^2 &= 1 \\ x^2 + y^2 &= 0 \\ x^2 + y^2 &= -1. \end{aligned}$$

The first two are circles of radius 2 and 1, respectively. The third is a single point, $(0, 0)$. The fourth curve is empty, as no (x, y) satisfies that equation.



Visually, a level curve is obtained by slicing the graph of f horizontally by planes of the form $z = c$ and projecting the intersection to the domain.

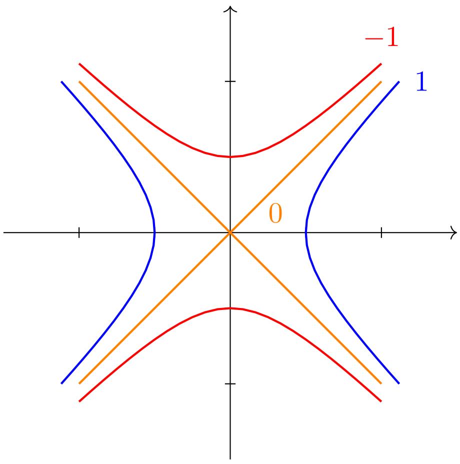


In the previous example we drew a collection of level curves together on the same picture of the domain. Such a picture is called a *contour plot*.

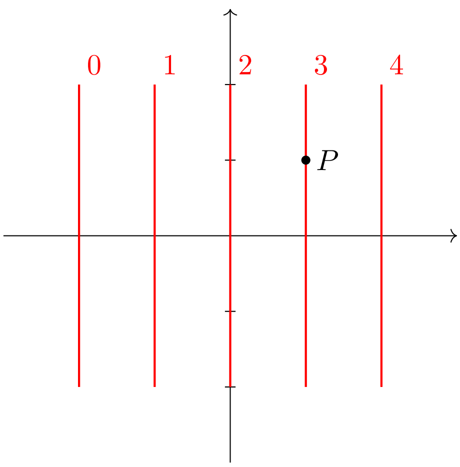
Definition 6.56. Let $f : \mathbb{R}^2 \to \mathbb{R}$. A **contour plot of** f is a collection of curves in $\mathbb{R}^2$ of the form $f(x, y) = c_k$ for some choice of constants $c_1, \ldots, c_N \in \mathbb{R}$.

Example 6.57. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x^2 - y^2$. We will draw a contour plot of f with level curves corresponding to the values $-1, 0, 1$.

The curve $x^2 - y^2 = 1$ is a hyperbola opening in the x directions, and the curve $x^2 - y^2 = -1$ is a hyperbola opening in the y directions. The curve $x^2 - y^2 = 0$ is equivalent to $x = \pm y$, which gives two lines. The contour plot is pictured below:

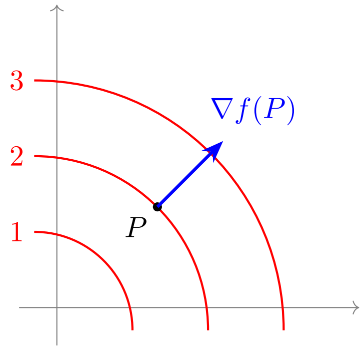


In general, a contour plot can reveal information about the nature of derivatives in various directions. For a simple example, consider the following contour plot of some function $f : \mathbb{R}^2 \to \mathbb{R}$.



Consider the point $P \in \mathbb{R}^2$ pictured above. Since P lies on a level curve corresponding to the value 3, the picture tells us that $f(P) = 3$. We can also make conclusions about the partial derivatives of f at P. For example, $f_x(P) > 0$ because moving in the $\mathbf{e}_1$ direction from P causes the values of the function to increase from 3 to 4. Likewise, it appears that $f_y(P) = 0$, since moving in the $\mathbf{e}_2$ direction from P keeps you on a level curve, which means the function does not change at all. You should be careful with these arguments, since calculus is an *infinitesimal* subject, but this is the basic idea.

In the previous section we learned that the gradient vector, pictured as a vector in the domain, points in the direction of greatest increase. We can interpret this in the context of a contour plot. At a given point P in the domain, travelling along the level curve $\{f(\mathbf{x}) = f(P)\}$ results in the function not changing *at all*. Thus, if we want to change *as much as possible*, we should probably move *off of the level curve as quickly as possible*. That is, we should move in a direction *orthogonal to the level curve*. Indeed, as we will now prove, the gradient vector is always orthogonal to level curves.



Proposition 6.58. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable, and let $K = \{f(x, y) = c\} \subset \mathbb{R}^2$ be a level curve. Let $P \in K$. Suppose that $\nabla f(P) \neq \mathbf{0}$. If $\mathbf{v} \in \mathbb{R}^2$ is a vector tangent to K at P, then*

$$\langle \nabla f(P), \mathbf{v} \rangle = 0.$$

In other words, the gradient vector is perpendicular to K.

Proof. There are a few subtleties in this proof that I will skim over for now. Suppose $\nabla f(P) \neq \mathbf{0}$. Let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ be any differentiable parametric curve with trace contained in K such that $\mathbf{r}(0) = P$. In words, $\mathbf{r}$ is a parametrization of the level curve K that passes through P at time 0. Suppose further that $\mathbf{r}'(0) \neq \mathbf{0}$. Then $\mathbf{v} = \lambda \mathbf{r}'(0)$ for some scalar λ, since $\mathbf{v}$ is tangent to K. Thus, it suffices to show that $\nabla f(P)$ is orthogonal to $\mathbf{r}'(0)$.

Since $g(t) := f(\mathbf{r}(t))$ is constant for all t, $g'(t) = 0$. By the Jacobian form of the chain rule, we have

$$g'(t) = 0 \quad \Rightarrow \quad f'(\mathbf{r}(t))\mathbf{r}'(t) = 0 \quad \Rightarrow \quad f'(P)\mathbf{r}'(0) = 0.$$

Since $f'(P) = [\nabla f(P)]^T$, this is equivalent to $\langle \nabla f(P), \mathbf{r}'(0) \rangle = 0$, as desired. □

6.6.3 Tangent planes

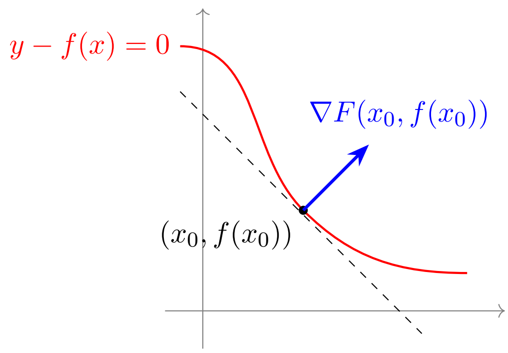
The next application of the gradient vector is the notion of a tangent (hyper)plane to a (hyper)surface. To motivate this, I want to return to a simple and hopefully very familiar situation. Suppose we have a function $f : \mathbb{R} \to \mathbb{R}$ as in single variable calculus and we wish to compute the tangent line to the graph at some point $(x_0, f(x_0))$. We already know how to do this: the tangent line is given by the graph of the function

$$T(x) = f(x_0) + f'(x_0)(x - x_0).$$

In words, the tangent line is the graph of the first order Taylor polynomial based at x_0. In even more words, the equation of the tangent line is

$$y = f(x_0) + f'(x_0)(x - x_0).$$

Here, I want to compute the tangent line to $y = f(x)$ in an absurd way: using level curves! Namely, consider the function $F : \mathbb{R}^2 \to \mathbb{R}$ defined by $F(x, y) = y - f(x)$. The graph of the one variable function, defined by $y = f(x)$, is then *the level curve of F corresponding to value* 0. Indeed, the equation $y = f(x)$ is equivalent to $y - f(x) = 0$, which is $F(x, y) = 0$. The gradient vector $\nabla F(x_0, f(x_0))$ is then orthogonal to this level curve, and the tangent line is the set of all points "orthogonal" to $\nabla F(x_0, f(x_0))$.



The word orthogonal is in quotes, because we have to shift appropriately in order for this to make sense, in complete analogy to how we define the equation of a plane. In particular, given any point (x, y) on the tangent line, we can consider the vector

$$(x, y) - (x_0, f(x_0)) = (x - x_0, y - f(x_0)).$$

This vector will be parallel to the tangent line, and hence orthogonal to the gradient vector $\nabla F(x_0, f(x_0)$. Note that we can actually compute this gradient vector:

$$\nabla F(x_0, f(x_0)) = \begin{pmatrix} -f'(x_0) \\ 1 \end{pmatrix}.$$

Thus, the equation of the tangent line is

$$\left\langle \begin{pmatrix} x - x_0 \\ y - f(x_0) \end{pmatrix}, \begin{pmatrix} -f'(x_0) \\ 1 \end{pmatrix} \right\rangle = 0$$

which simplifies to $-f'(x_0)(x - x_0) + y - f(x_0) = 0$ and thus

$$y = f(x_0) + f'(x_0)(x - x_0).$$

This confirms our familiar formula from calculus! I'll reiterate what we've done here: to compute the tangent line to the graph of a function $f(x)$, *we described the graph as the level surface of a two-variable function, and then used the gradient vector as a normal vector to get the equation of the line.*

We will generalize this to arbitrary dimensions. The fact that the gradient vector is orthogonal to level curves can be generalized to higher dimensions, and can further be used to compute tangent hyperplanes. We will explain this shortly; first, we record the appropriate generalizations of the above section.

Definition 6.59. Let $f : \mathbb{R}^n \to \mathbb{R}$. A **level hypersurface** or **level set** is a set

$$\{ \mathbf{x} \in \mathbb{R}^n \ : \ f(\mathbf{x}) = c \} \subset \mathbb{R}^n$$

for some $c \in \mathbb{R}$.

When $n = 2$, a level set is a curve. When $n = 3$, a level set (or level surface) is a surface in $\mathbb{R}^3$. For example, consider $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $f(x, y, z) = x^2 + y^2 + z^2$. Then the level set $f(x, y, z) = 1$ is the unit sphere in $\mathbb{R}^3$. In general, for nice enough values of c and nice enough functions $f : \mathbb{R}^n \to \mathbb{R}$, the level set $f(\mathbf{x}) = c$ will be an $n - 1$-dimensional object in $\mathbb{R}^n$ (whatever that means). We call such an object a *hypersurface*.

The next proposition is a direct generalization of the fact that gradient vectors are orthogonal to level curves. In words, the gradient vector is orthogonal to level sets in any dimension.

Proposition 6.60. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable, and let $K = \{f(x, y) = c\} \subset \mathbb{R}^n$ be a level set. Let $P \in K$. Assume $\nabla f(P) \neq \mathbf{0}$. If $\mathbf{v} \in \mathbb{R}^n$ is a vector tangent to K at P, then*

$$\langle \nabla f(P), \mathbf{v} \rangle = 0.$$

In other words, the gradient vector is perpendicular to the hypersurface K.

Our main use of this fact will be to compute tangent planes to surfaces in $\mathbb{R}^3$. For example, consider the unit sphere $x^2 + y^2 + z^2 = 1$. We could pick a point on the surface, say $(1, 0, 0)$, and seek the equation of the plane in $\mathbb{R}^3$ which is *tangent* to the surface at that point. Motivated by the discussion at the beginning of the section, we now define the notion of a tangent (hyper)surface in general.

Definition 6.61. Let $S \subset \mathbb{R}^n$ be a surface defined by the equation $f(\mathbf{x}) = c$ for some differentiable $f : \mathbb{R}^n \to \mathbb{R}$ and some $c \in \mathbb{R}$. Let $\mathbf{x}_0 \in S$ such that $\nabla f(\mathbf{x}_0) \neq \mathbf{0}$. The **tangent (hyper)plane to S at $\mathbf{x}_0$** is the plane with equation

$$\langle \mathbf{x} - \mathbf{x}_0, \nabla f(\mathbf{x}_0) \rangle = 0.$$

Example 6.62. Consider the unit 2-sphere S, i.e., the surface in $\mathbb{R}^3$ defined by $x^2 + y^2 + z^2 = 1$. Let's compute the tangent plane at the point $(1, 0, 0)$.

Let $f(x, y, z) = x^2 + y^2 + z^2$. Then S is the level surface $f(x, y, z) = 1$. Note that $\nabla f(x, y, z) = (2x, 2y, 2z)$. Thus, $\nabla f(1, 0, 0) = (2, 0, 0)$. According to the above definition, the equation of the tangent plane to the surface S at $(1, 0, 0)$ is given by

$$\left\langle \begin{pmatrix} x - 1 \\ y \\ z \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \right\rangle = 0$$

which simplifies to $2(x - 1) = 0$ and further to $x = 1$. This should make sense geometrically!

The next example is of fundamental importance. We motivated this whole discussion by wanting to find tangent (hyper)planes to the graphs of functions. The general process according to the above definition is then as follows: given a function $f : \mathbb{R}^n \to \mathbb{R}$ and a point $\mathbf{x}_0 \in \mathbb{R}^n$, we realize the graph $x_{n+1} = f(\mathbf{x})$ as the level surface $x_{n+1} - f(\mathbf{x})$ of the function $F : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by $F(\mathbf{x}, x_{n+1}) = x_{n+1} - f(\mathbf{x})$. The gradient of F then gives us a normal vector of the tangent (hyper)plane we seek.

Example 6.63. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x^2 + yx - 1$. Let's compute the equation of the tangent plane to the graph of f, the surface in $\mathbb{R}^3$ given by $z = f(x, y)$, at the point $(1, 1, 1)$.

First, let $F(x, y, z) = z - f(x, y) = z - x^2 - yx + 1$. Then

$$\nabla F(x, y, z) = \begin{pmatrix} -2x - y \\ -x \\ 1 \end{pmatrix}$$

so that

$$\nabla F(1, 1, 1) = \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}.$$

Thus, the equation of the tangent plane to the graph of f at $(1, 1, 1)$ is

$$\left\langle \begin{pmatrix} x - 1 \\ y - 1 \\ z - 1 \end{pmatrix}, \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} \right\rangle = 0$$

which simplifies to

$$-3(x - 1) - (y - 1) + (z - 1) = 0.$$

6.7 Exercises

1. Compute the partial derivatives f_x and f_y, wherever they make sense, for the following functions.
   (a) $f(x, y) = y^2 e^{x+y}$.
   (b) $f(x, y) = \sin(xy) + x^y$.
   (c) $f(x, y) = \frac{1}{\sqrt{x^2+y^2}}$.
2. Prove that the derivative operator is additive, i.e., if $f, g : \mathbb{R}^n \to \mathbb{R}^m$ are differentiable at $\mathbf{x}_0 \in \mathbb{R}^n$, then $f + g$ is differentiable and
$$D(f + g)(\mathbf{x}_0) = Df(\mathbf{x}_0) + Dg(\mathbf{x}_0).$$
3. For each of the following functions, compute the Jacobian ($f'(\mathbf{0})$ and $g'(\mathbf{0})$) and determine whether it is differentiable at the origin. Make sure you justify your work carefully!
   (a) The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by
$$f(x, y) = \begin{cases} \frac{x^3+y^3}{\sqrt{x^2+y^2}} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}.$$
   (b) The function $g : \mathbb{R}^2 \to \mathbb{R}$ defined by $g(x, y) = (|xy|)^{\frac{1}{4}}(x^2 + y^2)^{\frac{1}{4}}$.
4. Define $f : \mathbb{R}^2 \to \mathbb{R}$ as follows:
$$f(x, y) = \begin{cases} \frac{xy(y^2-x^2)}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}.$$

(a) Show that $\frac{\partial f}{\partial x}(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ exist and are continuous. Conclude that f is differentiable at all points in $\mathbb{R}^2$.

(b) Show that $\frac{\partial^2 f}{\partial x\,\partial y}(0,0)$ and $\frac{\partial^2 f}{\partial y\,\partial x}(0,0)$ exist but that

$$\frac{\partial^2 f}{\partial x\,\partial y}(0,0) \neq \frac{\partial^2 f}{\partial y\,\partial x}(0,0).$$

5. Let $A \in M_{n\times n}(\mathbb{R})$ be symmetric ($A^T = A$), $B \in M_{1\times n}(\mathbb{R})$, and $C \in \mathbb{R}$. Let $p : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$p(\mathbf{x}) = \langle A\mathbf{x}, \mathbf{x}\rangle + B\mathbf{x} + C.$$

Show that p is differentiable and compute $Dp(\mathbf{x}_0)$ for any $\mathbf{x}_0 \in \mathbb{R}^n$.

6. Define $f : \mathbb{R}^2 \to \mathbb{R}$ as follows:

$$f(x,y) = \begin{cases} \frac{xy(y^2-x^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}.$$

(a) Show that $\frac{\partial f}{\partial x}(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ exist and are continuous. Conclude that f is differentiable at all points in $\mathbb{R}^2$.

(b) Show that $\frac{\partial^2 f}{\partial x\,\partial y}(0,0)$ and $\frac{\partial^2 f}{\partial y\,\partial x}(0,0)$ exist but that

$$\frac{\partial^2 f}{\partial x\,\partial y}(0,0) \neq \frac{\partial^2 f}{\partial y\,\partial x}(0,0).$$

7. Find and classify all critical points of the function $f(x,y) = (x^2+y^2)e^{y^2-x^2}$.

8. Find and classify all critical points of the function $f(x,y) = y\cos x$.

9. Find the global minimum and global maximum value of the function $f(x,y) = 2x^3 + y^4$ on the domain $D = \{\,(x,y)\;:\; x^2+y^2 \le 1\,\}$.

10. Find global minimum and global maximum value of the function $f(x,y) = x^4 + y^4 - 4xy + 2$ on the domain $D = \{\,(x,y)\;:\; 0 \le x \le 3,\; 0 \le y \le 2\,\}$.

11. Suppose that the length of the diagonal of a rectangular box is L. What is the maximum volume of the box?

12. Let P be a plane in $\mathbb{R}^3$ described by the equation $x + 2y + z = 1$. Let $\mathbf{q} = (2,2,3)$. Compute the shortest distance from the point $\mathbf{q}$ to the plane P by minimizing a multivariable function.

13. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = 6xy^2 - 2x^3 - 3y^4$. Locate all critical points and classify each of them as either a local minimum, local maximum, or a saddle point. The graph of f is called *monkey saddle*.

[Hint: There will be three critical points. Two of them will be nondegenerate, but one of them will be degenerate and hence the second derivative test will not work.]

14. For each of the following functions, find the global minimum and global maximum on the given domain.

(a) The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = x^3 + x^2y + 2y^2$ on the domain

$$D = \left\{\,(x,y) \in \mathbb{R}^2 \;:\; x \ge 0,\; y \ge 0,\; x+y \le 1\,\right\}.$$

(b) The function $g : \mathbb{R}^2 \to \mathbb{R}$ defined by $g(x,y) = x^2 + 2y^2$ on the domain

$$E = \left\{ (x,y) \in \mathbb{R}^2 \ : \ (x-1)^2 + 2y^2 \leq 1 \right\}.$$

15. Suppose that you are in a vacuum and you have a flat metal sheet which is infinitely large, so that the sheet looks like $\mathbb{R}^2$. Let (x,y) denote the coordinates of the metal sheet. Next suppose you heat up the metal sheet in various places according to a function $f_0 : \mathbb{R}^2 \to \mathbb{R}$. That is, the temperature of the metal sheet at the point $(x,y) \in \mathbb{R}^2$ is the number $f(x,y) \in \mathbb{R}$. Because you have nothing better to do in a vacuum, you want to study how the temperature of the sheet changes over time. For example, if the sheet is very hot in some places and very cold in other places, it seems likely that the temperature will diffuse and level out over time. If $t \in \mathbb{R}$ denotes time, then we can model the temperature of the sheet using a function $f : \mathbb{R}^2_{(x,y)} \times \mathbb{R}_t \to \mathbb{R}$, that is, a function $f(x,y,t)$ so that $f(x,y,0) = f_0(x,y)$.

It turns out that such a function $f : \mathbb{R}^2_{(x,y)} \times \mathbb{R}_t \to \mathbb{R}$ will satisfy a partial differential equation called the *heat equation*:

$$f_{xx} + f_{yy} = f_t. \tag{6.4}$$

(a) Fix $\lambda \in \mathbb{R}$. Show that if f and g are both solutions of the heat equation (i.e., f and g both satisfy (6.4)) then $f+g$ and λf are also solutions of the heat equation.
A physicist would tell you that this is the *principle of superposition*. This principle appears in all sorts of physics, from quantum mechanics to the study of waves. It sounds fancy, but it's not: the principle of superposition is just the statement that some operator is *linear*. In the above case, the differential operator $D(f) = f_{xx} + f_{yy} - f_t$ that defines the heat equation is linear.

(b) Fix $\mathbf{a} \in \mathbb{R}^2$. Show that if $f(\mathbf{x},t)$ is a solution of the heat equation (here $\mathbf{x} = (x,y)$) then $g(\mathbf{x},t) := f(\mathbf{x} - \mathbf{a}, t)$ is also a solution of the heat equation. Think about why this makes sense in terms of heat diffusion.

(c) Show that $f(x,y,t) = e^{-t}\cos x$ is a solution of the heat equation. What happens as $t \to \infty$? Think about why this makes sense in terms of heat diffusion.

(d) Suppose now that $t > 0$. Let $r^2 = x^2 + y^2$ and define $f : \mathbb{R}^2 \times (0,\infty) \to \mathbb{R}$ by

$$f(x,y,t) = \frac{1}{t} e^{-\frac{r^2}{4t}}.$$

Let $f_1(x,y) := f(x,y,1) = e^{-\frac{r^2}{4}}$. The function f_1 is called a *Gaussian* function, and thus f is a time-dependent family of Gaussians.

(i) Compute $\lim_{r\to\infty} f_1(x,y)$.

(ii) Locate all of the critical points of $f_1(x,y)$. For each critical point, use the second derivative test to classify it.

(iii) Use (i) and (ii) to sketch a graph of f_1.
I know this might be difficult because the graph is a surface in $\mathbb{R}^3$. Just do your best and don't worry about the quality of the picture. If you can communicate to the reader that you understand what the graph should look like, you've succeeded.

(iv) Show that $f(x,y,t) = \frac{1}{t} e^{-\frac{r^2}{4t}}$ is a solution of the heat equation. What happens as $t \to \infty$? Think about why this makes sense in terms of heat diffusion.

(e) Suppose you want to look for a heat profile f_0 that *doesn't change over time*. In other words, you want an initial heat profile that is perfectly balanced and stable. A physicist would probably call this a *steady-state solution*. Mathematically, this means that we want a solution f such that $f_t = 0$ (the function does not change with time). In other words, we want a function $f : \mathbb{R}^2_{(x,y)} \to \mathbb{R}$ which satisfies

$$f_{xx} + f_{yy} = 0. \tag{6.5}$$

This equation is called *Laplace's equation* and its solutions are called *harmonic functions*. So, you can think about a harmonic function as a function which describes a steady-state / stable heat profile that will not change over time.

(i) Show that if $f(x,y) = c$ for some constant c, then f is harmonic. Think about why this makes sense in terms of heat diffusion.

(ii) Show that if $f(x,y) = ax+by$ for some constants $a, b \in \mathbb{R}$ then f is harmonic. Think about why this makes sense in terms of heat diffusion.

(iii) Show that if $f(x,y) = x^2 + y^2$, then f is *not* harmonic. Think about why this makes sense in terms of heat diffusion.

See if you can find more interesting examples of harmonic functions. The theory of harmonic functions is very rich, and they are of utmost importance to many areas of math and physics. For example, harmonic functions are intimately connected to imaginary numbers and complex analysis (see the optional challenge problem below). There are many amazing properties of harmonic functions that I can't even begin describe in these notes!

16. ($*$) In this problem, I'll give you a glimpse into the world of *complex analysis*. This is the study of the calculus of imaginary numbers.

Some complex arithmetic basics.

Recall that the set of complex numbers $\mathbb{C}$ is defined as follows:

$$\mathbb{C} = \left\{ a + bi \ : \ a, b \in \mathbb{R},\ i^2 = -1 \right\}.$$

Addition and multiplication on $\mathbb{C}$ is defined as you would expect. For example,

$$(1+2i) + (2+3i) = (1+2) + (2+3)i = 3 + 5i$$

and

$$(1+2i)(2+3i) = 1(2+3i) + 2i(2+3i) = 2 + 3i + 4i + 6i^2 = 2 + 7i - 6 = -4 + 7i.$$

Other important operations are the *conjugate* and *modulus*. The complex conjugate of $z = a+bi$ is the number $\bar{z} := a-bi$ and the modulus of $z = a+bi$ is $|z| := \sqrt{a^2+b^2} \in \mathbb{R}$.

(a) Show that $|z| = |\bar{z}|$ and that $|z|^2 = z\bar{z}$.

Using this, we can also define *division* of complex numbers. If $z, w \in \mathbb{C}$, we wish to define $\frac{z}{w}$. Multiply the top and bottom by $\bar{w}$ gives $\frac{z\bar{w}}{|w|^2}$, which is already well defined in terms of the above operations. Thus, we define $\frac{z}{w} := \frac{z\bar{w}}{|w|^2}$.

Finally, recall from a prior challenge problem that we can identify $\mathbb{C}$ with $\mathbb{R}^2$ in a natural way:

$$a + bi \in \mathbb{C} \quad \leftrightarrow \quad \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2.$$

Note that $|a+bi| = \|(a,b)\|$.

The complex derivative.

Suppose we have a function $f : \mathbb{C} \to \mathbb{C}$. We wish to define the *derivative* of such a function. In particular, we want to define what it means for such a function to be *differentiable.* Well, according to the above correspondence we can view f as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$, and we already know what it means for a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ to be differentiable. We could define *complex* differentiability as just differentiability viewing $f : \mathbb{R}^2 \to \mathbb{R}^2$, but it turns out that this isn't a good idea. On $\mathbb{C}$, we have *more* structure than on $\mathbb{R}^2$, because we can actually multiply and divide elements of $\mathbb{C}$! We cannot do this on $\mathbb{R}^2$. Thus, it turns out that the best thing to do on $\mathbb{C}$ is define the derivative of $f : \mathbb{C} \to \mathbb{C}$ like we did in high school calculus. Fix $z_0 \in \mathbb{C}$. We say that $f : \mathbb{C} \to \mathbb{C}$ is *differentiable at* z_0 if the following (complex!) limit exists:

$$\lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h}. \tag{6.6}$$

If the limit exists, we define $f'(z_0) := \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$ and called $f'(z_0) \in \mathbb{C}$ the *derivative of f at z_0*. This definition does not make sense for a function $f : \mathbb{R}^2 \to \mathbb{R}^2$, because in that case $h \in \mathbb{R}^2$ is a vector and we cannot divide by a vector. But in $\mathbb{C}$, the above expression makes sense.

At the end of the day, it turns out that complex differentiability of $f : \mathbb{C} \to \mathbb{C}$ is actually a much stronger condition than differentiability of $f : \mathbb{R}^2 \to \mathbb{R}^2$, although there are many connections.

(b) Suppose that $f : \mathbb{C} \to \mathbb{C}$ is complex differentiable. Writing $x + yi \in \mathbb{C}$, we can define $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ in the usual way by identifying $x + yi$ with $(x, y) \in \mathbb{R}^2$. Show that

$$i\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}.$$

[Hint: The derivative $\frac{\partial f}{\partial x}(z_0)$ is

$$\frac{\partial f}{\partial x}(z_0) = \lim_{s\to 0, s\in\mathbb{R}} \frac{f(z_0 + s) - f(z_0)}{s}$$

and the derivative $\frac{\partial f}{\partial y}(z_0)$ is

$$\frac{\partial f}{\partial y}(z_0) = \lim_{t\to 0, t\in\mathbb{R}} \frac{f(z_0 + ti) - f(z_0)}{t}.$$

Multiply the second limit on the top and bottom by i and use complex differentiability at z_0 to conclude that this limit has to equal $\frac{\partial f}{\partial x}(z_0)$.]

(c) Given $f : \mathbb{C} \to \mathbb{C}$, we can write $f(z) = u(z) + iv(z)$ where $u, v : \mathbb{C} \to \mathbb{R}$ are the real and imaginary parts of f. Viewing $\mathbb{C} \cong \mathbb{R}^2$, this means that $u(x, y)$ and $v(x, y)$ are functions $\mathbb{R}^2 \to \mathbb{R}$. Use (b) to show that if $f = u + iv$ is complex differentiable, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

These equations are called the *Cauchy-Riemann equations*.

(d) Use (c) to show that if $f = u + iv$ is complex differentiable, then $u : \mathbb{R}^2 \to \mathbb{R}$ and $v : \mathbb{R}^2 \to \mathbb{R}$ are both harmonic functions (as in problem 2 above).

This last conclusion is really quite amazing, though it may not seem like it if you don't know much about harmonic functions. According to the exercise above, harmonic functions correspond to heat profiles that are stable or perfectly balanced. This implies that a complex differentiable function is "perfectly balanced" in some vague sense.

Another nice fact that follows from the Cauchy-Riemann equations is that the Jacobian matrix of a complex differentiable function f, viewed a a map, $f : \mathbb{R}^2 \to \mathbb{R}^2$, has a special form.

(e) Suppose that $f = u + iv$ is complex differentiable. Show that the Jacobian matrix $f'(\mathbf{x}_0)$ of the map $f = (u, v) : \mathbb{R}^2 \to \mathbb{R}^2$ has the form

$$f'(\mathbf{x}_0) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

for some real numbers $a, b \in \mathbb{R}$.

This is another way to see how complex differentiability is a stronger condition than regular differentiability. This also makes sense for another reason: in a previous challenge problem you showed that multiplication by $a + bi$ on $\mathbb{C}$ corresponds to multiplying vectors in $\mathbb{R}^2$ by the matrix

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

All of the above then implies the following principle: *a complex differentiable function locally looks like multiplication by a complex number*. This jives with our understanding of *real* differentiability, where a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable if it locally looks like multiplication by a real number, and more generally multivariable differentiability: a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable if it locally looks like multiplication by a matrix (a "multivariable number").

Finally, here are some examples of complex differentiable functions yielding harmonic functions and some non-examples.

(f) Show that $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = z^2$ is complex differentiable. Show that if $f = u + iv$, then $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. Verify directly that u and v are harmonic.

(g) Show that $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (x^2, 0)$ is differentiable, but that $f : \mathbb{C} \to \mathbb{C}$ defined by $f(x + iy) = x^2 + 0i$ is not complex differentiable.

17. $(*)$ In this long and complicated exercise, I'll walk you through an even more sophisticated viewpoint of the derivative, one closer to the level of differential geometry. You'll need a bit more linear algebra than we've covered in the chapters so far (abstract vector spaces, bases, matrix representatives).

(a) First, a pure linear algebra exercise to put you in a healthy mindset. Let V be a (real) vector space. Define V^* to be the *dual space*, the set of all linear maps $V \to \mathbb{R}$. Prove that V^* is a vector space. Prove that if V has dimension n, then V^* also has dimension n.
[Hint: for the last statement, let $\{\mathbf{v}_j\}$ be a basis for V. Define $\mathbf{v}_j^* \in V^*$ by $\mathbf{v}_j^*(\mathbf{v}_j) = 1$ and $\mathbf{v}_j^*(\mathbf{v}_k) = 0$ for $k \neq j$.]

Next, start by considering the vector space $\mathbb{R}^n$ and fix some point $\mathbf{x}_0 \in \mathbb{R}^n$. Let $C^\infty(\mathbb{R}^n)$ denote the set of all infinitely differentiable functions $\mathbb{R}^n \to \mathbb{R}$. Define a *derivation at* $\mathbf{x}_0$ to be a map

$$X : C^\infty(\mathbb{R}^n) \to \mathbb{R}$$

such that, for all $f, g \in C^\infty(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$,

$$\begin{aligned} X(\lambda f) &= \lambda X(f) \\ X(f+g) &= X(f) + X(g) \\ X(fg) &= X(f) \cdot g + f \cdot X(g). \end{aligned}$$

In words, a derivation is an $\mathbb{R}$-linear map that also satisfies the "product rule." Finally, define the *tangent space to* $\mathbb{R}^n$ *at* $\mathbf{x}_0$, denoted $T_{\mathbf{x}_0}\mathbb{R}^n$, to be the set of all derivations at $\mathbf{x}_0$:

$$T_{\mathbf{x}_0}\mathbb{R}^n = \{ X : C^\infty(\mathbb{R}^n) \to \mathbb{R} \ : \ X \text{ is a derivation at } \mathbf{x}_0 \}.$$

(b) Prove that $T_{\mathbf{x}_0}\mathbb{R}^n$ is a vector space.

(c) Prove that $e_j : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ defined by

$$e_j(f) := \frac{\partial f}{\partial x_j}(\mathbf{x}_0)$$

is a derivation at $\mathbf{x}_0$.

(d) Prove that $T_{\mathbf{x}_0}\mathbb{R}^n$ has dimension n, so that $T_{\mathbf{x}_0}\mathbb{R}^n \cong \mathbb{R}^n$.
[Hint: show that the derivations from (c) give a basis of $T_{\mathbf{x}_0}\mathbb{R}^n$.]

Now we can define the correct notion of the derivative of a function $f : \mathbb{R}^n \to \mathbb{R}^m$. Fix $\mathbf{x}_0 \in \mathbb{R}^n$. Define the *differential of f at* $\mathbf{x}_0$, denoted f_*, to be the map

$$f_* : T_{\mathbf{x}_0}\mathbb{R}^n \to T_{f(\mathbf{x}_0)}\mathbb{R}^m$$

defined as follows. If $h \in C^\infty(\mathbb{R}^m)$, then

$$f_*(X)(h) = X(h \circ f).$$

This is a weird definition that takes some time to parse. But here is the point: given a function $f : \mathbb{R}^n \to \mathbb{R}^m$, f_* takes a derivation X at $\mathbf{x}_0$ on $\mathbb{R}^n$ to a derivation $f_*(X)$ at $f(\mathbf{x}_0)$ on $\mathbb{R}^m$. In other words, the differential is a map between tangent spaces.

(e) Show that $f_*(X)$ is in fact a derivation of $\mathbb{R}^m$ at $f(\mathbf{x}_0)$, so that the above definition makes sense.

(f) Show that f_* is a linear map.

Since f_* is a linear map between vector spaces $T_{\mathbf{x}_0}\mathbb{R}^n \to T_{f(\mathbf{x}_0)}\mathbb{R}^m$, we can pick bases and compute the matrix representative with respect to those bases! Indeed, if we do this with the natural choice of bases, we get the Jacobian matrix.

(g) Show that the matrix representative $[f_*]_\alpha^\beta$ of $f_* : T_{\mathbf{x}_0}\mathbb{R}^n \to T_{f(\mathbf{x}_0)}\mathbb{R}^m$ with respect to the bases

$$\begin{aligned} \alpha &= \{e_1, \ldots, e_n\} \\ \beta &= \{e_1, \ldots, e_m\} \end{aligned}$$

as defined in (c) is the usual Jacobian matrix.

All of this should make sense, since f_* is a map from an n-dimensional vector space to an m-dimensional vector space, so its standard matrix should be an $m \times n$ matrix.

Because I'm having fun, here is another differential geometric perspective of calculus. In particular, we can give precise meaning to the expression dx that appears in integrals.[1] An element of $T_{\mathbf{x}_0}\mathbb{R}^n$ is also called a *tangent vector to* $\mathbb{R}^n$ *at* $\mathbf{x}_0$. Here we are thinking about $\mathbb{R}^n$ as being a geometric blob, and $T_{\mathbf{x}_0}\mathbb{R}^n$ is the space of all tangent vectors to the blob at the point $\mathbf{x}_0$. We can further consider the dual space of $T_{\mathbf{x}_0}\mathbb{R}^n$, called the *cotangent space at* $\mathbf{x}_0$:

$$T^*_{\mathbf{x}_0}\mathbb{R}^n := (T_{\mathbf{x}_0}\mathbb{R}^n)^* = \{\, \phi : T_{\mathbf{x}_0}\mathbb{R}^n \to \mathbb{R} \;:\; \phi \text{ is linear} \,\}\,.$$

Naturally, elements of the cotangent space are called *covectors*. In words, a covector is a linear object which eats a tangent vector and returns a number. Now, consider a function $f : \mathbb{R}^n \to \mathbb{R}$. By the above portion of this exercise, its differential at $\mathbf{x}_0$ is a map $f_* : T_{\mathbf{x}_0}\mathbb{R}^n \to T_{f(\mathbf{x}_0)}\mathbb{R}$. By part (d), $T_{f(\mathbf{x}_0)}\mathbb{R} \cong \mathbb{R}$ as vector spaces given by $e_1 \mapsto 1$. Thus, using this isomorphism we can actually view the differential f_* as a map $T_{\mathbf{x}_0}\mathbb{R}^n \to \mathbb{R}$. When we view f_* in this way, we actually rewrite it as $df_{\mathbf{x}_0}$. That is, the differential of $f : \mathbb{R}^n \to \mathbb{R}$ at $\mathbf{x}_0$ is the linear map

$$df_{\mathbf{x}_0} : T_{\mathbf{x}_0}\mathbb{R}^n \to \mathbb{R}.$$

In other words, $df(\mathbf{x}_0)$ is a covector at $\mathbf{x}_0$!

The next step is to define df, the differential of f *everywhere*, and not just at a single point. To do this, we need to collect all of the cotangent spaces together. Let

$$T^*\mathbb{R}^n := \bigcup_{\mathbf{x}_0 \in \mathbb{R}^n} T^*_{\mathbf{x}_0}\mathbb{R}^n.$$

In words, $T^*\mathbb{R}^n$ is just the union of all of the individual cotangent spaces. We call this object the *cotangent bundle*.[2] If you've taken any physics, this may be more familiar to you as phase space of Hamiltonian dynamics. Given $f : \mathbb{R}^n \to \mathbb{R}$ we can then define its differential df as a function $df : \mathbb{R}^n \to T^*\mathbb{R}^n$ by $df(\mathbf{x}) := df_{\mathbf{x}} \in T^*_{\mathbf{x}}\mathbb{R}^n \subset T^*\mathbb{R}^n$. In words, df is the map which eats a point $\mathbf{x} \in \mathbb{R}^n$ and returns the differential of f at that point, an element of the cotangent space at that point. Anyway, here is the punchline: Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x$. Then its differential df is map from $\mathbb{R}$ to the cotangent bundle of $\mathbb{R}$. The expression dx that appears in integrals in calculus is actually df for this function f, interpreted appropriately!

18. A *vector field on* $\mathbb{R}^n$ is just a function $F : \mathbb{R}^n \to \mathbb{R}^n$. The informal perspective to take when thinking about a vector field is the following: at each point $\mathbf{x}_0 \in \mathbb{R}^n$, there is an associated vector $F(\mathbf{x}_0)$ emanating from the point $\mathbf{x}_0$.

    (a) Draw a sketch of each of the following vector fields $F, G, H : \mathbb{R}^2 \to \mathbb{R}^2$:

$$F(x,y) = \begin{pmatrix} x \\ y \end{pmatrix}$$
$$G(x,y) = \begin{pmatrix} x \\ -y \end{pmatrix}$$
$$H(x,y) = \begin{pmatrix} -y \\ x \end{pmatrix}.$$

[1] There are multiple ways to give precise meaning to an expression like dx in an integral; this is just one.

[2] To make sense of the cotangent bundle you really should put something called a *topology* on it, but we won't worry about that for now.

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable vector field. A *flowline of F* is a parametric curve $\mathbf{r} : \mathbb{R} \to \mathbb{R}^n$ such that $\mathbf{r}'(t) = F(\mathbf{r}(t))$. In words, this condition says "the tangent vector to $\mathbf{r}$ is equal to the vector field at that point."

(b) Let $F(x, y) = \begin{pmatrix} x \\ y \end{pmatrix}$ as in part (a). Compute the flowline $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ of F passing through the point $(1, 1)$. Draw a sketch of the trace of this flowline imposed over a sketch of the vector field F.

(c) Let $H(x, y) = \begin{pmatrix} -y \\ x \end{pmatrix}$ as in part (a). Compute the flowline $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ of F passing through the point $(1, 0)$. Draw a sketch of the trace of this flowline imposed over a sketch of the vector field H.

(d) Let $\tilde{F}(x, y) = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} \\ \frac{y}{\sqrt{x^2+y^2}} \end{pmatrix}$. Compute the flowline $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ of $\tilde{F}$ passing through the point $(1, 1)$. Draw a sketch of the trace of this flowline imposed over a sketch of the vector field $\tilde{F}$. How is this flowline similar to the flowline you computed in (b)? How is it different?

(e) Let $G(x, y) = \begin{pmatrix} x \\ -y \end{pmatrix}$ as in (a). Compute the flowline $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ of G passing through the point $(0, 0)$.

19. Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable. The gradient function $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ can be viewed as a vector field.

(a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x^2 + y^2$. Compute the gradient vector field $\nabla f(x, y)$ and sketch it.

For a function $f : \mathbb{R}^2 \to \mathbb{R}$, the *symplectic gradient* is the vector field $X_f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$X_f(x, y) := \begin{pmatrix} -f_y(x, y) \\ f_x(x, y) \end{pmatrix}.$$

This is in contrast to the usual gradient vector field $\nabla f(x, y) = \begin{pmatrix} f_x(x, y) \\ f_y(x, y) \end{pmatrix}$. Symplectic gradient vector fields are important in physics; in particular, in Hamiltonian mechanics (symplectic gradient vector fields are also called *Hamiltonian vector fields*). In this context, $\mathbb{R}^2$ is the *phase space* of a particle moving on a line: the x-coordinate of $\mathbb{R}^2$ describes the position, and the y-coordinate describes the particle's momentum. The function $f : \mathbb{R}^2 \to \mathbb{R}$ represents the total energy of a particle. That is, if a particle is at position x with momentum y, then the energy of that particle is $f(x, y) \in \mathbb{R}$. Given an energy function $f : \mathbb{R}^2 \to \mathbb{R}$, the symplectic gradient X_f describes how a particle in this system will evolve over time (Hamilton's equations of motion just describe flowlines of the symplectic gradient).

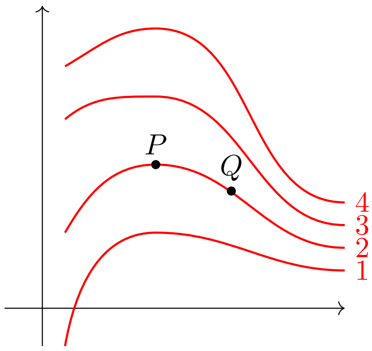
(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be $f(x, y) = x^2 + y^2$ as in part (a). Compute the symplectic gradient vector field $X_f(x, y)$ and sketch it.

(c) Describe level curves of the function $f(x, y) = x^2 + y^2$. Describe flowlines (you don't have to compute them if you don't want to, just describe them) of the symplectic gradient X_f. What do you notice?

(d) Now, let $f : \mathbb{R}^2 \to \mathbb{R}$ be a general differentiable function, let $X_f : \mathbb{R}^2 \to \mathbb{R}^2$ be the symplectic gradient, and let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ be a flowline of X_f. Prove that $f(\mathbf{r}(t))$ is constant.

In words, part (d) says that a flowline of the symplectic gradient describes a level curve of f, which hopefully confirms what you discovered in part (c). From a Hamiltonian mechanics perspective, this is describing *conservation of energy*: the energy of a particle does not change as it evolves over time.

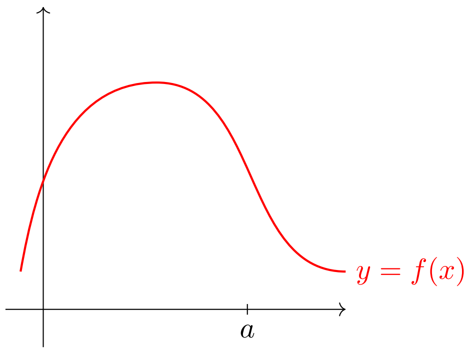
20. Consider the following contour plot of a function $f : \mathbb{R}^2 \to \mathbb{R}$.



For each of the following quantities, say whether it positive, negative, or 0. Give some explanation.

(a) $f_y(P)$
(b) $f_x(P)$
(c) $f_{yx}(Q)$
(d) $\nabla_{\mathbf{v}} f(P)$, where $\mathbf{v} = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

21. Consider the graph $y = f(x)$ of a function $f : \mathbb{R} \to \mathbb{R}$. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be defined by $F(x, y) = y - f(x)$, and let $a \in \mathbb{R}$ be the number depicted below.



(a) Draw $\nabla F(a, f(a))$ on the above figure.
(b) Draw $\nabla f(a)$ on the above figure.
(c) Let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ be a parametric curve defined by $\mathbf{r}(t) = (t, f(t))$. Let $\mathbf{v} = \frac{\mathbf{r}'(a)}{\|\mathbf{r}'(a)\|}$. Compute

$$\nabla_{\mathbf{v}} F(a, f(a)).$$

(d) Determine whether

$$\nabla_{-1} f(a)$$

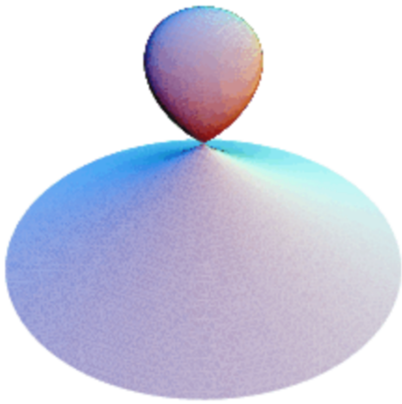
is positive, negative, or 0.

22. Here are two exercises involving tangent planes.

(a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x^4 y + e^{x^2 y} + 1$. Compute the equation of the tangent plane to the graph of f at the point $(1, 1, 2 + e)$. Use this to approximate the value of $f(1.1, 0.9)$.

(b) The *ding-dong surface* is the surface defined by

$$x^2 + y^2 = (1 - z)z^2.$$



Compute the equation of the tangent plane to the ding-dong surface at the point $(1, 1, -1)$.

23. Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable. Let $S \subseteq \mathbb{R}^{n+1}$ be the graph of f, i.e., the surface defined by $x_{n+1} = f(\mathbf{x})$. Fix $\mathbf{x}_0 \in \mathbb{R}^n$. Suppose that $\nabla f(\mathbf{x}_0) \neq \mathbf{0}$. Prove that the tangent (hyper)plane to S at $(\mathbf{x}_0, f(\mathbf{x}_0))$ is the graph of the function $T : \mathbb{R}^n \to \mathbb{R}$ defined by

$$T(\mathbf{x}) = f(\mathbf{x}_0) + Df(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0).$$

Chapter 7

The Inverse and Implicit Function Theorems

Given any kind of function $f : X \to Y$, an important question to ask is whether or not that function is *invertible*. Recall from Chapter 4 that such a function $f : X \to Y$ is *invertible* if there is a function $g : Y \to X$ such that $f \circ g = id_Y$ and $g \circ f = id_X$. Invertibility is desirable for many reasons, but the most primal reason is to solve equations. In particular, given $f : X \to Y$ and a fixed $y_0 \in Y$, you may be interested in finding solutions $x \in X$ that satisfy

$$f(x) = y_0.$$

If you happen to know that f is invertible, then $x = f^{-1}(y_0)$ is a solution. In fact, it is the *unique* solution! The point is: invertibility is a powerful tool. If you've take a linear algebra class, you already how important invertibility is, as a typical introductory linear algebra course dedicates a significant amount of energy to studying invertibility of matrices (with the goal of solving equation of the form $A\mathbf{x} = \mathbf{y}$).

In this chapter, we want to use the multivariable calculus tools that we have developed to investigate invertibility of functions $f : \mathbb{R}^n \to \mathbb{R}^n$. However, invertibility is a *global* question. Calculus is a *local* tool; the derivative gives information about a function *nearby a given point*, but in general will not say anything about the function's global behavior. For this reason, this chapter will begin with a quest to understand *local* invertibility. To formalize this, we first need to decide what we mean by *local*. To do this, I will give formal meaning to a phrase that I've probably casually used throughout the notes.

Definition 7.1. An **open ball** in $\mathbb{R}^n$ is any set of the form

$$U = \{\, \mathbf{x} \in \mathbb{R}^n \;:\; \|\mathbf{x} - \mathbf{x}_0\| < r \,\}$$

for some $\mathbf{x}_0 \in \mathbb{R}^n$ and $r > 0$.

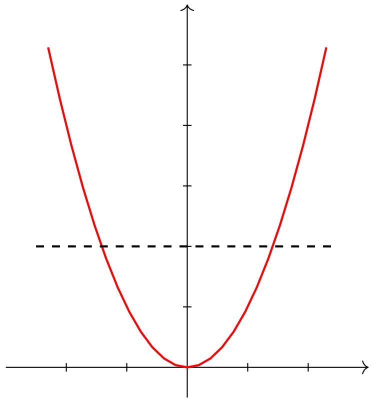
In words, an open ball is a set of points which lies within a distance of $r > 0$ from some fixed point $\mathbf{x}_0 \in \mathbb{R}^n$. For example, in $\mathbb{R}$, an open ball is an interval of the form $(x_0 - r, x_0 + r)$. We will use open balls to give meaning to what we mean by *local*. That is, if something happens *locally near* $\mathbf{x}_0$ then we mean that something is happening on an open ball containing $\mathbf{x}_0$. In particular, we can define what it means for a function to be *locally invertible* near a point.

Definition 7.2. A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is **locally invertible at $\mathbf{x}_0 \in \mathbb{R}^n$** if there is an open ball U containing $\mathbf{x}_0$ such that

$$f : U \to f(U)$$

is invertible.

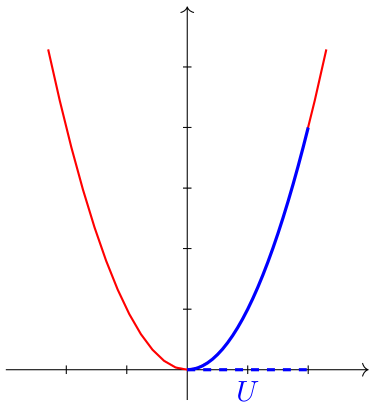
Let me demonstrate this concept with an example, using the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. You know very well that f is not (globally) invertible, and we can describe this on various levels. A high school calculus student would argue that f is not invertible because it fails the horizontal line test; indeed, this is a visual way of saying that f is not *injective*. In the language of these notes, we could use an argument like the following. Suppose that f were invertible. Then there is a function $g : \mathbb{R} \to \mathbb{R}$ such that $f \circ f = \mathrm{id}_{\mathbb{R}}$, and so $g(x^2) = x$ for all $x \in \mathbb{R}$. Plugging in $x = 1$ and $x = -1$ gives $g(1) = 1$ and $g(1) = -1$, which is a contradiction. Thus, no such function can exist.



However, I claim that $f(x) = x^2$ is *locally* invertible near many points. For example, consider $x_0 = 1$ and let $U = (0, 2)$ be the ball of radius 1 around $x_0 = 1$. If we restrict f to this ball, we obtain a function

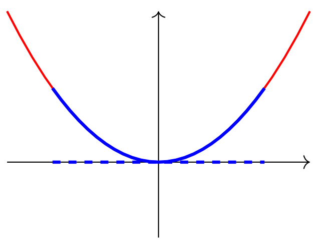
$$f : U \to f(U) = (0, 4)$$

which *is* invertible! Indeed, let $g : (0, 4) \to (0, 2)$ be defined by $g(x) = \sqrt{x}$. Then $f(g(x)) = x$ and $g(f(x)) = x$ precisely because $x \in U$ implies $x > 0$.



In the above figure, when we restrict to U (the dashed interval) the resulting function *does* pass the horizontal line test.

An important observation is that $f(x) = x^2$ is not locally invertible at every point. If $x_0 > 0$ or $x_0 < 0$, then you can always find a small enough ball around x_0 such that the graph of $f(x) = x^2$ passes the horizontal line test. But if $x_0 = 0$, *any* open ball containing x_0 will produce a graph which fails the horizontal line test.



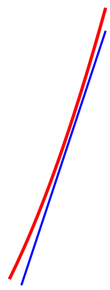
Thus, we informally conclude that $f(x) = x^2$ is locally invertible everywhere except $x_0 = 0$. There is a clear connection to calculus: $f'(x_0) = 0$ if $x_0 = 0$, and $f'(x_0) \neq 0$ if $x_0 \neq 0$. It seems as though the derivative being nonzero at x_0 is telling us that the function is locally invertible at x_0. This is indeed the case, for the following simple principle: if $f'(x_0) \neq 0$, then the derivative

$$Df(x_0) : \mathbb{R} \to \mathbb{R}$$

is invertible. Indeed, $Df(x_0)(h) = f'(x_0)h$, and if $f'(x_0)$ is a nonzero number, then one can check that $Df(x_0)^{-1}(h) = \frac{1}{f'(x_0)}h$. Since $Df(x_0)$ is heuristically a good approximation of f near x_0, then if $Df(x_0)$ is invertible it follows (non-rigorously, obviously) that f is invertible *near* x_0! This is essentially the statement of the single-variable inverse function theorem.

Theorem 7.3 (Single-variable inverse function theorem)**.** *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that $f'(x)$ is continuous. Fix $x_0 \in \mathbb{R}$. If $f'(x_0) \neq 0$, then f is locally invertible at x_0.*

Proof idea. If $f'(x_0) \neq 0$, then the tangent line to the graph of f at $(x_0, f(x_0))$ is not horizontal. Thus, the tangent line at this point passes the horizontal line test. Since the tangent line approximates the graph of f near of x_0, it follows that in a small neighborhood of x_0 the graph of f will pass the horizontal line test. Thus, f is invertible in a ball around x_0 as desired.



□

In this chapter we wish to generalize this statement to functions $f : \mathbb{R}^n \to \mathbb{R}^n$. It should be unsurprising from the preceding discussion that the general inverse function theorem says (informally) that if the linear map $Df(\mathbf{x}_0)$ is invertible, then f is locally invertible at $\mathbf{x}_0$. A more precise statement will be given below. After stating the inverse function theorem, and a close relative in the implicit function theorem, we will introduce the method of Lagrange multipliers as the main application. The proofs of the inverse and implicit function theorems are sophisticated and difficult, so they are included in Appendix B.

7.1 The inverse function theorem

Given the above discussion, the statement of the inverse function theorem should not be surprising.

Theorem 7.4 (Inverse function theorem)**.** *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable with continuous partial derivatives. Fix $\mathbf{x}_0 \in \mathbb{R}^n$. If $Df(\mathbf{x}_0)$ is invertible, then f is locally invertible at $\mathbf{x}_0$. Moreover, if $\mathbf{y}_0 := f(\mathbf{x}_0)$ then the locally defined function f^{-1} is differentiable at $\mathbf{y}_0$ and*

$$D(f^{-1})(\mathbf{y}_0) = [Df(\mathbf{x}_0)]^{-1}.$$

Remark 7.5. In this formal statement, there are a few more conclusions that are not included in the single-variable version from the introduction. Namely, f is locally invertible at $\mathbf{x}_0$, but also we can conclude that the (local) inverse f^{-1} is *differentiable* at $\mathbf{y}_0$, and we can compute the derivative of f^{-1}. The formula in the above statement is the generalization of the fact that

$$(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(\mathbf{x}_0))}$$

which should be familiar from single variable calculus.

Proof idea. As I mentioned earlier, a rigorously proving the inverse function theorem is a difficult and long journey, which is presented in Appendix B. For now, I will give a brief description of the main idea.

First, let's assume that $\mathbf{x}_0 = \mathbf{0}$ and that $f(\mathbf{0}) = \mathbf{0}$, so that $\mathbf{y}_0 = \mathbf{0}$. Here we are simply "shifting coordinates". Next, one of the assumptions of the theorem is that $Df(\mathbf{0})$ is invertible. By shifting our coordinate perspective again, we can assume that $Df(\mathbf{0})$ is actually the identity map, so that $Df(\mathbf{0})(\mathbf{h}) = \mathbf{h}$. Since $Df(\mathbf{x}_0)$ is the best linear approximation of f near $\mathbf{0}$, it follows that $f(\mathbf{x}) \approx \mathbf{x}$ near $\mathbf{0}$. In words, f is close to the identity near the origin. The identity map is certainly invertible, so therefore f is locally invertible near the origin.

Thus, we have an inverse function f^{-1} defined near $\mathbf{0}$. It turns out that f^{-1} is differentiable at $\mathbf{0}$. By differentiating both sides of the equation

$$f \circ f^{-1} = \mathrm{id}$$

at $\mathbf{0}$ and using the chain rule, we have

$$Df(\mathbf{0}) \circ D(f^{-1})(\mathbf{0}) = \mathrm{id}.$$

A similar computation using the reverse composition shows that $D(f^{-1})(\mathbf{0}) \circ Df(\mathbf{0}) = \mathrm{id}$, and therefore by definition of invertiblity we have

$$D(f^{-1})(\mathbf{0}) = [Df(\mathbf{0})]^{-1}$$

as desired.

□

7.2 The implicit function theorem

The *inverse* function theorem begets an equivalent theorem called the *implicit* function theorem. Stating and understanding the statement of the implicit function theorem requires a bit more effort than for the inverse function theorem, so I want to begin with some low-dimensional and familiar motivation.

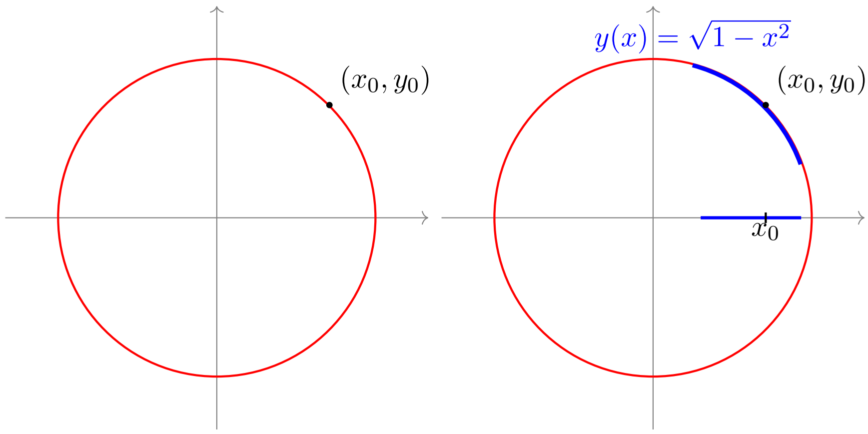
Motivation

Consider the equation $x^2 + y^2 = 1$. My motivating question is: can you solve for y as a function of x? A high school student might answer the question in the affirmative by saying *yes, you can:* $y = \sqrt{1-x^2}$; a better high school student might answer in the negative by saying *no, you can't:* $y = \pm\sqrt{1-x^2}$*, and this is not a function*. A good advanced multivariable calculus student would object that the question doesn't really make sense, and should be stated more precisely. The third answer is probably the best, so here is a more precise version of the question.

Fix a point (x_0, y_0) on the curve $x^2 + y^2 = 1$. Is there an open ball around $x_0 \in \mathbb{R}$ such that the values of y near y_0 can be described as a function of x on this ball?

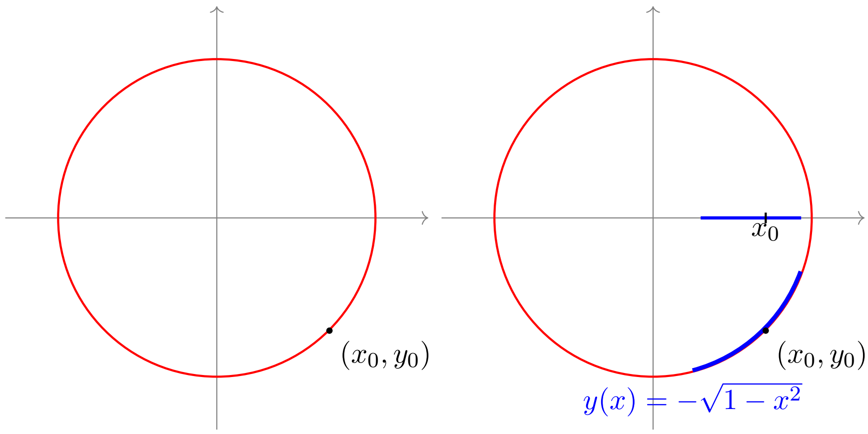
The answer is: *it depends on the value of y_0.*

Here are some pictures to elucidate both the question and answer. Suppose we considered a point (x_0, y_0) on the curve $x^2 + y^2 = 1$ with $y_0 > 0$; for example, at the point indicated below on the left.



In this case, since y is positive we can solve for y in terms of x via $y = \sqrt{1-x^2}$. More precisely, there is an open ball on the x-axis containing x_0 such that the function $y = \sqrt{1-x^2}$ describes the curve near (x_0, y_0) as y as a function of x on this ball; this is pictured on the right.

On the other hand, suppose that (x_0, y_0) is a point with $y_0 < 0$. Again, there will be an open ball around x_0 such that the values of y near y_0 can be described by a function on the ball, but this time the function is $y(x) = -\sqrt{1-x^2}$.



However, there are two points (x_0, y_0) where y *cannot* be solved as a function of x, namely the points $(1, 0)$ and $(-1, 0)$. Informally, any neighborhood on the curve around the point $(1, 0)$ will result in a curve that fails the vertical line test, since the curve $x^2 + y^2 = 1$ has a vertical tangency at $(1, 0)$. Likewise with $(-1, 0)$.

Finally, we could ask the dual question: can we solve for x as a function of y at various points? The answer is analogous: if $x_0 > 0$ or $x_0 < 0$, the answer is yes, given by $x(y) = \pm\sqrt{1-y^2}$, respectively. At the points $(0, \pm 1)$, the answer is no because of a horizontal tangency.

The implicit function theorem is a mathematical tool for answering such questions. It does so in the following way in our toy example: the curve $x^2 + y^2 = 1$ is a level curve of the function $g(x, y) = x^2 + y^2$, and so we can compute the derivative $Dg(x, y)$. In particular, we can compute the Jacobian:

$$g'(x, y) = \begin{pmatrix} 2x & 2y \end{pmatrix}.$$

The key observation is that at points (x_0, y_0) where $y_0 \neq 0$ (so that we *can* solve for y as a function of x), the second entry of $g'(x_0, y_0)$ is nonzero. However, at the points $(\pm 1, 0)$ where we *cannot* solve for y as a function of x, the Jacobian is $\begin{pmatrix} \pm 2 & 0 \end{pmatrix}$ and thus the second entry is 0. The analogous analysis holds for the x component and the ability to solve for x as a function of y.

Generalizing to the implicit function theorem

To generalize this, I want to ask a more general question with slightly different notation that mimics the statement of the theorem. Suppose we have variables $y_1, \ldots, y_r$ and variables $z_1, \ldots, z_k$, so that we have $r + k$ variables in total. Further, suppose we have a system of k algebraic equations relating all of these variables:

$$\begin{cases} g_1(y_1, \ldots, y_r, z_1, \ldots, z_k) = c_1 \\ \qquad\qquad\vdots \\ g_1(y_1, \ldots, y_r, z_1, \ldots, z_k) = c_k \end{cases} . \tag{7.1}$$

Here, each g_j is a function $\mathbb{R}^{r+k} \to \mathbb{R}$ and each c_j is a constant. It is important that the number of equations matches the number of z variables. Introducing some shorthand notation, we will write $\mathbf{y} = (y_1, \ldots, y_r) \in \mathbb{R}^r$ and $\mathbf{z} = (z_1, \ldots, z_k) \in \mathbb{R}^k$, so that each of the above algebraic equations can be written as $g_j(\mathbf{y}, \mathbf{z}) = c_j$. The question is then

Fix a solution $(\mathbf{y}_0, \mathbf{z}_0)$ of the above system of equations. Is there an open ball around $\mathbf{y}_0 \in \mathbb{R}^r$ such that the values of $\mathbf{z} = (z_1, \ldots, z_k) \in \mathbb{R}^k$ near $\mathbf{z}_0$ can be described as a function of $\mathbf{y}$ on this ball?

More informally, the question is: given a bunch of algebraic equations involving all of those variables, can we (locally) solve for $(z_1, \ldots, z_k)$ as a function of $(y_1, \ldots, y_r)$? Phrased even differently, the above system of algebraic equations cuts out some blob in $\mathbb{R}^{r+k}$. Can this blob be described as the graph of some function $\phi : \mathbb{R}^r \to \mathbb{R}^k$ near the point $(\mathbf{y}_0, \mathbf{z}_0)$?

This is the question that the implicit function theorem answers. To set up the statement of the implicit function theorem, we will actually rephrase the system of equations (7.1) in a more compact way. Given functions $g_j : \mathbb{R}^{r+k} \to \mathbb{R}$, we can view them as the component functions of a function $g : \mathbb{R}^{r+k} \to \mathbb{R}^k$ defined by

$$g(\mathbf{y}, \mathbf{z}) = \begin{pmatrix} g_1(\mathbf{y}, \mathbf{z}) \\ \vdots \\ g_k(\mathbf{y}, \mathbf{z}) \end{pmatrix} .$$

The system of equations (7.1) is then simply $g(\mathbf{y}, \mathbf{z}) = \mathbf{c}$, where $\mathbf{c} = (c_1, \ldots, c_k) \in \mathbb{R}^k$. Such a function $g : \mathbb{R}^{r+k} \to \mathbb{R}^k$ will have a Jacobian matrix $g'(\mathbf{y}, \mathbf{z})$ which is a $k \times (r + k)$ matrix. The k rows correspond to the k component functions $g_1, \ldots, g_k$, the first r columns correspond to the derivatives with respect to $\mathbf{y}$ (that is, with respect to $y_1, \ldots, y_r$), and the last k columns correspond to the derivatives with respect to $\mathbf{z}$. We can then describe the matrix $g'(\mathbf{y}, \mathbf{z})$ as a *block* matrix of the form

$$g'(\mathbf{y}, \mathbf{z}) = \begin{pmatrix} g'_{\mathbf{y}}(\mathbf{y}, \mathbf{z}) & g'_{\mathbf{z}}(\mathbf{y}, \mathbf{z}) \end{pmatrix}$$

where $g'_{\mathbf{y}}(\mathbf{y}, \mathbf{z}) \in M_{k \times r}(\mathbb{R})$ is the "restricted Jacobian matrix" of g with respect to only $y_1, \ldots, y_r$, and likewise $g'_{\mathbf{z}}(\mathbf{y}, \mathbf{z}) \in M_{k \times k}(\mathbb{R})$ is the "restricted Jacobian matrix" of g with respect to $z_1, \ldots, z_k$. Note that $g'_{\mathbf{z}}(\mathbf{y}, \mathbf{z})$ is a square matrix!

This is a lot of notation to absorb, so here is a concrete example. Consider the function $g : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$g(y_1, z_1, z_2) = \begin{pmatrix} y_1 z_1^2 + z_2 \\ y_1^2 - z_1 z_2 \end{pmatrix} .$$

Then the overall Jacobian is

$$g'(y_1, z_1, z_2) = \begin{pmatrix} z_1^2 & 2y_1z_1 & 1 \\ 2y_1 & -z_2 & -z_1 \end{pmatrix}.$$

Because there $\mathbf{y} = (y_1)$ and $\mathbf{z} = (z_1, z_2)$, we have

$$g'_{\mathbf{y}}(\mathbf{y}, \mathbf{z}) = \begin{pmatrix} z_1^2 \\ 2y_1 \end{pmatrix} \qquad \text{and} \qquad g'_{\mathbf{z}}(\mathbf{y}, \mathbf{z}) = \begin{pmatrix} 2y_1z_1 & 1 \\ -z_2 & -z_1 \end{pmatrix}.$$

Briefly, the implicit function theorem says that, given a point $\mathbf{x}_0 = (\mathbf{y}_0, \mathbf{z}_0)$ on the solution set $g(\mathbf{y}, \mathbf{z}) = \mathbf{c}$, if the square matrix $g'_{\mathbf{z}}(\mathbf{y}_0, \mathbf{z}_0)$ is invertible then you *can* locally solve for $\mathbf{z}$ as a function of $\mathbf{y}$ near $(\mathbf{y}_0, \mathbf{z}_0)$. This coincides with our example involving the unit circle $x^2 + y^2 = 1$. Switching to variable $y^2 + z^2 = 1$ for consistency, the overall Jacobian of the function $g(y, z) = y^2 + z^2$ was

$$g'(y, z) = \begin{pmatrix} 2y & 2z \end{pmatrix}.$$

Thus, $g_{\mathbf{z}}(y, z) = (2z)$. We saw that whenever this 1×1 matrix was nonzero (which means invertible in this dimension!) then we could locally solve for z as a function of y.

Finally, the formal statement of the implicit function theorem is as follows.

Theorem 7.6 (Implicit function theorem)**.** *Suppose that $g : \mathbb{R}^{r+k} \to \mathbb{R}^k$ is differentiable with continuous partial derivatives. Fix $\mathbf{x}_0 = (\mathbf{y}_0, \mathbf{z}_0) \in \mathbb{R}^{r+k}$. If $g'_{\mathbf{z}}(\mathbf{x}_0)$ is invertible, then there exists an open ball $\mathbf{y}_0 \in U \subset \mathbb{R}^r$ and a differentiable function $\phi : U \to \mathbb{R}^k$ such that $\phi(\mathbf{y}_0) = \mathbf{z}_0$ and*

$$\left\{ (\mathbf{y}, \mathbf{z}) \in \mathbb{R}^{r+k} \; : \; g(\mathbf{y}, \mathbf{z}) = 0 \right\} = \left\{ (\mathbf{y}, \mathbf{z}) \in \mathbb{R}^{r+k} \; : \; \mathbf{z} = \phi(\mathbf{y}) \right\}$$

for $(\mathbf{y}, \mathbf{z})$ near $\mathbf{x}_0 = (\mathbf{y}_0, \mathbf{z}_0)$. Moreover, the Jacobian of ϕ at $\mathbf{y}_0$ is

$$\phi'(\mathbf{y}_0) = -\left[g'_{\mathbf{z}}(\mathbf{y}_0, \mathbf{z}_0)\right]^{-1} g'_{\mathbf{y}}(\mathbf{y}_0, \mathbf{z}_0)$$

Remark 7.7. The implicit function theorem tells us that under certain conditions it is theoretically possible to solve for $\mathbf{z}$ as a function of $\mathbf{y}$ near $(\mathbf{y}_0, \mathbf{z}_0)$, given by $\mathbf{z} = \phi(\mathbf{y})$. However, it doesn't tell us *how* to do so. In other words, it doesn't give us a formula for the function ϕ. However, the "moreover" part of theorem tells us about the *derivative* of ϕ; thus, we can get a good linear approximation for ϕ as usual:

$$\begin{aligned} \phi(\mathbf{y}) &\approx \phi(\mathbf{y}_0) + \phi'(\mathbf{y}_0)(\mathbf{y} - \mathbf{y}_0) \\ &\approx \mathbf{z}_0 - \left[g'_{\mathbf{z}}(\mathbf{y}_0, \mathbf{z}_0)\right]^{-1} g'_{\mathbf{y}}(\mathbf{y}_0, \mathbf{z}_0)(\mathbf{y} - \mathbf{y}_0). \end{aligned}$$

Proof idea. For the moment, I want to give you a simple idea for why the implicit function theorem is true; a rigorous proof using the inverse function theorem is presented in Appendix B.

The philosophy of the inverse function theorem (and calculus in general) is: given a nonlinear problem, if the *linearization* of the problem can be solved at a point, then the original nonlinear problem can *locally* be solved near that point. The idea is the same here. Informally, we want to solve the equation

$$g(\mathbf{y}, \mathbf{z}) = \mathbf{c}$$

for $\mathbf{z}$ in terms of $\mathbf{y}$ near the point $(\mathbf{y}_0, \mathbf{z}_0)$. The function g could be highly nonlinear, so instead we *linearize* the problem by taking derivatives. This gives us

$$Dg(\mathbf{y}_0, \mathbf{z}_0)(\mathbf{h}) = \mathbf{0}$$

for some $\mathbf{h} \in \mathbb{R}^{r+k}$. Writing $\mathbf{h} = (\mathbf{h_y}, \mathbf{h_z})$ where $\mathbf{h_y} \in \mathbb{R}^r$ and $\mathbf{h_z} \in \mathbb{R}^k$ and writing the above equation in matrix form, we have

$$\begin{pmatrix} g'_{\mathbf{y}}(\mathbf{y}_0, \mathbf{z}_0) & g'_{\mathbf{z}}(\mathbf{y}_0, \mathbf{z}_0) \end{pmatrix} \begin{pmatrix} \mathbf{h_y} \\ \mathbf{h_z} \end{pmatrix} = \mathbf{0}.$$

Expanding the matrix vector product gives

$$g'_{\mathbf{y}}(\mathbf{y}_0, \mathbf{z}_0)\mathbf{h_y} + g'_{\mathbf{z}}(\mathbf{y}_0, \mathbf{z}_0)\mathbf{h_z} = \mathbf{0}.$$

In this linearized version of the equation, we now want to solve for $\mathbf{h_z}$, the linearized $\mathbf{z}$-variable, in terms of $\mathbf{h_y}$, the linearized $\mathbf{y}$ variable. By assumption, $g'_{\mathbf{z}}(\mathbf{y}_0, \mathbf{z}_0)$ is invertible! Thus, some simple matrix algebra gives

$$\mathbf{h_z} = -\left[g'_{\mathbf{z}}(\mathbf{y}_0, \mathbf{z}_0)\right]^{-1} g'_{\mathbf{y}}(\mathbf{y}_0, \mathbf{z}_0)\mathbf{h_y}.$$

We have solved the *linearized* equation for $\mathbf{z}$ in terms of $\mathbf{y}$ at $(\mathbf{y}_0, \mathbf{z}_0)$. Thus, we can *locally* solve for $\mathbf{z}$ in terms of $\mathbf{y}$ near $(\mathbf{y}_0, \mathbf{z}_0)$. □

Example 7.8. Consider the equation

$$xy - z \ln y + e^{xz} = 1.$$

One solution of this equation is $(0, 1, 1)$. Let's investigate whether we can solve for z as a function of x and y near this point. We will do so by applying the inverse function theorem to $g : \mathbb{R}^3 \to \mathbb{R}$ defined by $g(x, y, z) = xy - z \ln y + e^{xz}$.

Note that

$$g'(x, y, z) = \begin{pmatrix} y + ze^{xz} & x - \frac{z}{y} & -\ln y + xe^{xz} \end{pmatrix}.$$

At the point of interest,

$$g'(0, 1, 1) = \begin{pmatrix} 2 & -1 & 0 \end{pmatrix}.$$

Since we are interested in solving for z as a function of x and y, we want to consider the 1×1 matrix $g'_z(0, 1, 1) = (0)$. This matrix is *not* invertible, so we cannot solve for z as a function of x and y in a neighborhood of this point.

What about for y in terms of x and z? Let's rewrite the order of the variables to be consistent with the statement of the implicit function theorem. We then have

$$g'(x, z, y) = \begin{pmatrix} y + ze^{xz} & -\ln y + xe^{xz} & x - \frac{z}{y} \end{pmatrix}$$

and so

$$g'(x = 0, z = 1, y = 1) = \begin{pmatrix} 2 & 0 & -1 \end{pmatrix}.$$

The 1×1 matrix $g'_y(0, 1, 1) = (-1)$ *is* invertible, since it is a nonzero number. Thus, the implicit function theorem implies that we *can* solve for y as a function of x and z near this point: $y = \phi(x, z)$, for some function ϕ. We don't know what ϕ is, but we can write down the best linear approximation using the remark after the statement of the theorem:

$$\begin{aligned} \phi(x, z) &\approx \phi(0, 1) - g'_y(0, 1, 1)^{-1} g'_{x,z}(0, 1, 1) \begin{pmatrix} x - 0 \\ z - 1 \end{pmatrix} \\ &\approx 1 - (-1)^{-1} \begin{pmatrix} 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ z - 1 \end{pmatrix} \\ &\approx 1 + 2x. \end{aligned}$$

Example 7.9. Consider the following system of equations:

$$\begin{cases} xy^2 - xz + wy = 1 \\ yw^2 - z = 0 \\ wx + yz + w = 3 \end{cases}.$$

Let's investigate whether we can solve for (x, y, z) as a function of w near the point $(1, 1, 1, 1)$ (which is a particular solution of the system).

Let $g : \mathbb{R}^4 \to \mathbb{R}^3$ be defined by

$$g(w, x, y, z) = \begin{pmatrix} xy^2 - xz + wy \\ yw^2 - z \\ wx + yz + w \end{pmatrix}.$$

The Jacobian of g is

$$g'(w, x, y, z) = \begin{pmatrix} y & y^2 - z & 2xy + w & -x \\ 2yw & 0 & w^2 & -1 \\ x + 1 & w & z & y \end{pmatrix}.$$

At the point of interest,

$$g'(1, 1, 1, 1) = \begin{pmatrix} 1 & 0 & 3 & -1 \\ 2 & 0 & 1 & -1 \\ 2 & 1 & 1 & 1 \end{pmatrix}.$$

Since we want to solve for (x, y, z) in terms of w, we want to study invertibility of the matrix

$$g'_{(x,y,z)}(1, 1, 1, 1) = \begin{pmatrix} 0 & 3 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Now we're technically in territory outside of the scope of these notes, so you'll have to trust me a bit — or learn about *determinants*. In particular, we can compute the determinant of this matrix by cofactor expansion along the first column. This gives

$$\det g'_{(x,y,z)}(1, 1, 1, 1) = 1(-3 + 1) = -2.$$

Since $-2 \neq 0$, the matrix is invertible, and therefore we *can* solve for (x, y, z) as a function of w near $(1, 1, 1, 1)$: there exists some function $\phi(w) = (x(w), y(w), z(w))$. As before, we could find the best linear approximation of ϕ, but this would involve inverting the above matrix. For now, I'll just tell you: it turns out that

$$\begin{pmatrix} 0 & 3 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} -2 & 4 & 2 \\ 1 & -1 & 0 \\ 1 & -3 & 0 \end{pmatrix}.$$

Therefore,

$$\phi(w) \approx \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -2 & 4 & 2 \\ 1 & -1 & 0 \\ 1 & -3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} (w - 1) = \begin{pmatrix} 6 - 5w \\ \frac{1}{2} + \frac{1}{2}w \\ -\frac{3}{2} + \frac{5}{2}w \end{pmatrix}.$$

I find these applications of the implicit function theorem amazing. We are given highly nonlinear equations that are likely impossible to solve explicitly in the desired manner; however, if we can find a *single* solution, the implicit function theorem tells us whether we can *locally* solve the equations. Furthermore, we can get a degree 1 approximation of the solution.

I want to finish this section with some more geometric comments about the previous two examples. In the first example, the equation

$$xy - z\ln y + e^{xz} = 1$$

describes some sort of surface in $\mathbb{R}^3$. At the end of the example we determined that the best linear approximation of the function $y = \phi(x, z)$ at the point $(0, 1, 1)$ was given by $\phi(x, z) \approx 1 + 2x$. What we've done is compute the equation of the tangent plane to the surface at $(0, 1, 1)$! Explicitly, the equation of the tangent plane at this point is $y = 1 + 2x$; you can verify this using the techniques of Chapter 6. The application of the implicit function theorem in the case is essentially the computation of the tangent plane, and this should be clear from the computations.

Likewise, in the second example it turns out that the system of equations

$$\begin{cases} xy^2 - xz + wy = 1 \\ yw^2 - z = 0 \\ wx + yz + w = 3 \end{cases}$$

describes a *curve* in $\mathbb{R}^4$. The heuristic for this business is: 4 variables and 3 equations cuts out a $4 - 3 = 1$ dimensional object. A geometric description of what we found in the previous example is that, near the point $(1, 1, 1, 1)$, this curve can be parametrized by w. In particular, with our (unknown) function $\phi(w) = (x(w), y(w), z(w))$, we have described the curve parametrically as

$$\mathbf{r}(w) = \begin{pmatrix} w \\ x(w) \\ y(w) \\ z(w) \end{pmatrix}.$$

Furthermore, the best linear approximation that we found parametrizes the tangent line to the curve at the point $(1, 1, 1, 1)$!

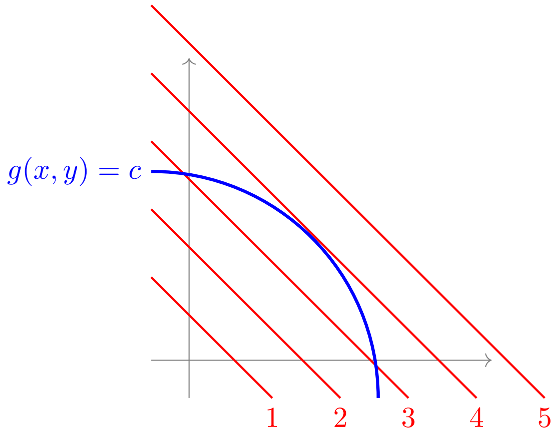
7.3 The method of Lagrange multipliers

The method of Lagrange multipliers is one of the most widely used techniques of multivariable calculus, ubiquitous in everything from economics to physics. The technique is for *constrained* optimization problems. In Chapter 6, we studied optimization of functions (finding local and global extrema) in a mostly *unconstrained* setting. Here, we are interested in problems of the following sort: given a function $f : \mathbb{R}^n \to \mathbb{R}$ and a *constraint equation* $g(\mathbf{x}) = c$ for some $g : \mathbb{R}^n \to \mathbb{R}$ and $c \in \mathbb{R}$, can we optimize the value of f subject to $g(\mathbf{x}) = c$?

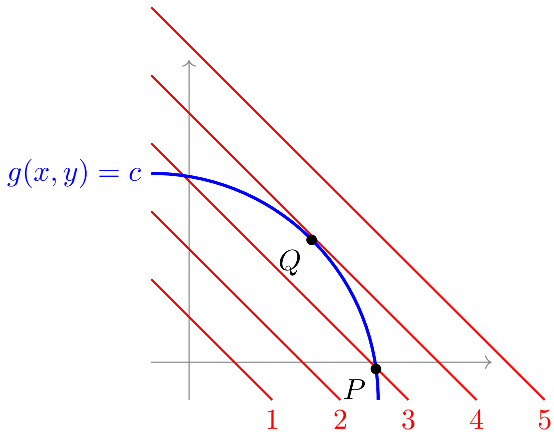
Here is a concrete, silly example to explain what I mean by this. Suppose that you and I own a bakery and we make three items: cookies (x), cakes (y), and scones (z). In a very simple model, our total profit P depends on how many of each we sell. Thus, we can view our total profit as a scalar-valued function $P(x, y, z)$. Presumably, we want to maximize our profit, so we wish to optimize our function $P(x, y, z)$. However, we don't have infinite resources: on any given day, we can only make a total of 100 items to sell. Thus, the number of cookies, cakes, and scones we sell is subject to the constraint $x + y + z = 100$. Thus, we have a *constrained optimization problem*: optimize the value of $P(x, y, z)$ subject to the constraint $x + y + z = 100$. Here $g(x, y, z) = x + y + z$ and $c = 100$.

Intuition

The method of Lagrange multipliers gives a computational way to solve such problems. Before stating the theorem, I want to give some intuition as to where the statement comes from and why the method works. Consider for the moment a function $f : \mathbb{R}^2 \to \mathbb{R}$ and a constraint $g(x, y) = c$. The constraint equation $g(x, y) = c$ describes a curve in $\mathbb{R}^2$; in particular, it is a level curve of the function $g(x, y)$. We can impose this level curve over a contour plot of f.



In the constrained optimization problem, we want to make the values of f as large as possible while restricting our movement to the blue curve. If you inspect the picture briefly, you would conclude that the largest value of f on the blue curve is 4, obtained at the point where the red level curve of f is tangent to the blue level curve of g Let's think about why this is the case. Suppose that you were at a point on the blue curve where the red level curves were *not* tangent, say at P below:



If our movement was unconstrained, we could increase the value of f as fast as possible by moving in the direction of $\nabla f(P)$, orthogonal to the red level curve. Our movement is constrained to the blue curve, but we can still move *off* of the red level curve to increase the value of f. At Q, the blue level curve is orthogonal to the $\nabla f(Q)$, so there is no direction we can walk in to increase the value of f.

All of this is to suggest that the optimum value of the constrained problem occurs when the level curves of f and g are tangent. Another way to phrase this is that the gradient vectors of f and g should parallel. In other words, at a constrained extremum $\mathbf{x}_0$ we should have

$$\nabla f(\mathbf{x}_0) = \lambda \nabla g(\mathbf{x}_0)$$

for some scalar $\lambda \in \mathbb{R}$. This is essentially the statement of the Lagrange multiplier theorem.

The formal statement

Here is the precise statement of the Lagrange multiplier method.

Theorem 7.10 (Lagrange multipliers)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ be differentiable functions with continuous partial derivatives. Suppose that f has a local extremum at $\mathbf{x}_0 \in \mathbb{R}^n$ subject to the constraint $g(\mathbf{x}) = c$ for some constant c and further suppose that $\nabla g(\mathbf{x}_0) \neq \mathbf{0}$. Then*

$$\nabla f(\mathbf{x}_0) = \lambda \nabla g(\mathbf{x}_0)$$

for some scalar λ.

Before we prove this theorem, we give a number of remarks.

Remark 7.11. The scalar λ above is called the *Lagrange multiplier*. In one of the exercises I describe a geometric interpretation of λ, but the utility of Theorem 7.10 is not based on any such interpretation; see the next remark.

Remark 7.12. In practice, here is how Theorem 7.10 is used. Consider for the moment the case that $f, g : \mathbb{R}^2 \to \mathbb{R}$. Then the constrained optimization problem of interest is to optimize $f(x, y)$ given $g(x, y) = c$. Theorem 7.10 suggest that you should compute $\nabla f(x, y)$ and $\nabla g(x, y)$ and try to solve the equation $\nabla f(x, y) = \lambda \nabla g(x, y)$ for some scalar λ, while simultaneously satisfying $g(x, y) = c$. That is, seek a solution (x, y, λ) to the following system of equations

$$\begin{cases} f_x(x,y) = \lambda g_x(x,y) \\ f_y(x,y) = \lambda g_y(x,y) \\ g(x,y) = c \end{cases}.$$

Note also that you need $\nabla g(x, y) \neq \mathbf{0}$! In general, solving this system will involve some nonlinear algebra. Furthermore, the variable λ is simply a tool for solving for x and y; you typically don't care about the actual value of λ, and only want the values of x and y. Once you've identified pairs (x, y) and associated numbers λ that solve the above system, you can then determine by hand if they are extrema.

Remark 7.13. If you're good, you will object that the above recipe abuses the statement of Theorem 7.10. Namely, the theorem says that *if* $\mathbf{x}_0$ is an extremum of the optimization problem, *then* the gradient condition is satisfied. The recipe above describes the converse: find points that satisfy the gradient condition, *then* conclude that these points are extremum.

This situation mimics that of the critical point theorem in Chapter 6. In particular, the critical point theorem says that if $\mathbf{x}_0$ is a local extremum of $f : \mathbb{R}^n \to \mathbb{R}$, then $\nabla f(\mathbf{x}_0) = \mathbf{0}$. The converse is *not* true: if $\nabla f(\mathbf{x}_0) = \mathbf{0}$, it is not necessarily the case that $\mathbf{x}_0$ is a local extremum. One way to determine whether it is a local extremum is to use the second derivative test.

It turns out that there is a kind of second derivative test for the Lagrange multiplier method involving something called a *bordered Hessian matrix*, but we will not discuss this here. There is an exercise at the end of the chapter that discusses this briefly. Typically, we will use the Lagrange multiplier method to seek *global* extremum, which allows us to abuse Theorem 7.10 as described above.

7.3.1 Proof of the Lagrange multiplier method

I will present two proofs of the Lagrange multiplier method. If you are primarily interested in just doing Lagrange multiplier problems, you can skip this section and proceed to the examples. The first "proof" is not actually a complete and rigorous proof, but it captures the spirit of the theorem. The second proof is actually rigorous and uses the implicit function theorem explicitly.

Imprecise proof 1. The first "proof" is the easiest to understand, albeit slightly imprecise. This argument is essentially the same as the argument to show that the gradient is perpendicular to level sets.

Namely, let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^n$ be any parametric curve such that the trace of $\mathbf{r}$ is contained in the hypersurface $g(\mathbf{x}) = c$, $\mathbf{r}(0) = \mathbf{x}_0$, and $\mathbf{r}'(0) \neq \mathbf{0}$.[1] Since $\mathbf{x}_0$ is a local extremum of f subject to the constraint $g(\mathbf{x}) = c$, it follows that $f(\mathbf{r}(t))$ has a local extremum at $t = 0$. Differentiating both sides and using the chain rule together with the critical point theorem, we find that $f'(\mathbf{x}_0)\mathbf{r}'(0) = 0$. Thus,

$$\left\langle \nabla f(\mathbf{x}_0), \mathbf{r}'(0) \right\rangle = 0.$$

Thus, $\nabla f(\mathbf{x}_0)$ is orthogonal to the curve $\mathbf{r}$ at $\mathbf{x}_0$. Since this computation holds for *any* such curve, we can pick enough curves such that the tangent vectors $\mathbf{r}'(0)$ generate the tangent plane to $g(\mathbf{x}) = c$ at $\mathbf{x}_0$. It follows that the gradient vector $\nabla f(\mathbf{x}_0)$ is orthogonal to the level set $g(\mathbf{x}) = c$. Since $\nabla g(\mathbf{x}_0)$ is also orthogonal to $g(\mathbf{x}_0) = c$ and since the orthogonal complement of the tangent hyperplane has a unique direction, it follows that $\nabla f(\mathbf{x}_0)$ is parallel to $\nabla g(\mathbf{x}_0)$. Thus, there is some scalar λ such that

$$\nabla f(\mathbf{x}_0) = \lambda \nabla g(\mathbf{x}_0)$$

as desired. □

Precise proof 2. The next proof is seemingly more complicated, but entirely correct as it uses the implicit function theorem explicitly and doesn't appeal to any implicit orthogonal complement dimension business. This proof also serves as a warm up for the proof of the more general Lagrange multiplier statement, Theorem 7.17.

Suppose that $\mathbf{x}_0 \in \mathbb{R}^n$ is a local extremum of the constrained optimization problem and that $\nabla g(\mathbf{x}_0) \neq \mathbf{0}$. The latter assumption implies that $g'(\mathbf{x}_0) \in M_{1\times n}(\mathbb{R})$ has rank 1. We can assume by reordering the coordinates if necessary that the last entry of $g'(\mathbf{x}_0)$ is nonzero.

Write $\mathbf{x} = (\mathbf{y}, z)$ where $\mathbf{y} \in \mathbb{R}^{n-1}$ and $z \in \mathbb{R}$, and likewise write $\mathbf{x}_0 = (\mathbf{y}_0, z_0)$. By the implicit function theorem, we can then find a ball $\mathbf{y}_0 \in U \subset \mathbb{R}^{n-1}$ and a function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $\phi(\mathbf{y}_0) = z_0$ and

$$\{\, (\mathbf{y}, z) \in \mathbb{R}^n \ : \ g(\mathbf{y}, z) = c \,\} = \{\, (\mathbf{y}, z) \in \mathbb{R}^n \ : \ z = \phi(\mathbf{y}) \,\}$$

for $(\mathbf{y}, z)$ near $\mathbf{x}_0$. In words, we have locally described the hypersurface $g(\mathbf{x}) = c$ as the graph of a function $\phi : U \subset \mathbb{R}^{n-1} \to \mathbb{R}$.

Since f has a constrained local extremum at $\mathbf{x}_0 = (\mathbf{y}_0, z_0)$, it follows that $F(\mathbf{y}) = f(\mathbf{y}, \phi(\mathbf{y}))$ has an unconstrained local extremum at $\mathbf{y}_0$, and so $F'(\mathbf{y}_0) \in M_{1\times n-1}(\mathbb{R})$ is the zero matrix. By the chain rule,

$$\frac{\partial F}{\partial x_j} = \frac{\partial f}{\partial x_j} + \frac{\partial f}{\partial x_n}\frac{\partial \phi}{\partial x_j}$$

for all $j = 1, \ldots, n-1$ and so

$$\frac{\partial f}{\partial x_j}(\mathbf{x}_0) + \frac{\partial f}{\partial x_n}(\mathbf{x}_0)\frac{\partial \phi}{\partial x_j}(\mathbf{y}_0) = 0$$

[1] The ability to find such a curve (specifically, a curve with $\mathbf{r}'(0) \neq \mathbf{0}$) uses the implicit function, but I will take this for granted for this proof.

for all $j = 1, \dots, n-1$. On the other hand, the fact that $g(\mathbf{y}, \phi(y)) = c$ implies by differentiating both sides with respect to x_j that

$$\frac{\partial g}{\partial x_j}(\mathbf{x}_0) + \frac{\partial g}{\partial x_n}(\mathbf{x}_0)\frac{\partial \phi}{\partial x_j}(\mathbf{y}_0) = 0$$

for $j = 1, \dots, n-1$. By assumption, $\frac{\partial g}{\partial x_n}(\mathbf{x}_0) \neq 0$. Thus, this equation implies

$$\frac{\partial \phi}{\partial x_j}(\mathbf{y}_0) = -\frac{g_{x_j}(\mathbf{x}_0)}{g_{x_n}(\mathbf{x}_0)}.$$

Plugging this into the preceding equation and rearranging gives

$$f_{x_j}(\mathbf{x}_0) = \frac{f_{x_n}(\mathbf{x}_0)}{g_{x_n}(\mathbf{x}_0)}\, g_{x_j}(\mathbf{x}_0)$$

for $j = 1, \dots, n-1$. Observe that this equation holds trivially for $j = n$. Thus, letting $\lambda = \frac{f_{x_n}(\mathbf{x}_0)}{g_{x_n}(\mathbf{x}_0)}$, we have shown that

$$\nabla f(\mathbf{x}_0) = \lambda \nabla g(\mathbf{x}_0)$$

as desired. □

There are other "proofs" of the Lagrange multiplier method that are often presented; for example, one often considers the *Lagrangian* function $\mathcal{L} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ given by

$$\mathcal{L}(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda(g(\mathbf{x}) - c).$$

See the exercises for a bit on this perspective.

7.3.2 Examples

Let's see a few simple examples of this method in action.

Example 7.14. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = x^2 + y^2$. Let's find the global minimum of $f(x, y)$ subject to the constraint $x + y = 1$.

Let $g(x, y) = x + y$. By the method of Lagrange multipliers, we seek solutions of the following system of equations:

$$\begin{cases} \nabla f(x, y) = \lambda \nabla g(x, y) \\ g(x, y) = 1 \end{cases}.$$

Since $\nabla f(x, y) = (2x, 2y)$ and $\nabla g(x, y) = (1, 1)$, this simplifies to the following system of equations:

$$\begin{cases} 2x = \lambda \\ 2y = \lambda \\ x + y = 1 \end{cases}.$$

Now, we just do enough algebra to find solutions. For example, the first two equations imply $2x = 2y$ and thus $x = y$. The third equation then implies $x + x = 1$ and so $x = \frac{1}{2}$. Thus, $y = \frac{1}{2}$. So the only point satisfying the Lagrange multiplier condition is $(\frac{1}{2}, \frac{1}{2})$. Since $\nabla g(x, y)$ is always nonzero, this is the only possible candidate for an extremum, so the global minimum of f subject to the constraint $x + y = 1$ is

$$f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}.$$

Note that this is in fact a minimum, as $\lim_{x \to \pm\infty} f(x, y) = \infty$.

There are number of comments to make about a simple example like the one above. The first is that the method of Lagrange multipliers is not necessary to solve the problem. In a single variable calculus course, you would solve the constraint equation $x + y = 1$ for y to get $y = 1 - x$, which you get then plug into f to get a single variable (unconstrained) optimization problem:

$$g(x) := f(x, 1-x) = x^2 + (1-x)^2 = 2x^2 - 2x + 1.$$

Then $g'(x) = 4x - 2 = 0$ exactly when $x = \frac{1}{2}$ which gives the same solution. An essentially identical solution in multivariable language would be to parametrize the constraint curve $x + y = 1$. For example, $\mathbf{r}(t) = (t, 1-t)$ for $t \in \mathbb{R}$ is such a parametrization. Then

$$g(t) := f(\mathbf{r}(t)) = f(t, 1-t) = 2t^2 - 2t + 1$$

is the single variable function to optimize, which again yields the same solution. The point here is that for some constrained optimization problems, the method of Lagrange multipliers is just another tool in the tool kit.

Example 7.15. Consider $f(x, y) = x^3 + xy + y^3$ subject to the constraint $x^3 - xy + y^3 = 1$. Let's investigate extrema for this constrained optimization problem.

Let $g(x, y) = x^3 - xy + y^3$. By the method of Lagrange multipliers, we seek solutions of the following system of equations:

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 1 \end{cases} \qquad \Rightarrow \qquad \begin{cases} 3x^2 + y = \lambda(3x^2 - y) \\ 3y^2 + x = \lambda(3y^2 - x) \\ x^3 - xy + y^3 = 1 \end{cases}.$$

First, we note that $3x^2 - y \neq 0$. Indeed, if $3x^2 - y = 0$ then the first equation implies $3x^2 + y = 0$, which further implies $3x^2 + 3x^2 = 0$, which implies $x = 0$. But then $3x^2 = y$ implies $y = 0$. This contradicts the fact that $x^3 - xy + y^3 = 1$. Thus, $3x^2 - y \neq 0$. By an identical argument, $3y^2 - x \neq 0$.

By these observations, we may solve for λ in the first two equations and equate them to get

$$\frac{3x^2 + y}{3x^2 - y} = \frac{3y^2 + x}{3y^2 - x}.$$

Simplifying gives

$$9x^2y^2 + 3y^3 - 3x^3 - xy = 9x^2y^2 + 3x^3 - 3y^3 - xy$$

which further simplifies to $6x^3 = 6y^3$ and thus $x = y$. Plugging this into the constraint equation gives $2x^3 - x^2 - 1 = 0$. Factoring gives

$$(x-1)(2x^2 + x + 1) = 0.$$

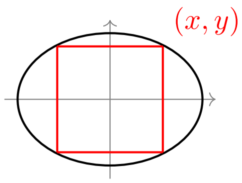
Thus, one solution is $x = 1$. Since the discriminant of the quadratic factor is $1^2 - 4(2)(1) < 0$, there are no other solutions. Thus, $(1, 1)$ is the only point satisfying the Lagrange condition.

It turns out that $(1, 1)$ is a global maximum for the constrained optimization problem. Along the constraint, $x^3 + y^3 = 1 + xy$ and so $f(x, y) = 1 + 2xy$. The constraint curve is unbounded. In particular, we need to consider the behavior of f as $x \to \pm\infty$. Since $y \to -\infty$ as $x \to \infty$ and vice versa, it follows that $f(x, y) \to -\infty$ as $x \to \pm\infty$, and thus $(1, 1)$ is a global max.

Example 7.16. Fix $a, b > 0$. Let's compute the largest possible area of a rectangle inscribed in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Consider a rectangle with sides parallel to the x and y axes with lengths $2x > 0$ and $2y > 0$ respectively, as pictured below. Here we are assuming $x, y \geq 0$ without loss of generality.



The area of such a rectangle is $A(x, y) = (2x)(2y) = 4xy$ and the values of x and y must satisfy $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. We will use the method of Lagrange multipliers to maximize A given this constraint.

Note that

$$\nabla A(x, y) = \begin{pmatrix} 4y \\ 4x \end{pmatrix}.$$

Let $g(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2}$. Then

$$\nabla A(x, y) = \begin{pmatrix} \frac{2}{a^2}x \\ \frac{2}{b^2}y \end{pmatrix}.$$

Next, we seek solutions (x, y, λ) to the system of equations

$$\begin{cases} \nabla A(x, y) = \lambda \nabla g(x, y) \\ g(x, y) = 1 \end{cases} \qquad \Rightarrow \qquad \begin{cases} 4y = \frac{2\lambda}{a^2}x \\ 4x = \frac{2\lambda}{b^2}y \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \end{cases}.$$

First, note that if either $x = 0$ or $y = 0$, then the first two equations imply that $x = y = 0$. The third equation implies that either $x \neq 0$ or $y \neq 0$. Thus, it follows that both x and y are nonzero; this makes sense geometrically, as otherwise, the area of the resulting rectangle would be 0. This also implies that $\nabla g(x, y) \neq \mathbf{0}$, so that by the Lagrange multiplier theorem a solution will yield a maximum of A.

Thus, solving for λ in the first two equations gives

$$\frac{2a^2 y}{x} = \frac{2b^2 x}{y} \qquad \Rightarrow \qquad y^2 = \frac{b^2}{a^2}x^2.$$

Plugging this into the third equation gives

$$\frac{x^2}{a^2} + \frac{x^2}{a^2} = 1 \qquad \Rightarrow \qquad x = \frac{a}{\sqrt{2}}.$$

Then $y = \frac{b}{\sqrt{2}}$. Thus, the maximum of f given the constraint $g(x, y) = 1$ occurs at $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$ and therefore the maximum area is

$$A\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right) = 4\left(\frac{a}{\sqrt{2}}\right)\left(\frac{b}{\sqrt{2}}\right) = 2ab.$$

7.3.3 Lagrange multipliers with multiple constraints

The method of Lagrange multipliers can be generalized to account for multiple constraint conditions. For example, you could wish to optimize a function $f(x, y, z)$ subject to the constraints

$$\begin{cases} g_1(x, y, z) = c_1 \\ g_2(x, y, z) = c_2 \end{cases}.$$

The formal statement is as follows.

Theorem 7.17 (Lagrange multipliers with multiple constraints)**.** *Fix $k < n$. Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g_j : \mathbb{R}^n \to \mathbb{R}$ for $1 \le j \le k$ be differentiable functions with continuous partial derivatives. Suppose that f has a local extremum at $\mathbf{x}_0 \in \mathbb{R}^n$ subject to the constraints*

$$\begin{cases} g_1(\mathbf{x}) = c_1 \\ \vdots \\ g_k(\mathbf{x}) = c_k \end{cases}$$

for some constants $c_1, \dots, c_k$ and further suppose that the matrix

$$\begin{pmatrix} \nabla g_1(\mathbf{x}_0) & \cdots & \nabla g_k(\mathbf{x}_0) \end{pmatrix} \in M_{n\times k}(\mathbb{R})$$

has rank k, i.e., has a $k \times k$ submatrix which is invertible. Then

$$\nabla f(\mathbf{x}_0) = \sum_{j=1}^{k} \lambda_j \nabla g_j(\mathbf{x}_0)$$

for some scalars $\lambda_1, \dots, \lambda_k$.

Proof. The proof is a fairly technical application of the implicit function theorem which is similar to Proof 2 of the single constraint theorem, and can safely be skipped. Define $G : \mathbb{R}^n \to \mathbb{R}^k$ by $G(\mathbf{x}) = (g_1(\mathbf{x}), \dots, g_k(\mathbf{x}))$ and let $\mathbf{c} := (c_1, \dots, c_k) \in \mathbb{R}^k$. Then the multiple constraint condition can be rephrased simply as

$$G(\mathbf{x}) = \mathbf{c}.$$

Furthermore, the condition that $\begin{pmatrix} \nabla g_1(\mathbf{x}_0) & \cdots & \nabla g_k(\mathbf{x}_0) \end{pmatrix}$ has rank k is equivalent to the condition that the Jacobian $G'(\mathbf{x}_0) \in M_{k\times n}(\mathbb{R})$ has rank k. Indeed,

$$G'(\mathbf{x}_0) = \begin{pmatrix} \nabla g_1(\mathbf{x}_0) & \cdots & \nabla g_k(\mathbf{x}_0) \end{pmatrix}^T$$

and rank is invariant under the transpose. By renumbering the g_j's if necessary, we may assume that the last k columns of $G'(\mathbf{x}_0)$ are linearly independent.

Let $r := n - k$. For any $\mathbf{x} \in \mathbb{R}^n$ we will write $\mathbf{x} = (\mathbf{y}, \mathbf{z})$ where $\mathbf{y} \in \mathbb{R}^r$ and $\mathbf{z} \in \mathbb{R}^k$. Likewise, write $\mathbf{x}_0 = (\mathbf{y}_0, \mathbf{z}_0)$. By the implicit function theorem, there is a ball $\mathbf{y}_0 \in U \subset \mathbb{R}^r$ and a function $\phi : U \to \mathbb{R}^k$ such that $\phi(\mathbf{y}_0) = \mathbf{z}_0$ and

$$\{\, (\mathbf{y}, \mathbf{z}) \in \mathbb{R}^n \;:\; G(\mathbf{y}, \mathbf{z}) = \mathbf{c} \,\} = \{\, (\mathbf{y}, \mathbf{z}) \in \mathbb{R}^n \;:\; \mathbf{z} = \phi(\mathbf{y}) \,\}$$

for $(\mathbf{y}, \mathbf{z})$ near $(\mathbf{y}_0, \mathbf{z}_0)$. In words, we have described the set $G(\mathbf{x}) = \mathbf{c}$ locally as the graph of some function $\phi : \mathbb{R}^r \to \mathbb{R}^k$ near the point of interest $\mathbf{x}_0 \in \mathbb{R}^n$.

Now consider the function $F : U \subset \mathbb{R}^r \to \mathbb{R}$ defined by $F(\mathbf{y}) = f(\mathbf{y}, \phi(\mathbf{y}))$. The fact that f has a local extremum at $\mathbf{x}_0 = (\mathbf{y}_0, \mathbf{z}_0)$ subject to the constraint $G(\mathbf{x}) = \mathbf{c}$ implies by the above work that F has an unconstrained local extremum at $\mathbf{y}_0 \in U \subset \mathbb{R}^r$. Thus, the Jacobian $F'(\mathbf{y}_0)$ is the 0 matrix. Note that $F(\mathbf{y}) = f(\mathrm{id}(\mathbf{y}), \phi(\mathbf{y})) = (F \circ (\mathrm{id}, \phi))(\mathbf{y})$. Using the Jacobian form of the chain rule, we then have

$$F'(\mathbf{y}_0) = 0 \in M_{1\times r}(\mathbb{R}) \qquad \Rightarrow \qquad f'(\mathbf{x}_0)(\mathrm{id}, \phi)'(\mathbf{y}) = 0$$

so that

$$f'(\mathbf{x}_0) \begin{pmatrix} I_r \\ \phi'(\mathbf{y}_0) \end{pmatrix} = 0.$$

As a sanity check, here I_r is the $r \times r$ identity matrix and $\phi'(\mathbf{y}_0) \in M_{k\times r}(\mathbb{R})$ is the Jacobian of ϕ at $\mathbf{y}_0$. Thus, the block matrix is $(r+k) \times r = n \times r$, and since $f'(\mathbf{x}_0)$ is $1 \times n$ the product about gives $0 \in M_{1\times r}(\mathbb{R})$, which agrees with the fact that $F'(\mathbf{y}_0) \in M_{1\times r}(\mathbb{R})$.

Next, write $f'(\mathbf{x}_0)$ as a block matrix

$$f'(\mathbf{x}_0) = \begin{pmatrix} f'_{\mathbf{y}}(\mathbf{x}_0) & f'_{\mathbf{z}}(\mathbf{x}_0) \end{pmatrix}.$$

Here we have denoted $f'_{\mathbf{y}}(\mathbf{x}_0) \in M_{1\times r}$ to be the Jacobian of f restricted to $\mathbb{R}^r$, where $\mathbf{y}$ is the variable, and likewise $f'_{\mathbf{z}}(\mathbf{x}_0) \in M_{1\times k}(\mathbb{R})$ is the Jacobian of the restriction of f to $\mathbb{R}^k$, where $\mathbf{z}$ is the variable. Then our chain rule computation shows that

$$\begin{pmatrix} f'_{\mathbf{y}}(\mathbf{x}_0) & f'_{\mathbf{z}}(\mathbf{x}_0) \end{pmatrix} \begin{pmatrix} I_r \\ \phi'(\mathbf{y}_0) \end{pmatrix} = 0 \qquad \Rightarrow \qquad f'_{\mathbf{y}}(\mathbf{x}_0) + f'_{\mathbf{z}}(\mathbf{x}_0)\phi'(\mathbf{y}_0) = 0.$$

On the other hand, we can differentiate $G(\mathbf{y}, \phi(\mathbf{y})) = \mathbf{c}$ in exactly the same way to get

$$\begin{pmatrix} G'_{\mathbf{y}}(\mathbf{x}_0) & G'_{\mathbf{z}}(\mathbf{x}_0) \end{pmatrix} \begin{pmatrix} I_r \\ \phi'(\mathbf{y}_0) \end{pmatrix} = 0 \qquad \Rightarrow \qquad G'_{\mathbf{y}}(\mathbf{x}_0) + G'_{\mathbf{z}}(\mathbf{x}_0)\phi'(\mathbf{y}_0) = 0.$$

This time, $G'_{\mathbf{y}}(\mathbf{x}_0) \in M_{k\times r}(\mathbb{R})$ and $G'_{\mathbf{z}}(\mathbf{x}_0) \in M_{k\times k}(\mathbb{R})$. Since $G'_{\mathbf{z}}(\mathbf{x}_0)$ is invertible, we have

$$\phi'(\mathbf{y}_0) = -\left[G'_{\mathbf{z}}(\mathbf{x}_0)\right]^{-1} G'_{\mathbf{y}}(\mathbf{x}_0).$$

Substituting this into the equation $f'_{\mathbf{y}}(\mathbf{x}_0) + f'_{\mathbf{z}}(\mathbf{x}_0)\phi'(\mathbf{y}_0) = 0$ from above gives

$$f'_{\mathbf{y}}(\mathbf{x}_0) - f'_{\mathbf{z}}(\mathbf{x}_0)\left[G'_{\mathbf{z}}(\mathbf{x}_0)\right]^{-1} G'_{\mathbf{y}}(\mathbf{x}_0) = 0$$

and thus

$$f'_{\mathbf{y}}(\mathbf{x}_0) = \left(f'_{\mathbf{z}}(\mathbf{x}_0)\left[G'_{\mathbf{z}}(\mathbf{x}_0)\right]^{-1}\right) G'_{\mathbf{y}}(\mathbf{x}_0). \tag{7.2}$$

Next, note that

$$f'_{\mathbf{z}}(\mathbf{x}_0) = f'_{\mathbf{z}}(\mathbf{x}_0)\left[G'_{\mathbf{z}}(\mathbf{x}_0)\right]^{-1} G'_{\mathbf{z}}(\mathbf{x}_0)$$

so that

$$f'_{\mathbf{z}}(\mathbf{x}_0) = \left(f'_{\mathbf{z}}(\mathbf{x}_0)\left[G'_{\mathbf{z}}(\mathbf{x}_0)\right]^{-1}\right) G'_{\mathbf{z}}(\mathbf{x}_0). \tag{7.3}$$

Combining (7.2) and (7.3) then gives

$$\begin{aligned} f'(\mathbf{x}_0) = \begin{pmatrix} f'_{\mathbf{y}}(\mathbf{x}_0) & f'_{\mathbf{z}}(\mathbf{x}_0) \end{pmatrix} &= \left(f'_{\mathbf{z}}(\mathbf{x}_0)\left[G'_{\mathbf{z}}(\mathbf{x}_0)\right]^{-1}\right) \begin{pmatrix} G'_{\mathbf{y}}(\mathbf{x}_0) & G'_{\mathbf{z}}(\mathbf{x}_0) \end{pmatrix} \\ &= \left(f'_{\mathbf{z}}(\mathbf{x}_0)\left[G'_{\mathbf{z}}(\mathbf{x}_0)\right]^{-1}\right) G'(\mathbf{x}_0). \end{aligned}$$

Let $\Lambda = f'_{\mathbf{z}}(\mathbf{x}_0)\left[G'_{\mathbf{z}}(\mathbf{x}_0)\right]^{-1}$. Note that, since $f'_{\mathbf{z}}(\mathbf{x}_0)$ is $1 \times k$ and $\left[G'_{\mathbf{z}}(\mathbf{x}_0)\right]^{-1}$ is $k \times k$, Λ is $1 \times k$ and hence

$$\Lambda = \begin{pmatrix} \lambda_1 & \cdots & \lambda_k \end{pmatrix}$$

for some scalars $\lambda_1, \ldots, \lambda_k$. We have shown that $f'(\mathbf{x}_0) = \Lambda G'(\mathbf{x}_0)$. Taking the transpose of this equation gives

$$\nabla f(\mathbf{x}_0) = \begin{pmatrix} \nabla g_1(\mathbf{x}_0) & \cdots & \nabla g_k(\mathbf{x}_0) \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{pmatrix} = \sum_{j=1}^{k} \lambda_j \nabla g(\mathbf{x}_0)$$

as desired. Whew! □

Because the proof is so technical, this theorem deserves a bit of informal exposition. Suppose we have a function $f(\mathbf{x})$ and multiple constraints given by

$$\begin{cases} g_1(\mathbf{x}) = c_1 \\ \vdots \\ g_k(\mathbf{x}) = c_k \end{cases}.$$

These constraints define a space of allowable movement, like. If we pick a point $\mathbf{x}_0$ in this space, the only direction we are allowed to move in is any direction which is contained in *all* of the level sets $\{g_j(\mathbf{x}) = c_j\}$ for $j = 1, \ldots, k$. If $\mathbf{v}$ is a vector based at $\mathbf{x}_0$ which represents the direction we wish to walk in, then it follows that

$$\langle \nabla g_j(\mathbf{x}_0), \mathbf{v} \rangle = 0$$

for all j. If $\mathbf{x}_0$ is a local extremum of the constrained problem, this also means that

$$\langle \nabla f(\mathbf{x}_0), \mathbf{v} \rangle = 0.$$

Otherwise, there would be a direction we could walk in to increase the value of f. Thus, at a local extremum it should be the case that $\nabla f(\mathbf{x}_0)$ is orthogonal to all allowable directions $\mathbf{v}$. By the above discussion, the space of allowable directions is exactly the set of directions orthogonal to all of the $\nabla g_j(\mathbf{x}_0)$'s, and thus it follows that $\nabla f(\mathbf{x}_0)$ should be some linear combination of the $\nabla g_j(\mathbf{x}_0)$'s:

$$\nabla f(\mathbf{x}_0) = \sum_{j=1}^{k} \lambda_j \nabla g_j(\mathbf{x}_0).$$

There is a more elegant way to describe this, using some extra language from linear algebra. Namely, the space of allowable directions is

$$\operatorname{Span}\left(\nabla g_1(\mathbf{x}_0), \ldots, \nabla g_k(\mathbf{x}_0)\right)^{\perp}.$$

At an extremum, the gradient $\nabla f(\mathbf{x}_0)$ should live in the orthogonal complement of *this* space:

$$\nabla f(\mathbf{x}_0) \in \left(\operatorname{Span}\left(\nabla g_1(\mathbf{x}_0), \ldots, \nabla g_k(\mathbf{x}_0)\right)^{\perp}\right)^{\perp} = \operatorname{Span}\left(\nabla g_1(\mathbf{x}_0), \ldots, \nabla g_k(\mathbf{x}_0)\right).$$

Example 7.18. As a simple example of a multiple constraint problem, let's optimize the function $f(x, y, z) = y + 2z$ subject to the constraints $2x + z = 4$ and $x^2 + y^2 = 1$.

Let $g_1(x, y, z) = 2x + z$ and $g_2(x, y, z) = x^2 + y^2$. By the method of Lagrange multipliers, we seek solutions to the system of equations

$$\begin{cases} \nabla f(x, y, z) = \lambda_1 \nabla g_1(x, y, z) + \lambda_2 \nabla g_2(x, y, z) \\ g_1(x, y, z) = 4 \\ g_2(x, y, z) = 1 \end{cases}.$$

This simplifies to

$$\begin{cases} 0 = 2\lambda_1 + 2\lambda_2 x \\ 1 = 2\lambda_2 y \\ 2 = \lambda_1 \\ 2x + z = 4 \\ x^2 + y^2 = 1 \end{cases}.$$

The third equation immediately gives us $\lambda_1 = 2$, so the first equation becomes $\lambda_2 x = -2$. Note that λ_2 is nonzero by this equation, so that

$$x = -\frac{2}{\lambda_2} \qquad \text{and} \qquad y = \frac{1}{2\lambda_2}.$$

Using the fifth equation, we then have

$$\left(-\frac{2}{\lambda_2}\right)^2 + \left(\frac{1}{2\lambda_2}\right)^2 = 1 \qquad \Rightarrow \qquad \frac{4}{\lambda_2^2} + \frac{1}{4\lambda_2^2} = 1$$

It follows that $\lambda_2 = \pm\frac{\sqrt{17}}{2}$.

Using this, we have $x = -\frac{4}{\sqrt{17}}$ and $y = \frac{1}{\sqrt{17}}$, or $x = \frac{4}{\sqrt{17}}$ and $y = -\frac{1}{\sqrt{17}}$. Using the fourth equation, when $x = \pm\frac{4}{\sqrt{17}}$ we have

$$2\left(\pm\frac{4}{\sqrt{17}}\right) + z = 4 \qquad \Rightarrow \qquad z = 4 - \left(\pm\frac{8}{\sqrt{17}}\right).$$

Thus, there are two points satisfying the Lagrange condition:

$$\left(-\frac{4}{\sqrt{17}}, \frac{1}{\sqrt{17}}, 4 + \frac{8}{\sqrt{17}}\right) \qquad \text{and} \qquad \left(\frac{4}{\sqrt{17}}, -\frac{1}{\sqrt{17}}, 4 - \frac{8}{\sqrt{17}}\right).$$

Since the constraint equations $x^2+y^2 = 1$ and $2x+z = 4$ describe a closed and bounded set (the first equation implies x and y are bounded, and since x is bounded the second implies z is bounded as well) and since the matrix

$$\begin{pmatrix} \nabla g_1(x, y, z) & \nabla g_2(x, y, z) \end{pmatrix} = \begin{pmatrix} 2 & 2x \\ 0 & 2y \\ 1 & 0 \end{pmatrix}$$

has rank 2 at any point satisfying the constraint equations, it follows that the two identified points above are the global maximum and global minimum points of f subject to the constraints, respectively.

7.4 Exercises

1. Is it possible to solve the equation
$$x^2 + y + \sin(xy) = 0$$
for y as a function of x near the point $(0, 0)$? Is possible to solve for x as a function of y near that point?

2. Consider the following system of equations:
$$\begin{cases} x^2 + y^2 = 4 \\ 2x^2 + y^2 + 8z^2 = 8 \end{cases}.$$
It is possible to solve for (y, z) as a function of x near $(2, 0, 0)$?

3. Consider the following system of equations:
$$\begin{cases} xy^2 + xzu + yv^2 = 3 \\ u^3yz + 2xz - u^2v^2 = 2 \end{cases}.$$
It is possible to solve for (u, v) as a function of (x, y, z) near $(1, 1, 1, 1, 1)$?

4. Use Lagrange multipliers to find the minimum and maximum value of $f(x, y) = e^{xy}$ on the curve $x^3 + y^3 = 16$.

5. Use Lagrange multipliers to find the minimum and maximum value of $f(x, y, z) = xyz$ on the surface $x^2 + 2y^2 + 3z^2 = 6$.

6. Use Lagrange multipliers to find the minimum and maximum value of $f(x, y, z) = x^2 - y - z$ on the surface $z = y^2 - x^2$.

7. For each of the following functions, sketch a contour plot.
   (a) $f(x, y) = x^2 + 2y^2$
   (b) $f(x, y) = x + y$
   (c) $f(x, y) = 3$
   (d) $f(x, y) = y^2 - x^2$

8. For each of the following functions, describe their level surfaces.
   (a) $f(x, y, z) = x^2 + 2y^2 + z^2$
   (b) $f(x, y, z) = y - x^2 - z^2$
   (c) $f(x, y, z) = x + y + z$
   (d) $f(x, y, z) = x^2 + y^2$

9. Find the nearest point on the ellipse $x^2 + 2y^2 = 1$ to the line $x + y = 4$.
[Hint: set this up as Lagrange multipler problem of a four-variable function with two constraints.]

10. Fix $\mathbf{n} \neq \mathbf{0} \in \mathbb{R}^n$ and let $P \subset \mathbb{R}^n$ be a *hyperplane* defined by the equation $\langle \mathbf{x}, \mathbf{n} \rangle = 0$. Let $\mathbf{x}_0 \in \mathbb{R}^n$. Compute the shortest distance from $\mathbf{x}_0$ to P as a Lagrange multiplier problem. See if you can do this without writing vectors out into components.

11. Compute the global extreme values of $f(x, y, z) = xy + xz$ on the domain $D = \{x^2 + y^2 + z^2 \leq 4\}$.

    [Hint: when you check the boundary, you can use Lagrange multipliers.]

12. Consider $f(x, y) = x$ with the constraint equation $g(x, y) = (x-1)^3 - y^2 = 0$. Show that there is no λ such that
$$\nabla f(1, 0) = \lambda \nabla g(1, 0)$$
    but that f constrained to $g(x, y) = 0$ has a minimum at $(1, 0)$. Think about why this doesn't contradict the statement of the Lagrange multiplier theorem.

13. In this exercise, we will study the scalar λ in the statement of the Lagrange multiplier theorem to give it some geometric meaning, as opposed to treating it purely as an algebraic tool. Consider a function $f : \mathbb{R}^n \to \mathbb{R}$ and consider a constraint equation $g(\mathbf{x}) = c$. Assume that f has a local extrema, say a maxima, subject to $g(\mathbf{x}) = c$ at a point P. This point P changes as c changes, so we may view $P : \mathbb{R}_c \to \mathbb{R}^n$ as a function (which we assume to be differentiable). Note that $g(P(c)) = c$.

    (a) Show that $\langle \nabla g(P(c)), P'(c) \rangle = 1$ by differentiating the equation $g(P(c)) = c$ with respect to c.

    (b) Use the fact that $\nabla f(P(c)) = \lambda(c) \nabla g(P(c)$ to show that
$$\frac{d}{dc} f(P(c)) = \lambda(c).$$

    The point here is that λ represents the rate of change of the maximum value of f subject to the constraint $g(x, y) = c$ as c changes. We can interpret this in a more down to earth, satisfying way as follows: view $g(x, y) = c$ as a budget constraint (so that your total budget is c) and f as a profit function. The problem of maximizing f given $g(x, y) = c$ is then viewed as the problem of maximizing profit given a certain budget. We can then ask: if we increase our budget by a little bit, how much will our maximum profit increase? The value of λ gives us the answer.

14. In this exercise, I will give you a glimpse into the second derivative test for constrained optimization problems involving something called the *bordered Hessian matrix*. Let $f : \mathbb{R}^n \to R$ and $g : \mathbb{R}^n \to \mathbb{R}$ and suppose we want to optimize $f(\mathbf{x})$ subject to the constraint $g(\mathbf{x}) = c$. Define a new function $\mathcal{L} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, the *Lagrangian* of the optimization problem, as follows:
$$\mathcal{L}(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda(g(\mathbf{x}) - c).$$

    (a) Show that if $(\lambda_0, \mathbf{x}_0)$ satisfies the Lagrange multiplier condition (i.e., $\nabla f(\mathbf{x}_0) = \lambda \nabla g(\mathbf{x}_0)$) then $(\lambda_0, \mathbf{x}_0)$ is a critical point of $\mathcal{L}$.

    Since the solution to our Lagrange multiplier condition is a critical point of the unconstrained, this motivates us to consider the Hessian matrix of $\mathcal{L}$. The *bordered Hessian matrix* of the given constrained optimization problem is by definition the usual Hessian matrix of $\mathcal{L}$, $\mathcal{L}''(\lambda, \mathbf{x})$.

    (b) Show that the bordered Hessian matrix is of the form
$$\mathcal{L}''(\lambda, \mathbf{x}) = \begin{pmatrix} 0 & g_{x_1}(\mathbf{x}) & \cdots & g_{x_n}(\mathbf{x}) \\ g_{x_1}(\mathbf{x}) & \mathcal{L}_{x_1 x_1}(\lambda, \mathbf{x}) & \cdots & \mathcal{L}_{x_1 x_n}(\lambda, \mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ g_{x_n}(\mathbf{x}) & \mathcal{L}_{x_n x_1}(\lambda, \mathbf{x}) & \cdots & \mathcal{L}_{x_n x_n}(\lambda, \mathbf{x}) \end{pmatrix}.$$

Now, we will restrict ourselves to the two-variable case. That is, we wish to optimize $f(x, y)$ subject to the constraint $g(x, y) = c$. In this case, the bordered Hessian matrix is (dropping the λ and (x, y) dependence for clarity)

$$\mathcal{L}'' = \begin{pmatrix} 0 & g_x & g_y \\ g_x & \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ g_y & \mathcal{L}_{xy} & \mathcal{L}_{yy} \end{pmatrix}$$

where we assume that f and g have continuous second partials to invoke Clairaut's theorem. In this case, here is the second derivative test for the constrained optimization problem:

Theorem 7.19. *Suppose that (λ_0, x_0, y_0) is a solution of the Lagrange multiplier problem.*

(a) If $-\det \mathcal{L}''(\lambda_0, x_0, y_0) > 0$, then (x_0, y_0) is a local minimum of f subject to $g(x, y) = c$.

(b) If $-\det \mathcal{L}''(\lambda_0, x_0, y_0) < 0$, then (x_0, y_0) is a local maximum of f subject to $g(x, y) = c$.

It is possible to state this is more generality, but it gets fairly complicated.

(c) Let $f(x, y) = x^2 + 4y^2$ and consider the constraint $x^2 + y^2 = 1$. Use the second derivative test stated above to classify the solutions of the Lagrange multiplier condition.

15. Another interesting interpretation of the Lagrange multiplier concerns the eigenvalues of a symmetric matrix A. In particular, in this exercise you will prove that a symmetric matrix has an eigenvector (without appealing to the fundamental theorem of algebra).

Let A be an $n \times n$ symmetric matrix and define $f : \mathbb{R}^n \to \mathbb{R}$ by $f(\mathbf{x}) = \langle A\mathbf{x}, \mathbf{x} \rangle$. Let $S = \{\|\mathbf{x}\| = 1\}$ be the unit sphere in $\mathbb{R}^n$. Since S is closed and bounded, by the extreme value theorem it follows that f has both a global maximum and minumum when restricted to S. Let $\mathbf{x}_0 \in S$ denote the global maximum and let $g(\mathbf{x}) = \|\mathbf{x}\|$.

(a) Show that

$$\nabla f(\mathbf{x}) = 2A\mathbf{x} \qquad \text{and} \qquad \nabla g(\mathbf{x}) = 2\frac{\mathbf{x}}{\|\mathbf{x}\|}.$$

(b) Conclude using the Lagrange multiplier theorem that

$$A\mathbf{x}_0 = \lambda \mathbf{x}$$

for some $\lambda \in \mathbb{R}$.

Thus, we have shown using the method of Lagrange multipliers that any symmetric matrix has a real eigenvalue! Furthermore, that eigenvalue λ is the Lagrange multiplier. Geometrically, you can view the function $\mathbf{x} \mapsto A\mathbf{x}$ as a map from the unit sphere S which stretches S into a (hyper)ellipsoid. The longest and shortest axes of the resulting (hyper)ellipsoid are eigendirections of A, and the corresponding eigenvalues are the Lagrange multipliers corresponding to the the global minimum and maximum described above.

16. In Chapter 2, I included a challenge problem to prove an inequality called *Holder's inequality*, a generalization of the Cauchy-Schwarz inequality. It is possible to prove Holder's inequality alternatively using Lagrange multipliers.

    (a) Fix $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Use the method of Lagrange multipliers with multiple constraints to maximize

    $$f(x_1, \ldots, x_n, y_1, \ldots, y_n) = \sum_{j=1}^{n} x_j y_j$$

    subject to the constraints

    $$\begin{cases} \sum_{j=1}^n x_j^p = 1 \\ \sum_{j=1}^n y_j^q = 1 \end{cases}.$$

    (b) Use this to prove Holder's inequality: for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

    $$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \left(\sum_{j=1}^{n} x_j^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} y_j^q \right)^{\frac{1}{q}}.$$

17. Another very well-known inequality is the *AM-GM inequality*, which compares the *arithmetic mean* of n numbers $x_1, \ldots, x_n \geq 0$ to the *geometric-mean*.

    (a) Use the method of Lagrange multipliers to optimize the function

    $$f(x_1, \ldots, x_n) = (x_1 \cdots x_n)^{\frac{1}{n}}$$

    subject to the constraint

    $$\frac{x_1 + \cdots + x_n}{n} = 1.$$

    Here $x_j \geq 0$.

    (b) Conclude the AM-GM inequality: for any $x_j \geq 0$,

    $$(x_1 \cdots x_n)^{\frac{1}{n}} \leq \frac{x_1 + \cdots + x_n}{n}.$$

Appendix A

More Linear Algebra

Appendix B

Proof of the Inverse and Implicit Function Theorems

B.1 Some preliminary results

B.1.1 The contraction mapping theorem

B.1.2 The mean value inequality

B.2 Proof of the inverse function theorem

B.3 Proof of the implicit function theorem

Finally, we can present a rigorous proof of the implicit function theorem using the inverse function theorem. I'll remind you of the set up: we have a differentiable function $g : \mathbb{R}^{r+k} \to \mathbb{R}^k$ with continuous partial derivatives, a point $(\mathbf{y}_0, \mathbf{z}_0)$ satisfying $g(\mathbf{y}_0, \mathbf{z}_0) = \mathbf{c}$ for some constant $\mathbf{c} \in \mathbb{R}^k$, and we assume that $g'_{\mathbf{z}}(\mathbf{y}_0, \mathbf{z}_0)$ is invertible. We want to show that there is a locally defined differentiable function $\phi : \mathbb{R}^r \to \mathbb{R}^k$ such that $\phi(\mathbf{y}_0) = \mathbf{z}_0$ and near the point $(\mathbf{y}_0, \mathbf{z}_0)$,

$$\{g(\mathbf{y}, \mathbf{z}) = \mathbf{c}\} = \{\mathbf{z} = \phi(\mathbf{y})\}.$$

Furthermore, we wish to show that the Jacobian of ϕ at $\mathbf{y}_0$ is

$$\phi'(\mathbf{y}_0) = -\left[g'_{\mathbf{z}}(\mathbf{y}_0, \mathbf{z}_0)\right]^{-1} g'_{\mathbf{y}}(\mathbf{y}_0, \mathbf{z}_0).$$

Proof of Theorem 7.6. I've mentioned that we will prove the implicit function theorem using the *inverse* function theorem. At the moment, we have a map

$$g : \mathbb{R}^{r+k} \to \mathbb{R}^k.$$

The inverse function theorem concerns functions $f : \mathbb{R}^n \to \mathbb{R}^n$. Thus, to have any hope of invoking the inverse function theorem, we need to extend g to a function $\tilde{g} : \mathbb{R}^{r+k} \to \mathbb{R}^{r+k}$. Furthermore, we want this extension to have invertible derivative at $(\mathbf{y}_0, \mathbf{z}_0)$. The most natural way to define such an extension is via the identity map! In particular, define

$$\tilde{g} : \mathbb{R}^{r+k} \to \mathbb{R}^{r+k}$$

by

$$\tilde{g}(\mathbf{y}, \mathbf{z}) := \begin{pmatrix} \mathbf{y} \\ g(\mathbf{y}, \mathbf{z}) \end{pmatrix} \in \mathbb{R}^{r+k}.$$

Note that $g'(\mathbf{y}, \mathbf{z})$ will be a square $(r+k) \times (r+k)$ matrix. We can write such a matrix in block form:

$$\tilde{g}'(\mathbf{y}, \mathbf{z}) = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \in M_{(r+k)\times(r+k)}(\mathbb{R})$$

where A is $r \times r$, B is $r \times k$, C is $k \times r$, and D is $k \times k$.

The matrix A will be the Jacobian of the first r components of $\tilde{g}$ with respect to the first r variables. Since $\tilde{g}(\mathbf{y}, \mathbf{z}) = (\mathbf{y}, g(\mathbf{y}, \mathbf{z}))$, this means the Jacobian of $\mathbf{y}$ with respect to $\mathbf{y}$. This is the identity matrix! Thus, $A = I_r$. Likewise, B will be the Jacobian of the first r components of $\tilde{g}$ with respect to the last k variables; this is Jacobian of $\mathbf{y}$ with respect to $\mathbf{z}$, which is the 0 matrix. Thus, $B = 0$. By the same logic, $C = g'_{\mathbf{y}}(\mathbf{y}, \mathbf{z})$ and $D = g'_{\mathbf{z}}(\mathbf{y}, \mathbf{z})$. Thus,

$$\tilde{g}'(\mathbf{y}_0, \mathbf{z}_0) = \begin{pmatrix} I_r & 0 \\ g'_{\mathbf{y}}(\mathbf{y}_0, \mathbf{z}_0) & g'_{\mathbf{z}}(\mathbf{y}_0, \mathbf{z}_0) \end{pmatrix}.$$

This is a "block lower triangular" matrix, in case those words resonate with you. Furthermore, since I_r is invertible and by assumption $g'_{\mathbf{z}}(\mathbf{y}_0, \mathbf{z}_0)$ is invertible, it turns out that $\tilde{g}'(\mathbf{y}_0, \mathbf{z}_0)$ is invertible. One way to argue this is to show that the determinant of a block triangular matrix is the product of the determinants of the diagonal block elements; however, since we haven't discussed much of the determinant in these notes, I will simply exhibit the inverse. Indeed, note that

$$\begin{pmatrix} I_r & 0 \\ g'_{\mathbf{y}}(\mathbf{y}_0, \mathbf{z}_0) & g'_{\mathbf{z}}(\mathbf{y}_0, \mathbf{z}_0) \end{pmatrix} \begin{pmatrix} I_r & 0 \\ -\left[g'_{\mathbf{z}}(\mathbf{y}_0, \mathbf{z}_0)\right]^{-1} g'_{\mathbf{y}}(\mathbf{y}_0, \mathbf{z}_0) & \left[g'_{\mathbf{z}}(\mathbf{y}_0, \mathbf{z}_0)\right]^{-1} \end{pmatrix} = \begin{pmatrix} I_r & 0 \\ 0 & I_k \end{pmatrix} = I_{r_k}.$$

You can verify that the reverse product also yields the identity. Thus, $\tilde{g}'(\mathbf{y}_0, \mathbf{z}_0)$ is invertible, and

$$\left[\tilde{g}'(\mathbf{y}_0, \mathbf{z}_0)\right]^{-1} = \begin{pmatrix} I_r & 0 \\ -\left[g'_{\mathbf{z}}(\mathbf{y}_0, \mathbf{z}_0)\right]^{-1} g'_{\mathbf{y}}(\mathbf{y}_0, \mathbf{z}_0) & \left[g'_{\mathbf{z}}(\mathbf{y}_0, \mathbf{z}_0)\right]^{-1} \end{pmatrix}.$$

By the inverse function theorem, there is an open ball U around $(\mathbf{y}_0, \mathbf{z}_0)$ in $\mathbb{R}^{r+k}$ such that $\tilde{g} : U \to \tilde{g}(U)$ is invertible. Note that $\tilde{g}(\mathbf{y}_0, \mathbf{z}_0) = (\mathbf{y}_0, \mathbf{c})$. Thus, $V := \tilde{g}(U)$ is some set set containing $(\mathbf{y}_0, \mathbf{c})$, and therefore $\tilde{g}^{-1} : V \to U$ is defined near $(\mathbf{y}_0, \mathbf{c})$.

We will use the function $\tilde{g}^{-1}(\mathbf{y}, \mathbf{z})$ to define the function $\phi(\mathbf{y})$ that satisfies the implicit function theorem. Writing $\tilde{g}^{-1}$ into it's $\mathbb{R}^r$ and $\mathbb{R}^k$ components, we have

$$\tilde{g}^{-1}(\mathbf{y}, \mathbf{z}) = \begin{pmatrix} \mathbf{y} \\ a(\mathbf{y}, \mathbf{z}) \end{pmatrix}$$

for some function $a : V \subset \mathbb{R}^{r+k} \to \mathbb{R}^k$. We can then produce a function $\phi(\mathbf{y})$ spitting out into $\mathbb{R}^k$ by restricting the $\mathbf{z}$ variable of a to be constant. Namely, define

$$\phi(\mathbf{y}) := a(\mathbf{y}, \mathbf{c}).$$

Since $\tilde{g}^{-1}$ is differentiable by the inverse function theorem, a is differentiable and thus ϕ is differentiable; also, note that $\phi(\mathbf{y}_0) = a(\mathbf{y}_0, \mathbf{c})$. Since $\tilde{g}(\mathbf{y}_0, \mathbf{z}_0) = (\mathbf{y}_0, \mathbf{c})$, we have $a(\mathbf{y}_0, \mathbf{c}) = \mathbf{z}_0$ and therefore $\phi(\mathbf{y}_0) = \mathbf{z}_0$ as desired.

It remains to do two things: show that, locally, the set $g(\mathbf{y}, \mathbf{z}) = \mathbf{c}$ is described by $\mathbf{z} = \phi(\mathbf{y})$, and to compute the Jacobian of ϕ. Towards the first goal, note that in the set U containing $(\mathbf{y}_0, \mathbf{z}_0)$ (where all the action is happening), a point $(\mathbf{y}, \mathbf{z})$ satisfying

$$g(\mathbf{y}, \mathbf{z}) = \mathbf{c}$$

further satisfies

$$\tilde{g}(\mathbf{y}, \mathbf{z}) = (\mathbf{y}, \mathbf{c}).$$

Applying $\tilde{g}^{-1}$ to both sides gives

$$(\mathbf{y}, \mathbf{z}) = \tilde{g}^{-1}(\mathbf{y}, \mathbf{c}) = (\mathbf{y}, a(\mathbf{y}, \mathbf{c})) = (\mathbf{y}, \phi(\mathbf{y})).$$

Thus, we have shown that the set of points satisfying $g(\mathbf{y}, \mathbf{z}) = \mathbf{c}$ are (locally) precisely the set of point satisfying $\mathbf{z} = \phi(\mathbf{y})$, as desired!

Finally, we compute the Jacobian of ϕ. This follows from the inverse function theorem and our earlier inverse matrix computation! In particular,

$$\begin{aligned}(\tilde{g}^{-1})'(\mathbf{y}_0, \mathbf{c}) &= \left[\tilde{g}'(\mathbf{y}_0, \mathbf{z}_0)\right]^{-1} \\ &= \begin{pmatrix} I_r & 0 \\ -\left[g'_{\mathbf{z}}(\mathbf{y}_0, \mathbf{z}_0)\right]^{-1} g'_{\mathbf{y}}(\mathbf{y}_0, \mathbf{z}_0) & \left[g'_{\mathbf{z}}(\mathbf{y}_0, \mathbf{z}_0)\right]^{-1}. \end{pmatrix}\end{aligned}$$

As we remarked, earlier, by matching up components, we know that the lower-left block matrix above represents the Jacobian of the last k components of $\tilde{g}^{-1}$ with respect to the first k components. This is exactly $a'_{\mathbf{y}}(\mathbf{y}_0, \mathbf{c})$, which is exactly $\phi'(\mathbf{y}_0)$. Therefore,

$$\phi'(\mathbf{y}_0) = \left[\tilde{g}'(\mathbf{y}_0, \mathbf{z}_0)\right]^{-1}$$

and we have proven the implicit function theorem.

$\square$

B.4 The constant rank theorem