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On the Generation of Sporadic Simple Group $O'N$ by $(2, 3, t)$ Generators

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Abstract

A finite group G is said to be $(2, 3, t)$-generated, if it is a quotient of triangle group $T(2, 3, t) := \langle x^2 = y^3 = (xy)^t = 1\rangle$. That is, G is $(2, 3, t)$-generated if can be generated by just two of its elements x and y such that x is an element of order 2, y is an element of order 3 and xy has order t. In this paper, we compute $(2, 3, t)$-generations for the sporadic simple group O'Nan, where t is a divisor of |O'N|. For computations, we make considerable use of the computer algebra system $\mathbb{GAP}$-Groups, Algorithms and Programming [20].

Mathematics Subject Classification: Primary 20D08, 20F05

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1 Introduction

There has been considerable amount of progress recently towards the computation of the $(2, 3, t)$-generations of the simple groups. A simple group G is

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$(2,3,t)$-generated if it can be generated by just two of its elements x and y such that $o(x) = 2$, $o(y) = 3$ and $o(xy) = t$. It is well known now that every finite simple group can be generated by two of its elements. The $(2,3,t)$-generated groups are the free product of two cyclic groups of order two and three. That is, such groups are the homomorphic images of the modular group $PSL(2,\mathbb{Z})$. Further, a $(2,3,7)$-generated group is called an *Hurwitz groups*. The problem of computing the genus of finite simple groups can be reduced to the generations of relevant simple groups.

It is well know that with few exceptions all simple groups are $(2,3)$ - generated. Woldar [23] showed that each sporadic simple groups, except M_{11}, M_{22}, M_{23} and McL, are $(2,3,t)$-generated. Further, most of the classical linear groups and Lie groups are $(2,3)$-generated. The problem of generating a group by a set of conjugate involutions of minimal size is also closely related to the $(2,3,t)$-generation of the group. In a series of articles [1, 2, 3, 4, 5], it has been shown that sporadic groups HS, McL, Co_1, Co_2, Co_3, Suz, Ru and Th can be generated by three conjugate involutions. The work of Liebeck and Shalev shows that all but finitely many simple classical groups can be generated by three involutions (see [17]). However, the problem of finding simple classical groups which can be generated by three conjugate involutions is still very much open.

Moori [18] computed all $(2,3,p)$-generations of the Fischer's sporadic group Fi_{22}, where p is a prime divisor of $|Fi_{22}|$. In [15], Ganief and Moori determined all $(2,3,t)$-generations of the third Janko group J_3. Drafsheh, Ashrafi and Moghani [14] computed all possible (p,q,r)-generations of the sporadic simple group $O'N$, where p, q and r are prime divisors of $|O'N|$. Recently, Ali [1] computed ranks (minimal conjugate generating sets) for the O'Nan's sporadic simple group $O'N$. More recently, the author in [6, 7] and with Ali [8, 9] determined $(2,3,t)$-generations for sporadic groups HS, McL, J_1, J_2, Co_2 and Co_3.

In the present article we compute the $(2,3,t)$-generations of the sporadic simple group $O'N$, where t is any divisor of $|O'N|$. We will also give the generating triples for Held group He. For more information regarding the study of $(2,3,t)$-generations and computational techniques used in this article to determine the generating pairs, we refer the reader to the references cited above [1], [4], [15], [18] and [23].

We follow the same notation as in [8] and [9]. Let G be a finite group and C_1, C_2 and C_3 are the conjugacy classes of elements of the group $O'N$, z is a fixed representative of C_3, then we denote $\Delta_G(C_1, C_2, C_3)$ by the number of distinct ordered pairs $(g_1, g_2) \in (C_1 \times C_2)$ such that $g_1g_2 = g_3$. We know that $\Delta_G(C_1, C_2, C_3)$ is structure constant of G for the conjugacy classes C_1, C_2, C_3 and can be calculated using the character table of the group G using the formula

$$\Delta_G(C_1, C_2, C_3) = \frac{|C_1||C_2|}{|G|} \times \sum_{i=1}^{m} \frac{\chi_i(g_1)\chi_i(g_2)\overline{\chi_i(g_3)}}{\chi_i(1_G)}$$

where $\chi_1, \chi_2, \cdots, \chi_m$ are the irreducible complex ordinary characters of the group G. Further let $\Delta^*_G(C_1, C_2, C_3)$ denotes the number of the distinct ordered pairs $(g_1, g_2) \in (C_1 \times C_2)$ such that $g_1g_2 = g_3$ and $G = \langle g_1, g_2 \rangle$. If $\Delta^*_G(C_1, C_2, C_3) > 0$, then G is said to be (C_1, C_2, C_3)-generated. For H any subgroup of the group G containing the fixed element $g_3 \in C_3$, then $\Sigma_H(C_1, C_2, C_3)$ denotes the number of distinct ordered pairs $(g_1, g_2) \in (C_1 \times C_2$ such that $g_1g_2 = g_3$ and $\langle g_1, g_2 \rangle \leq H$ where $\Sigma_H(C_1, C_2, C_3)$ is obtained by summing the structure constants $\Delta_H(c_1, c_2, c_3)$ of H over all H-conjugacy classes c_1, c_2 satisfying $c_i \subseteq H \cap C_i$ for $1 \leq i \leq 2$.

As in $\mathbb{ATLAS}$, a general conjugacy class of elements of order n in G will be denoted by nX. For examples, $3A$ represents the first conjugacy class of order 3 in the group G. We use the maximal subgroups of O'N as given in Table I quite [11] extensively.

The following result will be crucial in determining the non-generation of a triple in the finite group G.

Lemma 1.1. *([10]) Let G be a finite centerless group and suppose lX, mY, nZ are G-conjugacy classes for which $\Delta^*(G) = \Delta^*_G(lX, mY, nZ) < |C_G(nZ)|$. Then $\Delta^*(G) = 0$ and therefore G is not (lX, mY, nZ)-generated.* □

Theorem 1.2. *([24]) Let G be a finite group and H a subgroup of G containing a fixed element x such that $gcd(o(x), [N_G(H){:}H]) = 1$. Then the number h of conjugates of H containing x is $\chi_H(x)$, where χ_H is the permutation character of G with action on the conjugates of H. In particular,*

$$h = \sum_{i=1}^{m} \frac{|C_G(x)|}{|C_{N_G(H)}(x_i)|} ,$$

where $x_1, \ldots, x_m$ are representatives of the $N_G(H)$-conjugacy classes that fuse to the G-class $[x]_G$.

2 $(2,3,t)$-Generations of $O'N$

The simple group $O'N$ is O'Nan sporadic group of order

$$460815505920 = 2^9 \times 3^4 \times 5 \times 7^3 \times 11 \times 19 \times 31.$$

The group $O'N$ was discovered by M. O'Nan [19] in 1973 by considering the simple groups containing a subgroup 2^3 with normalizer $4^3.L_3(2)$ and an involution with centralizer $4.L_3(2).2$. Later, the existence and uniqueness of $O'N$ was proved by Sims and Andrilli using a computer. The O'Nan group $O'N$ has 13 classes of its maximal subgroups [21]. It has 30 conjugacy classes. It has a unique conjugacy class of elements of order 2 and elements of order 3, namely $2A$ and $3A$. This group acts on a set Λ of 122760 points and the point stabilizer is a group isomorphic $L_3(2){:}2$. For the basic properties of the group $O'N$ and other related information, we refer reader to [19] and [21].

Next, we compute the $(2,3,t)$-generations for the group $O'N$ where t is any divisor of order of the group $O'N$. By [10], if a simple non-abelian group G is $(2,3,t)$-generated then we must have $\frac{1}{2}+\frac{1}{3}+\frac{1}{t}<1$. That is, in order to investigate the $(2,3,t)$-generations we only need to consider $t \geq 7$. The $(2,3,p)$-generations, for p a prime, of the groups $O'N$ has been investigated in [14], so we are only concerned here in this article with the $(2,3,t)$-generations of the group $O'N$ when $t \geq 7$ and t is not a prime divisor of $|O'N|$. That is, $t \in \{8,10,12,14,16,20,28\}$.

Lemma 2.1. *The O'Nan sporadic simple group $O'N$ is $(2,3,8)$-generated.*

Proof. In the group $O'N$ we have two conjugacy classes of elements of order 8. So, we need to investigate two triples $(2A,3A,8A)$ and $(2A,3A,8B)$. Since the results obtained for the class $8A$ can be replaced by the results obtained for $8B$. Let $8X$ denote the conjugacy class $8A$ or $8B$ of the group $O'N$.

Simple computation in $\mathbb{GAP}$ shows that the structure constant $\Delta_{O'N}(2A,3A,8X)=848$. If z is a fixed element of order 8 in the group $O'N$ then there are 848 distinct ordered pairs (α,β) such $\{\alpha,\beta\}\in(2A\times 3A)$ and $\alpha\beta = z$. From the maximal subgroups of $O'N$ given in the $\mathbb{ATLAS}$ [11] (see Table I), we observe that up to isomorphism, $H_1 \cong L_3(7){:}2$, $H_4 \cong 4{\cdot}L_3(4){:}2$, $H_5 \cong (3^2{:}4\times A_6){\cdot}2$, $H_6 \cong 3^4{:}2_-^{1+4}{\cdot}D_{10}$, $H_7 \cong L_2(31)$ (two classes), $H_9 \cong 4^3{\cdot}L_3(2)$ and $H_{10} \cong M_{11}$ (two classes) are the only maximal subgroup of $O'N$ which may contain $(2A,3A,8X)$-generated proper subgroups. Further, by considering the fusions of conjugacy classes from these maximal subgroups into $O'N$-classes $2A$, $3A$ and $8X$ together with the values of h given in Table II, we compute that $\Sigma^*_{H_1}(2A,3A,8X) = 400$, $\Sigma^*_{H_4}(2A,3A,8X) = 80$, $\Sigma^*_{H_5}(2A,3A,8X) = 0$, $\Sigma^*_{H_6}(2A,3A,8X) = 0$, $\Sigma^*_{H_7}(2A,3A,8X) = 128$, $\Sigma^*_{H_9}(2A,3A,8X) = 32$ and $\Sigma^*_{H_{10}}(2A,3A,8X) = 32$. Hence, we have

$$\begin{aligned}
\Delta^*_{O'N}(2A,3A,8X) &\geq \Delta_{O'N}(2A,3A,8X) - \Sigma^*_{H_1}(2A,3A,8X) - \Sigma^*_{H_4}(2A,3A,8X) \\
&\quad -\Sigma^*_{H_5}(2A,3A,8X) - \Sigma^*_{H_6}(2A,3A,8X) - \Sigma^*_{H_7}(2A,3A,8X) \\
&\quad -\Sigma^*_{H_9}(2A,3A,8X) - \Sigma^*_{H_{10}}(2A,3A,8X) \\
&= 848 - 672 > 0.
\end{aligned}$$

Hence the the group $O'N$ is $(2,3,8)$-generated. □

Lemma 2.2. *The sporadic group $O'N$ is $(2,3,10)$- and $(2,3,14)$-generated.*

Proof. First we consider the triple $(2A,3A,10A)$. The sturcture contact $\Delta_{O'N}(2A,3A,10A) = 535$. Upto isomorphism, the proper subgroups of $O'N$ that admit $(2A,3A,10A)$-generation are contained in the maximal subgroups $H_3 \cong J_1$, $H_4 \cong 4{\cdot}L_3(4){:}2$, $H_5 \cong (3^2{:}4\times){\cdot}2$ and $H_6 \cong 3^4{:}2^{1+4}_{-}{\cdot}D_{10}$. We compute $\Sigma_{H_3}(2A,3A,10A) = 45$, $\Sigma_{H_4}(2A,3A,10A) = 15$, $\Sigma_{H_5}(2A,3A,10A) = 90$ and $\Sigma_{H_6}(2A,3A,10A) = 0$. Further, let z be a fixes element of order 10 in $O'N$ then from Table II we conclude

$$\Delta^*_{O'N}(2A,3A,10A)$$

$$\begin{aligned}
&\geq \Delta_{O'N}(2A,3A,10A) - 4\times\Sigma_{H_3}(2A,3A,10A) - 1\times\Sigma_{H_4}(2A,3A,10A) \\
&\quad -1\times\Sigma_{H_5}(2A,3A,10A) - 4\times\Sigma_{H_6}(2A,3A,10A) \\
&= 535 - 4(45) - 1(15) - 1(90) - 4(0) > 0,
\end{aligned}$$

proving that $(2A,3A,10A)$ is a generating triple of $O'N$.

Next, for the triple $(2A,3A,14A)$, we compute $\Delta_{O'N}(2A,3A,14A) = 1295$. The maximal subgroups of $O'N$ with order divisible by 14 are, up to isomorphims, $H_1 \cong L_3(7){:}2$, $H_2 \cong L_3(7){:}2$ and H_4 (see Table I). Now our calculations give

$$\Delta^*_{O'N}(2A,3A,14A) \geq 1295 - 378 - 378 - 7 > 0.$$

Therefore, $O'N$ is $(2A,3A,14A)$-generated.

□

Lemma 2.3. *The O'Nan group $O'N$ is $(2A,3A,15X)$-, $(2A,3A,20X)$-, and $(2A,3A,28X)$-generated, where $X \in \{A,B\}$.*

Proof. Since the results obtained for the conjugacy class $X = A$ or B, so let $X \in \{A, B\}$.

First consider the triple $(2A, 3A, 15X)$. Direct computation shows that $\Delta_{O'N}(2A, 3A, 15X)$
$= 820$. The only maximal subgroups of the group $O'N$ with order divisible by 15, up to isomorphim, are $H_3 \cong J_1$, $H_5 \cong (3^2{:}4\times A_6){\cdot}2$ and $H_7 \cong L_2(31)$ (two copies). We have $\Sigma_{H_3}(2A, 3A, 15X) = 60$, $\Sigma_{H_5}(2A, 3A, 15X) = 10$ and $\Sigma_{H_7}(2A, 3A, 15X) = 30$. Since a fixed element of order 15 in $O'N$ is contained three conjugate copies of H_3, six conugate copies of H_7 and in a unique conjugate of H_5, we have

$$\Delta^*_{O'N}(2A, 3A, 15X)$$

$$\begin{aligned}
&\geq \Delta_{O'N}(2A, 3A, 15X) - 3\times\Sigma_{H_3}(2A, 3A, 15X) - 1\times\Sigma_{H_5}(2A, 3A, 15X) \\
&\quad -12\times\Sigma_{H_7}(2A, 3A, 15X) \\
&= 820 - 3(180) - 1(10) - 12(30) > 0.
\end{aligned}$$

Thus $O'N$ $(2A, 3A, 15X)$-generated.

The maximal subgroups of $O'N$ with elements of order 20 that have non-empty intersection with the conjugacy classes $20A$, up to isomorphisms, are $H_4 \cong 4{\cdot}L_3(4){:}2$, H_5. We compute $\Sigma_{H_4}(2A, 3A, 20X) = 20$ and $\Sigma_{H_5}(2A, 3A, 20X) = 0$. Further a fixed element z of order 20 is contained in a unique conjugate of H_4. Hence, H_4 contributes at most 1×20 to $\Delta_{O'N}(2A, 3A, 20X)$. Now since $\Delta_{O'N}(2A, 3A, 20X) = 900 > 20$, we have $\Delta^*_{O'N}(2A, 3A, 20X) \geq 880$. Therefore, $O'N$ is $(2A, 3A, 20X)$-generated.

For the triple $(2A, 3A, 28X)$, we compute structute contantant
$\Delta_{O'N}(2A, 3A, 28X) = 854$. The only maximal subgroups of $O'N$ (see Table I) which may contain $(2A, 3A, 28X)$-generated proper subgroups are isomorphic to $H_1 \cong L_3(7){:}2$ (two copies) and H_4. By looking at the fusions from the maximal subgroups H_1 and H_4 into $O'N$ (see Table II), we have $\Sigma_{H_1}(2A, 3A, 28X) = 147$ and $\Sigma_{H_4}(2A, 3A, 28X) = 14$. Since a fixed element of order 28 in $O'N$ is contained in a unique conjugate of each H_1 and H_4 we obtain

$$\Delta^*_{O'N}(2A, 3A, 28X)$$

$$\begin{aligned}
&\geq \Delta_{O'N}(2A, 3A, 28X) - 2\times\Sigma_{H_1}(2A, 3A, 28X) - 1\times\Sigma_{H_4}(2A, 3A, 28X) \\
&= 854 - 2(147) - 1(14) > 0.
\end{aligned}$$

This shows that $(2A, 3A, 28X)$ is a generating triple of $O'N$. □

Lemma 2.4. *The O'Nan group $O'N$ is not $(2,3,12)$-generated.*

Proof. For the triple $(2A,3A,12A)$ we compute $\Delta_{O'N}(2A,3A,12A) = 980$ and $C_{O'N}(12A) = 36$. We show that the group $O'N$ is not $(2A,3A,12A)$-generations by using the random element generation. We constuct $O'N$ using its *standard generators* given in [22]. $O'N$ has 154-dimensional irreducible representation over $\mathbb{GF}(3)$. We see that $O'N \cong \langle a,b\rangle$, where a and b are 154×154 matrices over $\mathbb{GF}(3)$ with $a \in 2A$, $b \in 4A$ and ab has order 11. In $\mathbb{GAP}$ we use pseudo-random technique to produced elements c and d such that $c \in 2A$, $d \in 3A$ and $cd \in 12A$. Let $P = \langle c,d\rangle$ then $P < O'N$ and $|P| = 3753792$. Consequently, we obtain that $O'N$ is not $(2A,3A,12A)$-generated. $\square$

Lemma 2.5. *The group $O'N$ is $(2A,3A,16X)$-generated, where $X \in \{A,B,C,D\}$.*

Proof. The maximal subgroups of $O'N$ having non-empty intersection with the classes $2A$, $3A$ and $16X$, where $X \in \{A,B,C,D\}$, are isomorphic to $H_1 \cong L_3(7){:}2$, $H_2 \cong L_3(7){:}2$, $H_4 \cong 4{\cdot}L_3(4){:}2$, $H_7 \cong L_2(31)$ (2 copies) and $H_9 \cong 4^3{\cdot}L_3(2)$. Let $Y \in \{A,B\}$ and $Z \in \{C,D\}$ Using the fusions from these maximal subgroups to $O'N$ (Table II) we have

$$\begin{aligned}
\Delta^*_{O'N}(2A,3A,16Y) &\geq \Delta_{O'N}(2A,3A,16Y) - 2\times\Sigma_{H_1}(2A,3A,16Y) - 1\times\Sigma_{H_4}(2A,3A,16Y)\\
&\quad -2\times\Sigma_{H_7}(2A,3A,16Y) - 1\times\Sigma_{H_9}(2A,3A,16Y)\\
&= 896 - 256 - 32 - 64 - 32 > 0,\\
\Delta^*_{O'N}(2A,3A,16Z) &\geq \Delta_{O'N}(2A,3A,16Z) - 2\times\Sigma_{H_2}(2A,3A,16Z) - 1\times\Sigma_{H_4}(2A,3A,16Z)\\
&\quad -2\times\Sigma_{H_8}(2A,3A,16Z) - 1\times\Sigma_{H_9}(2A,3A,16Z)\\
&= 896 - 256 - 32 - 64 - 32 > 0.
\end{aligned}$$

Therefore, the group $O'N$ is $(2A,3A,16X)$-generated for $X \in \{A,B,C,D\}$. $\square$

We now summarize our main result of the article in the form the following theorem:

Theorem 2.6. *The O'Nan group $O'N$ is $(2,3,t)$-generated for any integer t except when $t = 12$.*

Proof. This follows from Lemmas 2.1 to 2.9. $\square$

Table I: The maximal subgroups of O'N

Group	Order	Group	Order
$L_3(7){:}2$ (2 classes)	$2^6.3^2.7^3.19$	J_1	$2^3.3.5.7.11.19$
$4.L_3(4){:}2$	$2^9.3^2.5.7$	$(3^2{:}4\times A_6).2$	$2^6.3^4.5$
$3^4{:}2_-^{1+4}.D_{10}$	$2^6.3^4.5$	$L_2(31)$ (2 classes)	$2^5.3.5.31$
$4^3{:}L_3(2)$	$2^9.3.7$	M_{11} (2 classes)	$2^4.3^2.5.11$
A_7 (2 classes)	$2^3.3^2.5.7$		

TABLE II: **Partial Fusion Maps into the group** $O'N$

$L_3(7){:}2$–class	$2a$	$2b$	$3a$	$8a$	$8b$	$8c$	$12a$	$14a$	$14b$	$16a$	$16b$
$\rightarrow O'N$	$2A$	$2A$	$3A$	$8A$	$8A$	$8A$	$12A$	$14A$	$14A$	$16A$	$16B$
h				1	1	2	3	1	2	1	1
$L_3(7){:}2$–class	$16c$	$16d$	$28a$								
$\rightarrow O'N$	$16B$	$16A$	$28A$								
h	1	1	1								
$L_3(7){:}2$–class	$2a$	$2b$	$3a$	$12a$	$14a$	$14b$	$16a$	$16b$	$16c$	$16d$	$28a$
$\rightarrow O'N$	$2A$	$2A$	$3A$	$12A$	$14A$	$14A$	$16C$	$16D$	$16D$	$16C$	$28A$
h				3	1	2	1	1	1	1	1
J_1–class	$2a$	$3a$	$10a$	$10b$	$15a$						
$\rightarrow O'N$	$2A$	$3A$	$10A$	$10A$	$15A$						
h			2	2	3						
$4{\cdot}L_3(4){:}2$–class	$2a$	$2b$	$2c$	$3a$	$8a$	$8b$	$8c$	$8d$	$10a$	$12a$	$14a$
$\rightarrow O'N$	$2A$	$2A$	$2A$	$3A$	$8A$	$8B$	$8A$	$8B$	$10A$	$12A$	$14A$
h					1	1	2	2	1	1	1
$4{\cdot}L_3(4){:}2$–class	$16a$	$16b$	$16c$	$16d$	$20a$	$28a$					
$\rightarrow O'N$	$16A$	$16B$	$16C$	$16D$	$20A$	$28A$					
h	1	1	1	1	1	1					
$(3^2{:}4 \times A_6){\cdot}2$–class	$2a$	$2b$	$2c$	$3a$	$3b$	$3c$	$3d$	$8a$	$8b$	$10a$	$12a$
$\rightarrow O'N$	$2A$	$2A$	$2A$	$3A$	$3A$	$3A$	$3A$	$8A$	$8A$	$10A$	$12A$
h								2	2	1	1
$(3^2{:}4 \times A_6){\cdot}2$–class	$12b$	$12c$	$15a$	$15b$	$20a$						
$\rightarrow O'N$	$12A$	$12A$	$15A$	$15D$	$20A$						
h	1	1	1	1	1						
$3^4{:}2_-^{1+4}{\cdot}D_{10}$–class	$2a$	$2b$	$3a$	$8a$	$10a$	$10b$	$12a$	$12b$			
$\rightarrow O'N$	$2A$	$2A$	$3A$	$8A$	$10A$	$10A$	$12A$	$12A$			
h				4	2	2	1	1			
$L_2(31)$–class	$2a$	$3a$	$8a$	$8b$	$15a$	$15c$	$16a$	$16b$	$16c$	$16d$	
$\rightarrow O'N$	$2A$	$3A$	$8A$	$8A$	$15A$	$15A$	$16A$	$16B$	$16A$	$16B$	
h			2	2	3	3	1	1	1	1	
$L_2(31)$–class	$2a$	$3a$	$8a$	$8b$	$15a$	$15c$	$16a$	$16b$	$16c$	$16d$	
$\rightarrow O'N$	$2A$	$3A$	$8B$	$8B$	$15A$	$15A$	$16A$	$16B$	$16A$	$16B$	
h			2	2	3	3	1	1	1	1	
$4^3{\cdot}L_3(2)$–class	$2a$	$3a$	$8a$	$12a$	$12b$	$16a$	$16b$	$16c$	$16d$		
$\rightarrow O'N$	$2A$	$3A$	$8A$	$12A$	$12A$	$16A$	$16B$	$16C$	$16D$		
h			1	3	3	1	1	1	1		
M_{11}–class	$2a$	$3a$	$8a$	$8b$							
$\rightarrow O'N$	$2A$	$3A$	$8A$	$8A$							
h			4	4							

References

[1] F. Ali, *On the ranks of O'N and Ly*, Discrete Applied Mathematics, **155** (2007), no. 3, 394–399.

[2] F. Ali and M. A. F. Ibrahim, On the ranks of Conway groups Co_2 and Co_3, *J. Algebra Appl.*, **4**, 2005, 557–565.

[3] F. Ali and M. A. F. Ibrahim, On the ranks of Conway group Co_1, *Proc. Japan Acad.*, **81A**, 2005, 95–98.

[4] F. Ali and M. A. F. Ibrahim, On the simple sporadic group He Generated by $(2,3,t)$ Generators, *Bull. Malays. Math. Sci. Soc.* (2)**35**(3) (2012), 745–753.

[5] F. Ali and M. A. F. Ibrahim, On the generation of Conway groups by conjugate involutions, *Algebras, Groups Geometries*, **24**, (2007), 203–210.

[6] M. A. Al-Kadhi, $(2,3,t)$*-generations of the Higman-Sims sporadic simple group* HS, Int. J. Pure Appl. Math., **46** (2008), 57–64.

[7] M. A. Al-Kadhi, *On* $(2,3,t)$*-Generations for the Janko groups* J_1 *and* J_2, Far East J. Math. Sci. (FJMS), **29** (2008), 379–388.

[8] M. A. Al-Kadhi and F. Ali, *(2,3,t)-Generations for the Conway group* Co_3, Int. J. Algebra, **4** (2010), 1341–1353.

[9] M. A. Al-Kadhi and F. Ali, *(2,3,t)-Generations for the Conway group* Co_2, J. Math. & Stat., **8**(2012), 339–341.

[10] M. D. E. Conder, R. A. Wilson and A. J. Woldar, *The symmetric genus of sporadic groups*, Proc. Amer. Math. Soc. **116** (1992), 653–663.

[11] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson, $\mathbb{ATLAS}$ *of Finite Groups*, Oxford Univ. Press (Clarendon), Oxford, 1985.

[12] L. Di Martino and C. Tamburini, *2-Generation of finite simple groups and some related topics*, Generators ans Relations in Groups and Geometry, A. Barlotti et al., Kluwer Acad. Publ., New York (1991), 195–233.

[13] L. Di Martino and N. Vavilov, $(2,3)$*-Generation of* $SL(n,q)$*. I. Cases* $n=5,6,7$, Comm. Algebra **22**(4) (1994), 1321–1347.

[14] M. R. Darafsheh, A. R. Ashrafi and G. A. Moghani, (p, q, r)-*Generations of the sporadic group* $O'N$, LMS Lecture Note Series **304**, Groups St. Anrewes 2001, Eds. C.M.Campbell, E.F.Robertson, G.C, Smith, CUP,101-109 (2003).

[15] S. Ganief and J. Moori, $(2, 3, t)$-*Generations for the Janko group* J_3, Comm. Algebra **23**(12) (1995), 4427–4437.

[16] I. M. Isaacs, *Character Theory of Finite Groups*, Dover, New-York, 1976.

[17] M. W. Liebeck and A. Shalev, *Classical groups, probabilistic methods and* $(2, 3)$-*generation problem*, Annals of Math, **144** (1996), 77–125.

[18] J. Moori, $(2, 3, p)$-*Generations for the Fischer group* F_{22}, Comm. Algebra **22**(11) (1994), 4597–4610.

[19] M. O'Nan, *Some evidence for the existance of a new simple group*, Proc. London Math. Soc. **32** (1976), 421–479.

[20] The GAP Group, $\mathbb{GAP}$ - *Groups, Algorithms and Programming, Version 4.3* , Aachen, St Andrews, 2002, (`http://www.gap-system.org`).

[21] R. A. Wilson, *The maximal subgroups of the O'Nan group*, J. Algebra **97** (1985), 467–473.

[22] R. A. Wilson et al., A world-wide-web *Atlas of Group Representations*, (`http://web.mat.bham.ac.uk/atlas`).

[23] A. J. Woldar, *On Hurwitz generation and genus actions of sporadic groups*, Illinois Math. J. (3) **33** (1989), 416–437.

[24] A. J. Woldar, *3/2-Generation of the sporadic simple groups*, Comm. Algebra **22** (1994), 675–685.

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