MATH 112
Section 2.3: Numeration

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Outline

Numbers vs. Numerals   Early Numeration Systems   Bases and Numeration   Conclusion

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Why "Numeration?"

Section 2.3 deals with various numeration systems. Why do we call them numeration systems?

Definitions

- **Number**

  A number is the concept of a certain quantity.

- **Numeral**

  A numeral is the word or symbol we use to express a given number.

Example

A friend tells you he has four movie tickets. Identify the number and numeral used and express the number with several different numerals.

Is the Difference Important?

Is the difference between a number and a numeral important?
Why should we even study different numeration systems?

Representation and Arithmetic

The way in which numbers are represented (called a numeration system) has important implications for how arithmetic works.
We study different numeration systems not only for their historical importance, but also to see advantages and disadvantages of our own system of numerals.

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Tally Marks

The Tally Mark System

One of the earliest numeration systems is the tally mark system.

Tally Marks

In the tally mark system, each number is represented by a list of tally marks equal in length to the number. Sometimes these marks were grouped so as to make counting easier.

Example

Translate between our numeration system and the tally mark system as appropriate.

<div align="center">

8 ||||| 17   ⦀⦀⦀ ⦀⦀⦀ ||

</div>

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Tally Marks

Properties of the Tally Mark System

While the tally mark system is very basic, there is at least one interesting property we should note.

Properties of the Tally Mark System

- **Additive**
  The tally mark system is additive because one adds symbol values together to get the number represented

Advantages and Disadvantages

What are some advantages of tally marks over our system?

- Advantages: simple, only one symbol needed, addition and subtraction are easy to model
- Disadvantages: large numbers are hard to represent, no way to write zero, multiplication and division are difficult to model

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Egyptian Numeration System

The Egyptian System

The Egyptian numeration system is an improvement on the tally mark system. It combines the concept of tally marks with different hieroglyph.

Egyptian System

The Egyptian system represents numbers with various hieroglyphs. Numerals are listed from highest value to lowest value, with preference given to using larger valued numerals where possible.

Staff	Heelbone	Scroll	Lotus Flower	Pointing Finger	Polliwog	Astonished Person
I	∩	ℓ	⚱	∫	⟍	👤
1	10	100	1,000	10,000	100,000	1,000,000

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Egyptian Numeration System

Examples and Properties of the Egyptian System

Example

Translate between our numeration system and the Egyptian numeration system as appropriate.

252   ∩∩∩   1522

Properties of the Egyptian System

- **Additive**
  Like the tally mark system, the Egyptian numeration system is additive.

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Egyptian Numeration System

Advantages and Disadvantages

The Egyptian system makes some significant additions to the older tally mark system, but how does it compare with our system?

Advantages and Disadvantages

What are some advantages and disadvantages of this system as compared to our numeration system?

- Advantages
  1. shorter representations for some numbers
  2. adding is easy but requires regrouping
  3. natural use of powers of 10

- Disadvantages
  1. some representations are much longer
  2. no way to write zero
  3. multiplication and division are not obvious
  4. multiple ways to represent a single number

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Roman Numeration System

The Roman System

Roman numerals are still used today in some circumstances. They are unique in our group of ancient numeration systems because they use letters from their alphabet to represent numbers as well.

Roman Numeration System

The following letters were used to represent the numbers shown.

I	V	X	L	C	D	M
1	5	10	50	100	500	1000

Numerals were listed starting with those representing the largest number on the left. If a numeral representing a smaller value appeared to the left of a larger-valued numeral, its value was subtracted from that of the larger numeral. Finally, even larger numbers can be represented using a line over the numerals below, which multiplies a numeral's value by 1000.

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Roman Numeration System

Examples and Properties of Roman Numerals

Example

Translate between our numeration system and Roman numerals as appropriate.

MXI   342   XMI   6419

Properties of the Roman System

- **Additive**
  The Roman system is additive as are the Egyptian and tally mark systems.

- **Subtractive**
  The Roman system is also subtractive, meaning one subtracts in certain situations to get the value of the numeral.

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Roman Numeration System

Advantages and Disadvantages

The Roman systems is able to represent a wider range of numbers with a small number of symbols. How does it compare with our system?

Advantages and Disadvantages

What are some advantages and disadvantages of this system as compared to our numeration system?

- Advantages
  1. uses only seven symbols
  2. makes use of "10's" and the in-between "5's"
  3. subtraction allows for shorter number representations
- Disadvantages
  1. difficult to add, subtract, multiply and divide
  2. no way to write zero
  3. multiple ways to represent the same number

The Babalyonian System

The final numeration we will examine in class is the Babalyonian system. This system was innovative in several ways.

Babalyonian Numeration System

The Babalyonian numeration system uses only two symbols. This is the first system we have studied which uses place value. However, the *base* is 60 instead of the more familiar 10.

▼	◀
1	10

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Babalyonian Numeration System

Examples and Properties of Babalyonian Numerals

Example

Translate between our numeration system and Babalyonian numerals as appropriate.

$$\blacktriangleleft\blacktriangleleft\blacktriangleleft\mathbf{\textbf{V V V V}} \quad 82 \qquad \blacktriangleleft\mathbf{\textbf{V}} \quad \blacktriangleleft\mathbf{\textbf{V}} \quad 1205$$

Properties of the Babalyonian System

- **Additive**
  The Babalyonian system is additive as have been the previous systems.

- **Place Value**
  Unline the systems seen previously, the value of a symbol depends on the place in which that symbol appears.

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Babalyonian Numeration System

Advantages and Disadvantages

One new aspect of the Babalyonian system is the use of place value. How does using place value affect a numeration system?

Advantages and Disadvantages

What are some advantages and disadvantages of this system as compared to our numeration system?

- Advantages
    1. uses only two symbols and a space
    2. very large numbers can be easily represented
    3. adding, subtracting, multiplication, and division are easier to accomplish
- Disadvantages
    1. the lack of a zero symbol is even more of a problem (why?)

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Base 10

Our Base 10 System

We are accustom to using place values in our numeration system
(the Hindu-Arabic system).

Base 10 Place Value

In base 10, each place represents a power of 10. For example:

$$\underline{\quad 3 \quad} \quad \underline{\quad 2 \quad} \quad \underline{\quad 1 \quad} \quad \underline{\quad 4 \quad}$$
$$\text{1000's} \quad \text{100's} \quad \text{10's} \quad \text{1's}$$

stands for the number:

$$(3 \times 10^3) + (2 \times 10^2) + (1 \times 10^1) + (4 \times 10^0)$$

Base 10 Representations

How can we represent this number concretely in a manner similar
to what we did with algebra tiles?

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Base 4

The Base 4 System

In the base 4 system place values are powers of 4.

Base 4 Place Value

The example below shows the first four place values for the base 4 system. In this example:

$$\frac{3}{64\text{'s}} \quad \frac{2}{16\text{'s}} \quad \frac{1}{4\text{'s}} \quad \frac{4}{1\text{'s}}$$

stands for the number:

$$(3 \times 4^3) + (2 \times 4^2) + (1 \times 4^1) + (4 \times 4^0)$$

Base n Representations

Do numbers represented in base 4 look smaller or larger than the same number represented in base 10? What about numbers written in base n?

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Base 4

Examples in the Base 4 System

To get a better understanding of base 4, let's try some exercises in base 4.

Example

Count out the first ten numerals in base 4 numerals and using base 4 blocks.

Example

Convert the number $321_{(4)}$ to base ten.

Example

Convert the number 73 to base 4.

Base 6

The Base 6 System

In the base 6 system, place values come in powers of 6.

Base 6 Place Value

The example below shows the first four place values for the base 6 system. In this example:

$$\underline{\quad 3 \quad} \quad \underline{\quad 2 \quad} \quad \underline{\quad 1 \quad} \quad \underline{\quad 4 \quad}$$
$$216\text{'s} \qquad 36\text{'s} \qquad 6\text{'s} \qquad 1\text{'s}$$

stands for the number:

$$(3 \times 6^3) + (2 \times 6^2) + (1 \times 6^1) + (4 \times 6^0)$$

Base n Numerals

Have you noticed that in base 4 we only use the digits 0-3, and in base 6 we only use digits from 0-5. How many digits would we need in base n?

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Base 6

Examples in the Base 6 System

To better understand the base 6 system, let's try some exercises in base 6.

Example

Count out the first ten numerals in base 6 numerals and using base 6 blocks.

Example

Convert $536_{(6)}$ to base ten.

Example

Convert 415 to base 6.

Example

Convert $125_{(6)}$ to base 4 without first converting to base 10.

Important Concepts

Things to Remember from Section 2.3

1. The difference between a number and a numeral

2. Early numeration systems

3. How to represent numbers in bases other than 10