2. Banach spaces

DEFINITION. Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. A *Banach space over* $\mathbb{K}$ is a normed $\mathbb{K}$-vector space $(\mathcal{X}, \| \,.\, \|)$, which is complete with respect to the metric

$$d(x,y) = \|x - y\|, \;\; x, y \in \mathcal{X}.$$

REMARK 2.1. Completeness for a normed vector space is a purely topological property. This means that, if $\| \,.\, \|$ is a norm on $\mathcal{X}$, such that $(\mathcal{X}, \| \,.\, \|)$ is a Banch space, then so is $(\mathcal{X}, \| \,.\, \|')$, where $\| \,.\, \|'$ is any norm equivalent to $\| \,.\, \|$. This is due to the fact that, if $\| \,.\, \|'$ is equivalent to $\| \,.\, \|$, then one has

$$C\|x\| \leq \|x\|' \leq D\|x\|, \;\; \forall\, x \in \mathcal{X},$$

for some constants $C, D > 0$. This clearly gives the fact that a sequence $(x_n)_{n=1}^\infty \subset \mathcal{X}$ is Cauchy with respect to $\| \,.\, \|$, if and only if it is Cauchy with respect to $\| \,.\, \|'$.

EXAMPLE 2.1. The field $\mathbb{K}$, equipped with the absolute value norm, is a Banach space. More generally, the vector space $\mathbb{K}^n$, equipped with the norm

$$\|(\lambda_1, \dots, \lambda_n)\|_\infty = \max\{|\lambda_1|, \dots, |\lambda_n|\},$$

is a Banach space.

REMARK 2.2. Since any two norms on a finite dimensional space are equivalent, by Proposition 1.4, it follows that *any finite dimensional normed vector space is a Banach space.*

Below is an interesting application of this fact.

PROPOSITION 2.1. *Let $\mathcal{X}$ be a normed vector space, and let $\mathcal{Y} \subset \mathcal{X}$ be a finite dimensional linear subspace.*

(i) *$\mathcal{Y}$ is closed.*

(ii) *More generally, if $\mathcal{Z} \subset \mathcal{X}$ is a closed subspace, then the linear subspace $\mathcal{Y} + \mathcal{Z}$ is also closed.*

PROOF. (i). Start with some sequence $(y_n)_{n=1}^\infty \subset \mathcal{Y}$, which is convergent to some point $x \in \mathcal{X}$, and let us show that $x \in \mathcal{Y}$. By the above Remark, when equipped with the norm coming from $\mathcal{X}$, the normed vector space $\mathcal{Y}$ is complete. Since obviously $(y_n)_{n=1}^\infty$ is Cauchy, it will converge in $\mathcal{Y}$ to some vector $y \in \mathcal{Y}$. Of course, this forces $x = y$, and we are done.

(ii). Let us consider the quotient space $\mathcal{X}/\mathcal{Z}$, equipped with the quotient norm $\| \,.\, \|_{\mathcal{X}/\mathcal{Z}}$, and the quotient map $P : \mathcal{X} \to \mathcal{X}/\mathcal{Z}$. By Proposition 1.5 we know that P is continuous. Since the linear subspace $\mathcal{V} = P(\mathcal{Y}) \subset \mathcal{X}/\mathcal{Z}$ is finite dimensional, by part (i) it follows that $\mathcal{V}$ is closed in $\mathcal{X}/\mathcal{Z}$. By continuity, its preimage $P^{-1}(\mathcal{V})$ is closed in $\mathcal{X}$. Now we are done since we obviously have the equality $P^{-1}(\mathcal{V}) = \mathcal{Y} + \mathcal{Z}$. □

REMARK 2.3. Using the facts from the general theory of metric spaces, we know that for a normed vector space $(\mathcal{X}, \| \,.\, \|)$, the following are equivalent:

(i) *$\mathcal{X}$ is a Banach space;*

(ii) *given any sequence $(x_n)_{n\geq 1} \subset \mathcal{X}$ with $\sum_{n=1}^\infty \|x_n\| < \infty$, the sequence $(y_n)_{n\geq 1}$ of partial sums, defined by $y_n = \sum_{k=1}^n x_k$, is convergent;*

(iii) *every Cauchy sequence in $\mathcal{X}$ has a convergent subsequence.*

This is pretty obvious, since the sequence of partial sums has the property that

$$d(y_{n+1}, y_n) = \|y_{n+1} - y_n\| = \|x_{n+1}\|, \ \ \forall\, n \geq 1.$$

There are several techniques for constructing new Banach space out of old ones. The result below is one example.

Proposition 2.2. *Let $\mathcal{X}$ be a Banach space, let $\mathcal{Y}$ be a normed vector space, and let $T : \mathcal{X} \to \mathcal{Y}$ be a surjective linear continuous map. Assume there exists some constant $C > 0$, such that*

- *for every $y \in \mathcal{Y}$, there exists $x \in \mathcal{X}$ with $Tx = y$, and $\|x\| \leq C\|y\|$.*

Then $\mathcal{Y}$ is a Banach space.

Proof. We are going to use the above characterization. Start with some sequence $(y_n)_{n=1}^\infty \subset \mathcal{Y}$, with $\sum_{n=1}^\infty \|y_n\| < \infty$. Define the sequence $(w_n)_{n=1}^\infty \subset \mathcal{Y}$ of partial sums $w_n = \sum_{k=1}^n y_n$, and let us prove that $(w_n)_{n=1}^\infty$ is convergent to some element in $\mathcal{Y}$. Use the hypothesis to find, for each $n \geq 1$, an element $x_n \in \mathcal{X}$, with $Tx_n = y_n$, and $\|x_n\| \leq C\|y_n\|$. In particular, we clearly have $\sum_{n=1}^\infty \|x_n\| \leq \sum_{n=1}^\infty C\|y_n\| < \infty$. Using the fact that $\mathcal{X}$ is a Banach space, it follows that the sequence $(z_n)_{n=1}^\infty$ defined by $z_n = \sum_{k=1}^n x_n$ is convergent to some $z \in \mathcal{X}$. Since we have $Tz_n = w_n$, $\forall\, n \geq 1$, by the continuity of T we get $\lim_{n\to\infty} w_n = Tz$. □

Corollary 2.1. *Let $\mathcal{X}$ be a Banach space, and let $\mathcal{Y}$ be a closed linear subspace of $\mathcal{X}$. When equipped with the quotient norm, the quotient space $\mathcal{X}/\mathcal{Y}$ is a Banach space.*

Proof. Use the notations from Section 1. Consider the quotient map

$$P : \mathcal{X} \ni x \longmapsto [x] \in \mathcal{X}/\mathcal{Y}.$$

We know that P is linear, continnuous, and surjective. Let us check that P satisfies the hypothesis in Proposition 2.2, with $C = 2$. Start with some vector $\boldsymbol{v} \in \mathcal{X}/\mathcal{Y}$, and let us show that there exists $x \in \mathcal{X}$ with $Px = \boldsymbol{v}$ (i.e. $x \in \boldsymbol{v}$), such that $\|x\| \leq 2\|\boldsymbol{v}\|_{\mathcal{X}/\mathcal{Y}}$. If $\boldsymbol{v} = 0$, there is nothing to prove, because we can take $x = 0$. If $\boldsymbol{v} \neq 0$, then we use the definition of the quotient norm

$$\|\boldsymbol{v}\|_{\mathcal{X}/\mathcal{Y}} = \inf_{x\in\boldsymbol{v}} \|x\|,$$

combined with $2\|\boldsymbol{v}\|_{\mathcal{X}/\mathcal{Y}} > \|\boldsymbol{v}\|_{\mathcal{X}/\mathcal{Y}}$. □

Exercise 1. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed vector spaces. Consider the product $\mathcal{X}\times\mathcal{Y}$, equipped with the natural vector space structure.

(i) Prove that $\|(x, y)\| = \|x\| + \|y\|$, $(x, y) \in \mathcal{X} \times \mathcal{Y}$ defines a norm on $\mathcal{X} \times \mathcal{Y}$.
(ii) Prove that, when equipped with the above norm, $\mathcal{X}\times\mathcal{Y}$ is a Banach space, if and only if both $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces.

Proposition 2.3. *Let $\mathcal{X}$ be a normed vector space, and let $\mathcal{Y}$ be a Banach space. Then $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is a Banach space, when equipped with the operator norm.*

Proof. Start with a Cauchy sequence $(T_n)_{n\geq1} \subset \mathcal{L}(\mathcal{X}, \mathcal{Y})$. This means that for every $\varepsilon > 0$, there exists some N_ε such that

$$\|T_m - T_n\| < \varepsilon, \ \ \forall\, m, n \geq N_\varepsilon. \tag{1}$$

Notice that, if one takes for example $\varepsilon = 1$, and we define

$$C = 1 + \max\{\|T_1\|, \|T_2\|, \dots, \|T_{N_1}\|\},$$

then we clearly have

$$\|T_n\| \leq C, \ \forall n \geq 1. \tag{2}$$

Notice that, using (1), we have

$$\|T_m x - T_n x\| \leq \varepsilon \|x\|, \ \forall m, n \geq N_\varepsilon, \ x \in \mathfrak{X}, \tag{3}$$

which proves that

- *for every $x \in \mathfrak{X}$, the sequence $(T_n x)_{n \geq 1} \subset \mathcal{Y}$ is Cauchy.*

Since $\mathcal{Y}$ is a Banach space, for each $x \in \mathfrak{X}$, the sequence $(T_n)_{n \geq 1}$ will be convergent. We define the map $T : \mathfrak{X} \to \mathcal{Y}$ by

$$Tx = \lim_{n \to \infty} T_n x, \ \ x \in \mathfrak{X}.$$

Using (2) we immediately get

$$\|Tx\| \leq C\|x\|, \ \forall x \in \mathfrak{X}.$$

Since T is obviously linear, this prove that T is continuous. Finally, if we fix $n \geq N_\varepsilon$ and we take $\lim_{m \to \infty}$ in (3), we get

$$\|T_n x - Tx\| \leq \varepsilon \|x\|, \ \forall n \geq N_\varepsilon, \ x \in \mathfrak{X},$$

which proves precisely that we have the inequality

$$\|T_n - T\| \leq \varepsilon, \ \forall n \geq N_\varepsilon,$$

hence $(T_n)_{n \geq 1}$ is convergent to T in the norm topology. □

Corollary 2.2. *If $\mathfrak{X}$ is a normed vector space, then its topological dual $\mathfrak{X}^* = \mathcal{L}(\mathfrak{X}, \mathbb{K})$ is a Banach space.*

Proof. Immediate from the fact that $\mathbb{K}$ is a Banach space. □

Example 2.2. Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$, and let J be a non-empty set. For every $p \in [1, \infty]$, the space $\ell^p_{\mathbb{K}}(J)$, introduced in Section 1, is a Banach space. This follows from the isometric linear isomorphisms $\ell^1 \simeq (c_0)^*$, and $\ell^p \simeq (\ell^q)^*$, where q is Hölder conjugate to p.

Proposition 2.4. *Let $\mathfrak{X}$ be a Banach space, and let $\mathfrak{Z} \subset \mathfrak{X}$ be a linear subspace. The following are equivalent:*

(i) *$\mathfrak{Z}$ is a Banach space, ehen equipped with the norm from $\mathfrak{X}$;*
(ii) *$\mathfrak{Z}$ is closed in $\mathfrak{X}$, in the norm topology.*

Proof. This is a particular case of a general result from the theory of complete metric spaces. □

Example 2.3. Let J be a non-empty set, and let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. Then $c_0^{\mathbb{K}}(J)$ is a Banach space, since it is a closed linear subspace in $\ell^\infty_{\mathbb{K}}(J)$.

The following results give examples of Banach spaces coming from topology.

Notation. Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and let Ω be a topological space. We define

$$C_b^{\mathbb{K}}(\Omega) = \{f : \Omega \to \mathbb{K} \, : \, f \text{ bounded and continuous}\}.$$

In the case when $\mathbb{K} = \mathbb{C}$ we use the notation $C_b(\Omega)$.

Proposition 2.5. *With the notations above, if we define*

$$\|f\| = \sup_{p\in\Omega} |f(p)|, \ \ \forall\, f \in C_b^{\mathbb{K}}(\Omega),$$

then $C_b^{\mathbb{K}}(\Omega)$ *is a Banach space.*

Proof. It is obvious that $C_b^{\mathbb{K}}(\Omega)$ is a linear subspace of $\ell^\infty_{\mathbb{K}}(\Omega)$, and the norm is precisely the one coming from $\ell^\infty_{\mathbb{K}}(\Omega)$. Therefore, it suffices to prove that $C_b^{\mathbb{K}}(\Omega)$ *is closed in* $\ell^\infty_{\mathbb{K}}(\Omega)$.

Start with some sequence $(f_n)_{n\geq 1} \subset C_b^{\mathbb{K}}(\Omega)$, which convergens in norm to some $f \in \ell^\infty_{\mathbb{K}}(\Omega)$, and let us prove that $f : \Omega \to \mathbb{K}$ is continuous (the fact that f is bounded is automatic).

Fix some point $p_0 \in \Omega$, and some $\varepsilon > 0$. We need to find some neighborhood V of p_0, such that

$$|f(p) - f(p_0)| < \varepsilon, \ \ \forall\, p \in V.$$

Start by choosing n such that $\|f_n - f\| < \frac{\varepsilon}{3}$. Use the fact that f_n is continuous, to find a neighborhood V of p_0, such that

$$|f_n(p) - f_n(p_0)| < \frac{\varepsilon}{3}, \ \ \forall\, \Omega \in V.$$

Suppose now $\Omega \in V$. We have

$$|f(p) - f(p_0)| \leq |f_n(p) - f(p)| + |f_n(p) - f_n(p_0)| + |f_n(p_0) - f(p_0)| \leq$$
$$|f_n(p) - f_n(p_0)| + 2\big[\sup_{q\in\Omega} |f_n(q) - f(q)|\big] < 2\frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

□

Notation. If X is a *compact* Hausdorff space, then every continuous function $f : X \to \mathbb{K}$ is automatically bounded. In this case, the Banach space $C_b^{\mathbb{K}}(X)$ will simply be denoted by $C^{\mathbb{K}}(X)$, again with the convention that when $\mathbb{K} = \mathbb{C}$, the superscript will be ommitted.

Example 2.4. Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$, and let Ω be a locally compact space, which is not compact. We define (see I.5) the space

$$C_0^{\mathbb{K}}(\Omega) = \big\{ f : X \to \mathbb{K} \,:\, f \text{ continuous, with limit 0 at } \infty \big\}.$$

In other words, for a continuous function $f : \Omega \to \mathbb{K}$, the condition $f \in C_0^{\mathbb{K}}(\Omega)$ is equivalent to

- *for every $\varepsilon > 0$, there exists some compact set $K_\varepsilon \subset \Omega$, such that*

$$|f(x)| < \varepsilon, \ \ \forall\, x \in \Omega \smallsetminus K_\varepsilon. \tag{4}$$

As before, when $\mathbb{K} = \mathbb{C}$, the superscript will be ommitted from the notation.

Proposition 2.6. *Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$, and let Ω be a locally compact space, which is not compact. Then $C_0^{\mathbb{K}}(\Omega)$ is a closed linear subspace of $C_b^{\mathbb{K}}(\Omega)$. In particular, when equipped with the supremum norm, $C_0^{\mathbb{K}}(\Omega)$ is a Banach space.*

Proof. Let $(f_n)_{n=1}^\infty \subset C_0^{\mathbb{K}}(\Omega)$ be a sequence, which converges in norm to some $f \in C_b^{\mathbb{K}}(\Omega)$, and let us show that $f \in C_0^{\mathbb{K}}(\Omega)$. Fix some $\varepsilon > 0$, and let us indicate how to construct a compact set K_ε, with the property (4). Start off by choosing $n \geq 1$, such that $\|f_n - f\| < \varepsilon/2$, and then choose the compact set $K_\varepsilon \subset \Omega$, such that

$$|f_n(x)| < \varepsilon, \ \ \forall\, x \in \Omega \smallsetminus K_\varepsilon. \tag{5}$$

Notice now that, if $x \in \Omega \smallsetminus K_\varepsilon$, then

$$|f(x)| \leq \|f(x) - f_n(x)| + |f_n(x)| \leq \|f_n - f\| + |f_n(x)|,$$

and then using the choice of n, combined with (5), the desired property (4) immediately follows. $\square$

The Banach space $C_0^{\mathbb{K}}(\Omega)$ has several other equivalent descriptions. One of them is based on the following notion.

Definitions. Let Ω be a locally compact space. If $\mathbb{K}$ is one of the fields $\mathbb{R}$ or $\mathbb{C}$, and $f : \Omega \to \mathbb{K}$ is a continuous function, we define the *support of f* by

$$\operatorname{supp} f = \overline{\{\omega \in \Omega : f(\omega) \neq 0\}}.$$

We define the space

$$C_c^{\mathbb{K}}(\Omega) = \big\{ f : \Omega \to \mathbb{K} : f \text{ continuous, with compact support} \big\}.$$

When $\mathbb{K} = \mathbb{C}$, this space will be denoted simply by $C_c(\Omega)$. Remark that, when equipped with pointwise addition and multiplication, the space $C_c^{\mathbb{K}}(\Omega)$ becomes a $\mathbb{K}$-algebra. One has obviously the inclusion $C_c^{\mathbb{K}}(\Omega) \subset C_b^{\mathbb{K}}(\Omega)$.

Proposition 2.7. *With the notations above, the Banach space $C_0^{\mathbb{K}}(\Omega)$ is the closure of $C_c^{\mathbb{K}}(\Omega)$ in $C_b^{\mathbb{K}}(\Omega)$.*

Proof. Start with some function $f \in C_0^{\mathbb{K}}(\Omega)$, and let us construct a sequence $(f_n)_{n=1}^\infty \subset C_c^{\mathbb{K}}(\Omega)$, which converges to f in the norm topology. For every integer $n \geq 1$ we choose some compact set $K_n \subset \Omega$, such that

$$|f(x)| < \frac{1}{n}, \ \ \forall\, x \in \Omega \smallsetminus K_n. \tag{6}$$

For each $n \geq 1$, we also choose some open set D_n with $D_n \supset K_n$, and $\overline{D}_n$ compact, and we choose (use Urysohn Lemma for locally compact spaces; see I.7) some continuous function $h_n : \Omega \to [0,1]$, such that $h_n\big|_{K_n} = 1$ and $h_n\big|_{\Omega \smallsetminus D_n} = 0$. Put $f_n = f h_n$, $\forall\, n \geq 1$. It is obvious that, since $h_n\big|_{\Omega \smallsetminus D_n} = 0$, we have $\operatorname{supp} h_n \subset \overline{D}_n$, and consequently we also have $\operatorname{supp} f_n \subset \overline{D}_n$, so we have $f_n \in C_c^{\mathbb{K}}(\Omega)$, $\forall\, n \geq 1$. Let us now estimate the norms $\|f - f_n\|$, $n \geq 1$. Fix for the moment n. On the one hand, we have $0 \leq 1 - h_n(x) \leq 1$, $\forall\, x \in \Omega$, which combined with (6), yields

$$|f(x) - f_n(x)| = |f(x)| \cdot |1 - h_n(x)| \leq \frac{1}{n}, \ \ \forall\, x \in \Omega \smallsetminus K_n.$$

On the other hand, we have $f(x) = f_n(x)$, $\forall\, x \in K_n$, so the above estimate actually gives

$$\|f - f_n\| = \sup_{x \in \Omega} |f(x) - f_n(x)| \leq \frac{1}{n}, \ \ \forall\, n \geq 1,$$

which in particular gives the fact that $\lim_{n\to\infty} f_n = f$, in the norm topology. $\square$

At this point, based on the above result we shall adopt the following.

Convention. If Ω is a *compact* Hausdorff space, the notation $C_0^{\mathbb{K}}(\Omega)$ designates the space $C^{\mathbb{K}}(\Omega)$. The reason for adopting this convention is the (trvial) fact that, when Ω is compact, every continuous function $f : \Omega \to \mathbb{K}$ has compact support. In this spirit, if we adopt the above characterization as the "working definition" of $C_0^{\mathbb{K}}(\Omega)$, then our convention is legitimate.

*Exercise 2**. Let $\mathcal{X}$ be an *infinite dimensional* Banach space, and let B be a linear basis for $\mathcal{X}$. Prove that B is uncountable.

HINT: If B is countable, say $B = \{b_n : n \in \mathbb{N}\}$, then

$$\mathcal{X} = \bigcup_{n=1}^{\infty} F_n,$$

where $F_n = \operatorname{Span}(b_1, b_2, \dots, b_n\}$. Since the F_n's are finite dimensional linear subspaces, they will be closed. Use Baire's Theorem to get a contradiction.

COMMENTS. A third method of constructing Banach spaces is the completion. If we start with a normed $\mathbb{K}$-vector space $\mathcal{X}$, when we regard $\mathcal{X}$ as a metric space, its completion $\tilde{\mathcal{X}}$ is constructed as follows. One defines

$$\mathrm{CS}(\mathcal{X}) = \big\{\boldsymbol{x} = (x_n)_{n\geq 1} : (x_n)_{n\geq 1} \text{ Cauchy sequence in } \mathcal{X}\big\}.$$

Two Cauchy sequences $\boldsymbol{x} = (x_n)_{n\geq 1}$ and $\boldsymbol{x}' = (x'_n)_{n\geq 1}$ are said to be equivalent, if $\lim_{n\to\infty} \|x_n - x'_n\| = 0$. In this case one writes $\boldsymbol{x} \sim \boldsymbol{x}'$. The completion $\tilde{\mathcal{X}}$ is then defined as the space

$$\tilde{\mathcal{X}} = \mathrm{CS}(\mathcal{X})/\sim$$

of equivalence classes. For $\boldsymbol{x} \in \mathrm{CS}(\mathcal{X})$, one denotes by $\tilde{\boldsymbol{x}}$ its equivalence class in $\tilde{\mathcal{X}}$. Finally for an element $x \in \mathcal{X}$ one denotes by $\langle x\rangle \in \tilde{\mathcal{X}}$ the equivalence class of the constant sequence x.

We know from general theory that $\tilde{\mathcal{X}}$ is a complete metric space, with the distance $\tilde{d}$ (correctly) defined by

$$\tilde{d}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{x}}') = \lim_{n\to\infty} \|x_n - x'_n\|,$$

for any two Cauchy sequences $\boldsymbol{x} = (x_n)_{n\geq 1}$ and $\boldsymbol{x}' = (x'_n)_{n\geq 1}$.

It turns out that, in our situation, the space $\mathrm{CS}(\mathcal{X})$ carries a natural vector space structure, defined by pointwise addition and scalar multiplication. Moreover, the space $\tilde{\mathcal{X}}$ is identified as a quotient vector space

$$\tilde{\mathcal{X}} = \mathrm{CS}(\mathcal{X})/\mathrm{NS}(\mathcal{X}),$$

where

$$\mathrm{NS}(\mathcal{X}) = \big\{\boldsymbol{x} = (x_n)_{n\geq 1} : (x_n)_{n\geq 1} \text{ sequence in } \mathcal{X} \text{ with } \lim_{n\to\infty} x_n = 0\big\}$$

is the linear subspace of null sequences. It then follows that $\tilde{\mathcal{X}}$ carries a natural vector space structure. More explicitly, if we start with a scalar $\lambda \in \mathbb{K}$, and with two elements $p, q \in \tilde{\mathcal{X}}$, which are represented as $p = \tilde{\boldsymbol{x}}$ and $q = \tilde{\boldsymbol{y}}$, for two Cauchy sequences $\boldsymbol{x} = (x_n)_{n\geq 1}$ and $\boldsymbol{y} = (y_n)_{n\geq 1}$ in $\mathcal{X}$, then the sequence

$$\boldsymbol{w} = (\lambda x_n + y_n)_{n\geq 1}$$

is Cauchy in $\mathcal{X}$, and the element $\lambda p + q \in \tilde{\mathcal{X}}$ is then defined as $\lambda p + q = \tilde{\boldsymbol{w}}$.

Finally, there is a natural norm on $\tilde{\mathcal{X}}$, (correctly) defined by

$$\|\tilde{\boldsymbol{x}}\| = \tilde{d}(\tilde{\boldsymbol{x}}, \langle 0\rangle) = \lim_{n\to\infty} \|x_n\|,$$

for all Cauchy sequences $\boldsymbol{x} = (x_n)_{n\geq 1}$. These considerations then prove that $\tilde{\mathcal{X}}$ is a Banach space, and the map

$$\mathcal{X} \ni x \longmapsto \langle x\rangle \in \tilde{\mathcal{X}}$$

is linear and isometric, in the sense that

$$\|\langle x\rangle\| = \|x\|, \ \forall\, x \in \mathcal{X}.$$

In the context of normed vector spaces, the universality property of the completion is stated as follows:

Proposition 2.8. *Let $\mathcal{X}$ be a normed vector space, let $\tilde{\mathcal{X}}$ denote its completion, and let $\mathcal{Y}$ be a Banach space. For every linear continuous map $T : \mathcal{X} \to \mathcal{Y}$, there exists a unique linear continuous map $\tilde{T} : \tilde{\mathcal{X}} \to \mathcal{Y}$, such that*

$$\tilde{T}\langle x\rangle = Tx, \ \forall\, x \in \mathcal{X}.$$

Moreover the map

$$\mathcal{L}(\mathcal{X}, \mathcal{Y}) \ni T \longmapsto \tilde{T} \in \mathcal{L}(\tilde{\mathcal{X}}, \mathcal{Y})$$

is an isometric linear isomorphism.

Proof. If $T : \mathcal{X} \to \mathcal{Y}$ is linear an continuous, then T is a Lipschitz map with Lipschitz constant $\|T\|$, because

$$\|Tx - Tx'\| \leq \|T\| \cdot \|x - x'\|, \ \forall\, x, x' \in \mathcal{X}.$$

We know, from the theory of metric spaces, that there exists a unique continuous map $\tilde{T} : \tilde{\mathcal{X}} \to \mathcal{Y}$, such that

$$\tilde{T}\langle x\rangle = Tx, \ \forall\, x \in \mathcal{X}.$$

We also know that $\tilde{T}$ is Lipschitz, with Lipschitz constant $\|T\|$. The only thing we need to prove is the fact that $\tilde{T}$ is linear. Start with two points $p, q \in \tilde{\mathcal{X}}$, represented as $p = \tilde{\boldsymbol{x}}$ and $q = \tilde{\boldsymbol{z}}$, for some Cauchy sequences $\boldsymbol{x} = (x_n)_{n\geq 1}$ and $\boldsymbol{z} = (z_n)_{n\geq 1}$ in $\mathcal{X}$. If $\lambda \in \mathbb{K}$, then $\lambda p + q = \tilde{\boldsymbol{w}}$, where $\boldsymbol{w} = (\lambda x_n + z_n)_{n\geq 1}$. We then have

$$\tilde{T}(\lambda p + q) = \lim_{n\to\infty} T(\lambda x_n + z + n) = \big[\lambda \cdot \lim_{n\to\infty} Tx_n\big] + \big[\lim_{n\to\infty} Tz_n\big] = \lambda \tilde{T}p + \tilde{T}q.$$

Let us prove now that $\|\tilde{T}\| = \|T\|$. Since $\tilde{T}$ is Lipschitz, with Lipschitz constant $\|T\|$, we will have $\|\tilde{T}\| \leq \|T\|$. To prove the other inequality, let us consider the sets

$$\mathcal{B}_0 = \{p \in \tilde{\mathcal{X}} \,:\, \|p\| \leq 1\},$$
$$\mathcal{B}_1 = \{\langle x\rangle \,:\, x \in \mathcal{X}, \ \|x\| \leq 1\}.$$

By definition, we have

$$\|\tilde{T}\| = \sup_{p\in\mathcal{B}_0} \|\tilde{T}p\|.$$

Since we clearly have $\mathcal{B}_0 \supset \mathcal{B}_1$, we get

$$\|\tilde{T}\| = \sup_{p\in\mathcal{B}_1} \|\tilde{T}p\| \geq \sup\big\{\|\tilde{T}\langle x\rangle\| \,:\, x \in \mathcal{X} \ \|x\| \leq 1\big\} =$$
$$= \sup\big\{\|Tx\| \,:\, x \in \mathcal{X} \ \|x\| \leq 1\big\} = \|T\|.$$

The fact that the map $\mathcal{L}(\mathcal{X}, \mathcal{Y}) \ni T \longmapsto \tilde{T} \in \mathcal{L}(\tilde{X}, \mathcal{Y})$ is linear is obvious.

To prove the surjectivity, start with some $S \in \mathcal{L}(\tilde{\mathcal{X}}, \mathcal{Y})$. Consider the map

$$\iota : \mathcal{X} \ni x \longmapsto \langle x\rangle \in \tilde{\mathcal{X}}.$$

Since ι is linear and isometric, in particular it is continuous, so the composition $T = S \circ \iota$ is linear and continuous. Notice that

$$S\langle x\rangle = S\big(\iota(x)\big) = (S \circ \iota)x = Tx, \ \forall\, x \in \mathcal{X},$$

so by uniqueness we have $S = \tilde{T}$. $\square$

COROLLARY 2.3. *Let $\mathcal{X}$ be a normed space, let $\mathcal{Y}$ be a Banach space, and let $T : \mathcal{X} \to \mathcal{Y}$ be an isometric linear map.*

(i) *Let $\tilde{T} : \tilde{\mathcal{X}} \to \mathcal{Y}$ be the linear continuous map defined in the previous result. Then $\tilde{T}$ is linear, isometric, and $\tilde{T}(\tilde{\mathcal{X}}) = \overline{T(\mathcal{X})}$.*

(ii) *$\mathcal{X}$ is complete, if and only of $T(\mathcal{X})$ is closed in $\mathcal{Y}$.*

PROOF. (i). The fact that $\tilde{T}$ is isometric, and has the range equal to $\overline{T(\mathcal{X})}$ is true in general (i.e. for $\mathcal{X}$ metric space, and $\mathcal{Y}$ complete metric space). The linearity follows from the previous result.

(ii). This is obvious. $\square$

EXAMPLE 2.5. Let $\mathcal{X}$ be a normed vector space. For every $x \in \mathcal{X}$ define the map $\epsilon_x : \mathcal{X}^* \to \mathbb{K}$ by

$$\epsilon_x(\phi) = \phi(x), \ \ \forall\, \phi \in \mathcal{X}^*.$$

Then ϵ_x is a linear and continuous. This is an immediate consequence of the inequality

$$|\epsilon_x(\phi)| = |\phi(x)| \leq \|x\| \cdot \|\phi\|, \ \ \forall\, \phi \in \mathcal{X}^*.$$

Notice that this also proves

$$\|\epsilon_x\| \leq \|x\|, \ \ \forall\, x \in \mathcal{X}.$$

Interestingly enough, we actually have

$$\|\epsilon_x\| = \|x\|, \ \ \forall\, x \in \mathcal{X}. \tag{7}$$

To prove this fact, we start with an arbitrary $x \in \mathcal{X}$, and we consider the linear subspace

$$\mathcal{Y} = \mathbb{K}x = \{\lambda x \,:\, \lambda \in \mathbb{K}\}.$$

If we define $\phi_0 : \mathcal{Y} \to \mathbb{K}$, by

$$\phi_0(\lambda x) = \lambda\|x\|, \ \ \forall\, \lambda \in \mathbb{K},$$

then it is clear that $\phi_0(x) = \|x\|$, and

$$|\phi_0(y)| \leq \|y\|, \ \ \forall\, y \in \mathcal{Y}.$$

Use then the Hahn-Banach Theorem to find $\phi : \mathcal{X} \to \mathbb{K}$ such that $\phi\big|_{\mathcal{Y}} = \phi_0$, and

$$|\phi(z)| \leq \|z\|, \ \ \forall\, z \in \mathcal{X}.$$

This will clearly imply $\|\phi\| \leq 1$, while the first condition will give $\phi(x) = \phi_0(x) = \|x\|$. In particular, we will have

$$\|x\| = |\phi(x)| = |\epsilon_x(\phi)| \leq \|\epsilon_x\| \cdot \|\phi\| \leq \|\epsilon_x\|.$$

Having proven (7), we now have a linear isometric map

$$E : \mathcal{X} \ni x \longmapsto \epsilon_x \in \mathcal{X}^{**}.$$

Since $\mathcal{X}^{**}$ is a Banach space, we now see that $\tilde{E} : \tilde{\mathcal{X}} \to \overline{E(\mathcal{X})}$ is an isometric linear isomorphism. In particular, $\mathcal{X}$ is Banach, if and only if $E(\mathcal{X})$ is closed in $\mathcal{X}^{**}$.

We continue with a series of results, which are often regarded as the “principles of Banach space theory.” These results are consequences of Baire Theorem.

THEOREM 2.1 (Uniform Boundedness Principle). *Let $\mathcal{X}$ be a Banach space, let $\mathcal{Y}$ be normed vector space, and let $\mathcal{M} \subset \mathcal{L}(\mathcal{X}, \mathcal{Y})$. The following are equivalent*

(i) $\sup\{\|T\| : T \in \mathcal{M}\} < \infty$;
(ii) $\sup\{\|Tx\| : T \in \mathcal{M}\} < \infty$, $\forall\, x \in \mathcal{X}$.

Proof. The implication $(i) \Rightarrow (ii)$ is trivial, because if we define

$$M = \sup\{\|T\| : T \in \mathcal{M}\},$$

then by the definition of the norm, we clearly have

$$\sup\{\|Tx\| : T \in \mathcal{M}\} \le M\|x\|, \;\; \forall\, x \in \mathcal{X}.$$

$(ii) \Rightarrow (i)$. Assume $\mathcal{M}$ satisfies condition (ii). For each integer $n \ge 1$, let us define the set

$$\mathcal{F}_n = \{x \in \mathcal{X} : \|Tx\| \le n, \; \forall\, T \in \mathcal{M}\}.$$

It is obvious that $\mathcal{F}_n$ is a closed subset of $\mathcal{X}$, for each $n \ge 1$. Moreover, by (ii) we clearly have $\bigcup_{n=1}^{\infty} \mathcal{F}_n = \mathcal{X}$. Using Baire's Theorem, there exists some $n \ge 1$, such that $\operatorname{Int}(\mathcal{F}_n) \ne \varnothing$. This means that there exists some $x_0 \in \mathcal{X}$ and some $r > 0$, such that

$$\mathcal{F}_n \supset \bar{\mathcal{B}}_r(x_0) = \{y \in \mathcal{X} : \|x - x_0\| \le r\}.$$

Put $M_0 = \sup\{\|Tx_0\| : T \in \mathcal{M}\}$. Fix for the moment some arbitrary $x \in \mathcal{X}$, with $\|x\| \le 1$, and some arbitrary element $T \in \mathcal{M}$. The vector $y = x_0 + rx$ clearly belongs to $\bar{\mathcal{B}}_r(x_0)$, so we have $\|Ty\| \le n$. We then get

$$\|Tx\| = \left\|T\left(\tfrac{1}{r}(y - x_0)\right)\right\| = \tfrac{1}{r}\|Ty - Tx_0\| \le \tfrac{1}{r}\left(\|Ty\| + \|Tx_0\|\right) \le \tfrac{1}{r}(n + M_0).$$

Keep T fixed, and use the above estimate, which gives

$$\sup\{\|Tx\| : x \in \mathcal{X}, \; \|x\| \le 1\} \le \frac{n + M_0}{r},$$

to conclude that $\|T\| \le \frac{n+M_0}{r}$. Since $T \in \mathcal{M}$ is arbitrary, we finally get

$$\sup\{\|T\| : T \in \mathcal{M}\} \le \frac{n + M_0}{r} < \infty. \quad \square$$

Theorem 2.2 (Inverse Mapping Theorem). *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, and let let $T : \mathcal{X} \to \mathcal{Y}$ be a bijective linear continuous map. Then the linear map $T^{-1} : \mathcal{Y} \to \mathcal{X}$ is also continuous.*

Proof. Let us denote by $\mathcal{A}$ the open unit ball in $\mathcal{X}$ centered at the origin, i.e.

$$\mathcal{A} = \{x \in \mathcal{X} : \|x\| < 1\}.$$

The first step in the proof is contained in the following.

Claim 1: *The closure $\overline{T(\mathcal{A})}$ is a neighborhood of 0 in $\mathcal{Y}$.*

Consider the sequence of closed sets $\left(k\overline{T(\mathcal{A})}\right)_{k=1}^{\infty}$. (Here we use the notation $k\mathcal{M} = \{kv : v \in \mathcal{M}\}$.) Since the map $v \longmapsto kv$ is a homeomorphism, one has the equalities

$$k\overline{T(\mathcal{A})} = \overline{kT(\mathcal{A})} = \overline{T(k\mathcal{A})}, \;\; \forall\, k \ge 1.$$

In particular, we have

$$\bigcup_{k=1}^{\infty} k\overline{T(\mathcal{A})} = \bigcup_{k=1} \overline{T(k\mathcal{A})} \supset \bigcup_{k=1}^{\infty} T(k\mathcal{A}) = T\Big(\bigcup_{k=1}^{\infty} [k\mathcal{A}]\Big).$$

Since we obviously have $\bigcup_{k=1}^{\infty} [k\mathcal{A}] = \mathcal{X}$, and T is surjective, the above equality shows that $\bigcup_{k=1}^{\infty} k\overline{T(\mathcal{A})} = \mathcal{Y}$. Using Baire's Theorem, there exists some $k \ge 1$, such that $\operatorname{Int}\left(k\overline{T(\mathcal{A})}\right) \ne \varnothing$. Again using the fact that $v \longmapsto kv$ is a homeomorphism, this gives $\text{Int}\Big(\overline{T(\mathcal{A})}\Big) \neq \varnothing$. Fix now some point $y \in \text{Int}\Big(\overline{T(\mathcal{A})}\Big)$, and some $r > 0$, such that $\overline{T(\mathcal{A})}$ contains the open ball

$$\mathcal{B}_r(y) = \big\{z \in \mathcal{Y} \,:\, \|z - y\| < r\big\}. \tag{8}$$

The proof of the Claim is then finished, once we prove the inclusion

$$\overline{T(\mathcal{A})} \supset \mathcal{B}_{\frac{r}{2}}(0).$$

To prove this inclusion, start with some arbitrary $v \in \mathcal{B}_{\frac{r}{2}}(0)$, i.e. $v \in \mathcal{Y}$ and $\|v\| < \frac{r}{2}$. Since $\|(2v+y)-y\| = 2\|v\| < r$, using (8) it follows that $2v+y \in \overline{T(\mathcal{A})}$. i.e. there exists a sequence $(x_n)_{n=1}^\infty \subset \mathcal{X}$ with $\|x_n\| < 1$, $\forall\, n \geq 1$, and $2v + y = \lim_{n\to\infty} Tx_n$. Since y itself belongs to $\overline{T(\mathcal{A})}$, there also exists some sequence $(z_n)_{n=1}^\infty \subset \mathcal{X}$, with $\|z_n\| < 1$, $\forall\, n \geq 1$, and $y = \lim_{n\to\infty} Tz_n$. On the one hand, if we consider the sequence $(u_n)_{n=1}^\infty \subset \mathcal{X}$ given by $u_n = \frac{1}{2}(x_n - z_n)$, then it is clear that

$$\|u_n\| \leq \tfrac{1}{2}\big(\|x_n\| + \|z_n\|\big) < 1, \ \ \forall\, n \geq 1,$$

i.e. $(u_n)_{n=1}^\infty \subset \mathcal{A}$. On the othe hand, we have

$$\lim_{n\to\infty} Tu_n = \lim_{n\to\infty} \tfrac{1}{2}\big(Tx_n - Tz_n\big) = \tfrac{1}{2}(2v + y - y) = v,$$

so v indeed belongs to $\overline{T(\mathcal{A})}$.

The next step is a slight (but crucial) improvement of Claim 1.

Claim 2: *$T(\mathcal{A})$ is a neighborhood of* 0.

Start off by choosing $\varepsilon > 0$, such that

$$\overline{T(\mathcal{A})} \supset \mathcal{B}_\varepsilon(0). \tag{9}$$

The Claim will follow, once we prove the inclusion

$$T(\mathcal{A}) \supset \mathcal{B}_{\frac{\varepsilon}{2}}(0). \tag{10}$$

To prove this inclusion, we start with some arbitrary $y \in \mathcal{B}_\varepsilon(0)$. We want to construct a sequence of vectors $(x_n)_{n=1}^\infty \subset \mathcal{A}$, such that, for every $n \geq 1$, we have the inequality

$$\Big\| y - \sum_{k=1}^n T\big(\tfrac{1}{2^k} x_k\big)\Big\| \leq \frac{\varepsilon}{2^{n+1}}. \tag{11}$$

This sequence is constructed inductively as follows. We start by using (9), and we pick $x_1 \in \mathcal{A}$ such that $\|2y - Tx_1\| < \frac{\varepsilon}{2}$. Once $x_1, \dots, x_p$ are constructed, such that (11) holds with $n = p$, we consider the vector

$$z = 2^{p+1}\big[y - \sum_{k=1}^p T\big(\tfrac{1}{2^k} Tx_k\big)\big] \in \mathcal{B}_\varepsilon(0),$$

and we use again (9) to find $x_{p+1} \in \mathcal{A}$, such that $\|z - Tx_{p+1}\| \leq \frac{\varepsilon}{2}$. We then claerly have

$$\Big\| y - \sum_{k=1}^{p+1} T\big(\tfrac{1}{2^k} x_k\big)\Big\| = \frac{\big\|z - Tx_{p+1}\big\|}{2^{p+1}} \leq \frac{\varepsilon}{2^{p+2}},$$

Consider now the series $\sum_{k=1}^\infty \frac{1}{2^k} x_k$. Since $\|x_k\| < 1$, $\forall\, k \geq 1$, and $\mathcal{X}$ is a Banacch space, by Remark 3.1, the sequence of $(w_n)_{n=1}^\infty \subset \mathcal{X}$ of partial sums

$$w_n = \sum_{k=1}^n \tfrac{1}{2^k} x_k, \ \ n \geq 1,$$

is convergent to some point $x \in \mathcal{X}$. Moreover, since we have

$$\|w_n\| \leq \sum_{k=1}^{n} \frac{\|x_k\|}{2^k} \leq \sum_{k=1}^{\infty} \frac{\|x_k\|}{2^k}, \ \ \forall\, n \geq 1,$$

we get the inequality

$$\|x\| \leq \sum_{k=1}^{\infty} \frac{\|x_k\|}{2^k} < 1,$$

which means that $x \in \mathcal{A}$. Note also that using these partial sums, the inequality (11) reads

$$\|y - Tw_n\| \leq \frac{\varepsilon}{2^{n+2}}, \ \ \forall\, n \geq 1,$$

so by the continuity of T, we have $y = Tx \in T(\mathcal{A})$.

Let us show now that T^{-1} is continuous. Use Claim 2, to find some $r > 0$ such that

$$T(\mathcal{A}) \supset \mathcal{B}_r(0), \tag{12}$$

and let $y \in \mathcal{Y}$ be an arbitrary vector with $\|y\| \leq 1$. Consider the vector $v = \frac{r}{2}y$, which has $\|v\| \leq \frac{r}{2} < r$. By (12), there exists $x \in \mathcal{A}$, such that $Tx = v$, which means that $T^{-1}y = \frac{2}{r}x$. This forces $\|T^{-1}y\| \leq \frac{2}{r}$. This argument shows that

$$\sup\big\{\|T^{-1}y\| \,:\, y \in \mathcal{Y},\ \|y\| \leq 1\big\} \leq \frac{2}{r} < \infty,$$

and the continuity of T^{-1} follows from Proposition 2.5. □

The following two exercises deal with two more "principles of Banach space theory."

Exercise 3$^\diamondsuit$. (Closed Graph Theorem). Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, and let $T : \mathcal{X} \to \mathcal{Y}$ be a linear map. Prove that the following are equivalent:

(i) T is continuous.

(ii) The *graph of* T

$$\mathcal{G}_T = \big\{(x, Tx) \,:\, x \in \mathcal{X}\big\}$$

is a closed subset of $\mathcal{X} \times \mathcal{Y}$, in the product topology.

Hint: For the implication *(ii)* ⇒ *(i)*, use Exercise 3, to get the fact that $\mathcal{G}_T$ is a Banach space. Then T is exactly the inverse of $\pi_{\mathcal{X}}\big|_{\mathcal{G}_T}$, where $\pi_{\mathcal{X}} : \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ is the projection onto the first coordinate. Use Theorem 3.2.

Exercise 4$^\diamondsuit$. (Open Mapping Theorem). Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, and let $T : \mathcal{X} \to \mathcal{Y}$ be a *surjective* linear continuous map. Prove that T is an *open* map, in the sense that

- *whenver* $\mathcal{D} \subset \mathcal{X}$ *is open, it follows that* $T(\mathcal{D})$ *is open in* $\mathcal{Y}$.

Hints: There are two possible proofs available.

First proof. Consider the linear map

$$S : \mathcal{X} \times \mathcal{Y} \ni (x, y) \longmapsto (x, Tx + y) \in \mathcal{X} \times \mathcal{Y}.$$

Prove that S is linear, continuous, bijective, hence by Theorem 3.2, it is a homeomorphism. Use this fact to prove that for every open set $\mathcal{D} \subset \mathcal{X}$, there exists some open set $\mathcal{E} \subset \mathcal{X} \times \mathcal{Y}$, such that $T(\mathcal{D}) = P_{\mathcal{Y}}(\mathcal{E})$, where $P_{\mathcal{Y}} : \mathcal{X} \times \mathcal{Y} \to \mathcal{Y}$ is the projection onto the second coordinate. This reduces the problem to proving the fact that $P_{\mathcal{Y}}$ is an open map.

Second proof. Take $\mathcal{N} = \operatorname{Ker} T$, and use the Factorization Theorem (Proposition 1.5) to write $T = \hat{T} \circ Q$, where $Q : \mathcal{X} \to \mathcal{X}/\mathcal{N}$ is the quotient map, and $\hat{T} : \mathcal{X}/\mathcal{N} \to \mathcal{Y}$ is some linear continuous map. Remark that $\hat{T}$ is bijective (so we can apply the Theorem 3.2 to it). Use the fact that Q is open (Section 1, Exercise 12).

We conclude with two useful results concerning Banach spaces of the form $C^{\mathbb{K}}(X)$, with X compact Hausdorff space. Additional results for these Banach spaces will be given in Section 6.

The following is an interesting continuity result.

Theorem 2.3. *Let T be a topological space, let X be compact Hausdorff spaces, and let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. For a map $F : T \times X \to \mathbb{K}$, the following are equivalent.*

(i) *F is continuous.*
(ii) *For every $t \in T$, the map $\phi_t : X \ni p \longmapsto F(t,p) \in \mathbb{K}$ is continuous, and moreover, the map*

$$\Phi : T \ni t \longmapsto \phi_t \in C^{\mathbb{K}}(X)$$

is continuous.

Proof. $(i) \Rightarrow (ii)$. Suppose F is continuous. The first assertion is clear, since, for $t \in T$, we can write $\phi_t = F \circ \iota_t$, where ι_t is the map

$$\iota_t : X \ni p \longmapsto (t,p) \in T \times X,$$

which is obviously continuous.

Next, we show that Φ is continuous, at every point $t \in T$. Fix $t \in T$. We need to prove that, for every $\varepsilon > 0$, there exists some neighborhood N of t in T, such that

$$\|\phi_v - \phi_t\| < \varepsilon, \ \ \forall\, v \in N.$$

By definition, the above condition reads:

$$\max_{p \in X} \big|F(v,p) - F(t,p)\big| < \varepsilon, \ \ \forall\, v \in N. \tag{13}$$

The neighborhood N is constructed as follows. Fix for the moment $p \in X$, and we use the continuity of $F : T \times X \to \mathbb{K}$ at the point (t,p), to find a neighborhood E_p of (t,p) in $T \times X$, such that

$$|F(v,q) - F(t,p)| < \frac{\varepsilon}{2}, \ \ \forall\, (v,q) \in E_p.$$

The using the definition of the product topology, there exist open sets $V_p \subset T$ and $D_p \subset X$, such that $(t,p) \subset V_p \times W_p \subset E_p$, so we have

$$|F(v,q) - F(t,p)| < \frac{\varepsilon}{2} \ \ \forall\, v \in V_p,\ q \in W_p.$$

Use now compactness to produce a finite sequence of points $p_1, \dots, p_n \in X$, such that $X = W_{p_1} \cup \dots \cup W_{p_n}$, and take $N = V_{p_1} \cap \dots \cap V_{p_n}$. To check the desired property, we start with some arbitrary $v \in N$. For each $p \in X$, there exists some $k \in \{1, \dots, n\}$ with $p \in W_{p_k}$, and then we have $N \times W_{p_k} \subset V_{p_k} \times W_{p_k} \subset E_{p_k}$. In particular, both (v,p) and (x,p) belong to E_{p_k}, so we get

$$|F(v,p) - F(t,p_k)| < \frac{\varepsilon}{2} \text{ and } |F(t,p) - F(t,p_k)| < \frac{\varepsilon}{2},$$

so we immediately get

$$|F(v,p) - F(t,p)| \leq |F(v,p) - F(t,p_k)| + |F(t,p) - F(t,p_k)| < \varepsilon.$$

Since the above inequality holds for all $p \in X$, it will force (13).

(ii) $\Rightarrow$ *(i)*. Assume condition (ii), and let us prove that F is continuous at every point $(t,p) \in T \times X$. Fix $(t,p) \in T \times X$, as well as some $\varepsilon > 0$. We need to find two open sets $V \subset T$ and $W \subset X$, with $(t,p) \in V \times W$, such that

$$|F(v,q) - F(t,q)| < \varepsilon, \ \forall\,(v,q) \in V \times W. \tag{14}$$

Using (ii), we find V such that

$$\max_{q\in X} |F(v,q) - F(t,q)| < \frac{\varepsilon}{2}, \ \forall\, v \in V,$$

so we get

$$|F(v,q)-F(t,p)| \leq |F(v,q)-F(t,q)|+|F(t,q)-F(t,p)| < \frac{\varepsilon}{2}+|\phi_t(q)-\phi_t(p)|. \tag{15}$$

Using the continuity of $\phi_t : X \to \mathbb{K}$, we can then find an open neighborhood W of p in X with

$$|\phi_t(q) - \phi_t(p)| < \frac{\varepsilon}{2}, \ \forall\, q \in W,$$

and then (15) forces (14). $\square$

Exercise 5$^\diamondsuit$. Let X be a compact Hausdorff space. Prove that, when we equip $X \times C^{\mathbb{K}}(X)$ with the product topology, the map

$$X \times C^{\mathbb{K}}(X) \ni (p,f) \longmapsto f(p) \in \mathbb{K}$$

is continuous. .

The following is a technical result which illustrates perfectly the advantage of working with Banach spaces.

Theorem 2.4 (Arzela-Ascoli Compactness Theorem). *Let X be a compact Hausdorff space, let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. For a subset $\mathcal{A} \subset C^{\mathbb{K}}(X)$, the following are equivalent.*

- (i) *The closure $\overline{\mathcal{A}}$ is compact, in the norm topology.*
- (ii) *$\mathcal{A}$ is bounded and equicontinuous, in the sense that it satifies the condition:*
  - (ec) *for every $p \in X$, and every $\varepsilon > 0$, there exists a neighborhood N of p in X, such that*

$$|f(q) - f(p)| < \varepsilon, \ \forall\, q \in N,\ f \in \mathcal{A}.$$

Proof. *(i)* $\Rightarrow$ *(ii)*. Assume (ii). In order to prove (i), we are going to employ Corollary I.6.?? (the fact that we work in $C^{\mathbb{K}}(X)$, which is a Banach space, is crucial here) which states that condition (i) is equivalent to:

- (i') *For every $\varepsilon > 0$, all ε-discrete subsets of $\mathcal{A}$ are finite.*

Recall that a set $\mathcal{F} \subset C^{\mathbb{K}}(X)$ is *ε-discrete*, if

$$\|f - g\| \geq \varepsilon, \ \forall\, f,g \in \mathcal{F}, \ f \neq g.$$

Fix $\varepsilon > 0$, as well as a ε-discrete set $\mathcal{F} \subset \mathcal{A}$. Start off by using the equicontinuity condition (ec), to construct, for each $p \in X$, an open neighborhood $D(p)$ of p in X, such that

$$|f(q) - f(p)| < \frac{\varepsilon}{3}, \ \forall\, q \in D(p),\ f \in \mathcal{A}. \tag{16}$$

Use compactness of X to find points $p_1, \dots, p_n$, such that $X = D(p_1) \cup \dots \cup D(p_n)$.

Consider the finite dimensional Banach space $\mathbb{K}^n$, equipped with the norm $\| \, . \, \|_\infty$, and the map

$$T : C^{\mathbb{K}}(X) \ni f \longmapsto \big(f(p_1), \dots, f(p_n)\big) \in \mathbb{K}^n.$$

It is obvious that T is linear, continuous, with $\|T\| \leq 1$. Remark also that, if we put $M = \sup\big\{\|f\| \, : \, f \in \mathcal{A}\big\}$, then we have the inclusion $T(\mathcal{A}) \subset B^n$, where $B = \{\lambda \in \mathbb{K} \, : \, |\lambda| \leq M\}$.

Claim: *One has*

$$\|Tf - Tg\|_\infty \geq \frac{\varepsilon}{3}, \;\; \forall \, f, g \in \mathcal{F}, \;\; f \neq g.$$

Start with $f, g \in \mathcal{F}$, with $f \neq g$. Since $\mathcal{F}$ is ε-discrete, we have $\|f - g\| \geq \varepsilon$, which means that there exists $q \in X$, with $|f(q) - g(q)| \geq \varepsilon$. If we choose $k \in \{1, \dots, n\}$, such that $D(p_k) \ni p$, then using (16), we have

$$|f(q) - f(p_k)| < \frac{\varepsilon}{3} \text{ and } |g(q) - g(p_k)| < \frac{\varepsilon}{3}.$$

We now have

$$\varepsilon \leq |f(q) - g(q)| \leq |f(q) - f(p_k)| + |f(p_k) - g(p_k)| + |g(q) - g(p_k)| \leq |f(p_k) - g(p_k)| + \frac{2\varepsilon}{3},$$

which forces $|f(p_k) - g(p_k)| \geq \frac{\varepsilon}{3}$, and we are done.

Having proven the Claim, let us observe that now we have the following features:

- *the map* $T\big|_{\mathcal{F}} : \mathcal{F} \to \mathbb{K}^n$ *is injective;*
- *the set* $T(\mathcal{F})$ *is* $\frac{\varepsilon}{3}$*-discrete in* $\mathbb{K}^n$.

Now we are done, since the fact that B^n is compact in $\mathbb{K}^n$, forces $T(\mathcal{F})$ to be finite.

$(i) \Rightarrow (ii)$. Assume $\overline{\mathcal{A}}$ is compact. Clearly this forces $\overline{\mathcal{A}}$ to be bounded, and so will be $\mathcal{A}$. Let us show that $\mathcal{A}$ is equicontinuous. Replacing $\mathcal{A}$ with $\overline{\mathcal{A}}$, we can assume that $\mathcal{A}$ itself is compact.

By Exercise 6, the map

$$X \times \mathcal{A} \ni (p, f) \longmapsto f(p) \in \mathbb{K}$$

is continuous, so by Theorem 2.3, for each $p \in X$, the map

$$\phi_p : \mathcal{A} \ni f \longmapsto f(p) \in \mathbb{K}$$

is continuous, and moreover, the map

$$\Phi : X \ni p \longmapsto \phi_p \in C^{\mathbb{K}}(\mathcal{A})$$

is also continuous.

Fix now $p \in X$, and $\varepsilon > 0$. By the continuity of Φ, there exists some open set $U \subset X$, such that

$$\|\Phi(q) - \Phi(p)\| < \varepsilon, \;\; \forall \, q \in U.$$

This reads

$$\sup_{f \in \mathcal{A}} |f(q) - f(p)| < \varepsilon, \;\; \forall \, q \in U,$$

which is precisely the equicontinuity condition. $\square$