Assignment 2

General Instructions

- Solutions due by
  25^{th} February 2025 for Group A
  27^{th} February 2025 for Group B
- Hand calculations can be done as homework and should be submitted together with the report.
- Write your name and roll-number on your report.

Hand calculations

1. We will examine the round-off errors in numerically solving linear systems using different pivoting techniques. The relative error between the exact and the numerical solution (using floating point arithmetic) is related to the *Growth factor*, defined as $\dfrac{\max_{i,j}|u_{ij}|}{\max_{i,j}|a_{ij}|}$. The smaller the growth factor, lesser will the number of digits you lose in the solution. Similarly, if the growth factor is larger, you will lose more digits in the solution, leading to higher error.

   (a) (10 points) Consider the matrix:
   $$\begin{pmatrix} \epsilon & 1 \\ 1 & 1 \end{pmatrix} \tag{1}$$
   where ϵ is a very small number. Determine the growth factor using Gaussian Elimination (i) without pivoting and (ii) with partial pivoting; Compute the growth factor as $\epsilon \to 0$. (10 points)

   (b) (10 points) In partial pivoting technique, while eliminating the k^{th} column, row swaps (with row index $i \geq k$) are performed to get the largest element in magnitude within the k^{th} column as the pivot. Whereas, in *complete pivoting* technique, while eliminating the k^{th} column, rows with index $j \geq k$ and columns with index $j \geq k$ are swapped to get the largest element in magnitude as the pivot.
   Consider the matrix:
   $$\begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{pmatrix} \tag{2}$$
   Find the growth factor using Gaussian Elimination with (i) partial pivoting and (ii) complete pivoting. What are your estimates of growth factors for a similar matrix with size $N \times N$?

2. (10 points) Calculate the 1-, ∞-, and Frobenius norms and the corresponding condition numbers for the following matrices
$$A = \begin{bmatrix} 2 & -2 \\ 1 & -3 \end{bmatrix}$$
$$B = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 5 & 6 \\ 2 & -1 & 3 \end{bmatrix}$$
$$C = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. (10 points) Calculate the condition number using Frobenius norm of the coefficient matrix of the system $Ax = b$, with,

$$A = \begin{bmatrix} 1 & 10^6 \\ 2 & 3 \end{bmatrix}$$

and $b = [1 \quad 3]^T$. Is this system well-conditioned? If not, multiply one of the equations through by a suitably chosen constant so as to make the system better conditioned. Calculate the condition number of the coefficient matrix in your new system of equations.

4. (10 points) In the Gauss-Jordan method to find the inverse of a non-singular matrix $\mathbf{A}$, we begin by augmenting $\mathbf{A}$ with the corresponding identity matrix to obtain $\left[\mathbf{A} \mid \mathbf{I}_\mathrm{n}\right]$. Subsequently, we perform necessary row operations on this augmented matrix to transform it into $\left[\mathbf{I}_\mathrm{n} \mid \mathbf{A}^{-1}\right]$.

   Now, consider the matrix

$$\mathbf{A} = \begin{bmatrix} 9 & 2 & 1 \\ 8 & 7 & 5 \\ 3 & 3 & 4 \end{bmatrix}. \tag{3}$$

   (a) Start with an augmented matrix $\left[\mathbf{A} \mid \mathbf{I}_\mathrm{n}\right]$ and obtain $\left[\mathbb{U} \mid \mathbf{B}\right]$, where $\mathbb{U}$ is an upper triangular matrix and $\mathbf{B}$ is the resultant of the row operations performed on $\mathbf{I}_\mathrm{n}$. Round off all calculations to *4 decimal places.*

   (b) Decompose $\mathbf{A}$ into $\mathbb{L}\mathbb{U}$ form using Gauss elimination and elementary matrices $\mathbf{E}$'s, where $\mathbb{L}$ is a *unit* lower triangular matrix and $\mathbb{U}$ is the same upper triangular matrix as in (a).

   (c) What is the relationship between $\mathbb{L}$ obtained in (b) and $\mathbf{B}$ obtained in (a)?

Programming

1. (10 points) *Effect of round-off*: Consider the recurrence $u_{n+2} = 3\,u_{n+1} - 2u_n$, where $n \geq 0$. Take $u_0 = u_1 = 2.9689$.

   (a) Determine u_n manually for $n = 2, .., 6$. What value does u_n take for any n?

   (b) Determine the values of u_n on a computer for $n = 2, \ldots, 64$. Do the values agree with (a)?

   (c) Repeat (b) when $u_0 = u_1 = 2.96875$. Do the values agree with (a)?

   (d) Explain your observations in (b) and (c)

2. (a) (20 points) Consider the following system,

$$\begin{pmatrix} 1 & 2 & 0 & . & . & 0 \\ 1 & 4 & 1 & 0 & . & 0 \\ 0 & 1 & 4 & 1 & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & 1 & 4 & 1 \\ 0 & . & . & 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ x_2 \\ . \\ . \\ . \\ y_{N-1} \\ y_N \end{pmatrix} = \begin{pmatrix} \frac{N}{3}\left(-\frac{5}{2}f(x_0) + 2f(x_1) + \frac{1}{2}f(x_2)\right) \\ N\left(f(x_2) - f(x_0)\right) \\ N\left(f(x_3) - f(x_1)\right) \\ . \\ . \\ . \\ N\left(f(x_n) - f(x_{N-2})\right) \\ \frac{N}{3}\left(\frac{5}{2}f(x_N) - 2f(x_{N-1}) - \frac{1}{2}f(x_{N-2})\right) \end{pmatrix}$$

   where $f(x_j) = \sin(5x_j)$ with $x_j = \frac{3j}{N}$, $j = 0, 1, ..., N$. Compute the solution y_j and plot y_j versus x_j for $N = 15, 25$ and 50 using LU decomposition algorithm (Doolittle algorithm).

   (b) (20 points) The matrix in the previous problem is called a scalar-tridiagonal system. It can be solved using the so-called Thomas algorithm, which is essentially a simplified form of Gaussian elimination. Convert the Gaussian elimination program (written for Assignment 1) to Thomas algorithm and calculate the solution for the scalar-tridiagonal system in problem (a).