Jay Prajapati ME21B143

8/4/25 ME5107 - Assignment 6

1) a) $y(t) = \left(y_0 + \frac{C_1}{\lambda} + \frac{C_2}{\lambda^2}\right) e^{\lambda t} - \frac{C_1}{\lambda} - \frac{C_2}{\lambda^2} - \frac{C_2 t}{\lambda}$ — ①

Differentiating,

$$y' = \lambda \left(y_0 + \frac{C_1}{\lambda} + \frac{C_2}{\lambda^2}\right) e^{\lambda t} - \frac{C_2}{\lambda}$$

From ①,

$$\lambda \left(y_0 + \frac{C_1}{\lambda} + \frac{C_2}{\lambda^2}\right) e^{\lambda t} = y + \frac{C_1}{\lambda} + \frac{C_2}{\lambda^2} + \frac{C_2 t}{\lambda}$$

$$y' = \lambda \left(y + \frac{C_1}{\lambda} + \frac{C_2}{\lambda^2} + \frac{C_2 t}{\lambda}\right) - \frac{C_2}{\lambda}$$

$$y' = \lambda y + C_1 + C_2 t$$

which is same as given

Hence proved.

b) $y_{n+1} = y_n + y' h$

$$y_{n+1} = y_n + (\lambda y + C_1 + C_2 t) h$$

$$t = n * h$$

$$y_{n+1} = y_n + (\lambda y_n + C_1 + C_2 n h) h$$

$$y_{n+1} = y_n (\lambda h + 1) + C_2 h^2 n + C_1 h$$

Therefore,

$$y_{n+1} = \alpha y_n + \beta n + \gamma$$

where $\alpha = \lambda h + 1$,

$\beta = C_2 h^2$, $\gamma = C_1 h$

c) $Z_n = y_{n+1} - y_n$

$Z_{n+1} = y_{n+2} - y_{n+1}$

$y_{n+1} = \alpha y_n + \beta n + \gamma$

$y_{n+1} - y_n = (\alpha - 1) y_n + \beta n + \gamma = Z_n$

$y_{n+2} = \alpha y_{n+1} + \beta(n+1) + \gamma$

$y_{n+2} - y_{n+1} = (\alpha - 1) y_{n+1} + \beta(n+1) + \gamma$
$= Z_{n+1}$

$Z_{n+1} - Z_n = (\alpha - 1)(y_{n+1} - y_n) + \beta$

$Z_{n+1} - Z_n = (\alpha - 1) Z_n + \beta$

$Z_{n+1} = \alpha Z_n + \beta$

d) $y_1 = \alpha y_0 + \gamma$

$y_2 = \alpha y_1 + \beta + \gamma$

$= \alpha^2 y_0 + \alpha\gamma + \beta + \gamma$

$y_3 = \alpha y_2 + 2\beta + \gamma$

$= \alpha^3 y_0 + \beta(\alpha + 2) + \gamma(\alpha^2 + \alpha + 1)$

Extending to k,

$y_k = \alpha^k y_0 + \beta(\alpha^{k-2} + 2\alpha^{k-3} + 3\alpha^{k-4} \dots k-1)$
$+ \gamma(\alpha^{k-1} + \alpha^{k-2} + \dots 1)$

Similarly,

$$y_{k+1} = \alpha^{k+1} y_0 + \beta(\alpha^{k-1} + \dots k)$$
$$+ \gamma(\alpha^{k} \dots 1)$$

$$z_k = y_{k+1} - y_k$$

$$z_k = \alpha^k y_0 (\alpha - 1) +$$
$$\beta(\alpha^{k-1} + \alpha^{k-2} \dots 1) + \gamma \alpha^k$$

$$\therefore z_k = \alpha^k y_0 (\alpha - 1) + \gamma \alpha^k + \beta \frac{(\alpha^k - 1)}{(\alpha - 1)}$$

e) $$y_n^{num} = \alpha^n y_0 + \beta(\alpha^{n-2} + 2\alpha^{n-3} + \dots n-1)$$
$$+ \gamma(\alpha^{n-1} + \alpha^{n-2} + \dots 1)$$

f) $$y_n^{analy} = \left(y_0 + \frac{C_1}{\lambda} + \frac{C_2}{\lambda^2}\right)\left(1 + \lambda t + \frac{\lambda^2 t^2}{2!} \dots (\lambda t)^n\right)$$
$$- \frac{C_1}{\lambda} - \frac{C_2}{\lambda^2} - \frac{C_2 t}{\lambda}$$

$$e_n = y_n^{num} = (\lambda h + 1)^n y_0 + C_2 h^2 (\alpha(\lambda h + 1)^{n-2} \dots n-1)$$
$$+ C_1 k \frac{((\lambda h + 1)^n - 1)}{\lambda k} - y_n^{analy}$$
$$= (\lambda h + 1)^n y_0 + C_1 \left[\frac{(\lambda h + 1)^n - \cancel{1}}{\lambda} - \frac{e^{\lambda t}}{\lambda} + \frac{\cancel{1}}{\lambda}\right]$$
$$+ C_2 \left[h^2 ((\lambda h + 1)^{n-2} \dots n-1) - \frac{e^{\lambda t}}{\lambda^2} + \frac{1}{\lambda^2} + \frac{t}{\lambda}\right]$$

$$\boxed{e_n = (\lambda h + 1)^n y_0}$$ $\therefore$ Hence

~~is~~ depends only on λ

2) $\Delta t = h, \quad \Delta x = k, \quad r = \frac{h}{k^2}$

$u_i^{n+1} = u + h u_t + \frac{h^2}{2} u_{tt} + \frac{h^3}{6} u_{ttt} + O(h^4)$

$u_i^{n-1} = u - h u_t + \frac{h^2}{2} u_{tt} - \frac{h^3}{6} u_{ttt} + O(h^4)$

$u_{i+1}^{n} = u + k u_x + \frac{k^2}{2} u_{xx} + \frac{k^3}{6} u_{xxx} + O(k^4)$

$u_{i-1}^{n} = u - k u_x + \frac{k^2}{2} u_{xx} - \frac{k^3}{6} u_{xxx} + O(k^4)$

Substituting,

$F(u_i^n) = \frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} - \frac{(u_{i+1}^n - u_i^{n+1} - u_i^{n-1} + u_{i-1}^n)}{\Delta x^2}$

$= \frac{2(h u_{tt} + \frac{h^3}{6} u_{ttt} + O(h^5))}{2h}$

$- \frac{2(u + \frac{k^2}{2} u_{xx} + O(k^4)) - u - \frac{h^2}{2} u_{tt} + O(h^4))}{k^2}$

$= u_t + \frac{h^2}{6} u_{ttt} + O(h^4) - u_{xx}$

$- \frac{h^2}{k^2} u_{tt} + O(k^2)$

$= \left(\frac{h^2}{6} u_{ttt} - \frac{h^2}{k^2} u_{tt}\right) + O(k^2, h^4)$

→ The above arises as $u_t = u_{xx}$ from transient heat conduction. Therefore equation is consistent as, for $h, k \to 0$, we get $F(u_i^n) \to 0$.

Using von-neumann stability,

$$u_i^n = A^n e^{ikp\Delta x}$$

$$\frac{A^{n+1} e^{ikp\Delta x} - A^{n-1} e^{ikp\Delta x}}{2\Delta t} =$$

$$\frac{A^n e^{ik(p+1)\Delta x} + A^n e^{ik(p-1)\Delta x} - A^{n+1} e^{ikp\Delta x} - A^{n-1} e^{ikp\Delta x}}{\Delta x^2}$$

$$\frac{A - A^{-1}}{2\Delta t} = \frac{e^{ik\Delta x} + e^{-ik\Delta x} - A - A^{-1}}{\Delta x^2}.$$

$$A - \frac{1}{A} = r\left(2\cos(k\Delta x) - A - \frac{1}{A}\right)$$

$$A(1+r) + \frac{1}{A}(r-1) - 2r\cos(k\Delta x) = 0.$$

$\cancel{A^{\_} - (2r\cos}$

$$A^2(r+1) - 2r\cos(k\Delta x)A + (r-1) = 0.$$

$$A = \frac{2r\cos k\Delta x \pm \sqrt{4r^2\cos^2(k\Delta x) - 4r^2 + 4}}{2(1+r)}$$

$$= \frac{r\cos k\Delta x \pm \sqrt{1 - r^2\sin^2(k\Delta x)}}{1+r}.$$

$$A = \frac{r\cos k\Delta x \mp i\sqrt{r^2\sin^2(k\Delta x) - 1}}{1+r}$$

$$|A| = \frac{1}{1+r}\sqrt{r^2\cos^2 k\Delta x + r^2\sin^2(k\Delta x) - 1}$$

$$= \frac{\sqrt{r^2 - 1}}{r+1} = \sqrt{\frac{r-1}{r+1}}.$$

$$|A| \le 1$$

$$\sqrt{\frac{r-1}{r+1}} \le 1.$$

$$1 - \frac{2}{r+1} \le 1$$

As the above is always true we find that scheme is stable.

→ Hence, Dufort - Frankel's method is consistent & stable.

3) $$u_i^{n+1} = u + hu_t + \frac{h^2}{2}u_{tt} + \frac{h^3}{6}u_{ttt} + O(h^4)$$

$$u_i^{n-1} = u - hu_t + \frac{h^2}{2}u_{tt} - \frac{h^3}{6}u_{ttt} + O(h^4)$$

$$u_{i+1}^{n} = u + ku_x + \frac{k^2}{2}u_{xx} - \frac{k^3}{6}u_{xxx} + O(k^4)$$

$$u_{i-1}^{n} = u - ku_x - \frac{k^2}{2}u_{xx} - \frac{k^3}{6}u_{xxx} + O(k^4)$$

$$F(u_i^n) = \frac{hu_t + \frac{h^3}{2}u_{tt} + \frac{h^3}{6}u_{ttt} + O(h^4)}{h} + \frac{2ku_x + k^3/6\, u_{xxx} + O(k^4)}{2k}$$

$$= u_t + u_x + \left(\frac{h}{2}u_{tt} + \frac{h^2}{6}u_{ttt} + \frac{k^2}{6}u_{xxx} + O(h^3, k^3)\right)$$

$$= \frac{h}{2}u_{tt} + \frac{h^2}{6}u_{ttt} + \frac{k^2}{6}u_{xxx} + O(h^3, k^3)$$

→ Therefore scheme is accurate to 3rd order in space & time. Additionally, when $h, k \to 0$ $F(u_i^n) \to 0$. Scheme is consistent.

A Stability $\Rightarrow$

$$\frac{A^{n+1} e^{ikp\Delta x} - A^n e^{ikp\Delta x}}{\Delta t} + c\,\frac{A^n e^{ik(p+1)\Delta x} - A^n e^{ik(p-1)\Delta x}}{2\Delta x} = 0$$

$$\frac{A-1}{\Delta t} + c\,\frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2\Delta x} = 0$$

$$A - 1 + i\,2cr\,\sin(k\Delta x) = 0$$

$$A = 1 - i\,2cr\,\sin(k\Delta x)$$

$$|A| = \sqrt{1 + 4c^2r^2\sin^2(k\Delta x)}$$

$$|A| > 1$$

$\rightarrow$ Explicit scheme is unconditionally unstable.

Lax Method :-

$$F(u_i^n) = \frac{u + hu_t + \frac{h^2}{2}u_{tt} + \frac{h^3}{6}u_{ttt} - u - \frac{k^2}{2}u_{xx} + O(h^4)}{h} + c\,\frac{2ku_x + \frac{k^3}{3}u_{xxx} + O(k^4)}{2k}$$

$$F(u_i^n) = u_t + cu_x + \frac{h}{2}u_{tt} + \frac{h^2}{6}u_{ttt} + O(h^3) - \frac{k^2}{2h}u_{xx} + O\left(\frac{k^4}{h}\right) + c\,\frac{k^3}{6}u_{xxx} + O(k^3))$$

$$F(u_i^n) = \frac{h}{2}u_{tt} + \frac{h^2}{6}u_{ttt} - \frac{k^2}{2h}u_{xx} + \frac{ck^3}{6}u_{xxx} + O(k^3, h^3)$$

→ This scheme is also 3rd order accurate in space & time when $h, k \to 0$ $F(u_i^n) \to 0$. So, it is consistent.

Stability ⇒

$$\frac{A^{n+1} e^{ikp\Delta x} - \frac{1}{2}\left(A^n e^{ik(p+1)\Delta x} + A^n e^{ik(p-1)\Delta x}\right)}{\Delta t}$$

$$+ c \frac{A^n e^{ik(p+1)\Delta x} - A^n e^{ik(p-1)\Delta x}}{2\Delta x} = 0$$

$$\frac{A - \frac{1}{2}\left(e^{ik\Delta x} + e^{-ik\Delta x}\right)}{\Delta t} +$$

$$c \frac{\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)}{\Delta x} = 0.$$

$$r = \frac{\Delta t}{2\Delta x}.$$

$$A - \cos k\Delta x + icr\sin(k\Delta x) = 0.$$

$$A = \cos k\Delta x - icr\sin k\Delta x$$

$$|A| = \sqrt{\cos^2 k\Delta x + c^2 r^2 \sin^2 k\Delta x} \leq 1$$

$$(c^2 r^2 - 1)\sin^2 k\Delta x \leq 0.$$

$$c^2 r^2 - 1 \leq 0.$$

$$\frac{\Delta t}{2\Delta x} \leq \frac{1}{c}.$$

Scheme is conditionally stable if

$$\Delta t \leq \frac{2\Delta x}{c}$$