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ME5107 - Assignment - 7

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1) $y_{i+1} = y_i + \omega_1 k_1 + \omega_2 k_2$

where

$$k_1 = hf(y_i)$$
$$k_0 = hf(y_i + \beta_0 k_1)$$
$$k_2 = hf(y_i + \beta_1 k_0)$$

$y'(x) = f(y)$
$y'' = f'(y)\, y'$
$y''' = f'' \times y'^2 + f'(y)\, y''$
$= f''(f)^2 + f(f')^2$

$$y_{i+1} = y(x+h)$$

$$y(x_i + h) = y_i + h y'_{y_i} + \frac{h^2}{2} y''_{y_i} + \frac{h^3}{6} y'''_{y_i} + O(h^4) \quad \text{--- (1)}$$

Now,

$$k_0 = h\left[f(y_i) + \beta_0 k_1 f'(y_i) + \frac{(\beta_0 k_1)^2}{2} f''(y_i) \ldots\right]$$

$$k_2 = h\left[f(y_i) + \beta_1 k_0 f'(y_i) + \frac{(\beta_1 k_0)^2}{2} f''(y_i) \ldots\right]$$

$$= h\Big[f(y_i) + \beta_1 h f'(y_i)\Big\{f(y_i) + \beta_0 k_1 f'(y_i) + \frac{(\beta_0 k_1)^2}{2} f''(y_i) \ldots\Big\}$$
$$+ h\frac{\beta_0^2}{2} f''(y_i)\Big\{f(y_i) + \beta_0 k_1 f'(y_i) + \frac{(\beta_0 k_1)^2}{2} f''(y_i) + \ldots\Big\}\Big]$$

$$k_2 = hf(y_i) + \beta_1 h^2 f(y_i) + \beta_0 \beta_1 h^3 f(y_i)(f'(y_i))^2$$
$$+ \frac{\beta_1^2 h^3}{2} f^2(y_i) \times f''(y_i) + \ldots$$

Substituting k_1 & k_2,

$$y_{i+1} = y_i + w_1 h f(y_i) + w_2 h f(y_i) + w_2 \beta_1 h^2 f(y_i) f'(y_i) + w_2 \beta_0 \beta_1 h^3 f(y_i) \times (f'(y_i))^2 + \frac{w_2 \beta_1^2 h^3}{2} f^2(y_i) f''(y_i) + \dots \quad (2)$$

Comparing (1) & (2),

$$w_1 + w_2 = 1$$

$$w_2 \beta_1 = \frac{1}{2}$$

$$w_2 \beta_0 \beta_1 = \frac{w_2 \beta_1^2}{2} = \frac{1}{6}$$

Now for any ratio of w_1 & w_2, we can get β_0 & β_1 but here we assume

$$w_1 = w_2 = \frac{1}{2}$$

$$\beta_1 = 1$$

$$\frac{1}{2} \times \beta_0 \times 1 + \frac{1}{2} \times \frac{1}{2} = \frac{1}{6}$$

$$\beta_0 = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}$$

$$\therefore w_1 = w_2 = \frac{1}{2} \quad \& \quad \beta_0 = -\frac{1}{6}, \quad \beta_1 = 1$$

$$\boxed{\beta_0 = \frac{1}{3}, \quad \beta_1 = \frac{2}{3}, \quad w_2 = \frac{3}{4}, \quad w_1 = \frac{1}{4}}$$

b) $k_1 = h \alpha y_i$

$k_0 = h \alpha (y_i + \beta_0 k_1)$
$= h \alpha (y_i + \beta_0 h \alpha y_i)$
$= h \alpha y_i (1 + \beta_0 h \alpha)$

$k_2 = h \alpha (y_i + \beta_1 k_0)$
$= h \alpha (y_i + \beta_1 h \alpha y_i (1 + \beta_0 h \alpha))$

$k_2 = h \alpha y_i (1 + \beta_1 h \alpha + \beta_0 \beta_1 h^2 \alpha^2)$

$y' = \alpha y. \qquad \alpha = ia$

$f(y_i) = i\,\overset{a}{\cancel{\alpha}}\,y \qquad$ (purely img).

$K_1 = h\,f(y_i) = i\,h\,a\,y.$

$K_0 = h\,f(y_i + \beta_0 K_1)$

$= h\,f(y_i + \frac{1}{3} \times i\,h\,a\,y_i)$

$= h\,i\,a\,y_i\,(1 + \frac{i\,h\,a}{3})$

$K_0 = h\,a\,y_i\,(i - \frac{h\,a}{3}).$

$K_2 = h\,f(y_i + \beta_1 K_0)$

$= h\,f(y_i + \frac{2}{3}\,y_i\,a\,h\,(i - \frac{h\,a}{3}))$

$= h\,i\,a\,y_i\,(1 - \frac{2a^2h^2}{9} + \frac{i\,2\,ah}{3})$

$y_{i+1} = y_i + w_1 K_1 + w_2 K_2$

$= y_i + \frac{1}{4}(i\,a\,h\,y_i) + \frac{3}{4}[h\,a\,y_i\,(-\frac{2\,ah}{3} + i\,(1 - \frac{2}{9}a^2h^2))]$

$= y_i\,[(1 - \frac{a^2h^2}{2}) + i\,(ah - \frac{a^3h^3}{6})]$

For stability,

$(1 - \frac{a^2h^2}{2})^2 + (ah - \frac{a^3h^3}{6})^2 \leq 1.$

$\frac{a^6h^6}{36} - \frac{a^4h^4}{12} \leq 0.$

$\boxed{h \leq \frac{\sqrt{3}}{a}}$

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c) $y' = \alpha y \quad \alpha < 0 \in R.$

$y' = \alpha y.$

$k_1 = h \alpha y_i$

$k_0 = h f(y_i + \frac{1}{3} h \alpha y_i)$

$= h \alpha y_i (1 + \frac{h\alpha}{3})$

$k_2 = h f(y_i + \frac{2}{3} h \alpha y_i (1 + \frac{h\alpha}{3}))$

$= h \alpha y_i (1 + \frac{2\alpha h}{3} + \frac{2\alpha^2 h^2}{9})$

$y_{i+1} = y_i + w_1 k_1 + w_2 k_2$

$= y_i + \frac{1}{4} h \alpha y_i + \frac{3 h \alpha y_i}{4} (1 + \frac{2\alpha h}{3} + \frac{2\alpha^2 h^2}{9}).$

$= y_i (1 + h\alpha + \frac{\alpha^2 h^2}{2} + \frac{\alpha^3 h^3}{6})$

For stability,

$|1 + \alpha h + \frac{\alpha^2 h^2}{2} + \frac{\alpha^3 h^3}{6}| \leq 1.$

$-2 \leq \alpha h + \frac{\alpha^2 h^2}{2} + \frac{\alpha^3 h^3}{6} \leq 0.$

Case 1 : $\alpha h + \frac{\alpha^2 h^2}{2} + \frac{\alpha^3 h^3}{6} \geq -2.$

$\alpha h \geq -2.5127$

$h \leq \frac{2.5127}{|\alpha|}$

Case 2 :- $\alpha h + \frac{\alpha^2 h^2}{2} + \frac{\alpha^3 h^3}{6} \leq 0$:

if $h < 0$.

$$\underset{+ve}{\alpha h} + \underset{+ve}{\alpha^2 h^2} + \underset{+ve}{\alpha^3 h^3} > 0.$$

$\therefore \quad h \geq 0.$

2) $y'(x_i) = x+y.$

$y''(x_i) = 1 + y'(x) = 1 + x + y$

$y'''(x) = 1 + y'(x) = 1 + x + y$

$y''''(x) = 1 + y'(x) = 1 + x + y.$

$$y(x_i+h) = y(x_i) + h\left[x_i + y_i + (1 + x_i + y_i) \times \left(\frac{h}{2} + \frac{h^2}{6} + \frac{h^3}{24}\right)\right]$$

x_i	$y(x_i)$	$y(x_i+h)$	y_exact	Error
0	1	1.11034	1	0
0.1	1.11034	1.2428	1.11034	3.37×10^{-12}
0.2	1.2428	1.3997	1.24281	3.64×10^{-11}
0.3	1.3997	1.5836	1.39972	3.1×10^{-10}
0.4	1.5836	1.79739	1.58365	2.44×10^{-9}
0.5	1.79739	2.04418	1.79744	2.76×10^{-9}
0.6	~~1.79~~ 2.04418	2.32744	2.04424	3.32×10^{-9}
0.7	2.32744	2.65101	2.3275	4.28×10^{-9}
0.8	2.65101	3.01913	2.65108	5.16×10^{-9}
0.9	3.01913	3.43648	3.0192	5.81×10^{-9}
1.0	3.43648	3.90824	3.43656	7×10^{-9}

$\boxed{\therefore\ y(x_i) = 3.43648}$ @ $x_i = 1.$

where Error $= (\text{y\_exact} - y(i))^2.$