Assignment 3

General Instructions

- Solutions due by 4^{th} March for Group A
- Solutions due by 6^{th} March for Group B

Hand calculations

1. (10 points) Gaussian elimination for banded matrices
   (a) Write down a modified form of the general Gaussian Elimination algorithm to solve for a scalar tri-diagonal system. This matrix has a bandwidth $B = 2$, *i.e.* 1 main diagonal + 1 super-diagonal (and sub-diagonal) populated with non-zeros.
   (b) Count the number of flops (multiplications and divisions) to solve for $\bar{x}$. Is this larger, smaller, or the same compared to that of the Thomas algorithm?
   (c) Generalize this algorithm to solve for a general banded system with bandwidth B
2. (10 points) Compute the LU decomposition for the matrix below with partial pivoting and the corresponding growth factor (the missing entries are zeros).

$$\begin{pmatrix} 1 & & & & 1 \\ -1 & 1 & & & 1 \\ -1 & -1 & 1 & & 1 \\ -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 \end{pmatrix} \tag{1}$$

Programming

1. (30 points) Convert your LU decomposition program to include partial pivoting. Using this program, experiment with solving 60×60 systems of equations $Ax = b$ with A having the form in the previous problem. Do you observe that the results are useless because of the growth factor of order 2^{60}? At your first attempt you may not observe this, because the integer entries of A may prevent any rounding errors from occurring. If so, find a way to modify your problem slightly so that the growth factor is the same or nearly so and catastrophic rounding errors really do take place.

2. (50 points) Write a program for LU decomposition of the block tridiagonal matrix in the system shown below for arbitrary N and N_{blk}. D_is are scalar tridiagonal matrices of size N_{blk} and has a structure shown in eq. (2) (shown for $N_{blk} = 5$). A_is and B_is are identity matrices of size N_{blk}. Use either Gaussian elimination or Thomas algorithm (tridiagonal matrix algorithm) for each block. Solve the block tridiagonal system for three right hand side vectors ($N = 10, 20, 30$) given by,

$$f_i = \begin{cases} 1 & j = 1 \\ 1/N_{blk} & 2 \le j \le (N_{blk} - 1) \\ 2 & j = N_{blk} \end{cases}$$

for $1 \le i \le N$. For all three cases, plot $\bar{x}_i$ vs j for $i = N/2$. Also plot the time taken for computation as a function of N, for $N = 10, 20,$ and 30 with $N_{blk} = 5$.

$$\begin{pmatrix} D_1 & A_1 & . & . & . & . \\ B_2 & D_2 & A_2 & . & . & . \\ . & B_3 & D_3 & A_3 & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & B_{N-1} & D_{N-1} & A_{N-1} \\ . & . & . & . & B_N & D_N \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \\ . \\ . \\ . \\ \bar{x}_{N-1} \\ \bar{x}_N \end{pmatrix} = \begin{pmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \bar{f}_3 \\ . \\ . \\ . \\ \bar{f}_{N-1} \\ \bar{f}_N \end{pmatrix}$$

$$D_i = \begin{bmatrix} -4 & 1 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 0 \\ 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix} \tag{2}$$