Assignment 7

General Instructions

- Solutions due by 22^{nd} April for Group A
- Solutions due by 24^{th} April for Group B

Hand calculations

1. (20 points) The following scheme has been proposed for solving $y' = f(y)$:

$$y_{i+1} = y_i + \omega_1 k_1 + \omega_2 k_2,$$

where

$$\begin{aligned} k_1 &= h\ f(y_i) \\ k_0 &= h\ f(y_i + \beta_0 k_1) \\ k_2 &= h\ f(y_i + \beta_1 k_0) \end{aligned}$$

with h being the step size.
(a) Determine the coefficients $\omega_1, \omega_2, \beta_0$ and β_1 that would maximize the order of accuracy of the method.
(b) Applying this method to $y' = \alpha y$, what is the maximum step size h for α pure imaginary?
(c) Applying this method to $y' = \alpha y$, what is the maximum step size h for α real negative?

2. (20 points) Consider solving the following first-order initial value problem:

$$y'(x) = \frac{dy}{dx} = f(x, y) = x + y, \quad y(0) = 1, \quad 0 \leq x \leq 1,$$

using a fourth-order accurate Taylor-series method given as follows:

$$y(x_i + h) = y(x_i) + h\ T_4(x_i, y_i, h)$$

where

$$T_4(x_i, y_i, h) = y'(x_i) + y''(x_i)\frac{h}{2!} + y'''(x_i)\frac{h^2}{3!} + y^{iv}(x_i)\frac{h^3}{4!}.$$

Starting with $x = 0$ using a step size of $h = 0.1$, calculate the solution at $x = 1$. Tabulate your results showing the computed solution, exact solution and error at each step. The exact analytical solution is given by $y(x) = 2e^x - x - 1$.

Programming

1. (60 points) Consider the differential equation governing simple harmonic motion: $y(t)$

$$\begin{gathered} y'' + \omega^2 y = 0 \text{ for } t > 0 \\ y(0) = y_0;\ y'(0) = 0. \end{gathered}$$

Use $y_0 = 1$ and $\omega = 4$.

(a) If the general form of the exact solution is $y(t) = A\sin(\omega t) + B\cos(\omega t)$, obtain the exact solution to the initial value problem subject to the given conditions
(b) Re-write the above second order differential equation as a system of first order differential equations.
(c) Solve this system using *leapfrog* method: $y^{(n+1)} = y^{(n-1)} + 2hf(y^{(n)}, t^{(n)})$ with time step size $h = 0.1$s for $0 \leq t \leq 9$ s. Use Euler explicit method to start the calculation.
(d) Plot the amplitude errors between the exact and numerical solution at specific time instances where the exact solution attains peak values.
(e) Do the observed amplitude errors agree with your expectation for leapfrog method? Explain why or why not.