Assignment 6

General Instructions

- Solutions due by 8^{th} April for Group A
- Solutions due by 11^{th} April for Group B

Hand calculations

1. (20 points) Consider the Euler explicit method applied to a differential equation $y^{'} = f(y,t)$ with initial condition $y(t = 0) = y_0$. In the lecture, to perform stability analysis, we linearized the differential equation to get: $y^{'} = \lambda y + c_1 + c_2 t$. Further, neglecting the inhomogenous terms, we obtained the model problem $y^{'} = \lambda y$, where $\lambda = \lambda_R + i\lambda_I$ with $\lambda_R \leq 0$. We will now study the effects of inhomogeous terms in the linearized equation on stability analysis.

   (a) Verify that $y(t) = \left(y_0 + \frac{c_1}{\lambda} + \frac{c_2}{\lambda^2}\right) e^{\lambda t} - \frac{c_1}{\lambda} - \frac{c_2}{\lambda^2} - \frac{c_2 t}{\lambda}$ is an exact solution to the differential equation

   (b) Apply Euler explicit method to derive a difference equation of the form: $y_{n+1} = \alpha y_n + \beta n + \gamma$. What are α, β, and γ?

   (c) Use transformation $z_n = y_{n+1} - y_n$ to derive the following difference equation: $z_{n+1} = \alpha z_n + \beta$.

   (d) Solve the above difference equation and obtain an expression for z_k in terms of $y_0, \alpha, \beta, \gamma$ for some k.

   (e) Express the numerical solution y_n^{num} in terms of y_0 using the result from part (d)

   (f) Show that the stability of the error, $e_n = y_n^{num} - y_n^{analy}$ depends only on λ.

2. (10 points) Calculate the truncation error and examine the consistency and stability of Dufort-Frankel's method for the solution of one-dimensional transient heat conduction equation given by

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = \frac{u_{i+1}^n - u_i^{n+1} - u_i^{n-1} + u_{i-1}^n}{\Delta x^2} \tag{1}$$

3. (10 points) Calculate the truncation error and examine the consistency and stability of the following discretizations for the solution of first-order wave equation give by

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \quad c > 0 \tag{2}$$

   (a) Euler explicit method

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\,\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = 0 \tag{3}$$

   (b) Lax method

$$\frac{u_i^{n+1} - \frac{1}{2}(u_{i+1}^n + u_{i-1}^n)}{\Delta t} + c\,\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = 0 \tag{4}$$

Programming

1. (60 points) The heat equation with a source term is

$$\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2} + S(x), \quad 0 \leq x \leq L \tag{5}$$

   The initial and boundary conditions are $T(x,0) = 0, T(0,t) = 0$ and $T(L,t) = T_{steady}(L)$. Taking $\alpha = 1, L = 15$ and $S(x) = -(x^2 - 4x + 2)e^{-x}$. The exact steady solution is

$$T_{steady}(x) = x^2 e^{-x}. \tag{6}$$

Solve the equation to steady state on a uniform grid with a grid spacing of $\Delta x = 1, 0.1$ and employing a time step of $\Delta t = 0.005$. For each of the following methods (i) plot the variation of error in $1-$norm with respect to Δx on a log-log plot for a $\Delta t = 0.005$ (ii) plot the exact and the numerical solutions for $\Delta x = 1$ and $\Delta t = 0.005$. Comment on the maximum time step you can use in each of the following methods and show that by employing a larger Δt you can take fewer iterations to arrive at the steady solution in case of implicit methods. (a) Explicit Euler time-advacement with second-order central difference scheme for spatial derivate (b) Implicit Euler time-advancement with second-order central difference scheme for spatial derivate (c) Crank-Nicolson method.

2. (60 points) Consider the following initial value problem

$$\frac{dy}{dt} = -\lambda\left(y - e^{-t}\right) - e^{-t}, \quad \lambda > 0$$

with the intial condition $y(t = 0) = y_0$. Solve this initial value problem for $\lambda = 10$ and $y_0 = 10$ using explicit Euler, implicit Euler, trapezoidal, second-order Runge-Kutta, fourth-order Runge-Kutta and Runge-Kutta-Fehlberg method. Use time step sizes $h = 0.1, 0.05, 0.025, 0.0125, 0.00625$ and integrate from $t = 0$ to 0.8. The exact analytical solution to this equation is

$$y = e^{-t} + (y_0 - 1)e^{-\lambda t}.$$

(a) For each method, plot the solutions obtained along with the exact solution from $t = 0$ to $t = 0.8$.
(b) Evaluate the error between the numerical and the exact solution at $t = 0.8$ as follows. If the order of the method is p, then the ratio of the error corresponding to a step size of h and $h/2$ should be 2^p. Tabulate this ratio for successive pairs of h and deduce the asymptotic value of p for each method. Use double precision arithmetic throghout. The formulation of the methods is given as follows,

- Explicit Euler Method

$$y_{i+1} = y_i + hf(x_i, y_i)$$

- Implicit Euler Method

$$y_{i+1} = y_i + hf(x_{i+1}, y_{i+1})$$

- Trapezoidal Method

$$y_{i+1} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1}, y_{i+1})\right]$$

- Second-Order Runge-Kutta Method

$$y_{i+1} = y_i + hk_2$$

where

$$\begin{aligned} k_1 &= f(x_i, y_i) \\ k_2 &= f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_1\right) \end{aligned}$$

- Fourth-Order Runge-Kutta Method

$$y_{i+1} = y_i + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

where

$$\begin{aligned} k_1 &= f(x_i, y_i) \\ k_2 &= f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_1\right) \\ k_3 &= f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_2\right) \\ k_4 &= f(x_i + h, y_i + hk_3) \end{aligned}$$

- Runge-Kutta-Fehlberg Method

$$y_{i+1} = y_i + \left(\frac{16}{135}k_1 + \frac{6656}{12825}k_3 + \frac{28561}{56430}k_4 - \frac{9}{50}k_5 + \frac{2}{55}k_6\right)$$

where

$$
\begin{aligned}
k_1 &= hf(x_i, y_i) \\
k_2 &= hf\left(x_i + \frac{h}{4}, y_i + \frac{k_1}{4}\right) \\
k_3 &= hf\left(x_i + \frac{3h}{8}, y_i + \frac{3k_1}{32} + \frac{9k_2}{32}\right) \\
k_4 &= hf\left(x_i + \frac{12h}{13}, y_i + \frac{1932k_1}{2197} - \frac{7200k_2}{2197} + \frac{7296k_3}{2197}\right) \\
k_5 &= hf\left(x_i + h, y_i + \frac{439k_1}{216} - 8k_2 + \frac{3680k_3}{513} - \frac{845k_4}{4104}\right) \\
k_6 &= hf\left(x_i + \frac{h}{2}, y_i - \frac{8k_1}{27} + 2k_2 - \frac{3544k_3}{2565} + \frac{1859k_4}{4104} - \frac{11k_5}{40}\right).
\end{aligned}
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