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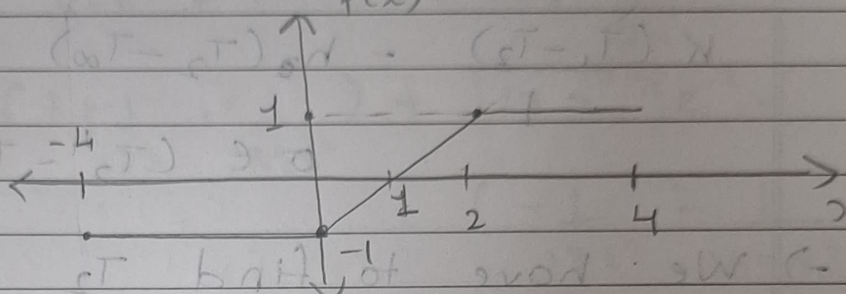
18/3/25 ME5107 - Assignment - 4

1) Given function $f(x)$ in $[-4, 4]$

$$f(x) = \begin{cases} -1 & , x \in [-4, 0] \\ x-1 & , x \in [0, 2] \\ 1 & , x \in [2, 4] \end{cases}$$




→ We cannot use Newton-Raphson method because it involves derivative of function and here function is continuous but not differentiable.

→ Therefore we will use secant method to solve for root of $f(x)$. Graphically or Analytically root is clearly $x = 1$.

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

| Itr. | $x_{n-1}$ | $x_n$ | $f(x_{n-1})$ | $f(x_n)$ | $x_{n+1}$ |
|---|---|---|---|---|---|
| 1 | -4 | 3 | -1 | 1 | -0.5 |
| 2 | 3 | -0.5 | 1 | -1 | 1.25 |
| 3 | -0.5 | 1.25 | -1 | 0.25 | 0.9 |
| 4 | 1.25 | 0.9 | 0.25 | -0.1 | 1 |
| 5 | 0.9 | 1 | -0.1 | 0 | 1 |

→ We can see that $x_{n+1}$ doesnt change after 4th & 5th iterations.

∴ $\boxed{x = 1}$

2) The equation is

$$\frac{K(T_1 - T_2)}{L} = h_2(T_2 - T_\infty) + \sigma \epsilon (T_2^4 - T_\infty^4)$$

→ We have to find $T_2$ with initial guess of 1000K for given $k, T_1, L, h, \epsilon$

$$f(T_2) = h(T_2 - T_\infty) + \sigma \epsilon (T_2^4 - T_\infty^4) - \frac{k(T_1 - T_2)}{L}$$

$$f'(T_2) = h + 4\sigma\epsilon T_2^3 + \frac{k}{L}$$

| Itr. | $T_i$ | $f(T_i)$ | $f'(T_i)$ | $T_{i+1}$ |
|---|---|---|---|---|
| 1 | 1000 | 62167.28 | 209.77 | 703.65 |
| 2 | 703.65 | 19530.16 | 91.54 | 490.31 |
| 3 | 490.31 | 4987.36 | 49.72 | 389.995 |
| 4 | 389.995 | 573.14 | 39.1 | 375.34 |
| 5 | 375.34 | 8.675 | 37.93 | 375.11 |

→ After 4th & 5th iteration, $T_2$ is converging & error is becoming small

∴ $\boxed{T_2 = 375.11\ K.}$

3)
$$f(x) = (2\pi)^{1/2} x^{x+1/2} e^{-x} \left(1 + \frac{1}{12x} + \frac{1}{288x^2}\right)$$

$$g(x) = f(x) - 1000$$

$$g(x) = \sqrt{2\pi}\, x^{x+1/2} e^{-x} \left(1 + \frac{1}{12x} + \frac{1}{288x^2}\right)$$

$$g'(x) = \sqrt{2\pi}\, x^{x+1/2} e^{-x} \left(\ln x - \frac{1}{2x}\right) \times \left(1 + \frac{1}{12x} + \frac{1}{288x^2}\right) + \sqrt{2\pi}\, x^{x+1/2} e^{-x} \times \left(-\frac{1}{12x^2} - \frac{1}{144x^3}\right)$$

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)} \rightarrow \text{NR method}$$

| Itr. | $x_i$ | $g(x_i)$ | $g'(x_i)$ | $x_{i+1}$ |
|---|---|---|---|---|
| 1 | 10 | 3627809.7036 | 8534059.7101 | 9.5749 |
| 2 | 9.5749 | 1346023.3355 | 3112238.4737 | 9.1424 |
| 3 | 9.1424 | 499599.1837 | 1134666.6662 | 8.7021 |
| 4 | 8.7021 | 185420.2455 | 413838.3638 | 8.2541 |
| 5 | 8.2541 | 68717.1125 | 151290.1534 | 7.7998 |
| 6 | 7.7998 | 25331.5738 | 55739.6808 | 7.3454 |
| 7 | 7.3454 | 9189.9905 | 20997.4709 | 6.9077 |
| 8 | 6.9077 | 3186.9225 | 8387.5578 | 6.5278 |
| 9 | 6.5278 | 975.2698 | 3853.1659 | 6.2746 |
| 10 | 6.2746 | 211.1506 | 2318.2468 | 6.1836 |
| 11 | 6.1836 | 18.0061 | 1934.7970 | 6.1743 |
| 12 | 6.1743 | 0.1648 | 1899.4972 | 6.1742 |
| 13 | 6.1742 | 0 | 1899.1713 | 6.1742 |

$$\therefore \boxed{x = 6.1742}$$

4)
$$x_{k+1} = \frac{x_0 f(x_k) - x_k f(x_0)}{f(x_k) - f(x_0)}$$

$$x_{k+1} = \frac{x_0 f(x_k) - x_k f(x_k) + x_k f(x_k) - x_k f(x_0)}{f(x_k) - f(x_0)}$$

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_0)}{f(x_k) - f(x_0)}$$

$$\epsilon_k = x_k - x_r$$

$$\epsilon_0 = x_0 - x_r$$

$$\epsilon_{k+1} = \epsilon_k - \frac{f(\epsilon_k + x_r)(\epsilon_k - \epsilon_0)}{f(x_r + \epsilon_k) - f(x_r + \epsilon_0)}$$

$$= \epsilon_k - \frac{(\epsilon_k - \epsilon_0)[\epsilon_k f'(x_r) + \frac{1}{2}\epsilon_k^2 \ldots]}{(\epsilon_k - \epsilon_0)[f'(x_r) + \frac{1}{2}(\epsilon_k + \epsilon_0) f''(x_r) \ldots]}$$

$$\epsilon_{k+1} = \epsilon_k - \frac{[\epsilon_k f'(x_r) + \frac{1}{2}\epsilon_k^2 f''(x_r) \ldots]}{[f'(x_r) + \frac{1}{2}(\epsilon_k + \epsilon_0) f''(x_r) \ldots]}$$

$$= \epsilon_k - [\epsilon_k f'(x_r) + \frac{1}{2}\epsilon_k^2 f''(x_r) \ldots] \times [f'(x_r) + \frac{1}{2}(\epsilon_k + \epsilon_0) f''(x_r)]^{-1} \ldots$$

$$= \frac{1}{2}\epsilon_k \epsilon_0 \frac{f''(x_r)}{f'(x_r)} + \ldots$$

$$\epsilon_{k+1} = C \epsilon_k \epsilon_0$$

$$\therefore \epsilon_{n+1} = C \epsilon_n \times \epsilon_0.$$

→ Now, $\epsilon_0$ is considered to be constant.

$\therefore \epsilon_{n+1} = A\,\epsilon_n^{1}$

$\boxed{p = 1}$

Rate of convergence = 1

→ ~~Now since error of successive term is proportional to product of two errors, order of convergence is quadratic i.e order is 2.~~

5) a)

$$\frac{d^2T}{dx^2} = \lambda (T - T_\infty)$$

Discretizing LHS,

$$\frac{T_{i+1} + T_{i-1} - 2T_i}{\Delta x^2} = \lambda (T_i - T_\infty)$$

$$-T_{i+1} + (2 + \lambda \Delta x^2) T_i - T_{i+1} = \lambda \Delta x^2 T_\infty$$

$L = 0.1$ m for 6 points.

$\Delta x = 0.02$ m.

$T_L = 100\,°C$

$$-T_1 + \left(2 + \frac{\lambda}{2500}\right) T_2 - T_3 = \frac{\lambda}{2500} T_\infty.$$

$$-T_2 + \left(2 + \frac{\lambda}{2500}\right) T_3 - T_4 = \frac{\lambda}{2500} T_\infty$$

$$-T_3 + \left(2 + \frac{\lambda}{2500}\right) T_4 - T_5 = \frac{\lambda}{2500} T_\infty$$

$$-T_4 + \left(2 + \frac{\lambda}{2500}\right) T_5 - T_R = \frac{\lambda}{2500} T_\infty.$$

$T_R = 30\,°C$

→ Converting it to matrix form

$$A(\lambda) \vec{T} = \vec{b}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2+\lambda/2500 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2+\lambda/2500 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2+\lambda/2500 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2+\lambda/2500 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_{6R} \end{bmatrix} \qquad b = \begin{bmatrix} 100 \\ \frac{\lambda}{2500} T_\infty \\ \frac{\lambda}{2500} T_\infty \\ \frac{\lambda}{2500} T_\infty \\ \frac{\lambda}{2500} T_\infty \\ 30 \end{bmatrix}$$

b) For absence of convection,

$\lambda = 0$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} [T] = \begin{bmatrix} 100 \\ 0 \\ 0 \\ 0 \\ 0 \\ 30 \end{bmatrix}$$

→ Solving this using LU decomposition we get

$$\vec{T}_2 = \begin{bmatrix} 100 \\ 86 \\ 72 \\ 58 \\ 42 \\ 30 \end{bmatrix}$$

c) $\lambda = 500$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2.2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2.2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2.2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2.2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T_L \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_R \end{bmatrix} = \begin{bmatrix} 100 \\ 6 \\ 6 \\ 6 \\ 6 \\ 30 \end{bmatrix}$$

→ Similarly solving this system & rounding off to 3 decimals we get

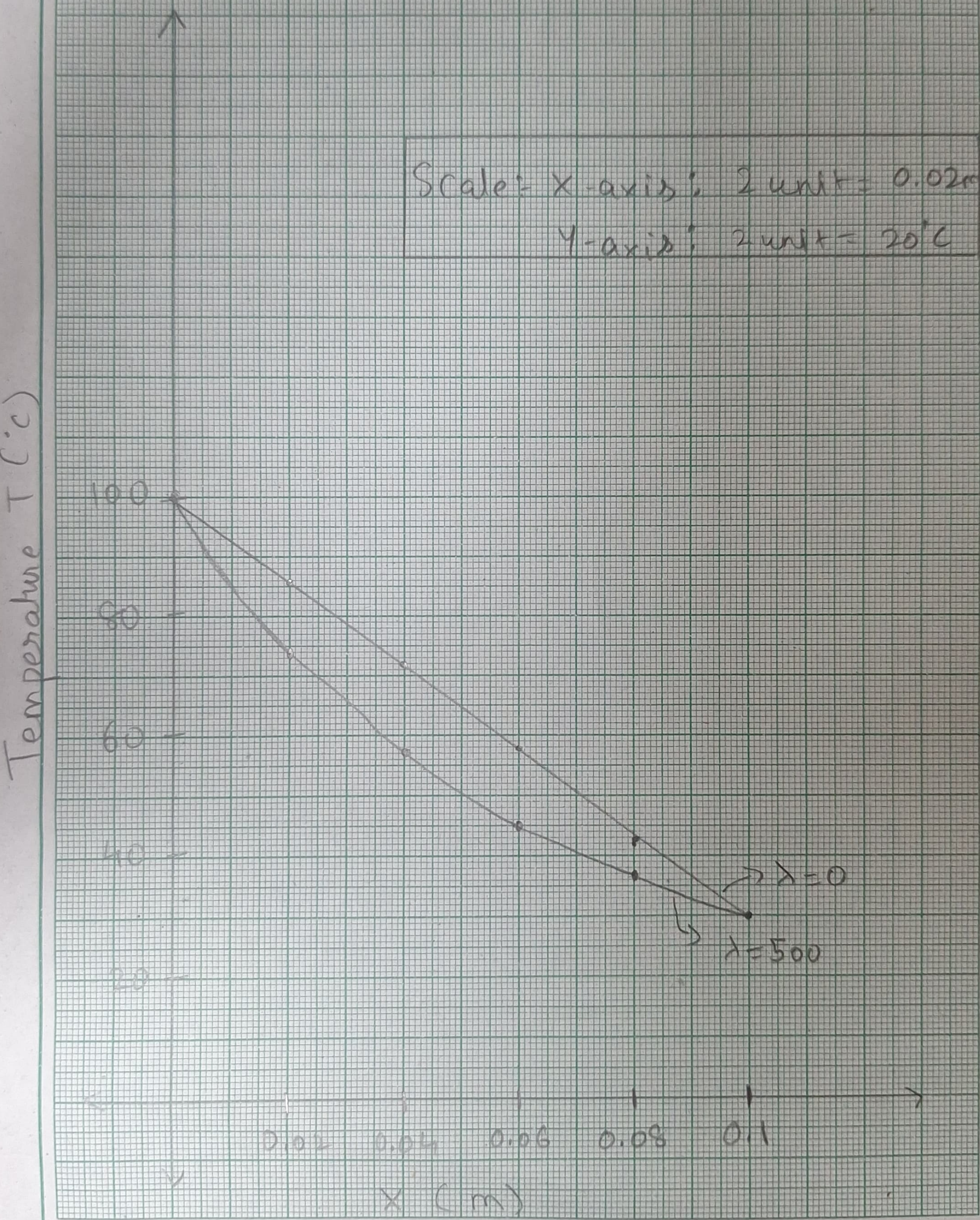
$$\vec{T} = \begin{bmatrix} 100 \\ 74.153 \\ 57.136 \\ 45.547 \\ 37.067 \\ 30 \end{bmatrix}$$

→ Plot for both temp. is in graph sheet attached. As we can see that $\lambda = 0$ case, graph is linear and $\lambda = 500$ case deviates below the first case before rejoining at boundary.

→ This is because there is heat loss through convection at all interior, non-boundary points in the rod. This results in ~~drop~~ proportional drop in temp. of node, which causes deviation.

→ If we vary the value of $\lambda$, we shall see that graph will lie between case 1 & 2 whereas greater than $\lambda = 500$ lies below case 2.

→ If $\lambda$ is negative, meaning rod gains heat through convective heat transfer, it rises above case 1



Scale: X-axis: 2 unit = 0.02m
Y-axis: 2 unit = 20°C
Temperature T (°C)
100
80
60
40
20
0.02
0.04
0.06
0.08
0.1
X (m)
λ = 0
λ = 500