ME5204 - Finite Element Analysis

Assignment 5 - Vector valued problems

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1 Strong Form with boundary conditions

The goal of this assignment is carry out static and free vibration analysis of the two water tanks: MSB and GymKhana (will refer it to as Gym in furthur analysis for simplicity). Both of these tasks can be done by solving Elastodynamic equation (Elastic wave equation). The tanks are to be modelled in 2D. The strong form for the same is given below.

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + b_x = \rho \frac{\partial^2 u}{\partial t^2} \tag{1}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + b_y = \rho \frac{\partial^2 v}{\partial t^2} \tag{2}$$

The following equation involves some assumptions which I have covered in later sections. The suitable boundary conditions here would be that tanks are rigid at the base i.e displacement would be zero for any node at the base. (Dirichlet boundary condition)

$$u(x, y = 0) = 0 \ in \ \ \Omega$$

assuming bottom most left point as origin.

The another boundary condition suitable here is the outer surface of tank is treated as free boundary with no external forces, allowing natural expansion represented by Neumann boundary condition.

2 Weak form

For writing the weak form, we will multiply the equations with arbitrary test functions and integral over that to be zero.

$$\int_\Omega g_1[\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + b_x - \rho \frac{\partial^2 u}{\partial t^2}] d\Omega + \tag{3}$$

$$g_2[\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + b_y - \rho \frac{\partial^2 v}{\partial t^2}] d\Omega = 0 \tag{4}$$

Now we will use the **Gauss-Divergence theorem**.

$$\underbrace{\int_\Omega \left(\frac{\partial g_1}{\partial x}\sigma_x + \frac{\partial g_1}{\partial y}\sigma_{xy} + \frac{\partial g_2}{\partial x}\sigma_{xy} + \frac{\partial g_2}{\partial y}\sigma_y\right) d\Omega}_{\textbf{Bilinear term}}$$

$$+ \underbrace{\int_\Omega \left(g_1\rho\frac{\partial^2 u}{\partial t^2} + g_2\rho\frac{\partial^2 v}{\partial t^2}\right) d\Omega}_{\textbf{Inertia term}} = \underbrace{\int_\Omega (g_1 b_x + g_2 b_y)\, d\Omega}_{\textbf{Body load term}} \tag{5}$$

$$+ \underbrace{\int_\Omega (g_1\sigma_x n_x + g_1\sigma_{xy}n_y + g_2\sigma_{xy}n_x + g_2\sigma_y n_y)}_{\textbf{Boundary term}} \tag{6}$$

Now rearranging the terms considering only bilinear term, we get

$$\int_\Omega \begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_2}{\partial y} & \frac{\partial g_1}{\partial y} + \frac{\partial g_2}{\partial x} \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix} d\Omega \tag{7}$$

We will use stress and strain relation to convert it to strains,

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix} = C_{3x3} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{pmatrix} \tag{8}$$

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix} = C_{3x3} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{pmatrix} \tag{9}$$

$$\int_\Omega \begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_2}{\partial y} & \frac{\partial g_1}{\partial y} + \frac{\partial g_2}{\partial x} \end{pmatrix} C \underbrace{\begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{pmatrix}}_{\mathbf{B}} d\Omega \tag{10}$$

Now assuming the test funcitons (u & v) to be summation of piece-wise functions.

$$u = \sum_I \phi_I u_I$$

$$v = \sum_I \phi_I v_I$$

Substituting this into above equations, we get

$$\varepsilon = BU$$

where B is strain displacement matrix and about C will be discussed in later sections So finally the bilinear term cna be written in matrix form as

$$Ku = \int_\Omega B^T C B \begin{pmatrix} u_1 \\ v_1 \\ \dots \\ v_n \end{pmatrix} \tag{11}$$

Similarly Interia term can be written as $M\ddot{u}$ and body load term as

$$F = \int_\Omega \phi^T \begin{pmatrix} b_x \\ b_y \end{pmatrix} d\Omega$$

Combining all we get the governing differential equation in matrix form as (Note we will include boundary conditions in F)

$$M\ddot{u} + Ku = F \tag{12}$$

For Static Deflection, transient term or the inertia term will be zero and hence we will solve for displacements.

$$Ku = F + BC \tag{13}$$

For Free vibration case, we discard the body load term and solve it as an eignevalue problem. Assuming an harmonic solution for u

$$M - K\omega^2 = 0 \tag{14}$$

where

$$u = u_0 e^{i\omega t}$$

3 Material properties used

For both the tanks, we are assuming it to be made of concrete and hence used its properties. (Source is wikipedia). Water and its properties are also listed below. For the Chetak Helicopter, the following source was used: ScienceDirect - Helicopter Structure. Below table summarizes all Young's modulus, poisson's ratio and density for all materials.

Table 1: Material Properties for Concrete, Water, and Helicopter

Material	Young's Modulus (E) [GPa]	Poisson's Ratio (ν)	Density [kg/m^3]
Concrete	210	0.3	2400
Water	0.126	0.49	1000
Helicopter	70	0.35	2800

where E for water is calculated using Bulk's modulus of 2.1 x 10^9 and nu as 0.49 (not 0.5 otherwise we cannot solve further)

$$E = 3K(1 - 2 * \nu)$$

4 Approach on getting geometry and mesh details

For getting the geometry, me and one of my friend (Shrey patel) went near the tank and measured using basic principles of trigonometry like angle of elevation and distance by walking foot by foot. And also used some apps, to get approx dimensions. I have used the dimensions to make the mesh in GMSH software. The following figure shows all details and dimensions.

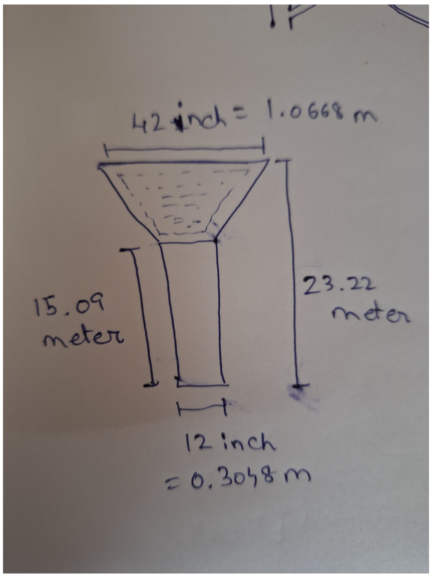


Figure 1: MSB Tank

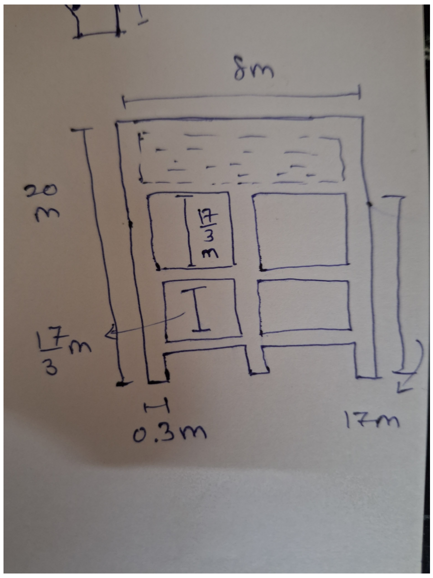


Figure 2: Gym Tank

The details for the helicopter's dimensions were taken from the link given in above section. Following reinstates that.

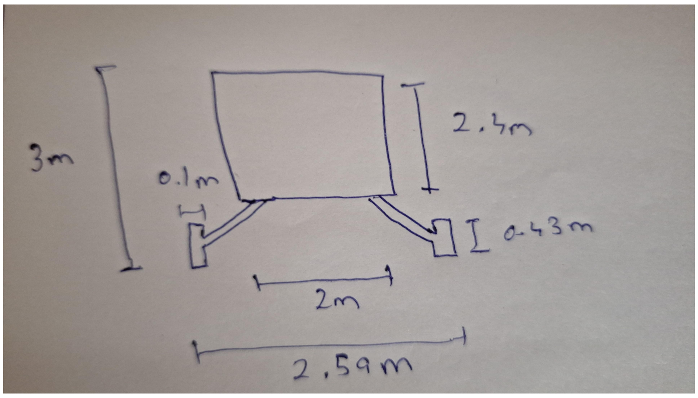


Figure 3: Chetak Helicopter's front view

5 Assumptions made

The following list of assumptions were made in solving:-

1) It is assumed that there will be small deformations, linear elastic, small strains.

2) The 3D problem is converted to 2D since there is symmetry. So, plane stress and plane strain condition can be applied i.e. stress and strain in z direction are zero.

$$\sigma_{xz} = \sigma_{yz} = \sigma_z = 0$$

3) Since we are applying Moment Equilibrium, Material symmetry and all materials are isotropic. There are only two constants. (E, γ)

$$Elasticity\ Matrix = (\frac{E}{1-\gamma^2}) \begin{bmatrix} 1 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & \frac{1-\gamma}{2} \end{bmatrix}$$

4) γ of water is 0.5 but for calculations purpose it is taken to be 0.49 so that matrix K can be converted.

5) Interanl structures like stairs, pipe are assumed to have negligible impact, allowing tank to be modelled as a hollow structure from inside.

6) The helicopter's effect is analyzed using a cross-section containing maximum structural information, despite the lack of perfect symmetry about the corss-sectional plane.

7) The helicopter structure is treated as a single, homogeneous material for simplification.

6 Mesh Convergence study

The K and F matrix are evaluated using Compute function, the formulas for which were given in weak form section.

The Dirichlet boundary condition is applied by putting u and v zero for all points with y = 0. and taking the rest of matrix and inverting it. We get the displacements of remaining points.

6.1 Static Deflection

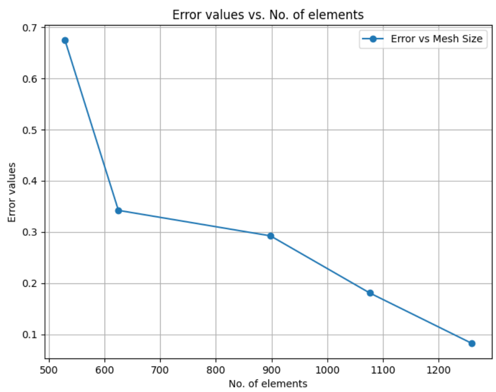
For Mesh convergence, displacement is calculated and error is evaluated at some check points (refer to code for better understanding). Error is given as below:-

$$Error = \sum_{i=1}^{t} \left(\frac{\sqrt{u_i^2 + v_i^2} - \sqrt{u_{i-1}^2 + v_{i-1}^2}}{\sqrt{u_{i-1}^2 + v_{i-1}^2}} \right)^2$$

The following are the 4 cases and mesh convergence details summarizes in the form of tables and graphs.

1) MSB Tank without Helicopter

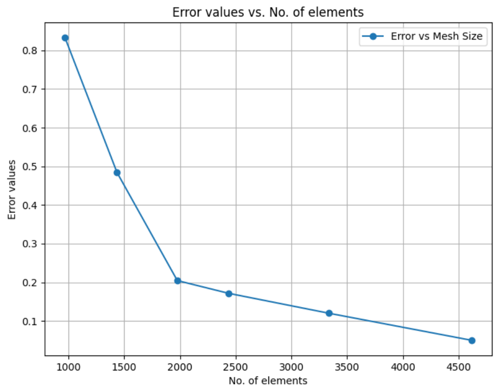
Nodes	Elements	Error
267	405	-
340	529	0.67489812
398	625	0.34223418
553	898	0.29239133
652	1076	0.18095085
755	1259	0.0826182



So, the optimal mesh for this case has **755** nodes and **1259** elements.

2) MSB Tank with Helicopter

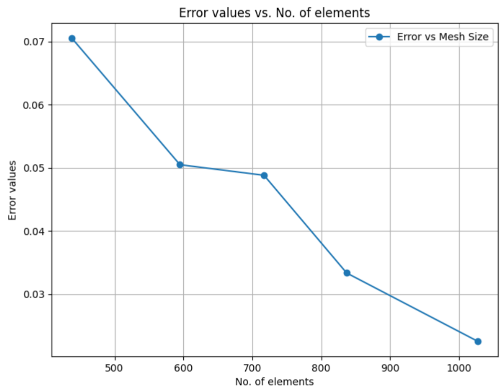
Nodes	Elements	Error
378	577	-
604	968	0.83307516
867	1435	0.48418114
1157	1976	0.20456837
1415	2436	0.17160977
1899	3332	0.12029223
2583	4615	0.05011922



So, the optimal mesh for this case has **2583** nodes and **4615** elements.

3) Gym Tank without Helicopter

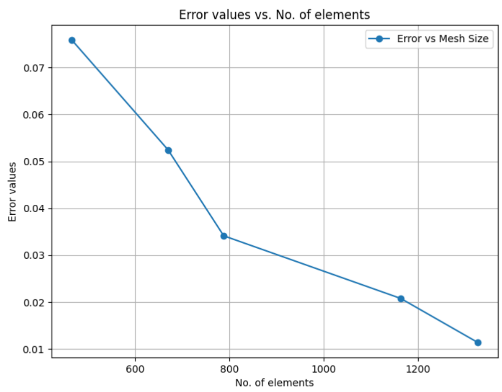
Nodes	Elements	Error
203	249	-
322	438	0.07056179
425	595	0.05051879
515	717	0.04884392
587	837	0.03337203
704	1027	0.02257178



So, the optimal mesh for this case has **704** nodes and **1027** elements.

4) Gym Tank with Helicopter

Nodes	Elements	Error
231	282	-
348	466	0.08699985
478	670	0.06142994
568	788	0.04557497
649	930	0.03592725
714	1029	0.01322422



So, the optimal mesh for this case has **714** nodes and **1029** elements.

6.2 Free Vibration

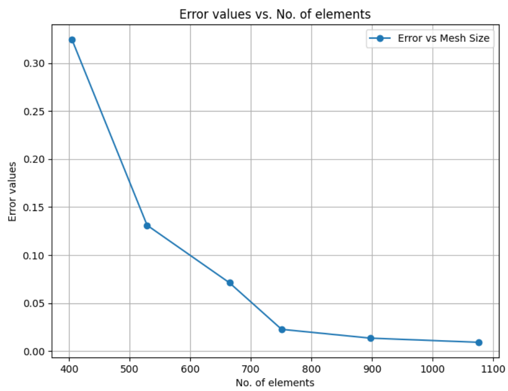
For Mesh Convergence, frequencies are calculated by solving eigenvalue problem and error is evaluated for first 2 fundamental frequencies. Error is given as below:-

$$Error = \sum_{i=1}^{2} \left(\frac{freq[i] - freq\_prev[i]}{freq\_prev[i]} \right)^2$$

The following are the 4 cases and mesh convergence details summarizes in the form of tables and graphs.

1) MSB Tank without Helicopter

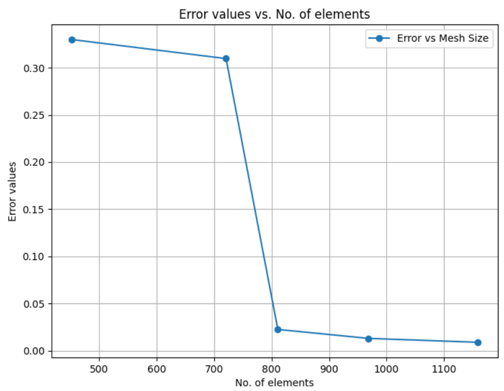
Nodes	Elements	Error
177	258	-
267	405	0.32474537
340	529	0.131118
420	665	0.07109899
471	751	0.02255868
553	898	0.01335891
652	1076	0.00905251



So, the optimal mesh for this case has **652** nodes and **1076** elements.

2) MSB Tank with Helicopter

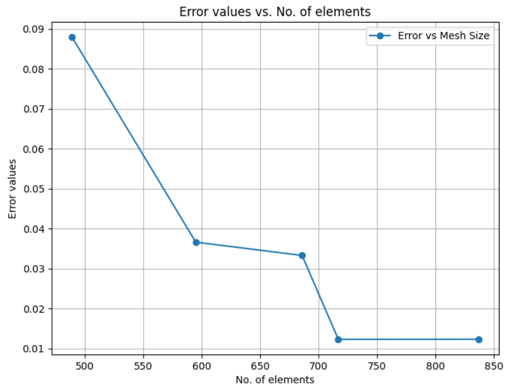
Nodes	Elements	Error
202	288	-
305	453	0.32985878
463	721	0.30965369
517	811	0.2261151
604	968	0.01314387
711	1158	0.00898908



So, the optimal mesh for this case has **711** nodes and **1158** elements.

3) Gym Tank without Helicopter

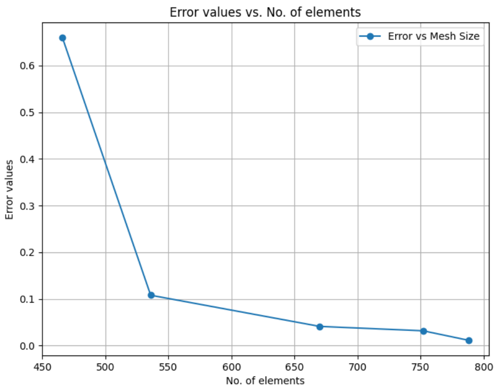
Nodes	Elements	Error
305	409	-
361	489	0.08796711
425	595	0.03660133
486	686	0.03326777
515	717	0.01226421
587	837	0.01225789



So, the optimal mesh for this case has **587** nodes and **837** elements.

4) Gym Tank with Helicopter

Nodes	Elements	Error
219	266	-
348	466	0.65986189
398	536	0.10806815
478	670	0.04128737
535	752	0.03188952
568	788	0.01155597



So, the optimal mesh for this case has **568** nodes and **788** elements.

7 Max Stress and Fundamental frequencies

The following table summarizes maximum stress and locations in different configurations.

Table 2: Maximum Stress and Coordinates for Different Configurations

Configuration	Maximum Stress (Pa)	Coordinates (x, y)
MSB Tank without Helicopter	97146.31055263	(-0.03252174 15.32202899)
MSB Tank with Helicopter	182677.79348607	(-1.69114545 18.38278788)
Gym Tank without Helicopter	36728.41233698	(4.19824561 19.9)
Gym Tank with Helicopter	58508.00418202	(7.9, 1.27678889)

1) MSB Tank without Helicopter

Table 3: First Five Frequencies

Index	Frequency
1	$-7.10495785 \times 10^{-7}$
2	$1.62217237 \times 10^{-7}$
3	$1.62217237 \times 10^{-7}$
4	2.44230401×10^{3}
5	6.34976564×10^{3}

2) MSB Tank with Helicopter

Table 4: First 5 Frequencies

Index	Frequency
1	$-6.40554233 \times 10^{-7}$
2	$1.06123059 \times 10^{-7}$
3	$4.32831564 \times 10^{-7}$
4	2.13645886×10^{3}
5	6.33950333×10^{3}

3) Gym Tank without Helicopter

Table 5: First 5 Frequencies

Index	Frequency
1	$-2.35516181 \times 10^{-7}$
2	$1.39259926 \times 10^{-8}$
3	$2.00364147 \times 10^{-6}$
4	1.65952927×10^{4}
5	1.90923365×10^{4}

4) Gym Tank with Helicopter

Table 6: First 5 Frequencies

Index	Frequency
1	$8.57774193 \times 10^{-8}$
2	$1.87481167 \times 10^{-8}$
3	$2.08230722 \times 10^{-8}$
4	2.42794561×10^{4}
5	2.49987258×10^{4}

8 Contour Plots

The following plots are for displacement field of magnitude.

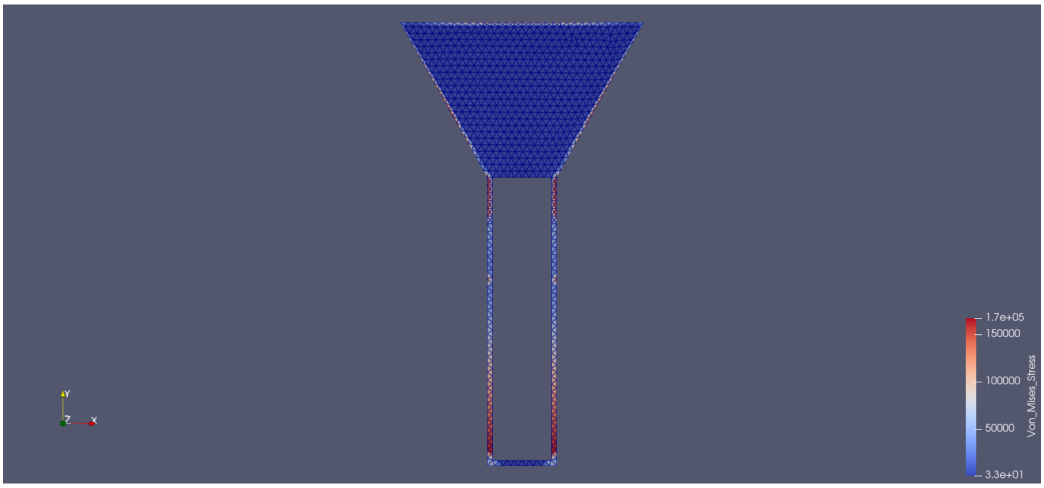


Figure 4: MSB Tank without Helicopter-von-mises equivalent stresses.

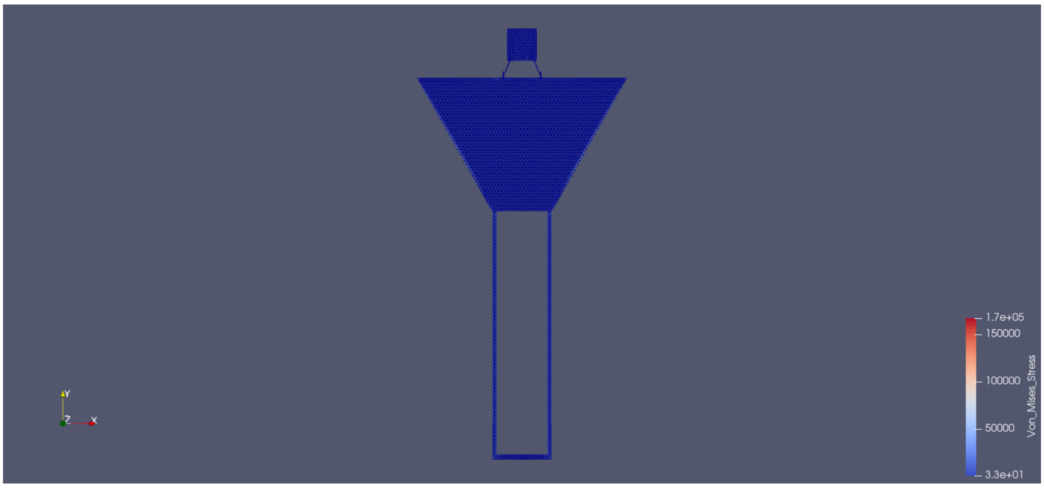


Figure 5: MSB Tank with Helicopter-von-mises equivalent stresses.

9 Mode shapes of tanks

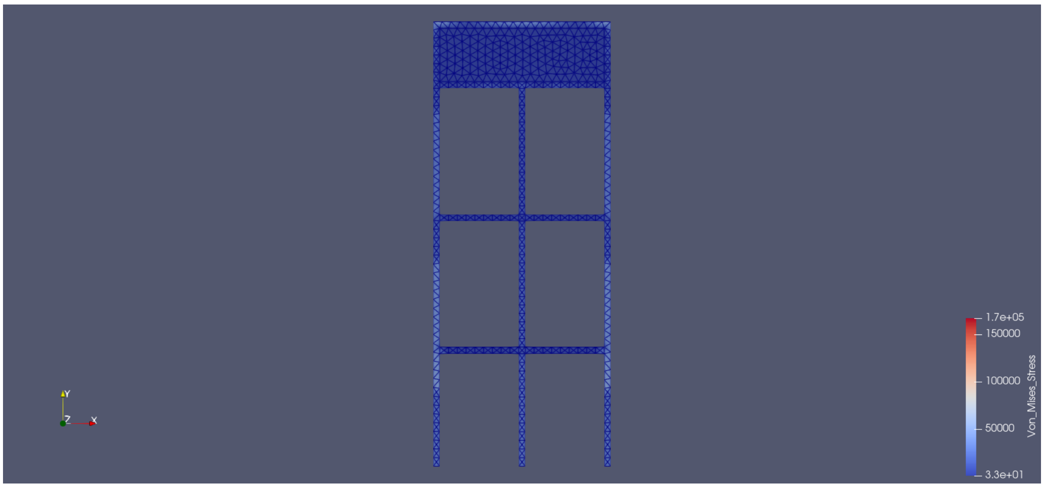


Figure 6: Gym Tank without Helicopter-von-mises equivalent stresses.

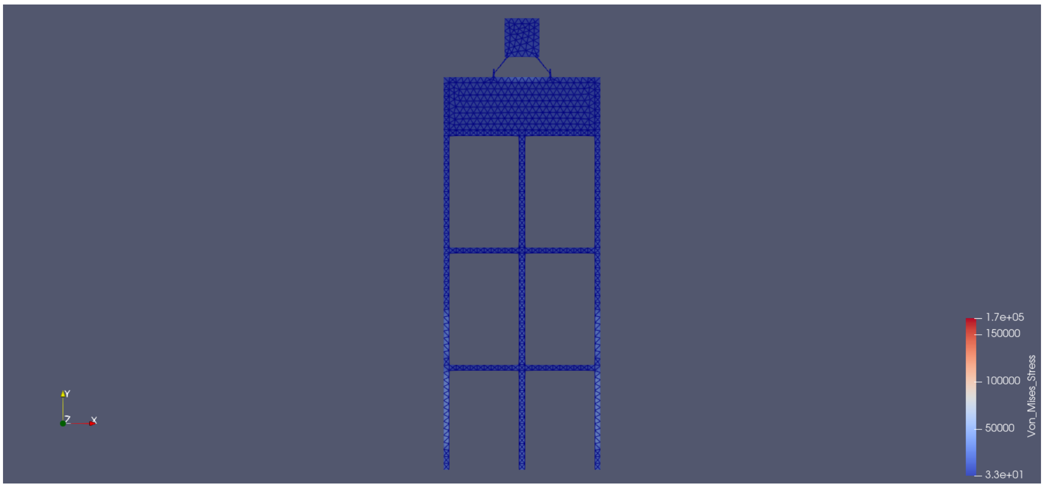


Figure 7: Gym Tank with Helicopter-von-mises equivalent stresses.

10 Impact of parameters due to Helicopter

1) Maximum Stress

For the MSB tank, the maximum stress has increased from 0.97 MPa to 1.82 MPa which twice as high as possible. Whereas for the gym tank, stress has increased from 0.367 MPa to 0.585 MPa which is better than MSB tank. This changes have occured due to added weight of Helicopter.

2) First Fundamental Frequency

For the MSB tank, the first fundamental frequency has increased by nearly 11 % whereas for the Gym tank, frequency has increased by about 4.64 times. Again this is the effect of dynamic effects of

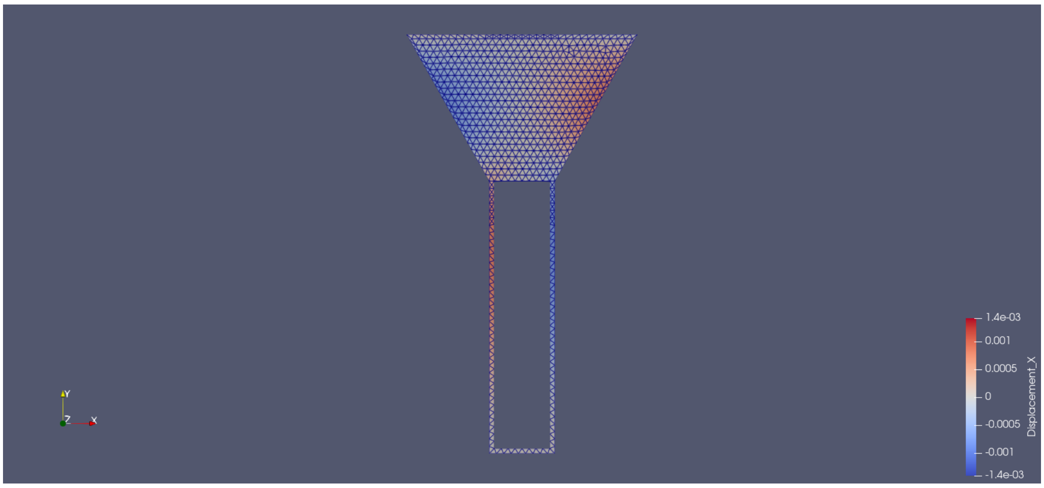


Figure 8: MSB Tank without Helicopter-disp in x direction

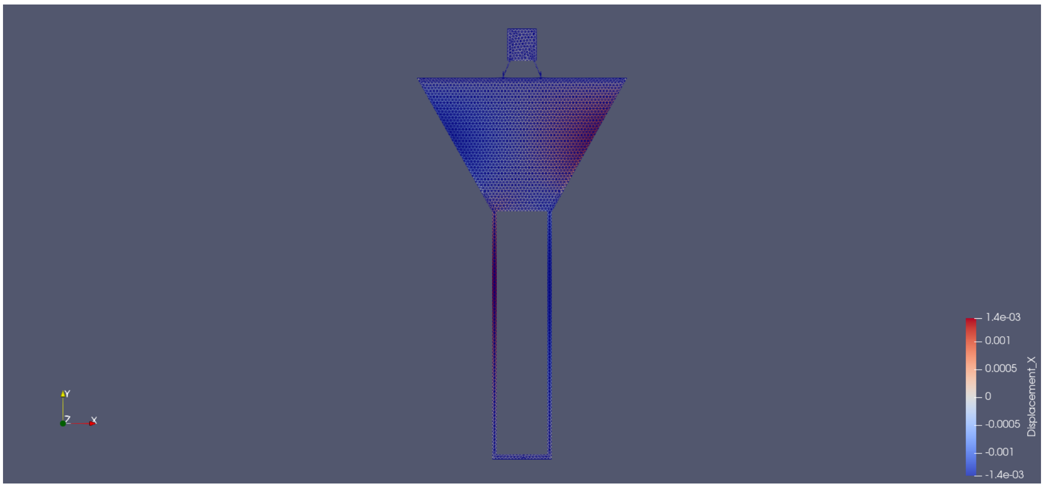


Figure 9: MSB Tank with Helicopter-disp in x direction

Helicopter. This increase in frequency trend is carried on for further nodes as well. So, Gym tank is more susceptible to reaching resonance frequency than MSB tank.

3) Mode Shape

Regarding the mode shapes, MSB tank has shown significant results or precisely increase in rotational motion after the Helicopter was added, indicating a shift in dynamic response of structure. Whereas Gym tank has shown a change in direction of rotation compared to its earlier mode shapes.

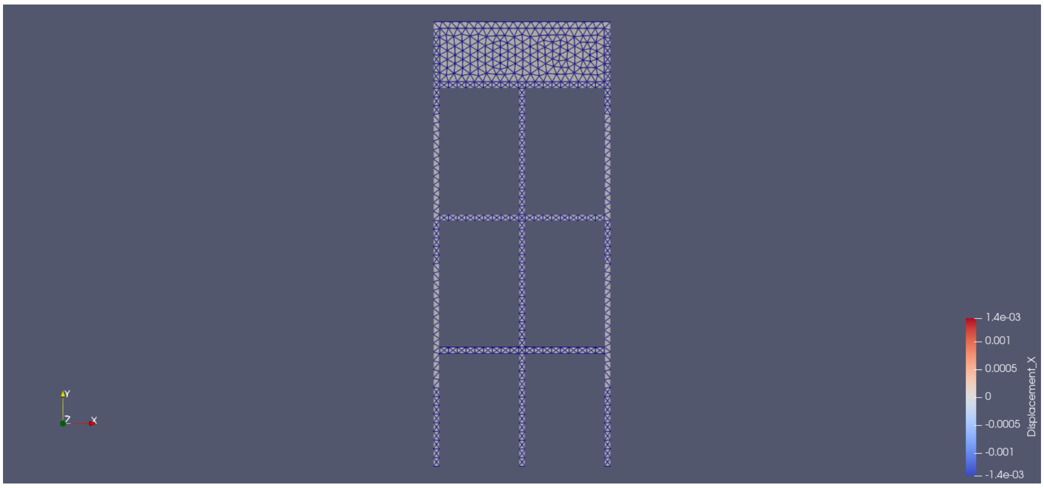


Figure 10: Gym Tank without Helicopter-disp in x direction

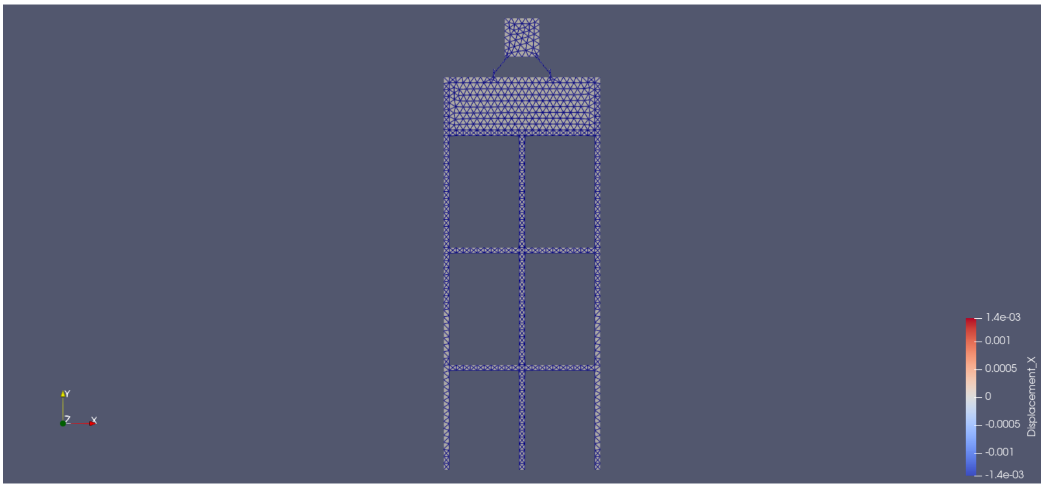


Figure 11: Gym Tank with Helicopter-disp in x direction

11 Choice of Water tank

11.1 Stress Analysis

The Gym Tank has shown significantly lower levels of stress with stress being near to 60 % on the other side stress level had gone up by 188 % in MSB Tank. This shows that Gym Tank is more efficient and less favourable to failure under given circumstances.

11.2 Accessibilty

In terms of Accessibilty, the MSB tank (42 m) has larger dimensions nearly five times the radius of Gym tank (8 m). This helps in the safe

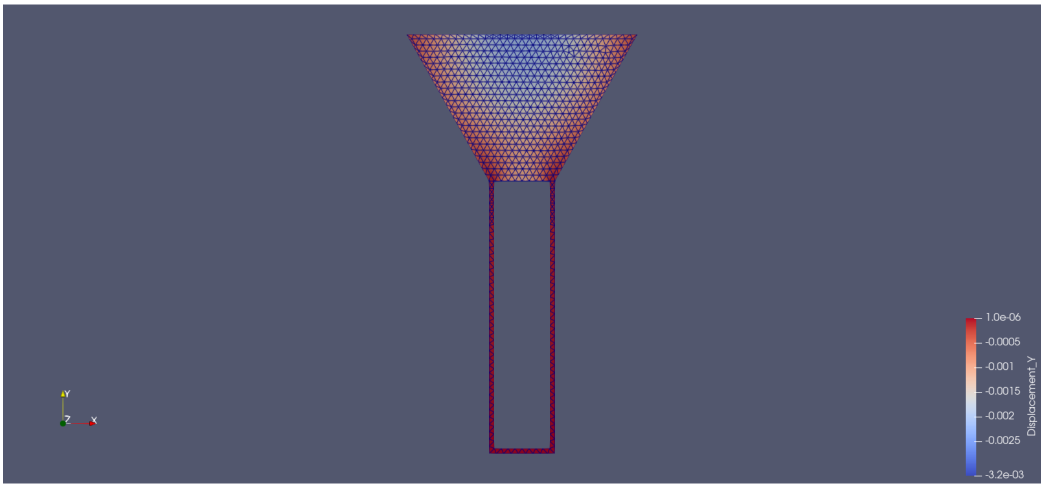


Figure 12: MSB Tank without Helicopter-disp in y direction

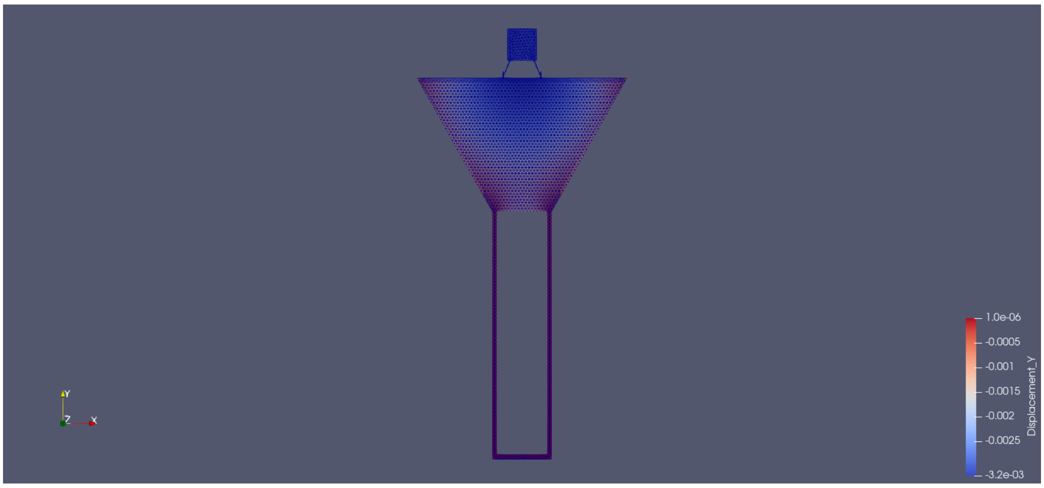


Figure 13: MSB Tank with Helicopter-disp in y direction

landing of Helicopter. But the helicopter's dimensions are small as well so gym tank might work as well.

11.3 Mode Shapes

We have seen that Gym tank is more viable to reach resonant frequency so MSB tank is favourable in that case.

In Conclusion, Gym tank can be a good choice by taking into considerations its stress management. And since Helicopter dimensions are comparable or smaller than Gym tank, it is a feasible choice from the point of view of accessibilty (not the best because that is MSB tank) as well. If only Accessibilty is our priority then, MSB tank turns out

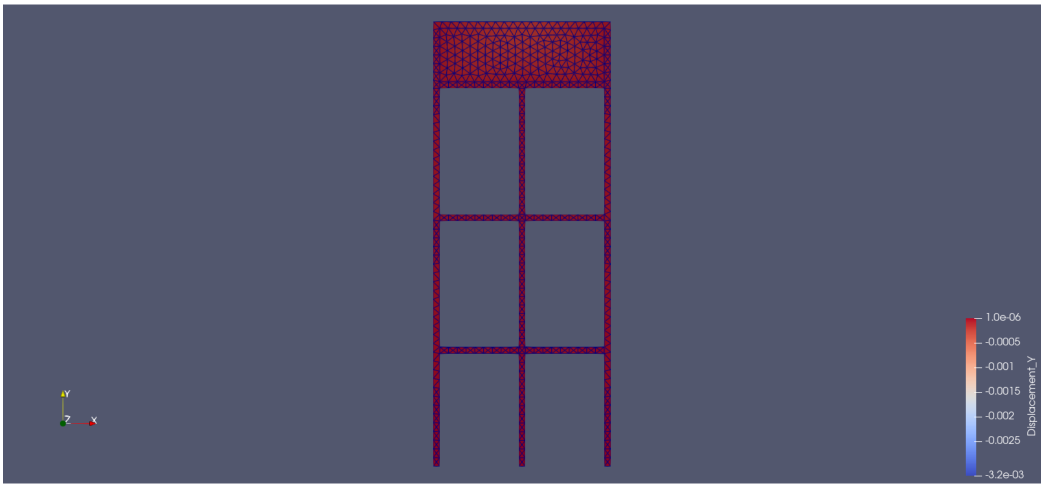


Figure 14: Gym Tank without Helicopter-disp in y direction

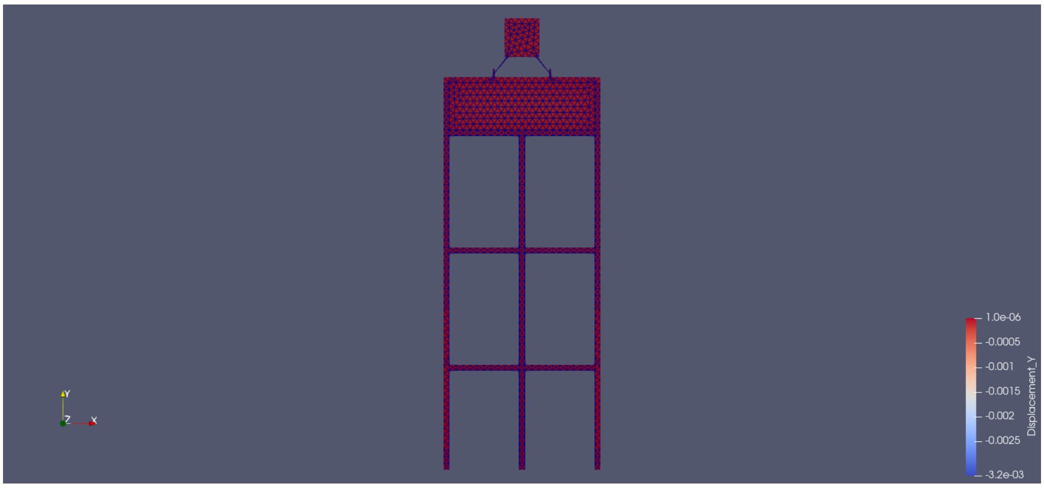


Figure 15: Gym Tank with Helicopter-disp in y direction

to be a good choice.

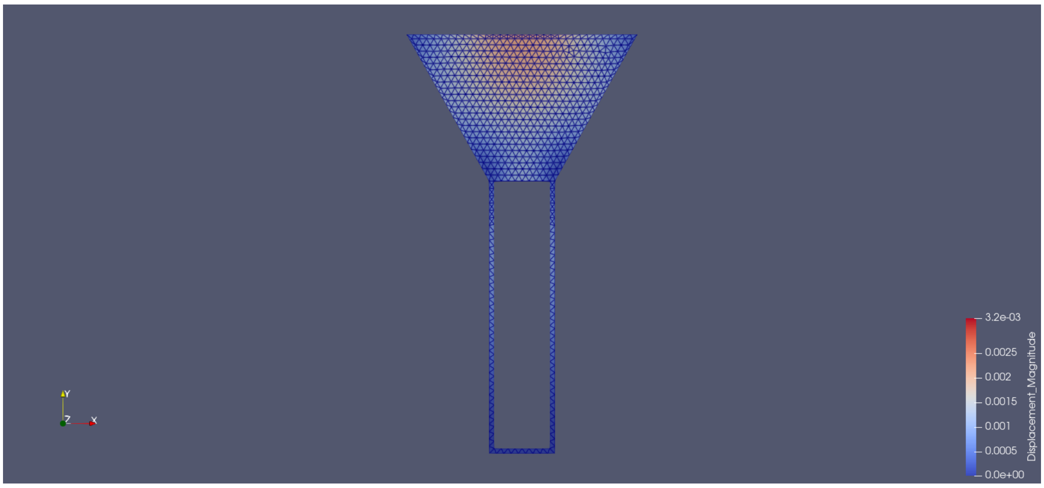


Figure 16: MSB Tank without Helicopter-disp magnitude

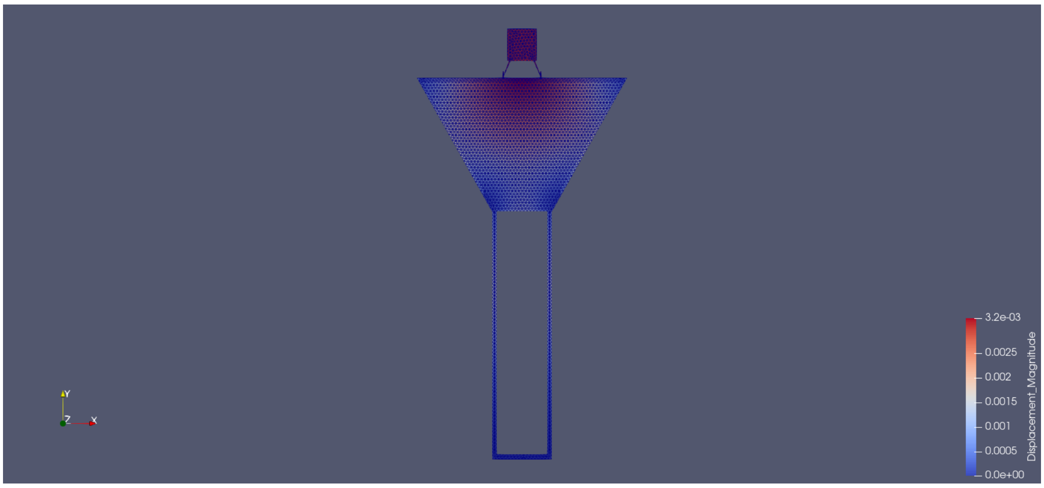


Figure 17: MSB Tank with Helicopter-disp magnitude

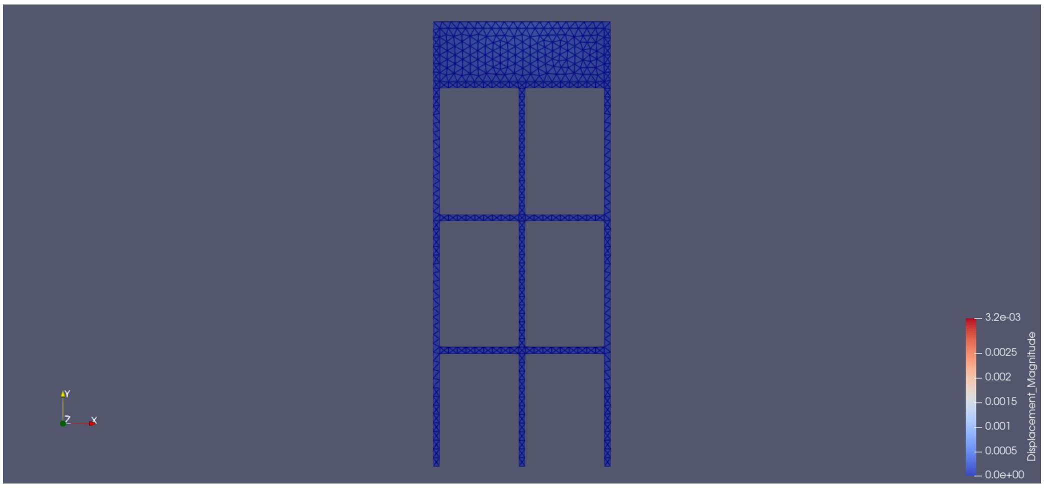


Figure 18: Gym Tank without Helicopter-disp magnitude

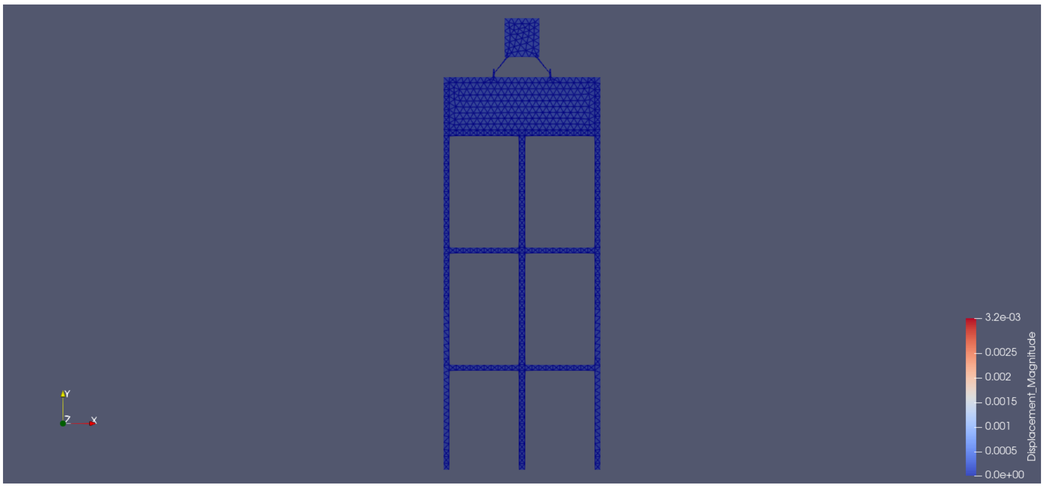


Figure 19: Gym Tank with Helicopter-disp magnitude

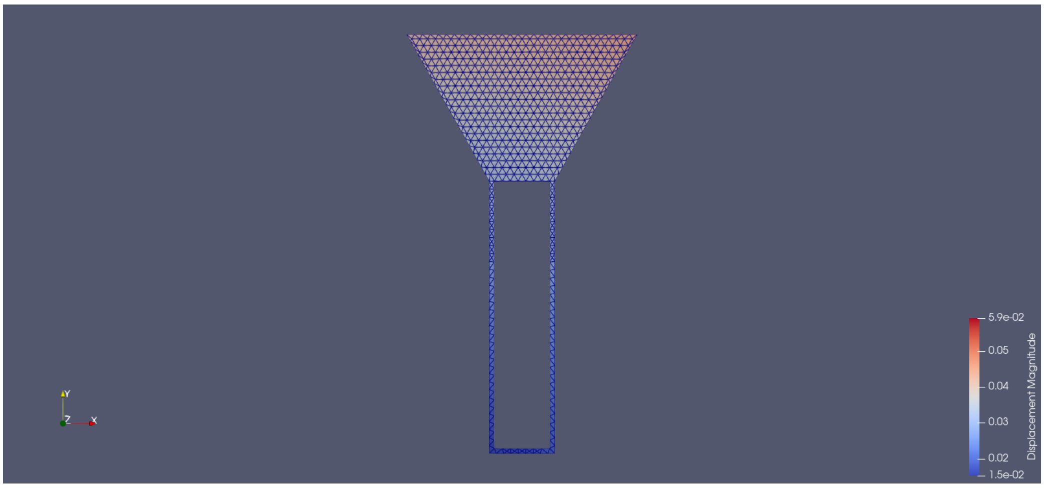


Figure 20: MSB Tank without Helicopter - 1st Frequency

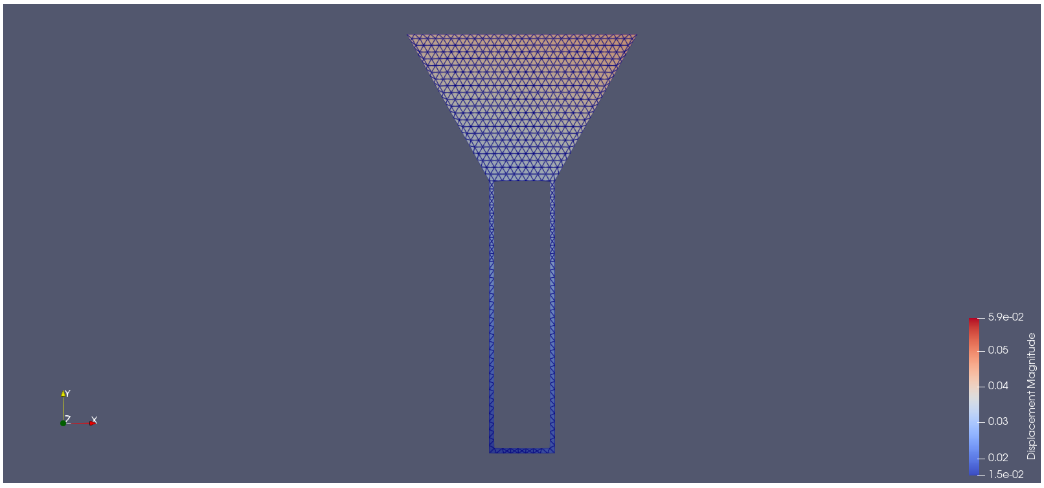


Figure 21: MSB Tank without Helicopter - 2nd Frequency

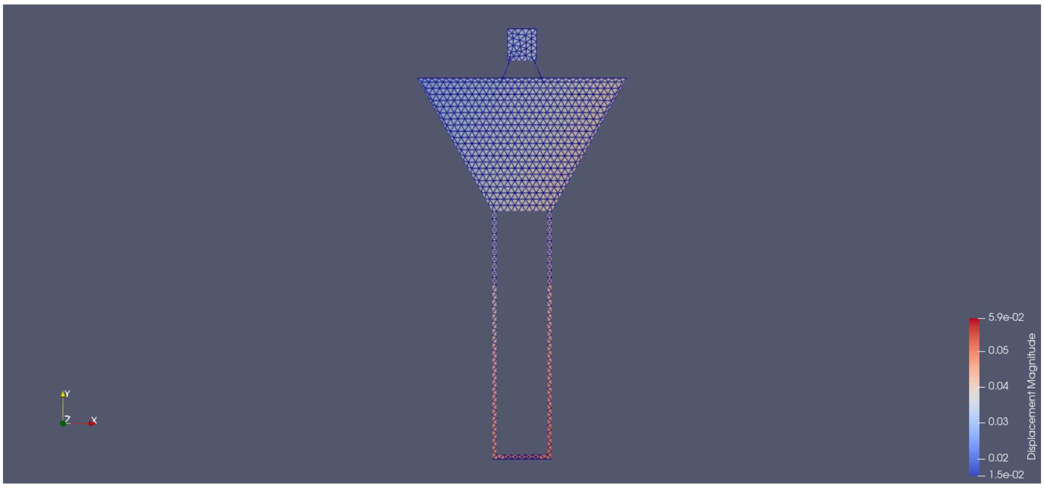


Figure 22: MSB Tank with Helicopter - 1st Frequency

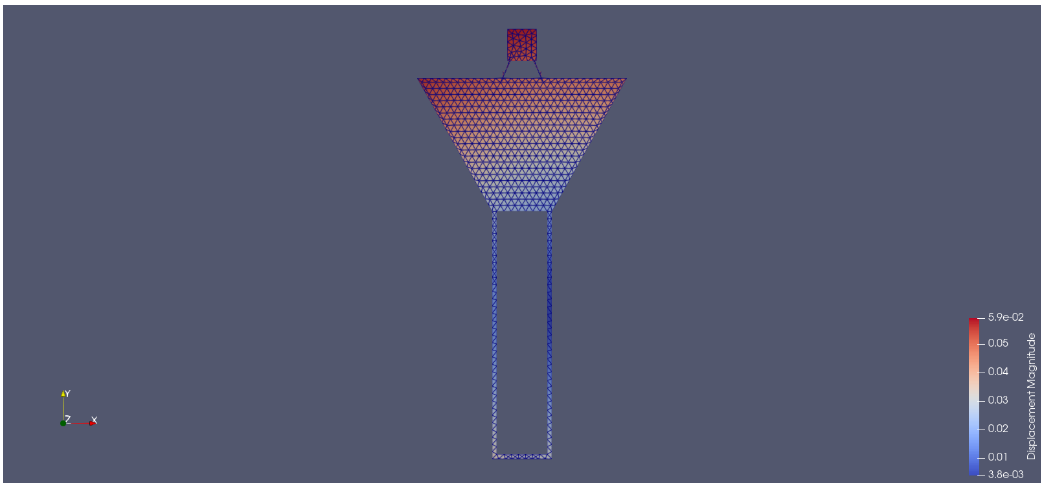


Figure 23: MSB Tank with Helicopter - 2nd Frequency

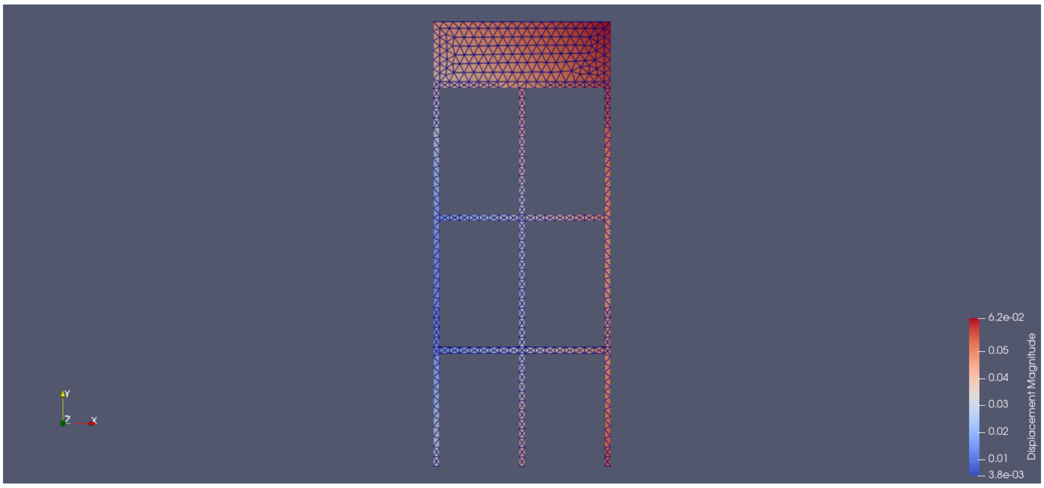


Figure 24: Gym Tank without Helicopter - 1st Frequency

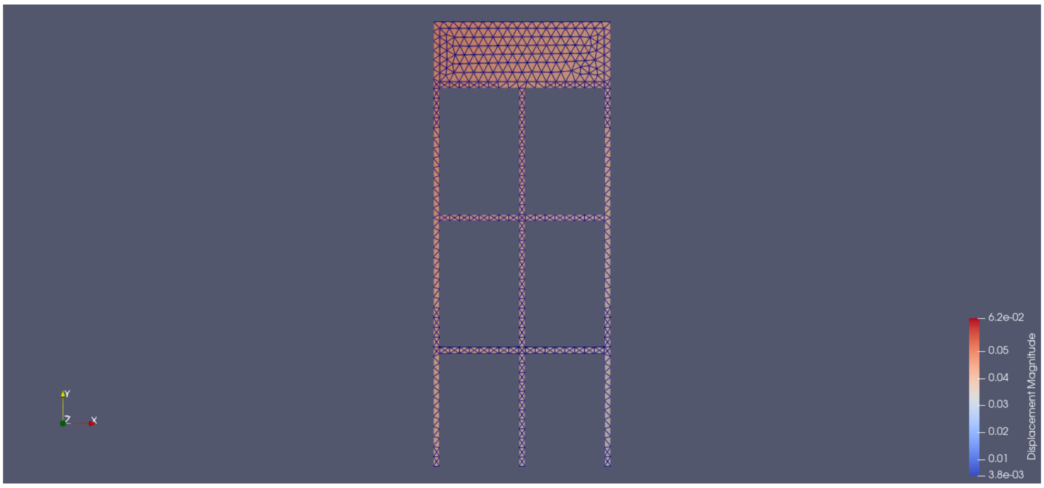


Figure 25: Gym Tank without Helicopter - 2nd Frequency

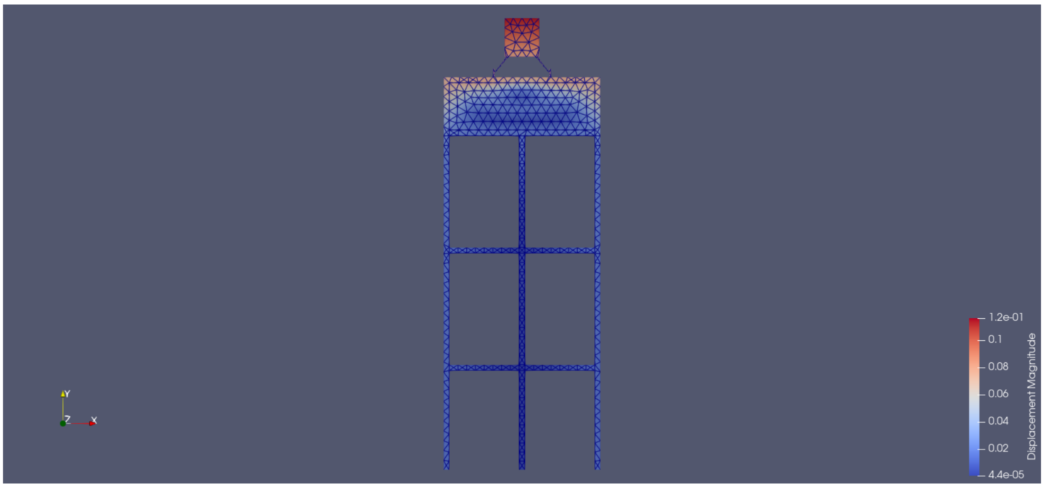


Figure 26: Gym Tank with Helicopter - 1st Frequency

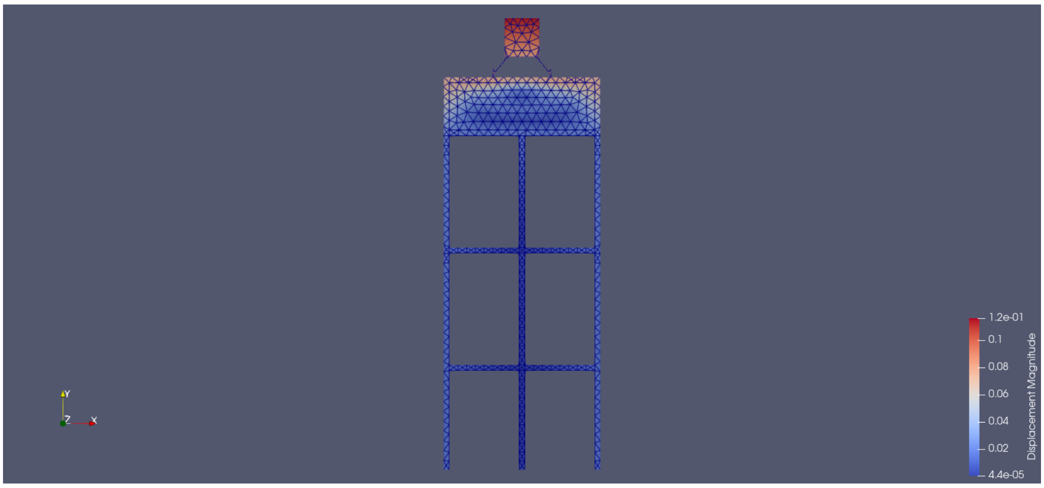


Figure 27: Gym Tank with Helicopter - 2nd Frequency