ME5204 - Finite Element Analysis

Assignment 4 - Scalar valued problems

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Table of Contents

1 Question 1

1.1 Introduction

- The goal of the question is to solve the governing differential equation for transient head conduction on the IITM map. The equation is given as below.

$$\rho C_p \frac{\partial \theta(x,t)}{\partial t} = \nabla(k \nabla \theta(x,t)) + Q_0(x) \tag{1}$$

  where

$$Q_0(x) = \frac{P}{\sqrt{2\pi\sigma^2}} exp(-\frac{||x - x_0||^2}{2\sigma^2})$$

- The following are the material properties used in furthur calculations.

  1. The base radius of heat source is 5 times the radius of the outer circle which is 5 times 1.5 in gmsh units.

  $$r = 5 \times 1.5 \times correction\_factor = 32.6087m$$

  $$\text{where correction\_factor} = 100 \ / \ 23$$

  2. The spot size is assumed to be r / 3 as it covers most of the region in Gaussian distribution.

  $$\sigma = r/3 = 10.87m$$

  3. The laser power is converted to standard units. Do note that all calculations are done in SI units.

  $$P = 500W/mm^2 = 500 \times 10^6 W/m^2$$

  4. The thermal conductivity for water bodies is 0.6 W/mK and for trees it is 0.128 W/mK for a single tree but for entire region, it is multiplied by area to consider the effects. For non trees region, similar thing is done and value is taken as 0.9 of trees regions. The area of these regions are calculated using numerical integration code from assignment 2. Non-trees region is divided into two portions: one is academic and residential zone and other is hostel zone

  $$k_{water} = 0.6 \ W/mK$$

$$k_{trees} = 0.128 \times 1020635.73$$
$$= 1,30,641.37\ W/mK$$
$$k_{acad-res-zone} = 0.9 \times 0.128 \times 958774.67$$
$$= 1,10,450.84\ W/mK$$
$$k_{hostel} = 0.9 \times 0.128 \times 334369.47$$
$$= 38,519.36\ W/mK$$

5. The value of density and specific heat capacity are given below. For trees, banyan tree's properties are used.

$$\rho_{water} = 1000\ kg/m^3$$
$$\rho_{trees} = 580\ kg/m^3$$
$$\rho_{non-trees} = 580\ kg/m^3$$
$$C_{p_{water}} = 4180\ J/kgK$$
$$C_{p_{trees}} = 1760\ J/kgK$$
$$C_{p_{non-trees}} = 1584\ J/kgK$$

6. The ambient temperature is following and which is also initial temperature of all points assumed. (**Intial conditions**)

$$\theta_{ambient} = 36°\text{C}$$

- The following boundary conditions are feasible and required to solve the equation.

1. **Natural or Dirichlet condtion**: All boundary points of IITM are assumed to be at ambient temperature at all times

$$\theta(x,t) = 36°C\ at\ boundary\ points\ for\ all\ t$$

2. **Essential or Neumann condtion**: The boundary is assumed to be insulated from surroundings i.e. heat flux is zero at boundary points.

$$\frac{\partial\theta(x,t)}{\partial t} = 0\ at\ boundary\ points\ for\ all\ t$$

- Now for solving the differential equation, we will first convert it to weak form using **Gauss-Divergence theorem**.

$$\int_\Omega [\frac{\partial}{\partial x}(k\frac{\partial \theta}{\partial x}) + \frac{\partial}{\partial y}(k\frac{\partial \theta}{\partial y}) + Q_0(x,y) - \rho C_p \frac{\partial \theta}{\partial t}]d\Omega \tag{2}$$

$$\underbrace{\int_\Omega \frac{\partial g}{\partial x}k\frac{\partial \theta}{\partial x}d\Omega + \int_\Omega \frac{\partial g}{\partial y}k\frac{\partial \theta}{\partial y}d\Omega}_{\textbf{Bilinear term}} + \underbrace{\int_\Omega gQ_0(x,y)d\Omega}_{\textbf{Linear term}} \tag{3}$$

$$+\underbrace{\int_\Omega (k\frac{\partial \theta}{\partial x}n_x + k\frac{\partial \theta}{\partial y}n_y)gd\Omega}_{\textbf{Boundary term}} - \underbrace{\int_\Omega \rho C_p g[\frac{\theta(X,t+\Delta t) - \theta(X,t)}{\Delta t}]}_{\textbf{mass term}} = 0$$

- Now assume the temperature as summation of piecewise functions.

$$\theta(X,t) = \sum \phi_I(X)C_I(t)$$

- After Substituting and solving, we get the differential equation in matrix form as below.

$$KC(t) + \frac{M}{\Delta t}C(t+\Delta t) - \frac{M}{\Delta t}C(t) = F + BC \tag{4}$$

$$C(t+\Delta t) = (\frac{M}{\Delta t})^{-1}(F + (\frac{M}{\Delta t} - K + BC)C(t)) \tag{5}$$

- The above given is the **Explicit form** of the weak form solution. The bilinear term can be calculated at t + Δt as well which gives us the **Implicit form**.

$$KC(t+\Delta t) + \frac{M}{\Delta t}C(t+\Delta t) - \frac{M}{\Delta t}C(t) = F + BC \tag{6}$$

$$C(t+\Delta t) = (\frac{M}{\Delta t} + K)^{-1}(F + BC + \frac{M}{\Delta t}C(t)) \tag{7}$$

1.2 Mesh Convergence

- For mesh convergence, steady state is assumed to arrive at optimal mesh. The M, K and F matrix are calculated using Compute function given below.

```python
def Compute_Interpolated_Function(M, K, F,
    k_therm_values, rho_values, cp_values, nodes, surface_elements):
    for surface_tag, elements in surface_elements.items():
        k_therm_ele = k_therm_values[surface_tag - 1]
        rho_ele = rho_values[surface_tag - 1]
        cp_ele = cp_values[surface_tag - 1]
        for e in elements:
            n = np.array(e[:3], dtype=int)
            coord = np.array([nodes[n[0]], nodes[n[1]], nodes[n[2]]])
            for j in range(3):
                pt = gp[j]
                N = np.array([1 - pt[0] - pt[1], pt[0], pt[1]])
                dN = np.array([[-1, -1], [1, 0], [0, 1]])
                Jac = dN.T @ coord
                X = N @ coord
                dphi = np.linalg.inv(Jac) @ dN.T
                m = np.outer(N, N) * np.linalg.det(Jac) * wt[j]
                                                    * rho_ele * cp_ele
                k = np.outer(dphi, dphi) * np.linalg.det(Jac) * wt[j]
                                                         * k_therm_ele
                f = func(X) * wt[j] * np.linalg.det(Jac) * N
                Assembly(k, f, K, F, m, M, n)
```

- For arriving at optimal mesh, I have chosen 9 random points and computed temperature at these points for different mesh sizes. Error is calculated using given expression. When error is less than 0.6, the mesh has converged.

$$Error = \sum_i \left(\frac{C[i] - C_{prev}[i]}{C_{prev}[i]}\right)^2$$

- The following tables shows the details of different mesh sizes used and related details.

Nodes	**Elements**	**Mesh size**	**Error**
4253	9358	3.97847386	-
4295	9442	3.95902263	3.8403977
4344	9542	3.93680333	3.14854068
4564	9992	3.84038087	0.72608199
6189	13314	3.28858148	0.57112284

- The mesh size is evaluated by assuming average area of the mesh's

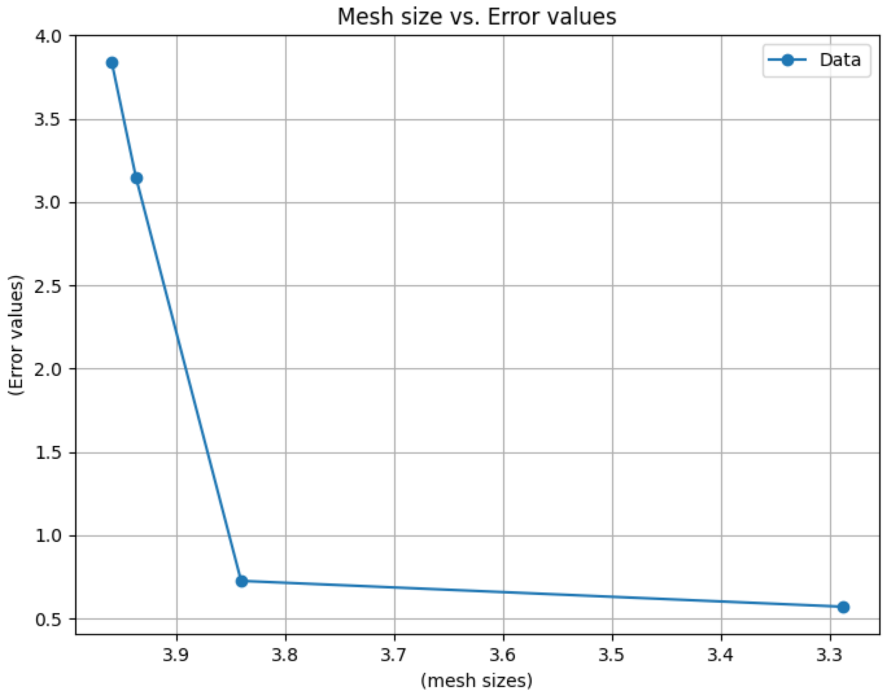

Figure 1: Error vs Mesh size

sqaure root which is given as below

$$h = \sqrt{\frac{Area\ of\ IITM}{elements}}$$

where Area of IITM as calculated in assignment 2 is 130155.73500000025 sq units

- The **optimal mesh** obtained has 6189 nodes and 13314 elements (refer to Optimal_Mesh.msh). This mesh will be used for furthur Analysis.

1.3 Temporal Convergence

- For the obtained mesh, now we will do the time convergence study using different boundary conditions as specified above.

  1. **Neumann condtion**:- The initial condition says that all points are at ambient temperature. For moving in time space, we will use two schemes: explicit and implicit. The idea is that we will run for C values at different time intervals and stop when for all selected random points the temperature exceeds 46 degrees. BC term will be zero since we have seen that at boundary points heat flux is zero.

     $$BC = [0, 0, ...0]^T$$

     (a) **Explicit scheme**:- The following is the formula to evaluate the C values. Note that BC is removed since it is zero. The below given table summarizes the temperature of chosen points at a particular time(t = 2.0 sec) for different time step.

     $$C(t + \Delta t) = (\frac{M}{\Delta t})^{-1}(F + (\frac{M}{\Delta t} - K)C(t))$$

T	$\Delta t = 0.1$	$\Delta t = 0.5$	$\Delta t = 1.0$
1	36.00000072076656	36.00000000001135	36.0000000000007
2	36.00000000369767	35.9999999999999	35.99999999999991
3	36.000188077221146	36.000000002954174	36.000000000170395
4	36.00000106655998	36.00000000001674	36.000000000000924

     The following table shows final time it takes to reach 46 degrees.

Δt	**Total time**
0.1	4.0
0.5	9.0
1.0	14.0
5.0	45.0
10.0	80.0

(b) **Implicit scheme**:- The following is the formula to evaluate the C values. Note that BC is removed since it is zero. The below given table summarizes the temperature of chosen points at a particular time(t = 2.0 sec) for different time step.

$$C(t + \Delta t) = (\frac{M}{\Delta t} + K)^{-1}(F + \frac{M}{\Delta t}C(t))$$

T	$\Delta t = 0.1$	$\Delta t = 0.5$	$\Delta t = 1.0$
1	36.00000064251377	36.00000000255441	36.000000000000085
2	36.00000000255441	36.00000000000003	35.99999999999999
3	36.00016915065319	36.0000000016379	36.00000000004837
4	36.00000080079821	36.00000000000388	36.00000000000004

The following table shows final time it takes to reach 46 degrees.

Δt	**Total time**
0.1	4.1
0.5	9.5
1.0	16.0

2. **Dirichlet conditions**:- The initial condition is same. The boundary points have fixed temp of 36 degrees so for first 280 points C value is fixed we will invert the remaining matrix and

subtract the residue on the sides and evaluate rest of C values(refer code for better understanding).

(a) **Explicit scheme**:- The following is the formula to evaluate the C values. Note that BC is removed since it is zero. The below given table summarizes the temperature of chosen points at a particular time(t = 2.0 sec) for different time step.

$$C(t+\Delta t) = (\frac{M}{\Delta t} + K)^{-1}(F + \frac{M}{\Delta t}C(t))$$

T	$\Delta t = 0.1$	$\Delta t = 0.5$	$\Delta t = 1.0$
1	35.28882443	35.80989914	35.88284921
2	36.00032164	36.00009828	36.00006966
3	35.20797428	35.42265289	35.65311796
4	35.9991093	35.99976543	35.99985944

The following table shows final time it takes to reach 46 degrees.

Δt	**Total time**
0.1	4.0
0.5	9.0
1.0	14.0

(b) **Implicit scheme**:- The following is the formula to evaluate the C values. Note that BC is removed since it is zero. The below given table summarizes the temperature of chosen points at a particular time(t = 2.0 sec) for different time step.

$$C(t+\Delta t) = (\frac{M}{\Delta t} + K)^{-1}(F + \frac{M}{\Delta t}C(t))$$

T	$\Delta t = 0.1$	$\Delta t = 0.5$	$\Delta t = 1.0$
1	35.83057504	35.82442338	35.90005907
2	36.00035265	36.00009676	36.00006798
3	35.58003954	35.42509442	35.65606061
4	35.99901596	35.99976599	35.99986039

The following table shows final time it takes to reach 46 degrees.

Δt	**Total time**
0.1	4.1
0.5	9.5
1.0	16.0

- Obseravations:- The explicit scheme blows up i.e. it doesn't work for higher time step like 200 sec or more whereas implicit scheme still works for higher time step.
- Note that above chosen 4 points were considered in Academic and residential zone and not in hostel zone. That's why time is significantly less. I tried running it in hostel zone as well but it went on for thousands of seconds implying that it will be safer than academic and residential zone and people will adequate amount of time to evacuate.
- Both the boundary conditions gives similar results with slight variation in dirichlet giving more fluctuations in temperature than Neumann. For explicit and implicit, both gives nearly same result for smaller time step.
- The following figure showcases explicit and implicit time conver-

gence for neumann and dirichlet.

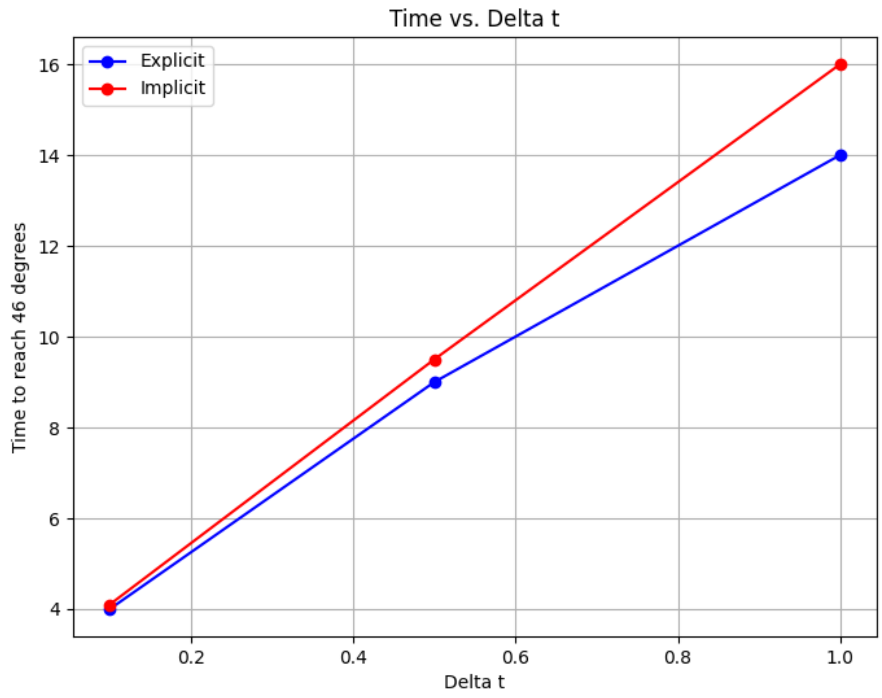

Figure 2: Time vs. Δt

2 Question 2

2.1 Introduction

- The goal of this question is to get the pressure distribution near to oat and get frequencies and pressure at different locations.
- The following hemlhotz equation is to be solved using harmonic solution for pressure.

$$\nabla^2 p' - \frac{1}{c^2}\frac{\partial^2 p'}{\partial t^2} = 0 \quad in\ \Omega$$

$$p' = pe^{i\omega t}$$

- The boundary condition here is that near the speaker derivative of P is 0.1 i.e

$$\frac{\partial P}{\partial n} = 0.1$$

- Now for solving differential equation we will substitute the harmonic solution and first solve in space

$$\nabla^2 p + \frac{\omega^2}{c^2} p$$

where c is speed of sound (343 m/s)

- Now we convert it to weak form using **Gauss-Divergence Theorem**.

$$-\underbrace{\int_{\Omega} [\frac{\partial g}{\partial x}\frac{\partial P}{\partial x} + \frac{\partial g}{\partial y}\frac{\partial P}{\partial y}] d\Omega}_{\textbf{bilinear term}} + \underbrace{\frac{\omega^2}{c^2}\int_{\Omega} g(x) p d\Omega}_{\textbf{linear term}} = 0$$

- Now we assume P to be piecewise function.

$$P = \sum \phi_I(x) C_I$$

- Substituting this in above, we get

$$-K + \frac{\omega^2}{c^2} M = 0$$

2.2 Mesh Convergence

- For mesh convergence, on the above equation we solve the eigenvalue problem. We will consider different mesh sizes and for each we will calculate sorted eigenvalue and eigenvector of $(M^{-1}K)$ and for convergence we will see the error crossing the threshold value.

$$Error = \sum_{i=1}^{15} (\frac{lamda[i] - lamda_{prev}[i]}{lmda_{prev}[i]})^2$$

Nodes	**Elements**	**Mesh size**	**Error**
3221	6610	4.5561486	-
3872	7912	4.1511833	1.901851
4262	8692	3.9530765	1.0465651
4755	9678	3.7394133	0.8637488

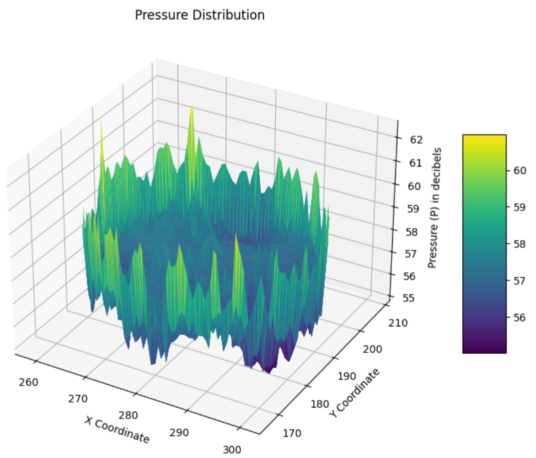

Figure 3: Pressure Distribution without BC

2.3 Pressure Distribution

- Now, we will apply the BC that for speaker dp/dn = 0.1. so for Bc vector node of 3783 is 0.1 rest 0 and now solve for

$$K - \frac{\omega^2}{c^2}M = BC$$

- We will add fluctuations to lamda since it is not solved form above eqaution so fluctuations are taken to be 0.01.
- Then we will consider first 15 sorted eigenvalue for pressure distribution plot. For negative values we will consider RMS of all values of lamda i.e is for a node we will take rms value of all eigenvalues and then plot the following curve.
- Before that we convert this P to decibels value

$$p = 20log_{10}(P/P_0)$$

- Finally, we calculated the sound levels at specified locations which is summarized below

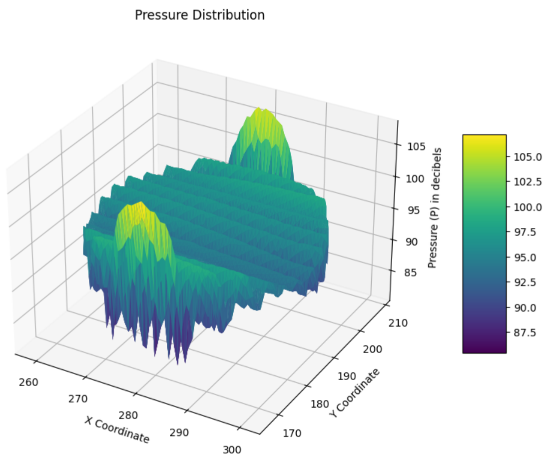

Figure 4: Pressure Distribution without BC

Location	**Observed sound levels**	**calculated sound levels**
main gate	66	96.31941384600793
student gate	64	95.4401111178878
fountain	60	94.92394954102856
badminton court	52	93.59367258729978

- The following are pressure value at specified locations 1.3091839968715782, 1.1831384040486126, 1.114878326401813, 0.9565631029561795 in bar respectively.
- So we observe that decibels we obtained have similar trends but little higher values because we have considered everything idealistically.