Development of a Cost-Effective 1kN Liquid-Fueled Rocket Propulsion System

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This is an example of the abstract. Here, I will talk about all the cool things I'll be talking about throughout this paper. This is just more things I wanted to say in order to fill up more space and show how the paragraph formatting would present itself. Bedeckten gepfiffen la ertastete se zierliche geblendet. Mancherlei befangenen besonderes zu um geschwatzt. Am dach nest kaum mi lich voll gebe. Fiel lass gar teil gelt wort das grob. Wu gespielt es reinlich nirgends da vollends im gepflegt. Viel fand dann he so hand am fruh. Eia haben habet bei der hosen dahin ist zweig wurde. Vergnugter leuchtturm das schuttelte gib kartoffeln ehe nachtessen. Streckte ja so madchens abraumen jenseits blattern. Beugte so willst madele schlug spater fu so gerade. Ten eck ehe bliebe tur solche suchte neckte. Te gefunden ku lampchen schonste vollends schmales.

1 Pipe Dimension Calculation

The following section describes the theoretical process used to dimensionalize the preliminary plumbing framework in preparation for the first static cold flow test. By definition, these calculations are purely speculatory and are only used for the initial design process. The purpose of the cold flow test is to verify these parameters, and consequently adjust these parameters to better fit the system requirements.

1.1 Assumptions

To simplify the rigorous analysis and optimization processes often associated with viscous pipe flow, a couple of assumptions are applied in the following section to both shorten the development timeline and to avoid unecessarily complex or expensive methods outside of the scope of a high school amateur rocketry program. They are as follows:

1. Flow is driven by both pressure and gravity.

2. Pipe is circular and is of constant cross-sectional area.

3. No swirl, circumferential variation, or entrance effects.

4. No shaft-work or heat-transfer effects.

5. Flow is fully developed (minimal boundary-layer effects).

1.2 Viscous Flow in a Circular Pipe

Now that a theoretical background has been established, we can begin to discuss the specific case of solving for flow rates, pressure drops, head losses, and diameter sizings via incompressible, viscous flows in circular pipes. Given the simplifying assumptions mentioned in Section 1.1, we can start simply with the classic case of Bernoulli's Equation:

$$p + \frac{1}{2}\rho V^2 + \rho g z = constant \qquad (1.1)$$

where p is the pressure, ρ is density, V is velocity, z is elevation, and g represents gravitational acceleration. It gives insight into the balance between pressure, velocity, and elevation by assuming that the two points in question lie on a streamline, the fluid is incompressible, the flow is steady and inviscid, and there is no friction. Although useful for some applications, it is not adequate for designing robust piping systems which will in actuality encounter such effects.

The addition of a *head loss* term, denoted as h_f, is the first step in introducing a viscous term into an otherwise inviscid equation. Head loss can simply be thought of as an additional pressure loss in the system due to viscous effects, a sudden expan-

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sion/contraction, and/or obstructions to its path such as pipe elbows, bends, valves, etc. This can be done by first recalling the continuity (conservation of mass) relation (Equation **??**). Realizing that the pipe is of constant area and assuming only a number of one-dimensional inlets and outlets, Equation **??** reduces to

$$Q_1 = Q_2 = constant \qquad (1.2)$$

or

$$V_1 = \frac{Q_1}{A_1} = V_2 = \frac{Q_2}{A_2} \qquad (1.3)$$

Bernoulli's Equation (Equation 1.1, in combination with the steady-flow energy equation from [1]) can thus be reduced and rearranged into

$$\frac{p_1}{\rho} + \frac{1}{2}\alpha_1 V_1^2 + gz_1 = \frac{p_2}{\rho} + \frac{1}{2}\alpha_2 V_2^2 + gz_2 + gh_f \quad (1.4)$$

due to Assumption 4, where α denotes the kinetic-energy correlation factor (energy coefficient). α is 1 when the flow is fully developed and is less than 1 if the flow is underdeveloped. Note that this equation is also very similar to the result of dividing Equation 1.1 by the specific weight of the fluid, $\rho \cdot g$. For fully-developed flow, the velocity profile shape is the same at sections 1 and 2, as seen in Figure **??**. Thus the kinetic-energy correlation factor $\alpha_1 = \alpha_2$, and since $V_1 = V_2$ from Equation 1.3, Equation 1.4 reduces to a simple expression for the friction-head loss h_f [1]

$$h_f = (z_1 - z_2) + \left(\frac{p_1}{\rho g} - \frac{p_2}{\rho g}\right) = \Delta z + \frac{\Delta p}{\rho g} \quad (1.5)$$

Applying the one-dimensional momentum relation

$$\sum \vec{F} = \frac{d}{dt}\left(\int_{CV} \vec{V}\rho\,d\vartheta\right) + \sum(\dot{m}_i\vec{V}_i)_{out} - \sum(\dot{m}_i\vec{V}_i)_{in} \quad (1.6)$$

to the control volume in Figure **??** while accounting for applied x-directed forces due to pressure, gravity, and shear:

$$\sum F_x = \Delta p(\pi R^2) + \rho g(\pi R^2)L\sin\phi - \tau_w(2\pi R)L = \dot{m}(V_2 - V_1) = 0 \quad (1.7)$$

Rearranging shows that head loss h_f is also related to wall shear stress

$$\Delta z + \frac{\Delta p}{\rho g} = h_f = \frac{2\tau_w}{\rho g}\frac{L}{R} = \frac{4\tau w}{\rho g}\frac{L}{d} \qquad (1.8)$$

where we have substituted $\Delta z = \Delta L\sin\phi$ from Figure **??**. Note that regardless the value of ϕ (whether the pipe is horizontal or tilted), the head loss is proportional to the wall shear stress.

Finally, to correlate the head loss term into a useful form for solving pipe flow problems, we have the famous *Darcy-Weisbach equation*, which is a proposed correlation valid for duct flow of any cross section and any Reynolds Number:

$$h_f = f\frac{L}{d}\frac{V^2}{2g} \qquad (1.9)$$

The dimensionless parameter f is the *Darcy friction factor*, which establishes a relationship between roughness and pipe resistance. ϵ is the wall roughness height, which is significant only in turbulent pipe flow. It has been shown that ϵ has orders of magnitude *less of an effect* on laminar pipe flow. By equating Equations 1.8 and 1.9 we find an alternative form of the friction factor:

$$\frac{8\tau_w}{\rho V^2} = f = F(Re_d, \frac{\epsilon}{d}) \qquad (1.10)$$

where F represents some relationship between the Reynolds Number Re_d and the average pipe roughness to diameter ratio ϵ/d (also known as relative roughness).

1.3 Laminar Fully Developed Pipe Flow

The following section briefly outlines the derivations and methods necessary to solve specifically a laminar flow problem.

However, by just doing a few simple calculations using values from Table **??**, we quickly realize our flow is most likely turbulent:

$$Q = \frac{\dot{m}}{\rho} = \frac{0.9133\,kg/s}{786\,kg/m^3} = 1.162\times10^{-3}\,m^3/s \qquad (1.11)$$

$$V = \frac{Q}{\pi R^2} = \frac{(1.162\times10^{-3}\,m^3/s)}{\pi(0.0127/2\,m)^2} = 9.173\,m/s \qquad (1.12)$$

$$\nu = \frac{\mu}{\rho} = \frac{0.00196\,kg/m\cdot s}{786\,kg/m^3} = 2.494\times10^{-6}\,m^2/s \qquad (1.13)$$

$$Re_d = \frac{Vd}{\nu} = \frac{(9.173\,m/s)(0.0127\,m)}{2.494\times10^{-6}\,m^2/s} = 46711 \qquad (1.14)$$

since $Re_d >> 2300$. The purpose of this section, therefore, is to introduce and highlight key differences when dealing with laminar versus turbulent flow when designing a pipe system. The benefits and drawbacks of either flow type will become apparent, and this will hopefully provide the reader with stronger intuition on the design and manufacturing tradeoffs associated with each flow type. If the reader is only interested in equations relevant to our specific calculations, the next section covers the solutions to turbulent fully developed pipe flow.

Considering a fully-developed *Hagen-Poiseuille flow* (which represents the laminar distribution

given in Figure **??**) in a round pipe of diameter d and radius R, complete analytical solutions can be easily derived from formulas given in White [1]'s Section 4.10. A compilation of those formulas is given below:

$$V = \frac{Q}{A} = \frac{u_{max}}{2} = \left(\frac{\Delta p + \rho g \Delta z}{L}\right)\frac{R^2}{8\mu} \quad (1.15)$$

$$Q = \int u\,dA = \pi R^2 V = \left(\frac{\Delta p + \rho g \Delta z}{L}\right)\frac{\pi R^4}{8\mu} \quad (1.16)$$

$$\tau_w = \frac{4\mu V}{R} = \frac{8\mu V}{d} = \frac{R}{2}\left(\frac{\Delta p + \rho g \Delta z}{L}\right) \quad (1.17)$$

$$h_f = \frac{32\mu L V}{\rho g d^2} = \frac{128\mu L Q}{\pi \rho g d^4} \quad (1.18)$$

To briefly explain the laminar distribution shown in Figure **??**, we start by evaluating off the assumption of the no-slip condition at the wall (i.e. $u = 0$ and $r = R$ in Figure **??**) to obtain the velocity solution for laminar fully-developed pipe flow

$$u = \frac{1}{4u}\left[-\frac{d}{dx}(p + \rho g z)\right](R^2 - r^2) \quad (1.19)$$

The detailed derivations of Equation 1.19 are not shown here, but it is notable that Equation 1.19 is based off the *hydraulic grade line* (HGL) equations while knowing that for laminar flow $\tau = \mu du/dr$ [1].

Therefore, the laminar flow profile is a paraboloid falling to zero at the wall and reaching a maximum at the axis

$$u_{max} = \frac{R^2}{4\mu}\left[-\frac{d}{dx}(p + \rho g z)\right] \quad (1.20)$$

The paraboloid profile, characteristic of Poiseuille flow, has an average velocity V which is one-half of the maximum velocity (also shown in Equation 1.15). The quantity Δp is the pressure *drop* in a pipe of length L. These formulas are valid whenever the pipe Reynolds number, $Re_d = \rho V d/\mu$, is less than about 2300 (this is the widely accepted threshold between laminar and turbulent flow). Note that τ_w is proportional to V (see Figure **??**, Equation 1.17) and is independent of density because the fluid acceleration is zero. Neither of these is true in turbulent flow [1].

If we know the wall shear stress, the Poiseuille flow friction factor is

$$f_{lam} = \frac{8\tau_{w,lam}}{\rho V^2} = \frac{8(8\mu V/d)}{\rho V^2} = \frac{64}{\rho V d/\mu} = \frac{64}{Re_d} \quad (1.21)$$

A clear relation between two terms in Equation 1.21 arises — the pipe friction factor decreases inversely with Reynolds Number.

1.4 Turbulent Pipe Flow

As discussed previously, there is a significant effect of pipe roughness on the head loss and pressure drop in turbulent flows. In fact, there are different sets of formulas applied when solving for turbulent flows, depending on the relative roughness of the pipe it is travelling through. For a smooth-walled pipe, a relation between friction factor and Reynolds Number for turbulent pipe flow (derived by Prandtl in 1935) is [1]

$$\frac{1}{f^{1/2}} = 2.0\,log(Re_d\,f^{1/2}) - 1.8 \quad (1.22)$$

Experimenting with a few values of f shows that f only drops by a factor of 5 over a 10,000 fold increase in Reynolds Number [1]. If Re_d is already known and f is wanted, an alternate form of equation 1.22 can be used to explicitly solve for f from Re_d:

$$f = \begin{cases} 0.316 Re_d^{-1/4} \\ \left(1.8\,log\frac{Re_d}{6.9}\right)^{-2} \end{cases} \quad \text{when } 4000 < Re_d < 10^5 \quad (1.23)$$

Blasius, a student of Prandtl, devised the first ever formula correlating pipe friction to Reynolds Number. Albeit for a limited range (low turbulent Reynolds Numbers) and for a horizontal pipe, from Equation 1.23:

$$h_f = \frac{\Delta p}{\rho g} = f\frac{L}{d}\frac{V^2}{2g} \approx 0.316\left(\frac{\mu}{\rho V d}\right)^{1/4}\frac{L}{d}\frac{V^2}{2g} \quad (1.24)$$

or

$$\Delta p \approx 0.158 L\rho^{3/4}\mu^{1/4}d^{-5/4}V^{7/4} \quad (1.25)$$

Note that Δp varies only slightly with viscosity, which is a characteristic of turbulent flow [1]. Substituting $Q = \frac{1}{4}\pi d^2 V$ into Equation 1.25 gives

$$\Delta p \approx 0.241 L\rho^{3/4}\mu^{1/4}d^{-4.75}Q^{1.75} \quad (1.26)$$

Analysing Equation 1.26 tells us that for a fixed flow rate Q, the turbulent pressure drop decreases with diameter even more sharply than in laminar flow (compare with Equation 1.16). Thus, the easiest way to reduce required pumping pressure is to increase the pipe diameter, although, of course, the larger the pipe the more expensive it is [1].

The formula relating mean velocity to maximum velocity is

$$\frac{V}{u_{max}} \approx (1 + 1.3\sqrt{f})^{-1} \quad (1.27)$$

and by plugging in some numerical values, we can see clearly that the ratio V/u_{max} varies with much greater sensitivity to Reynolds Number than the value of 0.5 predicted for all laminar pipe flow. Thus the turbulent velocity profile, as shown in

Figure **??**, is flat in the center and drops off sharply to zero at the wall [1].

When assessing turbulent pipe flow, it is worth noting that, due to its sensitivity to pipe roughness, there are generally three ranges of roughness when describing the effect of roughness and Reynolds Number on flow. In general for turbulent friction, after an *onset* point, increases monotonically with the roughness ratio ϵ/d. For *any* given ϵ/d at high Reynolds Numbers, however, the friction factor becomes constant (known as *fully rough flow*). These points of change are certain values of $\epsilon^+ = \epsilon u^*/\nu$ [1]:

$$\frac{\epsilon u^*}{\nu} < 5 = \ hydraulically\ smooth\ \text{walls, no effect of roughness on friction}$$

$$5 \leq \frac{\epsilon u^*}{\nu} \leq 70 = \ transitional\ \text{roughness, moderate Reynolds-Number effect}$$

$$\frac{\epsilon u^*}{\nu} > 70 = \ fully\ rough\ \text{flow, sublayer broken up and friction independent of } Re$$

For fully rough flows (independent of Reynolds Number):

$$\frac{1}{f^{1/2}} = -2.0\,log\frac{\epsilon/d}{3.7}\ \text{fully rough flow} \qquad (1.28)$$

To cover the transitionally rough range, in 1939 Colebrook combined the smooth wall (Equation 1.22) and fully rough (Equation 1.28) relations to a general interpolation formula [1]:

$$\frac{1}{f^{1/2}} = -2.0\,log\left(\frac{\epsilon/d}{3.7} + \frac{2.51}{Re_d\,f^{1/2}}\right) \qquad (1.29)$$

This is the accepted design formula for turbulent friction. It was first plotted by Moody in 1944 and is called the *Moody chart* for pipe friction (see Figure **??**).

However, Equation 1.29 is sometimes cumbersome to evaluate for f if Re_d is known. So, an alternate explicit formula was given by Haaland as

$$\frac{1}{f^{1/2}} \approx -1.8\,log\Big[\frac{6.9}{Re_d} + \left(\frac{\epsilon/d}{3.7}\right)^{1.11}\Big] \qquad (1.30)$$

and varies less than 2 percent from Equation 1.29 [1].

1.5 Example Calculation

1.5.1 Determining Head Loss and Pressure Drop Using the Moody Chart

1.6 Units and Symbols

All units are implied with accordance to the Metric system (seconds, kilograms, Pascals, etc.), but are defined explicitly along with common Greek symbols below for ease-of-use:

- Q = volumetric flow rate, m^3/s
- u = velocity, m/s
- $u(r), V$ = local mean velocity, m/s
- u_{max} = local maximum velocity, m/s
- μ = dynamic viscosity, $\frac{kg}{m \cdot s}$ or $Pa \cdot s$
- ν = kinematic viscosity, m^2/s
- R = radius, m
- Re_d = Reynolds Number, dimensionless

References

[1] Frank M. White. *Fluid Mechanics*. Fourth. McGraw-Hill Series in Mechanical Engineering. McGraw-Hill, Dec. 1998.