Development of a Cost-Effective 1kN Liquid-Fueled Rocket Propulsion System

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1 Pipe Dimension Calculation

The following section describes the theoretical process used to dimensionalize the preliminary plumbing framework in preparation for the first static cold flow test. By definition, these calculations are purely speculatory and are only used for the initial design process. The purpose of the cold flow test is to verify these parameters, and consequently adjust these parameters to better fit the system requirements.

1.1 Assumptions

To simplify the rigorous analysis and optimization processes often associated with viscous pipe flow, a couple of assumptions are applied in the following section to both shorten the development timeline and to avoid unecessarily complex or expensive methods outside of the scope of a high school amateur rocketry program. They are as follows:

1. Flow is driven by both pressure and gravity.

2. Pipe is circular and is of constant cross-sectional area.

3. No swirl, circumferential variation, or entrance effects.

4. No shaft-work or heat-transfer effects.

5. Flow is fully developed (minimal boundary-layer effects).

1.2 The Reynolds Transport Theorem

We can begin by introducing the concept of control-volume analysis — a powerful approach to solving fundamental fluid mechanics problems — in which a mathematical abstraction is employed to generate mathematical models of physical systems. That is, in an inertial frame of reference, the *control volume* is a volume fixed in space or moving with constant velocity with the fluid flow. The surface enclosing the control volume is referred to as the *control surface*. To convert a system analysis to a control-volume

analysis, we must shift our perspective such that our mathematics apply to a specific region rather than to individual bodies. This process is called the *Reynolds Transport Theorem*, and allows us to study certain properties (e.g. angular momentum L, enthalpy h, etc.) crossing the boundaries of the region. Essentially, we need to relate the time derivative of a system property of that property within a certain region.
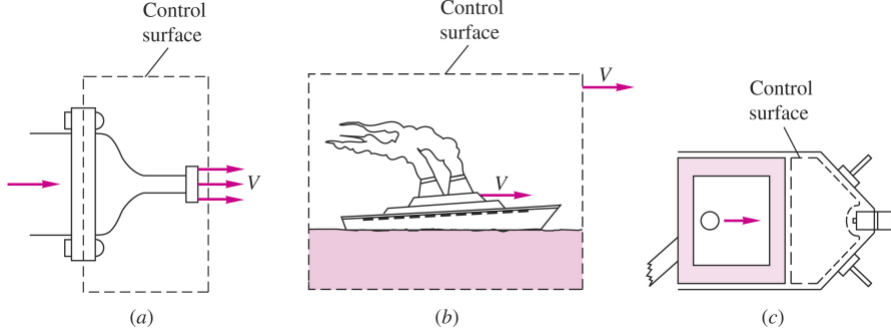

Figure 1: Fixed, moving, and deformable control volumes: (a) fixed control volume for nozzle stress analysis; (b) control volume moving at ship speed for drag force analysis; (c) control volume deforming within cylinder for transient pressure variation analysis.

The desired conversion formula differs slightly according to whether the control volume is fixed, moving, or deformable, but for the purposes of our application, we'll be focusing on a fixed control volume. The fixed control volume in Figure 1a encloses a stationary region of interest to a nozzle designer. The control surface is an abstract concept that does not hinder the flow in any way. It slices through the jet leaving the nozzle, circles around through the surrounding (ambient) atmosphere, and slices through the flange bolts and the fluid within the nozzle [2].

This particular control volume exposes the stresses in the flange bolts, which contribute to applied forces in the momentum analysis. In this sense, the control volume resembles the *free-body* concept, which is often applied in solid-mechanics analysis [2].

Figure 2 shows a generalized fixed control volume with an arbitrary flow pattern passing through. In general, each differential dA of surface will have a different velocity vector $\vec{V}$ making a different angle θ with the local normal to dA. This gives us the general Reynolds transport theorem equation for an arbitrary fixed control volume:

$$\frac{d}{dt}(B_{syst}) = \frac{d}{dt}\left(\int_{CV} \beta\rho \, d\vartheta\right) + \int_{CS} \beta\rho \, cos\theta \, dA_{out} - \int_{CS} \beta\rho \, cos\theta \, dA_{in} \qquad (1.1)$$

Asserting the property B to be either mass, momentum, angular momentum, or energy, all basic laws can be rewritten in control-volume form. Note that all three control volume integrals are a function of the intensive property β, and since the control volume is fixed in space, the elemental volumes $d\vartheta$ do not vary with time, and thus the time derivative of the volume integral disappears unless β or ρ varies with time (unsteady flow) [2].

However, realizing that if $\vec{n}$ is defined as the *outward* normal unit vector everywhere on the control surface, then $\vec{V}\cdot\vec{n} = V_n$ for outflow and $\vec{V}\cdot\vec{n} = -V_n$ for inflow. Therefore,
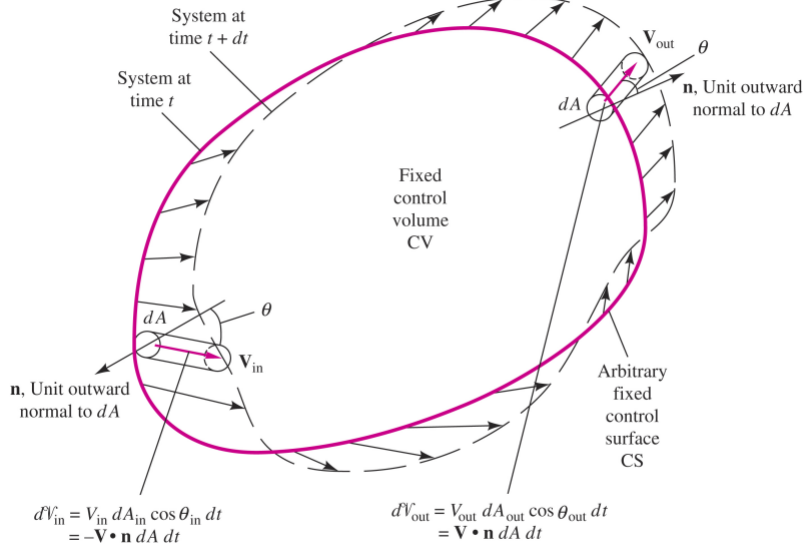
Figure 2: An arbitrary fixed control volume with an arbitrary flow pattern.

the flux terms can be combined and Equation 1.1 can be rewritten as

$$\frac{d}{dt}(B_{syst}) = \frac{d}{dt}\left(\int_{CV} \beta\rho\,d\vartheta\right) + \int_{CV} \beta\rho(\vec{V}\cdot\vec{n})\,dA \tag{1.2}$$

We are now able to apply the principles of conservation of mass to the Reynolds transport theorem by setting the arbitrary variable B equal to mass, and $\beta = dm/dm = 1$. Equation 1.2 becomes

$$\left(\frac{dm}{dt}\right)_{syst} = 0 = \frac{d}{dt}\left(\int_{CV} \rho\,d\vartheta\right) + \int_{CS} \rho(\vec{V_r}\cdot\vec{n})\,dA \tag{1.3}$$

This is the integral mass-conservation law for a deformable control volume. Adapting this for a fixed control volume and simplying gives

$$\int_{CV} \frac{\partial\rho}{\partial t}\,d\vartheta + \int_{CS} \rho(\vec{V}\cdot\vec{n})\,dA = 0 \tag{1.4}$$

Finally, knowing that we are dealing with steady, incompressible flow through the control volume ($\partial\rho/\partial t = 0$), Equation 1.4 reduces to

$$\int_{CS} \rho(\vec{V}\cdot\vec{n})\,dA = 0 \tag{1.5}$$

This equation explicitly states that in steady flow, the mass flows entering and exitting the control volume are exactly equal, while neglecting sources or sinks of mass which might be embedded in the control volume.

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1.3 Viscous Flow in a Circular Pipe

Now that a theoretical background has been established, we can begin to discuss the specific case of solving for flow rates, pressure drops, head losses, and diameter sizings via incompressible, viscous flows in circular pipes. Given the simplifying assumptions mentioned in Section 1.1, we can start simply with the classic case of Bernoulli's Equation:

$$p + \frac{1}{2}\rho V^2 + \rho g z = constant \tag{1.6}$$

where p is the pressure, ρ is density, V is velocity, z is elevation, and g represents gravitational acceleration. It gives insight into the balance between pressure, velocity, and elevation by assuming that the two points in question lie on a streamline, the fluid is incompressible, the flow is steady and inviscid, and there is no friction. Although useful for some applications, it is not adequate for designing robust piping systems which will in actuality encounter such effects.

The addition of a *head loss* term, denoted as h_f, is the first step in introducing a viscous term into an otherwise inviscid equation. Head loss can simply be thought of as an additional pressure loss in the system due to viscous effects, a sudden expansion/contraction, and/or obstructions to its path such as pipe elbows, bends, valves, etc. This can be done by first recalling the continuity (conservation of mass) relation (Equation 1.5). Realizing that the pipe is of constant area and assuming only a number of one-dimensional inlets and outlets, Equation 1.5 reduces to

$$Q_1 = Q_2 = constant \tag{1.7}$$

or

$$V_1 = \frac{Q_1}{A_1} = V_2 = \frac{Q_2}{A_2} \tag{1.8}$$

Bernoulli's Equation (Equation 1.6, in combination with the steady-flow energy equation from [2]) can thus be reduced and rearranged into

$$\frac{p_1}{\rho} + \frac{1}{2}\alpha_1 V_1^2 + gz_1 = \frac{p_2}{\rho} + \frac{1}{2}\alpha_2 V_2^2 + gz_2 + gh_f \tag{1.9}$$

due to Assumption 4, where α denotes the kinetic-energy correlation factor (energy coefficient). α is 1 when the flow is fully developed and is less than 1 if the flow is underdeveloped. Note that this equation is also very similar to the result of dividing Equation 1.6 by the specific weight of the fluid, $\rho \cdot g$. For fully-developed flow, the velocity profile shape is the same at sections 1 and 2, as seen in Figure 3. Thus the kinetic-energy correlation factor $\alpha_1 = \alpha_2$, and since $V_1 = V_2$ from Equation 1.8, Equation 1.9 reduces to a simple expression for the friction-head loss h_f [2]

$$h_f = (z_1 - z_2) + \left(\frac{p_1}{\rho g} - \frac{p_2}{\rho g} \right) = \Delta z + \frac{\Delta p}{\rho g} \tag{1.10}$$

Applying the one-dimensional momentum relation

$$\sum \vec{F} = \frac{d}{dt} \left(\int_{CV} \vec{V} \rho \, d\vartheta \right) + \sum (\dot{m}_i \vec{V}_i)_{out} - \sum (\dot{m}_i \vec{V}_i)_{in} \tag{1.11}$$
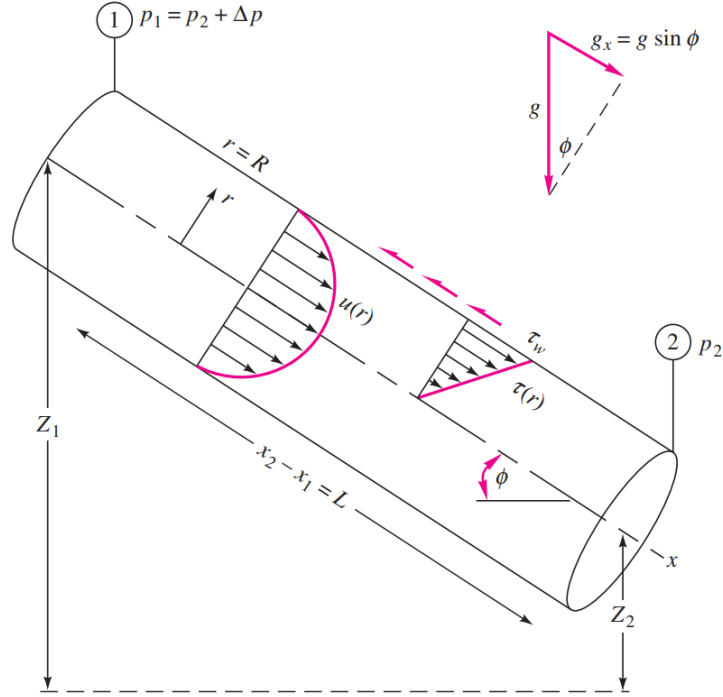
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Figure 3: Control volume of a steady, fully-developed flow between two sections of an inclined pipe.

to the control volume in Figure 3 while accounting for applied x-directed forces due to pressure, gravity, and shear:

$$\sum F_x = \Delta p(\pi R^2) + \rho g(\pi R^2)L\,sin\phi - \tau_w(2\pi R)L = \dot{m}(V_2 - V_1) = 0 \qquad (1.12)$$

Rearranging shows that head loss h_f is also related to wall shear stress

$$\Delta z + \frac{\Delta p}{\rho g} = h_f = \frac{2\tau_w}{\rho g}\frac{L}{R} = \frac{4\tau w}{\rho g}\frac{L}{d} \qquad (1.13)$$

where we have substituted $\Delta z = \Delta L\,sin\phi$ from Figure 3. Note that regardless the value of ϕ (whether the pipe is horizontal or tilted), the head loss is proportional to the wall shear stress.

Finally, to correlate the head loss term into a useful form for solving pipe flow problems, we have the famous *Darcy-Weisbach equation*, which is a proposed correlation valid for duct flow of any cross section and any Reynolds Number:

$$h_f = f\frac{L}{d}\frac{V^2}{2g} \qquad (1.14)$$

The dimensionless parameter f is the *Darcy friction factor*, which establishes a relationship between roughness and pipe resistance. ϵ is the wall roughness height, which is

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significant only in turbulent pipe flow. It has been shown that ϵ has orders of magnitude *less of an effect* on laminar pipe flow. By equating Equations 1.13 and 1.14 we find an alternative form of the friction factor:

$$\frac{8\tau_w}{\rho V^2} = f = F(Re_d, \frac{\epsilon}{d}) \tag{1.15}$$

where F represents some relationship between the Reynolds Number Re_d and the average pipe roughness to diameter ratio ϵ/d (also known as relative roughness).

1.4   Laminar Fully Developed Pipe Flow

The following section briefly outlines the derivations and methods necessary to solve specifically a laminar flow problem.

Design Parameters			
Name	Value	Unit	Uncertainty
Propellant	Isopropyl alcohol (95%)	N/A	N/A
μ, Dynamic viscosity	0.00196	$kg/m \cdot s$	$\pm 0.1\%$
ρ, Density	786	kg/m^3	$\pm 1\%$
L, Pipe length	8.0	m	$\pm 0.1\%$
d, Pipe diameter	12.7	mm	$\pm 0.1\%$
$\dot{m}$, Mass flow rate	0.9133	kg/s	$\pm 0.1\%$
Pipe material	Hard nylon	N/A	N/A
ϵ, Roughness	1.5 to 40.0	μm	$\pm 50\%$

Table 1: Design parameters and fluid properties required for pipe calculations.

However, by just doing a few simple calculations using values from Table 1, we quickly realize our flow is most likely turbulent:

$$Q = \frac{\dot{m}}{\rho} = \frac{0.9133\,kg/s}{786\,kg/m^3} = 1.162 \times 10^{-3}\,m^3/s \tag{1.16}$$

$$V = \frac{Q}{\pi R^2} = \frac{(1.162 \times 10^{-3}\,m^3/s)}{\pi(0.0127/2\,m)^2} = 9.173\,m/s \tag{1.17}$$

$$\nu = \frac{\mu}{\rho} = \frac{0.00196\,kg/m \cdot s}{786\,kg/m^3} = 2.494 \times 10^{-6}\,m^2/s \tag{1.18}$$

$$Re_d = \frac{Vd}{\nu} = \frac{(9.173\,m/s)(0.0127\,m)}{2.494 \times 10^{-6}\,m^2/s} = 46711 \tag{1.19}$$

since $Re_d >> 2300$. The purpose of this section, therefore, is to introduce and highlight key differences when dealing with laminar versus turbulent flow when designing a pipe system. The benefits and drawbacks of either flow type will become apparent, and this will hopefully provide the reader with stronger intuition on the design and manufacturing
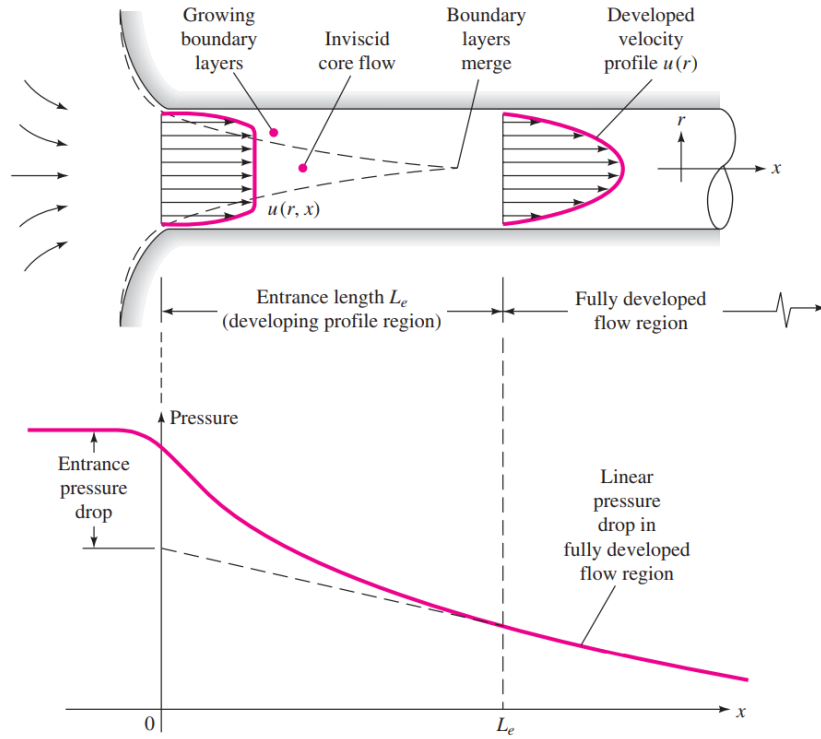
Figure 4: Developing velocity profiles and pressure changes in the entrance of a duct.
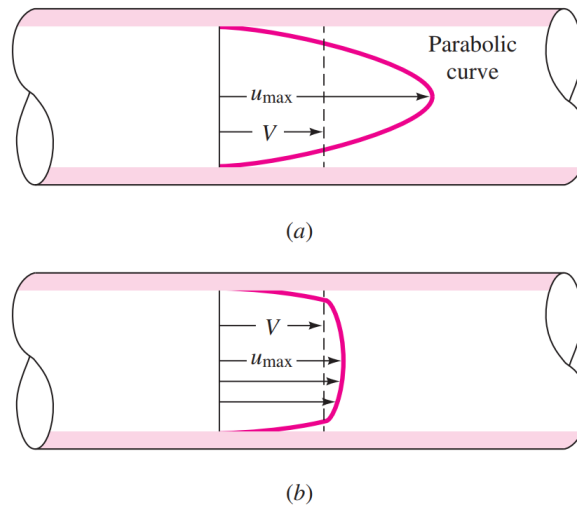

Figure 5: Comparison of laminar and turbulent pipe flow velocity profiles for the same volume flow: (a) laminar flow; (b) turbulent flow.

tradeoffs associated with each flow type. If the reader is only interested in equations relevant to our specific calculations, the next section covers the solutions to turbulent fully developed pipe flow.

Considering a fully-developed *Hagen-Poiseuille flow* (which represents the laminar distribution given in Figure 5) in a round pipe of diameter d and radius R, complete analytical solutions can be easily derived from formulas given in White [2]'s Section 4.10. A compilation of those formulas is given below:

$$V = \frac{Q}{A} = \frac{u_{max}}{2} = \left(\frac{\Delta p + \rho g \Delta z}{L}\right)\frac{R^2}{8\mu} \tag{1.20}$$

$$Q = \int u\,dA = \pi R^2 V = \left(\frac{\Delta p + \rho g \Delta z}{L}\right)\frac{\pi R^4}{8\mu} \tag{1.21}$$

$$\tau_w = \frac{4\mu V}{R} = \frac{8\mu V}{d} = \frac{R}{2}\left(\frac{\Delta p + \rho g \Delta z}{L}\right) \tag{1.22}$$

$$h_f = \frac{32\mu L V}{\rho g d^2} = \frac{128\mu L Q}{\pi \rho g d^4} \tag{1.23}$$

To briefly explain the laminar distribution shown in Figure 5, we start by evaluating off the assumption of the no-slip condition at the wall (i.e. $u = 0$ and $r = R$ in Figure 3) to obtain the velocity solution for laminar fully-developed pipe flow

$$u = \frac{1}{4u}\left[-\frac{d}{dx}(p + \rho g z)\right](R^2 - r^2) \tag{1.24}$$

The detailed derivations of Equation 1.24 are not shown here, but it is notable that Equation 1.24 is based off the *hydraulic grade line* (HGL) equations while knowing that for laminar flow $\tau = \mu du/dr$ [2].

Therefore, the laminar flow profile is a paraboloid falling to zero at the wall and reaching a maximum at the axis

$$u_{max} = \frac{R^2}{4\mu}\left[-\frac{d}{dx}(p + \rho g z)\right] \tag{1.25}$$

The paraboloid profile, characteristic of Poiseuille flow, has an average velocity V which is one-half of the maximum velocity (also shown in Equation 1.20). The quantity Δp is the pressure *drop* in a pipe of length L. These formulas are valid whenever the pipe Reynolds number, $Re_d = \rho V d/\mu$, is less than about 2300 (this is the widely accepted threshold between laminar and turbulent flow). Note that τ_w is proportional to V (see Figure 4, Equation 1.22) and is independent of density because the fluid acceleration is zero. Neither of these is true in turbulent flow [2].

If we know the wall shear stress, the Poiseuille flow friction factor is

$$f_{lam} = \frac{8\tau_{w,lam}}{\rho V^2} = \frac{8(8\mu V/d)}{\rho V^2} = \frac{64}{\rho V d/\mu} = \frac{64}{Re_d} \tag{1.26}$$

A clear relation between two terms in Equation 1.26 arises — the pipe friction factor decreases inversely with Reynolds Number.

1.5   Turbulent Pipe Flow

As discussed previously, there is a significant effect of pipe roughness on the head loss and pressure drop in turbulent flows. In fact, there are different sets of formulas applied when solving for turbulent flows, depending on the relative roughness of the pipe it is travelling through. For a smooth-walled pipe, a relation between friction factor and Reynolds Number for turbulent pipe flow (derived by Prandtl in 1935) is [2]

$$\frac{1}{f^{1/2}} = 2.0 \, log(Re_d \, f^{1/2}) - 1.8 \tag{1.27}$$

Experimenting with a few values of f shows that f only drops by a factor of 5 over a 10,000 fold increase in Reynolds Number [2]. If Re_d is already known and f is wanted, an alternate form of equation 1.27 can be used to explicitly solve for f from Re_d:

$$f = \begin{cases} 0.316Re_d^{-1/4} \\ \left(1.8 \, log\frac{Re_d}{6.9}\right)^{-2} \end{cases} \quad \text{when } 4000 < Re_d < 10^5 \tag{1.28}$$

Blasius, a student of Prandtl, devised the first ever formula correlating pipe friction to Reynolds Number. Albeit for a limited range (low turbulent Reynolds Numbers) and for a horizontal pipe, from Equation 1.28:

$$h_f = \frac{\Delta p}{\rho g} = f\frac{L}{d}\frac{V^2}{2g} \approx 0.316 \left(\frac{\mu}{\rho V d}\right)^{1/4} \frac{L}{d}\frac{V^2}{2g} \tag{1.29}$$

or

$$\Delta p \approx 0.158L\rho^{3/4}\mu^{1/4}d^{-5/4}V^{7/4} \tag{1.30}$$

Note that Δp varies only slightly with viscosity, which is a characteristic of turbulent flow [2]. Substituting $Q = \frac{1}{4}\pi d^2 V$ into Equation 1.30 gives

$$\Delta p \approx 0.241L\rho^{3/4}\mu^{1/4}d^{-4.75}Q^{1.75} \tag{1.31}$$

Analysing Equation 1.31 tells us that for a fixed flow rate Q, the turbulent pressure drop decreases with diameter even more sharply than in laminar flow (compare with Equation 1.21). Thus, the easiest way to reduce required pumping pressure is to increase the pipe diameter, although, of course, the larger the pipe the more expensive it is [2].

The formula relating mean velocity to maximum velocity is

$$\frac{V}{u_{max}} \approx (1 + 1.3\sqrt{f})^{-1} \tag{1.32}$$

and by plugging in some numerical values, we can see clearly that the ratio V/u_{max} varies with much greater sensitivity to Reynolds Number than the value of 0.5 predicted for all laminar pipe flow. Thus the turbulent velocity profile, as shown in Figure 5, is flat in the center and drops off sharply to zero at the wall [2].

When assessing turbulent pipe flow, it is worth noting that, due to its sensitivity to pipe roughness, there are generally three ranges of roughness when describing the effect of roughness and Reynolds Number on flow. In general for turbulent friction, after an *onset* point, increases monotonically with the roughness ratio ϵ/d. For *any* given ϵ/d at high Reynolds Numbers, however, the friction factor becomes constant (known as *fully rough flow*). These points of change are certain values of $\epsilon^+ = \epsilon u^*/\nu$ [2]:

$$\frac{\epsilon u^*}{\nu} < 5 = \quad hydraulically \ smooth \ \text{walls, no effect of roughness on friction}$$

$$5 \leq \frac{\epsilon u^*}{\nu} \leq 70 = \quad transitional \ \text{roughness, moderate Reynolds Number effect}$$

$$\frac{\epsilon u^*}{\nu} > 70 = \quad fully \ rough \ \text{flow, sublayer broken up and friction independent of } Re$$

For fully rough flows (independent of Reynolds Number):

$$\frac{1}{f^{1/2}} = -2.0 \, log \frac{\epsilon/d}{3.7} \text{fully rough flow} \tag{1.33}$$

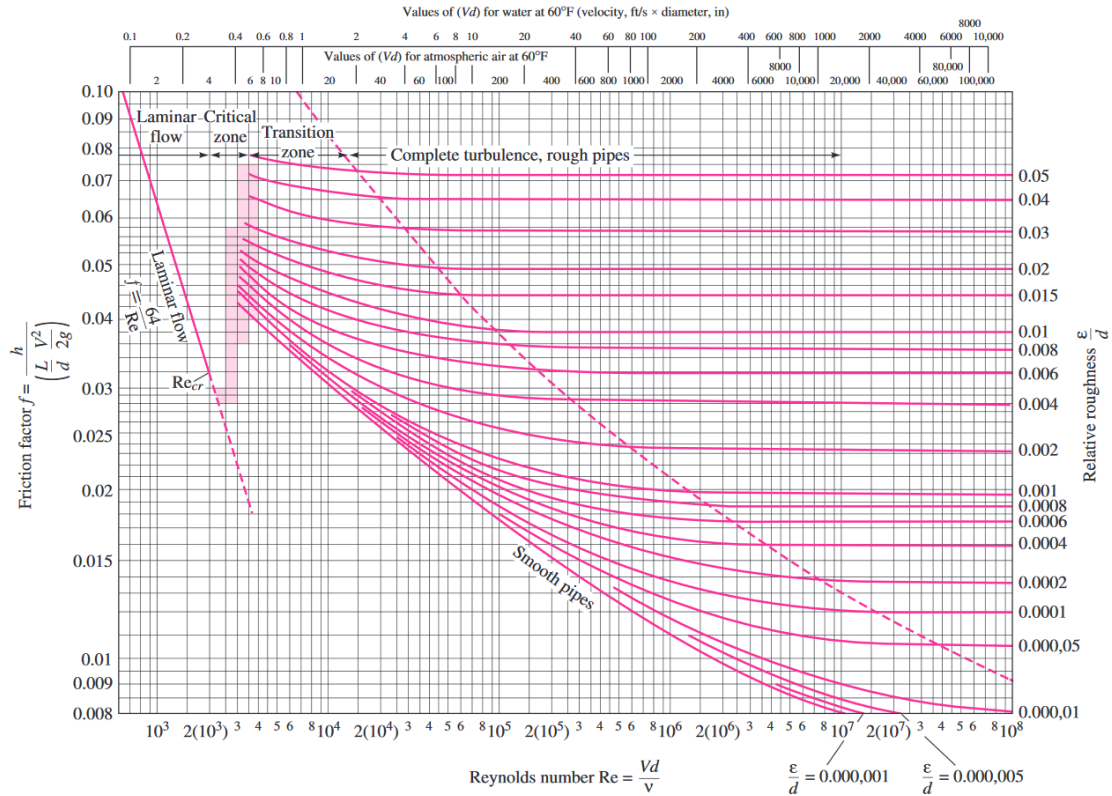To cover the transitionally rough range, in 1939 Colebrook combined the smooth wall

Figure 6: The Moody Chart.

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(Equation 1.27) and fully rough (Equation 1.33) relations to a general interpolation formula [2]:

$$\frac{1}{f^{1/2}} = -2.0 \, log \left(\frac{\epsilon/d}{3.7} + \frac{2.51}{Re_d \, f^{1/2}} \right) \tag{1.34}$$

This is the accepted design formula for turbulent friction. It was first plotted by Moody in 1944 and is called the *Moody chart* for pipe friction (see Figure 6).

However, Equation 1.34 is sometimes cumbersome to evaluate for f if Re_d is known. So, an alternate explicit formula was given by Haaland as

$$\frac{1}{f^{1/2}} \approx -1.8 \, log \Big[\frac{6.9}{Re_d} + \left(\frac{\epsilon/d}{3.7} \right)^{1.11} \Big] \tag{1.35}$$

and varies less than 2 percent from Equation 1.34 [2].

1.6   Example Calculation

1.6.1   Determining Head Loss and Pressure Drop Using the Moody Chart

1.7   Units and Symbols

All units are implied with accordance to the Metric system (seconds, kilograms, Pascals, etc.), but are defined explicity along with common Greek symbols below for ease-of-use:

- Q = volumetric flow rate, m^3/s

- u = velocity, m/s

- $u(r), V$ = local mean velocity, m/s

- u_{max} = local maximum velocity, m/s

- μ = dynamic viscosity, $\frac{kg}{m \cdot s}$ or $Pa \cdot s$

- ν = kinematic viscosity, m^2/s

- R = radius, m

- Re_d = Reynolds Number, dimensionless

References

[1]   F. F. Farshad and H. H. Rieke. "Surface-Roughness Design Values for Modern Pipes". In: *SPE Drilling and Completion, University of Louisiana at Lafayette* (Sept. 2006), pp. 212–215.

[2]   Frank M. White. *Fluid Mechanics*. Fourth. McGraw-Hill Series in Mechanical Engineering. McGraw-Hill, Dec. 1998.