hw_4

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1 AEROSP 536 Electric Propulsion: Homework 4

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1.1 Problem 1: Single Particle Motion in a Magnetic Field

1.1.1 Problem Statement

Consider an electron with charge $-q$ and mass m_e in a uniform magnetic field $\vec{B} = B_0\hat{z}$. The electron has initial velocity $\vec{v}_0 = v_{\perp 0}\hat{y} + v_{z0}\hat{z}$ where $5v_{z0} = v_{\perp 0}$. As with all charged particles in a magnetic field, the particle is subject to a Lorentz force $\vec{F}_L = -q\vec{v} \times \vec{B}$.

[2]:
```python
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D
from matplotlib.patches import FancyArrowPatch
from mpl_toolkits.mplot3d import proj3d

plt.rcParams['figure.dpi'] = 100
plt.rcParams['font.size'] = 10
```

1.1.2 Part (a) - Derivation of Electron Trajectory

Task: Show that the electron trajectory at time t is given by:

$$\vec{r} = r_L\cos(\omega_{ce}t)\hat{x} + r_L\sin(\omega_{ce}t)\hat{y} + v_{z0}t\hat{z} + \vec{x}_0 - r_L\hat{x}$$

First, we'll set up the equations of motion. The Lorentz force on an electron with charge $-q$ is:

$$\vec{F}_L = -q\vec{v} \times \vec{B}$$

Newton's second law gives:

$$m_e\frac{d\vec{v}}{dt} = -q\vec{v} \times \vec{B}$$

Let $\vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$ and $\vec{B} = B_0\hat{z}$.

We want to find the equations in component form. The cross product is:

$$\vec{v} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ v_x & v_y & v_z \\ 0 & 0 & B_0 \end{vmatrix} = v_y B_0 \hat{x} - v_x B_0 \hat{y}$$

and so:

$$m_e \frac{dv_x}{dt} = -q v_y B_0$$
$$m_e \frac{dv_y}{dt} = -q(-v_x B_0) = q v_x B_0$$
$$m_e \frac{dv_z}{dt} = 0$$

By definition, the electron cyclotron frequency is:

$$\omega_{ce} = \frac{qB_0}{m_e}$$

$$\frac{dv_x}{dt} = -\omega_{ce} v_y$$
$$\frac{dv_y}{dt} = \omega_{ce} v_x$$
$$\frac{dv_z}{dt} = 0$$

Since $\frac{dv_z}{dt} = 0$, we have:

$$v_z(t) = v_{z0} = \text{constant}$$

Differentiating the first equation:

$$\frac{d^2 v_x}{dt^2} = -\omega_{ce} \frac{dv_y}{dt} = -\omega_{ce}(\omega_{ce} v_x) = -\omega_{ce}^2 v_x$$

This is a simple harmonic oscillator equation with solution:

$$v_x(t) = A\cos(\omega_{ce} t) + B\sin(\omega_{ce} t)$$

From the first equation:

$$v_y(t) = -\frac{1}{\omega_{ce}} \frac{dv_x}{dt} = A\sin(\omega_{ce} t) - B\cos(\omega_{ce} t)$$

Now, we can apply the initial conditions.

At $t = 0$: $\vec{v}(0) = v_{\perp 0}\hat{y} + v_{z0}\hat{z}$, so $v_x(0) = 0$ and $v_y(0) = v_{\perp 0}$.

From $v_x(0) = 0$: $A = 0$

From $v_y(0) = v_{\perp 0}$: $-B = v_{\perp 0}$, so $B = -v_{\perp 0}$

Therefore:

$$v_x(t) = -v_{\perp 0} \sin(\omega_{ce} t)$$
$$v_y(t) = v_{\perp 0} \cos(\omega_{ce} t)$$
$$v_z(t) = v_{z0}$$

So now, we can integrate each component to find the position function.

$$x(t) = \int v_x \, dt = -v_{\perp 0} \int \sin(\omega_{ce} t) \, dt = \frac{v_{\perp 0}}{\omega_{ce}} \cos(\omega_{ce} t) + C_x$$
$$y(t) = \int v_y \, dt = v_{\perp 0} \int \cos(\omega_{ce} t) \, dt = \frac{v_{\perp 0}}{\omega_{ce}} \sin(\omega_{ce} t) + C_y$$
$$z(t) = \int v_z \, dt = v_{z0} t + C_z$$

The Larmor radius by definition is:

$$r_L = \frac{v_{\perp 0}}{\omega_{ce}} = \frac{m_e v_{\perp 0}}{q B_0}$$

So:

$$x(t) = r_L \cos(\omega_{ce} t) + C_x$$
$$y(t) = r_L \sin(\omega_{ce} t) + C_y$$
$$z(t) = v_{z0} t + C_z$$

Once again we need to find the integration constants. Let the initial position be $\vec{x}_0 = x_0 \hat{x} + y_0 \hat{y} + z_0 \hat{z}$.

At $t = 0$:

$$x(0) = r_L + C_x = x_0 \quad \Rightarrow \quad C_x = x_0 - r_L$$
$$y(0) = 0 + C_y = y_0 \quad \Rightarrow \quad C_y = y_0$$
$$z(0) = 0 + C_z = z_0 \quad \Rightarrow \quad C_z = z_0$$

So finally, the trajectory over time is:

$$\vec{r}(t) = [r_L \cos(\omega_{ce} t) + x_0 - r_L] \hat{x} + [r_L \sin(\omega_{ce} t) + y_0] \hat{y} + [v_{z0} t + z_0] \hat{z}$$

This can be written as:

$$\boxed{\vec{r}(t) = r_L \cos(\omega_{ce} t) \hat{x} + r_L \sin(\omega_{ce} t) \hat{y} + v_{z0} t \hat{z} + \vec{x}_0 - r_L \hat{x}}$$

This makes sense from what we learned in class. The physical interpretation is that the electron follows a helical path with circular motion in the x-y plane (perpendicular to $\vec{B}$) and uniform motion along the z-axis (parallel to $\vec{B}$). The center of the circular motion is at $(x_0 - r_L, y_0, z_0)$.

1.1.3 Part (b) - 3D Visualization of Electron Trajectory

Task: Suppose the initial position $\vec{x}(0) = r_L\hat{x}$. Draw in three dimensions the trajectory from part (a) from $t = 0$ to $t = 10 \times (2\pi/\omega_{ce})$. Label the Larmor radius r_L and denote with an arrow the direction of travel of the electron.

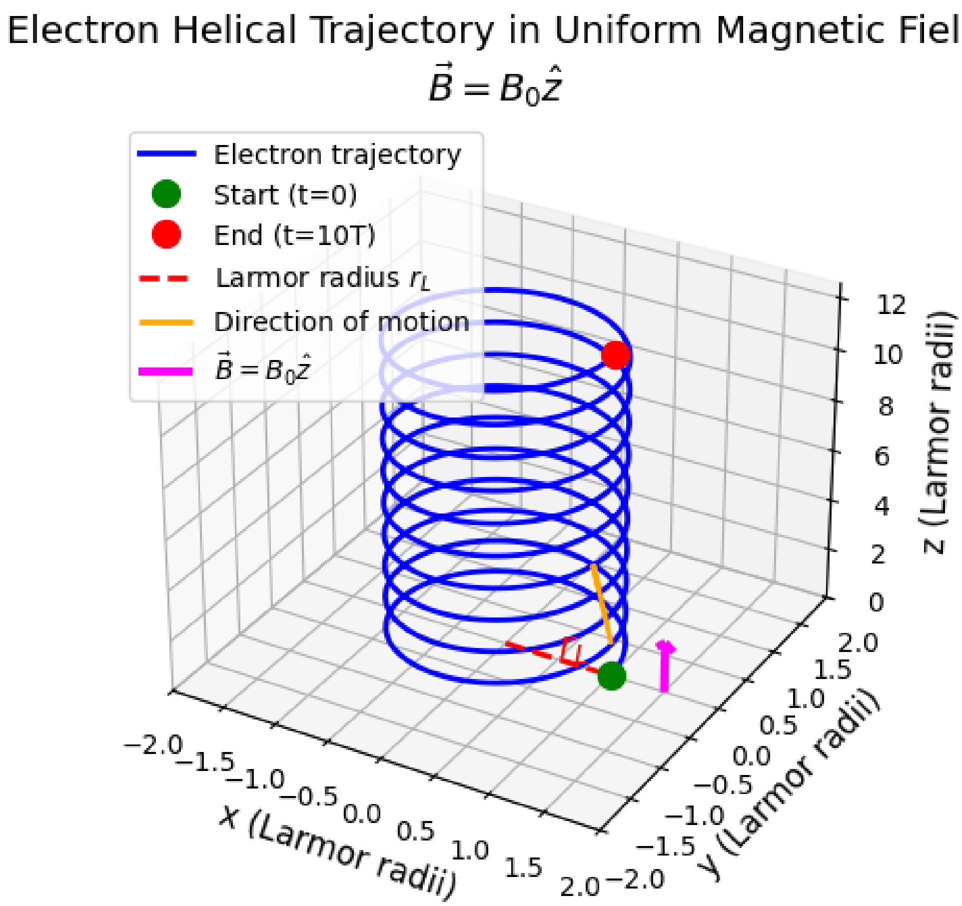
With $\vec{x}_0 = r_L\hat{x}$, the trajectory simplifies to:

$$\vec{r}(t) = r_L \cos(\omega_{ce}t)\hat{x} + r_L \sin(\omega_{ce}t)\hat{y} + v_{z0}t\hat{z}$$

Note the center of gyration is at the origin, and that the direction of motion is indicated by the start/stop points.

[ ]:
```
# Define parameters (normalized units for visualization)
r_L = 1.0  # Larmor radius (arbitrary units)
omega_ce = 1.0  # Cyclotron frequency (arbitrary units)
v_perp_0 = r_L * omega_ce  # Perpendicular velocity
v_z0 = v_perp_0 / 5  # Parallel velocity (given: 5*v_z0 = v_perp_0)
T_ce = 2 * np.pi / omega_ce  # Cyclotron period
t = np.linspace(0, 10 * T_ce, 1000)

# Calculate trajectory
x = r_L * np.cos(omega_ce * t)
y = r_L * np.sin(omega_ce * t)
z = v_z0 * t



fig = plt.figure()
ax = fig.add_subplot(111, projection='3d')
# Plot the helical trajectory
ax.plot(x, y, z, 'b-', linewidth=2, label='Electron trajectory')
# Mark starting point
ax.plot([x[0]], [y[0]], [z[0]], 'go', markersize=10, label='Start (t=0)',␣
↪zorder=5)
# Mark ending point
ax.plot([x[-1]], [y[-1]], [z[-1]], 'ro', markersize=10, label='End (t=10T)',␣
↪zorder=5)
# Draw Larmor radius at t=0
ax.plot([0, x[0]], [0, y[0]], [0, 0], 'r--', linewidth=2, label=f'Larmor radius␣
↪$r_L$')
# Add direction arrow (at about t = T_ce)
idx = len(t) // 10
arrow_start = [x[idx], y[idx], z[idx]]
arrow_end = [x[idx+20], y[idx+20], z[idx+20]]
ax.quiver(arrow_start[0], arrow_start[1], arrow_start[2],
          arrow_end[0] - arrow_start[0],
          arrow_end[1] - arrow_start[1],
          arrow_end[2] - arrow_start[2],
```

```
          color='orange', arrow_length_ratio=0.3, linewidth=2, label='Direction␣
↪of motion')

# Draw magnetic field direction
ax.quiver(1.5*r_L, 0, 0, 0, 0, 2*r_L,
          color='magenta', arrow_length_ratio=0.15, linewidth=3,
          label=r'$\vec{B} = B_0\hat{z}$')

# Labels and formatting
ax.set_xlabel('x (Larmor radii)', fontsize=12)
ax.set_ylabel('y (Larmor radii)', fontsize=12)
ax.set_zlabel('z (Larmor radii)', fontsize=12)
ax.set_title('Electron Helical Trajectory in Uniform Magnetic Field\n' +
             r'$\vec{B} = B_0\hat{z}$', fontsize=14)
ax.legend(loc='upper left', fontsize=10)
ax.grid(True, alpha=0.3)

ax.set_xlim(-2, 2)
ax.set_ylim(-2, 2)
ax.set_zlim(0, z.max())

# Add text annotation for Larmor radius
ax.text(x[0]/2, y[0]/2, 0, '$r_L$', fontsize=14, color='red', fontweight='bold')
plt.tight_layout()
plt.show()
```



1.1.4 Part (c) - Larmor Radius in Variable Magnetic Field

Task: Consider the case where the magnetic field has a variable magnitude $B_0(z)$. For magnetized electrons in regions where the change in magnetic field is sufficiently weak compared to the Larmor radius ($r_L|\nabla B/B| \ll 1$), the magnetic moment is constant:

$$\mu = \frac{m_e v_\perp^2}{2B_0}$$

Suppose a particle starts at position $z = 0$ with Larmor radius r_{L0}. Find an expression for the Larmor radius $r_L(z)$ at position z as a function of the initial Larmor radius r_{L0}, the initial magnetic field magnitude $B_0(0)$, and the local magnetic field amplitude $B_0(z)$.

First, we write the magnetic moment at $z = 0$. At the initial position $z = 0$:

$$\mu = \frac{m_e v_{\perp 0}^2}{2B_0(0)}$$

where $v_{\perp 0}$ is the initial perpendicular velocity.

Next, we can express $v_{\perp 0}$ in terms of r_{L0}. The Larmor radius is defined (again) as:

$$r_{L0} = \frac{m_e v_{\perp 0}}{qB_0(0)}$$

Solving for $v_{\perp 0}$:

$$v_{\perp 0} = \frac{qB_0(0) r_{L0}}{m_e}$$

So, substiuting:

$$\mu = \frac{m_e}{2B_0(0)} \left(\frac{qB_0(0) r_{L0}}{m_e} \right)^2 = \frac{m_e}{2B_0(0)} \cdot \frac{q^2 B_0(0)^2 r_{L0}^2}{m_e^2}$$

$$\mu = \frac{q^2 B_0(0) r_{L0}^2}{2m_e}$$

Since μ is conserved, the magnetic moment at position z:

$$\mu = \frac{m_e v_\perp(z)^2}{2B_0(z)} = \frac{q^2 B_0(0) r_{L0}^2}{2m_e}$$

At position z:

$$r_L(z) = \frac{m_e v_\perp(z)}{qB_0(z)}$$

So:

$$v_\perp(z) = \frac{qB_0(z) r_L(z)}{m_e}$$

Substituting and solving for $r_L(z)$

$$\frac{m_e}{2B_0(z)} \left(\frac{qB_0(z) r_L(z)}{m_e} \right)^2 = \frac{q^2 B_0(0) r_{L0}^2}{2m_e}$$

$$\frac{m_e}{2B_0(z)} \cdot \frac{q^2 B_0(z)^2 r_L(z)^2}{m_e^2} = \frac{q^2 B_0(0) r_{L0}^2}{2m_e}$$

$$\frac{q^2 B_0(z) r_L(z)^2}{2m_e} = \frac{q^2 B_0(0) r_{L0}^2}{2m_e}$$

$$B_0(z) r_L(z)^2 = B_0(0) r_{L0}^2$$

$$r_L(z)^2 = \frac{B_0(0)}{B_0(z)} r_{L0}^2$$

$$\boxed{r_L(z) = r_{L0}\sqrt{\frac{B_0(0)}{B_0(z)}}}$$

Notes on the solution: - As the electron moves into regions of stronger magnetic field ($B_0(z) > B_0(0)$), the Larmor radius decreases - As the electron moves into regions of weaker magnetic field ($B_0(z) < B_0(0)$), the Larmor radius increases - This is consistent with the conservation of the magnetic moment μ

1.1.5 Part (d) - Plot of Larmor Radius Ratio

Task: Suppose the magnetic field magnitude is given by $B_0(z) = B_{min} \cdot (1 + z^2)$, where $B_{min} = B_0(0)$. Plot the ratio $r_L(z)/r_{L0}$ from $z = 0$ to $z = 10$.

Using the result from Part (c):

$$\frac{r_L(z)}{r_{L0}} = \sqrt{\frac{B_0(0)}{B_0(z)}}$$

With $B_0(z) = B_{min}(1 + z^2)$ and $B_{min} = B_0(0)$:

$$B_0(z) = B_0(0)(1 + z^2)$$

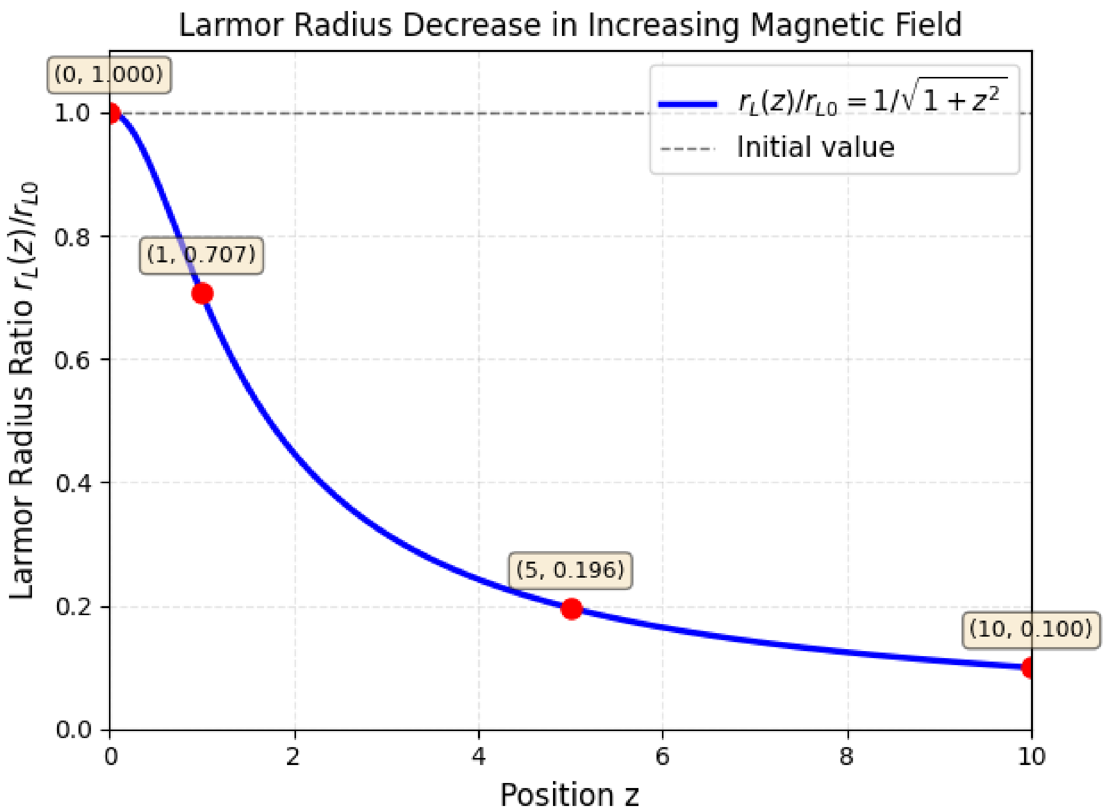
Therefore:

$$\frac{r_L(z)}{r_{L0}} = \sqrt{\frac{B_0(0)}{B_0(0)(1+z^2)}} = \sqrt{\frac{1}{1+z^2}} = \frac{1}{\sqrt{1+z^2}}$$

[37]:

```
z = np.linspace(0, 10, 500)
# Calculate Larmor radius ratio
r_L_ratio = 1 / np.sqrt(1 + z**2)

plt.figure()
plt.plot(z, r_L_ratio, 'b-', linewidth=2.5, label=r'$r_L(z)/r_{L0} = 1/
 ↪\sqrt{1+z^2}$')
plt.axhline(y=1, color='k', linestyle='--', linewidth=1, alpha=0.5,␣
 ↪label='Initial value')
plt.axhline(y=0, color='k', linestyle='-', linewidth=0.5, alpha=0.3)

# Mark some specific points
z_points = [0, 1, 5, 10]
for zp in z_points:
    rp = 1 / np.sqrt(1 + zp**2)
    plt.plot(zp, rp, 'ro', markersize=8)
    plt.text(zp, rp + 0.05, f'({zp:.0f}, {rp:.3f})',
             ha='center', fontsize=9, bbox=dict(boxstyle='round',␣
 ↪facecolor='wheat', alpha=0.5))

# Labels and formatting
```

```python
plt.xlabel('Position z', fontsize=12)
plt.ylabel(r'Larmor Radius Ratio $r_L(z)/r_{L0}$', fontsize=12)
plt.title(r'Larmor Radius Decrease in Increasing Magnetic Field')
plt.grid(True, alpha=0.3, linestyle='--')
plt.legend(fontsize=11, loc='upper right')
plt.xlim(0, 10)
plt.ylim(0, 1.1)

plt.tight_layout()
plt.show()
```



1.1.6 Part (e) - Magnetic Mirror Turning Point

Task: Although the Larmor radius decreases in regions of higher magnetic field, the electron will not propagate indefinitely. As it spirals from low to high magnetic field, the parallel velocity gradually slows until the electron reverses direction. This is the "magnetic mirroring" effect used in ion discharge chambers to trap electrons.

Consider the case where the electron starts at $z = 0$, $v_{\perp 0} = 2000$ km/s, and $v_{z0} = 400$ km/s. Find the location z^* where the electron turns around.

Hint: Magnetic fields do no work, so kinetic energy $KE = \frac{m_e}{2}(v_\perp^2 + v_z^2)$ is conserved.

First, we will start with conservation laws. Two quantities are conserved:

1. Magnetic moment: $\mu = \frac{m_e v_\perp^2}{2B_0}$ = constant
2. Kinetic energy: $KE = \frac{m_e}{2}(v_\perp^2 + v_z^2)$ = constant

Writing these at $z = 0$ and $z = z^*$:

At $z = 0$:

$$\mu = \frac{m_e v_{\perp 0}^2}{2B_0(0)}$$

$$KE = \frac{m_e}{2}(v_{\perp 0}^2 + v_{z0}^2)$$

At the turning point $z = z^*$, the parallel velocity becomes zero: $v_z(z^*) = 0$

$$\mu = \frac{m_e v_\perp(z^*)^2}{2B_0(z^*)}$$

$$KE = \frac{m_e}{2} v_\perp(z^*)^2$$

From μ conservation:

$$\frac{m_e v_{\perp 0}^2}{2B_0(0)} = \frac{m_e v_\perp(z^*)^2}{2B_0(z^*)}$$

$$v_\perp(z^*)^2 = v_{\perp 0}^2 \frac{B_0(z^*)}{B_0(0)}$$

From KE conservation:

$$v_{\perp 0}^2 + v_{z0}^2 = v_\perp(z^*)^2$$

Then, we combine the two equations. Substituting the expression for $v_\perp(z^*)^2$:

$$v_{\perp 0}^2 + v_{z0}^2 = v_{\perp 0}^2 \frac{B_0(z^*)}{B_0(0)}$$

$$1 + \frac{v_{z0}^2}{v_{\perp 0}^2} = \frac{B_0(z^*)}{B_0(0)}$$

Now, we solve for $B_0(z^*)$

$$\frac{B_0(z^*)}{B_0(0)} = 1 + \frac{v_{z0}^2}{v_{\perp 0}^2}$$

And applying the specific magnetic field profile:

Given $B_0(z) = B_0(0)(1 + z^2)$:

$$\frac{B_0(0)(1 + z^{*2})}{B_0(0)} = 1 + \frac{v_{z0}^2}{v_{\perp 0}^2}$$

$$1 + z^{*2} = 1 + \frac{v_{z0}^2}{v_{\perp 0}^2}$$

$$z^{*2} = \frac{v_{z0}^2}{v_{\perp 0}^2}$$

$$z^* = \frac{v_{z0}}{v_{\perp 0}}$$

Given: - $v_{\perp 0} = 2000$ km/s - $v_{z0} = 400$ km/s

$$z^* = \frac{400}{2000} = 0.2$$

$$\boxed{z^* = 0.2 \text{ (in units where the magnetic field is } B_0(z) = B_0(0)(1 + z^2))}$$

1.1.7 Part (f) - Mirror Ratio for Particle Trapping

Task: Not all particles are trapped by a magnetic mirror. Consider the case where the walls are located at $z = \pm L$. If the electrons have finite velocity along the magnetic field at these locations, they will escape to the walls. Show that for an electron to remain trapped it must satisfy the "mirror ratio":

$$\frac{v_{\perp 0}}{|\vec{v}_0|} > \sqrt{\frac{B_0(0)}{B_0(L)}}$$

For a particle to be trapped, it must turn around (i.e., $v_z = 0$) before reaching the wall at $z = L$. This means the turning point z^* must satisfy:

$$z^* < L$$

If $z^* \geq L$, the particle will hit the wall and escape. From Part (e), we found that at the turning point:

$$\frac{B_0(z^*)}{B_0(0)} = 1 + \frac{v_{z0}^2}{v_{\perp 0}^2} = \frac{v_{\perp 0}^2 + v_{z0}^2}{v_{\perp 0}^2} = \frac{|\vec{v}_0|^2}{v_{\perp 0}^2}$$

where $|\vec{v}_0| = \sqrt{v_{\perp 0}^2 + v_{z0}^2}$ is the initial speed.

Therefore:

$$B_0(z^*) = B_0(0) \frac{|\vec{v}_0|^2}{v_{\perp 0}^2}$$

For the particle to turn around before reaching the wall (trapping condition), we need:

$$z^* < L$$

Since $B_0(z)$ is monotonically increasing with z, this is equivalent to:

$$B_0(z^*) < B_0(L)$$

Substituting the expression for $B_0(z^*)$:

$$B_0(0)\frac{|\vec{v}_0|^2}{v_{\perp 0}^2} < B_0(L)$$

$$\frac{|\vec{v}_0|^2}{v_{\perp 0}^2} < \frac{B_0(L)}{B_0(0)}$$

Taking the square root of both sides:

$$\frac{|\vec{v}_0|}{v_{\perp 0}} < \sqrt{\frac{B_0(L)}{B_0(0)}}$$

Taking the reciprocal (and reversing the inequality):

$$\frac{v_{\perp 0}}{|\vec{v}_0|} > \sqrt{\frac{B_0(0)}{B_0(L)}}$$

So:

$$\boxed{\frac{v_{\perp 0}}{|\vec{v}_0|} > \sqrt{\frac{B_0(0)}{B_0(L)}}}$$

This is the criterion for particle trapping (mirror ratio). For a high mirror ratio, more particles are trapped. A stronger magnetic field at the mirrors is more effective at reflecting particles. For a ratio near 1, fewer particles are trapped. The magnetic field variation is too weak to reflect most particles.