# Notes for MATH 561 Differential Geometry

Zhan Cheng

Spring 2025

## 1 Lecture 1

### 1.1 Felix Klein's Erlangen-Nürnberg Theorem

Let $(X, G)$ be a pair where:

- $X$ is a set, and

- $G$ is a group with operation $\cdot$.

A group $G$ is defined as a set with:

1. A binary operation $G \times G \to G$, $\forall g_1, g_2, g_3 \in G$, $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$.

2. An identity element $e \in G$ such that $g \cdot e = g = e \cdot g$, $\forall g \in G$.

3. An inverse $g^{-1} \in G$ for every $g \in G$, satisfying $g \cdot g^{-1} = g^{-1} \cdot g = e$.

**Lemma 1.1.** *The group $(G, \cdot)$ satisfies:*

1. *The identity element is unique.*

2. *The inverse $g^{-1}$ is unique for each $g \in G$.*

*Proof.*    1. Assume there are two identity elements $e_1$ and $e_2$. By the definition of the identity element:
$$e_1 = e_1 \cdot e_2 = e_2.$$

Hence, the identity element is unique.

2. Let $g \in G$ and suppose $g$ has two inverses $g_1$ and $g_2$. By the definition of the inverse:

$$g_1 = g_1 \cdot e = g_1 \cdot (g \cdot g_2) = (g_1 \cdot g) \cdot g_2 = e \cdot g_2 = g_2.$$

Therefore, the inverse of $g$ is unique. $\qquad\square$

### 1.2 Transformation Group

**Definition 1.1** (Transformation Group)**.** *A group $G$ is called a transformation group on $X$ if $G$ is a set of bijections $\phi : X \to X$ and the group multiplication is given by the function composition.*

## Example 1: Euclidean Isometries

Let $X = \mathbb{R}^n$ with the Euclidean distance $d_0(x, y) = \|x - y\|$. A Euclidean isometry is a mapping $\phi : \mathbb{R}^n \to \mathbb{R}^n$ such that:

$$d_0(\phi(x), \phi(y)) = d_0(x, y), \quad \forall x, y \in \mathbb{R}^n.$$

The group of isometries of $\mathbb{R}^n$ is:

$$\text{Isom}(\mathbb{R}^n) = \{\phi \mid \phi(x) = Ax + b,\ b \in \mathbb{R}^n,\ A \in O(n)\}.$$

Here:

1. $O(n)$ is the set of all $n \times n$ orthogonal matrices, i.e., $A^T A = I$.

2. $b \in \mathbb{R}^n$ is a translation vector.

### Question

Why do Euclidean isometries use orthogonal matrices?

### Answer

Orthogonal matrices preserve the length of vectors and angles. For $\phi(x) = Ax + b$, $d_0(\phi(x), \phi(y)) = \|A(x-y)\| = \|x-y\|$, which holds only if $A^\top A = I$. Geometrically, orthogonal matrices represent rotations and reflections.

### Example

1. Rotation Matrix:
$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$
This rotates vectors by angle $\theta$ around the origin.
2. Reflection Matrix:
$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$
This reflects vectors across the $x$-axis.

## Example 2: Affine Transformations

Affine transformations extend Euclidean isometries by allowing general linear mappings and translations. An affine transformation $\phi$ is defined as:

$$\phi(x) = Ax + b, \quad A \in \text{GL}(n, \mathbb{R}), b \in \mathbb{R}^n,$$

where:

1. $A \in \text{GL}(n, \mathbb{R})$, the group of all invertible $n \times n$ matrices.

2. $b \in \mathbb{R}^n$ is a translation vector.

**Question**

Why must the matrix in an affine transformation be invertible?

**Answer**

Invertibility ensures:
1. No collapse to lower dimensions (e.g., projecting $\mathbb{R}^2$ to a line).
2. Bijectivity: every point has a unique preimage, enabling the inverse transformation:
$$\phi^{-1}(x) = A^{-1}(x - b).$$

**Example**

1. Non-invertible matrix:
$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

This projects all points onto the $x$-axis, losing information about the $y$-coordinate.
2. Invertible matrix:
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

This preserves the full dimensionality of the space.

### 1.3  Metric Space

**Definition 1.2** (Metric Space). *A metric space is a pair $(X, d)$ where $X$ is a set and $d : X \times X \to [0, \infty)$ is a distance function satisfying:*

*1. $d(p, q) \geq 0$, and $d(p, q) = 0 \iff p = q$, $\forall p, q \in X$.*

*2. $d(p, q) = d(q, p)$, $\forall p, q \in X$.*

*3. $d(p, q) \leq d(p, w) + d(w, q)$, $\forall p, q, w \in X$.*

**Lemma 1.2.** *For $(\mathbb{R}^n, d_0)$, $K \subset \mathbb{R}^n$, Two True, Force the Third:*

*1. $K$ is compact.*

*2. $K$ is sequentially compact.*

*3. $K$ is bounded and closed.*

**Question**

What is compactness, sequential compactness, and the relationship with bounded and closed sets?

**Example**

1. $[0, 1]$: - Compact: Yes (bounded and closed). - Sequentially compact: Yes.

2. $(0, 1)$: - Compact: No (not closed). - Sequentially compact: No (e.g., sequence $x_n = \frac{1}{n}$ converges to 0, which is not in $(0, 1)$).

3. $\mathbb{R}$: - Compact: No (not bounded).

## 1.4 Arc Length

**Definition (Arc Length):** Let $(X, d)$ be a metric space and $\gamma : [0, 1] \to X$ be a curve. Then the arc length of $\gamma$ is:

$$L(\gamma) = \sup_{k \in \mathbb{Z}_+} \sum_{j=1}^{k} d(\gamma(t_{j-1}), \gamma(t_j)),$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_{k-1} < t_k = 1$ and $\{t_j\}_{j=0}^{k}$ is a partition of $[0, 1]$. If $L(\gamma) < \infty$, the curve $\gamma$ is called *rectifiable*.

**Question**

Why does the definition of arc length use sup instead of inf?

**Answer**

Arc length is defined as the supremum over all possible partitions of the sum of distances between consecutive points on a curve. Using sup ensures that finer partitions can better approximate the true curve length. If inf were used, coarse partitions might underestimate the length, failing to capture the curve's geometry.

**Example**

For the parabola $\gamma(t) = (t, t^2), t \in [0, 1]$:
- Actual arc length: $L(\gamma) > 1.478$.
- Coarse partition (e.g., $t_0 = 0, t_1 = 1$): Straight-line length $\sqrt{2} \approx 1.414$, which underestimates the true length.

## 1.5 Length Metric

**Definition (Length Metric):** Let $(X, d)$ be a metric space, the length metric $d^{\#}(p, q)$ is defined as:

$$d^{\#}(p, q) = \inf L(\gamma),$$

where $\gamma : [0,1] \to X$ is a rectifiable curve connecting $p$ and $q$.

---

**Question**

Why does the length metric use inf?

---

**Answer**

The length metric $d^{\#}(p,q)$ is defined as the infimum of arc lengths over all curves connecting $p$ and $q$. This ensures that the metric captures the shortest possible distance, avoiding unnecessary detours or longer paths.

---

**Example**

1. In $\mathbb{R}^2$, the length metric $d^{\#}(p,q)$ equals the Euclidean distance $d(p,q)$ because the straight line minimizes the length.
2. On a sphere, $d^{\#}(p,q)$ corresponds to the length of the shortest great circle arc between $p$ and $q$.

---

# 2 Lecture 2

## 2.1 Curve

**Definition (Curve):** A mapping $\gamma : I \to X$, where $I \subseteq \mathbb{R}$ is an interval, is said to be a **curve** if $\gamma$ is continuous.

**Lemma 2.1.** *If $(X, d)$ is a metric space, then $(X, d^{\#})$ is also a metric space.*

**Theorem 2.1.** *$(\mathbb{R}^n, d_0)$ is a connected and complete metric space (and is also path-connected).*

## 2.2 Connectedness

**Definition (Connectedness):** A metric space $(X, d)$ is said to be **connected** if $X$ does not admit any separation:

$$X = A \cup B, \quad A, B \neq \varnothing, \quad A \cap B = \varnothing, \quad \text{with } A, B \text{ both open.}$$

**Definition (Path Connectedness):** A metric space $X$ is said to be **path connected** if $\forall p, q \in X$, there exists a continuous curve $\gamma$ connecting $p$ and $q$.

**Lemma 2.2.** *Path connected implies connected.*

## 2.3 Parameterization by Arc Length

Let $\gamma : [0,1] \to \mathbb{R}^n$ be a $C^1$-curve. Denote:

- $\gamma'(t)$: tangent vector at $t$,

- $\|\gamma'(t)\|$: speed at $t$.

The length of $\gamma$ is given by:

$$L(\gamma) = \int_0^1 \|\gamma'(t)\| dt.$$

Define the arc length function:

$$s(t) = \int_0^t \|\gamma'(\tau)\| d\tau, \quad s'(t) = \|\gamma'(t)\| \geq 0.$$

**Definition (Parameterization by Arc Length):** Let $\gamma : [0, L(\gamma)] \to \mathbb{R}^n$ and $L(\gamma\big|_{[0,s]}) = s$.

**Lemma 2.3.** $\|\gamma'(s)\| = 1$.

**Lemma 2.4.** *Any $C^1$ curve meets a constant speed parameterization.*

## 2.4  Coincidence of Distance Structures

**Proposition 2.1.** *The two distance structures $d_0$ and $d_0^\#$ coincide on $(\mathbb{R}^n, d_0)$, where $d_0^\#$ is realized by $\|p - q\|$.*

*Proof.* Let $\gamma_0 : [0, 1] \to \mathbb{R}^n$ be a smooth curve connecting $p$ and $q$, with constant speed $v$. We define the variation of a curve $\gamma_0$ as:

$$V(t, s) = \gamma_0(t) + s\psi(t),$$

where $-\epsilon \leq s \leq \epsilon$, $\psi$ is a smooth function with $\psi(0) = \psi(1) = 0$.

Thus, $V(0, s) = p$, $V(1, s) = q$, and $L(s) = L(V(\cdot, s)) = \int_0^1 \|\partial_t V(t, s)\| dt$.

We compute $L'(0)$ as follows:

$$
\begin{aligned}
L'(0) &= \frac{d}{ds}\bigg|_{s=0} \int_0^1 \|\partial_t V(t, s)\| \, dt \\
&= \int_0^1 \frac{\partial}{\partial s}\bigg|_{s=0} \|\partial_t V(t, s)\| \, dt \\
&= \int_0^1 \frac{\frac{\partial}{\partial s}\langle \partial_t V(t, s), \partial_t V(t, s)\rangle^{1/2}}{\|\partial_t V(t, s)\|}\bigg|_{s=0} \, dt \\
&= \int_0^1 \frac{1}{2}\langle \partial_t V(t, s), \partial_t V(t, s)\rangle^{-1/2} \cdot 2\langle \frac{\partial^2 V}{\partial s \partial t}, \partial_t V(t, s)\rangle\bigg|_{s=0} \, dt \\
&= \int_0^1 \frac{\langle \frac{\partial^2 V}{\partial s \partial t}, \partial_t V(t, s)\rangle}{\|\partial_t V(t, s)\|}\bigg|_{s=0} \, dt \\
&= \int_0^1 v^{-1}\langle \psi'(t), \gamma_0'(t)\rangle \, dt \\
&= v^{-1}\langle \psi(t), \gamma_0'(t)\rangle\bigg|_0^1 - v^{-1}\int_0^1 \langle \psi(t), \gamma_0''(t)\rangle \, dt \\
&= -v^{-1}\int_0^1 \langle \psi(t), \gamma_0''(t)\rangle \, dt,
\end{aligned}
$$

where we used the fact that $\psi(0) = \psi(1) = 0$ to eliminate the boundary terms.

6

Thus, combining these results, we find that the variation $L'(0)$ vanishes when $\gamma_0$ is a straight line connecting $p$ and $q$, since in this case $\gamma_0''(t) = 0$ for all $t$. This implies that the shortest path connecting $p$ and $q$ in the distance structure $d_0^{\#}$ is realized by the Euclidean norm $\|p - q\|$.

Finally, since $d_0$ also corresponds to the Euclidean distance, it follows that $d_0$ and $d_0^{\#}$ coincide on $(\mathbb{R}^n, d_0)$. $\qquad\square$

# 3 Lecture 3

## 3.1 Length Space

**Definition (Length Space):** A metric space $(X, d)$ is called a **length space** if $d^{\#} = d$.

**Theorem 3.1.** $(\mathbb{R}^n, d_0)$ *is a length space. In other words, for all* $p, q \in \mathbb{R}^n$, *we have* $d_0(p, q) = d_0^{\#}(p, q)$. *Moreover, the shortest curve connecting* $p$ *and* $q$ *is the line segment* $pq$.

**Remark:**

$$d_0(p, q) \le d_0^{\#}(p, q) \le \inf\{\ell(\gamma) \mid \gamma \text{ is a smooth curve connecting } p, q\} = d_0(p, q)$$

$$\Rightarrow d_0(p, q) = d(p, q).$$

## 3.2 Isometry

**Definition (Isometry):** A function $F : \mathbb{R}^n \to \mathbb{R}^n$ is called an **isometry** if $F$ preserves distances:
$$d_0(F(x), F(y)) = d_0(x, y), \quad \forall x, y \in \mathbb{R}^n.$$

**Lemma 3.1.** *If* $F(x) = Ax + b$, *where* $A \in O(n)$ *and* $b \in \mathbb{R}^n$, *then* $F$ *is an isometry.*

*Proof.*
$$[d_0(F(x), F(y))]^2 = \|Ax - Ay\|^2$$
$$= \langle A(x - y), A(x - y) \rangle$$
$$= \langle A^T A(x - y), x - y \rangle$$
$$= \|x - y\|^2 = [d_0(x, y)]^2.$$
Thus, $d_0(F(x), F(y)) = d_0(x, y)$. $\qquad\square$

**Lemma 3.2.**    *1. The set of all Euclidean isometries is a group with respect to composition, denoted by* $Isom(\mathbb{R}^n)$

   *2. The set of all affine mappings* $x \to Ax + b$, *where* $A \in O(n)$ *and* $b \in \mathbb{R}^n$, *is also a group with respect to composition.*

**Definition (Isometry):** More generally, given two metric spaces $(X, d_X)$ and $(Y, d_Y)$, a function $F : X \to Y$ is called an *isometry* if:

1. It is an **isometric embedding**:

$$d_Y(F(p), F(q)) = d_X(p, q), \quad \forall p, q \in X.$$

7

2. $F$ is subjective.

**Lemma 3.3.** *The following are equivalent:*

1. $F : \mathbb{R}^n \to \mathbb{R}^n$ *is an isometry and* $F(0) = 0$.

2. $F : \mathbb{R}^n \to \mathbb{R}^n$ *preserves inner products* $\langle \cdot, \cdot \rangle$.

*Proof.* We prove $(2) \Rightarrow (1)$ here.

$$[d_0(F(x), F(y))]^2 = ||F(x) - F(y)||^2$$

$$= ||F(x)||^2 + ||F(y)||^2 - 2\langle F(x), F(y) \rangle.$$

Also, by

$$||F(0)||^2 = \langle F(0), F(0) \rangle = \langle 0, 0 \rangle = 0,$$

we get

$$||F(x)||^2 = [d_0(F(x), F(0))]^2 = \langle F(x), F(0) \rangle = ||x||^2,$$

$$||F(y)||^2 = ||y||^2,$$

Since

$$\langle F(x), F(y) \rangle = \langle x, y \rangle,$$

we prove that

$$d_0(F(x), F(y)) = d_0(x, y).$$

$\square$

**Proposition 3.1.** *If* $\{e_j\}_{j=1}^n$ *is an orthonormal basis of* $\mathbb{R}^n$ *and* $F : \mathbb{R}^n \to \mathbb{R}^n$ *is an isometry satisfying:*

1. $F(0) = 0$.

2. $F(e_j) = e_j, \quad \forall 1 \le j \le n.$

*Then* $F$ *must be the identity.*

*Proof.* For any $x = \sum_{j=1}^n x_j e_j$, we want to show:

$$\langle F(x), e_j \rangle = \langle x, e_j \rangle, \quad \forall 1 \le j \le n.$$

By Lemma 3.3, $F$ preserves inner products:

$$\langle F(x), e_j \rangle = \langle F(x), F(e_j) \rangle = \langle x, e_j \rangle.$$

$\square$

**Theorem 3.2.** *The following are equivalent:*

1. $F : \mathbb{R}^n \to \mathbb{R}^n$ *is an isometry.*

2. $F$ *is an affine mapping, where* $F(x) = Ax + b$, *where* $A \in O(n)$ *and* $b \in \mathbb{R}^n$

*Proof.* We prove $(1) \Rightarrow (2)$ here since $(2) \Rightarrow (1)$ is proved by Lemma 3.1.

Define $b = F(0) \in \mathbb{R}^n$ and set $\hat{F}(x) = F(x) - b$ so that $\hat{F}(0) = 0$. Let $\{e_j\}_{j=1}^n$ be an orthonormal basis. Define: $\hat{e}_j = \hat{F}(e_j), \quad \forall 1 \le j \le n$.

Since $\{\hat{e}_j\}$ is an orthonormal basis, there exists a unique $\hat{G} \in O(n)$ such that:

$$\hat{G}(\hat{e}_j) = e_j, \quad \forall 1 \le j \le n.$$

Thus, the mapping $\hat{G} \circ \hat{F} : \mathbb{R}^n \to \mathbb{R}^n$ preserves the origin and satisfies:

$$\hat{G} \circ \hat{F}(e_j) = e_j \quad \forall 1 \le j \le n,$$

which implies that

$$\hat{G} \circ \hat{F} = \mathrm{Id}$$

Therefore, $F(x) = \hat{G}^{-1}(x) + b$. $\qquad\square$

### 3.3  Local Geometry of Euclidean Curves

**Definition (Regular):** A smooth curve $\gamma : [0,1] \to \mathbb{R}^n$ has a **regular** parameterization if:

$$\gamma'(t) \ne 0.$$

Let $\gamma : (0, l) \to \mathbb{R}^n$ be parameterized by arc length:

$$\|\gamma'(s)\| = 1, \quad \forall s \in (0, l).$$

We denote $T(s) = \gamma'(s)$, $N(s) \equiv \{\text{Unit vector in the direction of } \gamma''(s)\}$.

**Definition (Curvature):** The **curvature** of a curve is:

$$\kappa(s) = \|T'(s)\| = \|\gamma''(s)\|, \text{ where } \gamma''(s) = \kappa(s)N(s).$$

**Example:**

The curve $\gamma(s) = (\cos(s), \sin(s))$ has a constant curvature $\kappa = 1$.

## 4  Lecture 4

**Lemma 4.1.** *A regular curve $\gamma : [0,1] \to \mathbb{R}^n$ with $F : \mathbb{R}^n \to \mathbb{R}^n$, $F \in Isom(\mathbb{R}^n)$, we have*

$$\forall t_0 \in [0,1] \quad \|\frac{d}{dt}\Big|_{t=t_0}(F \circ \gamma)\| = \|\frac{d}{dt}\Big|_{t=t_0}\gamma\|$$

*Proof.* Let $\bar{\gamma} = F \circ \gamma$, we compute

$$\frac{d\bar{\gamma}}{dt}\Big|_{t=t_0} = \mathrm{DF}(\frac{d\gamma}{dt}\Big|_{t=t_0}) \quad \text{, where DF is the Jacobian matrix of } F,$$

given by

$$DF = \begin{bmatrix} \frac{\partial f_1}{\partial t_1} & \cdots & \frac{\partial f_1}{\partial t_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial t_1} & \cdots & \frac{\partial f_n}{\partial t_n} \end{bmatrix}$$

Then

$$LHS = \|\mathrm{DF}(\gamma')\| = \|A\gamma'\| = \|\gamma'\| = RHS,$$

where $F(x) = Ax + b$, $A \in O(n), b \in \mathbb{R}^n$. $\qquad\square$

Note that Lemma 4.1 actually tells us that isometry doesn't change the curvature in $\mathbb{R}^n$.

**Lemma 4.2.** *Let $\gamma : [0, l] \rightarrow \mathbb{R}^n$ has an arc-length parameter $s$, with an isometry $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $F \in \text{Isom}(\mathbb{R}^n)$, then*

$$\forall 0 \leq s \leq l \quad \kappa(\tilde{\gamma}(s)) = \kappa(\gamma(s)), \quad \text{where } \tilde{\gamma} = F \circ \gamma$$

*Proof.*

$$\kappa(\tilde{\gamma}(s)) = \left\| \frac{d}{ds} \frac{d}{ds}(F \circ \gamma) \right\| = \left\| \frac{d}{ds}\bar{T}(s) \right\| = \left\| \frac{d}{ds}(\text{DF}(\gamma'(s))) \right\| = \|A\gamma''(s)\| = \|\gamma''(s)\|$$

$\square$

Let $\gamma : [0, 1] \rightarrow \mathbb{R}^n$ be a regular curve, then $\gamma(s)$ can be written as $\gamma(t(s))$. Thus,

$$\frac{d\gamma}{ds} = (\frac{ds}{dt})\frac{d\gamma}{dt} = \frac{1}{\|\gamma'(t)\|}\frac{d\gamma}{dt}.$$

We note that

$$s(t) = \int_0^t \|\gamma'(\tau)\|d\tau \Rightarrow \frac{ds}{dt} = \|\gamma'(t)\|.$$

Also,

$$\frac{d^2\gamma}{ds^2} = \frac{d}{ds}(\frac{\gamma'(t)}{\|\gamma'(t)\|}) = \frac{d}{dt}(\frac{\gamma'(t)}{\|\gamma'(t)\|})\frac{dt}{ds} = \frac{1}{\|\gamma'(t)\|}\frac{d}{dt}(\frac{\gamma'(t)}{\|\gamma'(t)\|}),$$

which implies that

$$\kappa(\gamma(s)) = \frac{\|\frac{d}{dt}(\frac{\gamma'(t)}{\|\gamma'(t)\|})\|}{\|\gamma'(t)\|}$$

## 4.1   Regular Curves in $\mathbb{R}^3$

Let $\gamma : [0, 1] \rightarrow \mathbb{R}^3$ be a regular curve, having an arc-length parameter $s$. As mentioned above, we denote $T(s) = \gamma'(s)$, $\frac{dT}{ds} = \|T'(s)\| = \|\gamma''(s)\|$, where $\gamma''(s) = \kappa(s)N(s)$.

**Definition (Wedge product):** The **Wedge product** between vector $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ is given by :

$$u \wedge v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

We denote that $\beta(s) = T(s) \wedge N(s)$. Since $\|\gamma'(s)\| = 1$ by definition, then

$$0 \equiv \frac{d}{ds}\langle \gamma', \gamma' \rangle = 2\langle \gamma'', \gamma' \rangle,$$

which implies that $\gamma'' \perp \gamma'$. Here $N \perp T$.

We can naturally give out that $\{T, N, \beta\}$ is an orthogonal basis on $\mathbb{R}^3$. Moreover,

$$T \wedge N = \beta, N \wedge \beta = T, \beta \wedge T = N.$$

This is known as **Frenet Frame**.

10

**Theorem 4.1.** *Let $\{T, N, \beta\}$ be the Frenet frame at $\gamma(s)$, then*

$$\begin{cases} \dfrac{dT}{ds} = \kappa N \\[2mm] \dfrac{dN}{ds} = -\tau\beta - \kappa T \\[2mm] \dfrac{d\beta}{ds} = \tau N \end{cases}$$

*Here $\tau$ is called the torsion of $\gamma$, given by $\beta'(s) = \tau(s)N(s)$. Moreover, it can be written as*

$$\begin{bmatrix} T \\ N \\ \beta \end{bmatrix}' = \Phi \begin{bmatrix} T \\ N \\ \beta \end{bmatrix},$$

*where*

$$\Phi = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix}.$$

*Proof.* Firstly, $\frac{dT}{ds} = \kappa N$ is proved by definition. Since $\langle N, N \rangle = 1$, we have

$$0 \equiv \frac{d}{ds}\langle N, N \rangle = 2\langle \frac{dN}{ds}, N \rangle \Rightarrow \frac{dN}{ds} \perp N \Rightarrow \frac{dN}{ds} \in \mathrm{Span}\{\beta, T\}.$$

Since $\beta = T \wedge N$, then

$$\frac{d\beta}{ds} = T'(s) \wedge N(s) + T(s) \wedge N'(s) = T(s) \wedge N'(s) \Rightarrow \frac{d\beta}{ds} \perp T \Rightarrow \frac{d\beta}{ds} \in \mathrm{Span}\{\beta, N\}.$$

Also,

$$0 \equiv \frac{d}{ds}\langle \beta, \beta \rangle = 2\langle \frac{d\beta}{ds}, \beta \rangle \Rightarrow \frac{d\beta}{ds} \perp \beta$$

Thus,

$$\frac{d\beta}{ds} \parallel N,$$

and we write $\beta'(s) = \tau(s)N(s)$.

Additionally,

$$\frac{dN}{ds} = \frac{d\beta}{ds} \wedge T + \beta \wedge \frac{dT}{ds} = \tau N \wedge T + \beta \wedge \kappa N = -\tau\beta - \kappa T.$$

$\square$

# 5   Lecture 5

## 5.1   *Multi-variable Calculus: A quick review*

Consider the following functions:

$$f : \mathbb{R}^m \to \mathbb{R}^n \quad \gamma : [0, 1] \to \mathbb{R}^n$$

we have

1. $F \circ \gamma : [0, 1] \to \mathbb{R}^n$

2. $DF : \mathbb{R}^m \to \mathbb{R}^n$   (Jacobi matrix)

Additionally, given a simple example with $f : \mathbb{R} \to \mathbb{R}$, its differential is given by $df = f'(x)dx$, and $f(x)$ can be written as

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + O(|x - x_0|^2).$$

Consider a regular curve $\gamma(t) = (t, f(t))$,   $\gamma'(t) = (1, f'(t))$, we can quickly write out the tangent line at $(t_0, f(t_0))$:

- Passing through $(t_0, f(t_0))$

- Velocity at $t_0$ is $(1, f'(t_0))$

And the length is given by

$$L(t) = (t, f'(t_0)(t - t_0) + f(t_0)).$$

## 5.2   Fundamental Theorem of Regular Curves

**Theorem 5.1.** *Let $A(s)$ be an $(n \times n)$-matrix, and each entry is continuous on interval $I \subset \mathbb{R}$. Consider the system:*

$$\frac{d}{ds} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = A(s) \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \quad s \in I.$$

*Then,*

1. *Given any $(v_1, \ldots, v_n) \in \mathbb{R}^n$, there exists a unique solution*

$$V(s) = (V_1(s), \ldots, V_n(s))$$

   *such that*
$$V(0) = (v_1, \ldots, v_n)$$

2. *The set $S$ of all solutions of the equation above is a vector space of $n$ dimensions, so that*
$$\Phi : \mathbb{R}^n \to S, \quad (v_1, \ldots, v_n) \mapsto (V_1(s), \ldots, V_n(s))$$
   *is an isomorphism.*

**Theorem 5.2.** *Given two continuous functions $\kappa(s)$, $\tau(s)$, $s \in I$, then there exists a unique regular curve*
$$\gamma : I \to \mathbb{R}^3,$$
*(up to a Euclidean isometry) such that $\gamma(s)$ has curvature $\kappa(s)$ and torsion $\tau(s)$.*

*Proof.* Let $\gamma(s)$ be a regular curve parameterized by arc-length $s$, meaning that $\|\gamma'(s)\| = 1$. The Frenet frame consists of $\{T, N, \beta\}$, which satisfy the Frenet equations:

$$\begin{bmatrix} T \\ N \\ \beta \end{bmatrix}' = \Phi \begin{bmatrix} T \\ N \\ \beta \end{bmatrix},$$

where

$$\Phi = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix}.$$

Given $\kappa(s)$ and $\tau(s)$, we need to construct a frame $\{T, N, \beta\}$ satisfying the Frenet system. This system is a first-order linear ODE for the orthonormal frame:

$$\frac{d}{ds}Y(s) = A(s)Y(s), \quad \text{where } Y(s) = \begin{bmatrix} T(s) \\ N(s) \\ \beta(s) \end{bmatrix}, \quad A(s) = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix}.$$

By the existence and uniqueness theorem for linear ODEs, for any given initial orthonormal frame $(T_0, N_0, \beta_0)$ at $s_0$, there exists a unique solution $(T(s), N(s), \beta(s))$ satisfying this system.

Since $T(s) = \gamma'(s)$, the curve $\gamma(s)$ itself can be recovered by integrating:

$$\gamma(s) = \gamma(s_0) + \int_{s_0}^{s} T(u)\, du.$$

Thus, given an initial point $\gamma(s_0)$ and an initial Frenet frame $(T_0, N_0, \beta_0)$, we can uniquely reconstruct the curve $\gamma(s)$.

Suppose there exist two different curves $\gamma(s)$ and $\bar{\gamma}(s)$ with the same curvature $\kappa(s)$ and torsion $\tau(s)$. The system governed by $A(s)$ has a total of 9 degrees of freedom. However, specifying the initial position $\gamma(s_0)$ reduces the freedom by 3 dimensions, the unit tangent vector $T(s_0)$ by 2 dimensions, and the principal normal vector $N(s_0)$ by 1 dimension. Consequently, a unique Euclidean isometry $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ exists such that

$$\varphi(\gamma(s_0)) = \bar{\gamma}(s_0), \quad \varphi(T_0, N_0, \beta_0) = (\bar{T}_0, \bar{N}_0, \bar{\beta}_0).$$

This implies that any two solutions differing only by a Euclidean transformation must be identical up to an isometry.

Thus, we have proven that for given functions $\kappa(s)$ and $\tau(s)$, there exists a unique regular curve up to a Euclidean isometry. $\qquad\square$

# 6 Lecture 6

**Corollary 6.1.** *Any curve in $\mathbb{R}^3$ with constant curvature and constant torsion must be either a circle ($\tau \equiv 0$) or a helix ($\tau \equiv constant > 0$).*

**Definition (Tangent Curve):** Given a curve $\gamma : [0, L] \to \mathbb{R}^n$, the tangent line at $\gamma(t_0)$ is the unique line

$$l_{t_0}(t) = \gamma'(t_0)(t - t_0) + \gamma(t_0) \quad l'_{t_0}(t) = \gamma'(t_0).$$

13

Fix $\varepsilon > 0$, and let $s_0 \in [0, L]$. Define

$$\gamma_{\varepsilon, s_0}(s) = \frac{\gamma(\varepsilon s + s_0) - \gamma(s_0)}{\varepsilon},$$

which satisfies:

1. $\gamma_{\varepsilon, s_0}(0) = 0$.

2. $\lim_{\varepsilon \to 0} \gamma_{\varepsilon, s_0}(s) = \gamma'(s_0) s$.

Here if $\gamma$ is smooth, then

$$\|\gamma'_{\varepsilon, s_0}(s)\| = 1 \quad \kappa(\gamma_{\varepsilon, s_0}(s)) = \varepsilon \kappa(\gamma(\varepsilon s + s_0)).$$

Then we define the **Tangent Curve** of $\gamma$ at $s = s_0$ as

$$\gamma_{s_0}(s) \equiv \lim_{\varepsilon \to 0} \gamma_{\varepsilon, s_0}(s) + \gamma(s_0).$$

**Definition (Asymptotic Curve):** Beyond local behavior, we consider the asymptotic limit as $s \to \infty$. For any $\varepsilon \in (0, 1)$, we define

$$\gamma_{\varepsilon^{-1}, s_0}(s) \equiv \frac{\gamma(\varepsilon^{-1} s + s_0) - \gamma(s_0)}{\varepsilon^{-1}},$$

where $\gamma_{\varepsilon^{-1}, s_0}(0) = 0$, and if $\gamma$ is smooth, then $\|\gamma'_{\varepsilon^{-1}, s_0}(s)\| = 1$.

The limiting curve:
$$\gamma_\infty(s) = \lim_{\varepsilon \to 0} \gamma_{\varepsilon^{-1}, s_0}(s) + \gamma(s_0)$$

is called the **asymptotic curve** of $\gamma$ centered at $\gamma(s_0)$. Here

$$\|\gamma'_{\varepsilon^{-1}, s_0}(s)\| = \varepsilon^{-1} \kappa(\gamma(\varepsilon^{-1} s + s_0)).$$

If the curvature satisfies:
$$\kappa(\gamma(s)) = o(s),$$

then the asymptotic curve $\gamma_\infty$ has curvature equal to zero when $s \neq 0$, which is ensured by

$$\lim_{\varepsilon \to 0} \kappa(\gamma_{\varepsilon^{-1}, s_0}(s)) = \lim_{\varepsilon \to 0} \frac{\kappa(\gamma(\varepsilon^{-1} s + s_0))}{\varepsilon} = 0$$

# 7 Lecture 7

**Lemma 7.1.** *If a rectifiable curve $\gamma$ realizes $d_0(x, y)$, then it must be $\gamma = \overline{xy}$.*

*Proof.* Suppose $\exists t_0 \in (0, 1)$ s.t. $\gamma(t_0) \notin \overline{xy}$, then

$$d_0(x, \gamma(t_0)) = L(\gamma|_{[0, t_0]})$$

$$d_0(\gamma(t_0), y) = L(\gamma|_{[t_0, 1]})$$

Taking the sum,

$$d_0(x, y) < d_0(x, \gamma(t_0)) + d_0(\gamma(t_0), y) = L(\gamma|_{[0, t_0]}) + L(\gamma|_{[t_0, 1]}) = L(\gamma) = d_0(x, y)$$

which is a contradiction. $\qquad\square$

**Definition (Riemannian patch):** Let $U \subseteq \mathbb{R}^n$ be open and connected. A function $g : U \to \mathcal{B}(\mathbb{R}^n)$ is said to be a **Riemannian structure (metric)** on $U$ if

1. $\forall x \in U, g_x$ is an inner product with $g_x(u, v) = \langle u, G_x v \rangle$

2. $G_x$ is smooth on $U$.

Here $\mathcal{B}(\mathbb{R}^n)$ is the space of all bilinear functions on $\mathbb{R}^n$. In this situation, $(U, g)$ is called a **Riemannian patch**.

Consider a function $f : U \to \mathbb{R}$, where $U \subseteq \mathbb{R}^n$ is open and connected, and $\Gamma(f) \equiv \{(x, f(x)) : x \in U\} \subseteq \mathbb{R}^n \times \mathbb{R}$. Let $\gamma : [0, 1] \to U$ be a curve

$$\hat{\gamma}(t) \equiv (\gamma(t), f(\gamma(t))) \in \Gamma(f).$$

Compute $L(\hat{\gamma})$ and $\frac{d\hat{\gamma}}{dt}$:

$$L(\hat{\gamma}) = \int |\hat{\gamma}'(t)| dt, \quad \frac{d\hat{\gamma}}{dt} = \left( \frac{d\gamma}{dt}, \langle \nabla f, \frac{d\gamma}{dt} \rangle \right).$$

Then

$$\left\langle \frac{d\hat{\gamma}}{dt}, \frac{d\hat{\gamma}}{dt} \right\rangle = \| \frac{d\gamma}{dt} \|^2 + \langle \nabla f, \frac{d\gamma}{dt} \rangle \langle \nabla f, \frac{d\gamma}{dt} \rangle$$

We define

$$g_x(u, v) \equiv \langle u, v \rangle + \langle \nabla f, u \rangle \langle \nabla f, v \rangle,$$

then $\forall x \in U$, $g_x$ is an inner product.

**Recall:** Given any orthonormal basis of $\mathbb{R}^n$, any inner product uniquely corresponds to a positive definite matrix.

$$g_x(u, v) \equiv \langle u, G_x v \rangle,$$

where $G_x$ is a positive definite matrix, satisfying

1. $G_x^T = G_x$

2. $x^T G_x x > 0, \forall x \in \mathbb{R}^n$

**Example:** Consider Riemannian patch $(B_1(0), g_k)$, we have

$$G_k(x) = \frac{4}{(1 + k|x|^2)^2} \mathrm{Id}_{n \times n}$$

Specifically, if $k = -1$, consider

$$\gamma(t) = tv, \quad ||v||_g = 1, \quad \gamma'(t) = v.$$

Then $g_{-1}$ is given by

$$g_{-1}(\gamma'(t), \gamma'(t)) = g_{\gamma(t)}(v, v) = \langle v, G_{\gamma(t)} v \rangle = \frac{4v}{(1 - ||\gamma(t)||^2)^2} = \frac{4}{(1 - ||tv||^2)^2}.$$

Thus,

$$L(\gamma_{[0,r]}) = \int_0^r \left( \frac{4}{(1 - t^2)^2} \right)^{\frac{1}{2}} dt = \ln \left( \frac{1 + r}{1 - r} \right)$$

# 8 Lecture 8

## 8.1 Length structure on a Riemannian patch

Consider a Riemannian patch $(U, g)$, where $U \subseteq \mathbb{R}^n$ is a domain, and we have

1. $\forall x \in U, g_x(u, v) = \langle u, G_x v \rangle$


2. $g_x$ is smooth in $x \Leftrightarrow G_x$ is smooth on $U$.

$\gamma : [0, 1] \to U$. For a $C^1$-curve $\gamma : [0, 1] \to U$, we define its arc length and length metric by

1. $L_g(\gamma) = \int_0^1 \left( g_{\gamma(t)}(\gamma'(t), \gamma'(t)) \right)^{\frac{1}{2}} dt$


2. $d_g(x, y) \equiv \inf\{L_g(\gamma) : \gamma$ is a piecewise $C^1$ curve connecting $x, y\}$

**Theorem 8.1.** $(U, g)$ *is a Riemannian patch* $\Rightarrow (U, d_g)$ *is a metric space.*

*Proof.* We only prove $d_g(x, y) = 0 \Rightarrow x = y$ here.

   **Claim 1:** If $d_g(x, y) = 0$, then $y \in B_\varepsilon(x, g_0)$, where $\varepsilon_0$ is the number that satisfies $\bar{B}_\varepsilon^{g_0}(x, g_0)$ is compact in $U$. Suppose not, $y \notin B_\varepsilon(x, g_0)$, $d_g(y, x) \geq \varepsilon$, then $\forall \gamma : [0, 1] \to U$ with $\gamma(0) = x, \gamma(1) = y$ s.t. $L_0(\gamma) \geq \varepsilon$. Suppose $\gamma \subseteq B_\varepsilon(x, g_0)$, then

$$L_g(\gamma) \geq \lambda_0 L_0(\gamma) \geq \lambda_0 \varepsilon.$$

Suppose $\gamma \not\subseteq B_\varepsilon(x, g_0)$, then

$$L_g(\gamma) \geq \lambda_0 L_0(\gamma|_{[0,t_0]}) \geq \lambda_0 \varepsilon.$$

Both of the cases imply that

$$d_g(x, y) \geq \lambda_0 \varepsilon > 0 \Rightarrow \text{Contradiction.}$$

Thus, If $d_g(x, y) = 0$, then $y \in B_\varepsilon(x, g_0)$.

   **Claim 2:** Inside $\bar{B}_\varepsilon(x, g_0)$, $g$ is equivalent to $g_0$, that is,

$$\forall w \in B_\epsilon(x, g_0) \quad \exists \Lambda_0 > \lambda_0 > 0 \quad s.t. \lambda_0 d_0(x, w) \leq d_g(x, w) \leq \Lambda_0 d_0(x, w).$$

Since $g_w$ is continuous in $w \in U$, and $\bar{B}_\epsilon(x, g_0)$ is compact in $U$, we have

$$\forall w \in B_\epsilon(x, g_0), u, v \in \mathbb{R}^n \quad \exists \Lambda_0 > \lambda_0 > 0 \quad s.t. \lambda_0 g_0(u, v) \leq g_w(u, v) \leq \Lambda_0 g_0(u, v).$$

Consider a curve $\gamma : [0, 1] \to U$ with $\gamma(0) = x, \gamma(1) = w \in B_\epsilon(x, g_0)$. Suppose $\gamma \subseteq B_\varepsilon(x, g_0)$, then

$$L_g(\gamma) = \int_0^1 \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))} dt \geq \lambda_0 \int_0^1 ||\gamma'(t)|| dt = \lambda_0 L_0(\gamma).$$

Also, suppose $\gamma \not\subseteq B_\varepsilon(x, g_0)$, then define

$$t_0 \equiv \min\{t \in [0, 1], \gamma(t) \notin B_\varepsilon(x, g_0)\},$$

which implies $\forall 0 \leq t \leq t_0$, $\gamma(t) \in B_\varepsilon(x, g_0)$. Thus

$$L_g(\gamma) \geq \int_0^{t_0} \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))} dt \geq \lambda_0 \int_0^{t_0} ||\gamma'(t)|| dt = \lambda_0 L_0(\gamma_{[0,t_0]}) \geq \varepsilon$$

Since $d_g(x, w) = \inf\{L_g(\gamma) : \gamma : [0, 1] \to U \text{ connects } x, w\}$, by both cases discussed above, we finally have

$$d_g(x, w) \geq \lambda_0 d_0(x, w).$$

Additionally, we have

$$d_0(x, w) = L_0(\overline{xw}) \geq \Lambda_0^{-1} L_g(\overline{xw}) \geq \Lambda_0^{-1} d_g(x, w).$$

Finally, we prove that

$$\forall w \in B_\epsilon(x, g_0) \quad \exists \Lambda_0 > \lambda_0 > 0 \quad s.t. \lambda_0 d_0(x, w) \leq d_g(x, w) \leq \Lambda_0 d_0(x, w).$$

By claim 1 and claim 2, we can prove that $d_g(x, y) = 0 \Rightarrow d_0(x, y) = 0 \Rightarrow x = y$. $\quad \square$

**Definition (Normal Geodesic):** $\gamma : [a, b] \to U$ is called a **Normal Geodesic** if

$$\forall t \in (a, b), \exists \varepsilon > 0, \text{ s.t. } \forall t_1, t_2 \in (t - \varepsilon, t + \varepsilon),$$

we have

$$d_g(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|,$$

which is equivalent to

$$d_g(\gamma(t_1), \gamma(t_2)) = L_g(\gamma_{[t_1,t_2]}).$$

**Definition (Minimal Geodesic):** $\gamma : [a, b] \to U$ is called a **Minimal Geodesic** if

$$L_g(\gamma) = d_g(\gamma(a), \gamma(b)) = |b - a|.$$

# 9 Lecture 9

**Lemma 9.1.** $(U, d_g)$ *is a length metric space, i.e.* $d_g = d_g^\#$.

*Proof.* We define $d_g(p, q) \equiv \inf\{L(\gamma)|\gamma \in \mathcal{C}_{p,q}\}$ and $d_g^\# \equiv \inf\{L(\gamma)|\gamma \in \mathcal{R}_{p,q}\}$, where

$$\mathcal{C}_{p,q} \equiv \text{All piecewise } C^1 \text{ curves connecting } p, q,$$

$$\mathcal{R}_{p,q} \equiv \text{All rectifiable curves connecting } p, q.$$

Then $\mathcal{C}_{p,q} \subseteq \mathcal{R}_{p,q}$ since there are more elements in $\mathcal{C}_{p,q}$, from which we get $d_g(p, q) \geq d_g^\#(p, q)$. Also by definition, $d_g(p, q) \leq d_g^\#(p, q)$. Finally, we obtain $d_g(p, q) = d_g^\#(p, q)$. $\quad \square$

**Definition (Lower Semi-continuity)** : A function $f(x)$ has a **Lower Semi-continuity** in $x_0$ if

$$f(x_0) \leq \lim_{x \to x_0} \inf f(x).$$

**Lemma 9.2.** *Let* $\gamma_j : [0, 1] \to U$ *be rectifiable curves that pointwise converge to* $\gamma_\infty : [0, 1] \to U$, *then*

$$L(\gamma_\infty) \leq \lim_{j \to \infty} \inf L(\gamma_j).$$

*Proof.* Fix $\varepsilon > 0$, s.t. there is a partition $p = \{0 = t_0 < t_1 < \cdots < t_N = 1\}$ of $[0, 1]$, s.t.

$$\sum_{k=1}^{N} d_g(\gamma_\infty(t_k), \gamma_\infty(t_{k-1})) \geq L(\gamma_\infty) - \varepsilon.$$

For any $1 \leq k \leq N$, $\exists M_k = M_k(\varepsilon) > 0$, s.t.

$$d_g(\gamma_\infty(t_k), \gamma_j(t_k)) \leq \varepsilon \quad \forall j \geq M_k.$$

Take $M_0 = \max_{1 \leq k \leq N} M_k$, then $\forall j \geq M_0(\varepsilon)$,

$$d_g(\gamma_\infty(t_k), \gamma_j(t_k)) \leq \varepsilon \quad \forall 1 \leq k \leq N.$$

Thus

$$\sum_{k=1}^{N} d_g(\gamma_\infty(t_k), \gamma_\infty(t_{k-1})) \leq N\varepsilon + \sum_{k=1}^{N} d_g(\gamma_j(t_k), \gamma_j(t_{k-1})).$$

We further get

$$L(\gamma_\infty) \leq (N+1)\varepsilon + \sum_{k=1}^{N} d_g(\gamma_j(t_k), \gamma_j(t_{k-1})) = (N+1)\varepsilon + L(\gamma_j).$$

Finally,

$$L(\gamma_\infty) \leq \liminf_{j \to \infty} L(\gamma_j) + (N+1)\varepsilon \Rightarrow L(\gamma_\infty) \leq \liminf_{j \to \infty} L(\gamma_j).$$

$\square$

## 10 Lecture 10

**Lemma 10.1.** *Let $(X, d)$ be a length space. Given $p, q \in X$, if one can find a sequence of rectifiable curves $\gamma_j : [0, 1] \to X$, s.t.*

*1. $L(\gamma_j) \to d(p, q)$,*

*2. Assume $\{\gamma_j\}$ pointwise converges to $\gamma_\infty : [0, 1] \to X$ that connects $p, q$,*

*then $\gamma_\infty$ is a minimal geodesic s.t.*

$$L(\gamma_\infty) = d(p, q).$$

*Proof.*

$$d(p, q) \leq L(\gamma_\infty) \leq \lim_j L(\gamma_j) = d(p, q).$$

$\square$

**Definition ($Q_0$-Lipschitz curve):** In metric space $(X, d)$, a curve $\gamma : [0, 1] \to X$ is $Q_0$-Lipschitz curve if

$$\forall t_1, t_2 \in [0, 1], \quad d(\gamma(t_1), \gamma(t_2)) \leq Q_0|t_1 - t_2|.$$

**Lemma 10.2** (Arzela-Ascoli theorem). *Let $(X, d)$ be a compact metric space, and $\gamma_j :$ $[0, 1] \to X$ be a sequence of $Q_0$-Lipschitz curves. Then there exists a sequence $\{\gamma_{j_k}\}_{k \in Z^+}$ that converges uniformly to some $Q_0$-Lipschitz curve $\gamma_\infty$, i.e.*

$$\sup_{t \in [0,1]} d(\gamma_{j_k}(t), \gamma_\infty(t)) \to 0 \quad as \ k \to \infty.$$

**Lemma 10.3.** *If $(U, g)$ is a Riemannian patch, then*

$$\overline{B_r(p)} = \{q \in U | d_g(q, p) \le r\}$$

*Proof.* We can easily verify that $\overline{B_r(p)} \subseteq$ R.H.S. Denote $S_r(p) = \{y \in U : d_g(y, q) = r\}$, then we will show $\forall y \in S_r(p), y \in \overline{B_r(p)}$. Suppose not, then $\exists y \in S_r(p), \varepsilon_0 > 0$, s.t.

$$B_{\varepsilon_0}(y) \cap B_r(p) = \emptyset.$$

For $j \to \infty, \exists \gamma_j : [0, 1] \to U$ connecting $y, q$, s.t.

$$L(\gamma_j) \le r + j^{-1},$$

where $\gamma_j(0) = y, \gamma_j(1) = p$. Let $t_0 \in [0, 1]$ satisfying

$$L_g(\gamma_j|_{[0,t_0]}) = \frac{\varepsilon_0}{10},$$

then

$$d_g(\gamma_j(0), \gamma_j(t_0)) \le \frac{\varepsilon_0}{10} < \varepsilon_0 \Rightarrow \gamma_j(t_0) \in B_{\varepsilon_0}(y).$$

Also,

$$L_g(\gamma_j|_{[t_0,1]}) = L_g(\gamma_j) - L_g(\gamma_j|_{[0,t_0]}) \le r + j^{-1} - \frac{\varepsilon_0}{10} < r,$$

we get

$$d_g(\gamma_j(1), \gamma_j(t_0)) \le L_g(\gamma_j|_{[t_0,1]}) < r \Rightarrow \gamma_j(t_0) \in B_r(p).$$

We get a contradiction here, which implies that $\forall y \in S_r(p), y \in \overline{B_r(p)}$. We further prove this lemma by $\overline{B_r(p)} =$ R.H.S. $\qquad\square$

**Lemma 10.4.** *If $(U, g)$ is a Riemannian patch, then $\forall x \in U, \exists \varepsilon = \varepsilon_x > 0$, s.t. $\overline{B_\varepsilon(x)}$ is compact in $U$.*

**Proposition 10.1.** *If $(U, g)$ is a Riemannian patch, assume $\overline{B_R(p)}$ is compact in $U$, then $\forall x \in \overline{B_r(p)}$, there exists a minimal geodesic $\gamma : [0, 1] \to \overline{B_R(p)}$ with $\gamma(0) = p, \gamma(1) = x$.*

*Proof.* Letting $r = d_g(x, p) < R$, then $\exists \delta > 0$, s.t. $r \le R - \delta$.

Given a minimizing sequence $\{\gamma_j : [0, 1] \to U\}$ with constant speed, s.t. $L_g(\gamma_j) \to d_g(x, p)$. Then for $j$ sufficiently large, we have

$$\|\gamma_j'\|_g \le 2d_g(x, p),$$

and

$$\gamma_j \subset \overline{B_R(p)} : \quad \forall t \in [0, 1], d_g(p, \gamma_j(t)) \le L(\gamma_j|_{[0,t]}) \le L(\gamma_j) \le r + j^{-1} < R.$$

Thus, $\gamma_j$ converges uniformly to $\gamma_\infty$. $\qquad\square$