

# Number Theory and Cryptology

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## Part I

## Number Theory

**Definition 1** (Binary Operation). A binary operation on a set  $S$  is a function from  $S \times S$  to  $S$ .

Eg:  $A : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ , i.e.,  $(a, b) \mapsto a + b$

**Definition 2** (Domain). A domain is triple  $(D, +, \cdot)$ , where  $|D| > 1$  and  $+$  and  $\cdot$  are two operations on  $D$  such that :

i)  $a + b = b + a$  and  $a \cdot b = b \cdot a, \forall a, b \in D$

ii)  $(a + b) + c = a + (b + c)$  and  $(a \cdot b) \cdot c = a \cdot (b \cdot c), \forall a, b, c \in D$

iii)  $\exists 0, 1 \in D, a + 0 = a$  and  $a \cdot 1 = a, \forall a \in D$

iv)  $a \cdot (b + c) = a \cdot b + a \cdot c, \forall a, b, c \in D$

v)  $\forall a \in D, \exists a', a + a' = 0$

vi)  $a \cdot b = 0 \implies$  either  $a = 0$  or  $b = 0$

Eg:  $(\mathbb{Z}, +, \cdot)$  and  $(\mathbb{R}[X], +, \cdot)$ , where  $\mathbb{R}[X]$  is the Set of real polynomials

**Definition 3** (Field). If every non-zero elements of a domain  $D$  has an inverse, i.e., units are  $D - \{0\}$ , then  $D$  is called a field.

## Division Algorithm

**Theorem 1.** Let  $a \in \mathbb{Z}$  and  $b \in \mathbb{N}$ . Then  $\exists$  unique  $q, r \in \mathbb{Z}$  such that

$$a = bq + r, 0 \leq r < b$$

*Proof.* If  $a = 0$  (trivial). Let's prove for  $a \in \mathbb{N}$  by induction. If  $a = 1$ , take  $r = 1$  and  $q = 0$  (Base Case). Assume the statement is true  $\forall n \in \mathbb{N}, n < a$ , then we prove the statement for  $a$ . If  $a \geq b$  then  $a - b < a$ . Then by induction, we have

$$a - b = qb + r, 0 \leq r < b \implies a = (q + 1)b + r$$

If  $a < b$ , then take  $q = 0$  and  $r = a$ . Hence the theorem is proved for  $a \in \mathbb{N}$ .  
Now let  $a \in \mathbb{Z}_-$ . Then  $-a \in \mathbb{N}$ .

$$\begin{aligned} \exists q \text{ and } r, -a &= bq + r, 0 \leq r < b \\ \implies a &= (-q)b + (-r) \\ \implies a &= (-q-1)b + (b-r), \text{ where } 0 \leq b-r < b \end{aligned}$$

This ends the existence proof.

Now we prove the uniqueness. Let  $(q, r)$  and  $(q', r')$  be two pairs that satisfy the theorem. Then,

$$\begin{aligned} a &= bq + r, 0 \leq r < b \\ a &= bq' + r', 0 \leq r' < b \end{aligned}$$

WLOG, assume  $r' \geq r$ , then

$$\begin{aligned} \implies 0 &\leq r' - r < b \\ \implies bq + r &= bq' + r' \\ \implies b(q - q') &= r' - r \\ \implies b \mid (r' - r) \\ \implies r' &= r \text{ and } q' = q \quad (\text{since } r' - r < b) \end{aligned}$$

This completes the uniqueness proof.  $\square$

**Lemma 2** (Modified Division Algorithm). *Let  $a \in \mathbb{Z}$  and  $b \in \mathbb{N}$ . Then  $\exists$  unique  $q, r \in \mathbb{Z}$  such that*

$$a = bq + r, |r| \leq \frac{b}{2}$$

**Theorem 3.** *Let  $a(X), b(X) \in \mathbb{R}[X]$ . Then  $\exists q(X), r(X) \in \mathbb{R}[X]$  such that*

$$a(X) = b(X)q(X) + r(X), \text{ either } r(X) = 0 \text{ or } \deg(r(X)) < \deg(b(X))$$

*Proof.* Proof by induction on  $\deg(a(X))$ . If  $\deg(a(X)) < \deg(b(X))$ , then take  $q(X) = 0$  and  $r(X) = a(X)$ . If  $\deg(b(X)) = 0$ , i.e.,  $b(X) = b_0$ , then take  $q(X) = b_0^{-1}a(X)$  and  $r(X) = 0$ .

Now assume  $\deg(b(X)) > 0$  and  $\deg(a(X)) \geq \deg(b(X))$  and also assume the theorem is true  $\forall h(X) \in \mathbb{R}[X], \deg(h(X)) < \deg(a(X))$ .

Then if  $\deg(a(X)) = m$  and  $\deg(b(X)) = n$ ,

$$\begin{aligned} \implies a(X) &= a_0 + a_1X + \cdots + a_mX^m \\ \text{and } b(X) &= b_0 + b_1X + \cdots + b_nX^n, \quad (m \geq n) \end{aligned}$$

Now consider the polynomial  $g(X) = a(X) - b_n^{-1}a_mX^{m-n}b(X)$ . It can be easily verified that  $\deg(g(X)) < m$ . Then,

$$\begin{aligned} \exists q(X), r(X) &\in \mathbb{R}[X], g(X) = b(X)q(X) + r(X), \\ \text{where } r(X) &= 0 \text{ or } \deg(r(X)) \leq \deg(b(X)) \end{aligned}$$

$$\begin{aligned} \implies a(X) - b_n^{-1}a_mX^{m-n}b(X) &= b(X)q(X) + r(X) \\ \implies a(X) &= b(X)(q(X) + b_n^{-1}a_mX^{m-n}) + r(X), \end{aligned}$$

where  $r(X) = 0$  or  $\deg(r(X)) \leq \deg(b(X))$

□

**Definition 4** (Unit). *The multiplicatively invertible elements in a domain are called units of a domain.*

*Eg: Units in  $\mathbb{Z} = \{\pm 1\}$  and Units in  $\mathbb{R}[X] = \{c \mid c \in \mathbb{R} - \{0\}\}$*

**Definition 5** (Prime).  *$a$  is prime if  $a = uv \implies$  either  $u$  or  $v$  is a unit, but not both.*

**Definition 6** (Associate).  *$b$  is an associate of  $a$  if  $a \mid b$  and  $b \mid a$  or equivalently  $a = ub$ , where  $u$  is a unit.*

**Theorem 4.** *If  $x$  is a prime and  $u$  is a unit, then  $ux$  is also a prime.*

*Proof.* Suppose  $ux = st$ . Since  $u$  is a unit,  $x = (u^{-1}s)t$ . But we know,  $x$  is a prime, then either of  $u^{-1}s$  or  $t$  is a unit. If  $t$  is unit, proof is completed. Else  $u^{-1}s$  must be a unit. We know that the product of two units is again a unit. So is  $uu^{-1}s$ , i.e,  $s$  is a unit. □

**Definition 7** (Greatest Common Divisor).  *$d$  is said to be gcd of  $a$  and  $b$  if  $d \mid a$  and  $d \mid b$  and every common divisor  $c$  of  $a$  and  $b$  must divide  $d$ , i.e, if  $c \mid a$  and  $c \mid b$ , then  $c \mid d$ . It is written as  $d = (a, b)$ .*

**Remark.** *If  $d$  is a gcd  $a$  and  $b$  and then an associate of  $d$  is also a gcd of  $a$  and  $b$ , i.e, if  $u$  is a unit, then  $d = (a, b) = ud$ .*

**Definition 8.** *If  $a$  and  $b \in \mathbb{Z}$ , then we define*

$$a\mathbb{Z} + b\mathbb{Z} = \{ax + by \mid x, y \in \mathbb{Z}\}$$

**Remark.** *It can be seen that  $a, b \in a\mathbb{Z} + b\mathbb{Z}$  and if  $s_1$  and  $s_2 \in a\mathbb{Z} + b\mathbb{Z}$  then  $s_1x + s_2y \in a\mathbb{Z} + b\mathbb{Z}$ ,  $\forall x, y \in \mathbb{Z}$ . Therefore  $a\mathbb{Z} + b\mathbb{Z} \cap \mathbb{N} \neq \emptyset$ .*

**Theorem 5.** *If  $a, b \in \mathbb{Z}$ , then  $\exists d \in \mathbb{Z}$ ,  $a\mathbb{Z} + b\mathbb{Z} = d\mathbb{Z}$ , where  $d = (a, b)$ .*

*Proof.* We first prove the existence of such a  $d$ . Since  $a\mathbb{Z} + b\mathbb{Z} \cap \mathbb{N} \neq \emptyset$ , let  $d$  be least natural number in  $a\mathbb{Z} + b\mathbb{Z}$ . Then  $d\mathbb{Z} \subseteq a\mathbb{Z} + b\mathbb{Z}$ . Now let  $s \in a\mathbb{Z} + b\mathbb{Z}$ , then by division algorithm on  $\mathbb{Z}$ ,

$$\exists q, r \in \mathbb{Z}, s = qd + r, 0 \leq r < d.$$

$$\implies r = s - qd \in \mathbb{Z}$$

$$\implies r = 0, \text{ i.e, } s = qd$$

$$\implies a\mathbb{Z} + b\mathbb{Z} \subseteq d\mathbb{Z}$$

$$\text{Therefore, } a\mathbb{Z} + b\mathbb{Z} = d\mathbb{Z}.$$

Now we prove that  $d = (a, b)$ . Since  $a, b \in a\mathbb{Z} + b\mathbb{Z}$ ,  $d \mid a$  and  $d \mid b$ . But  $d \in a\mathbb{Z} + b\mathbb{Z}$ , so  $d = ax + by$  for some  $x, y \in \mathbb{Z}$ . Suppose  $c \mid a$  and  $c \mid b$ , then  $a = a_1c$  and  $b = b_1c$ . Then  $d = c(xa_1 + yb_1)$ , implies  $c \mid d$ . □

**Corollary 5.1.** *If  $a \mid bc$  and  $(a, b) = 1$ , then  $a \mid c$ .*

**Theorem 6.**  $\mathbb{Z}$  is a UFD (Unique factorization Domain), i.e, every non-zero, non-unit can be written as product of primes and this factorization is unique upto order and association, i.e, if  $n$  is a non-zero, non-unit in  $\mathbb{Z}$ , and  $n = p_1 p_2 \cdots p_r = q_1 q_2 \cdots q_s$ , where  $p_i$ 's and  $q_i$ 's are primes, then  $r = s$  and every  $p_i$  is an associate of some  $q_j$  and vice versa.

*Proof.* The existence of such factorization can be proved by using strong induction for non-negative integers and using this result, we can multiply by a -1 (unit) and show it's true for negative integers as well.

Now, we prove the uniqueness by induction. Suppose  $n$  is a non-zero, non-unit.

$$\text{Suppose } n = p_1 p_2 \cdots p_r = q_1 q_2 \cdots q_s.$$

If  $r = 1$  (Base Case), then  $n = p_1 = q_1 q_2 \cdots q_s$ . But  $p_1$  is a prime, therefore,  $s = 1$  and  $n = p_1 = u q_1$ , where  $u$  is a unit. Assume the statement is true  $\forall a \in \mathbb{N}$ ,  $a < n$ . Now we prove the statement for  $n$ .

$$p_r \mid n, \text{ i.e, } p_r \mid q_1(q_2 \cdots q_s).$$

$$\text{If } (p_r, q_1) = 1 \implies p_r \mid q_2(q_3 \cdots q_s)$$

This way, we get some  $q_j$  which is an associate of  $p_r$ . WLOG, we can assume  $p_r$  is an associate of  $q_s$ , i.e,  $u p_r = q_s$ .

$$\begin{aligned} \implies p_1 p_2 \cdots p_r - u q_1 q_2 \cdots q_{s-1} p_r &= 0 \\ \implies p_r (p_2 \cdots p_{r-1} - u q_1 q_2 \cdots q_{s-1}) &= 0 \\ \implies p_2 \cdots p_{r-1} = u q_1 q_2 \cdots q_{s-1} &< n \end{aligned}$$

□

**Definition 9** ( $\mathbb{Z}[\omega]$ ).  $\mathbb{Z}[\omega] = \{a + b\omega \mid a, b \in \mathbb{Z}\} \subset \mathbb{C}$ , where  $\omega = \frac{-1 \pm i\sqrt{3}}{2}$ .

$$\text{and } N(\alpha) = \alpha \bar{\alpha}.$$

**Remark.** If  $\alpha = a + b\omega$ , then

$$\begin{aligned} N(a + b\omega) &= (a + b\omega)(\overline{a + b\omega}) \\ &= (a + b\omega)(a + b\omega^2) \\ &= a^2 - ab + b^2 \\ &= \frac{(2a - b)^2 + 3b^2}{4} \end{aligned}$$

**Remark.** The only element whose norm is 0 is 0.

**Proposition.**  $\alpha \in \mathbb{Z}[\omega]$  is a unit iff  $N(\alpha) = 1$ .

*Proof.* Suppose  $N(\alpha) = 1$ , then  $\alpha \bar{\alpha} = 1$ . Therefore  $\alpha$  is a unit in  $\mathbb{Z}[\omega]$ . Conversely, suppose  $\alpha$  is a unit in  $\mathbb{Z}[\omega]$ .

$$\begin{aligned} \exists \alpha' \in \mathbb{Z}[\omega], \alpha \alpha' &= 1 \\ \implies N(\alpha \alpha') &= 1 \\ \implies N(\alpha) N(\alpha') &= 1 \\ \implies N(\alpha) &= 1 \quad (\text{since, } N(\alpha) \in \mathbb{N}, \forall \alpha \in \mathbb{Z}[\omega]). \end{aligned}$$

□

**Theorem 7.** *The units in  $\mathbb{Z}[\omega]$  are  $\pm 1, \pm\omega, \pm\omega^2$ .*

**Theorem 8.** *There is no element in  $\mathbb{Z}[\omega]$  with norm 2.*

**Theorem 9.** *The only elements in  $\mathbb{Z}[\omega]$  with norm 3 are  $\pm\pi, \pm\pi\omega, \pm\pi\omega^2$ , where  $\pi = 1 - \omega$ .*

**Theorem 10.**  *$\mathbb{Z}[\omega]$  is a Euclidean Domain, i.e.,*

$$\forall \alpha, \beta \in \mathbb{Z}[\omega], \beta \neq 0, \exists \gamma, \delta \in \mathbb{Z}[\omega], \alpha = \beta\gamma + \delta, N(\delta) < N(\beta).$$

*Proof.* Let  $\alpha = a + b\omega, \beta = c + d\omega, a, b, c, d \in \mathbb{Z}, \beta \neq 0$ , then  $c, d \neq 0$ .

Case i) Let  $d = 0$ . Then by Modified Division Algorithm, we have

$$\begin{aligned} a &= cq_1 + r_1, & (q_1, r_1 \in \mathbb{Z} \text{ and } |r_1| \leq \frac{c}{2}) \\ b &= cq_2 + r_2, & (q_2, r_2 \in \mathbb{Z} \text{ and } |r_2| \leq \frac{c}{2}) \\ \implies \alpha &= a + b\omega = c(q_1 + q_2\omega) + (r_1 + r_2\omega) \\ \implies N(\delta) &= N(r_1 + r_2\omega) \\ &= r_1^2 - r_1r_2 + r_2^2 \\ &\leq |r_1|^2 + |r_1||r_2| + |r_2|^2 \\ &= \frac{c^2}{4} + \frac{c^2}{4} + \frac{c^2}{4} \\ &= \frac{3c^2}{4} < c^2 = N(b) = N(\beta) \end{aligned}$$

Case ii) If  $d \neq 0$ , consider  $\alpha' = \alpha\bar{\beta}, \beta' = \beta\bar{\beta}$ , then  $\beta' \in \mathbb{Z}$ , then by Case i),

$$\exists \gamma', \delta' \in \mathbb{Z}[\omega], \alpha' = \beta'\gamma' + \delta', N(\delta') < N(\beta') = (N(\beta))^2.$$

Let  $\delta = \alpha - \beta\gamma$ , then  $\delta\bar{\beta} = \alpha\bar{\beta} - \beta\bar{\beta}\gamma = \delta'. N(\delta\bar{\beta}) = N(\delta') < (N(\beta))^2$

$$\begin{aligned} \implies N(\delta)N(\beta) &< (N(\beta))^2 \\ \implies N(\delta) &< N(\beta). \end{aligned}$$

□

**Theorem 11.** *If  $\alpha, \beta \in \mathbb{Z}[\omega]$ , then  $\exists \delta \in \mathbb{Z}[\omega], \alpha\mathbb{Z}[\omega] + \beta\mathbb{Z}[\omega] = \delta\mathbb{Z}[\omega]$ , where  $\delta = (\alpha, \beta)$ .*

**Definition 10.** *If  $a, b, m \in \mathbb{Z}$  and  $m \neq 0$ , we say that  $a$  is congruent to  $b$  modulo  $m$  if  $m \mid b - a$ . This relation is written  $a \equiv b \pmod{m}$ .*

**Definition 11**  $(\mathbb{Z}_n, +_n, \cdot_n)$ . -FILL IN-

**Theorem 12.** *If  $a \in \mathbb{Z}_n - \{0\}$  is a unit iff  $(a, n) = 1$ .*

*Proof.* Let  $a \in \mathbb{Z}_n - \{0\}$  be a unit. Then  $\exists a' \in \mathbb{Z}_n - \{0\}$ , such that  $a \cdot_n a' = 1$ , i.e.,  $\exists q, aa' = qn + 1$ .

$$\implies (a, n) = 1.$$

Now let  $(a, n) = 1$ , then  $\exists u, v \in \mathbb{Z}$ ,  $au + nv = 1$ . By Division Algorithm,  $\exists q, r$ , such that  $u = qn + r$ ,  $r \in \mathbb{Z}_n$ .

$$\begin{aligned} &\implies a(qn + r) + nv = 1 \\ &\implies ar = n(-aq - v) + 1 \\ &\implies a \cdot_n r = 1 \quad (\text{Since, } a, r \in \mathbb{Z}_n). \end{aligned}$$

Therefore,  $a$  is a unit in  $\mathbb{Z}_n$ .  $\square$

**Definition 12.** We define  $U_n$  to be the set of all units in  $\mathbb{Z}_n$  and  $\phi(n)$  to be the cardinality of  $U_n$ , where  $\phi_n$  is called Euler totient function, i.e.,

$$U_n = \{a \in \mathbb{Z}_n - \{0\} \mid (a, n) = 1\}, \quad \phi(n) = |U_n|.$$

We define  $\phi(1) = 1$ .

**Remark.** If  $n = p$ ,  $p$  is prime, then every element is relatively prime to  $p$ , i.e.,  $U_p = \mathbb{Z}_p - \{0\} = \{1, 2, \dots, p-1\}$ . And also  $(\mathbb{Z}_p, +_p, \cdot_p)$  is a field. If  $n = p^t$ ,  $\phi(n) = p^{t-1}(p-1)$ . If  $n = pq$ ,  $\phi(n) = (p-1)(q-1)$ .

**Theorem 13** (Euler's Theorem). If  $(a, n) = 1$ , then  $a^{\phi(n)} \equiv 1 \pmod{n}$ .

*Proof.* Let's prove it for elements in  $U_n$  first and then for any element in general. Let  $U_n = \{a_1, a_2, \dots, a_{\phi(n)}\}$  and let  $a \in U_n$ . Then,

$$a \cdot_n U_n = \{a \cdot_n a_1, a \cdot_n a_2, \dots, a \cdot_n a_{\phi(n)}\} \subseteq U_n$$

**Claim.** All elements of  $a \cdot_n U_n$  are distinct, i.e.,  $a \cdot_n U_n = U_n$ .

We prove this by contradiction. Assume,  $a \cdot_n a_i = a \cdot_n a_j$ , such that  $i \neq j$ . Then  $a^{-1} \cdot_n a \cdot_n a_i = a^{-1} \cdot_n a \cdot_n a_j$ , hence  $a_i = a_j$ . Therefore,  $a \cdot_n U_n = U_n$ .

$$\begin{aligned} &\implies \prod_{i=1}^{\phi(n)} a \cdot_n a_i = \prod_{j=1}^{\phi(n)} a_j \\ &\implies a^{\phi(n)} \left( \prod_{i=1}^{\phi(n)} a_i \right) = \prod_{j=1}^{\phi(n)} a_j \\ &\implies a^{\phi(n)} b = b, \text{ where } b = \prod_{i=1}^{\phi(n)} a_i \in U_n \\ &\implies a^{\phi(n)} = 1 \text{ in } (\mathbb{Z}_n, +_n, \cdot_n). \end{aligned}$$

Now, let's prove the theorem for any  $a \in \mathbb{Z}$ , such that  $(a, n) = 1$ . By Division Algorithm,  $\exists q, r$ , such that  $a = qn + r$ ,  $r \in \mathbb{Z}_n$ . Since  $(a, n) = 1$ , we have  $(r, n) = 1$ .

$$\begin{aligned} &\implies a^{\phi(n)} = (qn + r)^{\phi(n)} \\ &\quad = r^{\phi(n)} + \binom{\phi(n)}{1} (nq) + \dots + (nq)^{\phi(n)} \\ &\quad = r^{\phi(n)} + nk \\ &\implies a^{\phi(n)} - 1 = r^{\phi(n)} - 1 + nk \\ &\text{But } n \mid r^{\phi(n)} - 1, \text{ then } n \mid a^{\phi(n)} - 1 \\ &\implies a^{\phi(n)} \equiv 1 \pmod{n} \end{aligned}$$

$\square$

**Notation:**  $\mathbb{Z}_p^x = \mathbb{Z}_p - \{0\}$  and  $\mathbb{Z}_p^{x^2}$  to be set of elements in  $\mathbb{Z}_p^x$  which are square. Here  $p$  is a prime.

**Proposition.**  $|\mathbb{Z}_p^{x^2}| = \frac{p-1}{2}$ , therefore  $\exists u \in \mathbb{Z}_p^x$  which is a non-square. Then  $u\mathbb{Z}_p^{x^2}$  will be the set of all non-square in  $\mathbb{Z}_p^x$ .

*Proof.* First, we prove that  $|\mathbb{Z}_p^{x^2}| = \frac{p-1}{2}$ . Consider the following mapping:

$$\begin{aligned}\mathbb{Z}_p^x &\mapsto \mathbb{Z}_p^{x^2} \\ x &\mapsto x^2 \\ \implies p-x &\mapsto (p-x)^2 = p^2 - 2px + x^2 = x^2 + pk \\ \implies p-x &\mapsto x^2 \text{ in } (\mathbb{Z}_p, +_p, \cdot_p)\end{aligned}$$

Therefore this mapping is a 2-1 mapping and hence  $|\mathbb{Z}_p^{x^2}| = \frac{p-1}{2}$ . There are  $\frac{p-1}{2}$  non-square elements in  $\mathbb{Z}_p^x$ . Let  $u$  be a non-square. Then consider the following mapping:

$$\begin{aligned}\mathbb{Z}_p^{x^2} &\mapsto u\mathbb{Z}_p^{x^2} \\ x^2 &\mapsto ux^2\end{aligned}$$

We prove that this mapping is bijective. It is enough to show that all the elements in  $u\mathbb{Z}_p^{x^2}$  are distinct and non-squares. Consider two elements  $ux^2, uy^2 \in u\mathbb{Z}_p^{x^2}$ .

$$\begin{aligned}\text{If } ux^2 &= uy^2 \\ \implies u^{-1}ux^2 &= u^{-1}uy^2 \quad (\text{since } \mathbb{Z}_p \text{ is a field}) \\ \implies x^2 &= y^2\end{aligned}$$

This shows that all the elements of  $u\mathbb{Z}_p^{x^2}$  are distinct. Now we show that elements of  $u\mathbb{Z}_p^{x^2}$  are all the non-square elements in  $\mathbb{Z}_p^x$ . Suppose some element in  $u\mathbb{Z}_p^{x^2}$  is a square, i.e,

$$\begin{aligned}\implies ux^2 &= y^2 \\ \implies ux^2x^{-2} &= y^2x^{-2} \\ \implies u &= (yx^{-1})^2 \in \mathbb{Z}_p^{x^2}\end{aligned}$$

But  $u$  is a non-square, which is a contradiction. Therefore, this mapping is not just a bijection, but none of the elements in one set belongs to other. Hence,  $u\mathbb{Z}_p^{x^2}$  is the set of all non-squares in  $\mathbb{Z}_p^x$ .  $\square$

**Remark.** From the above proposition it can be concluded that

$$\mathbb{Z}_p^x = u\mathbb{Z}_p^{x^2} \oplus \mathbb{Z}_p^{x^2}.$$

**Definition 13.** We define a mapping such that,

$$\begin{aligned}\mathbb{Z} &\rightarrow \mathbb{Z}_n \\ x &\mapsto \bar{x}, \bar{x} = x(\text{mod } n)\end{aligned}$$

**Theorem 14.** *The following properties hold for  $x, y \in \mathbb{Z}$ :*

$$i) \overline{x+y} = \bar{x} +_n \bar{y} \qquad ii) \overline{xy} = \bar{x} \cdot_n \bar{y}$$

*Define the following mapping from  $\mathbb{Z}_{mn} \rightarrow \mathbb{Z}_n$  in the similar way as above. Then the following properties hold:*

$$i) \overline{x+_{mn}y} = \bar{x} +_n \bar{y} \qquad ii) \overline{x \cdot_{mn} y} = \bar{x} \cdot_n \bar{y}$$

**Theorem 15** (Chinese Remainder Theorem). *Suppose  $(m, n) = 1$ . Let  $\bar{x} = x(\text{mod } m)$  and  $\bar{\bar{x}} = x(\text{mod } n)$ ,  $x \in \mathbb{Z}$ . Then the following mapping is bijection which preserves operation:*

$$\begin{aligned} \mathbb{Z}_{mn} &\rightarrow \mathbb{Z}_m \times \mathbb{Z}_n \\ x &\mapsto (\bar{x}, \bar{\bar{x}}) \end{aligned}$$

*Proof.* Since the sets are finite, by pigeon hole principle, it is enough to show the mapping  $f : \mathbb{Z}_{mn} \rightarrow \mathbb{Z}_m \times \mathbb{Z}_n, x \mapsto (\bar{x}, \bar{\bar{x}})$  is onto for it to be bijective, i.e, if  $\forall (u, v) \in \mathbb{Z}_m \times \mathbb{Z}_n \exists x \in \mathbb{Z}_{mn}$ , such that  $\bar{x} = x(\text{mod } m)$  and  $\bar{\bar{x}} = x(\text{mod } n)$ . We will first show that  $\exists x \in \mathbb{Z}$  satisfying the above conditions. Since  $(m, n) = 1$ ,  $\exists M, N \in \mathbb{Z}, Mm + nN = 1$ . Let  $x = mMv + nNu$ , then  $x - u = mMv + (nN - 1)u = mM(v - u) = mq \implies x = mq + u \implies \bar{x} = u$ . Similarly,  $\bar{\bar{x}} = v$ . So  $\exists x \in \mathbb{Z}, \bar{x} = u, \bar{\bar{x}} = v$ , then by division algorithm,  $\exists r \in \mathbb{Z}_{mn}, q, x = mnq + r$ . Notice that  $u = \bar{x} = \overline{mnq + r} = \overline{mnq} +_m \bar{r} = \bar{r}$  and similarly  $\bar{\bar{r}} = v$ . Therefore,

$$\exists x \in \mathbb{Z}, \bar{x} = x(\text{mod } m) \text{ and } \bar{\bar{x}} = x(\text{mod } n).$$

We know,

$$\begin{aligned} i) \overline{x+_{mn}y} &= \bar{x} +_m \bar{y} & ii) \overline{x \cdot_{mn} y} &= \bar{x} \cdot_m \bar{y} \\ i) \overline{\overline{x+_{mn}y}} &= \bar{\bar{x}} +_n \bar{\bar{y}} & ii) \overline{\overline{x \cdot_{mn} y}} &= \bar{\bar{x}} \cdot_n \bar{\bar{y}} \end{aligned}$$

$$\begin{aligned} f(x+_{mn}y) &= (\overline{x+_{mn}y}, \overline{\overline{x+_{mn}y}}) \\ &= (\bar{x} +_m \bar{y}, \bar{\bar{x}} +_n \bar{\bar{y}}) \\ &= (\bar{x}, \bar{\bar{x}}) + (\bar{y}, \bar{\bar{y}}) \\ &= f(x) + f(y). \end{aligned}$$

Similarly,  $f(x \cdot_{mn} y) = f(x) \times f(y)$ . Here addition(+) and multiplication( $\times$ ) are component-wise. Therefore, f is onto and hence bijective(???).  $\square$

**Theorem 16.** *If  $(m, n) = 1$ , then  $\phi(m) = \phi(m)\phi(n)$ .*

*Proof.* First we show that under this bijection mapping defined above the elements of  $U_{mn}$  group map bijectively to the elements of  $U_m \times U_n$ , i.e,

- i) if  $x \in U_{mn}$ , then  $\bar{x} \in U_m$  and  $\bar{\bar{x}} \in U_n$ . If  $x \in \mathbb{Z}_{mn}$ , then  $(x, mn) = 1$ , i.e,  $ax + bmn = 1$ , this implies  $(x, m) = 1$  and  $(x, n) = 1$ , i.e,  $\bar{x} \in U_m$  and  $\bar{\bar{x}} \in U_n$ .
- ii) if  $\bar{x} \in U_m$  and  $\bar{\bar{x}} \in U_n$ , then  $x \in U_{mn}$ . Then  $\exists u \in \mathbb{Z}_m$  and  $v \in \mathbb{Z}_n$ , such that  $\bar{x} \cdot_m u = 1$  and  $\bar{\bar{x}} \cdot_n v = 1$ . Since f is onto,  $\exists y \in \mathbb{Z}_{mn}, \bar{y} = u$  and  $\bar{\bar{y}} = v$ . Then  $\bar{x} \cdot_{mn} \bar{y} = \bar{x} \cdot_m \bar{y} = 1$  and  $\overline{\bar{x} \cdot_{mn} \bar{y}} = \bar{\bar{x}} \cdot_n \bar{\bar{y}} = 1$ . Therefore,  $f(x \cdot_{mn} y) = (\bar{1}, \bar{\bar{1}}) = f(1)$ . Since f is one-to-one,  $x \cdot_{mn} y = 1$ , i.e,  $x \in U_{mn}$ .



Therefore,

$$\begin{aligned} U_{mn} &\leftrightarrow U_m \times U_n \\ \implies |U_{mn}| &= |U_m||U_n| \\ \implies \phi(mn) &= \phi(m)\phi(n). \end{aligned}$$

□

**Theorem 17.** *Some important facts from group theory that are used later.*

- i) *If order of an element in the group is equal to order of the group, then the group is cyclic.*
- ii) *Let  $(G, *)$  be a finite group and  $(H, *)$  be a subgroup of  $G$ , then  $|H| \mid |G|$ .*
- iii) *In a finite group the order of an element must divide the order of the group.*
- iv) *If  $a \in G$ ,  $(G, *)$  be a finite group, then  $a^{|G|} = e$ .*
- v) *If  $a \in G$ ,  $(G, *)$  be a finite group and if  $a^i = e$ , then  $o(a) \mid i$ .*
- vi) *If  $a \in G$ ,  $(G, *)$  be a finite group, then  $o(a^i) = \frac{o(a)}{(i, o(a))}$  and  $o(a^i) = o(a)$  iff  $(i, o(a)) = 1$ . Therefore, there are  $\phi(n)$  generators in a cyclic group of order  $n$ .*

**Lemma 18.**  $\sum_{d|n} \phi(d) = n \quad \forall n \in \mathbb{N}$ .

*Proof.* Define  $f(n) = \sum_{d|n} \phi(d)$ . We show that if  $(m, n) = 1$ , then  $f(mn) = f(m)f(n)$ . Let  $m = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$  and  $n = q_1^{e_1} q_2^{e_2} \cdots q_s^{e_s}$ . If  $d \mid mn$ , then  $d = (p_1^{l_1} p_2^{l_2} \cdots p_r^{l_r})(q_1^{r_1} q_2^{r_2} \cdots q_s^{r_s}) = d_1 d_2$ , such that  $(d, m) = d_1$  and  $(d, n) = d_2$ . Now,

$$\begin{aligned} f(mn) &= \sum_{d|mn} \phi(d) = \sum_{d_1|m, d_2|n} \phi(d_1 d_2) = \sum_{d_1|m, d_2|n} \phi(d_1) \phi(d_2) \\ &= \sum_{d_1|m} \phi(d_1) \sum_{d_2|n} \phi(d_2) = f(m)f(n) \end{aligned}$$

Let  $n$  be a non-zero, such that  $n > 1$ . Let  $n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ , where  $p_i$ 's are distinct primes.

$$\begin{aligned} f(p_1^{e_1}) &= \sum_{d|p_1^{e_1}} \phi(d) = \phi(1) + \phi(p_1) + \phi(p_1^2) + \cdots + \phi(p_1^{e_1}) \\ &= 1 + p_1 + \cdots + p_1^{e_1-1}(p_1 - 1) = p_1^{e_1} \\ \text{Then, } f(n) &= f(p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}) = f(p_1^{e_1}) f(p_2^{e_2} \cdots p_r^{e_r}) \\ &= p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} = n = \sum_{d|n} \phi(d). \end{aligned}$$

□

**Theorem 19.** *Let  $(\mathbb{F}, +, \cdot)$  be a field and  $G$  a finite subgroup of  $(\mathbb{F} - \{0\}, \cdot)$ , then  $G$  is a cyclic group.*

*Proof.* Given that  $G \subseteq (\mathbb{F} - \{0\}, \cdot)$  is finite, i.e,  $|G| < \infty$ . Let  $|G| = n$ . If  $d \mid n$  define  $G_d$  as the set containing all the elements in  $G$  of order  $d$ . We have  $G = \coprod_{d \mid n} G_d$ , then  $|G| = \sum_{d \mid n} |G_d|$ . If  $G_d = \phi$ , then  $|G_d| = 0$ . Suppose  $|G_d| \neq 0$ , let  $a \in G_d$ , then  $o(a) = d$ . Consider  $H = \{1, a, a^2, \dots, a^{d-1}\}$ ,  $a^d = 1$ . Then  $X^d - 1$  is a polynomial in  $\mathbb{F}[X]$ . Notice that all the elements of  $H$  are the roots of the polynomial and these are the only roots of  $X^d - 1$  in  $\mathbb{F}$ . Therefore,  $G_d \subseteq H$ . Notice that the number of elements in  $H$  of order  $d$  are  $\phi(d)$ , since  $o(a^i) = o(a) = d$  iff  $(i, d) = 1$ , then  $|G_d| = \phi(d)$ . Hence,  $G_d = \phi(d)$  or 0. We know,  $\sum_{d \mid n} \phi(d) = n$  and  $n = \sum_{d \mid n} |G_d|$ , hence  $|G_d|$  is never 0, i.e,  $G_d$  is never empty  $\forall d \mid n$ . In particular,  $G_n \neq \phi$ , therefore there exists an element of order  $n$ . Hence,  $G$  is cyclic.  $\square$

**Corollary 19.1.**  $(\mathbb{Z}_p - \{0\}, \cdot_p)$  is cyclic group in  $F = (\mathbb{Z}_p - \{0\}, +, \cdot_p)$ .

**Definition 14** (Legendre Symbol). Let  $c \in \mathbb{Z}$ ,  $p$  is an odd prime. Then we define:

$$\left(\frac{c}{p}\right) = \begin{cases} 0, & \text{if } p \mid c \\ 1, & \text{if } \exists x \in \mathbb{Z}, x^2 \equiv c \pmod{p} \\ -1, & \text{otherwise} \end{cases}$$

**Theorem 20.** The properties of Legendre Symbol are listed below:

- i) If  $a \equiv b \pmod{p}$ , then  $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$ .
- ii)  $\left(\frac{xy}{p}\right) = \left(\frac{x}{p}\right)\left(\frac{y}{p}\right)$ .
- iii)  $\left(\frac{a}{p}\right) = \bar{a}^{\left(\frac{p-1}{2}\right)}$ , where  $\bar{a} = a \pmod{p}$ .
- iv)