

# Differential Geometry

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# 1 Smooth Manifolds

**Definition 1.** A topological space  $\mathcal{M}$  is said to be **locally Euclidean of dimension  $n$**  if every point of  $\mathcal{M}$  has a neighborhood in  $\mathcal{M}$  that is homeomorphic to an open subset of  $\mathbb{R}^n$ .

**Lemma 1.** A topological space  $\mathcal{M}$  is locally Euclidean of dimension  $n$  if and only if either of the following properties holds:

- (a) Every point of  $\mathcal{M}$  has a neighborhood homeomorphic to an open ball in  $\mathbb{R}^n$ .
- (b) Every point of  $\mathcal{M}$  has a neighborhood homeomorphic to  $\mathbb{R}^n$ .

**Proof Sketch.** For (a) one need to show that any open subset of  $\mathbb{R}^n$  is homeomorphic to the unit open ball in  $\mathbb{R}^n$ . Note that the unit open ball is homeomorphic to any open ball in  $\mathbb{R}^n$ . For (b) it suffices to show that unit open ball in  $\mathbb{R}^n$  is homeomorphic to  $\mathbb{R}^n$ .

**Definition 2.** Suppose  $\mathcal{M}$  is a topological space.  $\mathcal{M}$  is a **topological manifold of dimension  $n$**  or a **topological  $n$ -manifold** if it has the following properties:

- (a)  $\mathcal{M}$  is a **Hausdorff space**.
- (b)  $\mathcal{M}$  is a **second-countable**.
- (c)  $\mathcal{M}$  is **locally Euclidean of dimension  $n$** .

A **coordinate chart** (or just a **chart**) on  $\mathcal{M}$  is a pair  $(U, \varphi)$ , where  $U$  is an open subset of  $\mathcal{M}$  and  $\varphi : U \rightarrow \hat{U}$  is a homeomorphism from  $U$  to an open subset  $\hat{U} = \varphi(U) \subseteq \mathbb{R}^n$ . The set  $U$  is called a **coordinate domain** or a **coordinate neighborhood** of each of its points. The map  $\varphi$  is called a **(local) coordinate map**, and the component functions  $(x^1, \dots, x^n)$  of  $\varphi$ , defined by  $\varphi(p) = (x^1(p), \dots, x^n(p))$ , are called **local coordinates** on  $U$ .

**Proposition 1.** A nonempty  $n$ -dimensional topological manifold cannot be homeomorphic to an  $m$ -dimensional manifold unless  $m = n$ .

**Example.** Here are some examples of topological manifolds.

- (i) Open subset of a topological  $n$ -manifold.
- (ii) Graphs of Continuous Functions.
- (iii) Spheres.
- (iv) Projective Spaces.
- (v) Product Manifolds.

**Definition 3.** Let  $\mathcal{M}$  be a topological  $n$ -manifold. If  $(U, \varphi), (V, \psi)$  are two charts such that  $U \cap V \neq \emptyset$ , the composite map  $\psi \circ \varphi^{-1} : \varphi(U \cap V) \rightarrow \psi(U \cap V)$  is called **transition map from  $\varphi$  to  $\psi$** . Two charts  $(U, \varphi), (V, \psi)$  are said to be **smoothly compatible** if either  $U \cap V = \emptyset$  or the transition map  $\psi \circ \varphi^{-1}$  is a  $(\mathcal{C}^\infty)$  diffeomorphism.

**Remark 1.** In the above definition, since  $\psi(U \cap V)$  and  $\varphi(U \cap V)$  are open subsets of  $\mathbb{R}^n$ , smoothness of the transition map  $\psi \circ \varphi^{-1}$  can be interpreted in the ordinary sense of having continuous partial derivatives of all orders.

**Definition 4.** We define an **atlas for  $\mathcal{M}$**  to be a collection of charts whose domains cover  $\mathcal{M}$ . An atlas  $\mathcal{A}$  is called a **smooth atlas** if any two charts in  $\mathcal{A}$  are smoothly compatible with each other.

**Remark 2.** To show that an atlas is smooth, we need only verify that each transition map  $\psi \circ \varphi^{-1}$  is smooth whenever  $(U, \varphi), (V, \psi)$  are charts in  $\mathcal{A}$  such that  $U \cap V \neq \emptyset$ ; once we have proved this, it follows that  $\psi \circ \varphi^{-1}$  is a diffeomorphism because its inverse  $(\psi \circ \varphi^{-1})^{-1} = \varphi \circ \psi^{-1}$  is one of the transition maps we have already shown to be smooth. Alternatively, given two particular charts  $(U, \varphi), (V, \psi)$ , it is often easiest to show that they are smoothly compatible by verifying that  $\psi \circ \varphi^{-1}$  is smooth and injective with nonsingular Jacobian at each point, and appealing to a variant inverse function theorem([1, Corollary C.36]).

**Definition 5.** Let  $\mathcal{M}$  be a topological manifold. A smooth atlas  $\mathcal{A}$  on  $\mathcal{M}$  is **maximal** if it is not properly contained in any larger smooth atlas.

**Remark 3.** If  $\mathcal{A}$  is a maximal smooth atlas on  $\mathcal{M}$ , then any chart that is smoothly compatible with every chart in  $\mathcal{A}$  is already in  $\mathcal{A}$ .

**Definition 6.** Let  $\mathcal{M}$  be a topological manifold. A **smooth structure on  $\mathcal{M}$**  is a maximal smooth atlas. A **smooth manifold** is a pair  $(\mathcal{M}, \mathcal{A})$  where  $\mathcal{M}$  is a topological manifold and  $\mathcal{A}$  is a smooth structure on  $\mathcal{M}$ . Any chart  $(U, \varphi)$  in  $\mathcal{A}$  is called a **smooth chart** and the corresponding coordinate map  $\varphi$  and the domain  $U$  of  $\varphi$  are called **smooth coordinate map** and **smooth coordinate domain** or **smooth coordinate neighborhood** respectively.

**Theorem 2.** Let  $\mathcal{M}$  be a topological manifold.

- (a) Every smooth atlas  $\mathcal{A}$  on  $\mathcal{M}$  is contained in a unique maximal smooth atlas, called the **smooth structure determined by  $\mathcal{A}$** .
- (b) Two smooth atlases for  $\mathcal{M}$  determine the same smooth structure iff their union is a smooth atlas.

**Example.** Here are some examples of smooth manifolds.

- (i) Euclidean Spaces.
- (ii) Finite-Dimensional Vector Spaces.
- (iii) Space of Matrices.
- (iv) Open Submanifolds.
- (v) The General Linear Group.
- (vi) Matrices of Full Rank.
- (vii) Spaces of Linear Maps.
- (viii) Graphs of Continuous Functions.
- (ix) Spheres.
- (x) Level Sets.
- (xi) Projective Spaces.
- (xii) Smooth Product Manifolds.
- (xiii) Grassmann Manifolds.

Solve the exercise questions 1-1 to 1-10 from [1, Ch 1].

## 2 Smooth Maps

**Note 1.** For the sake of convenience, we reserve the word **function** for a map whose codomain is  $\mathbb{R}$  (a **real-valued function**) or  $\mathbb{R}^k$  for some  $k > 1$  (a **vector-valued function**). Either of the words **map** or **mapping** can mean any type of map, such as a map between arbitrary manifolds.

**Definition 7.** Suppose  $\mathcal{M}$  is a smooth  $n$ -manifold,  $k$  is a nonnegative integer, and  $f : \mathcal{M} \rightarrow \mathbb{R}^k$  is any function. We say that  $f$  is a **smooth function** if for every  $p \in \mathcal{M}$ , there exists a smooth chart  $(U, \varphi)$  for  $\mathcal{M}$  whose domain contains  $p$  and such that the composite function  $f \circ \varphi^{-1}$  is smooth on the open subset  $\hat{U} = \varphi(U) \subseteq \mathbb{R}^n$ .

**Note 2.** The most important special case is that of smooth real-valued functions  $f : \mathcal{M} \rightarrow \mathbb{R}$ ; the set of all such functions is denoted by  $C^\infty(\mathcal{M})$ . Because sums and constant multiples of smooth functions are smooth,  $C^\infty(\mathcal{M})$  is a vector space over  $\mathbb{R}$ .

**Proposition 2.** Let  $\mathcal{M}$  be a smooth manifold, and suppose  $f : \mathcal{M} \rightarrow \mathbb{R}^k$  is a smooth function. Then  $f \circ \varphi^{-1} : \varphi(U) \rightarrow \mathbb{R}^k$  is smooth for every smooth chart  $(U, \varphi)$  for  $\mathcal{M}$ .

**Definition 8.** Given a function  $f : \mathcal{M} \rightarrow \mathbb{R}^k$ , and a chart  $(U, \varphi)$  for  $\mathcal{M}$ , the function  $\hat{f} : \varphi(U) \rightarrow \mathbb{R}^k$  defined by  $\hat{f}(x) = f \circ \varphi^{-1}(x)$  is called the **coordinate representation of  $f$** .

**Remark 4.** By Def.7,  $f$  is smooth iff its coordinate representation is smooth in some smooth chart around each point. By Prop.2, smooth functions have smooth coordinate representations in every smooth chart.

**Proposition 3.** Let  $U$  be an open submanifold of  $\mathbb{R}^n$  with its standard smooth manifold structure. Then a function  $f : U \rightarrow \mathbb{R}^k$  is smooth in the sense of Def.7 iff it is smooth in the sense of ordinary calculus.

**Definition 9.** Let  $\mathcal{M}, \mathcal{N}$  be smooth manifolds, and let  $F : \mathcal{M} \rightarrow \mathcal{N}$  be any map. We say that  $F$  is a **smooth map** if for every  $p \in \mathcal{M}$ , there exist smooth charts  $(U, \varphi)$  containing  $p$  and  $(V, \psi)$  containing  $F(p)$  such that  $F(U) \subseteq V$  and the composite map  $\psi \circ F \circ \varphi^{-1}$  is smooth from  $\varphi(U)$  to  $\psi(V)$ .

**Note 3.** Def.7 can be viewed as a special case of Def.9 by taking  $\mathcal{N} = V = \mathbb{R}^k$  and  $\psi = \text{Id} : \mathbb{R}^k \rightarrow \mathbb{R}^k$ .

**Proposition 4.** Every smooth map is continuous.

**Proposition 5 (Equivalent Characterizations of Smoothness).** Suppose  $\mathcal{M}$  and  $\mathcal{N}$  are smooth manifolds, and  $F : \mathcal{M} \rightarrow \mathcal{N}$  is a map. Then  $F$  is smooth iff either of the following conditions is satisfied:

- (a) For every  $p \in \mathcal{M}$ , there exists smooth charts  $(U, \varphi)$  containing  $p$  and  $(V, \psi)$  containing  $F(p)$  such that  $U \cap F^{-1}(V)$  is open in  $\mathcal{M}$  and the composite map  $\psi \circ F \circ \varphi^{-1}$  is smooth from  $\varphi(U \cap F^{-1}(V))$  to  $\psi(V)$ .
- (b)  $F$  is continuous and there exist smooth atlases  $\{(U_\alpha, \varphi_\alpha)\}$  and  $\{(V_\beta, \psi_\beta)\}$  for  $\mathcal{M}$  and  $\mathcal{N}$ , respectively, such that for each  $\alpha$  and  $\beta$ ,  $\psi_\beta \circ F \circ \varphi_\alpha^{-1}$  is smooth from  $\varphi_\alpha(U_\alpha \cap F^{-1}(V_\beta))$  to  $\psi_\beta(V_\beta)$ .

**Proposition 6 (Smoothness is Local).** Let  $\mathcal{M}, \mathcal{N}$  be smooth manifolds, and let  $F : \mathcal{M} \rightarrow \mathcal{N}$  be a map.

- (a) If every point  $p \in \mathcal{M}$  has a neighborhood  $U$  such that the restriction  $F|_U$  is smooth, then  $F$  is smooth.

(b) Conversely, if  $F$  is smooth, then its restriction to every open subset is smooth.

**Proposition 7 (Gluing Lemma for Smooth Maps).** Let  $\mathcal{M}, \mathcal{N}$  be smooth manifolds, and let  $\{U_\alpha\}_{\alpha \in A}$  be an open cover of  $\mathcal{M}$ . Suppose that for each  $\alpha \in A$ , we are given a smooth map  $F_\alpha : U_\alpha \rightarrow \mathcal{N}$  such that the maps agree on overlaps:  $F_\alpha|_{U_\alpha \cap U_\beta} = F_\beta|_{U_\alpha \cap U_\beta}$  for all  $\alpha$  and  $\beta$ . Then there exists a unique smooth map  $F : \mathcal{M} \rightarrow \mathcal{N}$  such that  $F|_{U_\alpha} = F_\alpha$  for each  $\alpha \in A$ .

**Definition 10.** Given a map  $F : \mathcal{M} \rightarrow \mathcal{N}$ , and smooth charts  $(U, \varphi)$  and  $(V, \psi)$  for  $\mathcal{M}$  and  $\mathcal{N}$ , respectively, the function  $\hat{F} : \varphi(U \cap F^{-1}(V)) \rightarrow \psi(V)$  defined by  $\hat{F}(x) = \psi \circ F \circ \varphi^{-1}(x)$  is called the **coordinate representation of  $F$** .

**Proposition 8.** Suppose  $F : \mathcal{M} \rightarrow \mathcal{N}$  is a smooth map between smooth manifolds. Then the coordinate representation of  $F$  with respect to every pair of smooth charts for  $\mathcal{M}$  and  $\mathcal{N}$  is smooth.

**Proposition 9.** Let  $\mathcal{M}, \mathcal{N}$ , and  $\mathcal{P}$  be smooth manifolds.

- (a) Every constant map  $c : \mathcal{M} \rightarrow \mathcal{N}$  is smooth.
- (b) The identity map of  $\mathcal{M}$  is smooth.
- (c) If  $U \subseteq \mathcal{M}$  is an open submanifold, then the inclusion map  $U \hookrightarrow \mathcal{M}$  is smooth.
- (d) If  $F : \mathcal{M} \rightarrow \mathcal{N}$  and  $G : \mathcal{N} \rightarrow \mathcal{P}$  are smooth, then so is  $G \circ F : \mathcal{M} \rightarrow \mathcal{P}$ .

**Proposition 10.** Suppose  $\mathcal{M}_1, \dots, \mathcal{M}_k$  and  $\mathcal{N}$  are smooth manifolds. For each  $i$ , let  $\pi_i : \mathcal{M}_1 \times \dots \times \mathcal{M}_k \rightarrow \mathcal{M}_i$  denote the projection onto the  $\mathcal{M}_i$  factor. A map  $F : \mathcal{N} \rightarrow \mathcal{M}_1 \times \dots \times \mathcal{M}_k$  is smooth iff each of the component maps  $F_i = \pi_i \circ F : \mathcal{N} \rightarrow \mathcal{M}_i$  is smooth.

### 3 Partitions of Unity

### 4 Tangent Vectors

**Definition 11.** Given a point  $x \in \mathbb{R}^n$ , the **geometric tangent space** to  $\mathbb{R}^n$  at  $x$ , denoted by  $\mathbb{R}_x^n$ , is the set

$$\mathbb{R}_x^n = \{x\} \times \mathbb{R}^n = \{(x, v) : v \in \mathbb{R}^n\}.$$

A **geometric tangent vector** in  $\mathbb{R}^n$  is an element of  $\mathbb{R}_x^n$  for some  $x \in \mathbb{R}^n$ . As a matter of notation, we abbreviate  $(x, v)$  as  $v_x$  or  $v|_x$ . We think of  $v_x$  as the vector  $v$  with its initial point at  $x$ .

**Note 4.** The set  $\mathbb{R}_x^n$  is a real vector space under the natural operations

$$v_x + w_x = (v + w)_x, \quad c(v_x) = (cv)_x.$$

Consequently, the vectors  $e_i|_x, i = 1, \dots, n$ , are a basis for  $\mathbb{R}_x^n$ .

**Definition 12.** If  $x$  is a point of  $\mathbb{R}^n$ , a map  $w : \mathcal{C}^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}$  is called a **derivation at  $x$**  if it is linear over  $\mathbb{R}$  and satisfies the following product rule:

$$w(fg) = f(x)wg + g(x)wf.$$

Let  $T_x\mathbb{R}^n$  denote the set of all derivation of  $\mathcal{C}^\infty(\mathbb{R}^n)$  at  $x$ .

**Note 5.** Clearly,  $T_x\mathbb{R}^n$  is a vector space under the operations

$$(w_1 + w_2)f = w_1f + w_2f, \quad (cw)f = c(wf).$$

**Note 6.** For any geometric tangent vector  $v_x \in \mathbb{R}_x^n$  we define a derivation to be a map which takes the directional derivative of any  $f \in \mathcal{C}^\infty(\mathbb{R}^n)$  in the direction  $v$  at  $x$ :

$$D_v|_x f = Df(x)[v] = \left. \frac{d}{dt} \right|_{t=0} f(x + tv) = \lim_{t \rightarrow 0} \frac{f(x + tv) - f(x)}{t}.$$

It is indeed true that it is linear over  $\mathbb{R}$  since for any  $f, g \in \mathcal{C}^\infty(\mathbb{R}^n)$  and  $\alpha, \beta \in \mathbb{R}$ , we have

$$\begin{aligned} D_v|_x (\alpha f + \beta g) &= D(\alpha f + \beta g)(x)[v] = \lim_{t \rightarrow 0} \frac{\alpha f(x + tv) + \beta g(x + tv) - \alpha f(x) - \beta g(x)}{t} \\ &= \alpha \lim_{t \rightarrow 0} \frac{f(x + tv) - f(x)}{t} + \beta \lim_{t \rightarrow 0} \frac{g(x + tv) - g(x)}{t} \\ &= \alpha Df(x)[v] + \beta Dg(x)[v] = \alpha D_v|_x f + \beta D_v|_x g. \end{aligned}$$

One can also note that this map satisfies the product rule (or chain rule):

$$D_v|_x (fg) = f(x) D_v|_x g + g(x) D_v|_x f.$$

If  $v_a = \sum_{i=1}^n v^{(i)} e_i|_a$  in terms of the standard basis, then by the chain rule  $D_v|_a f$  can be written more concretely as

$$D_v|_a f = \sum_{i=1}^n v^{(i)} \frac{\partial f}{\partial x^{(i)}}(a).$$

**Lemma 3 (Properties of Derivations).** Suppose  $x \in \mathbb{R}^n$ ,  $w \in T_x \mathbb{R}^n$ , and  $f, g \in \mathcal{C}^\infty(\mathbb{R}^n)$ .

(a) If  $f$  is a constant function, then  $wf = 0$ .

(b) If  $f(x) = g(x) = 0$ , then  $w(fg) = 0$ .

**Proposition 11.** Let  $x \in \mathbb{R}^n$ .

(a) For each geometric tangent vector  $v_x \in \mathbb{R}_x^n$ , the map  $D_v|_x : \mathcal{C}^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}$  defined in Note 6 is a derivation at  $x$ .

(b) The map  $v_x \mapsto D_v|_x$  is an isomorphism from  $\mathbb{R}_x^n$  onto  $T_x \mathbb{R}^n$ .

**Corollary 3.1.** For any  $a \in \mathbb{R}^n$ , the  $n$  derivations

$$\left. \frac{\partial}{\partial x^{(1)}} \right|_a, \dots, \left. \frac{\partial}{\partial x^{(n)}} \right|_a \text{ defined by } \left. \frac{\partial}{\partial x^{(i)}} \right|_a f = \frac{\partial f}{\partial x^{(i)}}(a)$$

form a basis for  $T_a \mathbb{R}^n$ , which therefore has dimension  $n$ .

**Definition 13.** Let  $\mathcal{M}$  be a smooth manifold, and let  $p$  be a point of  $\mathcal{M}$ . A linear map  $v : \mathcal{C}^\infty(\mathcal{M}) \rightarrow \mathbb{R}$  is called a **derivation at  $p$**  if it satisfies

$$v(fg) = f(p)v g + g(p)v f, \text{ for all } f, g \in \mathcal{C}^\infty(\mathcal{M}).$$

The set of all derivations of  $\mathcal{C}^\infty(\mathcal{M})$  at  $p$ , denoted by  $T_p \mathcal{M}$ , is a vector space called the **tangent space to  $\mathcal{M}$  at  $p$** . An element of  $T_p \mathcal{M}$  is called a **tangent vector at  $p$** .

**Lemma 4 (Properties of Tangent Vectors on Manifolds).** Suppose  $\mathcal{M}$  is a smooth manifold,  $p \in \mathcal{M}$ ,  $v \in T_p \mathcal{M}$ , and  $f, g \in \mathcal{C}^\infty(\mathcal{M})$ .

(a) If  $f$  is a constant function, then  $v f = 0$ .

(b) If  $f(p) = g(p) = 0$ , then  $v(fg) = 0$ .

**Definition 14.** If  $\mathcal{M}$  and  $\mathcal{N}$  are smooth manifolds and  $F : \mathcal{M} \rightarrow \mathcal{N}$  is a smooth map, for each  $p \in \mathcal{M}$  we define a map  $dF_p : T_p\mathcal{M} \rightarrow T_{F(p)}\mathcal{N}$ , called the **differential of  $F$  at  $p$** , as follows. Given  $v \in T_p\mathcal{M}$ , we let  $dF_p(v)$  be the derivation at  $F(p)$  that acts on  $f \in C^\infty(\mathcal{N})$  by the rule

$$dF_p(v)(f) = v(f \circ F).$$

**Remark 5.** The operator  $dF_p : C^\infty(\mathcal{N}) \rightarrow \mathbb{R}$  is linear because  $v$  is, and is a derivation at  $F(p)$  because for any  $f, g \in C^\infty(\mathcal{N})$  we have

$$\begin{aligned} dF_p(v)(fg) &= v((fg) \circ F) = v((f \circ F)(g \circ F)) \\ &= f(F(p))v(g \circ F) + g(F(p))v(f \circ F) \\ &= f(F(p))dF_p(v)(g) + g(F(p))dF_p(v)(f). \end{aligned}$$

**Proposition 12 (Properties of Differentials).** Let  $\mathcal{M}, \mathcal{N}$ , and  $\mathcal{P}$  be smooth manifolds, let  $F : \mathcal{M} \rightarrow \mathcal{N}$  and  $G : \mathcal{N} \rightarrow \mathcal{P}$  be smooth maps, and let  $p \in \mathcal{M}$ ,

1.  $dF_p : T_p\mathcal{M} \rightarrow T_{F(p)}\mathcal{N}$  is linear.
2.  $d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_p\mathcal{M} \rightarrow T_{G \circ F(p)}\mathcal{P}$ .
3.  $d(Id_{\mathcal{M}}) = Id_{T_p\mathcal{M}} : T_p\mathcal{M} \rightarrow T_p\mathcal{M}$ .
4. If  $F$  is a diffeomorphism, then  $dF_p : T_p\mathcal{M} \rightarrow T_{F(p)}\mathcal{N}$  is an isomorphism, and  $(dF_p)^{-1} = d(F^{-1})_{F(p)}$ .

**Proposition 13 (Tangent vectors act locally).** Let  $\mathcal{M}$  be a smooth manifold,  $p \in \mathcal{M}$  and  $v \in T_p\mathcal{M}$ . If  $f, g \in C^\infty(\mathcal{M})$  agree on some neighborhood of  $p$ , then  $vf = vg$ .

**Proposition 14 (The Tangent Space to an Open Submanifold).** Let  $\mathcal{M}$  be a smooth manifold, let  $U \subseteq \mathcal{M}$  be an open subset, and let  $\iota : U \rightarrow \mathcal{M}$  be the inclusion map. For every  $p \in U$ , the differential  $d\iota_p : T_pU \rightarrow T_p\mathcal{M}$  is an isomorphism.

**Proposition 15 (Dimension of the Tangent Space).** If  $\mathcal{M}$  is an  $n$ -dimensional smooth manifold, then for each  $p \in \mathcal{M}$ , the tangent space  $T_p\mathcal{M}$  is an  $n$ -dimensional vector space.

**Proposition 16 (The Tangent Space to a Vector Space).** Suppose  $V$  is a finite-dimensional vector space with its standard smooth structure. For any vector  $v \in V$ , we define a map  $D_v|_a : C^\infty(V) \rightarrow \mathbb{R}$  by

$$D_v|_a f = \left. \frac{d}{dt} \right|_{t=0} f(a + tv).$$

Then, the map  $v \mapsto D_v|_a$  defined above is an isomorphism from  $v$  to  $T_aV$ , such that for any linear map  $L : V \rightarrow W$  we have  $dL_a(D_v|_a) = D_{Lv}|_{La}$ .

**Proposition 17 (The Tangent Space to a Product Manifold).** Let  $\mathcal{M}_1, \dots, \mathcal{M}_k$  be smooth manifolds, and for each  $j$ , let  $\pi_j : \mathcal{M}_1 \times \dots \times \mathcal{M}_k \rightarrow \mathcal{M}_j$  denote the projection onto the  $\mathcal{M}_j$  factor. For any point  $p = (p_1, \dots, p_k) \in \mathcal{M}_1 \times \dots \times \mathcal{M}_k$ , the map

$$\alpha : T_p(\mathcal{M}_1 \times \dots \times \mathcal{M}_k) \rightarrow T_{p_1}\mathcal{M}_1 \oplus \dots \oplus T_{p_k}\mathcal{M}_k$$

defined by

$$\alpha(v) = (d(\pi_1)_p(v), \dots, d(\pi_k)_p(v))$$

is an isomorphism.

**Proposition 18.** Let  $\mathcal{M}$  be a smooth  $n$ -manifold, and let  $p \in \mathcal{M}$ . Then  $T_p\mathcal{M}$  is an  $n$ -dimensional vector space, and for any smooth chart  $(U, (x^{(i)}))$  containing  $p$ , the coordinate vectors  $\partial/\partial x^{(1)}|_p, \dots, \partial/\partial x^{(n)}|_p$  form a basis for  $T_p\mathcal{M}$ .

## References

- [1] John M. Lee, Introduction to Smooth Manifolds.