Differential Geometry

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1   Smooth Manifolds

Definition 1. *A topological space $\mathcal{M}$ is said to be **locally Euclidean of dimension** n if every point of $\mathcal{M}$ has a neighborhood in $\mathcal{M}$ that is homeomorphic to an open subset of $\mathbb{R}^n$.*

Lemma 1. *A topological space $\mathcal{M}$ is locally Euclidean of dimension n if and only if either of the following properties holds:*

(a) Every point of $\mathcal{M}$ has a neighborhood homeomorphic to an open ball in $\mathbb{R}^n$.

(b) Every point of $\mathcal{M}$ has a neighborhood homeomorphic to $\mathbb{R}^n$.

Proof Sketch. *For (a) one need to show that any open subset of $\mathbb{R}^n$ is homeomorphic to the unit open ball in $\mathbb{R}^n$. Note that the unit open ball is homeomorphic to any open ball in $\mathbb{R}^n$. For (b) it suffices to show that unit open ball in $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$.*

Definition 2. *Suppose $\mathcal{M}$ is a topological space. $\mathcal{M}$ is a **topological manifold of dimension** n or a **topological n-manifold** if it has the following properties:*

*(a) $\mathcal{M}$ is a **Hausdorff space**.*

*(b) $\mathcal{M}$ is a **second-countable**.*

*(c) $\mathcal{M}$ is **locally Euclidean of dimension** n.*

*A **coordinate chart**(or just a **chart**) on $\mathcal{M}$ is a pair (U, φ), where U is an open subset of $\mathcal{M}$ and $\varphi : U \to \hat{U}$ is a homeomorphism from U to an open subset $\hat{U} = \varphi(U) \subseteq \mathbb{R}^n$. The set U is called a **coordinate domain** or a **coordinate neighborhood** of each of its points. The map φ is called a **(local) coordinate map**, and the component functions $(x^1, \ldots, x^n)$ of φ, defined by $\varphi(p) = (x^1(p), \ldots, x^n(p))$, are called **local coordinates** on U.*

Proposition 1. *A nonempty n-dimensional topological manifold cannot be homeomorphic to an m-dimensional manifold unless $m = n$.*

Example. *Here are some examples of topological manifolds.*

(i) Open subset of a topological n-manifold.

(ii) Graphs of Continuous Functions.

(iii) Spheres.

(iv) Projective Spaces.

(v) Product Manifolds.

Definition 3. *Let $\mathcal{M}$ be a topological n-manifold. If $(U, \varphi), (V, \psi)$ are two charts such that $U \cap V \neq \emptyset$, the composite map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is called **transition map from φ to ψ**. Two charts $(U, \varphi), (V, \psi)$ are said to be **smoothly compatible** if either $U \cap V = \emptyset$ or the transition map $psi \circ \varphi^{-1}$ is a ($\mathcal{C}^\infty$)diffeomorphism.*

Remark 1. *In the above definition, since $\psi(U \cap V)$ and $\varphi(U \cap V)$ are open subsets of $\mathbb{R}^n$, smoothness of the transition map $\psi \circ \varphi^{-1}$ can be interpreted in the ordinary sense of having continuous partial derivatives of all orders.*

Definition 4. *We define an **atlas for $\mathcal{M}$** to be a collection of charts whose domains cover $\mathcal{M}$. An atlas $\mathcal{A}$ is called a **smooth atlas** if any two charts in $\mathcal{A}$ are smoothly compatible with each other.*

Remark 2. *To show that an atlas is smooth, we need only verify that each transition map $\psi \circ \varphi^{-1}$ is smooth whenever $(U, \varphi), (V, \psi)$ are charts in $\mathcal{A}$ such that $U \cap V \neq \emptyset$; once we have proved this, it follows that $\psi \circ \varphi^{-1}$ is a diffeomorphism because its inverse $(\psi \circ \varphi^{-1})^{-1} = \varphi \circ \psi^{-1}$ is one of the transition maps we have already shown to be smooth. Alternatively, given two particular charts $(U, \varphi), (V, \psi)$, it is often easiest to show that they are smoothly compatible by verifying that $\psi \circ \varphi^{-1}$ is smooth and injective with nonsingular Jacobian at each point, and appealing to a variant inverse function theorem([1, Corollary C.36]).*

Definition 5. *Let $\mathcal{M}$ be a topological manifold. A smooth atlas $\mathcal{A}$ on $\mathcal{M}$ is **maximal** if it is not properly contained in any larger smooth atlas.*

Remark 3. *If $\mathcal{A}$ is a maximal smooth atlas on $\mathcal{M}$, then any chart that is smoothly compatible with every chart in $\mathcal{A}$ is already in $\mathcal{A}$.*

Definition 6. *Let $\mathcal{M}$ be a topological manifold. A **smooth structure on $\mathcal{M}$** is a maximal smooth atlas. A **smooth manifold** is a pair $(\mathcal{M}, \mathcal{A})$ where $\mathcal{M}$ is a topological manifold and $\mathcal{A}$ is a smooth structure on $\mathcal{M}$. Any chart (U, φ) in $\mathcal{A}$ is called a **smooth chart** and the corresponding coordinate map φ and the domain U of φ are called **smooth coordinate map** and **smooth coordinate domain** or **smooth coordinate neighborhood** respectively.*

Theorem 2. *Let $\mathcal{M}$ be a topological manifold.*

(a) *Every smooth atlas $\mathcal{A}$ on $\mathcal{M}$ is contained in a unique maximal smooth atlas, called the **smooth structure determined by** $\mathcal{A}$.*

(b) *Two smooth atlases for $\mathcal{M}$ determine the same smooth structure iff their union is a smooth atlas.*

Example. *Here are some examples of smooth manifolds.*

  (i) *Euclidean Spaces.*

  (ii) *Finite-Dimensional Vector Spaces.*

  (iii) *Space of Matrices.*

  (iv) *Open Submanifolds.*

  (v) *The General Linear Group.*

  (vi) *Matrices of Full Rank.*

  (vii) *Spaces of Linear Maps.*

 (viii) *Graphs of Continuous Functions.*

  (ix) *Spheres.*

  (x) *Level Sets.*

  (xi) *Projective Spaces.*

  (xii) *Smooth Product Manifolds.*

 (xiii) *Grassmann Manifolds.*

Solve the exercise questions 1-1 to 1-10 from [1, Ch 1].

2 Smooth Maps

Note 1. *For the sake of convenience, we reserve the word **function** for a map whose codomain is $\mathbb{R}$ (a **real-valued function**) or $\mathbb{R}^k$ for some $k > 1$ (a **vector-valued function**). Either of the words **map** or **mapping** can mean any type of map, such as a map between arbitrary manifolds.*

Definition 7. *Suppose $\mathcal{M}$ is a smooth n-manifold, k is a nonnegative integer, and $f : \mathcal{M} \to \mathbb{R}^k$ is any function. We say that f is a **smooth function** if for every $p \in \mathcal{M}$, there exists a smooth chart (U, φ) for $\mathcal{M}$ whose domain contains p and such that the composite function $f \circ \varphi^{-1}$ is smooth on the open subset $\hat{U} = \varphi(U) \subseteq \mathbb{R}^n$.*

Note 2. *The most important special case is that of smooth real-valued functions $f : \mathcal{M} \to \mathbb{R}$; the set of all such functions is denoted by $\mathcal{C}^\infty(\mathcal{M})$. Because sums and constant multiples of smooth functions are smooth, $\mathcal{C}^\infty(\mathcal{M})$ is a vector space over $\mathbb{R}$.*

Proposition 2. *Let $\mathcal{M}$ be a smooth manifold, and suppose $f : \mathcal{M} \to \mathbb{R}^k$ is a smooth function. Then $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^k$ is smooth for every smooth chart (U, φ) for $\mathcal{M}$.*

Definition 8. *Given a function $f : \mathcal{M} \to \mathbb{R}^k$, and a chart (U, φ) for $\mathcal{M}$, the function $\hat{f} : \varphi(U) \to \mathbb{R}^k$ defined by $\hat{f}(x) = f \circ \varphi^{-1}(x)$ is called the **coordinate representation of f**.*

Remark 4. *By Def.7, f is smooth iff its coordinate representation is smooth in some smooth chart around each point. By Prop.2, smooth functions have smooth coordinate representations in every smooth chart.*

Proposition 3. *Let U be an open submanifold of $\mathbb{R}^n$ with its standard smooth manifold structure. Then a function $f : U \to \mathbb{R}^k$ is smooth in the sense of Def.7 iff it is smooth in the sense of ordinary calculus.*

Definition 9. *Let $\mathcal{M}, \mathcal{N}$ be smooth manifolds, and let $F : \mathcal{M} \to \mathcal{N}$ be any map. We say that F is a **smooth map** if for every $p \in \mathcal{M}$, there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subseteq V$ and the composite map $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U)$ to $\psi(V)$.*

Note 3. *Def.7 can be viewed as a special case of Def.9 by taking $\mathcal{N} = V = \mathbb{R}^k$ and $\psi = Id : \mathbb{R}^k \to \mathbb{R}^k$.*

Proposition 4. *Every smooth map is continuous.*

Proposition 5 (*Equivalent Characterizations of Smoothness*). *Suppose $\mathcal{M}$ and $\mathcal{N}$ are smooth manifolds, and $F : \mathcal{M} \to \mathcal{N}$ is a map. Then F is smooth iff either of the following conditions is satisfied:*

(a) For every $p \in \mathcal{M}$, there exists smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $U \cap F^{-1}(V)$ is open in $\mathcal{M}$ and the composite map $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U \cap F^{-1}(V))$ to $\psi(V)$.

(b) F is continuous and there exist smooth atlases $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ for $\mathcal{M}$ and $\mathcal{N}$, respectively, such that for each α and β, $\psi_\beta \circ F \circ \varphi_\alpha^{-1}$ is smooth from $\varphi_\alpha(U_\alpha \cap F^{-1}(V_\beta))$ to $\psi_\beta(V_\beta)$.

Proposition 6 (*Smoothness is Local*). *Let $\mathcal{M}, \mathcal{N}$ be smooth manifolds, and let $F : \mathcal{M} \to \mathcal{N}$ be a map.*

(a) If every point $p \in \mathcal{M}$ has a neighborhood U such that the restriction $F|_U$ is smooth, then F is smooth.

(b) Conversely, if F is smooth, then its restriction to every open subset is smooth.

Proposition 7 (**Gluing Lemma for Smooth Maps**). *Let $\mathcal{M}, \mathcal{N}$ be smooth manifolds, and let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of $\mathcal{M}$. Suppose that for each $\alpha \in A$, we are given a smooth map $F_\alpha : U_\alpha \to \mathcal{N}$ such that the maps agree on overlaps: $F_\alpha|_{U_\alpha \cap U_\beta} = F_\beta|_{U_\alpha \cap U_\beta}$ for all α and β. Then there exists a unique smooth map $F : \mathcal{M} \to \mathcal{N}$ such that $F|_{U_\alpha} = F_\alpha$ for each $\alpha \in A$.*

Definition 10. *Given a map $F : \mathcal{M} \to \mathcal{N}$, and smooth charts (U, φ) and (V, ψ) for $\mathcal{M}$ and $\mathcal{N}$, respectively, the function $\hat{F} : \varphi(U \cap F^{-1}(V)) \to \psi(V)$ defined by $\hat{F}(x) = \psi \circ F \circ \varphi^{-1}(x)$ is called the **coordinate representation of F**.*

Proposition 8. *Suppose $F : \mathcal{M} \to \mathcal{N}$ is a smooth map between smooth manifolds. Then the coordinate representation of F with respect to every pair of smooth charts for $\mathcal{M}$ and $\mathcal{N}$ is smooth.*

Proposition 9. *Let $\mathcal{M}$, $\mathcal{N}$, and $\mathcal{P}$ be smooth manifolds.*

(a) Every constant map $c : \mathcal{M} \to \mathcal{N}$ is smooth.

(b) The identity map of $\mathcal{M}$ is smooth.

(c) If $U \subseteq \mathcal{M}$ is an open submanifold, then the inclusion map $U \hookrightarrow \mathcal{M}$ is smooth.

(d) If $F : \mathcal{M} \to \mathcal{N}$ and $G : \mathcal{N} \to \mathcal{P}$ are smooth, then so is $G \circ F : \mathcal{M} \to \mathcal{P}$.

Proposition 10. *Suppose $\mathcal{M}_1, \ldots, \mathcal{M}_k$ and $\mathcal{N}$ are smooth manifolds. For each i, let $\pi_i : \mathcal{M}_1 \times \ldots \times \mathcal{M}_k \to \mathcal{M}_i$ denote the projection onto the $\mathcal{M}_i$ factor. A map $F : \mathcal{N} \to \mathcal{M}_1 \times \ldots \times \mathcal{M}_k$ is smooth iff each of the component maps $F_i = \pi_i \circ F : \mathcal{N} \to \mathcal{M}_i$ is smooth.*

3 Partitions of Unity

4 Tangent Vectors

Definition 11. *Given a point $x \in \mathbb{R}^n$, the **geometric tangent space** to $\mathbb{R}^n$ at x, denoted by $\mathbb{R}^n_x$, is the set*
$$\mathbb{R}^n_x = \{x\} \times \mathbb{R}^n = \{(x, v) : v \in \mathbb{R}^n\}.$$

*A **geometric tangent vector** in $\mathbb{R}^n$ is an element of $\mathbb{R}^n_x$ for some $x \in \mathbb{R}^n$. As a matter of notation, we abbreviate (x, v) as v_x or $v|_x$. We think of v_x as the vector v with its initial point at x.*

Note 4. *The set $\mathbb{R}^n_x$ is a real vector space under the natural operations*
$$v_x + w_x = (v + w)_x, \quad c(v_x) = (cv)_x.$$

Consequently, the vectors $e_i|_x$, $i = 1, \ldots, n$, are a basis for $\mathbb{R}^n_x$.

Definition 12. *If x is a point of $\mathbb{R}^n$, a map $w : \mathcal{C}^\infty(\mathbb{R}^n) \to \mathbb{R}$ is called a **derivation at x** if it is linear over $\mathbb{R}$ and satisfies the following product rule:*
$$w(fg) = f(x)wg + g(x)wf.$$

Let $T_x\mathbb{R}^n$ denote the set of all derivation of $\mathcal{C}^\infty(\mathbb{R}^n)$ at x.

Note 5. *Clearly, $T_x\mathbb{R}^n$ is a vector space under the operations*
$$(w_1 + w_2)f = w_1 f + w_2 f, \quad (cw)f = c(wf).$$

Note 6. *For any geometric tangent vector $v_x \in \mathbb{R}^n_x$ we define a derivation to be a map which takes the directional derivative of any $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ in the direction v at x:*

$$D_v|_x f = Df(x)[v] = \frac{d}{dt}\bigg|_{t=0} f(x + tv) = \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}.$$

It is indeed true that it is linear over $\mathbb{R}$ since for any $f, g \in \mathcal{C}^\infty(\mathbb{R}^n)$ and $\alpha, \beta \in \mathbb{R}$, we have

$$
\begin{aligned}
D_v|_x (\alpha f + \beta g) = D(\alpha f + \beta g)(x)[v] \ =& \lim_{t \to 0} \frac{\alpha f(x + tv) + \beta g(x + tv) - \alpha f(x) - \beta g(x)}{t} \\
=& \alpha \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t} + \beta \lim_{t \to 0} \frac{g(x + tv) - g(x)}{t} \\
=& \alpha Df(x)[v] + \beta Dg(x)[v] = \alpha \ D_v|_x f + \beta \ D_v|_x g.
\end{aligned}
$$

One can also note that this map satisfies the product rule(or chain rule):

$$D_v|_x (fg) = f(x) \ D_v|_x g + g(x) \ D_v|_x f.$$

If $v_a = \sum_{i=1}^n v^{(i)} e_i|_a$ in terms of the standard basis, then by the chain rule $D_v|_a f$ can be written more concretely as

$$D_v|_a f = \sum_{i=1}^n v^{(i)} \frac{\partial f}{\partial x^{(i)}}(a).$$

Lemma 3 (*Properties of Derivations*). *Suppose $x \in \mathbb{R}^n, w \in T_x\mathbb{R}^n$, and $f, g \in \mathcal{C}^\infty(\mathbb{R}^n)$.*

(a) If f is a constant function, then $wf = 0$.

(b) If $f(x) = g(x) = 0$, then $w(fg) = 0$.

Proposition 11. *Let $x \in \mathbb{R}^n$.*

(a) For each geometric tangent vector $v_x \in \mathbb{R}^n_x$, the map $D_v|_x : \mathcal{C}^\infty(R^n) \to \mathbb{R}$ defined in Note 6 is a derivation at x.

(b) The map $v_x \mapsto D_v|_x$ is an isomorphism from $\mathbb{R}^n_x$ onto $T_x\mathbb{R}^n$.

Corollary 3.1. *For any $a \in \mathbb{R}^n$, the n derivations*

$$\frac{\partial}{\partial x^{(1)}}\bigg|_a, \ldots, \frac{\partial}{\partial x^{(n)}}\bigg|_a \quad \text{defined by} \quad \frac{\partial}{\partial x^{(i)}}\bigg|_a f = \frac{\partial f}{\partial x^{(i)}}(a)$$

form a basis for $T_a\mathbb{R}^n$, which therefore has dimension n.

Definition 13. *Let $\mathcal{M}$ be a smooth manifold, and let p be a point of $\mathcal{M}$. A linear map $v : \mathcal{C}^\infty(\mathcal{M}) \to \mathbb{R}$ is called a **derivation at p** if it satisfies*

$$v(fg) = f(p)vg + g(p)vf, \quad \text{for all } f, g \in \mathcal{C}^\infty(\mathcal{M}).$$

*The set of all derivations of $\mathcal{C}^\infty(\mathcal{M})$ at p, denoted by $T_p\mathcal{M}$, is a vector space called the **tangent space to $\mathcal{M}$ at p**. An element of $T_p\mathcal{M}$ is called a **tangent vector at p**.*

Lemma 4 (*Properties of Tangent Vectors on Manifolds*). *Suppose $\mathcal{M}$ is a smooth manifold, $p \in \mathcal{M}$, $v \in T_p\mathcal{M}$, and $f, g \in \mathcal{C}^\infty(\mathcal{M})$.*

(a) If f is a constant function, then $vf = 0$.

(b) If $f(p) = g(p) = 0$, then $v(fg) = 0$.

Definition 14. *If $\mathcal{M}$ and $\mathcal{N}$ are smooth manifolds and $F : \mathcal{M} \to \mathcal{N}$ is a smooth map, for each $p \in \mathcal{M}$ we define a map $dF_p : T_p\mathcal{M} \to T_{F(p)}\mathcal{N}$, called the **differential of F at p**, as follows. Given $v \in T_p\mathcal{M}$, we let $dF_p(v)$ be the derivation at $F(p)$ that acts on $f \in \mathcal{C}^\infty(\mathcal{N})$ by the rule*

$$dF_p(v)(f) = v(f \circ F).$$

Remark 5. *The operator $dF_p : \mathcal{C}^\infty(\mathcal{N}) \to \mathbb{R}$ is linear because v is, and is a derivation at $F(p)$ because for any $f, g \in \mathcal{C}^\infty(\mathcal{N})$ we have*

$$
\begin{aligned}
dF_p(v)(fg) &= v((fg) \circ F) = v((f \circ F)(g \circ F)) \\
&= f(F(p))v(g \circ F) + g(F(p))v(f \circ F) \\
&= f(F(p))dF_p(v)(g) + g(F(p))dF_p(v)(f).
\end{aligned}
$$

Proposition 12 (*Properties of Differentials*). *Let $\mathcal{M}, \mathcal{N}$, and $\mathcal{P}$ are smooth manifolds, let $F : \mathcal{M} \to \mathcal{N}$ and $G : \mathcal{N} \to \mathcal{P}$ be smooth maps, and let $p \in \mathcal{M}$,*

1. *$dF_p : T_p\mathcal{M} \to T_{F(p)}\mathcal{N}$ is linear.*

2. *$d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_p\mathcal{M} \to T_{G \circ F(p)}\mathcal{P}$.*

3. *$d(Id_\mathcal{M}) = Id_{T_p\mathcal{M}} : T_p\mathcal{M} \to T_p\mathcal{M}$.*

4. *If F is a diffeomorphism, then $dF_p : T_p\mathcal{M} \to T_{F(p)}\mathcal{N}$ is an isomorphism, and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.*

Proposition 13 (*Tangent vectors act locally*). *Let $\mathcal{M}$ be a smooth manifold, $p \in \mathcal{M}$ and $v \in T_p\mathcal{M}$. If $f, g \in \mathcal{C}^\infty(M)$ agree on some neighborhood of p, then $vf = vg$.*

Proposition 14 (*The Tangent Space to an Open Submanifold*). *Let $\mathcal{M}$ be a smooth manifold, let $U \subseteq \mathcal{M}$ be an open subset, and let $\iota : U \to \mathcal{M}$ be the inclusion map. For every $p \in U$, the differential $d\iota_p : T_pU \to T_p\mathcal{M}$ is an isomorphism.*

Proposition 15 (*Dimension of the Tangent Space*). *If $\mathcal{M}$ is an n-dimensional smooth manifold, then for each $p \in \mathcal{M}$, the tangent space $T_p\mathcal{M}$ is an n-dimensional vector space.*

Proposition 16 (*The Tangent Space to a Vector Space*). *Suppose V is a finite-dimensional vector space with its standard smooth structure. For any vector $v \in V$, we define a map $D_v|_a : \mathcal{C}^\infty(V) \to \mathbb{R}$ by*

$$D_v|_a f = \left. \frac{d}{dt} \right|_{t=0} f(a + tv).$$

Then, the map $v \mapsto D_v|_a$ defined above is an isomorphism from v to T_aV, such that for any linear map $L : V \to W$ we have $dL_a(D_v|_a) = D_{Lv|_{La}}$.

Proposition 17 (*The Tangent Space to a Product Manifold*). *Let $\mathcal{M}_1, \dots, \mathcal{M}_k$ be smooth manifolds, and for each j, let $\pi_j : \mathcal{M}_1 \times \dots \times \mathcal{M}_k \to \mathcal{M}_j$ denote the projection onto the $\mathcal{M}_j$ factor. For any point $p = (p_1, \dots, p_k) \in \mathcal{M}_1 \times \dots \times \mathcal{M}_k$, the map*

$$\alpha : T_p(\mathcal{M}_1 \times \dots \times \mathcal{M}_k) \to T_{p_1}\mathcal{M}_1 \oplus \dots \oplus T_{p_k}\mathcal{M}_k$$

defined by

$$\alpha(v) = (d(\pi_1)_p(v), \dots, d(\pi_k)_p(v))$$

is an isomorphism.

Proposition 18. *Let $\mathcal{M}$ be a smooth n-manifold, and let $p \in \mathcal{M}$. Then $T_p\mathcal{M}$ is an n-dimensional vector space, and for any smooth chart $(U, (x^{(i)}))$ containing p, the coordinate vectors $\partial/\partial x^{(1)}|_p, \dots, \partial/\partial x^{(n)}|_p$ form a basis for $T_p\mathcal{M}$.*

References

[1] John M. Lee, Introduction to Smooth Manifolds.