# Real Analysis

Jayadev Naram

## Contents



## 1 The Real Numbers

| | |
|---|---|
| $\mathbb{R}$ | Set of all real numbers |
| $\mathbb{C}$ | Set of all complex numbers |
| $\mathbb{N}$ | Set of all natural numbers |
| $\mathbb{Q}$ | Set of all rational numbers |
| $\mathbb{R} - \mathbb{Q}$ | Set of all irrational numbers |

Table 1: Standard notation for various sets.

**Definition 1.1.** A subset $A$ of $\mathbb{R}$ is said to be ***bounded above*** if there is some $x \in \mathbb{R}$ such that $a \leq x$ for all $a \in A$. Any such number $x$ is called an ***upper bound*** for $A$.

**Axiom 1.2** (The Least Upper Bound Axiom or The Completeness Axiom)**.** *Any nonempty set of real numbers with an upper bound has a least upper bound.*

**Definition 1.3.** Let $A \subseteq \mathbb{R}$ be a nonempty set that is bounded above. Then the ***supremum*** of the set, denoted by $\sup A$, is the least upper bound, i.e., $\sup A = s \in \mathbb{R}$ such that

(i) $s$ is an upper bound for $A$;

(ii) if $x$ is any upper bound for $A$, then $s \leq x$.

**Remark 1.4.** If $A \subseteq \mathbb{R}$ is unbounded above, then $\sup A = +\infty$ and if $A = \emptyset$, then $\sup A = -\infty$ as every real number is an upper bound for $A$.

**Proposition 1.5** (Characterization of the Supremum)**.** *Let $A \subseteq \mathbb{R}$ be a nonempty set that is bounded above. Then the following statements about $s = \sup A$ are all equivalent:*

*(a) if $x$ is any upper bound for $A$, then $s \leq x$;*

*(b) if $y < s$, then we must have $y < a \leq s$ for some $a \in A$;*

*(c) for every $\varepsilon > 0$, there is an $a \in A$ such that $a > s - \varepsilon$.*

*Proof.* (a) $\implies$ (b) Suppose for some $y < s$, there is no $a \in A$ such that $y < a \leq s$. Then for all $a \in A$, we have $a \leq y < s$. But then $y$ is an upper bound with $y < s$ which contradicts (a).
(b) $\implies$ (c) Consider an arbitrary $y \in \mathbb{R}$ such that $y < s$. Setting $\varepsilon = s - y$ and using (b) we have that there is some $a \in A$ such that $s - \varepsilon = y < a$.
(c) $\implies$ (a) Let $x$ be an upper bound for $A$, i.e., $a \leq x$ for all $a \in A$ and suppose that $x < s$. By taking $\varepsilon = s - x$ and applying (c), there must be some $a \in A$, such that $s - \varepsilon = x < a$ which is a contradiction that $x$ is an upper bound for $A$. Then $s \leq x$. $\square$

**Theorem 1.6.** *If $A$ is a nonmepty subset of $\mathbb{R}$ that is bounded below, then $A$ has a greatest lower bound called the **infimum** of $A$ which is denoted by* $\inf A$, *i.e., there is a number $m \in \mathbb{R}$ satisfying:*

*(i) $m$ is a lower bound for $A$;*

*(ii) if $x$ is a lower bound for $A$, then $x \leq m$.*

*Proof.* Conisder the set $-A = \{-a \ : \ a \in A\}$ which is bounded above as $A$ is bounded below. By the completeness axiom, $-A$ must have a least upper bound $m = \sup(-A)$. Note that $-m = -\sup(-A)$ then the greatest lower bound for $A$, so that $\inf A = -\sup(-A)$. □

**Remark 1.7.** As we have established that $\inf A = -\sup(-A)$, we have $\inf A = -\infty$ if $A$ is unbounded below, and $\inf \emptyset = +\infty$. In case a set $A$ is both bounded above and bounded below, we simply say that $A$ is ***bounded***.

**Proposition 1.8** (Characterization of the Infimum)**.** *Let $A \subseteq \mathbb{R}$ be a nonempty set that is bounded below. Then the following statements about $m = \inf A$ are all equivalent:*

*(a) if $x$ is any lower bound for $A$, then $x \leq m$;*

*(b) if $y > m$, then we must have $m \leq a < y$ for some $a \in A$;*

*(c) for every $\varepsilon > 0$, there is an $a \in A$ such that $a < m + \varepsilon$.*

Proof similar to that of Prop. 1.5.

**Proposition 1.9.** *Let $A$ be a bounded subset of $\mathbb{R}$ containing at least two points. Then*

*(a)* $-\infty < \inf A < \sup A < +\infty$.

*(b) If $B$ is a nonempty subset of $A$, then* $\inf A \leq \inf B \leq \sup B \leq \sup A$.

*(c) If $B$ is the set of all upper bounds for $A$, then $B$ is nonmepty, bounded below and* $\inf B = \sup A$.

*Proof.* (a) The boundedness of $A$ implies that $-\infty < \inf A \leq \sup A < +\infty$. Since there are at least two points in $A$, $\inf A \neq \sup A$.
(b) and (c) trivally hold from the definitions of infimum and supremum. □

**Definition 1.10.** A sequence $(x_n)$ of real numbers is said to ***converge*** to $x \in \mathbb{R}$ if, for every $\varepsilon > 0$, there is a positive integer $N$ such that

$$n \geq N \implies |x_n - x| \leq \varepsilon.$$

In this case, we call $x$ the limit of the sequence $(x_n)$ and write $x = \lim_{n\to\infty} x_n$.

**Proposition 1.11.** *Let $A$ be a nonempty subset of $\mathbb{R}$ that is bounded above. Then there is a sequence $(x_n)$ of elements of $A$ that converges to* $\sup A$.

*Proof.* Using Prop. 1.5(c), we set $\varepsilon_n = 1/n$ and $y_n = \sup A - \varepsilon_n$. Then for all $n \in N$, there exists an element $x_n \in A$ such that $x_n > \sup A - \varepsilon = y_n$. But then $|x_n - \sup A| < |s - y_n| = \varepsilon_n = 1/n$. This shows that $\lim_{n\to\infty} |x_n - \sup A| = 0$, i.e., $\lim_{n\to\infty} x_n = \sup A$. □

**Lemma 1.12** (Archimedean property in $\mathbb{R}$)**.** *If $x$ and $y$ are positive real numbers, then there is some positive integer $n$ such that $nx > y$.*

*Proof.* Suppose that no such $n$ existed, i.e., suppose that $nx \le y$ for all $n \in \mathbb{N}$. Then $A = \{nx : n \in \mathbb{N}\}$ is bounded above by $y$, and so $s = \sup A$ is finite. Now, since $s - x < s$, we must have some element of $A$ in between, i.e., $s - x < nx \le s$ for some $n \in \mathbb{N}$. But then $s < (n+1)x \in A$ which is a contradiction, hence there is some $n \in \mathbb{N}$ such that $nx > y$. $\square$

**Theorem 1.13.** *If $a$ and $b$ are real numbers with $a < b$, then there is a rational number $r \in \mathbb{Q}$ with $a < r < b$.*

*Proof.* We set $x = b - a > 0$, $y = 1$, and apply Lemma 1.12 to get a positive integer $q$ such that $q(b-a) > 1$, i.e., $qb > qa + 1$. But if $qa$ and $qb$ differ by more than 1, there must be some integer in between, i.e., there is some $p \in \mathbb{Z}$ with $qa < p < qb$. Thus, $a < \dfrac{p}{q} < b$. $\square$

**Corollary 1.13.1.** *If $a$ and $b$ are real numbers with $a < b$, then there is also an irrational number $x \in \mathbb{R} - \mathbb{Q}$ with $a < x < b$.*

*Proof.* We set $x = \dfrac{1}{\sqrt{2}}(b-a) > 0$, $y = 1$, and apply Lemma 1.12 to get a positive integer $q$ such that $\dfrac{1}{\sqrt{2}}qb > \dfrac{1}{\sqrt{2}}qa + 1$. But then there is some $p \in \mathbb{Z}$ with $\dfrac{1}{\sqrt{2}}qa < p < \dfrac{1}{\sqrt{2}}qb$. Thus, $a < \dfrac{\sqrt{2}p}{q} < b$. $\square$

**Corollary 1.13.2.** *Given $a < b$, there are, in fact, infinitely many distinct rationals between $a$ and $b$. The same goes for irrationals, too.*

*Proof.* $\square$

**Remark 1.14.** The least upper bound axiom holds in $\mathbb{Z}$ since for any nonempty set that is bounded above, there exists a least upper bound which is the maximum value of the set itself. But this axiom does not hold in $\mathbb{Q}$. Consider the set $A = \{p/q \in \mathbb{Q} : p^2 < 2q^2\}$. This is bounded above in $\mathbb{Q}$ as 2 is an upper bound. But $\sup A = \sqrt{2} \notin \mathbb{Q}$ so $A$ has not least upper bound in $\mathbb{Q}$.

**Proposition 1.15.** *The following statements are all equivalent:*

(a) *(The least upper bound property). Any nonempty set of real numbers with an upper bound has a least upper bound.*

(b) *A monotone, bounded sequence of real numbers converges.*

(c) *(The nested interval propoerty). If $(I_n)$ is a sequence of closed, bounded, nonempty intervals in $\mathbb{R}$ with $I_1 \supset I_2 \supset \cdots$, then $\cap_{n=1}^{\infty} I_n \neq \emptyset$. If, in addition, length$(I_n) \to 0$, then $\cap_{n=1}^{\infty} I_n$ contains precisely one point.*

*Proof.* (a) $\implies$ (b) Let $(x_n) \subset \mathbb{R}$ be monotone and bounded. We first suppose that $(x_n)$ is increasing. Now, since $(x_n)$ is bounded, we may set $x = \sup_n x_n \in \mathbb{R}$. Suppose $\varepsilon > 0$, then from (a) we must have $x_N > x - \varepsilon$ for some $N$. But then, for any $n \ge N$, we have $x - \varepsilon < x_N \le x_n \le x$, i.e., $|x - x_n| < \varepsilon$ for all $n \ge N$. Consequently, $(x_n)$ converges and $x = \sup_n x_n = \lim_{n\to\infty} x_n$. Finally, if $(x_n)$ is decreasing, consider the increasing sequence $(-x_n)$. From the previous arguments, $(-x_n)$ converges to $\sup_n(-x_n) = -\inf_n x_n$. It then follows that $(x_n)$ converges to $\inf_n x_n$.
(b) $\implies$ (c) Let $I_n = [a_n, b_n]$. Then $I_n \supset I_{n+1}$ means that $a_n \le a_{n+1} \le b_{n+1} \le b_n$ for all $n$. Then from (b), we have $a = \lim_{n\to\infty} a_n = \sup_n a_n$ and $b = \lim_{n\to\infty} b_n = \inf_n b_n$ both exist and satisfy $a \le b$. Thus we must have $\cap_{n=1}^{\infty} I_n = [a,b]$. Finally, if $b_n - a_n = \text{length}(I_n) \to 0$, then $a = b$ and so $\cap_{n=1}^{\infty} I_n = \{a\}$.

(c) $\Longrightarrow$ (a) Let $A$ be a nonempty subset of $\mathbb{R}$ that is bounded above. Specifically, let $a_1 \in A$ and let $b_1$ be an upper bound for $A$. For later reference, set $I_1 = [a_1, b_1]$. Now consider the point $x_1 = (a_1 + b_1)/2$. If $x_1$ is an upper bound for $A$, we set $I_2 = [a_1, x_1]$; otherwise, there is an element $a_2 \in A$ with $a_2 > x_1$. In this case, set $I_2 = [a_2, b_1]$. In either event, $I_2$ is a closed subinterval of $I_1$ of the form $[a_2, b_2]$, where $a_2 \in A$ and $b_2$ is an upper bound for $A$. Moreover, $\text{length}(I_2) \leq \text{length}(I_1)/2$. We now start the process all over again, using $I_2$ in the place of $I_1$, and obtain $I_3 = [a_3, b_3] \supset I_2$, where $a_3 \in A$ and $b_3$ is an upper bound for $A$, with $\text{length}(I_3) \leq \text{length}(I_2)/2 \leq \text{length}(I_1)/4$. By induction, we get a sequence of nested clsoed intervals $I_n = [a_n, b_n]$, where $a_n \in A$ and $b_n$ is an upper bound for $A$, with $\text{length}(I_n) \leq \text{length}(I_1)/2^{n-1} \to 0$ as $n \to \infty$. The single point $b \in \cap_{n=1}^{\infty} I_n$ is the least upper bound for $A$ since $b = \sup_n a_n = \inf_n b_n$. $\square$

## References

[1] Carothers, N.-L. Real analysis. Cambridge University Press, 2000.