Set Theory $\bigcap$ Functional Analysis

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1 Basic Definitions

Definition 1. *A* ***choice function*** *f on a collection $\mathcal{C}$ of set X is a function such that for all $A \in \mathcal{C}$, $f(A) \in A$.*

Example. *Consider a collection $\{\{1,2\},\{3,4\}\}$ and a function f defined as $f(\{1,2\}) = 2$ and $f(\{3,4\}) = 3$. Then f is a choice function.*

Definition 2. *Suppose I is a set, called as the* **index set***, and with each $i \in I$ we associate a set A_i. Then, $\{A_i : i \in I\}$ is defined as the* **family of sets***. This can also be denoted by $\{A_i\}_{i\in I}$*

Definition 3. *A* **partially ordered set** *is a set together with a partial order on it $(X, \preccurlyeq)$ where partial order on X is defined as a relation $\preccurlyeq$ in X such that, for all $x, y, z \in X$ it follows*

1. ***Reflexive.*** *$x \preccurlyeq x$*

2. ***Anti-symmetric.*** *If $x \preccurlyeq y$ and $y \preccurlyeq x$ then $x = y$*

3. ***Transitive.*** *If $x \preccurlyeq y$ and $y \preccurlyeq z$, then $x \preccurlyeq z$*

remark. *If $x \preccurlyeq y$ and $x \neq y$, then we write $x \prec y$ and say that x is* **smaller than** *y. It is not necessary for all $x, y \in X$ to have a partial order defined between them.*

Definition 4. *A set together with a total order on it is a* **chain** *or* **totally ordered set** *where a relation $\preccurlyeq$ is* **total order** *if for every $x, y \in X$ either $x \preccurlyeq y$ or $y \preccurlyeq x$, consequently the set is called as a totally ordered set.*

Definition 5. *Let X be a partially ordered set, then an element $a \in X$ is the* **upper bound** *of a subset $E \subseteq X$ if $x \preccurlyeq a$ for all $x \in E$.*

Definition 6. *Let X be a partially ordered set, then an element $a \in X$ is* **maximal** *if $a \preccurlyeq x$ implies $x = a$.*

Definition 7. *Let X be a partially ordered set, then an element $a \in X$ is* ***maximum (or largest)*** *if $x \preccurlyeq a \ \forall x \in X$.*

Example. *Consider the set $W = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}\}$ with set inclusion $\subseteq$ as a partial ordering. The maximal elements are $\{1,2\}$ and $\{3\}$. If we view W as a subset of the power set of $\{1,2,3\}$, then the upper bound of W is the element $\{1,2,3\}$.*

Definition 8. *A poset P is called* **well-ordered** *if it is a chain, and every non-empty subset $S \subseteq P$ has a minimum.*

Definition 9. *For a vector space X, a set $B \subseteq X$ is called a* **basis** *(or* **Hamel basis***) if B is a linearly independent set and span$(B) = X$.*

Definition 10. *Let X be a linear space. A function $p : X \to \mathbb{R}$ is a* ***Sublinear Functional*** *if the following properties hold*

1. **Subadditive.** *$p(x + y) \leq p(x) + p(y) \ \forall x, y \in X$.*

2. **Nonnegatively Homogeneous.** *$p(\lambda x) = \lambda p(x) \ \forall \lambda \geq 0$ where $\lambda \in \mathbb{R}, x \in X$*

Example. *Norm is an example of sublinear functional which is not linear.*

2 Theorem statements

Two formulations of ***Axiom of Choice*** are given.

- The Cartesian product of a non-empty family of non-empty sets is non-empty.
- For every non-empty set X, there exists a choice function f defined on X.

remark. *The equivalence proof of the above mentioned variants is skipped. In the further discussion regarding Axiom of Choice, we will be using the choice function formulation.*

Zorn's Lemma. *If X is a non-empty partially ordered set such that every chain in X has an upper bound, then X contains a maximal element.*

Well-ordering principle. *Every set has a well ordering.*

Existence of Hamel Basis. *Every vector space $X \neq \{0\}$ has a basis.*

Hahn-Banach Theorem. *Let X be a real vector space and p a sublinear functional on X. Furthermore, let f be a linear functional which is defined on a subspace Z of X and satisfies*

$$f(x) \leq p(x) \ \forall x \in Z$$

Then f has a linear extension of $\tilde{f}$ from Z to X satisfying

$$\tilde{f}(x) \leq p(x) \ \forall x \in X$$

that is, $\tilde{f}$ is a linear functional on X, satisfying above inequality on X and $\tilde{f}(x) = f(x)$ for every $x \in Z$.

3 Proof of Equivalences

Zorn's Lemma $\iff$ Axiom of Choice $\iff$ Well-Ordering Principle

Theorem 1. *Zorn's Lemma $\implies$ Axiom of Choice.*

Proof. Let X be any non-empty set. Consider a set P

$$P = \{(Y, f) : \ Y \subseteq X \text{ and f is choice function on } Y\}.$$

We define a relation $\preccurlyeq$ on P as $(Y, f) \preccurlyeq (Y', f')$ whenever $Y \subseteq Y'$ and $f = f'|_Y$. It is easy to see that $(P, \preccurlyeq)$ is a poset. Note that P is non-empty as for any element $x \in X$, $\{x\} \mapsto x$ is a choice function, consequently $(\{x\}, \{x\} \mapsto x) \in P$.

Consider a chain C in P. We define $\tilde{Y} = \bigcup_{(Y,f)\in C} Y$ and $\tilde{f}(S) = f(S)$ for any S such that f is defined on S. Notice that $(\tilde{Y}, \tilde{f})$ is an upper bound for C by construction. Since C was chosen arbitrarily and was shown to have an upper bound, then by Zorn's Lemma there is some maximal element in P, say (Y^*, f^*).

Now we show that $Y^* = X$. Suppose not then there is some element $x \in X \setminus Y^*$. We can extend f^* to f^{**} from Y^* to $Y^* \cup \{x\}$ by defining $f^{**}(S) = x$, if $x \in S$ and $f^{**}(S) = f^*(S)$, if $S \subseteq Y^*$. Then we note that $(Y^*, f^*) \preccurlyeq (Y^* \cup \{x\}, f^{**})$ which is a contradiction. Hence, $Y^* = X$ and f^* is the required choice function on X. $\square$

Theorem 2. *Zorn's Lemma $\implies$ Well-ordering principle.*

Proof. The proof is similar to that of Theorem 1. Let X be any non-empty set. Consider a relation $(P, \preccurlyeq)$, where P is defined as

$$P = \{(Y, \leq_Y) : \; Y \subseteq X \text{ and } \; \leq_Y \; \text{ is a well-ordering on } Y\},$$

and $(Y, \leq_Y) \preccurlyeq (Y', \leq_{Y'})$ whenever $Y \subseteq Y'$ and $\leq_Y$ and $\leq_{Y'}$ agree on Y. It is easy to see that $(P, \preccurlyeq)$ is a poset. Note that P is non-empty as every singleton set is well-ordered.

On similar lines of Theorem 1 proof, we conclude that there exists some maximal element in P, say $(Y^*, \leq_{Y^*})$.

Now we show that $Y^* = X$. Suppose not then there is some element $x \in X \setminus Y^*$. We can extend $(Y^*, \leq_{Y^*})$ to $Y^* \cup \{x\}$ by defining x to be greater than every element in Y^*. This is a contradiction that $(Y^*, \leq_{Y^*})$ is maximal element. Hence, $Y^* = X$ and $\leq_{Y^*}$ is the required well-ordering on X. ◻

Theorem 3. *Well-ordering principle* $\implies$ *Axiom of choice.*

Proof. Suppose X is a non-empty set, and $\leq$ is a well-ordering of X. Then $f(S) = \min S$, defines a choice function on X which is guaranteed to exist for any set S by Well-ordering principle. ◻

Theorem 4. *Axiom of Choice* $\implies$ *Zorn's Lemma.*

Proof. Let's assume there exist a non-empty partially ordered set P such that every chain in P has an upper bound, but does not contain a maximal element.

Considering axiom of choice is true, there must exist a choice function f on P, and let $x_0 := f(P)$.

Also, let the set of *strict* upper bounds on a chain C in P be

$$Upp(C) := \{u \notin C : \forall x \in C, x \prec u\}$$

Lemma 5. *For any chain C, the set $Upp(C)$ is non-empty.*

Proof. As C is a chain in P, therefore there exists an upper bound u for C. There can be two cases,

1. C does not have any maximum element, then $u \notin C$ and $u \in Upp(C)$ must be true by definitions.

2. C contains a maximum element, let's say m. Since P has no maximal element (assumed), there exist a u greater than m. Then $x \preccurlyeq m \prec u$ for each $x \in C$, and hence $u \in Upp(C)$.

Hence, in both cases $Upp(C)$ is non-empty for any chain C in P. ◻

A sub-chain C' is an initial segment of a chain C such that $x \in C, y \in C'$ and $x \prec y$ implies $c \in C'$. **Intuition:** For all $y \in C \setminus C'$, and for all $x \in C'$, $x \prec y$. Now, let's define a function g, such that for any chain C,

$$g(C) := f(Upp(C))$$

Also, let's define an ***attempt*** as a well ordered set $A \subset P$ satisfying following:

1. $\min A = x_0$

2. For every proper initial segment $C \subset A$, $\min A \setminus C = g(C)$

Lemma 6. *If A and A' are two attempts, then either $A \subseteq A'$ or $A' \subseteq A$.*

Proof. Let's assume that both $A \subseteq A'$ and $A' \subseteq A$ does not apply, and let $z = \min A \setminus A'$ and $z' = \min A' \setminus A$. As both A and A' are attempts (and hence well-ordered by definition). Since $z \neq z'$, $z \preccurlyeq z'$ and $z' \preccurlyeq z$ can not be true together. So, let's assume wlog $z' \not\preccurlyeq z$. Let's define a set $C = \{x \in A : x \prec z\}$. From the definitions of z it follows that $C \subseteq A$. It is clear from this that $z = \min A \setminus C$, and so $z = g(C)$. There are now two cases possible.

1. $C = A'$. Now, as $C \subseteq A$ therefore $A' \subseteq A$. Hence, the given lemma is true in this case.
2. $C \neq A'$. If $z' \preccurlyeq x$ for some $x \in C$, then transitivity of partial order implies $z' \prec z$, which is a contradiction. So, since A' is a chain (as it is well-ordered), $x \preceq z'$ $\forall x \in C$. therefore C is a proper initial segment of A' which implies $g(C) \in A'$. But, $g(C) = z \notin A'$. Therefore a contradiction. Hence,the given lemma is true.

$\square$

As, for any two attempts A, A' either $A \subseteq A'$ or $A' \subseteq A$, therefore $A \cup A'$ is either A or A' which is an attempt. Let $\mathcal{A}$ be the set of all attempts then $A := \bigcup_{\tilde{A} \in \mathcal{A}} \tilde{A}$. Then A is also an attempt.

However, $A \cup \{g(A)\}$ is also an attempt and must have belonged in the previous set of attempts $\mathcal{A}$, and also $A \subseteq A \cup \{g(A)\}$ therefore $A \cup \{g(A)\} := \bigcup_{A \in \mathcal{A}} A$ but this is not the case, therefore a contradiction. And, hence there must exist a maximal element of P. $\square$

Theorem 7. *Zorn's Lemma $\implies$ "Every vector space $X \neq \{0\}$ has a basis".*

Proof. Let X be a non-empty vector space. We define a relation $(P, \preccurlyeq)$ where P is the set of subsets of X which are linearly independent and for every $B, B' \in P$, $B \preccurlyeq B'$ whenever $B \subseteq B'$. It is easy to note that $(P, \subseteq)$ is a poset. Notice that $P \neq \emptyset$ as $X \neq \{0\}$, there is some non-zero element $x \in X$, consequently $B = \{x\} \in P$.

Consider a chain C in P. Define $\tilde{B} = \bigcup_{B \in C} B$. Notice that $\tilde{B}$ is an upper bound for C, hence by Zorn's Lemma there exists a maximal element in P, say B^*.

We show that $\operatorname{span}(B^*) = X$. Suppose not, then there is an element $x \in X \setminus \operatorname{span}(B^*)$ and $x \neq 0$. Then extend the set B^* by including x in it. Notice that the extended set is an element in P which is greater than B^* under the subset relation. This is a contradiction. Hence $\operatorname{span}(B^*) = X$ and B^* is a linearly independent set, thus B^* is a basis for X. $\square$

remark. *A variant of converse of the above result is also true which is*

"Every vector space $X \neq \{0\}$ has a basis" $\implies$ Axiom of Choice,

thus establishing equivalence between the two. The proof can be found in **[1]** *which shows the implication for* ***Axiom of Multiple Choice*** *instead. It is known that Axiom of Multiple Choice is equivalent to Axiom of Choice. We skip this proof as it is quite involved.*

Theorem 8. *Zorn's Lemma $\implies$ Hahn-Banach Theorem*

Proof. Let's proof this in 3 parts,

(A) Let's define M as the partial order set of pairs (Z, f_Z) where

 (a) Z is a subspace of X containing Y.

 (b) $f_Z : Z \to \mathbb{R}$ is a linear functional extending f, satisfying

$$f_Z(z) \leq p(z) \ \forall \ z \in Z$$

with partial ordering defined as $(Z_1, f_{Z_1}) \preccurlyeq (Z_2, f_{Z_2})$ if $Z_1 \subset Z_2$ and $(f_{Z_2})|_{Z_1} = f_{Z_1}$. Since, $(Y, f) \in M$, M is a non-empty set. Let's choose any arbitrary chain $C = \{(Z_\alpha, f_{Z_\alpha})\}_{\alpha\in\Lambda}$ in M, with Λ being some indexing set.

Lemma 9. *C has an upper bound in M.*

Proof. Let $W = \bigcup_{\alpha\in\Lambda} Z_\alpha$ and construct a functional $f_W : W \implies \mathbb{R}$ defined as follow: If $w \in W$, then $w \in Z_\alpha$ for some $\alpha \in \Lambda$ and we set $f_W(w) = f_{Z_\alpha}(w)$ for that particular α.

- This definition is well-defined. Indeed, suppose $w \in Z_\alpha$ and $w \in Z_\beta$. If $Z_\alpha \subset Z_\beta$, then $f_{Z_\beta}|_{Z_\alpha} = f_{Z_\alpha}$, since they are a part of chain.
- W clearly contains Y, and we show that W is a subspace of X and f_W is a linear functional on W. Choose any $w_1, w_2 \in W$, then $w_1 \in Z_{\alpha_1}, w_2 \in Z_{\alpha_2}$ for some $\alpha_1, \alpha_2 \in \lambda$. If $Z_{\alpha_1} \subset Z_{\alpha_2}$, say, then for any scalars $\beta, \gamma \in \mathbb{R}$ we have

$$w_1, w_2 \in Z_{\alpha_2} \implies \beta w_1 + \gamma w_2 \in Z_{\alpha_2} \subset W$$

  Also, with $f_W(u) = f_{Z_{\alpha_1}}(u)$ and $f_W(v) = f_{Z_{\alpha_2}}(v)$,

$$\begin{aligned} f_w(\beta u + \gamma v) &= f_{Z_{\alpha_2}}(\beta u + \gamma v) \\ &= \beta f_{Z_{\alpha_2}}(u) + \gamma f_{Z_{\alpha_2}}(v) \text{ linearity} \\ &= \beta f_{Z_{\alpha_1}}(u) + \gamma f_{Z_{\alpha_2}}(v) \text{ because in same chain} \\ &= \beta f_{Z_W}(u) + \gamma f_{Z_W}(v) \end{aligned}$$

  The case $Z_{\alpha_2} \subset Z_{\alpha_1}$ follows from a symmetric argument.
- Choose any $w \in W$,then $w \in Z_\alpha$ for some $\alpha \in \Lambda$ and

$$f_W(w) = f_{Z_\alpha}(w) \le p(w) \text{ since } (w, Z_\alpha) \in M$$

Hence, (W, f_W) is an element of M and an upper bound of C since $(Z_\alpha, f_{Z_\alpha}) \le (W, f_W)$ for all $\alpha in \Lambda$. Since C was an arbitrary chain in M, by Zorn's lemma, M has a maximal element $(Z, f_Z) \in M$, and f_Z is (by definition) a linear extension of f satisfying $f_Z(z) \le p(z)$ for all $z \in Z$. □

(B) The proof is complete if we can show that $Z = X$. Suppose not, then there exists an $\theta \in X\backslash Z$; note $\theta \neq 0$ since Z is a subspace of X. Consider the subspace $Z_\theta = \text{span}\{Z, \{\theta\}\}$. Any $x \in Z_\theta$ has a unique representation $x = z + \alpha\theta$, $z \in Z$, $\alpha \in \mathbb{R}$. Indeed, if

$$x = z_1 + \alpha_1\theta = z_2 + \alpha_2\theta, z_1, z_2 \in Z, \alpha_1, \alpha_2 \in \mathbb{R}$$

then $z_1 - z_2 = (\alpha_2 - \alpha_1)\theta \in Z$ since Z is a subspace of X. Since $\theta \notin Z$, we must have $\alpha_2 - \alpha_1 = 0$ and $z_1 - z_2 = \theta$. Next, we construct a functional $f_{Z_\theta} : Z_\theta \to \mathbb{R}$ defined by

$$f_{Z_\theta}(x) = f_{Z_\theta}(z + \alpha\theta) = f_Z(z) + \alpha\delta, \text{ ... } (1)$$

where δ is any real number. It can be shown that f_{Z_θ} is linear and f_{Z_θ} is a proper linear extension of f_Z ; indeed, we have, for $\alpha = 0, f_{Z_\theta}(x) = f_{Z_\theta}(z) = f_Z(x)$. Consequently, if we can show that

$$f_{Z_\theta}(x) \le p(x) \ \forall x \in Z_\theta \text{ ... } (2)$$

then $(Z_\theta, f_{Z_\theta}) \in M$ satisfying $(Z, f_Z) \le (Z_\theta, f_{Z_\theta})$, thus contradicting the maximality of (Z, f_Z).

(C) From (1), observe that (2) is trivial if $\alpha = 0$, so suppose $\alpha \neq 0$. We do have a single degree of freedom, which is the parameter δ in (1), thus the problem reduces to showing the existence of a suitable $\delta \in \mathbb{R}$ such that (2) holds. Consider any $x = z + \alpha\theta \in Z_\theta, z \in Z, \alpha \in \mathbb{R}$. Assuming $\alpha > 0$,(2) is equivalent to

$$\begin{aligned} f_Z(z) + \alpha\delta &\leq p(z + \alpha\theta) = \alpha p(z/\alpha + \theta) \\ f_Z(z/\alpha) + \delta &\leq p(z/\alpha + \theta) \\ \delta &\leq p(z/\alpha + \theta) - f_Z(z/\alpha) \end{aligned}$$

Since the above must holds for all $z \in Z, \alpha \in \mathbb{R}$, we need to choose δ such that

$$\delta \leq \inf_{z_1 \in Z} (p(z_1 + \theta) - f_Z(z_1)) = m_1 \text{ ... } (3)$$

Assuming $\alpha < 0$, (2) is equivalent to

$$\begin{aligned} f_Z(z) + \alpha\delta &\leq p(z + \alpha\theta) = -\alpha p(-z/\alpha - \theta) \\ -f_Z(z/\alpha) - \delta &\leq p(-z/\alpha - \theta) \\ \delta &\geq -p(-z/\alpha - \theta) - f_Z(z/\alpha) \end{aligned}$$

Since the above must holds for all $z \in Z, \alpha \in \mathbb{R}$, we need to choose δ such that

$$\delta \geq \sup_{z_2 \in Z} (-p(z_2 + \theta) - f_Z(z_2)) = m_0 \text{ ... } (4)$$

We are left with showing condition (3), (4) are compatible, i.e

$$-p(-z_2 - \theta) - f_Z(z_2) \leq p(z_1 + \theta) - f_Z(z_1) \ \forall z_1, z_2 \in Z$$

The inequality above is trivial if $z_1 = z_2$, so suppose not. We have that

$$\begin{aligned} p(z_1 + \theta) - f_Z(z_1) + p(-z_2 - \theta) + f_Z(z_2) &= p(z_1 + \theta) + p(-z_2 - \theta) + f_Z(z_2 - z_1) \\ &\geq f_Z(z_2 - z_1) + p(z_1 + \theta - z_2 - \theta) \\ &= f_Z(z_2 - z_1) + p(z_1 - z_2) \\ &= -f_Z(z_1 - z_2) + p(z_1 - z_2) \geq 0 \end{aligned}$$

where linearity of f_Z and subadditivity of p are used. Hence, the required condition on δ is $m_0 \leq \delta \leq m_1$

Therefore, by using Zorn's Lemma (as using in 1st part) we proved Hahn-Banach Theorem. $\square$

remark. *The converse of the above theorem is not true, i.e, Hahn-Banach Theorem $\not\Rightarrow$ Zorn's Lemma. It is known that Hahn-Banach Theorem is equivalent to a statement which is strictly weaker than Axiom of Choice.[3]*

References

[1] Existence of Basis implies AC

[2] Axiom of Choice equivalents

[3] Wikipedia, Hahn-Banach Theorem

[4] P. Halmos, Naive set theory. New York, 1974.

[5] Hahn-Banach Theorem