Real Analysis

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1 The Real Numbers **1**

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$\mathbb{R}$	Set of all real numbers
$\mathbb{C}$	Set of all complex numbers
$\mathbb{N}$	Set of all natural numbers
$\mathbb{Q}$	Set of all rational numbers
$\mathbb{R} - \mathbb{Q}$	Set of all irrational numbers

Table 1: Standard notation for various sets.

Definition 1.1. A subset A of $\mathbb{R}$ is said to be ***bounded above*** if there is some $x \in \mathbb{R}$ such that $a \leq x$ for all $a \in A$. Any such number x is called an ***upper bound*** for A.

Axiom 1.2 (The Least Upper Bound Axiom or The Completeness Axiom)**.** *Any nonempty set of real numbers with an upper bound has a least upper bound.*

Definition 1.3. Let $A \subseteq \mathbb{R}$ be a nonempty set that is bounded above. Then the ***supremum*** of the set, denoted by $\sup A$, is the least upper bound, i.e., $\sup A = s \in \mathbb{R}$ such that

(i) s is an upper bound for A;

(ii) if x is any upper bound for A, then $s \leq x$.

Remark 1.4. If $A \subseteq \mathbb{R}$ is unbounded above, then $\sup A = +\infty$ and if $A = \emptyset$, then $\sup A = -\infty$ as every real number is an upper bound for A.

Proposition 1.5 (Characterization of the Supremum)**.** *Let $A \subseteq \mathbb{R}$ be a nonempty set that is bounded above. Then the following statements about $s = \sup A$ are all equivalent:*

(a) if x is any upper bound for A, then $s \leq x$;

(b) if $y < s$, then we must have $y < a \leq s$ for some $a \in A$;

(c) for every $\varepsilon > 0$, there is an $a \in A$ such that $a > s - \varepsilon$.

Proof. (a) $\implies$ (b) Suppose for some $y < s$, there is no $a \in A$ such that $y < a \leq s$. Then for all $a \in A$, we have $a \leq y < s$. But then y is an upper bound with $y < s$ which contradicts (a).
(b) $\implies$ (c) Consider an arbitrary $y \in \mathbb{R}$ such that $y < s$. Setting $\varepsilon = s - y$ and using (b) we have that there is some $a \in A$ such that $s - \varepsilon = y < a$.
(c) $\implies$ (a) Let x be an upper bound for A, i.e., $a \leq x$ for all $a \in A$ and suppose that $x < s$. By taking $\varepsilon = s - x$ and applying (c), there must be some $a \in A$, such that $s - \varepsilon = x < a$ which is a contradiction that x is an upper bound for A. Then $s \leq x$. $\square$

Theorem 1.6. *If A is a nonmepty subset of $\mathbb{R}$ that is bounded below, then A has a greatest lower bound called the* ***infimum*** *of A which is denoted by $\inf A$, i.e., there is a number $m \in \mathbb{R}$ satisfying:*

(i) m is a lower bound for A;

(ii) if x is a lower bound for A, then $x \leq m$.

Proof. Conisder the set $-A = \{-a \ : \ a \in A\}$ which is bounded above as A is bounded below. By the completeness axiom, $-A$ must have a least upper bound $m = \sup(-A)$. Note that $-m = -\sup(-A)$ then the greatest lower bound for A, so that $\inf A = -\sup(-A)$. □

Remark 1.7. As we have established that $\inf A = -\sup(-A)$, we have $\inf A = -\infty$ if A is unbounded below, and $\inf \emptyset = +\infty$. In case a set A is both bounded above and bounded below, we simply say that A is ***bounded***.

Proposition 1.8 (Characterization of the Infimum)**.** *Let $A \subseteq \mathbb{R}$ be a nonempty set that is bounded below. Then the following statements about $m = \inf A$ are all equivalent:*

(a) if x is any lower bound for A, then $x \leq m$;

(b) if $y > m$, then we must have $m \leq a < y$ for some $a \in A$;

(c) for every $\varepsilon > 0$, there is an $a \in A$ such that $a < m + \varepsilon$.

Proof similar to that of Prop. 1.5.

Proposition 1.9. *Let A be a bounded subset of $\mathbb{R}$ containing at least two points. Then*

(a) $-\infty < \inf A < \sup A < +\infty$.

(b) If B is a nonempty subset of A, then $\inf A \leq \inf B \leq \sup B \leq \sup A$.

(c) If B is the set of all upper bounds for A, then B is nonmepty, bounded below and $\inf B = \sup A$.

Proof. (a) The boundedness of A implies that $-\infty < \inf A \leq \sup A < +\infty$. Since there are at least two points in A, $\inf A \neq \sup A$.
(b) and (c) trivally hold from the definitions of infimum and supremum. □

Definition 1.10. A sequence (x_n) of real numbers is said to ***converge*** to $x \in \mathbb{R}$ if, for every $\varepsilon > 0$, there is a positive integer N such that

$$n \geq N \implies |x_n - x| \leq \varepsilon.$$

In this case, we call x the limit of the sequence (x_n) and write $x = \lim_{n\to\infty} x_n$.

Proposition 1.11. *Let A be a nonempty subset of $\mathbb{R}$ that is bounded above. Then there is a sequence (x_n) of elements of A that converges to $\sup A$.*

Proof. Using Prop. 1.5(c), we set $\varepsilon_n = 1/n$ and $y_n = \sup A - \varepsilon_n$. Then for all $n \in N$, there exists an element $x_n \in A$ such that $x_n > \sup A - \varepsilon = y_n$. But then $|x_n - \sup A| < |s - y_n| = \varepsilon_n = 1/n$. This shows that $\lim_{n\to\infty} |x_n - \sup A| = 0$, i.e., $\lim_{n\to\infty} x_n = \sup A$. □

Lemma 1.12 (Archimedean property in $\mathbb{R}$)**.** *If x and y are positive real numbers, then there is some positive integer n such that $nx > y$.*

Proof. Suppose that no such n existed, i.e., suppose that $nx \leq y$ for all $n \in \mathbb{N}$. Then $A = \{nx : n \in \mathbb{N}\}$ is bounded above by y, and so $s = \sup A$ is finite. Now, since $s - x < s$, we must have some element of A in between, i.e., $s - x < nx \leq s$ for some $n \in \mathbb{N}$. But then $s < (n+1)x \in A$ which is a contradiction, hence there is some $n \in \mathbb{N}$ such that $nx > y$. □

Theorem 1.13. *If a and b are real numbers with $a < b$, then there is a rational number $r \in \mathbb{Q}$ with $a < r < b$.*

Proof. We set $x = b - a > 0$, $y = 1$, and apply Lemma 1.12 to get a positive integer q such that $q(b-a) > 1$, i.e., $qb > qa + 1$. But if qa and qb differ by more than 1, there must be some integer in between, i.e., there is some $p \in \mathbb{Z}$ with $qa < p < qb$. Thus, $a < \dfrac{p}{q} < b$. □

Corollary 1.13.1. *If a and b are real numbers with $a < b$, then there is also an irrational number $x \in \mathbb{R} - \mathbb{Q}$ with $a < x < b$.*

Proof. We set $x = \dfrac{1}{\sqrt{2}}(b-a) > 0$, $y = 1$, and apply Lemma 1.12 to get a positive integer q such that $\dfrac{1}{\sqrt{2}}qb > \dfrac{1}{\sqrt{2}}qa + 1$. But then there is some $p \in \mathbb{Z}$ with $\dfrac{1}{\sqrt{2}}qa < p < \dfrac{1}{\sqrt{2}}qb$. Thus, $a < \dfrac{\sqrt{2}p}{q} < b$. □

Corollary 1.13.2. *Given $a < b$, there are, in fact, infinitely many distinct rationals between a and b. The same goes for irrationals, too.*

Proof. □

Remark 1.14. The least upper bound axiom holds in $\mathbb{Z}$ since for any nonempty set that is bounded above, there exists a least upper bound which is the maximum value of the set itself. But this axiom does not hold in $\mathbb{Q}$. Consider the set $A = \{p/q \in \mathbb{Q} : p^2 < 2q^2\}$. This is bounded above in $\mathbb{Q}$ as 2 is an upper bound. But $\sup A = \sqrt{2} \notin \mathbb{Q}$ so A has not least upper bound in $\mathbb{Q}$.

Proposition 1.15. *The following statements are all equivalent:*

(a) (The least upper bound property). Any nonempty set of real numbers with an upper bound has a least upper bound.

(b) A monotone, bounded sequence of real numbers converges.

(c) (The nested interval propoerty). If (I_n) is a sequence of closed, bounded, nonempty intervals in $\mathbb{R}$ with $I_1 \supset I_2 \supset \cdots$, then $\cap_{n=1}^{\infty} I_n \neq \emptyset$. If, in addition, length$(I_n) \to 0$, then $\cap_{n=1}^{\infty} I_n$ contains precisely one point.

Proof. (a) $\implies$ (b) Let $(x_n) \subset \mathbb{R}$ be monotone and bounded. We first suppose that (x_n) is increasing. Now, since (x_n) is bounded, we may set $x = \sup_n x_n \in \mathbb{R}$. Suppose $\varepsilon > 0$, then from (a) we must have $x_N > x - \varepsilon$ for some N. But then, for any $n \geq N$, we have $x - \varepsilon < x_N \leq x_n \leq x$, i.e., $|x - x_n| < \varepsilon$ for all $n \geq N$. Consequently, (x_n) converges and $x = \sup_n x_n = \lim_{n\to\infty} x_n$. Finally, if (x_n) is decreasing, consider the increasing sequence $(-x_n)$. From the previous arguments, $(-x_n)$ converges to $\sup_n(-x_n) = -\inf_n x_n$. It then follows that (x_n) converges to $\inf_n x_n$.
(b) $\implies$ (c) Let $I_n = [a_n, b_n]$. Then $I_n \supset I_{n+1}$ means that $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$ for all n. Then from (b), we have $a = \lim_{n\to\infty} a_n = \sup_n a_n$ and $b = \lim_{n\to\infty} b_n = \inf_n b_n$ both exist and satisfy $a \leq b$. Thus we must have $\cap_{n=1}^{\infty} I_n = [a, b]$. Finally, if $b_n - a_n = \text{length}(I_n) \to 0$, then $a = b$ and so $\cap_{n=1}^{\infty} I_n = \{a\}$.

(c) $\Longrightarrow$ (a) Let A be a nonempty subset of $\mathbb{R}$ that is bounded above. Specifically, let $a_1 \in A$ and let b_1 be an upper bound for A. For later reference, set $I_1 = [a_1, b_1]$. Now consider the point $x_1 = (a_1 + b_1)/2$. If x_1 is an upper bound for A, we set $I_2 = [a_1, x_1]$; otherwise, there is an element $a_2 \in A$ with $a_2 > x_1$. In this case, set $I_2 = [a_2, b_1]$. In either event, I_2 is a closed subinterval of I_1 of the form $[a_2, b_2]$, where $a_2 \in A$ and b_2 is an upper bound for A. Moreover, $\text{length}(I_2) \leq \text{length}(I_1)/2$. We now start the process all over again, using I_2 in the place of I_1, and obtain $I_3 = [a_3, b_3] \supset I_2$, where $a_3 \in A$ and b_3 is an upper bound for A, with $\text{length}(I_3) \leq \text{length}(I_2)/2 \leq \text{length}(I_1)/4$. By induction, we get a sequence of nested clsoed intervals $I_n = [a_n, b_n]$, where $a_n \in A$ and b_n is an upper bound for A, with $\text{length}(I_n) \leq \text{length}(I_1)/2^{n-1} \to 0$ as $n \to \infty$. The single point $b \in \cap_{n=1}^{\infty} I_n$ is the least upper bound for A since $b = \sup_n a_n = \inf_n b_n$. $\square$

References

[1] Carothers, N.-L. Real analysis. Cambridge University Press, 2000.