

Differential Geometry

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1 Smooth Manifolds

Definition 1. A topological space \mathcal{M} is said to be **locally Euclidean of dimension n** if every point of \mathcal{M} has a neighborhood in \mathcal{M} that is homeomorphic to an open subset of \mathbb{R}^n .

Lemma 1. A topological space \mathcal{M} is locally Euclidean of dimension n if and only if either of the following properties holds:

- (a) Every point of \mathcal{M} has a neighborhood homeomorphic to an open ball in \mathbb{R}^n .
- (b) Every point of \mathcal{M} has a neighborhood homeomorphic to \mathbb{R}^n .

Proof Sketch. For (a) one need to show that any open subset of \mathbb{R}^n is homeomorphic to the unit open ball in \mathbb{R}^n . Note that the unit open ball is homeomorphic to any open ball in \mathbb{R}^n . For (b) it suffices to show that unit open ball in \mathbb{R}^n is homeomorphic to \mathbb{R}^n .

Definition 2. Suppose \mathcal{M} is a topological space. \mathcal{M} is a **topological manifold of dimension n** or a **topological n -manifold** if it has the following properties:

- (a) \mathcal{M} is a **Hausdorff space**.
- (b) \mathcal{M} is a **second-countable**.
- (c) \mathcal{M} is **locally Euclidean of dimension n** .

A **coordinate chart** (or just a **chart**) on \mathcal{M} is a pair (U, φ) , where U is an open subset of \mathcal{M} and $\varphi : U \rightarrow \hat{U}$ is a homeomorphism from U to an open subset $\hat{U} = \varphi(U) \subseteq \mathbb{R}^n$. The set U is called a **coordinate domain** or a **coordinate neighborhood** of each of its points. The map φ is called a **(local) coordinate map**, and the component functions (x^1, \dots, x^n) of φ , defined by $\varphi(p) = (x^1(p), \dots, x^n(p))$, are called **local coordinates** on U .

Proposition 1. A nonempty n -dimensional topological manifold cannot be homeomorphic to an m -dimensional manifold unless $m = n$.

Example. Here are some examples of topological manifolds.

- (i) Open subset of a topological n -manifold.
- (ii) Graphs of Continuous Functions.
- (iii) Spheres.
- (iv) Projective Spaces.
- (v) Product Manifolds.

Definition 3. Let \mathcal{M} be a topological n -manifold. If $(U, \varphi), (V, \psi)$ are two charts such that $U \cap V \neq \emptyset$, the composite map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \rightarrow \psi(U \cap V)$ is called **transition map from φ to ψ** . Two charts $(U, \varphi), (V, \psi)$ are said to be **smoothly compatible** if either $U \cap V = \emptyset$ or the transition map $\psi \circ \varphi^{-1}$ is a (\mathcal{C}^∞) diffeomorphism.

Remark 1. In the above definition, since $\psi(U \cap V)$ and $\varphi(U \cap V)$ are open subsets of \mathbb{R}^n , smoothness of the transition map $\psi \circ \varphi^{-1}$ can be interpreted in the ordinary sense of having continuous partial derivatives of all orders.

Definition 4. We define an **atlas for \mathcal{M}** to be a collection of charts whose domains cover \mathcal{M} . An atlas \mathcal{A} is called a **smooth atlas** if any two charts in \mathcal{A} are smoothly compatible with each other.

Remark 2. To show that an atlas is smooth, we need only verify that each transition map $\psi \circ \varphi^{-1}$ is smooth whenever $(U, \varphi), (V, \psi)$ are charts in \mathcal{A} such that $U \cap V \neq \emptyset$; once we have proved this, it follows that $\psi \circ \varphi^{-1}$ is a diffeomorphism because its inverse $(\psi \circ \varphi^{-1})^{-1} = \varphi \circ \psi^{-1}$ is one of the transition maps we have already shown to be smooth. Alternatively, given two particular charts $(U, \varphi), (V, \psi)$, it is often easiest to show that they are smoothly compatible by verifying that $\psi \circ \varphi^{-1}$ is smooth and injective with nonsingular Jacobian at each point, and appealing to a variant inverse function theorem([1, Corollary C.36]).

Definition 5. Let \mathcal{M} be a topological manifold. A smooth atlas \mathcal{A} on \mathcal{M} is **maximal** if it is not properly contained in any larger smooth atlas.

Remark 3. If \mathcal{A} is a maximal smooth atlas on \mathcal{M} , then any chart that is smoothly compatible with every chart in \mathcal{A} is already in \mathcal{A} .

Definition 6. Let \mathcal{M} be a topological manifold. A **smooth structure on \mathcal{M}** is a maximal smooth atlas. A **smooth manifold** is a pair $(\mathcal{M}, \mathcal{A})$ where \mathcal{M} is a topological manifold and \mathcal{A} is a smooth structure on \mathcal{M} . Any chart (U, φ) in \mathcal{A} is called a **smooth chart** and the corresponding coordinate map φ and the domain U of φ are called **smooth coordinate map** and **smooth coordinate domain** or **smooth coordinate neighborhood** respectively.

Theorem 2. Let \mathcal{M} be a topological manifold.

- (a) Every smooth atlas \mathcal{A} on \mathcal{M} is contained in a unique maximal smooth atlas, called the **smooth structure determined by \mathcal{A}** .
- (b) Two smooth atlases for \mathcal{M} determine the same smooth structure iff their union is a smooth atlas.

Example. Here are some examples of smooth manifolds.

- (i) Euclidean Spaces.
- (ii) Finite-Dimensional Vector Spaces.
- (iii) Space of Matrices.
- (iv) Open Submanifolds.
- (v) The General Linear Group.
- (vi) Matrices of Full Rank.
- (vii) Spaces of Linear Maps.
- (viii) Graphs of Continuous Functions.
- (ix) Spheres.
- (x) Level Sets.
- (xi) Projective Spaces.
- (xii) Smooth Product Manifolds.
- (xiii) Grassmann Manifolds.

Solve the exercise questions 1-1 to 1-10 from [1, Ch 1].

2 Smooth Maps

Note 1. For the sake of convenience, we reserve the word **function** for a map whose codomain is \mathbb{R} (a **real-valued function**) or \mathbb{R}^k for some $k > 1$ (a **vector-valued function**). Either of the words **map** or **mapping** can mean any type of map, such as a map between arbitrary manifolds.

Definition 7. Suppose \mathcal{M} is a smooth n -manifold, k is a nonnegative integer, and $f : \mathcal{M} \rightarrow \mathbb{R}^k$ is any function. We say that f is a **smooth function** if for every $p \in \mathcal{M}$, there exists a smooth chart (U, φ) for \mathcal{M} whose domain contains p and such that the composite function $f \circ \varphi^{-1}$ is smooth on the open subset $\hat{U} = \varphi(U) \subseteq \mathbb{R}^n$.

Note 2. The most important special case is that of smooth real-valued functions $f : \mathcal{M} \rightarrow \mathbb{R}$; the set of all such functions is denoted by $C^\infty(\mathcal{M})$. Because sums and constant multiples of smooth functions are smooth, $C^\infty(\mathcal{M})$ is a vector space over \mathbb{R} .

Proposition 2. Let \mathcal{M} be a smooth manifold, and suppose $f : \mathcal{M} \rightarrow \mathbb{R}^k$ is a smooth function. Then $f \circ \varphi^{-1} : \varphi(U) \rightarrow \mathbb{R}^k$ is smooth for every smooth chart (U, φ) for \mathcal{M} .

Definition 8. Given a function $f : \mathcal{M} \rightarrow \mathbb{R}^k$, and a chart (U, φ) for \mathcal{M} , the function $\hat{f} : \varphi(U) \rightarrow \mathbb{R}^k$ defined by $\hat{f}(x) = f \circ \varphi^{-1}(x)$ is called the **coordinate representation of f** .

Remark 4. By Def.7, f is smooth iff its coordinate representation is smooth in some smooth chart around each point. By Prop.2, smooth functions have smooth coordinate representations in every smooth chart.

Proposition 3. Let U be an open submanifold of \mathbb{R}^n with its standard smooth manifold structure. Then a function $f : U \rightarrow \mathbb{R}^k$ is smooth in the sense of Def.7 iff it is smooth in the sense of ordinary calculus.

Definition 9. Let \mathcal{M}, \mathcal{N} be smooth manifolds, and let $F : \mathcal{M} \rightarrow \mathcal{N}$ be any map. We say that F is a **smooth map** if for every $p \in \mathcal{M}$, there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subseteq V$ and the composite map $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U)$ to $\psi(V)$.

Note 3. Def.7 can be viewed as a special case of Def.9 by taking $\mathcal{N} = V = \mathbb{R}^k$ and $\psi = \text{Id} : \mathbb{R}^k \rightarrow \mathbb{R}^k$.

Proposition 4. Every smooth map is continuous.

Proposition 5 (Equivalent Characterizations of Smoothness). Suppose \mathcal{M} and \mathcal{N} are smooth manifolds, and $F : \mathcal{M} \rightarrow \mathcal{N}$ is a map. Then F is smooth iff either of the following conditions is satisfied:

- (a) For every $p \in \mathcal{M}$, there exists smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $U \cap F^{-1}(V)$ is open in \mathcal{M} and the composite map $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U \cap F^{-1}(V))$ to $\psi(V)$.
- (b) F is continuous and there exist smooth atlases $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ for \mathcal{M} and \mathcal{N} , respectively, such that for each α and β , $\psi_\beta \circ F \circ \varphi_\alpha^{-1}$ is smooth from $\varphi_\alpha(U_\alpha \cap F^{-1}(V_\beta))$ to $\psi_\beta(V_\beta)$.

Proposition 6 (Smoothness is Local). Let \mathcal{M}, \mathcal{N} be smooth manifolds, and let $F : \mathcal{M} \rightarrow \mathcal{N}$ be a map.

- (a) If every point $p \in \mathcal{M}$ has a neighborhood U such that the restriction $F|_U$ is smooth, then F is smooth.

(b) Conversely, if F is smooth, then its restriction to every open subset is smooth.

Proposition 7 (Gluing Lemma for Smooth Maps). Let \mathcal{M}, \mathcal{N} be smooth manifolds, and let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of \mathcal{M} . Suppose that for each $\alpha \in A$, we are given a smooth map $F_\alpha : U_\alpha \rightarrow \mathcal{N}$ such that the maps agree on overlaps: $F_\alpha|_{U_\alpha \cap U_\beta} = F_\beta|_{U_\alpha \cap U_\beta}$ for all α and β . Then there exists a unique smooth map $F : \mathcal{M} \rightarrow \mathcal{N}$ such that $F|_{U_\alpha} = F_\alpha$ for each $\alpha \in A$.

Definition 10. Given a map $F : \mathcal{M} \rightarrow \mathcal{N}$, and smooth charts (U, φ) and (V, ψ) for \mathcal{M} and \mathcal{N} , respectively, the function $\hat{F} : \varphi(U \cap F^{-1}(V)) \rightarrow \psi(V)$ defined by $\hat{F}(x) = \psi \circ F \circ \varphi^{-1}(x)$ is called the **coordinate representation of F** .

Proposition 8. Suppose $F : \mathcal{M} \rightarrow \mathcal{N}$ is a smooth map between smooth manifolds. Then the coordinate representation of F with respect to every pair of smooth charts for \mathcal{M} and \mathcal{N} is smooth.

Proposition 9. Let \mathcal{M}, \mathcal{N} , and \mathcal{P} be smooth manifolds.

- (a) Every constant map $c : \mathcal{M} \rightarrow \mathcal{N}$ is smooth.
- (b) The identity map of \mathcal{M} is smooth.
- (c) If $U \subseteq \mathcal{M}$ is an open submanifold, then the inclusion map $U \hookrightarrow \mathcal{M}$ is smooth.
- (d) If $F : \mathcal{M} \rightarrow \mathcal{N}$ and $G : \mathcal{N} \rightarrow \mathcal{P}$ are smooth, then so is $G \circ F : \mathcal{M} \rightarrow \mathcal{P}$.

Proposition 10. Suppose $\mathcal{M}_1, \dots, \mathcal{M}_k$ and \mathcal{N} are smooth manifolds. For each i , let $\pi_i : \mathcal{M}_1 \times \dots \times \mathcal{M}_k \rightarrow \mathcal{M}_i$ denote the projection onto the \mathcal{M}_i factor. A map $F : \mathcal{N} \rightarrow \mathcal{M}_1 \times \dots \times \mathcal{M}_k$ is smooth iff each of the component maps $F_i = \pi_i \circ F : \mathcal{N} \rightarrow \mathcal{M}_i$ is smooth.

3 Partitions of Unity

4 Tangent Vectors

Definition 11. Given a point $x \in \mathbb{R}^n$, the **geometric tangent space** to \mathbb{R}^n at x , denoted by \mathbb{R}_x^n , is the set

$$\mathbb{R}_x^n = \{x\} \times \mathbb{R}^n = \{(x, v) : v \in \mathbb{R}^n\}.$$

A **geometric tangent vector** in \mathbb{R}^n is an element of \mathbb{R}_x^n for some $x \in \mathbb{R}^n$. As a matter of notation, we abbreviate (x, v) as v_x or $v|_x$. We think of v_x as the vector v with its initial point at x .

Note 4. The set \mathbb{R}_x^n is a real vector space under the natural operations

$$v_x + w_x = (v + w)_x, \quad c(v_x) = (cv)_x.$$

Consequently, the vectors $e_i|_x, i = 1, \dots, n$, are a basis for \mathbb{R}_x^n .

Definition 12. If x is a point of \mathbb{R}^n , a map $w : \mathcal{C}^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}$ is called a **derivation at x** if it is linear over \mathbb{R} and satisfies the following product rule:

$$w(fg) = f(x)wg + g(x)wf.$$

Let $T_x\mathbb{R}^n$ denote the set of all derivation of $\mathcal{C}^\infty(\mathbb{R}^n)$ at x .

Note 5. Clearly, $T_x\mathbb{R}^n$ is a vector space under the operations

$$(w_1 + w_2)f = w_1f + w_2f, \quad (cw)f = c(wf).$$

Note 6. For any geometric tangent vector $v_x \in \mathbb{R}_x^n$ we define a derivation to be a map which takes the directional derivative of any $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ in the direction v at x :

$$D_v|_x f = Df(x)[v] = \left. \frac{d}{dt} \right|_{t=0} f(x + tv) = \lim_{t \rightarrow 0} \frac{f(x + tv) - f(x)}{t}.$$

It is indeed true that it is linear over \mathbb{R} since for any $f, g \in \mathcal{C}^\infty(\mathbb{R}^n)$ and $\alpha, \beta \in \mathbb{R}$, we have

$$\begin{aligned} D_v|_x (\alpha f + \beta g) &= D(\alpha f + \beta g)(x)[v] = \lim_{t \rightarrow 0} \frac{\alpha f(x + tv) + \beta g(x + tv) - \alpha f(x) - \beta g(x)}{t} \\ &= \alpha \lim_{t \rightarrow 0} \frac{f(x + tv) - f(x)}{t} + \beta \lim_{t \rightarrow 0} \frac{g(x + tv) - g(x)}{t} \\ &= \alpha Df(x)[v] + \beta Dg(x)[v] = \alpha D_v|_x f + \beta D_v|_x g. \end{aligned}$$

One can also note that this map satisfies the product rule (or chain rule):

$$D_v|_x (fg) = f(x) D_v|_x g + g(x) D_v|_x f.$$

If $v_a = \sum_{i=1}^n v^{(i)} e_i|_a$ in terms of the standard basis, then by the chain rule $D_v|_a f$ can be written more concretely as

$$D_v|_a f = \sum_{i=1}^n v^{(i)} \frac{\partial f}{\partial x^{(i)}}(a).$$

Lemma 3 (Properties of Derivations). Suppose $x \in \mathbb{R}^n, w \in T_x \mathbb{R}^n$, and $f, g \in \mathcal{C}^\infty(\mathbb{R}^n)$.

(a) If f is a constant function, then $wf = 0$.

(b) If $f(x) = g(x) = 0$, then $w(fg) = 0$.

Proposition 11. Let $x \in \mathbb{R}^n$.

(a) For each geometric tangent vector $v_x \in \mathbb{R}_x^n$, the map $D_v|_x : \mathcal{C}^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}$ defined in Note 6 is a derivation at x .

(b) The map $v_x \mapsto D_v|_x$ is an isomorphism from \mathbb{R}_x^n onto $T_x \mathbb{R}^n$.

Corollary 3.1. For any $a \in \mathbb{R}^n$, the n derivations

$$\left. \frac{\partial}{\partial x^{(1)}} \right|_a, \dots, \left. \frac{\partial}{\partial x^{(n)}} \right|_a \text{ defined by } \left. \frac{\partial}{\partial x^{(i)}} \right|_a f = \frac{\partial f}{\partial x^{(i)}}(a)$$

form a basis for $T_a \mathbb{R}^n$, which therefore has dimension n .

Definition 13. Let \mathcal{M} be a smooth manifold, and let p be a point of \mathcal{M} . A linear map $v : \mathcal{C}^\infty(\mathcal{M}) \rightarrow \mathbb{R}$ is called a **derivation at p** if it satisfies

$$v(fg) = f(p)v g + g(p)v f, \text{ for all } f, g \in \mathcal{C}^\infty(\mathcal{M}).$$

The set of all derivations of $\mathcal{C}^\infty(\mathcal{M})$ at p , denoted by $T_p \mathcal{M}$, is a vector space called the **tangent space to \mathcal{M} at p** . An element of $T_p \mathcal{M}$ is called a **tangent vector at p** .

Lemma 4 (Properties of Tangent Vectors on Manifolds). Suppose \mathcal{M} is a smooth manifold, $p \in \mathcal{M}$, $v \in T_p \mathcal{M}$, and $f, g \in \mathcal{C}^\infty(\mathcal{M})$.

(a) If f is a constant function, then $v f = 0$.

(b) If $f(p) = g(p) = 0$, then $v(fg) = 0$.

Definition 14. If \mathcal{M} and \mathcal{N} are smooth manifolds and $F : \mathcal{M} \rightarrow \mathcal{N}$ is a smooth map, for each $p \in \mathcal{M}$ we define a map $dF_p : T_p\mathcal{M} \rightarrow T_{F(p)}\mathcal{N}$, called the **differential of F at p** , as follows. Given $v \in T_p\mathcal{M}$, we let $dF_p(v)$ be the derivation at $F(p)$ that acts on $f \in C^\infty(\mathcal{N})$ by the rule

$$dF_p(v)(f) = v(f \circ F).$$

Remark 5. The operator $dF_p : C^\infty(\mathcal{N}) \rightarrow \mathbb{R}$ is linear because v is, and is a derivation at $F(p)$ because for any $f, g \in C^\infty(\mathcal{N})$ we have

$$\begin{aligned} dF_p(v)(fg) &= v((fg) \circ F) = v((f \circ F)(g \circ F)) \\ &= f(F(p))v(g \circ F) + g(F(p))v(f \circ F) \\ &= f(F(p))dF_p(v)(g) + g(F(p))dF_p(v)(f). \end{aligned}$$

Proposition 12 (Properties of Differentials). Let \mathcal{M}, \mathcal{N} , and \mathcal{P} be smooth manifolds, let $F : \mathcal{M} \rightarrow \mathcal{N}$ and $G : \mathcal{N} \rightarrow \mathcal{P}$ be smooth maps, and let $p \in \mathcal{M}$,

1. $dF_p : T_p\mathcal{M} \rightarrow T_{F(p)}\mathcal{N}$ is linear.
2. $d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_p\mathcal{M} \rightarrow T_{G \circ F(p)}\mathcal{P}$.
3. $d(Id_{\mathcal{M}}) = Id_{T_p\mathcal{M}} : T_p\mathcal{M} \rightarrow T_p\mathcal{M}$.
4. If F is a diffeomorphism, then $dF_p : T_p\mathcal{M} \rightarrow T_{F(p)}\mathcal{N}$ is an isomorphism, and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.

Proposition 13 (Tangent vectors act locally). Let \mathcal{M} be a smooth manifold, $p \in \mathcal{M}$ and $v \in T_p\mathcal{M}$. If $f, g \in C^\infty(\mathcal{M})$ agree on some neighborhood of p , then $vf = vg$.

Proposition 14 (The Tangent Space to an Open Submanifold). Let \mathcal{M} be a smooth manifold, let $U \subseteq \mathcal{M}$ be an open subset, and let $\iota : U \rightarrow \mathcal{M}$ be the inclusion map. For every $p \in U$, the differential $d\iota_p : T_pU \rightarrow T_p\mathcal{M}$ is an isomorphism.

Proposition 15 (Dimension of the Tangent Space). If \mathcal{M} is an n -dimensional smooth manifold, then for each $p \in \mathcal{M}$, the tangent space $T_p\mathcal{M}$ is an n -dimensional vector space.

Proposition 16 (The Tangent Space to a Vector Space). Suppose V is a finite-dimensional vector space with its standard smooth structure. For any vector $v \in V$, we define a map $D_v|_a : C^\infty(V) \rightarrow \mathbb{R}$ by

$$D_v|_a f = \left. \frac{d}{dt} \right|_{t=0} f(a + tv).$$

Then, the map $v \mapsto D_v|_a$ defined above is an isomorphism from v to T_aV , such that for any linear map $L : V \rightarrow W$ we have $dL_a(D_v|_a) = D_{Lv}|_{L_a}$.

Proposition 17 (The Tangent Space to a Product Manifold). Let $\mathcal{M}_1, \dots, \mathcal{M}_k$ be smooth manifolds, and for each j , let $\pi_j : \mathcal{M}_1 \times \dots \times \mathcal{M}_k \rightarrow \mathcal{M}_j$ denote the projection onto the \mathcal{M}_j factor. For any point $p = (p_1, \dots, p_k) \in \mathcal{M}_1 \times \dots \times \mathcal{M}_k$, the map

$$\alpha : T_p(\mathcal{M}_1 \times \dots \times \mathcal{M}_k) \rightarrow T_{p_1}\mathcal{M}_1 \oplus \dots \oplus T_{p_k}\mathcal{M}_k$$

defined by

$$\alpha(v) = (d(\pi_1)_p(v), \dots, d(\pi_k)_p(v))$$

is an isomorphism.

Proposition 18. Let \mathcal{M} be a smooth n -manifold, and let $p \in \mathcal{M}$. Then $T_p\mathcal{M}$ is an n -dimensional vector space, and for any smooth chart $(U, (x^{(i)}))$ containing p , the coordinate vectors $\partial/\partial x^{(1)}|_p, \dots, \partial/\partial x^{(n)}|_p$ form a basis for $T_p\mathcal{M}$.

References

- [1] John M. Lee, Introduction to Smooth Manifolds.