

# Set Theory $\cap$ Functional Analysis

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## 1 Basic Definitions

**Definition 1.** A **choice function**  $f$  on a collection  $\mathcal{C}$  of set  $X$  is a function such that for all  $A \in \mathcal{C}$ ,  $f(A) \in A$ .

**Example.** Consider a collection  $\{\{1, 2\}, \{3, 4\}\}$  and a function  $f$  defined as  $f(\{1, 2\}) = 2$  and  $f(\{3, 4\}) = 3$ . Then  $f$  is a choice function.

**Definition 2.** Suppose  $I$  is a set, called as the **index set**, and with each  $i \in I$  we associate a set  $A_i$ . Then,  $\{A_i : i \in I\}$  is defined as the **family of sets**. This can also be denoted by  $\{A_i\}_{i \in I}$ .

**Definition 3.** A **partially ordered set** is a set together with a partial order on it  $(X, \preceq)$  where partial order on  $X$  is defined as a relation  $\preceq$  in  $X$  such that, for all  $x, y, z \in X$  it follows

1. **Reflexive.**  $x \preceq x$
2. **Anti-symmetric.** If  $x \preceq y$  and  $y \preceq x$  then  $x = y$
3. **Transitive.** If  $x \preceq y$  and  $y \preceq z$ , then  $x \preceq z$

**Note.** If  $x \preceq y$  and  $x \neq y$ , then we write  $x \prec y$  and say that  $x$  is **smaller than**  $y$ . It is not necessary for all  $x, y \in X$  to have a partial order defined between them.

**Definition 4.** A set together with a total order on it is a **chain** or **totally ordered set** where a relation  $\preceq$  is **total order** if for every  $x, y \in X$  either  $x \preceq y$  or  $y \preceq x$ , consequently the set is called as a totally ordered set.

**Definition 5.** Let  $X$  be a partially ordered set, then an element  $a \in X$  is the **upper bound** of a subset  $E \subseteq X$  if  $x \preceq a$  for all  $x \in E$ .

**Definition 6.** Let  $X$  be a partially ordered set, then an element  $a \in X$  is **maximal** if  $a \preceq x$  implies  $x = a$ .

**Definition 7.** Let  $X$  be a partially ordered set, then an element  $a \in X$  is **maximum (or largest)** if  $x \preceq a \forall x \in X$ .

**Example.** Consider the set  $W = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}\}$  with set inclusion  $\subseteq$  as a partial ordering. The maximal elements are  $\{1, 2\}$  and  $\{3\}$ . If we view  $W$  as a subset of the power set of  $\{1, 2, 3\}$ , then the upper bound of  $W$  is the element  $\{1, 2, 3\}$ .

**Definition 8.** A poset  $P$  is called **well-ordered** if it is a chain, and every non-empty subset  $S \subseteq P$  has a minimum.

**Definition 9.** For a vector space  $X$ , a set  $B \subseteq X$  is called a **basis** (or **Hamel basis**) if  $B$  is a linearly independent set and  $\text{span}(B) = X$ .

**Definition 10.** Let  $X$  be a linear space. A function  $p : X \rightarrow \mathbb{R}$  is a **Sublinear Functional** if the following properties hold

1. **Subadditive.**  $p(x + y) \leq p(x) + p(y) \forall x, y \in X$ .
2. **Nonnegatively Homogeneous.**  $p(\lambda x) = \lambda p(x) \forall \lambda \geq 0$  where  $\lambda \in \mathbb{R}, x \in X$

**Example.** Norm is an example of sublinear functional which is not linear.

## 2 Theorem statements

Two formulations of **Axiom of Choice** are given.

- The Cartesian product of a non-empty family of non-empty sets is non-empty.
- For every non-empty set  $X$ , there exists a choice function  $f$  defined on  $X$ .

**Note.** *The equivalence proof of the above mentioned variants is skipped. In the further discussion regarding Axiom of Choice, we will be using the choice function formulation.*

**Zorn's Lemma.** *If  $X$  is a non-empty partially ordered set such that every chain in  $X$  has an upper bound, then  $X$  contains a maximal element.*

**Well-ordering principle.** *Every set has a well ordering.*

**Existence of Hamel Basis.** *Every vector space  $X \neq \{0\}$  has a basis.*

**Hahn-Banach Theorem.** *Let  $X$  be a real vector space and  $p$  a sublinear functional on  $X$ . Furthermore, let  $f$  be a linear functional which is defined on a subspace  $Z$  of  $X$  and satisfies*

$$f(x) \leq p(x) \quad \forall x \in Z$$

*Then  $f$  has a linear extension of  $\tilde{f}$  from  $Z$  to  $X$  satisfying*

$$\tilde{f}(x) \leq p(x) \quad \forall x \in X$$

*that is,  $\tilde{f}$  is a linear functional on  $X$ , satisfying above inequality on  $X$  and  $\tilde{f}(x) = f(x)$  for every  $x \in Z$ .*

## 3 Proof of Equivalences

**Zorn's Lemma  $\iff$  Axiom of Choice  $\iff$  Well-Ordering Principle**

**Theorem 1.** *Zorn's Lemma  $\implies$  Axiom of Choice.*

*Proof.* Let  $X$  be any non-empty set. Consider a set  $P$

$$P = \{(Y, f) : Y \subseteq X \text{ and } f \text{ is choice function on } Y\}.$$

We define a relation  $\preceq$  on  $P$  as  $(Y, f) \preceq (Y', f')$  whenever  $Y \subseteq Y'$  and  $f = f'|_Y$ . It is easy to see that  $(P, \preceq)$  is a poset. Note that  $P$  is non-empty as for any element  $x \in X$ ,  $\{x\} \mapsto x$  is a choice function, consequently  $(\{x\}, \{x\} \mapsto x) \in P$ .

Consider a chain  $C$  in  $P$ . We define  $\tilde{Y} = \bigcup_{(Y, f) \in C} Y$  and  $\tilde{f}(S) = f(S)$  for any  $S$  such that  $f$  is defined on  $S$ . Notice that  $(\tilde{Y}, \tilde{f})$  is an upper bound for  $C$  by construction. Since  $C$  was chosen arbitrarily and was shown to have an upper bound, then by Zorn's Lemma there is some maximal element in  $P$ , say  $(Y^*, f^*)$ .

Now we show that  $Y^* = X$ . Suppose not then there is some element  $x \in X \setminus Y^*$ . We can extend  $f^*$  to  $f^{**}$  from  $Y^*$  to  $Y^* \cup \{x\}$  by defining  $f^{**}(S) = x$ , if  $x \in S$  and  $f^{**}(S) = f^*(S)$ , if  $S \subseteq Y^*$ . Then we note that  $(Y^*, f^*) \preceq (Y^* \cup \{x\}, f^{**})$  which is a contradiction. Hence,  $Y^* = X$  and  $f^*$  is the required choice function on  $X$ .  $\square$

**Theorem 2.** *Zorn's Lemma  $\implies$  Well-ordering principle.*

*Proof.* The proof is similar to that of Theorem 1. Let  $X$  be any non-empty set. Consider a relation  $(P, \preceq)$ , where  $P$  is defined as

$$P = \{(Y, \leq_Y) : Y \subseteq X \text{ and } \leq_Y \text{ is a well-ordering on } Y\},$$

and  $(Y, \leq_Y) \preceq (Y', \leq_{Y'})$  whenever  $Y \subseteq Y'$  and  $\leq_Y$  and  $\leq_{Y'}$  agree on  $Y$ . It is easy to see that  $(P, \preceq)$  is a poset. Note that  $P$  is non-empty as every singleton set is well-ordered.

On similar lines of Theorem 1 proof, we conclude that there exists some maximal element in  $P$ , say  $(Y^*, \leq_{Y^*})$ .

Now we show that  $Y^* = X$ . Suppose not then there is some element  $x \in X \setminus Y^*$ . We can extend  $(Y^*, \leq_{Y^*})$  to  $Y^* \cup \{x\}$  by defining  $x$  to be greater than every element in  $Y^*$ . This is a contradiction that  $(Y^*, \leq_{Y^*})$  is maximal element. Hence,  $Y^* = X$  and  $\leq_{Y^*}$  is the required well-ordering on  $X$ .  $\square$

**Theorem 3.** *Well-ordering principle  $\implies$  Axiom of choice.*

*Proof.* Suppose  $X$  is a non-empty set, and  $\leq$  is a well-ordering of  $X$ . Then  $f(S) = \min S$ , defines a choice function on  $X$  which is guaranteed to exist for any set  $S$  by Well-ordering principle.  $\square$

**Theorem 4.** *Axiom of Choice  $\implies$  Zorn's Lemma.*

*Proof.* Let's assume there exist a non-empty partially ordered set  $P$  such that every chain in  $P$  has an upper bound, but does not contain a maximal element.

Considering axiom of choice is true, there must exist a choice function  $f$  on  $P$ , and let  $x_0 := f(P)$ .

Also, let the set of *strict* upper bounds on a chain  $C$  in  $P$  be

$$Upp(C) := \{u \notin C : \forall x \in C, x \prec u\}$$

**Lemma 5.** *For any chain  $C$ , the set  $Upp(C)$  is non-empty.*

*Proof.* As  $C$  is a chain in  $P$ , therefore there exists an upper bound  $u$  for  $C$ . There can be two cases,

1.  $C$  does not have any maximum element, then  $u \notin C$  and  $u \in Upp(C)$  must be true by definitions.
2.  $C$  contains a maximum element, let's say  $m$ . Since  $P$  has no maximal element (assumed), there exist a  $u$  greater than  $m$ . Then  $x \prec m \prec u$  for each  $x \in C$ , and hence  $u \in Upp(C)$ .

Hence, in both cases  $Upp(C)$  is non-empty for any chain  $C$  in  $P$ .  $\square$

A sub-chain  $C'$  is an initial segment of a chain  $C$  such that  $x \in C, y \in C'$  and  $x \prec y$  implies  $c \in C'$ . **Intuition:** For all  $y \in C \setminus C'$ , and for all  $x \in C'$ ,  $x \prec y$ . Now, let's define a function  $g$ , such that for any chain  $C$ ,

$$g(C) := f(Upp(C))$$

Also, let's define an **attempt** as a well ordered set  $A \subset P$  satisfying following:

1.  $\min A = x_0$
2. For every proper initial segment  $C \subset A$ ,  $\min A \setminus C = g(C)$

**Lemma 6.** *If  $A$  and  $A'$  are two attempts, then either  $A \subseteq A'$  or  $A' \subseteq A$ .*

*Proof.* Let's assume that both  $A \subseteq A'$  and  $A' \subseteq A$  does not apply, and let  $z = \min A \setminus A'$  and  $z' = \min A' \setminus A$ . As both  $A$  and  $A'$  are attempts (and hence well-ordered by definition). Since  $z \neq z'$ ,  $z \preceq z'$  and  $z' \preceq z$  can not be true together. So, let's assume wlog  $z' \not\preceq z$ . Let's define a set  $C = \{x \in A : x \prec z\}$ . From the definitions of  $z$  it follows that  $C \subseteq A$ . It is clear from this that  $z = \min A \setminus C$ , and so  $z = g(C)$ . There are now two cases possible.

1.  $C = A'$ . Now, as  $C \subseteq A$  therefore  $A' \subseteq A$ . Hence, the given lemma is true in this case.
2.  $C \neq A'$ . If  $z' \preceq x$  for some  $x \in C$ , then transitivity of partial order implies  $z' \prec z$ , which is a contradiction. So, since  $A'$  is a chain (as it is well-ordered),  $x \preceq z' \forall x \in C$ . therefore  $C$  is a proper initial segment of  $A'$  which implies  $g(C) \in A'$ . But,  $g(C) = z \notin A'$ . Therefore a contradiction. Hence, the given lemma is true. □

As, for any two attempts  $A, A'$  either  $A \subseteq A'$  or  $A' \subseteq A$ , therefore  $A \cup A'$  is either  $A$  or  $A'$  which is an attempt. Let  $\mathcal{A}$  be the set of all attempts then  $A := \bigcup_{\tilde{A} \in \mathcal{A}} \tilde{A}$ . Then  $A$  is also an attempt.

However,  $A \cup \{g(A)\}$  is also an attempt and must have belonged in the previous set of attempts  $\mathcal{A}$ , and also  $A \subseteq A \cup \{g(A)\}$  therefore  $A \cup \{g(A)\} := \bigcup_{A \in \mathcal{A}} A$  but this is not the case, therefore a contradiction. And, hence there must exist a maximal element of  $P$ . □

**Theorem 7.** *Zorn's Lemma*  $\implies$  “Every vector space  $X \neq \{0\}$  has a basis”.

*Proof.* Let  $X$  be a non-empty vector space. We define a relation  $(P, \preceq)$  where  $P$  is the set of subsets of  $X$  which are linearly independent and for every  $B, B' \in P$ ,  $B \preceq B'$  whenever  $B \subseteq B'$ . It is easy to note that  $(P, \subseteq)$  is a poset. Notice that  $P \neq \emptyset$  as  $X \neq \{0\}$ , there is some non-zero element  $x \in X$ , consequently  $B = \{x\} \in P$ .

Consider a chain  $C$  in  $P$ . Define  $\tilde{B} = \bigcup_{B \in C} B$ . Notice that  $\tilde{B}$  is an upper bound for  $C$ , hence by Zorn's Lemma there exists a maximal element in  $P$ , say  $B^*$ .

We show that  $\text{span}(B^*) = X$ . Suppose not, then there is an element  $x \in X \setminus \text{span}(B^*)$  and  $x \neq 0$ . Then extend the set  $B^*$  by including  $x$  in it. Notice that the extended set is an element in  $P$  which is greater than  $B^*$  under the subset relation. This is a contradiction. Hence  $\text{span}(B^*) = X$  and  $B^*$  is a linearly independent set, thus  $B^*$  is a basis for  $X$ . □

**Note.** A variant of converse of the above result is also true which is

**“Every vector space  $X \neq \{0\}$  has a basis”  $\implies$  Axiom of Choice,**

thus establishing equivalence between the two. The proof can be found in [1] which shows the implication for **Axiom of Multiple Choice** instead. It is known that Axiom of Multiple Choice is equivalent to Axiom of Choice. We skip this proof as it is quite involved.

**Theorem 8.** *Zorn's Lemma*  $\implies$  *Hahn-Banach Theorem*

*Proof.* Let's proof this in 3 parts,

(A) Let's define  $M$  as the partial order set of pairs  $(Z, f_Z)$  where

- (a)  $Z$  is a subspace of  $X$  containing  $Y$ .
- (b)  $f_Z : Z \rightarrow \mathbb{R}$  is a linear functional extending  $f$ , satisfying

$$f_Z(z) \leq p(z) \forall z \in Z$$

with partial ordering defined as  $(Z_1, f_{Z_1}) \preceq (Z_2, f_{Z_2})$  if  $Z_1 \subset Z_2$  and  $(f_{Z_2})|_{Z_1} = f_{Z_1}$ . Since,  $(Y, f) \in M$ ,  $M$  is a non-empty set. Let's choose any arbitrary chain  $C = \{(Z_\alpha, f_{Z_\alpha})\}_{\alpha \in \Lambda}$  in  $M$ , with  $\Lambda$  being some indexing set.

**Lemma 9.**  $C$  has an upper bound in  $M$ .

*Proof.* Let  $W = \bigcup_{\alpha \in \Lambda} Z_\alpha$  and construct a functional  $f_W : W \Rightarrow \mathbb{R}$  defined as follow: If  $w \in W$ , then  $w \in Z_\alpha$  for some  $\alpha \in \Lambda$  and we set  $f_W(w) = f_{Z_\alpha}(w)$  for that particular  $\alpha$ .

- This definition is well-defined. Indeed, suppose  $w \in Z_\alpha$  and  $w \in Z_\beta$ . If  $Z_\alpha \subset Z_\beta$ , then  $f_{Z_\beta}|_{Z_\alpha} = f_{Z_\alpha}$ , since they are a part of chain.
- $W$  clearly contains  $Y$ , and we show that  $W$  is a subspace of  $X$  and  $f_W$  is a linear functional on  $W$ . Choose any  $w_1, w_2 \in W$ , then  $w_1 \in Z_{\alpha_1}, w_2 \in Z_{\alpha_2}$  for some  $\alpha_1, \alpha_2 \in \Lambda$ . If  $Z_{\alpha_1} \subset Z_{\alpha_2}$ , say, then for any scalars  $\beta, \gamma \in \mathbb{R}$  we have

$$w_1, w_2 \in Z_{\alpha_2} \implies \beta w_1 + \gamma w_2 \in Z_{\alpha_2} \subset W$$

Also, with  $f_W(u) = f_{Z_{\alpha_1}}(u)$  and  $f_W(v) = f_{Z_{\alpha_2}}(v)$ ,

$$\begin{aligned} f_W(\beta u + \gamma v) &= f_{Z_{\alpha_2}}(\beta u + \gamma v) \\ &= \beta f_{Z_{\alpha_2}}(u) + \gamma f_{Z_{\alpha_2}}(v) \text{ linearity} \\ &= \beta f_{Z_{\alpha_1}}(u) + \gamma f_{Z_{\alpha_2}}(v) \text{ because in same chain} \\ &= \beta f_{Z_W}(u) + \gamma f_{Z_W}(v) \end{aligned}$$

The case  $Z_{\alpha_2} \subset Z_{\alpha_1}$  follows from a symmetric argument.

- Choose any  $w \in W$ , then  $w \in Z_\alpha$  for some  $\alpha \in \Lambda$  and

$$f_W(w) = f_{Z_\alpha}(w) \leq p(w) \text{ since } (w, Z_\alpha) \in M$$

Hence,  $(W, f_W)$  is an element of  $M$  and an upper bound of  $C$  since  $(Z_\alpha, f_{Z_\alpha}) \leq (W, f_W)$  for all  $\alpha \in \Lambda$ . Since  $C$  was an arbitrary chain in  $M$ , by Zorn's lemma,  $M$  has a maximal element  $(Z, f_Z) \in M$ , and  $f_Z$  is (by definition) a linear extension of  $f$  satisfying  $f_Z(z) \leq p(z)$  for all  $z \in Z$ .  $\square$

- (B) The proof is complete if we can show that  $Z = X$ . Suppose not, then there exists an  $\theta \in X \setminus Z$ ; note  $\theta \neq 0$  since  $Z$  is a subspace of  $X$ . Consider the subspace  $Z_\theta = \text{span}\{Z, \{\theta\}\}$ . Any  $x \in Z_\theta$  has a unique representation  $x = z + \alpha\theta$ ,  $z \in Z$ ,  $\alpha \in \mathbb{R}$ . Indeed, if

$$x = z_1 + \alpha_1\theta = z_2 + \alpha_2\theta, z_1, z_2 \in Z, \alpha_1, \alpha_2 \in \mathbb{R}$$

then  $z_1 - z_2 = (\alpha_2 - \alpha_1)\theta \in Z$  since  $Z$  is a subspace of  $X$ . Since  $\theta \notin Z$ , we must have  $\alpha_2 - \alpha_1 = 0$  and  $z_1 - z_2 = \theta$ . Next, we construct a functional  $f_{Z_\theta} : Z_\theta \rightarrow \mathbb{R}$  defined by

$$f_{Z_\theta}(x) = f_{Z_\theta}(z + \alpha\theta) = f_Z(z) + \alpha\delta, \dots (1)$$

where  $\delta$  is any real number. It can be shown that  $f_{Z_\theta}$  is linear and  $f_{Z_\theta}$  is a proper linear extension of  $f_Z$ ; indeed, we have, for  $\alpha = 0$ ,  $f_{Z_\theta}(x) = f_{Z_\theta}(z) = f_Z(x)$ . Consequently, if we can show that

$$f_{Z_\theta}(x) \leq p(x) \quad \forall x \in Z_\theta \dots (2)$$

then  $(Z_\theta, f_{Z_\theta}) \in M$  satisfying  $(Z, f_Z) \leq (Z_\theta, f_{Z_\theta})$ , thus contradicting the maximality of  $(Z, f_Z)$ .

(C) From (1), observe that (2) is trivial if  $\alpha = 0$ , so suppose  $\alpha \neq 0$ . We do have a single degree of freedom, which is the parameter  $\delta$  in (1), thus the problem reduces to showing the existence of a suitable  $\delta \in \mathbb{R}$  such that (2) holds. Consider any  $x = z + \alpha\theta \in Z_\theta, z \in Z, \alpha \in \mathbb{R}$ . Assuming  $\alpha > 0$ , (2) is equivalent to

$$\begin{aligned} f_Z(z) + \alpha\delta &\leq p(z + \alpha\theta) = \alpha p(z/\alpha + \theta) \\ f_Z(z/\alpha) + \delta &\leq p(z/\alpha + \theta) \\ \delta &\leq p(z/\alpha + \theta) - f_Z(z/\alpha) \end{aligned}$$

Since the above must hold for all  $z \in Z, \alpha \in \mathbb{R}$ , we need to choose  $\delta$  such that

$$\delta \leq \inf_{z_1 \in Z} (p(z_1 + \theta) - f_Z(z_1)) = m_1 \dots (3)$$

Assuming  $\alpha < 0$ , (2) is equivalent to

$$\begin{aligned} f_Z(z) + \alpha\delta &\leq p(z + \alpha\theta) = -\alpha p(-z/\alpha - \theta) \\ -f_Z(z/\alpha) - \delta &\leq p(-z/\alpha - \theta) \\ \delta &\geq -p(-z/\alpha - \theta) - f_Z(z/\alpha) \end{aligned}$$

Since the above must hold for all  $z \in Z, \alpha \in \mathbb{R}$ , we need to choose  $\delta$  such that

$$\delta \geq \sup_{z_2 \in Z} (-p(z_2 + \theta) - f_Z(z_2)) = m_0 \dots (4)$$

We are left with showing condition (3), (4) are compatible, i.e

$$-p(-z_2 - \theta) - f_Z(z_2) \leq p(z_1 + \theta) - f_Z(z_1) \quad \forall z_1, z_2 \in Z$$

The inequality above is trivial if  $z_1 = z_2$ , so suppose not. We have that

$$\begin{aligned} p(z_1 + \theta) - f_Z(z_1) + p(-z_2 - \theta) + f_Z(z_2) &= p(z_1 + \theta) + p(-z_2 - \theta) + f_Z(z_2 - z_1) \\ &\geq f_Z(z_2 - z_1) + p(z_1 + \theta - z_2 - \theta) \\ &= f_Z(z_2 - z_1) + p(z_1 - z_2) \\ &= -f_Z(z_1 - z_2) + p(z_1 - z_2) \geq 0 \end{aligned}$$

where linearity of  $f_Z$  and subadditivity of  $p$  are used. Hence, the required condition on  $\delta$  is  $m_0 \leq \delta \leq m_1$

Therefore, by using Zorn's Lemma (as using in 1st part) we proved Hahn-Banach Theorem.  $\square$

**Note.** The converse of the above theorem is not true, i.e, Hahn-Banach Theorem  $\nRightarrow$  Zorn's Lemma. It is known that Hahn-Banach Theorem is equivalent to a statement which is strictly weaker than Axiom of Choice.[3]

## References

- [1] Existence of Basis implies AC
- [2] Axiom of Choice equivalents
- [3] Wikipedia, Hahn-Banach Theorem
- [4] P. Halmos, Naive set theory. New York, 1974.
- [5] Hahn-Banach Theorem