

# Differential Calculus

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## 1 Total Derivatives in Finite-Dimensional Vector spaces

**Definition 1.** Let  $V, W$  be finite-dimensional vector spaces, which we may assume to be endowed with norms. If  $U \subseteq V$  is an open subset  $a \in U$ , a map  $F : U \rightarrow W$  is said to be **differentiable at  $a$**  if there exists a linear map  $L : V \rightarrow W$  such that

$$\lim_{v \rightarrow 0} \frac{\|F(a+v) - F(a) - Lv\|}{\|v\|} = 0. \quad (1)$$

**Remark 1.** It can be easily seen that

$$\lim_{v \rightarrow 0} \frac{\|F(a+v) - F(a) - Lv\|}{\|v\|} = 0 \Leftrightarrow \lim_{v \rightarrow 0} \frac{F(a+v) - F(a) - Lv}{\|v\|} = 0. \quad (2)$$

**Proposition 1.** Suppose  $F : U \rightarrow W$  is differentiable at  $a \in U$ . Then the linear map  $L$  satisfying (1) is unique.

*Proof.* Let  $L_1 : V \rightarrow W$  and  $L_2 : V \rightarrow W$  satisfy (1). Define  $A = L_1 - L_2$ , then we have

$$\begin{aligned} \|Av\| &= \|(F(a+v) - F(a) - L_2v) - (F(a+v) - F(a) - L_1v)\| \\ &\leq \|F(a+v) - F(a) - L_2v\| + \|F(a+v) - F(a) - L_1v\| \end{aligned}$$

Dividing by  $\|v\|$  and taking limit we get

$$\Rightarrow \lim_{v \rightarrow 0} \frac{\|Av\|}{\|v\|} \leq \lim_{v \rightarrow 0} \left( \frac{\|F(a+v) - F(a) - L_1v\|}{\|v\|} + \frac{\|F(a+v) - F(a) - L_2v\|}{\|v\|} \right) = 0$$

Therefore we have  $Av = 0 \forall v \in V$  which implies that  $A = 0$ , i.e.,  $L_1 = L_2$ .  $\square$

**Definition 2.** If  $F$  is differentiable at  $a$ , the linear map  $L$  satisfying (1) is denoted by  $DF(a)$  and is called the **total derivative of  $F$  at  $a$** .

**Remark 2.** Condition (1) can also be written as

$$F(a+v) - F(a) = DF(a)v + R(v), \quad (3)$$

where the remainder term  $R(v)$  satisfies  $\|R(v)\|/\|v\| \rightarrow 0$  as  $v \rightarrow 0$ . Thus the total derivative represents the “best linear approximation” to  $F(a+v) - F(a)$  near  $a$ . Note that  $\|R(v)\|/\|v\| \rightarrow 0$  implies that eventually  $\|R(v)\|/\|v\| \leq 1$ , i.e.,  $\|R(v)\| \leq \|v\|$ .

**Proposition 2.** Suppose  $V, W$  are finite-dimensional vector spaces,  $U \subseteq V$  is an open subset,  $a \in U$ , and  $F : U \rightarrow W$  is a map. If  $F$  is differentiable at  $a$ , then it is continuous at  $a$ .

*Proof.* In (3) take norm and apply limit  $v \rightarrow 0$  on both sides

$$\begin{aligned} 0 \leq \lim_{v \rightarrow 0} \|F(a+v) - F(a)\| &= \lim_{v \rightarrow 0} \|DF(a)v + R(v)\| \leq \lim_{v \rightarrow 0} (\|DF(a)v\| + \|R(v)\|) \\ &\leq \lim_{v \rightarrow 0} (\|DF(a)\| \|v\| + \|v\|) = 0, \end{aligned}$$

where  $\|DF(a)\|$  is the operator norm. Thus  $F$  is continuous at  $a$ .  $\square$

**Proposition 3.** Suppose  $V, W, X$  are finite-dimensional vector spaces. Then

- (a) If  $T : V \rightarrow W$  is a linear map, then  $T$  is differentiable at every point  $v \in V$ , with total derivative equal to  $T$  itself:  $DT(v) = T$ .
- (b) If  $B : V \times W \rightarrow X$  is a bilinear map, then  $B$  is differentiable at every point  $(v, w) \in V \times W$ , and  $DB(v, w)(x, y) = B(v, y) + B(x, w)$ .

*Proof.* (a) Setting  $L = T$  in (1) and using the linearity of  $T$ , we see that  $T$  is differentiable everywhere with the total derivative equal to  $T$  itself.

(b) We use (3) to show that bilinear map is differentiable. Note that

$$B(v+x, w+y) = B(v, w) + B(v, y) + B(x, w) + B(x, y).$$

But  $B(x, y) \leq \|B\| \|x\| \|y\|$  where  $\|B\|$  is operator norm which is finite by continuity of  $B$ . Then comparing with (3)

$$B(v+x, w+y) - B(v, w) = DB(v, w)(x, y) + R(x, y),$$

where  $DB(v, w)(x, y) = B(v, y) + B(x, w)$  and  $R(x, y) \leq \|B\| \|x\| \|y\| \rightarrow 0$  as  $(x, y) \rightarrow 0$ .  $\square$

**Proposition 4 (The Chain Rule for Total Derivatives).** Suppose  $V, W, X$  are finite-dimensional vector spaces,  $U \subseteq V$  and  $\tilde{U} \subseteq W$  are open subsets, and  $F : U \rightarrow \tilde{U}$  and  $G : \tilde{U} \rightarrow X$  are maps. If  $F$  is differentiable at  $a \in U$  and  $G$  is differentiable at  $F(a) \in \tilde{U}$ , then  $G \circ F$  is differentiable at  $a$  and  $D(G \circ F)(a) = DG(F(a)) \circ DF(a)$ .

*Proof.* Let  $A = DF(a)$  and  $B = DG(F(a))$ . We need to show that

$$\lim_{v \rightarrow 0} \frac{\|G(F(a+v)) - G(F(a)) - BAv\|}{\|v\|} = 0. \quad (4)$$

Let us write  $b = F(a)$  and  $w = F(a+v) - F(a)$ . With these substitutions, we can rewrite the quotient in (4) as

$$\begin{aligned} \frac{\|G(b+w) - G(b) - BAv\|}{\|v\|} &= \frac{\|G(b+w) - G(b) - Bw + Bw - BAv\|}{\|v\|} \\ &\leq \frac{\|G(b+w) - G(b) - Bw\|}{\|v\|} + \frac{\|B(w - Av)\|}{\|v\|} \quad (\dagger) \end{aligned}$$

The differentiability of  $F$  at  $a$  means that for any  $\epsilon > 0$ , we can ensure that

$$\|w - Av\| = \|F(a+v) - F(a) - Av\| \leq \epsilon \|v\|$$

as long as  $v$  lies in a small enough neighborhood of 0. Moreover, as  $v \rightarrow 0$ ,  $\|v\| = \|F(a+v) - F(a)\| \rightarrow 0$  by continuity of  $F$ . Therefore by differentiability of  $G$  at  $b$  means that by making  $\|v\|$  even smaller if necessary, we can also achieve

$$\|G(b+w) - G(b) - Bw\| \leq \epsilon \|w\|.$$

Also note that  $\|B(w - Av)\| \leq \|B\|\|w - Av\|$ . Putting all of these estimates together, we see that for  $\|v\|$  sufficiently small,  $(\dagger)$  is bounded by

$$\begin{aligned} \epsilon \frac{\|w\|}{\|v\|} + \|B\| \frac{\|w - Av\|}{\|v\|} &= \epsilon \frac{\|w - Av + Av\|}{\|v\|} + \|B\| \frac{\|w - Av\|}{\|v\|} \\ &\leq \epsilon \frac{\|w - Av\|}{\|v\|} + \frac{\|Av\|}{\|v\|} + \|B\| \frac{\|w - Av\|}{\|v\|} \\ &\leq \epsilon^2 + \epsilon\|A\| + \epsilon\|B\|, \end{aligned}$$

which can be made as small as desired.  $\square$

**Lemma 1.** *The addition operation  $+: \mathbb{R}^2 \rightarrow \mathbb{R}$  defined as  $(+)(x, y) = x + y$  is differentiable and  $D(+)(a, b) = +$ . The multiplication operation  $\times: \mathbb{R}^2 \rightarrow \mathbb{R}$  defined as  $(\times)(x, y) = xy$  is differentiable and  $D(\times)(a, b)(x, y) = bx + ay$ . The reciprocal operation  $h: \mathbb{R} \rightarrow \mathbb{R}$  defined as  $h(x) = 1/x$  is differentiable and  $Dh(x) = -\frac{1}{x^2}$ .*

**Proposition 5.** *Suppose  $V, W$  are finite-dimensional vector spaces,  $U \subseteq V$  is an open subset,  $a$  is a point in  $U$ , and  $F, G: U \rightarrow W$  and  $f, g: U \rightarrow \mathbb{R}$  are maps. Then*

(a) *If  $F$  is a constant map, then  $F$  is differentiable at  $a$  and  $DF(a) = 0$ .*

(b) *If  $F$  and  $G$  are differentiable at  $a$ , then  $F + G$  is also, and*

$$D(F + G)(a) = DF(a) + DG(a).$$

(c) *If  $f$  and  $g$  are differentiable at  $a$ , then  $fg$  is also, and*

$$D(fg)(a) = f(a)Dg(a) + g(a)Df(a).$$

(d) *If  $f$  and  $g$  are differentiable at  $a$  and  $g(a) \neq 0$ , then  $f/g$  is differentiable at  $a$ , and*

$$D(f/g)(a) = \frac{g(a)Df(a) - f(a)Dg(a)}{g(a)^2}.$$

*Proof.* (a) Let  $F(v) = c \in W$  for all  $v \in V$ . Then setting  $L = 0$  in (1) satisfies the equation showing that  $F$  is differentiable at  $a$  and  $DF(a) = 0$ .

(b) Note that  $F + G = (+) \circ (F, G)$ , then using chain rule

$$\begin{aligned} D(F + G)(a) &= D((+) \circ (F, G))(a) = D(+)(F(a), G(a)) \circ D(F, G)(a) \\ &= D((+)(F(a), G(a))) \circ (DF(a), DG(a)) \\ &= (+)(DF(a), DG(a)) = DF(a) + DG(a). \end{aligned}$$

(c) Note that  $fg = (\times) \circ (f, g)$ , then using chain rule

$$\begin{aligned} D(fg)(a) &= D((\times) \circ (f, g))(a) = D((\times)(f(a), g(a))) \circ D(f, g)(a) \\ &= D((\times)(f(a), g(a))) \circ (Df(a), Dg(a)) \\ &= g(a)Df(a) + f(a)Dg(a) \end{aligned}$$

(d) Note that  $f/g = (\times) \circ (f, 1/g)$  and  $1/g = h \circ g$ , so by chain rule

$$D(1/g)(a) = -\frac{1}{g(a)^2}Dg(a).$$

Then

$$\begin{aligned}
D(f/g)(a) &= D((\times) \circ (f, 1/g))(a) = D(\times)(f(a), 1/g(a)) \circ D(f, 1/g)(a) \\
&= D(\times)(f(a), 1/g(a))(Df(a), D(1/g)(a)) \\
&= \frac{1}{g(a)} Df(a) - f(a) \frac{1}{g(a)^2} Dg(a) \\
&= \frac{g(a) Df(a) - f(a) Dg(a)}{g(a)^2}.
\end{aligned}$$

□

## 2 Total and Partial Derivatives in $\mathbb{R}^n$

**Definition 3.** Suppose  $U \subseteq \mathbb{R}^n$  is open and  $f : U \rightarrow \mathbb{R}$  is a real-valued function. For any  $a = (a^1, \dots, a^n) \in U$  and any  $j \in \{1, \dots, n\}$ , the **j-th partial derivative of f at a** is defined to be the ordinary derivative of  $f$  w.r.t.  $x^j$  while holding the other variables fixed:

$$\frac{\partial f}{\partial x^j}(a) = \lim_{h \rightarrow 0} \frac{f(a + he_j) - f(a)}{h}$$

if the limit exists.

**Definition 4.** For a vector-valued function  $F : U \rightarrow \mathbb{R}^m$ , we can write the coordinates of  $F(x)$  as  $F(x) = (F^1(x), \dots, F^m(x))$ . This defines  $m$  functions  $F^1, \dots, F^m : U \rightarrow \mathbb{R}$  called the **component functions of F**. The partial derivatives of  $F$  are defined simply to be the partial derivatives  $\partial F^i / \partial x^j$  of its component functions. The matrix  $(\partial F^i / \partial x^j)$  of partial derivatives is called the **Jacobian matrix of F**, and its determinant is called the **Jacobian determinant of F**.

**Definition 5.** If  $F : U \rightarrow \mathbb{R}^m$  is a function for which each partial derivative exists at each point in  $U$  and the functions  $\partial F^i / \partial x^j : U \rightarrow \mathbb{R}$  so defined are all continuous, then  $F$  is said to be of class  $C^1$  or **continuously differentiable**. If this is the case, we can differentiate the functions  $\partial F^i / \partial x^j$  to obtain **second-order partial derivatives**

$$\frac{\partial^2 F^i}{\partial x^k \partial x^j} = \frac{\partial}{\partial x^k} \left( \frac{\partial F^i}{\partial x^j} \right),$$

if they exist. Continuing this way leads to higher-order partial derivatives: the **partial derivatives of F of order k** are the (first) partial derivatives of those of order  $k - 1$ , when they exist.

**Definition 6.** If  $U \subseteq \mathbb{R}^n$  is an open subset and  $k \geq 0$ , a function  $F : U \rightarrow \mathbb{R}^m$  is said to be of class  $C^k$  or **k times continuously differentiable** if all the partial derivatives of  $F$  of order less than or equal to  $k$  exist and are continuous functions on  $U$ .

**Remark 3.** Thus a function of class  $C^0$  is just a continuous function. Because existence and continuity of derivatives are local properties, clearly  $F$  is  $C^k$  iff it has the property in a neighborhood of each point in  $U$ .

**Definition 7.** A function that is of class  $C^k$  for every  $k \geq 0$  is said to be of class  $C^\infty$ , or **smooth**, or **infinitely differentiable**. If  $U$  and  $V$  are open subsets of Euclidean spaces, a function  $F : U \rightarrow V$  is called a **diffeomorphism** if it is smooth and bijective and its inverse function is also smooth.

**Proposition 6.** Suppose  $U \subseteq \mathbb{R}^n$  and  $V \subseteq \mathbb{R}^m$  are open subsets and  $F : U \rightarrow V$  is a diffeomorphism. Then  $m = n$ , and for each  $a \in U$ , the total derivative is invertible, with  $DF(a)^{-1} = D(F^{-1})(F(a))$ .

*Proof.* Because  $F^{-1} \circ F = Id_U$ , the chain rule implies that for each  $a \in U$ ,

$$Id_{\mathbb{R}^n} = D(Id_U)(a) = D(F^{-1} \circ F)(a) = D(F^{-1})(F(a)) \circ DF(a).$$

Similarly,  $F \circ F^{-1} = Id_V$  implies that for each  $F(a) \in V$ , we have

$$DF(F^{-1}(F(a))) \circ D(F^{-1})(F(a)) = DF(a) \circ D(F^{-1})(F(a)) = Id_{\mathbb{R}^m}.$$

This implies that  $DF(a)$  is invertible with inverse  $D(F^{-1})(F(a))$ , and therefore  $m = n$ .  $\square$

**Definition 8 (Smoothness on Arbitrary Domains).** If  $A \subseteq \mathbb{R}^n$  is an arbitrary subset, a function  $F : A \rightarrow \mathbb{R}^m$  is said to be **smooth on  $A$**  if it admits a smooth extension to an open neighborhood of each point, or more precisely, if for every  $x \in A$ , there exists an open neighborhood  $U_x \subseteq \mathbb{R}^n$  and a smooth function  $\tilde{F} : U_x \rightarrow \mathbb{R}^m$  that agrees with  $F$  on  $U_x \cap A$ . The notion of diffeomorphism extends to arbitrary subsets in the obvious way: given arbitrary subsets  $A, B \subseteq \mathbb{R}^n$ , a **diffeomorphism from  $A$  to  $B$**  is a smooth bijective map  $f : A \rightarrow B$  with smooth inverse.

**Definition 9.** If  $U \subseteq \mathbb{R}^n$  is open, the set of all real-valued functions of class  $C^k$  on  $U$  is denoted by  $C^k(U)$ , and the set of all smooth real-valued functions by  $C^\infty(U)$ . Sums, constant multiples, and products of functions are defined pointwise: for  $f, g : U \rightarrow \mathbb{R}$  and  $c \in \mathbb{R}$ ,

$$\begin{aligned}(f + g)(x) &= f(x) + g(x), \\ (cf)(x) &= c(f(x)), \\ (fg)(x) &= f(x)g(x).\end{aligned}$$

**Proposition 7 (Equality of Mixed Partial Derivatives).** If  $U$  is an open subset of  $\mathbb{R}^n$  and  $F : U \rightarrow \mathbb{R}^m$  is a function of class  $C^2$ , then the mixed second-order partial derivatives of  $F$  do not depend on the order of differentiation:

$$\frac{\partial^2 F^i}{\partial x^j \partial x^k} = \frac{\partial^2 F^i}{\partial x^k \partial x^j}.$$

**Corollary.** If  $F : U \rightarrow \mathbb{R}^m$  is smooth, then the mixed partial derivatives of  $F$  of any order are independent of the order of differentiation.

**Proposition 8.** Let  $U \subseteq \mathbb{R}^n$  be open, and suppose  $F : U \rightarrow \mathbb{R}^m$  is differentiable at  $a \in U$ . Then all of the partial derivatives of  $F$  at  $a$  exist, and  $DF(a)$  is the linear map whose matrix is the Jacobian of  $F$  at  $a$ :

$$DF(a) = \left( \frac{\partial F^j}{\partial x^i}(a) \right).$$

*Proof.* Let  $B = DF(a)$ , and for  $v \in \mathbb{R}^n$  small enough that  $a + v \in U$ , let  $R(v) = F(a + v) - F(a) - Bv$ . The fact that  $F$  is differentiable at  $a$  implies that each component of the vector-valued function  $R(v)/\|v\|$  goes to zero as  $v \rightarrow 0$ . The  $i$ -th partial derivative of  $F^j$  at  $a$ , if it exists, is

$$\frac{\partial F^j}{\partial x^i}(a) = \lim_{t \rightarrow 0} \frac{F^j(a + te_i) - F^j(a)}{t} = \lim_{t \rightarrow 0} \frac{B_i^j t + R^j(te_i)}{t} = B_i^j + \lim_{t \rightarrow 0} \frac{R^j(te_i)}{t}.$$

The norm of the quotient on the right above is  $\|R^j(te_i)\|/\|te_i\|$ , which approaches zero as  $t \rightarrow 0$ . It follows that  $\partial F^j/\partial x^i(a)$  exists and is equal to  $B_i^j$  as claimed.  $\square$

**Proposition 9.** Suppose  $U \subseteq \mathbb{R}^n$  is open. Then  $F : U \rightarrow \mathbb{R}^m$  is differentiable at  $a \in U$  iff each of its component functions  $F^1, \dots, F^m$  is differentiable at  $a$  and

$$DF(a) = \begin{pmatrix} DF^1(a) \\ \vdots \\ DF^m(a) \end{pmatrix}$$

*Proof.* Using the fact that  $y = (y_1, \dots, y_m) \rightarrow 0 \Leftrightarrow y_i \rightarrow 0$  for each  $i$ , from Remark 1 we see that

$$\begin{aligned} \lim_{v \rightarrow 0} \frac{\|F(a+v) - F(a) - DF(a)v\|}{\|v\|} = 0 &\Leftrightarrow \lim_{v \rightarrow 0} \frac{F(a+v) - F(a) - DF(a)v}{\|v\|} = 0 \\ &\Leftrightarrow \lim_{v \rightarrow 0} \frac{F^i(a+v) - F^i(a) - DF^i(a)v}{\|v\|} = 0, \forall i \\ &\Leftrightarrow \lim_{v \rightarrow 0} \frac{\|F^i(a+v) - F^i(a) - DF^i(a)v\|}{\|v\|} = 0, \forall i. \end{aligned}$$

□

**Proposition 10.** Let  $U \subseteq \mathbb{R}^n$  be open. If  $F : U \rightarrow \mathbb{R}^n$  is of class  $C^1$ , then it is differentiable at each point of  $U$ .

**Proposition 11.** Let  $U \subseteq \mathbb{R}^n$  be an open subset, and suppose  $f, g \in C^\infty(U)$  and  $c \in \mathbb{R}$ .

(a) Then  $f + g$ ,  $cf$ , and  $fg$  are smooth.

(b) If  $g$  never vanishes on  $U$ , then  $f/g$  is smooth.

*Proof.* The result follows immediately by noting that each of the partial derivatives of  $f + g$ ,  $cf$ ,  $fg$  and  $f/g$  of any order are continuous as they can be written as sum, product, quotient of partial derivatives of  $f$  and  $g$  which are assumed to be continuous. □

**Proposition 12 (The Chain Rule for Partial Derivatives).** Let  $U \subseteq \mathbb{R}^n$  and  $\tilde{U} \subseteq \mathbb{R}^m$  be open subsets, and let  $x = (x^1, \dots, x^n)$  denote the standard coordinates on  $U$  and  $y = (y^1, \dots, y^m)$  those on  $\tilde{U}$ .

(a) A composition of  $C^1$  functions  $F : U \rightarrow \tilde{U}$  and  $G : \tilde{U} \rightarrow \mathbb{R}^p$  is again of class  $C^1$ , with partial derivatives given by

$$\frac{\partial(G^i \circ F)}{\partial x^j}(x) = \sum_{k=1}^m \frac{\partial G^i}{\partial y^k}(F(x)) \frac{\partial F^k}{\partial x^j}(x).$$

(b) If  $F$  and  $G$  are smooth, then  $G \circ F$  is smooth.

*Proof.* (a) From the chain rule of total derivative (Prop. 4) and the Jacobian matrix formulation of total derivative (Prop. 8), the matrix of  $D(G \circ F)$  will be the product of the Jacobian matrices of  $G$  and  $F$ . Since  $H = G \circ F : U \rightarrow \mathbb{R}^p$ , the components of  $H = (H^1, \dots, H^p)$  can be written as  $H^i = G^i \circ F : U \rightarrow \mathbb{R}$ . Then we have

$$\begin{aligned} (\partial H^i / \partial x^j) &= (\partial G^i / \partial y^k)(\partial F^k / \partial x^j), \\ \Rightarrow \frac{\partial H^i}{\partial x^j}(x) &= \frac{\partial(G^i \circ F)}{\partial x^j}(x) = \sum_{k=1}^m \frac{\partial G^i}{\partial y^k}(F(x)) \frac{\partial F^k}{\partial x^j}(x). \end{aligned}$$

Then each component  $\partial H^i / \partial x^j$  is continuous because it is sum of product of continuous functions. Thus  $G \circ F$  is also  $C^1$ .

(b) Repeated application of chain rule shows that  $G \circ F$  is smooth. □

**Proposition 13.** Suppose  $A \subseteq \mathbb{R}^n$  and  $B \subseteq \mathbb{R}^m$  are arbitrary subsets, and  $F : A \rightarrow \mathbb{R}^m$  and  $G : B \rightarrow \mathbb{R}^p$  are smooth maps (according to Def. 8) such that  $F(A) \subseteq B$ . Then  $G \circ F : A \rightarrow \mathbb{R}^p$  is smooth.

*Proof.* Let  $x \in A$ , then by smoothness of  $F$ , there exists a neighborhood  $U$  of  $x$  and a smooth map  $\tilde{F} : U \rightarrow \mathbb{R}^m$  such that  $\tilde{F}|_{U \cap A} = F$ . But  $F(x) \in B$ , so by smoothness of  $G$ , we find a neighborhood  $V$  of  $F(x)$  and a smooth map  $\tilde{G} : V \rightarrow \mathbb{R}^p$  such that  $\tilde{G}|_{V \cap B} = G$ . Define  $\tilde{U} = U \cap A \cap F^{-1}(V \cap B)$ . Then  $\tilde{U}$  is a neighborhood of  $x$ , and  $\tilde{G} \circ \tilde{F} : \tilde{U} \rightarrow \mathbb{R}^p$  is a smooth map (by Prop. 12) such that  $\tilde{G} \circ \tilde{F}|_{\tilde{U}} = G \circ F$ .  $\square$

**Definition 10.** Suppose  $f : U \rightarrow \mathbb{R}$  is a smooth real-valued function on an open subset  $U \subseteq \mathbb{R}^n$  and  $a \in U$ . For each vector  $v \in \mathbb{R}^n$ , we define the **directional derivative of  $f$  in the direction  $v$  at  $a$**  to be the number

$$D_v f(a) = \left. \frac{d}{dt} \right|_{t=0} f(a + tv). \quad (5)$$

**Remark 4.** This definition makes sense for any vector  $v$ ; we do not require  $v$  to be a unit vector as one sometimes does in elementary calculus.

**Remark 5.** Since  $D_v f(a)$  is the ordinary derivative of the composite function  $t \mapsto a + tv \mapsto f(a + tv)$ , by chain rule it can be written more concretely as

$$D_v f(a) = \sum_{i=1}^n v^i \frac{\partial f}{\partial x^i}(a) = Df(a)v.$$

**Proposition 14 (Differentiation Under an Integral Sign).** Let  $U \subseteq \mathbb{R}^n$  be an open subset, let  $a, b \in \mathbb{R}$ , and let  $f : U \times [a, b] \rightarrow \mathbb{R}$  be continuous function such that partial derivatives  $\partial f / \partial x^i : U \times [a, b] \rightarrow \mathbb{R}$  exist and are continuous on  $U \times [a, b]$  for  $i = 1, \dots, n$ . Define  $F : U \rightarrow \mathbb{R}$  by

$$F(x) = \int_a^b f(x, t) dt.$$

Then  $F$  is of class  $C^1$ , and its partial derivatives can be computed by differentiating under the integral sign:

$$\frac{\partial F}{\partial x^i}(x) = \int_a^b \frac{\partial f}{\partial x^i}(x, t) dt.$$

For any  $m$ -tuple  $I = (i_1, \dots, i_m)$  of indices with  $1 \leq i_j \leq n$ , we let  $|I| = m$  denote the number of indices in  $I$ , and

$$\partial_I = \frac{\partial^m}{\partial x^{i_1} \dots \partial x^{i_m}},$$

$$(x - a)^I = (x^{i_1} - a^{i_1}) \dots (x^{i_m} - a^{i_m}).$$

**Proposition 15 (Taylor's Theorem).** Let  $U \subseteq \mathbb{R}^n$  be an open subset, and let  $a \in U$  be fixed. Suppose  $f \in C^{k+1}(U)$  for some  $k \geq 0$ . If  $W$  is any convex subset of  $U$  containing  $a$ , then for all  $x \in W$ ,

$$f(x) = P_k(x) + R_k(x), \quad (6)$$

where  $P_k$  is the  **$k$ -th order Taylor polynomial of  $f$  at  $a$** , defined by

$$P_k(x) = f(a) + \sum_{m=1}^k \frac{1}{m!} \sum_{I: |I|=m} \partial_I f(a) (x - a)^I, \quad (7)$$

and  $R_k$  is the  **$k$ -th remainder term**, given by

$$R_k(x) = \frac{1}{k!} \sum_{I: |I|=k+1} (x-a)^I \int_0^1 (1-t)^k \partial_I f(a+t(x-a)) dt. \quad (8)$$

*Proof.* For  $k = 0$  (where we interpret  $P_0$  to mean  $f(a)$ ), this is just the fundamental theorem of calculus (Prop. 14) applied to the function  $u(t) = f(a+t(x-a))$ , together with the chain rule. Assume the result holds for some  $k$ , integration by parts applied to the integral in the remainder term yield

$$\begin{aligned} & \int_0^1 (1-t)^k \partial_I f(a+t(x-a)) dt \\ &= \left[ -\frac{(1-t)^{k+1}}{k+1} \partial_I f(a+t(x-a)) \right]_{t=0}^{t=1} + \int_0^1 \frac{(1-t)^{k+1}}{k+1} \frac{\partial}{\partial t} (\partial_I f(a+t(x-a))) dt \\ &= \frac{1}{k+1} \partial_I f(a) + \frac{1}{k+1} \sum_{j=1}^n (x^j - a^j) \int_0^1 (1-t)^{k+1} \frac{\partial}{\partial x^j} \partial_I f(a+t(x-a)) dt. \end{aligned}$$

When we insert this into (6), we obtain the analogous formula with  $k$  replaced by  $k+1$ .  $\square$

**Corollary.** Suppose  $U \subseteq \mathbb{R}^n$  is an open subset,  $a \in U$ , and  $f \in C^{k+1}(U)$  for some  $k \geq 0$ . If  $W$  is a convex subset of  $U$  containing  $a$  on which all of the  $(k+1)$ -st partial derivatives of  $f$  are bounded in absolute value by a constant  $M$ , then for all  $x \in W$ ,

$$|f(x) - P_k(x)| \leq \frac{n^{k+1} M}{(k+1)!} |x-a|^{k+1},$$

where  $P_k$  is the  $k$ -th Taylor polynomial of  $f$  at  $a$ , defined by (7).

*Proof.* There are  $n^{k+1}$  terms on the right-hand side of (8), each term is bounded in absolute value by  $(1/(k+1)!)|x-a|^{k+1}M$ .  $\square$

**Proposition 16 (Lipschitz Estimate for  $C^1$  Functions).** Let  $U \subseteq \mathbb{R}^n$  be an open subset, and suppose  $F : U \rightarrow \mathbb{R}^m$  is of class  $C^1$ . Then  $F$  is Lipschitz continuous on every compact convex subset  $K \subseteq U$ . The Lipschitz constant can be taken to be  $\sup_{x \in K} \|DF(x)\|$ .

*Proof.* Since  $\|DF(x)\|$  is a continuous function of  $x$ , it is bounded on the compact set  $K$ . Let  $M = \sup_{x \in K} \|DF(x)\|$ . For arbitrary  $a, b \in K$ , we have  $a+t(b-a) \in K$  for all  $t \in [0, 1]$  because  $K$  is convex. By the fundamental theorem of calculus applied to each component of  $F$ , together with the chain rule,

$$\begin{aligned} F(b) - F(a) &= \int_0^1 \frac{d}{dt} F(a+t(b-a)) dt \\ &= \int_0^1 DF(a+t(b-a))(b-a) dt. \\ \implies \|F(b) - F(a)\| &\leq \int_0^1 \|DF(a+t(b-a))\| \|b-a\| dt \\ &\leq \int_0^1 M \|b-a\| dt = M \|b-a\|. \end{aligned}$$

$\square$

**Corollary.** If  $U \subseteq \mathbb{R}^n$  is an open subset and  $F : U \rightarrow \mathbb{R}^m$  is of class  $C^1$ , then  $f$  is locally Lipschitz continuous.

*Proof.* Each point of  $U$  is contained in a ball whose closure is contained in  $U$ , and Prop. 16 shows that the restriction of  $F$  to such a ball is Lipschitz continuous.  $\square$



### 3 The Inverse Function Theorem and Related Results

**Definition 11.** Let  $(X, d)$  be a metric space. A map  $G : X \rightarrow X$  is said to be a **contraction** if there is a constant  $\lambda \in (0, 1)$  such that  $d(G(x), G(y)) \leq \lambda d(x, y)$  for all  $x, y \in X$ . A **fixed point** of a map  $G : X \rightarrow X$  is a point  $x \in X$  such that  $G(x) = x$ .

**Remark 6.** Clearly, every contraction is continuous.

**Proposition 17 (Contraction Lemma).** Let  $X$  be a nonempty complete metric space. Every contraction  $G : X \rightarrow X$  has a unique fixed point.

*Proof.* Uniqueness is immediate, for if  $x, x'$  are both fixed points of  $G$ , the contraction property implies that  $d(x, x') = d(G(x), G(x')) \leq \lambda d(x, x')$ , which is possible only if  $x = x'$ .

To prove the existence of a fixed point, let  $x_0$  be an arbitrary point in  $X$ , and define a sequence  $(x_n)_{n=0}^\infty$  inductively by  $x_{n+1} = G(x_n)$ . For any  $i \geq 1$  we have  $d(x_i, x_{i+1}) = d(G(x_{i-1}), G(x_i)) \leq \lambda d(x_{i-1}, x_i)$ , and therefore by induction  $d(x_i, x_{i+1}) \leq \lambda^i d(x_0, x_1)$ . If  $N$  is a positive integer and  $j \geq i \geq N$ ,

$$\begin{aligned} d(x_i, x_j) &\leq d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) + \dots + d(x_{j-1}, x_j) \\ &\leq (\lambda^i + \dots + \lambda^{j-1})d(x_0, x_1) \leq \lambda^i \left( \sum_{n=0}^{\infty} \lambda^n \right) d(x_0, x_1) \\ &\leq \lambda^N \left( \frac{1}{1-\lambda} \right) d(x_0, x_1), \end{aligned}$$

where we have used that  $\lambda^N \geq \lambda^i$  for  $i \geq N$ . Since the last expression can be made as small as desired by choosing  $N$  large, the sequence  $(x_n)$  is Cauchy and therefore converges to a limit  $x \in X$ . Because  $G$  is continuous,

$$x_n \rightarrow x \implies G(x_n) \rightarrow G(x), \text{ but } G(x) = \lim_{n \rightarrow \infty} G(x_n) = \lim_{n \rightarrow \infty} x_{n+1} = x,$$

so  $x$  is the desired fixed point. □

**Proposition 18 (Inverse Function Theorem).** Suppose  $U, V$  are open subsets of  $\mathbb{R}^n$ , and  $F : U \rightarrow V$  is a smooth function. If  $DF(a)$  is invertible, i.e., Jacobian determinant is nonzero, at some point  $a \in U$ , then there exists connected neighborhoods  $U_0 \subseteq U$  of  $a$  and  $V_0 \subseteq V$  of  $F(a)$  such that  $F|_{U_0} : U_0 \rightarrow V_0$  is a diffeomorphism.

*Proof.* We begin by making some simple modifications to the function  $F$  to streamline the proof. First, the function  $F_1$  defined by  $F_1(x) = F(x + a) - F(a)$  is smooth on a neighborhood of 0 and satisfies  $F_1(0) = 0$  and  $DF_1(0) = DF(a)$ ; clearly,  $F$  is a diffeomorphism on a connected neighborhood of  $a$  iff  $F_1$  is a diffeomorphism on a connected neighborhood of 0. Second, the function  $F_2 = DF_1(0)^{-1} \circ F_1$  is smooth on the same neighborhood of 0 and satisfies  $F_2(0) = 0$  and  $DF_2(0) = I_n$ ; and  $F_2$  is a diffeomorphism on a connected neighborhood of 0 iff  $F_1$  is a diffeomorphism and therefore also  $F$ . Henceforth, replacing  $F$  by  $F_2$ , we assume that  $F$  is defined in a neighborhood  $U$  of 0,  $F(0) = 0$  and  $DF(0) = I_n$ . Because the determinant of  $DF(x)$  is a continuous function of  $x$ , by shrinking  $U$  if necessary, we may assume that  $DF(x)$  is invertible for each  $x \in U$ .

Let  $H(x) = x - F(x)$  for each  $x \in U$ . Then  $DH(0) = I_n - I_n = 0$ . Because the matrix entries of  $DH(x)$  are continuous functions of  $x$ , there is a number  $\delta > 0$  such that  $\mathbb{B}_0(\delta) \subseteq U$  and for all  $x \in \mathbb{B}_0(\delta)$ , we have  $\|DH(x)\| \leq \frac{1}{2}$ . If  $x, x' \in \mathbb{B}_0(\delta)$ , the Lipschitz estimate for smooth functions (Prop. 16) implies that

$$\|H(x) - H(x')\| \leq \frac{1}{2} \|x - x'\|. \quad (9)$$

In particular, taking  $x' = 0$ , this implies

$$\|H(x)\| \leq \frac{1}{2}\|x\|. \quad (10)$$

Since  $x' - x = F(x') - F(x) + H(x') - H(x)$ , it follows that

$$\|x' - x\| \leq \|F(x') - F(x)\| + \|H(x') - H(x)\| \leq \|F(x') - F(x)\| + \frac{1}{2}\|x' - x\|,$$

and rearranging gives

$$\|x' - x\| \leq 2\|F(x') - F(x)\| \quad (11)$$

for all  $x, x' \in \bar{\mathbb{B}}_0(\delta)$ . In particular, this shows that  $F$  is injective on  $\bar{\mathbb{B}}_0(\delta)$ .

Now let  $y \in \mathbb{B}_0(\delta/2)$  be arbitrary. We will show that there exists a unique point  $x \in \mathbb{B}_0(\delta)$  such that  $F(x) = y$ . Let  $G(x) = y + H(x) = y + x - F(x)$ , so that  $G(x) = x$  iff  $F(x) = y$ . If  $\|x\| \leq \delta$ , (10) implies

$$\|G(x)\| \leq \|y\| + \|H(x)\| < \frac{\delta}{2} + \frac{1}{2}\|x\| \leq \delta, \quad (12)$$

so  $G$  maps  $\bar{\mathbb{B}}_0(\delta)$  to itself. It follows from (9) that  $\|G(x') - G(x)\| = \|H(x) - H(x')\| \leq \frac{1}{2}\|x - x'\|$ , so  $G$  is a contraction. Since  $\bar{\mathbb{B}}_0(\delta)$  is a complete metric space, the contraction lemma implies that  $G$  has a unique fixed point  $x \in \bar{\mathbb{B}}_0(\delta)$ . From (12),  $\|x\| = \|G(x)\| < \delta$ , so in fact  $x \in \mathbb{B}_0(\delta)$ , thus proving the claim.

Let  $V_0 = \mathbb{B}_0(\delta/2)$  and  $U_0 = \mathbb{B}_0(\delta) \cap F^{-1}(V_0)$ . Then  $U_0$  is open in  $\mathbb{R}^n$ , and the argument above shows that  $F : U_0 \rightarrow V_0$  is bijective, so  $F^{-1} : V_0 \rightarrow U_0$  exists. Substituting  $x = F^{-1}(y)$  and  $x' = F^{-1}(y')$  into (11) shows that  $F^{-1}$  is continuous. Thus  $F : U_0 \rightarrow V_0$  is a homeomorphism, and it follows that  $U_0$  is connected because  $V_0$  is.

The only thing that remains to be proved is that  $F^{-1}$  is smooth. If we knew it were smooth, Prop. 6 would imply that  $D(F^{-1})(y) = DF(x)^{-1}$ , where  $x = F^{-1}(y)$ . We begin by showing that  $F^{-1}$  is differentiable to each point of  $V_0$ , with total derivative given by this formula.

Let  $y \in V_0$  be arbitrary, and set  $x = F^{-1}(y)$  and  $L = DF(x)$ . We need to show that

$$\lim_{y' \rightarrow y} \frac{F^{-1}(y') - F^{-1}(y) - L^{-1}(y' - y)}{\|y' - y\|} = 0.$$

Given  $y' \in V_0 - \{y\}$ , write  $x' = F^{-1}(y') \in U_0 - \{x\}$ . Then

$$\begin{aligned} \frac{F^{-1}(y') - F^{-1}(y) - L^{-1}(y' - y)}{\|y' - y\|} &= L^{-1} \left( \frac{L(x' - x) - (y' - y)}{\|y' - y\|} \right) \\ &= \frac{\|x' - x\|}{\|y' - y\|} L^{-1} \left( - \frac{F(x') - F(x) - L(x' - x)}{\|x' - x\|} \right). \end{aligned}$$

The factor  $\|x' - x\|/\|y' - y\|$  above is bounded due to (11), and because  $L^{-1}$  is linear and therefore bounded,  $\|L^{-1}\|$  is bounded. As  $y' \rightarrow y$ , it follows that  $x' \rightarrow x$  by continuity of  $F^{-1}$ , and then the term in the bracket of last equation goes to zero because  $L = DF(x)$  and  $F$  is differentiable. This complete the proof that  $F^{-1}$  is differentiable.

By Prop. 8, the partial derivatives of  $F^{-1}$  are defined at each point  $y \in V_0$ . Observe that the formula  $D(F^{-1})(y) = DF(F^{-1}(y))^{-1}$  implies that the matrix-valued function  $y \mapsto D(F^{-1})(y)$  can be written as the composition

$$y \xrightarrow{F^{-1}} F^{-1}(y) \xrightarrow{DF} DF(F^{-1}(y)) \xrightarrow{i} DF(F^{-1}(y))^{-1}, \quad (13)$$

where  $i$  is the matrix inversion. In the composition,  $F^{-1}$  is continuous;  $DF$  is smooth because its component functions are the partial derivatives of  $F$ ; and  $i$  is smooth because Cramer's rule expresses the entries of an inverse matrix as rational functions of entries of the matrix. Because

$D(F^{-1})$  is composition of continuous functions, it is continuous. Thus, the partial derivatives of  $F^{-1}$  are continuous, so  $F^{-1}$  is of class  $C^1$ .

Now assume by induction that we have shown that  $F^{-1}$  is of class  $C^k$ . This means that each of the functions in (13) is of class  $C^k$ . Because  $D(F^{-1})$  is a composition of  $C^k$  functions, it is itself  $C^k$ ; this implies that partial derivatives of  $F^{-1}$  are of class  $C^k$ , so  $F^{-1}$  itself is of class  $C^{k+1}$ . Continuing by induction, we conclude that  $F^{-1}$  is smooth.  $\square$

**Corollary.** Suppose  $U \subseteq \mathbb{R}^n$  is an open subset, and  $F : U \rightarrow \mathbb{R}^m$  is a smooth function whose Jacobian determinant is nonzero at every point in  $U$ . Then

(a)  $F$  is an open map.

(b) If  $F$  is injective, then  $F : U \rightarrow F(U)$  is a diffeomorphism.

*Proof.* (a) For each  $a \in U$ , the fact that the Jacobian determinant of  $F$  is nonzero implies that  $DF(a)$  is invertible, so the inverse function theorem implies that there exists open subsets  $U_a \subseteq U$  containing  $a$  and  $V_a \subseteq F(U)$  containing  $F(a)$  such that  $F$  restricts to a diffeomorphism  $F|_{U_a} : U_a \rightarrow V_a$ . In particular, this means that each point of  $F(U)$  has a neighborhood contained in  $F(U)$ , so  $F(U)$  is open. If  $U_0 \subseteq U$  is an arbitrary open subset, the same argument with  $U$  replaced by  $U_0$  shows that  $F(U_0)$  is also open.

(b) If  $F$  is injective, then the inverse map  $F^{-1} : F(U) \rightarrow U$  exists; on a neighborhood of each point  $F(a) \in F(U)$   $F^{-1}$  defined above is equal to the inverse of  $F|_{U_a}$ , so it is smooth.  $\square$

**Proposition 19 (Implicit Function Theorem).** Let  $U \subseteq \mathbb{R}^n \times \mathbb{R}^k$  be an open subset, and let  $(x, y) = (x^1, \dots, x^n, y^1, \dots, y^k)$  denote the standard coordinates on  $U$ . Suppose  $\Phi : U \rightarrow \mathbb{R}^k$  is a smooth function,  $(a, b) \in U$ , and  $c = \Phi(a, b)$ . If the  $k \times k$  matrix  $(\partial\Phi^i(a, b)/\partial y^j)$  is nonsingular, then there exists neighborhoods  $V_0 \subseteq \mathbb{R}^n$  of  $a$  and  $W_0 \subseteq \mathbb{R}^k$  of  $b$  and a smooth function  $F : V_0 \rightarrow W_0$  such that  $\Phi^{-1}(c) \cap (V_0 \times W_0)$  is the graph of  $F$ , i.e.,  $\Phi(x, y) = c$  for  $(x, y) \in V_0 \times W_0$  iff  $y = F(x)$ .

*Proof.* Consider the smooth function  $\Psi : U \rightarrow \mathbb{R}^n \times \mathbb{R}^k$  defined by  $\Psi(x, y) = (x, \Phi(x, y))$ . Its total derivative at  $(a, b)$  is

$$D\Psi(a, b) = \begin{pmatrix} I_n & 0 \\ \frac{\partial\Phi^i}{\partial x^j}(a, b) & \frac{\partial\Phi^i}{\partial y^j}(a, b) \end{pmatrix},$$

which is nonsingular because it is block lower triangular and the two blocks on the main diagonal are nonsingular. Thus by inverse function theorem there exists connected neighborhood  $U_0$  of  $(a, b)$  and  $Y_0$  of  $(a, c)$  such that  $\Psi : U_0 \rightarrow Y_0$  is a diffeomorphism. Since  $\Psi : U_0 \rightarrow Y_0$  is defined by  $\Psi(x, y) = (x, \Phi(x, y))$ , the inverse map  $\Psi^{-1} : Y_0 \rightarrow U_0$  will be of the form  $\Psi^{-1}(x, y) = (x, B(x, y))$  for smooth function  $B : Y_0 \rightarrow \mathbb{R}^k$ . Shrinking  $U_0$  and  $Y_0$  if necessary, we may assume that  $U_0 = V \times W$  is a product neighborhood.

The two compositions  $\Psi \circ \Psi^{-1}$  and  $\Psi^{-1} \circ \Psi$  give

$$\begin{aligned} (x, y) &= (\Psi \circ \Psi^{-1})(x, y) = \Psi(x, B(x, y)) = (x, \Phi(x, B(x, y))), \forall (x, y) \in Y_0 \\ (x, y) &= (\Psi^{-1} \circ \Psi)(x, y) = \Psi^{-1}(x, \Phi(x, y)) = (x, B(x, \Phi(x, y))) \forall (x, y) \in U_0. \end{aligned} \quad (14)$$

If  $\Phi(x, y) = c$ , then the second equation of (14) gives  $y = B(x, c)$ . This suggests that we define  $F(x) = B(x, c)$  for all  $x \in \mathbb{R}^n$  for which  $(x, c) \in Y_0$ . Now let  $V_0 = \{x \in V : (x, c) \in Y_0\}$  and  $W_0 = W$ , then  $F : V_0 \rightarrow W_0$  defined by  $F(x) = B(x, c)$ .

Let  $x \in V_0$ . If  $\Phi(x, y) = c$  then  $y = B(x, c) = F(x)$ , so the graph of  $F$  is contained in  $\Phi^{-1}(c)$ . Conversely, suppose  $y = F(x)$  and in the first equation of (14) we set  $(x, y) = (x, c)$ , then  $c = \Phi(x, B(x, c)) = \Phi(x, F(x)) = \Phi(x, y)$ . This completes the proof.  $\square$

**Proposition 20.** *The implicit function theorem is equivalent to the inverse function theorem.*

*Proof.* ( $\implies$ ) Already shown above.

( $\impliedby$ ) Let  $F : U \rightarrow V$  be a smooth map defined such that  $U, V \subseteq \mathbb{R}^n$  are open subsets such that at some point  $p \in U$  the Jacobian determinant is nonzero. Finding a local inverse for  $y = F(x)$  near  $p$  amounts to solving the equation  $G(x, y) = F(x) - y = 0$  for  $x$  in terms of  $y$  near  $(p, F(p))$ . Note that  $\partial G^i / \partial x^j = \partial F^i / \partial x^j$ . Hence,

$$\det \left[ \frac{\partial G^i}{\partial x^j}(p, F(p)) \right] = \det \left[ \frac{\partial F^i}{\partial x^j}(p, F(p)) \right] \neq 0.$$

By the implicit function theorem,  $x$  can be expressed in terms of  $y$  locally near  $(p, F(p))$ , i.e., there is a smooth function  $x = H(y)$  defined in a neighborhood of  $F(p)$  in  $\mathbb{R}^n$  such that  $G(x, y) = F(x) - y = F(H(y)) - y = 0$ . Thus,  $y = F(H(y))$ . Since  $y = F(x)$ ,  $x = H(y) = H(F(x))$ . Therefore,  $F$  and  $H$  are inverse functions defined near  $p$  and  $F(p)$  respectively and  $H$  is smooth by implicit function theorem.  $\square$

## References

- [1] John M. Lee, Introduction to Smooth Manifolds.