# Set Theory $\bigcap$ Functional Analysis

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## 1 Basic Definitions

**Definition 1.** *A* ***choice function*** *$f$ on a collection $\mathcal{C}$ of set $X$ is a function such that for all $A \in \mathcal{C}$, $f(A) \in A$.*

**Example.** *Consider a collection $\{\{1,2\},\{3,4\}\}$ and a function $f$ defined as $f(\{1,2\}) = 2$ and $f(\{3,4\}) = 3$. Then $f$ is a choice function.*

**Definition 2.** *Suppose $I$ is a set, called as the* ***index set****, and with each $i \in I$ we associate a set $A_i$. Then, $\{A_i : i \in I\}$ is defined as the* ***family of sets****. This can also be denoted by $\{A_i\}_{i \in I}$*

**Definition 3.** *A* ***partially ordered set*** *is a set together with a partial order on it $(X, \preccurlyeq)$ where partial order on $X$ is defined as a relation $\preccurlyeq$ in $X$ such that, for all $x, y, z \in X$ it follows*

1. ***Reflexive.*** *$x \preccurlyeq x$*
2. ***Anti-symmetric.*** *If $x \preccurlyeq y$ and $y \preccurlyeq x$ then $x = y$*
3. ***Transitive.*** *If $x \preccurlyeq y$ and $y \preccurlyeq z$, then $x \preccurlyeq z$*

**remark.** *If $x \preccurlyeq y$ and $x \neq y$, then we write $x \prec y$ and say that $x$ is* ***smaller than*** *$y$. It is not necessary for all $x, y \in X$ to have a partial order defined between them.*

**Definition 4.** *A set together with a total order on it is a* ***chain*** *or* ***totally ordered set*** *where a relation $\preccurlyeq$ is* ***total order*** *if for every $x, y \in X$ either $x \preccurlyeq y$ or $y \preccurlyeq x$, consequently the set is called as a totally ordered set.*

**Definition 5.** *Let $X$ be a partially ordered set, then an element $a \in X$ is the* ***upper bound*** *of a subset $E \subseteq X$ if $x \preccurlyeq a$ for all $x \in E$.*

**Definition 6.** *Let $X$ be a partially ordered set, then an element $a \in X$ is* ***maximal*** *if $a \preccurlyeq x$ implies $x = a$.*

**Definition 7.** *Let $X$ be a partially ordered set, then an element $a \in X$ is* ***maximum (or largest)*** *if $x \preccurlyeq a$ $\forall x \in X$.*

**Example.** *Consider the set $W = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}\}$ with set inclusion $\subseteq$ as a partial ordering. The maximal elements are $\{1,2\}$ and $\{3\}$. If we view $W$ as a subset of the power set of $\{1,2,3\}$, then the upper bound of $W$ is the element $\{1,2,3\}$.*

**Definition 8.** *A poset $P$ is called* ***well-ordered*** *if it is a chain, and every non-empty subset $S \subseteq P$ has a minimum.*

**Definition 9.** *For a vector space $X$, a set $B \subseteq X$ is called a* ***basis*** *(or* ***Hamel basis****) if $B$ is a linearly independent set and span$(B) = X$.*

**Definition 10.** *Let $X$ be a linear space. A function $p : X \to \mathbb{R}$ is a* ***Sublinear Functional*** *if the following properties hold*

1. ***Subadditive.*** *$p(x + y) \leq p(x) + p(y)$ $\forall x, y \in X$.*
2. ***Nonnegatively Homogeneous.*** *$p(\lambda x) = \lambda p(x)$ $\forall \lambda \geq 0$ where $\lambda \in \mathbb{R}, x \in X$*

**Example.** *Norm is an example of sublinear functional which is not linear.*

## 2 Theorem statements

Two formulations of ***Axiom of Choice*** are given.

- The Cartesian product of a non-empty family of non-empty sets is non-empty.
- For every non-empty set X, there exists a choice function f defined on X.

**remark.** *The equivalence proof of the above mentioned variants is skipped. In the further discussion regarding Axiom of Choice, we will be using the choice function formulation.*

**Zorn's Lemma.** *If $X$ is a non-empty partially ordered set such that every chain in $X$ has an upper bound, then $X$ contains a maximal element.*

**Well-ordering principle.** *Every set has a well ordering.*

**Existence of Hamel Basis.** *Every vector space $X \neq \{0\}$ has a basis.*

**Hahn-Banach Theorem.** *Let $X$ be a real vector space and $p$ a sublinear functional on $X$. Furthermore, let $f$ be a linear functional which is defined on a subspace $Z$ of $X$ and satisfies*

$$f(x) \leq p(x) \ \forall x \in Z$$

*Then $f$ has a linear extension of $\tilde{f}$ from $Z$ to $X$ satisfying*

$$\tilde{f}(x) \leq p(x) \ \forall x \in X$$

*that is, $\tilde{f}$ is a linear functional on $X$, satisfying above inequality on $X$ and $\tilde{f}(x) = f(x)$ for every $x \in Z$.*

## 3 Proof of Equivalences

**Zorn's Lemma $\iff$ Axiom of Choice $\iff$ Well-Ordering Principle**

**Theorem 1.** *Zorn's Lemma $\implies$ Axiom of Choice.*

*Proof.* Let $X$ be any non-empty set. Consider a set $P$

$$P = \{(Y, f) : \ Y \subseteq X \text{ and f is choice function on } Y\}.$$

We define a relation $\preccurlyeq$ on P as $(Y, f) \preccurlyeq (Y', f')$ whenever $Y \subseteq Y'$ and $f = f'|_Y$. It is easy to see that $(P, \preccurlyeq)$ is a poset. Note that P is non-empty as for any element $x \in X$, $\{x\} \mapsto x$ is a choice function, consequently $(\{x\}, \{x\} \mapsto x) \in P$.

Consider a chain $C$ in $P$. We define $\tilde{Y} = \bigcup_{(Y,f)\in C} Y$ and $\tilde{f}(S) = f(S)$ for any $S$ such that $f$ is defined on $S$. Notice that $(\tilde{Y}, \tilde{f})$ is an upper bound for $C$ by construction. Since $C$ was chosen arbitrarily and was shown to have an upper bound, then by Zorn's Lemma there is some maximal element in P, say $(Y^*, f^*)$.

Now we show that $Y^* = X$. Suppose not then there is some element $x \in X \setminus Y^*$. We can extend $f^*$ to $f^{**}$ from $Y^*$ to $Y^* \cup \{x\}$ by defining $f^{**}(S) = x$, if $x \in S$ and $f^{**}(S) = f^*(S)$, if $S \subseteq Y^*$. Then we note that $(Y^*, f^*) \preccurlyeq (Y^* \cup \{x\}, f^{**})$ which is a contradiction. Hence, $Y^* = X$ and $f^*$ is the required choice function on $X$. $\square$

**Theorem 2.** *Zorn's Lemma $\implies$ Well-ordering principle.*

*Proof.* The proof is similar to that of Theorem 1. Let $X$ be any non-empty set. Consider a relation $(P, \preccurlyeq)$, where P is defined as

$$P = \{(Y, \leq_Y) : \ Y \subseteq X \text{ and } \ \leq_Y \ \text{ is a well-ordering on } Y\},$$

and $(Y, \leq_Y) \preccurlyeq (Y', \leq_{Y'})$ whenever $Y \subseteq Y'$ and $\leq_Y$ and $\leq_{Y'}$ agree on $Y$. It is easy to see that $(P, \preccurlyeq)$ is a poset. Note that P is non-empty as every singleton set is well-ordered.

On similar lines of Theorem 1 proof, we conclude that there exists some maximal element in P, say $(Y^*, \leq_{Y^*})$.

Now we show that $Y^* = X$. Suppose not then there is some element $x \in X \setminus Y^*$. We can extend $(Y^*, \leq_{Y^*})$ to $Y^* \cup \{x\}$ by defining $x$ to be greater than every element in $Y^*$. This is a contradiction that $(Y^*, \leq_{Y^*})$ is maximal element. Hence, $Y^* = X$ and $\leq_{Y^*}$ is the required well-ordering on $X$. □

**Theorem 3.** *Well-ordering principle $\implies$ Axiom of choice.*

*Proof.* Suppose $X$ is a non-empty set, and $\leq$ is a well-ordering of $X$. Then $f(S) = \min S$, defines a choice function on $X$ which is guaranteed to exist for any set $S$ by Well-ordering principle. □

**Theorem 4.** *Axiom of Choice $\implies$ Zorn's Lemma.*

*Proof.* Let's assume there exist a non-empty partially ordered set $P$ such that every chain in $P$ has an upper bound, but does not contain a maximal element.

Considering axiom of choice is true, there must exist a choice function $f$ on $P$, and let $x_0 := f(P)$.

Also, let the set of *strict* upper bounds on a chain $C$ in $P$ be

$$Upp(C) := \{u \notin C : \forall x \in C, x \prec u\}$$

**Lemma 5.** *For any chain $C$, the set $Upp(C)$ is non-empty.*

*Proof.* As $C$ is a chain in $P$, therefore there exists an upper bound $u$ for $C$. There can be two cases,

1. $C$ does not have any maximum element, then $u \notin C$ and $u \in Upp(C)$ must be true by definitions.

2. $C$ contains a maximum element, let's say $m$. Since $P$ has no maximal element (assumed), there exist a $u$ greater than $m$. Then $x \preccurlyeq m \prec u$ for each $x \in C$, and hence $u \in Upp(C)$.

Hence, in both cases $Upp(C)$ is non-empty for any chain $C$ in $P$. □

A sub-chain $C'$ is an initial segment of a chain $C$ such that $x \in C, y \in C'$ and $x \prec y$ implies $c \in C'$. **Intuition:** For all $y \in C \setminus C'$, and for all $x \in C'$, $x \prec y$. Now, let's define a function $g$, such that for any chain $C$,

$$g(C) := f(Upp(C))$$

Also, let's define an ***attempt*** as a well ordered set $A \subset P$ satisfying following:

1. $\min A = x_0$

2. For every proper initial segment $C \subset A$, $\min A \setminus C = g(C)$

**Lemma 6.** *If $A$ and $A'$ are two attempts, then either $A \subseteq A'$ or $A' \subseteq A$.*

*Proof.* Let's assume that both $A \subseteq A'$ and $A' \subseteq A$ does not apply, and let $z = \min A \setminus A'$ and $z' = \min A' \setminus A$. As both $A$ and $A'$ are attempts (and hence well-ordered by definition). Since $z \neq z'$, $z \preccurlyeq z'$ and $z' \preccurlyeq z$ can not be true together. So, let's assume wlog $z' \not\preccurlyeq z$. Let's define a set $C = \{x \in A : x \prec z\}$. From the definitions of $z$ it follows that $C \subseteq A$. It is clear from this that $z = \min A \setminus C$, and so $z = g(C)$. There are now two cases possible.

1. $C = A'$. Now, as $C \subseteq A$ therefore $A' \subseteq A$. Hence, the given lemma is true in this case.
2. $C \neq A'$. If $z' \preccurlyeq x$ for some $x \in C$, then transitivity of partial order implies $z' \prec z$, which is a contradiction. So, since $A'$ is a chain (as it is well-ordered), $x \preceq z'$ $\forall x \in C$. therefore $C$ is a proper initial segment of $A'$ which implies $g(C) \in A'$. But, $g(C) = z \notin A'$. Therefore a contradiction. Hence,the given lemma is true.

□

As, for any two attempts $A, A'$ either $A \subseteq A'$ or $A' \subseteq A$, therefore $A \cup A'$ is either $A$ or $A'$ which is an attempt. Let $\mathcal{A}$ be the set of all attempts then $A := \bigcup_{\tilde{A} \in \mathcal{A}} \tilde{A}$. Then $A$ is also an attempt.

However, $A \cup \{g(A)\}$ is also an attempt and must have belonged in the previous set of attempts $\mathcal{A}$, and also $A \subseteq A \cup \{g(A)\}$ therefore $A \cup \{g(A)\} := \bigcup_{A \in \mathcal{A}} A$ but this is not the case, therefore a contradiction. And, hence there must exist a maximal element of $P$. □

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**Theorem 7.** *Zorn's Lemma* $\implies$ *"Every vector space $X \neq \{0\}$ has a basis".*

*Proof.* Let $X$ be a non-empty vector space. We define a relation $(P, \preccurlyeq)$ where $P$ is the set of subsets of $X$ which are linearly independent and for every $B, B' \in P$, $B \preccurlyeq B'$ whenever $B \subseteq B'$. It is easy to note that $(P, \subseteq)$ is a poset. Notice that $P \neq \emptyset$ as $X \neq \{0\}$, there is some non-zero element $x \in X$, consequently $B = \{x\} \in P$.

Consider a chain $C$ in $P$. Define $\tilde{B} = \bigcup_{B \in C} B$. Notice that $\tilde{B}$ is an upper bound for $C$, hence by Zorn's Lemma there exists a maximal element in $P$, say $B^*$.

We show that $\text{span}(B^*) = X$. Suppose not, then there is an element $x \in X \setminus \text{span}(B^*)$ and $x \neq 0$. Then extend the set $B^*$ by including $x$ in it. Notice that the extended set is an element in $P$ which is greater than $B^*$ under the subset relation. This is a contradiction. Hence $\text{span}(B^*) = X$ and $B^*$ is a linearly independent set, thus $B^*$ is a basis for $X$. □

**remark.** *A variant of converse of the above result is also true which is*

***"Every vector space $X \neq \{0\}$ has a basis"*** $\implies$ ***Axiom of Choice,***

*thus establishing equivalence between the two. The proof can be found in **[1]** which shows the implication for **Axiom of Multiple Choice** instead. It is known that Axiom of Multiple Choice is equivalent to Axiom of Choice. We skip this proof as it is quite involved.*

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**Theorem 8.** *Zorn's Lemma* $\implies$ *Hahn-Banach Theorem*

*Proof.* Let's proof this in 3 parts,

(A) Let's define $M$ as the partial order set of pairs $(Z, f_Z)$ where

   (a) $Z$ is a subspace of $X$ containing $Y$.

   (b) $f_Z : Z \to \mathbb{R}$ is a linear functional extending $f$, satisfying

$$f_Z(z) \leq p(z) \ \forall \ z \in Z$$

with partial ordering defined as $(Z_1, f_{Z_1}) \preccurlyeq (Z_2, f_{Z_2})$ if $Z_1 \subset Z_2$ and $(f_{Z_2})|_{Z_1} = f_{Z_1}$. Since, $(Y, f) \in M$, $M$ is a non-empty set. Let's choose any arbitrary chain $C = \{(Z_\alpha, f_{Z_\alpha})\}_{\alpha \in \Lambda}$ in $M$, with $\Lambda$ being some indexing set.

**Lemma 9.** *$C$ has an upper bound in $M$.*

*Proof.* Let $W = \bigcup_{\alpha \in \Lambda} Z_\alpha$ and construct a functional $f_W : W \implies \mathbb{R}$ defined as follow: If $w \in W$, then $w \in Z_\alpha$ for some $\alpha \in \Lambda$ and we set $f_W(w) = f_{Z_\alpha}(w)$ for that particular $\alpha$.

- This definition is well-defined. Indeed, suppose $w \in Z_\alpha$ and $w \in Z_\beta$. If $Z_\alpha \subset Z_\beta$, then $f_{Z_\beta}|_{Z_\alpha} = f_{Z_\alpha}$, since they are a part of chain.
- $W$ clearly contains $Y$, and we show that $W$ is a subspace of $X$ and $f_W$ is a linear functional on $W$. Choose any $w_1, w_2 \in W$, then $w_1 \in Z_{\alpha_1}, w_2 \in Z_{\alpha_2}$ for some $\alpha_1, \alpha_2 \in \lambda$. If $Z_{\alpha_1} \subset Z_{\alpha_2}$, say, then for any scalars $\beta, \gamma \in \mathbb{R}$ we have

  $$w_1, w_2 \in Z_{\alpha_2} \implies \beta w_1 + \gamma w_2 \in Z_{\alpha_2} \subset W$$

  Also, with $f_W(u) = f_{Z_{\alpha_1}}(u)$ and $f_W(v) = f_{Z_{\alpha_2}}(v)$,

  $$\begin{aligned} f_w(\beta u + \gamma v) &= f_{Z_{\alpha_2}}(\beta u + \gamma v) \\ &= \beta f_{Z_{\alpha_2}}(u) + \gamma f_{Z_{\alpha_2}}(v) \text{ linearity} \\ &= \beta f_{Z_{\alpha_1}}(u) + \gamma f_{Z_{\alpha_2}}(v) \text{ because in same chain} \\ &= \beta f_{Z_W}(u) + \gamma f_{Z_W}(v) \end{aligned}$$

  The case $Z_{\alpha_2} \subset Z_{\alpha_1}$ follows from a symmetric argument.
- Choose any $w \in W$,then $w \in Z_\alpha$ for some $\alpha \in \Lambda$ and

  $$f_W(w) = f_{Z_\alpha}(w) \leq p(w) \text{ since } (w, Z_\alpha) \in M$$

Hence, $(W, f_W)$ is an element of $M$ and an upper bound of $C$ since $(Z_\alpha, f_{Z_\alpha}) \leq (W, f_W)$ for all $\alpha in \Lambda$. Since $C$ was an arbitrary chain in $M$, by Zorn's lemma, $M$ has a maximal element $(Z, f_Z) \in M$, and $f_Z$ is (by definition) a linear extension of $f$ satisfying $f_Z(z) \leq p(z)$ for all $z \in Z$. □

(B) The proof is complete if we can show that $Z = X$. Suppose not, then there exists an $\theta \in X \backslash Z$; note $\theta \neq 0$ since $Z$ is a subspace of $X$. Consider the subspace $Z_\theta = \text{span}\{Z, \{\theta\}\}$. Any $x \in Z_\theta$ has a unique representation $x = z + \alpha\theta$, $z \in Z$, $\alpha \in \mathbb{R}$. Indeed, if

$$x = z_1 + \alpha_1\theta = z_2 + \alpha_2\theta, z_1, z_2 \in Z, \alpha_1, \alpha_2 \in \mathbb{R}$$

then $z_1 - z_2 = (\alpha_2 - \alpha_1)\theta \in Z$ since $Z$ is a subspace of $X$. Since $\theta \notin Z$, we must have $\alpha_2 - \alpha_1 = 0$ and $z_1 - z_2 = \theta$. Next, we construct a functional $f_{Z_\theta} : Z_\theta \to \mathbb{R}$ defined by

$$f_{Z_\theta}(x) = f_{Z_\theta}(z + \alpha\theta) = f_Z(z) + \alpha\delta, \ \ldots \ (1)$$

where $\delta$ is any real number. It can be shown that $f_{Z_\theta}$ is linear and $f_{Z_\theta}$ is a proper linear extension of $f_Z$ ; indeed, we have, for $\alpha = 0, f_{Z_\theta}(x) = f_{Z_\theta}(z) = f_Z(x)$. Consequently, if we can show that

$$f_{Z_\theta}(x) \leq p(x) \ \forall x \in Z_\theta \ \ldots \ (2)$$

then $(Z_\theta, f_{Z_\theta}) \in M$ satisfying $(Z, f_Z) \leq (Z_\theta, f_{Z_\theta})$, thus contradicting the maximality of $(Z, f_Z)$.

(C) From (1), observe that (2) is trivial if $\alpha = 0$, so suppose $\alpha \neq 0$. We do have a single degree of freedom, which is the parameter $\delta$ in (1), thus the problem reduces to showing the existence of a suitable $\delta \in \mathbb{R}$ such that (2) holds. Consider any $x = z + \alpha\theta \in Z_\theta, z \in Z, \alpha \in \mathbb{R}$. Assuming $\alpha > 0$,(2) is equivalent to

$$\begin{aligned} f_Z(z) + \alpha\delta &\leq p(z + \alpha\theta) = \alpha p(z/\alpha + \theta) \\ f_Z(z/\alpha) + \delta &\leq p(z/\alpha + \theta) \\ \delta &\leq p(z/\alpha + \theta) - f_Z(z/\alpha) \end{aligned}$$

Since the above must holds for all $z \in Z, \alpha \in \mathbb{R}$, we need to choose $\delta$ such that

$$\delta \leq \inf_{z_1 \in Z} (p(z_1 + \theta) - f_Z(z_1)) = m_1 \ ... \ (3)$$

Assuming $\alpha < 0$, (2) is equivalent to

$$\begin{aligned} f_Z(z) + \alpha\delta &\leq p(z + \alpha\theta) = -\alpha p(-z/\alpha - \theta) \\ -f_Z(z/\alpha) - \delta &\leq p(-z/\alpha - \theta) \\ \delta &\geq -p(-z/\alpha - \theta) - f_Z(z/\alpha) \end{aligned}$$

Since the above must holds for all $z \in Z, \alpha \in \mathbb{R}$, we need to choose $\delta$ such that

$$\delta \geq \sup_{z_2 \in Z} (-p(z_2 + \theta) - f_Z(z_2)) = m_0 \ ... \ (4)$$

We are left with showing condition (3), (4) are compatible, i.e

$$-p(-z_2 - \theta) - f_Z(z_2) \leq p(z_1 + \theta) - f_Z(z_1) \ \forall z_1, z_2 \in Z$$

The inequality above is trivial if $z_1 = z_2$, so suppose not. We have that

$$\begin{aligned} p(z_1 + \theta) - f_Z(z_1) + p(-z_2 - \theta) + f_Z(z_2) &= p(z_1 + \theta) + p(-z_2 - \theta) + f_Z(z_2 - z_1) \\ &\geq f_Z(z_2 - z_1) + p(z_1 + \theta - z_2 - \theta) \\ &= f_Z(z_2 - z_1) + p(z_1 - z_2) \\ &= -f_Z(z_1 - z_2) + p(z_1 - z_2) \geq 0 \end{aligned}$$

where linearity of $f_Z$ and subadditivity of $p$ are used. Hence, the required condition on $\delta$ is $m_0 \leq \delta \leq m_1$

Therefore, by using Zorn's Lemma (as using in 1st part) we proved Hahn-Banach Theorem. □

**remark.** *The converse of the above theorem is not true, i.e, Hahn-Banach Theorem $\not\Rightarrow$ Zorn's Lemma. It is known that Hahn-Banach Theorem is equivalent to a statement which is strictly weaker than Axiom of Choice.[3]*

## References


[1] Existence of Basis implies AC

[2] Axiom of Choice equivalents

[3] Wikipedia, Hahn-Banach Theorem

[4] P. Halmos, Naive set theory. New York, 1974.

[5] Hahn-Banach Theorem