Topology

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Contents

1 Topological Spaces **2**

2 Bases and Countability **5**

3 Hausdorff Spaces and Convergent Sequences **7**

4 Continuous Maps **8**

5 Subspaces **10**

6 Product Spaces **13**

7 Disjoint Union Spaces **15**

8 Quotient Spaces and Quotient Maps **16**

9 Open and Closed Maps **18**

10 Connectedness **20**

11 Compactness **25**

1 Topological Spaces

Definition 1. *Let X be a set. A* ***topology on X*** *is a collection $\mathcal{T}$ of subsets of X, called the* ***open subsets****, satisfying*

1. *X and $\emptyset$ are open.*
2. *The union of any family of open subsets is open.*
3. *The intersection of any finite family of open subsets is open.*

The pair $(X, \mathcal{T})$ consisting of a set X together with a topology $\mathcal{T}$ on X is called a ***topological space****.*

When the topology is understood, one omits mention of it and simply says "X is a topological space".

Definition 2. *Suppose X is a topological space, $p \in X$, and $S \subseteq X$. A* ***neighborhood of p*** *is an open subset containing p. Similarly, a* ***neighborhood of the set S*** *is an open subset containing S.*

Proposition 1 (***Local criterion for open set***)**.** *Let X be a topological space. A subset U is open iff for every $p \in U$, there is an open set V such that $p \in V \subseteq U$.*

Proof. ($\implies$)For any $p \in U$, take $V = U$ since U is open.
($\impliedby$)For every $p \in U$ there exists an open set $V_p \subseteq U$. Then $U = \bigcup_{p \in U} V_p$ which is open. □

Definition 3. *Suppose X is a topological space, and $S \subseteq X$. S is said to be* ***closed*** *if $X \setminus S$ is open.*

Definition 4. *Suppose X is a topological space, and $S \subseteq X$. The* ***interior of S****, denoted by Int S, is the union of all open subsets of X contained in S.*

Definition 5. *Suppose X is a topological space, and $S \subseteq X$. The* ***closure of S****, denoted by $\bar{S}$, is the intersection of all closed subsets of X containing S.*

Proposition 2 (***Local criterion of closure***)**.** *Let S be a subset of a topological space X. A point $p \in X$ is in $\bar{S}$ iff every neighborhood of p contains a point of S.*

Proof. We will prove the contrapositive of the statement, i.e., $p \notin \bar{S}$ iff there is a neighborhood of p disjoint from S.

($\implies$)$p \notin \bar{S} = \bigcap\{F : S \subseteq F \text{ and } F \text{ is closed}\}$. Then $p \notin F$ for atleast one closed set containing S. It follows that $p \in X \setminus F$, an open set disjoint from S.

($\impliedby$)Suppose $p \in U$, an open set disjoint from S. Then the complement $X \setminus U$ is a closed set containing S and not p, therefore $p \notin \bar{S}$. □

Example 1 (***Discrete Spaces***)**.** *If X is an arbitrary set, the* ***discrete topology*** *on X is the topology defined by declaring every subset of X to be open. Any space that has the discrete topology is called the* ***discrete space****.*

Example 2 (***Metric Spaces***)**.** *A* ***metric space*** *is a set M endowed with a* ***metric*** *$d : M \times M \to \mathbb{R}$ satisfying the following properties for all $x, y, z \in M$:*

(i) POSITIVITY: $d(x, y) \geq 0$, with equality if and only if $x = y$.

(ii) SYMMETRY: $d(x, y) = d(y, x)$.

(iii) TRIANGLE INEQUALITY: $d(x, z) \leq d(x, y) + d(y, z)$.

If M is a metric space, $x \in M$, and $r > 0$, the **open ball of radius r around x** *is the set*

$$\mathbb{B}_x(r) = \{y \in M : d(x, y) < r\},$$

and the **closed ball of radius r** *is*

$$\bar{\mathbb{B}}_x(r) = \{y \in M : d(x, y) \leq r\}.$$

The **metric topology on M** *is defined by declaring a subset $S \subseteq M$ to be open if for every $x \in S$, there is some $r > 0$ such that $\mathbb{B}_x(r) \subseteq S$.*

If M is a metric space and S is any subset of M, the restriction of metric to the pair of points in S turns S into a metric space and thus a topological space. We use the following standard terminology for metric spaces:

(i) A subset S of M is **bounded** *if there exists a positive number R such that $d(x, y) < R$ for all x,y in S.*

(ii) If S is a nonempty subset of M, the **diameter of S** *is the number*

$$\delta(S) = sup\{d(x, y) : x, y \in S\}.$$

Example 3 (***Euclidean Spaces***)**.** *For each integer $n \geq 1$, the set $\mathbb{R}^n$ of ordered n-tuples of real numbers is called* **n-dimensional Euclidean space***. We denote a point in $\mathbb{R}^n$ by $(x^1, \cdots, x^n), (x^i)$, or x; the numbers x^i are called* **components** *or* **coordinates of x***. By convention, $\mathbb{R}^0$ is the one element set $\{0\}$.*

For each $x \in \mathbb{R}^n$, the **Euclidean norm of x** *is the nonnegative real number*

$$\|x\| = \sqrt{(x^1)^2 + \cdots + (x^n)^2},$$

and for any $x, y \in \mathbb{R}^n$, the **Euclidean metric** *is defined by $d(x, y) = \|x - y\|$. The resulting metric topology on $\mathbb{R}^n$ is called the* **Euclidean topology***.*

Example 4 (***Subsets of Euclidean Spaces***)**.** *Every subset of $\mathbb{R}^n$ becomes a metric space, and thus a topological space, when endowed with the Euclidean metric. Here are some standard subsets of Euclidean spaces that we work with frequently:*

(i) The **unit interval** *is the subset $I \subseteq \mathbb{R}$ defined by*

$$I = [0, 1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}.$$

(ii) The **(open) unit ball of dimension n** *is the subset $\mathbb{B}_0(1) \subseteq \mathbb{R}^n$ defined by*

$$\mathbb{B}_0(1) = \{x \in \mathbb{R}^n : \|x\| < 1\}.$$

(iii) The **closed unit ball of dimension n** *is the subset $\bar{\mathbb{B}}_0(1) \subseteq \mathbb{R}^n$ defined by*

$$\bar{\mathbb{B}}_0(1) = \{x \in \mathbb{R}^n : \|x\| \leq 1\}.$$

(iv) For $n \geq 0$, the **(unit) n-sphere** *is the subset $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ defined by*

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}.$$

(v) The **(unit) circle** *is the 1-sphere $\mathbb{S}^1$, considered as a subset of $\mathbb{R}^2$.*

Definition 6. *Suppose X is a topological space, $p \in X$, and $S \subseteq X$. A point $p \in X$ (not necessarily in S) is said to be a* ***limit point of S*** *or* ***accumulation point of S*** *if every neighborhood of p contains at least one point of S other than p. The set of all accumulation points of S is denoted by $ac(S)$.*

Proposition 3. *Let S be subset of a topological space X. Then $\bar{S} = S \cup ac(S)$.*

Proof. Suppose $p \in \bar{S}$. Then $p \in S$ or $p \notin S$. If $p \in S$, then $p \in S \cup ac(S)$. If $p \notin S$, then by Prop. 2, every neighborhood of p contains a point of S. Since $p \notin S$, there is atleast one point from S other than p. Thus $p \in ac(S)$. In which case, $p \in ac(S) \cup S$. Thus, $\bar{S} \subseteq S \cup ac(S)$.

To prove the converse, note that $S \subseteq \bar{S}$. Also note that $ac(S) \subseteq \bar{S}$ by Prop. 2. Thus $S \cup ac(S) \subseteq \bar{S}$. □

Proposition 4. *Let S be subset of a topological space X. S is closed iff $S = \bar{S}$.*

Proposition 5. *If $A \subseteq B$ in a topological space X, then $\bar{A} \subseteq \bar{B}$.*

Proof. Since $B \subseteq \bar{B}$, it also contains A. As a closed set containing A, it also contains $\bar{A}$ by definition. □

Proposition 6 (***Unions and Intersections of Closures***)**.** *Let A, B and A_α be subsets of a topological space X. Then,*

(a) $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

(b) $\overline{\bigcup A_\alpha} \supseteq \bigcup \bar{A}_\alpha$.

(c) $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$.

(d) $\overline{\bigcap A_\alpha} \subseteq \bigcap \bar{A}_\alpha$.

Proof. (a) Since $A \subseteq \bar{A}$ and $B \subseteq \bar{B}$, then $A \cup B \subseteq \bar{A} \cup \bar{B}$ which is closed. Then $\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$. Now we prove the converse. Note that $A \subseteq A \cup B$ and $B \subseteq A \cup B$. Then by Prop. 5, $\bar{A} \subseteq \overline{A \cup B}$ and $\bar{B} \subseteq \overline{A \cup B}$. Then $\bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$. Thus, $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

(b) Let $x \in \bigcup \bar{A}_\alpha$, then $x \in \bar{A}_\beta$ for some β. Let U be neighborhood of x, then U intersects A_β at some point $y \in A_\beta \subseteq \bigcup A_\alpha$, so U intersects $\bigcup A_\alpha$ at y. Then $x \in \overline{\bigcup A_\alpha}$.

(c) Since $A \subseteq \bar{A}$ and $B \subseteq \bar{B}$, then $A \cap B \subseteq \bar{A} \cap \bar{B}$ which is closed. Then $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$.

(d) Let $x \in \overline{\bigcap A_\alpha}$, then $x \in \bar{A}_\alpha$ for all α. For any neighborhood U of x, we find a $y \in U \cap A_\alpha$ for all α. Then $x \in \bigcap \bar{A}_\alpha$. □

Remark 1. *Converse of (b) is not true in general. Let $A_n = (1/n, 2], n \in \mathbb{Z}_+$ as a subset of $\mathbb{R}$ with Euclidean topology. Then $\bigcup A_n = (0, 2]$ and $\bigcup \bar{A}_n = [0, 2]$. But $0 \notin \bar{A}_n$ for all n because $(-1, 1/n)$ is a neighborhood of 0 which does not intersect any A_n. Then $\overline{\bigcup A_n} \nsubseteq \bigcup \bar{A}_n$.*

Converse of (c) and (d) is also not true in general. Consider the example of $A = (-1, 0)$ and $B = (0, 1)$ in the $\mathbb{R}$ with Euclidean topology. Then $\{0\} = \bar{A} \cap \bar{B} \nsubseteq \overline{A \cap B} = \emptyset$.

Definition 7. *Suppose X is a topological space, and $S \subseteq X$. S is said to be* ***dense in X*** *if $\bar{S} = X$, or equivalently if every nonempty open subset of X contains at least one point of S.*

Definition 8. *Suppose X is a topological space, and $S \subseteq X$. S is said to be* ***nowhere dense in X*** *if $\bar{S}$ contains no nonempty open subset.*

Definition 9. *A topological space X is said to be* ***separable*** *if it contains a countable dense subset.*

Definition 10. *Suppose X is a topological space, $S \subseteq X$. A point $p \in S$ is said to be an* ***isolated point of S*** *if p has a neighborhood $U \subseteq X$ such that $U \cap S = \{p\}$.*

2 Bases and Countability

Definition 11. *A subcollection $\mathcal{B}$ of a topology $\mathcal{T}$ on a topological space X is a* ***basis for the topology $\mathcal{T}$*** *if given an open set U and a point $p \in U$, there is an open set $B \in \mathcal{B}$ such that $p \in B \subseteq U$. We also say that* ***$\mathcal{B}$ generates topology $\mathcal{T}$*** *or that $\mathcal{B}$ is a* ***basis for the topological space X****.*

Proposition 7. *A collection $\mathcal{B}$ of open sets of X is a basis iff every open set in X is a union of sets in $\mathcal{B}$.*

Proof. ($\implies$)Suppose $\mathcal{B}$ is a basis for X. Then for any open set U, and for any $p \in U$, there exists an open set $B_p \in \mathcal{B}$ such that $p \in B_p \subseteq U$. Then note that $U = \bigcup_{p \in U} B_p$ which is indeed the union of sets in $\mathcal{B}$.

($\impliedby$)Suppose every open set in X is a union of set in $\mathcal{B}$, then we need to show that $\mathcal{B}$ is a basis for X. For any open set U in X, we know that it can be written as $U = \bigcup_i B_i$, $B_i \in \mathcal{B}$ for all i. Then note that any point $p \in U$ must belong to atleast one of the sets B_i, i.e., for any $p \in U$, there is a set B_i in $\mathcal{B}$ such that $p \in B_i \subseteq U$, thus $\mathcal{B}$ is a basis for X. □

Proposition 8. *A collection $\mathcal{B}$ of subsets of a set X is a basis for some topology $\mathcal{T}$ on X iff*

(a) X is the union of all set in $\mathcal{B}$, and

(b) given any two set $B_1, B_2 \in \mathcal{B}$, and a point $p \in B_1 \cap B_2$, there is a set $B \in \mathcal{B}$ such that $p \in B \subseteq B_1 \cap B_2$.

Proof. ($\implies$)Suppose $\mathcal{B}$ is a basis for X. Then X is an open set and every open set is union of set in $\mathcal{B}$ from Prop. 7. Similarly, for any two sets $B_1, B_2 \in \mathcal{B}$, they are open, so $B_1 \cap B_2$ is also an open set. Again using Prop. 7, for any $p \in B_1 \cap \mathcal{B}_2$, there exists a set $B \in \mathcal{B}$ with the required property.

($\impliedby$)Suppose for a collection $\mathcal{B}$, (a) and (b) hold. Then we will show that the set $\mathcal{T}$ formed by taking unions of sets in $\mathcal{B}$ is a topology on X. It is easy to note that $X, \emptyset \in \mathcal{T}$. It is also easy to note that the sets in $\mathcal{T}$ are closed under arbitrary union. To show that $\mathcal{T}$ is closed under finite intersection, let $U = \bigcup_\alpha B_\alpha$ and $V = \bigcup_\beta B_\beta$ be in $\mathcal{T}$, where $B_\alpha, B_\beta \in \mathcal{B}$. Then $U \cap V = \left(\bigcup_\alpha B_\alpha\right) \cap \left(\bigcup_\beta B_\beta\right) = \bigcup_{\alpha,\beta}(B_\alpha \cap B_\beta)$. Now note that by (b), there exists a set B_p for every $p \in B_\alpha \cap B_\beta$, such that $p \in B_p \subseteq B_\alpha \cap B_\beta$, so that $U \cap V$ can be written as union of sets in $\mathcal{B}$. Thus $\mathcal{T}$ is a topology. □

Proposition 9. *Let M be a metric space. Then the collection $\mathcal{B}$ of all open balls $\mathbb{B}_x(r)$, for $x \in M$ and $r > 0$, is a basis for metric topology defined in Eg. 2.*

Proof. We show that the collection $\mathcal{B}$ satisfies both the conditions in Prop. 8.
(a) For every $x \in M$, the open ball $\mathbb{B}_x(r)$, $r > 0$ is a element of $\mathcal{B}$ containing x.
(b) Let $B_1, B_2 \in \mathcal{B}$, and that $y \in B_1 \cap B_2$. Suppose $B_1 = \mathbb{B}_x(r)$, for some $x \in M$ and $r > 0$. Then setting $r_1 = r - d(x, y) > 0$, we have that $\mathbb{B}_y(r_1) \subseteq \mathbb{B}_x(r) = B_1$ because for any $z \in \mathbb{B}_y(r_1)$, $d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + r - d(x, y) = r$. Similarly we can choose $r_2 > 0$ such that $\mathbb{B}_y(r_2) \subseteq B_2$. Then taking $r_0 = \min\{r_1, r_2\}$, we see that $y \in \mathbb{B}_y(r_0) \subseteq B_1 \cap B_2$. □

We say that a point in $\mathbb{R}^n$ is ***rational*** if all its coordinates are rational numbers.

Lemma 1. *Every open interval in $\mathbb{R}$ contains a rational point.*

Lemma 2. *Every open set in $\mathbb{R}^n$ contains a rational point.*

Proof. An open set U in $\mathbb{R}^n$ containing $p \in \mathbb{R}^n$ contains an open ball $\mathbb{B}_p(r)$ for some r. It in turn contains an open cube $\prod_{i=1}^n I_i$, where $I_i = (p^{(i)} - r/\sqrt{n}, p^{(i)} + r/\sqrt{n})$. For each i, let $q^{(i)}$ be a rational number(by Lemma 1) in I_i. Then $(q^{(1)}, \ldots, q^{(n)})$ is a rational point in $\prod_{i=1}^n I_i \subseteq \mathbb{B}_p(r) \subseteq U$. □

Proposition 10. *The collection $\mathcal{B}_{rat}$ of all open balls in $\mathbb{R}^n$ with rational centers and rational radii is a basis for $\mathbb{R}^n$.*

Proof. To show that $\mathcal{B}_{rat}$ is a basis for $\mathbb{R}^n$, we will show that for any open set U in $\mathbb{R}^n$, for $p \in U$, there exists $B \in \mathcal{B}$ such that $p \in B \subseteq U$. Since U is an open set and $p \in U$, $\mathbb{B}_p(r') \subseteq U$ for some real number $r' > 0$. There exists a rational $r \in (0, r')$ and $\mathbb{B}_p(r) \subseteq U$. By Lemma 2, there exists a rational point $q \in \mathbb{B}_p(r/2)$, then $p \in \mathbb{B}_q(r/2)$. For any $x \in \mathbb{B}_q(r/2)$, we have $d(x,p) \leq d(x,q) + d(q,p) < r/2 + r/2 = r$, so that $\mathbb{B}_q(r/2) \subseteq \mathbb{B}_p(r)$. Thus we have found a set $\mathbb{B}_q(r/2) \in \mathcal{B}_{rat}$ such that for an open set U and $p \in U$, $p \in \mathbb{B}_q(r/2) \subseteq U$. □

Definition 12. *A topological space is said to be* ***second countable*** *if it has a countable basis.*

Proposition 11. *The set of all open balls in $\mathbb{R}^n$ whose radii are rational and whose centers have rational coordinates is a countable basis for the Euclidean topology, and thus $\mathbb{R}^n$ is second-countable.*

Definition 13. *Let X be a topological space. A* ***cover of X*** *is a collection $\mathcal{U}$ of subsets of X whose union is X; it is called an* ***open cover*** *if each of the sets in $\mathcal{U}$ is open. A* ***subcover of*** *$\mathcal{U}$ is a subcollection of U that is still a cover.*

Proposition 12. *Let X be a second-countable topological space. Every open cover of X has a countable subcover.*

Proof. Let $\mathcal{B}$ be the countable basis for X and let $\mathcal{U}$ be any arbitrary open cover of X. Consider a collection $\mathcal{U}' = \{U \in \mathcal{U} : \exists\ B \in \mathcal{B}, B \subseteq U\}$. Then $\mathcal{U}'$ is countable. Now consider a point x in X. There exists $V \in \mathcal{U}$ such that $x \in V$. But as $\mathcal{B}$ is basis, there exists a $B \in \mathcal{B}$ such that $x \in B \subseteq V$, which implies that $V \in \mathcal{U}'$. Thus $\mathcal{U}'$ also covers X. □

Definition 14. *If X is a topological space and $p \in X$, a* ***neighborhood basis at p*** *is a collection $\mathcal{B}_p$ of neighborhoods of p such that every neighborhood of p contains at least one $B \in \mathcal{B}_p$. A topological space X is said to be* ***first-countable*** *if there is a countable neighborhood basis at each point.*

Remark 2. *Suppose p is a point in a first countable topological space and $\{V_i\}_{i=1}^{\infty}$ is a countable neighborhood basis at p. By taking $U_j = \bigcap_{i=1}^{j} V_i$, we obtain a countable descending sequence $U_1 \supset U_2 \supset \ldots$ that is also a neighborhood basis at p. Thus, in definition of first countability, we may assume that at every point the countable neighborhood basis at the point is a descending sequence of open sets.*

Example 5. *Every metric space M is first-countable as for any point $x \in M$, the set $\mathcal{B} = \{\mathbb{B}_x(1/n) : n \in \mathbb{Z}_+\}$ is countable and forms a neighborhood basis at x. But there exists metric spaces that are not second countable. For example, consider an uncountable set X with the discrete metric defined as:*

$$d(x,y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{otherwise.} \end{cases}$$

By considering the metric topology on X, we note that for any $x \in X$, $\mathbb{B}_x(1) = \{x\}$ should be an open set. Thus every singleton set is an open set on X. A base for this topology must include all these singleton sets, so it is uncountable.

Remark 3. *Since a countable basis for X contains a countable neighborhood basis at each point, second-countability implies first-countability.*

Proposition 13. *Every second countable space is separable. Every separable metric space is second countable.*

Proof. Suppose X is a second countable space. Then there exists a countable basis $\mathcal{B}$ such that any open subset of X can be written as union of a subcollection of $\mathcal{B}$. If $\mathcal{B}$ contains $\emptyset$, discard it. Let $A = \{x : x \text{ is any element of } B, B \in \mathcal{B}\}$. Then A is countable since we take one element each from a countable collection of basis elements. Moreover for every nonempty open subset U of X, $U = \cup B$, for some $B \in \mathcal{B}$. Since $A \cap B \neq \emptyset$, we have that $A \cap U \neq \emptyset$ showing that A is dense in X. Thus X is separable.

Suppose (X, d) is a separable metric space. Let A be the countable dense subset in X. The set $\mathbb{Q}$ of rational numbers is countable, so $\mathcal{B} = \{\mathbb{B}_x(r) : x \in D, 0 < r \in \mathbb{Q}\}$ is a countable collection of open balls in X. We show that $\mathcal{B}$ is a basis for the metric topology. Let U be an open subset in X, and $x \in U$. By openness of U, there is some $\epsilon > 0$ such that $\mathbb{B}_x(\epsilon) \subseteq U$. But $x \in \bar{D}$ and $\mathbb{B}_x(\epsilon/2)$ is a neighborhood of x, then there is some $y \in \mathbb{B}_x(\epsilon/2) \cap D$. For any $z \in \mathbb{B}_y(\epsilon/2)$, $d(x, z) \leq d(x, y) + d(y, z) < \epsilon$, i.e., $x \in \mathbb{B}_y(\epsilon/2) \subseteq \mathbb{B}_x(\epsilon)$. Since $\epsilon/2 > 0$, there is a rational number $0 < r < \epsilon/2$ by Lemma 1. Then we have $x \in \mathbb{B}_y(r) \subseteq \mathbb{B}_x(\epsilon), \mathbb{B}_y(r) \in \mathcal{B}$. □

3 Hausdorff Spaces and Convergent Sequences

Definition 15. *Given a sequence $(p_i)_{i=1}^{\infty}$ of points in X and a point $p \in X$, the sequence is said to* ***converge to p*** *if for every neighborhood U of p, there exists a positive integer N such that $p_i \in U$ for all $i \geq N$. In this case, we write $p_i \to p$ or $\underset{i\to\infty}{Lt}\, p_i = p$.*

Topological spaces allows us to describe a wide variety of concepts if "spaces." But for the purposes of manifold theory, arbitrary topological spaces are far too general, because they can have some unpleasant properties, as the next example illustrates.

Example 6. *Let X be any set. Then $\{X, \emptyset\}$ is a topology on X, called* ***trivial topology****. When X is endowed with this topology, every sequence in X converges to every point of X, and every map from a topological space into X is continuous. For a better example consider the topology $\mathcal{T} = \{[a, +\infty) : a \in \mathbb{R}\} \cup \{\emptyset\}$ on $\mathbb{R}$. It is easy to verify that it is indeed a topology on $\mathbb{R}$. Then consider a constant sequence $\{a\}$. Applying the standard definition of limit of a sequence, the sequence converges to $a + r$ for any $r \geq 0$.*

To avoid pathological cases like this, which results when X does not have sufficiently many open subsets, we often restrict our attention to topological spaces satisfying the following special condition.

Definition 16. *A topological space X is said to be a* ***Hausdorff space*** *if for every pair of distinct points p, q in X, there exist disjoint open subsets $U, V \subseteq X$, such that $p \in U$ and $q \in V$.*

Proposition 14. *Every metric space is Hausdorff in the metric topology.*

Proof. For any distinct points x, y in a metric space, setting $r = \frac{1}{2}d(x, y)$ and $U = \mathbb{B}_x(r), V = \mathbb{B}_y(r)$ proves that the metric space is Hausdorff. □

Proposition 15. *Let X be a Hausdorff space. Then each finite subset of X is closed, and that each convergent sequence in X has a unique limit.*

Proof. To prove that a finite subset of X is closed, it suffices to prove that a singleton set $\{x\}$, x in X is closed. Consider the set $X \setminus \{x\}$ and let p be a point in it. Since it is distinct from x, there exists a disjoint open neighborhood U_p of p in X, such that $U_p \subseteq X \setminus \{x\}$. Then $\bigcup_{p \neq x} U_p = X \setminus \{x\}$, which makes it open, thus singleton set is closed.

Now consider a convergent sequence $(x_i)_{i=1}^{\infty}$ converging to two points x,y in X. Then for every neighborhoods U, V of x and y, there exists positive numbers N_1, N_2 respectively such that for all $i \geq N_1, j \geq N_2, x_i \in U, x_j \in V$. WLOG suppose that $N_1 > N_2$, then $i > N_1$, we have

$x_i \in U, x_i \in V$. Since U and V were chosen arbitrarily, this must hold for every neighborhoods of U and V, which can happen only if $x = y$. □

Remark 4 (T_1-axiom)**.** *It states that for each pair of distinct points in X, each has a neighborhood not containing the other. This axiom is strictly weaker than Hausdorff axiom. This axiom guarantees that finite point sets are closed, but not the uniqueness of limit of convergent sequences. Therefore, we are not interested in this axiom.*

Proposition 16. *Suppose X is a Hausdorff space and $A \subseteq X$. If $p \in X$ is a limit point of A, then every neighborhood of p contains infinitely many points of A.*

Proof. We prove the contrapositive. Suppose there is a neighborhood U of p containing only finitely many points $x_1 \ldots, x_k \in A$. Since X is Hausdorff, there exist disjoint neighborhoods U_i and V_i of x_i and p respectively for each i. Then note that $V = \cap_{i=1}^{k} V_i$ is a neighborhood of p which disjoint from each of U_i. Then p is not a limit point of A. □

4 Continuous Maps

Definition 17. *A map $F : X \to Y$ is said to be **continuous** if for every open subset $U \subseteq Y$, the preimage $F^{-1}(U)$ is open in X.*

Remark 5. *It is easy to check that a map $F : X \to Y$ is continuous iff for every basis element B of Y, $F^{-1}(B)$ is open in X.*

Proposition 17. *Suppose (M_1, d_1) and (M_2, d_2) are metric spaces, and $F : M_1 \to M_2$ is a map. Then F is continuous(by Def. 17) iff for all $x, y \in M_1$ and every $\epsilon > 0$, there exists a $\delta > 0$ such that $d_1(x, y) < \delta$ implies $d_2(F(x), F(y)) < \epsilon$.*

Proof. ($\implies$)Let $x_0 \in M_1$ and $\epsilon > 0$. Consider the open ball $\mathbb{B}_{F(x_0)}(\epsilon)$ containing $F(x_0)$ of radius ϵ. Then by continuity of F, we have $F^{-1}(\mathbb{B}_{F(x_0)}(\epsilon))$ is open in M_1. Note that $x_0 \in F^{-1}(\mathbb{B}_{F(x_0)}(\epsilon))$. Then there exists a open ball $\mathbb{B}_{x_0}(\delta) \subseteq F^{-1}(\mathbb{B}_{F(x_0)}(\epsilon))$ for some $\delta > 0$. It follows that $F(\mathbb{B}_{x_0}(\delta)) \subseteq F(F^{-1}(\mathbb{B}_{F(x_0)}(\epsilon))) \subseteq \mathbb{B}_{F(x_0)}(\epsilon)$ by elementary set theory, i.e., for all $x \in M_1$ such that $d_1(x, x_0) < \delta$ implies $d_2(F(x), F(y)) < \epsilon$.

($\impliedby$)Let $V \subseteq Y$ be open and $x_0 \in F^{-1}(V)$. Then for some $\epsilon > 0$ we have $\mathbb{B}_{F(x_0)}(\epsilon) \subseteq V$. Then by assumption there exists a $\delta > 0$ such that for all $x \in \mathbb{B}_{x_0}(\delta)$, we have $F(x) \in \mathbb{B}_{F(x_0)}(\epsilon)$, i.e., $F(\mathbb{B}_{x_0}(\delta)) \subseteq \mathbb{B}_{F(x_0)}(\epsilon) \subseteq V$ which implies that $F^{-1}(V) \supseteq F^{-1}(F(\mathbb{B}_{x_0}(\delta))) \supseteq \mathbb{B}_{x_0}(\delta)$ by elementary set theory. So $F^{-1}(V)$ is open in X. □

Proposition 18. *Let $F : X \to Y$ be a map between topological spaces. Then all of the following properties are equivalent:*

(a) F is continuous.

(b) For every $p \in X$ and a neighborhood V of $f(p)$ in Y, there is a neighborhood U of p in X such that $F(U) \subseteq V$.

(c) For every subset $B \subseteq Y$, $F^{-1}(Int\ B) \subseteq Int\ F^{-1}(B)$.

(d) For every subset $A \subseteq X$, $F(\bar{A}) \subseteq \overline{F(A)}$.

(e) For every closed subset $C \subseteq Y$, $F^{-1}(C)$ is closed subset in X.

Proof. We show that (a) iff (b), (a) iff (c) and (a) $\implies$ (d) $\implies$ (e) $\implies$ (a).
(a) $\implies$ (b) Suppose F is continuous and $p \in X$. Let V be a neighborhood of $f(p)$ in Y. Then by continuity of F, $F^{-1}(V)$ is open in X which contains p. Then set $U = F^{-1}(V)$ since $F(F^{-1}(V)) \subseteq V$.

(b) $\implies$ **(a)** Let $p \in X$, such that V is any open set in Y containing $f(p)$. Then there is a neighborhood $U \subseteq X$ containing p such that $F(U) \subseteq V \implies U \subseteq F^{-1}(V)$. Then by Prop. 1, $F^{-1}(V)$ is open.
(a) $\implies$ **(c)** For any subset B in Y, $Int\ B = \cup_i U_i$, where $U_i \subseteq B$ are open in Y. Then $F^{-1}(Int\ B) = F^{-1}(\cup_i U_i) = \cup_i F^{-1}(U_i)$. Since F is continuous, $\cup_i F^{-1}(U_i)$ is open in X. Then $\cup_i F^{-1}(U_i) = Int\ \cup_i F^{-1}(U_i) = IntF^{-1}(\cup_i U_i) \subseteq IntF^{-1}(B)$.
(c) $\implies$ **(a)** Suppose B is open in Y such that $F^{-1}(IntB) \subseteq IntF^{-1}(B)$. Then $F^{-1}(B) = F^{-1}(IntB) \subseteq IntF^{-1}(B)$. So $F^{-1}(B)$ is open in X.
(a) $\implies$ **(d)** Assume F is continuous and A is a subset of X. Let $x \in \bar{A}$. Let V be a neighborhood of $F(x)$. Then $F^{-1}(V)$ is an open set containing x; it must intersect A in some point y. Then V intersects $F(A)$ in the point $F(y)$, so that $F(x) \in F(\bar{A})$.
(d) $\implies$ **(e)** Let C be a closed set in Y and let $B = F^{-1}(C)$. Then $F(B) = F(F^{-1}(C)) \subseteq C$. Therefore, if $x \in \bar{B}$, then $F(x) \in F(\bar{B}) \subseteq \overline{F(B)} \subseteq \bar{C} = C$, so that $x \in F^{-1}(C) = B$. Then $\bar{B} \subseteq B$ or $B = F^{-1}(C)$ is closed.
(e) $\implies$ **(a)** Let V be an open set of Y. Then $F^{-1}(Y - V)$ is a closed set. But $F^{-1}(Y - V) = F^{-1}(Y) - F^{-1}(V) = X - F^{-1}(V)$, which is a closed set. Then $F^{-1}(V)$ is open in X, thus F is continuous. □

Proposition 19. *Let X, Y and Z be topological spaces. Then the following maps are continuous:*

(a) The ***identity map*** $Id_X : X \to X$, *defined by* $Id_X(x) = x$ *for all* $x \in X$.

(b) Any ***constant map*** $F : X \to Y$ *(i.e., a map such that* $F(x) = F(y)$ *for all* $x, y \in X$*).*

(c) Any composition $G \circ F : X \to Z$ *of continuous map* $F : X \to Y$ *and* $G : Y \to Z$.

Proof. (a) For every open set U in X, $Id_X^{-1}(U) = U$, which is open.
(b) Let $F(x) = k$, for all $x \in X$. Then for every open set U in Y, either $F^{-1}(U) = X$, when $k \in U$ or $F^{-1}(U) = \emptyset$. In both cases the preimage of U is open in X.
(c) Let U be open in Z. Since G is continuous from Y to Z, $G^{-1}(U)$ is open in Y. By the continuity of F, $F^{-1}(G^{-1}(U)) = (G \circ F)^{-1}(U)$ is open in X. □

Definition 18. *A continuous bijective map* $F : X \to Y$ *with continuous inverse is called a* ***homeomorphism****.*

Definition 19. *A continuous map* $F : X \to Y$ *is said to be a* ***local homeomorphism*** *if every point* $p \in X$ *has a neighborhood* $U \subseteq X$ *such that* $F(U)$ *is open in* Y *and* F *restricts to a homeomorphism from* U *to* $F(U)$.

To say that a sequence is ***eventually in a subset*** means that all but finitley many terms of the sequence are in the subset.

Proposition 20 (***Sequence Lemma***)**.** *Let* X *be a first-countable space, let* $A \subseteq X$ *be any subset, and let* $x \in X$.

(a) $x \in \bar{A}$ *if and only if* x *is a limit of a sequence of points in* A.

(b) $x \in Int\ A$ *if and only if every sequence in* X *converging to* x *is eventually in* A.

(c) A *is closed in* X *if and only if* A *contains every limit of every convergent sequence of points in* A.

(d) A *is open in* X *if and only if every sequence in* X *converging to a point of* A *is eventually in* A.

Proof. (a) Suppose x is a limit of a sequence of points in A and let $x \notin \bar{A}$. Then there exists at least one closed set C in X containing A such that $x \notin C$. Then $x \in X \setminus C$ which is open. Then there exists an open set U in X such that $x \in U, U \subseteq X \setminus C$. Notice that this open set contains no point of the sequence which is a contradiction. ***(First-countability is not need to prove this.)***

Let $x \in \bar{A}$ and X be first-countable. Then there exists a countable neighborhood basis $\mathcal{B}_x$ at x. Let $U_1 \in \mathcal{B}_x$ and since $x \in \bar{A}$, $U_1 \cap A$ is nonempty. (If it were empty then $X \setminus U_1$ is a closed set containing A but not x, a contradiction!) Let $x_1 \in U_1 \cap A$. Similarly, for every positive number i, if $U_i \in \mathcal{B}_x$, then $U_i' = (\bigcap_i U_i) \cap A$ is also nonempty. Then we get a sequence (x_i) such that $x_i \in U_i' \in A$ for each i. This sequence converges to x.

(b) Let $x \in Int\ A$, and suppose there exists a sequence (x_i) converging to x in X. Since Int A is itself a neighborhood of x, there exists a positive integer N such that for all $n \geq N, x_n \in IntA \subseteq A$, so (x_i) eventually converges in A.

Suppose that every sequence in X converging to x is eventually in A and let $x \in X \setminus IntA$. Then there exists a countable neighborhood basis $\{U_n : n \in \mathbb{N}\}$ at x. For each $n \in \mathbb{N}$, define $B_n = \cap_{i=1}^n U_i$ and $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$. Then $\mathcal{B}$ is a neighborhood basis at x. Since $x \notin IntA$, $B_n \setminus A \neq \emptyset$ for all $n \in \mathbb{N}$ (otherwise $x \in B_n \subseteq A$ implies $x \in IntA$). So for each $n \in \mathbb{N}$, we can pick $x_n \in B_n \setminus A$ and form the sequence (x_i). Let V be a neighborhood of x, as $\mathcal{B}$ is a neighborhood basis at x, there exists $m \in \mathbb{N}$ such that $B_m \subseteq V$. Notice that for all $n \geq m$, we have $x_n \in V$. Thus $x_n \to x$ but (x_i) is never in A.

(c) Follows from (a) and the fact that for a closed set $A = \bar{A}$.

(d) Follows from (b) and the fact that for an open set $A = Int\ A$. □

Proposition 21. *Let X and Y be topological spaces and suppose $F : X \to Y$ is continuous. Then for every convergent sequence $x_n \to x$ in X, $F(x_n)$ converges to $F(x)$ in Y. The converse holds if X is first-countable.*

Proof. Suppose $x_n \to x$ is a convergent sequence in X and let U be a neighborhood of $F(x)$. Since F is continuous, $x \in F^{-1}(U)$ which is open in X. Then there exists a positive integer N such that $x_n \in F^{-1}(U)$ for all $n \geq N$. Since $F(F^{-1}(U)) \subseteq U$, $F(x_n) \in U$ for all $n \geq N$. Thus, $F(x_n) \to F(x)$.

To prove the converse assume that the convergent sequence condition is satisfied. Let A be a subset of X. We show that $F(\bar{A}) \subseteq \overline{F(A)}$. If $x \in \bar{A}$, then there is a sequence $\{x_n\}$ of points of A converging to x(by Prop. 20). By assumption the sequence $\{F(x_n)\}$ converges to $F(x)$. Since $F(x_n) \in F(A)$, by Prop. 20 we have $F(x) \in \overline{F(A)}$. Thus, $F(\bar{A}) \subseteq \overline{F(A)}$. □

5 Subspaces

Probably the simplest way to obtain new topological spaces from the old ones is by taking subsets of other spaces.

Definition 20. *Let X be a topological space and $S \subseteq X$ any arbitrary subset. We define the* ***subspace topology on S*** *as follows:*

$$\mathcal{T}_S = \{U \subseteq S : U = V \cap S \text{ for some } V \in \mathcal{T}\}.$$

It is easy to see that subspace topology is a topology on S. Any subset X endowed with the subspace topology is said to be a ***subspace of X***.

Definition 21. *Let X and Y be topological spaces. A continuous injective map $F : X \to Y$ is called a* ***topological embedding*** *if it is a homeomorphism onto its image $F(X) \subseteq Y$ in the subspace topology.*

The most important properties of the subspace topology are summarized in the next proposition.

Proposition 22 (***Properties of the Subspace Topology***)**.** *Let X be a topological space and let S be a subspace of X.*

(a) (CHARACTERISTIC PROPERTY). If Y is a topological space, a map $F : Y \to S$ is continuous if and only if the composition $\iota_S \circ F : Y \to X$ is continuous, where $\iota_S : S \hookrightarrow X$ is the inclusion map (the restriction of the identity map of X to S).

(b) The inclusion map $\iota_S : S \hookrightarrow X$ is continuous.

(c) The subspace topology is the unique topology on S for which the characteristic property holds.

(d) A subset $K \subseteq S$ is closed in S if and only if there exists a closed subset : $L \subseteq X$, such that $K = L \cap S$.

(e) The inclusion map $\iota_S : S \hookrightarrow X$ is a topological embedding.

(f) If Y is a topological space and $F : X \to Y$ is continuous, then $F|_S : S \to Y$ (the restriction of F to S) is continuous.

(g) If $\mathcal{B}$ is a basis for the topology of X, then $\mathcal{B}_S = \{B \cap S : B \in \mathcal{B}\}$ is a basis for the subspace topology on S.

(h) If X is Hausdorff, then so is S.

(i) If X is first-countable, then so is S.

(j) If X is second-countable, then so is S.

Proof. (a) ($\implies$)Suppose F is continuous. Let U be an open set in X, then

$$(\iota_S \circ F)^{-1}(U) = F^{-1}(\iota_S^{-1}(U)) = F^{-1}(U \cap S),$$

which is open in Y as $S \cap U$ is open in S.
($\impliedby$)Suppose $\iota_S \circ F$ is continuous. Let V be an open set in S, then there is an open set U in X such that $V = S \cap U = \iota_S^{-1}(U)$. Consider

$$F^{-1}(V) = F^{-1}(\iota_S^{-1}(U)) = (\iota_S \circ F)^{-1}(U),$$

which is open in Y as $\iota_S \circ F$ is continuous.

(b) In (a) set F to be $Id_S : S \to S$, such that $Id_S(x) = x,\ x \in S$. Then we have Id_S is continuous iff $\iota_S \circ Id_S = \iota_S$ is continuous. Since Id_S is continuous, we have ι_S is also continuous.

(c) See [1, Thm. 3.24].

(d) ($\implies$)Suppose K is closed in S. Then $S \setminus K$ is open in S, so that $S \setminus K = S \cap U$, for some open set U in X. Then $K = S \setminus (S \setminus K) = S \setminus (S \cap U) = S \cap (X \setminus U)$, which is required.
($\impliedby$)Suppose a set $K \subseteq S, K = S \cap C$, for some closed set C in X. Then $S \setminus K = S \setminus (S \cap C) = S \cap (X \setminus C)$ which is open in S as $X \setminus C$ is open in X. Thus K is closed in S.

(e) The image of the inclusion map is S and ι_S when defined onto S, it is essentially Id_S which is a homeomorphism. Thus, ι_S is a topological embedding.

(f) Take note that $F|_S = F \circ \iota_S$ which is continuous as F and ι_S are continuous.

(g) Suppose U is an open set in S and $p \in U$. Then $U = V \cap S$, for some open set V in X and $p \in V$. Since $\mathcal{B}$ is a basis for X, there exists a $B \in \mathcal{B}$ such that $p \in B \subseteq V$. Then $p \in B \cap S \subseteq V \cap S = U$.

(h) For any two distinct points $x, y \in S \subseteq X$, since X is Hausdorff, there exists two disjoint open sets $U, V \subseteq X$ such that $x \in U, y \in V$. Then note that $x \in U \cap S, y \in V \cap S$ which are disjoint.

(i) For a point $p \in S$, there exists a countable neighborhood basis $\mathcal{B} = \{B_\alpha\}$ in X. Let U be a neighborhood of p in S. Then, $U = S \cap U'$ for some open set U' in X, so that there exists a set $B_\alpha \in \mathcal{B}$ such that $p \in B_\alpha \subseteq U'$. Then $p \in B_\alpha \cap S \subseteq U' \cap S = U$.

(j) For a countable basis $\mathcal{B}$ for X, we know that $\mathcal{B}_S = \{B \cap S : B \in \mathcal{B}\}$ is basis for S which is countable. □

If X and Y are topological space and $F : X \to Y$ is a continuous map, part (f) of the preceding proposition guarantees that the restriction of F to every subspace of X is continuous(in the subspace topology). We can also ask the converse question.: If we know that the restriction of F to certain subspaces of X is continuous, is F itself continuous? The next two propositions express two somewhat different answers to this question.

Proposition 23 (***Continuity is Local property***)**.** *Let $F : X \to Y$ be a map between topological spaces. If every point $p \in X$ has a neighborhood U on which the restriction $F|_U$ is continuous, then F is continuous.*

Proof. Given that every point $p \in X$ has a neighborhood U_x, then X can be written as union of these open sets U_x, such that $F|_{U_x}$ is continuous. Let V be an open set in Y. Then $F^{-1}(V) \cap U_x = (F|_{U_x})^{-1}(V)$, because both the expressions represents the set of points x lying in U_x for which $F(x) \in V$. Since $F|_{U_x}$ is continuous this set is open in U_x and hence open in X. But $F^{-1}(V) = \bigcup_x (F^{-1}(V) \cap U_x)$. □

Proposition 24 (***Gluing Lemma for Continuous Maps***)**.** *Let X, Y be topological spaces, and suppose one of the following holds:*

(a) $B_1, \ldots, B_n$ are finitely many closed subsets of X whose union is X, i.e., $\{B_i\}$ is a finite closed cover of X.

(b) $\{B_i\}_{i \in A}$ is a collection of open subsets of X whose union is X, i.e., $\{B_i\}$ is an arbitrary open cover of X.

Suppose that for all i we are given continuous maps $F_i : B_i \to Y$ that agree on overlaps: $F_i|_{B_i \cap B_j} = F_j|_{B_i \cap B_j}$. Then there exists a unique continuous map $F : X \to Y$ whose restriction to each B_i is equal to F_i.

Proof. (a) Suppose $X = \bigcup_{i=1}^n B_n$, where each B_i is a closed set. Define $F(x) = F_i(x)$, if $x \in B_i$, so that $F|_{B_i} = F_i$. We show that F is continuous from X to Y. Let C be a closed set in Y. Then $F^{-1}(C) = \bigcup_{i=1}^n F_i(C)$, by elementary set theory. But each $F_i^{-1}(C)$ is closed in B_i as F_i is continuous. Since B_i is closed in X, $F_i^{-1}(C)$ is also closed in X. Then $F^{-1}(C)$ is closed in X.
(b) Suppose $X = \bigcup_\alpha B_\alpha$, where each B_α is open in X. Define $F(x) = F_\alpha(x)$, if $x \in B_\alpha$, so that $F|_{B_\alpha} = F_\alpha$. We show that F is continuous from X to Y. Let V be an open set in Y. Then $F^{-1}(V) = \bigcup_\alpha F_\alpha(V)$, by elementary set theory. But each $F_\alpha^{-1}(V)$ is open in B_α as F_α is continuous. Since B_α is open in X, $F_\alpha^{-1}(V)$ is also closed in X. Then $F^{-1}(V)$ is open in X. □

Proposition 25. *Let X be a topological space, and suppose X admits a countable open cover $\{U_i\}$ such that each set U_i is second-countable in the subspace topology. Then X is second-countable.*

Proof. □

Proposition 26. *Let $A \subseteq X$. If d is a metric for the topology of X, then $d|_{A \times A}$ is a metric for the subspace topology on A.*

Proof. We show that metric topology induced by $d' \equiv d|_{A\times A}$ is same as the subspace topology. Any basis element of metric topology induced by d' is $\mathbb{B}'_x(r)$, for some $x \in A$ and $r > 0$, the corresponding basis element induced by subspace topology of M is $\mathbb{B}_x(r) \cap A$. We show that $\mathbb{B}'_x(r) = \mathbb{B}_x(r) \cap A$. Any $y \in \mathbb{B}'_x(r) \Leftrightarrow y \in A$ and $y \in \mathbb{B}_x(r) \Leftrightarrow y \in \mathbb{B}_x(r) \cap A$. Then we see that for any $x \in A$, $x \in \mathbb{B}'_x(r) \subseteq \mathbb{B}_x(r) \cap A$ and $x \in \mathbb{B}_x(r) \cap A \subseteq \mathbb{B}'_x(r)$ showing that each topology is finer than the other. □

6 Product Spaces

Definition 22. *Suppose $X_1, \ldots, X_n$ are finitely many sets. Then their* ***Cartesian product*** *is the set $X_1 \times \ldots \times X_n$ consisting of all ordered n-tuples of the form $(x_1, \ldots, x_n)$ with $x_i \in X_i$ for each i. The* ***i-th projection map*** *is the map $\pi_i : X_1 \times \ldots \times X_n \to X_i$ defined by $\pi_i(x_1, \ldots, x_n) = x_i$.*

Proposition 27. *$\mathcal{B} = \{U_1 \times \ldots \times U_n : U_i \subseteq X_i$ is open, $i = 1, \ldots, n\}$ forms a basis for a topology on $X_1 \times \ldots \times X_n$, called the* ***product topology****.*

Proof. We use Prop. 8. Since X_i is open in X_i, we have a set of the form $X_1 \times \ldots \times X_n \in \mathcal{B}$ thus $\bigcup_{B\in\mathcal{B}} B = X_1 \times \ldots \times X_n$. Now suppose $B_1, B_2 \in \mathcal{B}$. Then $B_1 = U_1 \times \ldots \times U_n$ and $B_2 = V_1 \times \ldots \times V_n$ for some open sets U_i, V_i in X_i. Let $p = (p^{(1)}, \ldots, p^{(n)}) \in B_1 \cap B_2$. Then for each i, $p^{(i)} \in U_i \cap V_i$ which is also open in X_i. Therefore, $p \in B \subseteq B_1 \cap B_2$, where $B = B_1 \cap B_2$. □

Proposition 28 (*Properties of the Product Topology*). *Suppose $X_1 \times \ldots \times X_n$ are topological spaces, and let $X_1 \times \ldots \times X_n$ be their product space.*

(a) (CHARACTERISTIC PROPERTY). If B is a topological space, a map $F : B \to X_1 \times \ldots \times X_n$ is continuous iff each of its component functions $F_i = \pi_i \circ F : B \to X_i$ is continuous.

(b) The product topology is the unique topology on $X_1 \times \ldots \times X_n$ for which the characteristic property holds.

(c) Each projection map $\pi_i : X_1 \times \ldots \times X_n \to X_i$ is continuous.

(d) Given any continuous map $F_i : X_i \to Y_i$ for $i = 1, \ldots, n$ the ***product map*** *$F_1 \times \ldots \times F_n : X_1 \times \ldots \times X_n \to Y_1 \times \ldots \times Y_n$ is continuous, where*

$$F_1 \times \ldots \times F_n(x_1, \ldots, x_n) = (F_1(x_1), \ldots, F_n(x_n)).$$

(e) If S_i is a subspace of X_i for $i = 1, \ldots, n$ the product topology and the subspace topology on $S_1 \times \ldots \times S_n \subseteq X_1 \times \ldots \times X_n$ coincide.

(f) For any $i \in \{1, \ldots, n\}$ and any choices of points $a_j \in X_j$ for $j \neq i$, the map $x \mapsto (a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n)$ is a topological embedding of X_i into the product space $X_1 \times \ldots \times X_n$.

(g) If $\mathcal{B}_i$ is a basis for the topology of X_i for $i = 1, \ldots, n$, then the collection

$$\mathcal{B} = \{B_1 \times \ldots \times B_k : B_i \in \mathcal{B}_i\}$$

is a basis for the topology of $X_1 \times \ldots \times X_n$.

(h) Every finite product of Hausdorff spaces is Hausdorff.

(i) Every finite product of first-countable spaces is first-countable.

(j) Every finite product of second-countable spaces is second-countable.

Proof. (a) ($\implies$)Suppose F is continuous and let U_i is an open set in X_i. Then consider the set $U = X_1 \times \ldots \times U_i \times \ldots \times X_n$ which is open in the product space. Then

$$F^{-1}(U) = F^{-1}(X_1 \times \ldots \times U_i \times \ldots \times X_n) = \Big(\bigcap_{j\neq i} F_j^{-1}(X_j)\Big) \cap F_i^{-1}(U_i) = F_i^{-1}(U_i)$$

is open in B, since $F_j^{-1}(X_j) = B$. Thus, F_i is continuous for each i.
($\impliedby$)Suppose F_i is continuous for each i. Consider the set $U_1 \times \ldots \times U_n$, an open set in the product space, for some open sets $U_i \subseteq X_i$. Then $F^{-1}(U_1 \times \ldots \times U_n) = \bigcap_{i=1}^n F_i^{-1}(U_i)$ by elementary set theory, which is a finite intersection of open sets in B as F_i is continuous for each i. Thus, F is continuous.

(b) See [1, Thm. 3.30].

(c) In the characteristic property of the product space, set $B = X_1 \times \ldots \times X_n$ and $F = Id$ on the product space, which is continuous map. Then $F_i = \pi_i \circ Id = \pi_i$ is continuous on X_i.

(d) Let $U_1 \times \ldots \times U_n$ is open in $Y_1 \times \ldots \times Y_n$. Then its inverse image under the product map is

$$\begin{aligned}(F_1 \times \ldots \times F_n)^{-1}(U_1 \times \ldots \times U_n) &= \{(x_1, \ldots, x_n) : (F_1(x_1), \ldots, F_n(X_n) \in U_1 \times \ldots \times U_n\} \\ &= \bigcap_{i=1}^n \{(x_1, \ldots, x_n) : F_i(x_i) \in U_i)\} \\ &= \bigcap_{i=1}^n X_1 \times \ldots \times F_i(U_i) \times \ldots \times X_n = \bigcap_{i=1}^n \bar{V}_i,\end{aligned}$$

where $\bar{V}_i = X_1 \times \ldots \times F_i(U_i) \times \ldots \times X_n$. Note that $\bar{V}_i$ is open in $X_1 \times \ldots \times X_n$, and $(F_1 \times \ldots \times F_n)^{-1}(U_1 \times \ldots \times U_n)$ is finite intersection of open sets which is again open in $X_1 \times \ldots \times X_n$.

(e) Considering subspace topology, any open subset of $S_1 \times \ldots \times S_n$ can be written as $U = S_1 \times \ldots \times S_n \cap U_1 \times \ldots \times U_n$, for some open set $U_i \subseteq X_i$. But note that $(S_1 \cap U_1) \times \ldots \times (S_n \cap U_n)$, which is open set with respect to the product topology on $S_1 \times \ldots \times S_n$, since $S_i \cap U_i$ is open in S_i.

(f) ($\implies$)Considered as a map from X_i to $V = \{(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n) : x \in X_i\}$, it is bijective. It is continuous by using characteristic property with F being the given function and $B = X_i$. Then note that $F_j(x) = a_j$, for $j \neq i$, which is a constant function and $F_i(x) = x$, which is an identity function. For each j, F_j is continuous on X_i, thus F is continuous on X_i.
($\impliedby$)Notice that the inverse function is just the i-th projection map, which is again continuous on domain restricted X.

(g) Since $\mathcal{B}_i$ is a basis for X_i, we have that $\bigcup_{B_i \in \mathcal{B}_i} B_i = X_i$, so that $\bigcup_{B \in \mathcal{B}} B = X_1 \times \ldots \times X_n$. Suppose $B' = \prod_{i=1}^n B_i'$, and $B'' = \prod_{i=1}^n B_i''$, $B_i', B_i'' \in \mathcal{B}_i$. Let $p \in B' \cap B''$. Then note that

$$B' \cap B'' = \Big(\prod_{i=1}^n B_i'\Big) \cap \Big(\prod_{i=1}^n B_i''\Big) = \prod_{i=1}^n (B_i' \cap B_i''),$$

so that there exists a basic open set $\bar{B}_i \subseteq B_i' \cap B_i''$, such that $p \in \prod_{i=1}^n \bar{B}_i \subseteq B' \cap B''$.

(h) Let p, q be two distinct points in the product space. Then there exists i, such that $p^{(i)} \neq q^{(i)}$. Then consider the neighborhoods $U = X_1 \times \ldots X_{i-1} \times U_i \times X_{i+1} \times \ldots \times X_n$ and $V = X_1 \times \ldots X_{i-1} \times V_i \times X_{i+1} \times \ldots \times X_n$, where $U_i, V_i \subseteq X_i$ are disjoint neighborhoods of $p^{(i)}$ and $q^{(i)}$. Then we have $U \cap V = X_1 \times \ldots X_{i-1} \times (U_i \cap V_i) \times X_{i+1} \times \ldots \times X_n = \emptyset$.

(i)

(j) □

Lemma 3. *The addition, subtraction, and multiplication operations are continuous functions from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$; and the quotient operation is continuous from $\mathbb{R} \times (\mathbb{R} - \{0\})$ into $\mathbb{R}$.*

Proposition 29. *If X is a topological space, and if $F, G : X \to \mathbb{R}$ are continuous, then $F+G, F-G, F \cdot G$ are continuous. If $G(x) \neq 0$ for all x, then F/G is continuous.*

Proof. The map $H : X \to \mathbb{R} \times \mathbb{R}$ defined by $H(x) = (F(x), G(x))$ is continuous by Prop. 28. Then $F+G$ is continuous as is $F = + \circ H$, where $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the addition function which is continuous. Similar arguments apply for $F - G, F \cdot G$ and F/G. □

Proposition 30. *Let (M, d) be a metric space. Then $d : M \times M \to \mathbb{R}$ is continuous in the metric topology.*

Proof. We show that inverse image of every basis element(open interval) of $\mathbb{R}$ is open in $M \times M$. Consider an open interval $I = (a..b), 0 < a < b$. Then the inverse image of I is $U = d^{-1}(I) = \{(x, y) \in M \times M : a < d(x, y) < b\}$. Now choose $r > 0$ such that $\mathbb{B}_{d(x,y)}(2r) \subseteq I$. We show that $\mathbb{B}_x(r) \times \mathbb{B}_y(r) \subseteq U$ which completes the proof. For any $(x', y') \in \mathbb{B}_x(r) \times \mathbb{B}_y(r)$, we have $d(x', y') \leq d(x', x) + d(x, y) + d(y, y') < d(x, y) + 2r$ so that $d(x', y') < d(x, y) + 2r$ and similarly, $d(x, y) \leq d(x, x') + d(x', y') + d(y', y) < d(x', y') + 2r$ so that $d(x, y) - 2r < d(x', y')$. Combining the two inequalities, we have $a < d(x, y) - 2r < d(x', y') < d(x, y) + 2r < b$ such that $(x', y') \in U$. □

7 Disjoint Union Spaces

Definition 23. *If $\{X_\alpha\}_{\alpha \in J}$ is an indexed family of sets, their **disjoint union** is the set $\coprod_{\alpha \in J} X_\alpha = \{(x, \alpha) : \alpha \in J, x \in X_\alpha\}$. For each α there is a canonical injective map $\iota_\alpha : X_\alpha \to \coprod_{\alpha \in J} X_\alpha$ given by $\iota_\alpha(x) = (x, \alpha)$, and the images of these maps for different values of α are disjoint.*

Typically, we implicitly identify X_α with its image in the disjoint union, thereby viewing X_α as a subset of $\coprod_{\alpha \in J} X_\alpha$.

Definition 24. *Given an indexed family of topological spaces $\{X_\alpha\}_{\alpha \in J}$, we define the **disjoint union topology** on $\coprod_{\alpha \in J} X_\alpha$ by declaring a subset of $\coprod_{\alpha \in J} X_\alpha$ to be open iff its intersection with each X_α is open in X_α.*

Proposition 31 (*Properties of the Disjoint Union Topology*). *Suppose $\{X_\alpha\}_{\alpha \in J}$ is an indexed family of topological spaces, and $\coprod_{\alpha \in J} X_\alpha$ is endowed with the disjoint union topology.*

(a) (CHARACTERISTIC PROPERTY). If Y is a topological space, a map $F : \coprod_{\alpha \in J} X_\alpha \to Y$ is continuous iff $F \circ \iota_\alpha : X_\alpha \to Y$ is continuous for each $\alpha \in J$.

(b) The disjoint union topology is the unique topology on $\coprod_{\alpha \in J} X_\alpha$ for which the characteristic property holds.

(c) A subset of $\coprod_{\alpha \in J} X_\alpha$ is closed iff its intersection with each X_α is closed.

(d) Each injection $\iota_\alpha : X_\alpha \to \coprod_{\alpha \in J} X_\alpha$ is a topological embedding.

(e) Every disjoint union of Hausdorff spaces is Hausdorff.

(f) Every disjoint union of first-countable spaces is first-countable.

(g) Every disjoint union of countably many second-countable spaces is second-countable.

Proof. (a) Suppose $U \subseteq Y$ is an open set. Then note that $F^{-1}(U)$ is open in $\coprod_\alpha X_\alpha$ iff the restriction of $F^{-1}(U)$ to X_α is open in X_α for each α iff $\iota^{-1}(F^{-1}(U)) = (F \circ \iota_\alpha)^{-1}(U)$ is open in X_α for each $\alpha \in J$.

(b)

(c) Suppose $C \subseteq \coprod_\alpha X_\alpha$ is closed. Then $\coprod_\alpha X_\alpha - C$ is open so that $\iota_\alpha^{-1}(\coprod_\alpha X_\alpha - C) = X_\alpha - \iota_\alpha^{-1}(C)$ is open for each $\alpha \in J$.

(d) Define the map $\iota'_\alpha = \iota_\alpha : X_\alpha \to \iota_\alpha(X_\alpha)$. The map ι'_α is a bijection. The continuity of the map is implied from the definition of disjoint union topology. For any open subset $U \subseteq X_\alpha$, consider $\iota'_\alpha(U)$. For $\alpha \neq \beta$ we have $\iota_\beta^{-1}(\iota'_\alpha(U)) = \emptyset$ which is open in X_β and $\iota_\alpha^{-1}(\iota'_\alpha(U)) = U$ which is open in X_α by assumption. Then $\iota'_\alpha(U)$ is open in $\coprod_\alpha X_\alpha$. Thus ι'_α is a topological embedding.

(e) Suppose (x, α) and (y, β) be two distinct points of $\coprod_\alpha X_\alpha$. Then $\iota_\alpha^{-1}(x, \alpha) = x \in X_\alpha$ and $\iota_\beta^{-1}(y, \beta) = y \in X_\beta$. If $\alpha \neq \beta$, then any neighborhood $\iota_\alpha(U)$ of (x, α) and $\iota_\beta(V)$ of (y, β) are disjoint, where U is open in X_α and V is open in X_β. If $\alpha = \beta$, then there exist disjoint neighborhoods of $U, V \subseteq X_\alpha$ of x and y respectively. Then $\iota_\alpha(U)$ and $\iota_\alpha(V)$ are disjoint neighborhood of (x, α) and (y, α) since $\iota_\gamma^{-1}(\iota_\alpha(U)) = \iota_\gamma^{-1}(\iota_\alpha(V)) = \emptyset$ for $\gamma \neq \alpha$ and $\iota_\alpha^{-1}(\iota_\alpha(U)) = U$, $\iota_\alpha^{-1}(\iota_\alpha(V)) = V$ which are all disjoint.

(f)

(g) □

8 Quotient Spaces and Quotient Maps

Definition 25. *Let $\pi : X \to Y$ be a map between two topological spaces. A subset $U \subseteq X$ is said to be* ***saturated with respect to π*** *if U is the entire preimage of its image, i.e., if $U = \pi^{-1}(\pi(U))$. Given $y \in Y$, the* ***fiber of π over y*** *is the set $\pi^{-1}(\{y\})$.*

Proposition 32. *Let $\pi : X \to Y$ be a map between two topological spaces and U be a subset of X. The following are equivalent:*

(a) U is saturated w.r.t. π.

(b) There exists a subset $V \subseteq Y$, such that $U = \pi^{-1}(V)$.

(c) If $\pi^{-1}(\{y\}) \cap U \neq \emptyset$ for any $y \in Y$, then $\pi^{-1}(\{y\}) \subseteq U$.

Proof. We show that $(a) \implies (b), (b) \implies (c)$ and $(c) \implies (a)$.
(a) $\implies$ (b) Take $V = \pi(U)$.
(b) $\implies$ (c) Let $U = \pi^{-1}(V)$ for some $V \subseteq Y$ and let $y \in Y$ be any point. If $u \in \pi^{-1}(\{y\}) \cap U$, then $u \in U$ and $y = \pi(u)$ so that $y \in V$. Then $\pi^{-1}(\{y\}) \subseteq \pi^{-1}(V) = U$.
(c) $\implies$ (a) We need to show that assuming (c) holds, $U = \pi^{-1}(\pi(U))$. But $U \subseteq \pi^{-1}(\pi(U))$ by elementary set theory, so we show the converse to hold. If $x \in \pi^{-1}(\pi(U))$, then $\pi(x) \in \pi(U)$. Then there is a $u \in U$ such that $\pi(x) = \pi(u)$. But then $u \in \pi^{-1}(\{\pi(x)\}) \cap U$ and so $x \in \pi^{-1}(\{\pi(x)\}) \subseteq U$ by (c). □

Remark 6. *We can restate the condition (c) of Prop. 32 as follows: U is saturated w.r.t. π iff it is a union of fibers.*

Definition 26. *Let X and Y be topological spaces; let $\pi : X \to Y$ be a surjective map. The map π is said to be a* ***quotient map*** *provided a subset U of Y is open in Y iff $\pi^{-1}(U)$ is open in X.*

Remark 7. *An equivalent condition for a quotient map given in Def. 26 is to require that a subset A of Y is closed in Y iff $\pi^{-1}(A)$ is closed in X. This follows from the fact that $\pi^{-1}(Y - A) = X - \pi^{-1}(A)$.*

Proposition 33. *Let X and Y be topological spaces and suppose that $F : X \to Y$ is a surjective continuous map. Then the following are equivalent:*

(a) F is a quotient map.

(b) F takes saturated open subsets to open subsets.

(c) F takes saturated closed subsets to closed subsets.

Proof. We prove that $(a) \implies (b)$ and $(b) \implies (a)$.
(a) $\implies$ (b) Let F be a quotient map and U is a saturated open subset of X such that $U = F^{-1}(F(U))$. But then $V = F(U)$ is open in Y since $F^{-1}(V)$ is open in X.
(b) $\implies$ (a) Let V be open an subset of Y. Then by continuity of F, $F^{-1}(V)$ is open in X. Now suppose that $U = F^{-1}(V)$ is an open subset of X. Then U saturated by Prop. 32(b) and $F(U) = F(F^{-1}(V)) = V$ since F is surjective. But $F(U)$ is open in Y by assumption.

Similar arguments show that (c) is equivalent to (a) by using the alternate definition of quotient maps in terms of closed set(Remark 7). □

Proposition 34. *If X is a space and Y is a set and if $\pi : X \to Y$ is a surjective map, then there exists exactly one topology $\mathcal{T}$ on Y relative to which π is a quotient map; it is called the* ***quotient topology on Y induced by π****.*

Proof. Let $\mathcal{T} = \{V \subseteq Y : \pi^{-1}(V) \text{ is open in } X\}$ be the topology on Y. Then the map π is a quotient map. We show that it is indeed a topology. Since $\pi^{-1}(\emptyset) = \emptyset$ is open in X, $\emptyset \in \mathcal{T}$. By surjectivity of π, $\pi^{-1}(Y) = X$, which is open in X, so that $Y \in \mathcal{T}$. Now suppose $\{V_\alpha\}_{\alpha \in J}$ and $\{U_i\}_{i=1}^n$ be collections of open subsets in Y, where J is an arbitrary set. Then $\pi^{-1}(\cup_{\alpha \in J} V_\alpha) = \cup_{\alpha \in J} \pi^{-1}(V_\alpha)$ which is union of a arbitrary collection of open subsets in X and $\pi^{-1}(\cap_{i=1}^n U_i) = \cap_{i=1}^n \pi^{-1}(U_i)$ which is finite intersection of open subsets in X. Thus $\cup_{\alpha \in J} V_\alpha, \cap_{i=1}^n U_i \in \mathcal{T}$. This shows the existence of such topology on Y.

To prove the unqiueness note that since any additional or less open sets in set Y would mean that π is not (by definition) a quotient map. □

A relation $\sim$ on a set X is called an ***equivalence relation*** if it is ***reflexive***($x \sim x$ for all $x \in X$), ***symmetric***($x \sim y \implies y \sim x$), and ***transitive***($x \sim y$ and $y \sim z \implies x \sim z$). If $\sim$ is an equivalence relation on X, then for each $x \in X$, the ***equivalence class of x***, denoted by $[x]$, is the set of all $y \in X$ such that $x \sim y$. The set of all equivalence classes, denoted by $X/\sim$, is a ***partition of X***, which is a disjoint non-empty collection of subsets of X whose union is X.

Definition 27. *Suppose X is a topological space and $\sim$ is an equivalence relation on X. Let $\pi : X \to X/\sim$ be the natural projection sending each point to its equivalence class. Endowed with the quotient topology determined by π, the space $X/\sim$ is called the* ***quotient space*** *(or* ***identification space****) of X determined by $\sim$.*

Remark 8. *We can describe the topology of $X/\sim$ in another way. A subset U of $X/\sim$ is a collection of equivalence classes, and the set $\pi^{-1}(U)$ is the union of the equivalence classes belonging to U. Thus the typical open set of $X/\sim$ is a collection of equivalence classes whose union is an open set of X.*

Proposition 35 (***Properties of Quotient Maps***)**.** *Let $\pi : X \to Y$ be a quotient map.*

(a) (CHARACTERISTIC PROPERTY). If B is a topological space, a map $F : Y \to B$ is continuous iff $F \circ \pi : X \to B$ is continuous.

(b) The quotient topology is the unique topology on Y for which the characteristic property holds.

(c) If π is injective, then it is a homeomorphism.

(d) If $U \subseteq X$ is a saturated open or closed subset, then $\pi|_U : U \to \pi(U)$ is a quotient map.

(e) Any composition of π with another quotient map is again a quotient map.

Proof. (a) Let $U \subseteq B$ be an open subset. Then the result follows immediately by noting that $F^{-1}(U)$ is open in Y iff $\pi^{-1}(F^{-1}(U)) = (F \circ \pi)^{-1}(U)$ is open in X.

(b) The results follows from Prop. 34.

(c) If π is innjective, then π is bijective map such that $V \subseteq Y$ is open iff $\pi^{-1}(V)$ is an open subset of X. Thus π is a homeomorphism.

(d) Let $V \subseteq U$ be a saturated open subset of w.r.t. $\pi' \equiv \pi|_U$. Then $V = \pi'^{-1}(\pi'(V)) = \pi^{-1}(\pi(V))$ since U is saturated w.r.t. π. So V is saturated w.r.t. π. Then $\pi(V) = \pi'(V) \subseteq \pi(U)$ is open in Y. But $\pi(U)$ is open in Y so that $\pi'(V) = \pi(V) \cap \pi(U)$ is open in $\pi(U)$.

(e) Suppose $\pi' : Y \to Z$ is a quotient map. Then for any subset $V \in Z$, V is open in Z iff $\pi'^{-1}(V)$ is open in Y iff $\pi^{-1}(\pi'^{-1}(V))$ is open in X. Then $(\pi' \circ \pi)^{-1}(V)$ is open iff V is open, which shows that $\pi' \circ \pi$ is a quotient map. □

Proposition 36 (***Passing to the Quotient***)**.** *Suppose $\pi : X \to Y$ is a quotient map, B is a topological space, and $F : X \to B$ is a map that is constant on the fibers of π (i.e., $\pi(p) = \pi(q) \implies F(p) = F(q)$). Then F induces a unique map $\tilde{F} : Y \to B$ such that $F = \tilde{F} \circ \pi$. The induced map $\tilde{F}$ is continuous iff F is continuous.*

Proof. For each $y \in Y$, $F(\pi^{-1}(\{y\}))$ is a singleton set in B since F is constant on the fibers of π. If we let $\tilde{F}(y)$ denote this point, then we have defined a map $\tilde{F} : Y \to B$ such that for each $x \in X$, $\tilde{F}(\pi(x)) = F(x)$. If $\tilde{G}$ is also induced by F, then for all $x \in X$, $\tilde{F}(\pi(x)) = \tilde{G}(\pi(x)) = F(x)$, but then $\tilde{G} = \tilde{F}$. This show that existence and uniqueness of the induced map. The continuity of $\tilde{F}$ is implied by Prop. 35(a). □

Proposition 37 (***Uniqueness of Quotient Spaces***)**.** *If $\pi_1 : X \to Y_1$ and $\pi_2 : X \to Y_2$ are quotient maps that are constant on each other's fibers (i.e., $\pi_1(p) = \pi_1(q)$ iff $\pi_2(p) = \pi_2(q)$), then there exists a unique homeomorphism $\varphi : Y_1 \to Y_2$ such that $\varphi \circ \pi_1 = \pi_2$.*

Proof. Applying the Prop. 36 to the quotient map $\pi_1 : X \to Y_1$, we see that π_2 passes to the quotient, yielding a continuous map $\tilde{\pi}_2 : Y_1 \to Y_2$ such that $\tilde{\pi}_2 \circ \pi_1 = \pi_2$. Applying the same argument with roles of π_1 and π_2 reversed, there is a continuous map $\tilde{\pi}_1 : Y_2 \to Y_1$ such that $\tilde{\pi}_1 \circ \pi_2 = \pi_1$. Together, these identities imply that $\tilde{\pi}_2 \circ \tilde{\pi}_1 \circ \pi_2 = \pi_2$. Apply the Prop. 36 again with π_2 playing the roles of both π and F, we see that both $\tilde{\pi}_2 \circ \tilde{\pi}_1$ and Id_{Y_2} are obtained from π_2 by passing to the quotient, so the uniqueness assertion of Prop. 36 implies that $\tilde{\pi}_2 \circ \tilde{\pi}_1 = Id_{Y_2}$. A similar argument shows that $\tilde{\pi}_1 \circ \tilde{\pi}_2 = Id_{Y_1}$, so that $\tilde{\pi}_2$ is the desired homeomorphism. □

9 Open and Closed Maps

Definition 28. *A map $F : X \to Y$ (continuous or not) is said to be an* ***open map*** *if for every open subset $U \subseteq X$, the image set $F(U)$ is open in Y, and a* ***closed map*** *if for every closed subset $K \subseteq X$, the image set $F(K)$ is closed in Y.*

Example 7. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = c$, where $c \in \mathbb{R}$ is a constant. Then for any closed interval $I_1 = [a, b]$ in $\mathbb{R}$, $f(I_1) = \{c\}$ which is closed in $\mathbb{R}$, thus f is closed. But note that for an open interval $I_2 = (a, b) \subseteq \mathbb{R}$, $f(I_2) = \{c\}$, which is closed, so that f is not open. Thus a closed map is not necessarily open.*

Proposition 38. *Suppose $X_1, \ldots, X_k$ are topological spaces. Then each projection $\pi_i : X_1 \times \ldots \times X_k \to X_i$ is an open map.*

Proof. The result follows immediately by noting that $U = U_1 \times \ldots \times U_k$ is open in $X_1 \times \ldots \times X_k$ iff $U_i = \pi_i(U)$ is open in X_i for all $i = 1, \ldots, k$. □

Example 8. *Let $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ be the projection onto the first coordinate, i.e., $\pi_1(x_1, x_2) = x_1$. Then the preceding proposition shows that π_1 is open. We show that π_1 is not closed. Let $S = \{(x_1, x_2) : x_1 x_2 = 1\}$ be the hyperbola. Then note that S is closed in $\mathbb{R}^2$ since it is the inverse image of $\{1\}$ under $(x_1, x_2) \mapsto x_1 x_2$ (which is continuous). Now note that $\pi_1(S) = (0, \infty)$ which not closed, so that π_1 is not closed. Thus an open map is not necessarily closed.*

Proposition 39. *Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces. Then each injection $\iota_\alpha : X_\alpha \to \coprod_{\alpha \in J} X_\alpha$ is both open and closed.*

Proof. Follows from Def. 24 and Prop. 31(c). □

Proposition 40. *Every local homeomorphism is an open map.*

Proof. Follows from the definition of local homeomorphism(Def. 19). □

Proposition 41. *Every bijective local homeomorphism is a homeomorphism.*

Proof. Local homeomorphism is a continuous map and by the preceding proposition and bijection property, its inverse is also continuous. □

Proposition 42. *Suppose X is Hausdorff and that $\pi : X \to Y$ is an open quotient map. Then Y is Hausdorff iff the set $\mathcal{R} = \{(x_1, x_2) : \pi(x_1) = \pi(x_2)\}$ is closed in $X \times X$.*

Proof. ($\Longrightarrow$)Define $O = X \times X - \mathcal{R} = \{(x_1, x_2), \pi(x_1) \neq \pi(x_2)\}$. Then for any $(x_1, x_2) \in O$, we have $\pi(x_1) \neq \pi(x_2)$ so that there exist disjoint neighborhoods U and V of $\pi(x_1)$ and $\pi(x_2)$ respectively. By the definition of the quotient map, $\pi^{-1}(U)$ and $\pi^{-1}(V)$ are open disjoint in X. Consider the set $W = \pi^{-1}(U) \times \pi^{-1}(V)$. If $(a, b) \in W$, then $\pi(a) \in U$ and $\pi(b) \in V$, so $\pi(a) \neq \pi(b)$, i.e., $(a, b) \in O$ which implies that $W \subseteq O$. Also note that W is an open set. It follows that O is also open in $X \times X$, then $\mathcal{R}$ is closed.

($\Longleftarrow$)Suppose $x, y \in Y$, such that $x \neq y$. Then there exists $(a, b) \in O$ such that $x = \pi(a)$ and $y = \pi(b)$. Since O is open and X is Hausdorff, there exist disjoint neighborhoods U and V of a and b respectively such that $U \times V \subseteq O$. Since π is open, $\pi(U)$ and $\pi(V)$ are open in Y containing x and y respectively. Then note that $\pi(U) \cap \pi(V) = \emptyset$, thus Y is Hausdorff. □

Proposition 43. *Let X and Y be topological spaces, and let $F : X \to Y$ be a map. Then*

(a) F is closed iff for every $A \subseteq X$, $F(\bar{A}) \supseteq \overline{F(A)}$.

(b) F is open iff for every $B \subseteq Y$, $F^{-1}(Int\, B) \supseteq Int\, F^{-1}(B)$.

Proof. (a) ($\Longrightarrow$)Since $\bar{A}$ is closed in X, $F(\bar{A})$ is closed in Y. But $A \subseteq \bar{A}$, then $F(A) \subseteq F(\bar{A})$. So $F(\bar{A})$ is a closed set in Y containing $F(A)$, then $F(\bar{A}) \supseteq \overline{F(A)}$.
($\Longleftarrow$)Suppose $A \subseteq X$ is a closed set. Then $F(\bar{A}) = F(A)$, but $F(\bar{A}) = F(A) \supseteq \overline{F(A)}$ which implies that $F(A)$ is closed in Y.

(b) ($\Longrightarrow$)Let $p \in Int\, F^{-1}(B)$, then $F(p) \in B$ and there exists a neighborhood U of p such that $U \subseteq F^{-1}(B)$. But F is open, then $F(U)$ is a neigborhood of $F(p)$ such that $F(U) \subseteq F(F^{-1}(B)) \subseteq B$ which means $F(p) \in Int\, B$. Then $p \in F^{-1}(Int\, B)$.
($\Longleftarrow$)Suppose $U \subseteq X$ is open. Then $F(U) \subseteq Y$ satisfies $F^{-1}(Int\, F(U)) \supseteq Int\, F^{-1}(F(U)) \supseteq Int\, U = U$, i.e., $F^{-1}(Int\, F(U)) \supseteq U$. Then $F(F^{-1}(Int\, F(U))) \supseteq F(U)$. But $Int\, F(U) \supseteq F(F^{-1}(Int\, F(U)))$, so we have $Int\, F(U) \supseteq F(U)$, which means $F(U)$ is open in Y. □

Proposition 44. *Suppose X and Y are topological spaces, and $F : X \to Y$ is a continuous map that is either open or closed.*

(a) If F is surjective, then it is a quotient map.

(b) If F is injective, then it is a topological embedding.

(c) If F is bijective, then it is a homeomorphism.

Proof. (a) Suppose F is surjective. If it is open, it certainly takes saturated open subsets to open subsets. Similarly, if it is closed, it takes saturated closed subsets to closed subsets. Thus it is a quotient map.

(b) Now suppose F is open and injective. Then $F : X \to F(X)$ is bijective, so $F^{-1} : F(X) \to X$ exists. If $U \subseteq X$ is open, then $(F^{-1})^{-1}(U) = F(U)$ is open in Y by assumption, and therefore is also open in $F(X)$. This proves that F^{-1} is continuous, so that F is a homeomorphism onto its image. If F is closed, the same argument goes through with "open" replaced by "closed".

(c) This immediately follows from (b) by setting $F(X) = Y$. □

10 Connectedness

Definition 29. *A topological space X is said to be **disconnected** if it has two disjoint nonempty open subsets whose union is X in which case we say a **separation** of X is a pair U, V of disjoint nonempty open subsets such that $U \cup V = X$. It is **connected** if there does not exist a separation of X.*

Proposition 45. *A topological space X is connected iff the only subsets of X that are both open and closed are $\emptyset, X$.*

Proof. ($\implies$)Suppose X is connected and that $U \subseteq X$ which is both open and closed. Then the set $V = X \setminus U$ is both open and closed in X such that $U \cup V = X$ and $U \cap V = \emptyset$. Since X is connected, either U or V must be empty, which gives $X, \emptyset$ are only subsets of X which are both open and closed.
($\impliedby$)We prove the contrapositive of the statement, i.e., suppose X is disconnected, then there exist a set $U \subseteq X$ not equal to $X, \emptyset$ which is both open and closed. Since X is disconnected, there exists disjoint open set $U, V \subseteq X$ which are nonempty and $U \cup V = X$. Then note that U, V are also closed. □

Definition 30. *Let X be a topological space. A subset A of X is called a **connected subset of X** if A is a connected space when endowed with the subspace topology.*

Proposition 46. *If Y is a subspace of X, a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y, neither of which contains a limit point of the other. The space Y is connected if there exists no separation.*

Proof. ($\implies$)Suppose A and B form a separation of Y. Then A is both open and closed in Y. The closure of A in Y is $\bar{A} \cap Y$, where $\bar{A}$ is closure of A in X. But since A is closed in Y, we have $A = \bar{A} \cap Y$ or $\bar{A} \cap B = \emptyset$. Then B contains no limit points of A. A similar argument show that A contains no limit points of B.

($\impliedby$)Suppose A and B are disjoint nonempty sets whose union is Y, neither of which contains a limit point of the other. Then $\bar{A} \cap B = \emptyset$ and $A \cap \bar{B} = \emptyset$; therefore we conclude that $\bar{A} \cap Y = A$ and $\bar{B} \cap Y = B$. Thus both A and B are closed in Y, and since $A = Y - B$ and $B = Y - A$, they are open in Y as well. □

Definition 31. *Let X be a topological space. A **component of X** is a maximal nonempty connected subset of X.*

Remark 9. *By definition empty set contains no components.*

Proposition 47 (*Properties of Connected Spaces*). *Let X and Y be topological spaces.*

(a) If $F : X \to Y$ is continuous and X is connected, then $F(X)$ is connected.

(b) ***(Topological Invariance of Connectedness).*** *Every space homeomorphic to a connected space is connected.*

(c) *Suppose X is any topological space and U, V are disjoint open subsets of X. If A is a connected subset of X contained in $U \cup V$, then either $A \subseteq U$ or $A \subseteq V$.*

(d) *A union of connected subspaces of X with a point in common is connected.*

(e) *If X contains a dense connected subset, then X is connected.*

(f) *Suppose $A \subseteq X$ is connected. Then $\bar{A}$ is connected, as is any subset B such that $A \subseteq B \subseteq \bar{A}$. This can be rephrased as: "If B is formed by adjoining to the connected subspace A some or all of its limit points, then B is connected".*

(g) *Every product of finitely many connected spaces is connected.*

(h) *Every quotient space of a connected space is connected.*

(i) *The components of X form a partition of X.*

(j) *Each component of X is closed in X.*

(k) *Every connected subset is contained in a single component of X.*

(l) *If S is a subset of X that is both open and closed, then S is a union of components of X.*

Proof. ***(a)*** By replacing Y with $F(X)$, we may as well assume that f is surjective. We prove the contrapositive, i.e., if Y is disconnected, then X is also disconnected. Since Y is disconnected it is union of two nonempty disjoint open subsets U, V. Then $F^{-1}(U), F^{-1}(V)$ must be nonempty, since F is surjective and open in X, since F is continuous. They must also be closed as U, V are closed in Y. Thus X is disconnected.

(b) If X and Y are homeomorphic, then there exists a homeomorphism between them. Since it is bijective and continuous both sides X and Y are both connected.

(c) Let A contained points in both U and V. Consider two open subset of A, $U' = A \cap U$ and $V' = A \cap V$. Then $U' \cap V' = (A \cap U) \cap (A \cap V) = A \cap (U \cap V) = \emptyset$ and $U' \cup V' = A$, so that A is disconnected, which is a contradiction.

(d) Let $\{B_\alpha\}_{\alpha \in A}$ be a collection of connected subspaces of X with a point p in commmon. Suppose U, V are disjoint open subsets of $\bigcup_{\alpha \in A} B_\alpha$. WLOG, suppose $p \in U$. By (c), each of the B_α is entirely contained in U and so is their union. Thus, $\bigcup_{\alpha \in A} B_\alpha$ is connected.

(e) Let $A \subseteq X$ be a dense connected subset, and assume for the sake of contradiction that X is not connnected. Then it is a disjoint union of open subsets U, V, which are also closed. Then (c) shows that A is contained in one of these sets, say U. Then it follows that $X = \bar{A} \subseteq \bar{U} = U$, since U is closed. Then $X = U$ and $V = \emptyset$, which is a contradiction.

(f) Suppose A is connected and $A \subseteq B \subseteq \bar{A}$. Then note that A is dense in $\bar{A}$ and similarly, A is dense in B, since $\bar{A} \cap B = B$. Then (e) shows that B is connected. Applying this with $B = \bar{A}$ then show that $\bar{A}$ is connected.

(g) Since $X_1 \times \ldots \times X_n = (X_1 \times \ldots \times X_{n-1}) \times X_n$, by induction it suffices to consider a product of two spaces. Thus let X, Y be connected spaces and suppose that there are disjoint open subsets U, V whose union is $X \times Y$. Let (x_0, y_0) be a point in U. The set $\{x_0\} \times Y$ is connected because it is homeomorphic to Y. Since it contains $(x_0, y_0) \in U$, it must be entirely contained in U by (c). For each $y \in Y$, the set $X \times \{y\}$ is also connected and has the point $(x_0, y_0) \in U$, so it must be contained in U. Since $X \times Y$ is the union of the sets $X \times \{y\}$ for $y \in Y$, it follows that $U = X \times Y$ and V is empty.

(h) It follows from (a) and the fact that quotient maps are surjective.

(i) We need to show that the components are disjoint and their union is X. To see that distinct components are disjoint, suppose U, V are components that are not disjoint. Then there

is a point in common, and (d) implies that $U \cup V$ is connected. By maximality, $U \cup V = U = V$, so U and V are not distinct.

To see that union of the components is X, let $x \in X$ be arbitrary. There is atleast one connected set containing x, namely $\{x\}$. If U is union of all connected sets containing x, then U is connected by (d), and it certainly is maximal, so it is a component containing x.

(j) If B is any component of X, it follows from (e) that $\bar{B}$ is a connected set containing B. Since components are maximal connected sets, $\bar{B} = B$, so B is closed.

(k) Suppose $A \subseteq X$ is connected. Because the components cover X, if A is nonempty, it has a point in common with some component B. By (d), $A \cup B$ is connected, so by maximality of B, $A \cup B = B$, which implies that $A \subseteq B$.

(l) If $S \subseteq X$ is both open and closed in X. If X is connected, then S is either $\emptyset$ or X. In either case it is union of components of X. Hence, suppose that X is not connected and there it can be partitioned by it components C_i as $X = \bigcup_i C_i$. Consider $S \cap C_i$ for each i. As S is both open and closed, $S \cap C_i$ is also both open and closed in C_i; since C_i is connected, $S \cap C_i$ is either $\emptyset$ or C_i, thus $C_i \subseteq S$ or $S \cap C_i = \emptyset$. Since distinct components are disjoint, we have $S = \bigcup_i S \cap C_i$, which is union of components of X. □

Remark 10. *By Prop. 47(j), any component of X is a closed set. If X is union of only finitely many components, then each component C of X is open in X, since its complement is union of finite collection of closed sets. But in general the components of X need not be open in X.*

Lemma 4. *$\mathbb{R}$ is connected in its standard topology and so are any intervals and rays in $\mathbb{R}$.*

Proposition 48 (Intermediate value theorem)**.** *Let $F : X \to \mathbb{R}$ be a continuous map, where X is a connected space and $\mathbb{R}$ is adjoined with its standard topology. If a and b are two points of X and if $r \in \mathbb{R}$ such that $F(a) < r < F(b)$, then there exists a point $c \in X$ such that $F(c) = r$.*

Proof. Suppose for the sake of contradiction there is no $c \in X$ such that $F(c) = r$, then $A = F(X) \cap (-\infty, r)$ and $B = F(X) \cap (r, +\infty)$ for a separation of $F(X)$ which contradicts the fact that the image of a connected space under a continuous map is connected. □

Definition 32. *Let X be a topological space and $p, q \in X$. A* ***path in X from p to q*** *is a continuous map $\gamma : [0,1] \to X$ such that $\gamma(0) = p$ and $\gamma(1) = q$. We say that X is* ***path-connected*** *if for every $p, q \in X$, there is a path X from p to q. The* ***path components of X*** *are its maximal path-connected subsets.*

Remark 11. *The domain of the path can be any closed interval [a,b] of $\mathbb{R}$ such that $a < b$, since $[a, b]$ is homeomorphic to $[0, 1]$.*

Proposition 49. *Path connectedness implies connectedness.*

Proof. Suppose X is path-connected, and fix $p \in X$. For each $q \in X$, let B_q be the image of a path in X from p to q. Since $[0, 1]$ is connected and B_q is image set of a continuous map over $[0, 1]$, B_q is connected. Thus by Prop. 47(d), $X = \bigcup_{q \in X} B_q$ is connected. □

Proposition 50. *Each path component of X is entirely contained within a component of X.*

Proof. If two points x and y in X are path-connected by $f : [0, 1] \to X$, then they are both contained in the connected subspace $f([0, 1]) \subseteq X$. □

Proposition 51 (***Properties of Path-Connected Spaces***)**.** *Let X and Y be topological spaces.*

(a) If $F : X \to Y$ is continuous and X is path-connected, then $F(X)$ is path-connected.

(b) ***(Topological Invariance of Path-Connectedness).*** *Every space homeomorphic to a path-connected space is path-connected.*

(c) A union of path-connected subspaces of X with a point in common is path-connected.

(d) Every product of finitely many path-connected spaces is path-connected.

(e) Every quotient space of a path-connected space is path-connected.

(f) The path-components of X form a partition of X.

(g) Every path-connected subset is contained in a single path-component of X.

(h) If $S \subseteq X$ is both open and closed, then S is a union of path-components of X.

Proof. ***(a)*** Consider $x, y \in F(X)$ such that $F(a) = x$ and $F(b) = y$ for some $a, b \in X$. Since X is path-connected, there exists a continuous map $\gamma : [0,1] \to X$ such that $\gamma(0) = a$ and $\gamma(1) = b$. Then note that $\varphi = F \circ \gamma : [0,1] \to Y$ is continuous map such that $\varphi(0) = x$ and $\varphi(1) = y$. Thus $F(X)$ is path-connected.

(b) If X and Y are homeomorphic, then there exists a homeomorphism between them. Since it is bijective and continuous both sides X and Y are both path-connected.

(c) Suppose $X = \cup_k C_k$, where each C_k is path-connected subspace of X such that $\cap_k C_k = \{q\}$ and $p_k \in C_k$. Let $i \neq j$. Then there is a path $\gamma_i : I_i = [0,1] \to X$ such that $\gamma_i(0) = p_i$ and $\gamma_i(1) = q$ and γ_i is continuous. Similarly there is a path $\gamma_j : I_j = [1,2] \to X$ such that $\gamma_j(1) = q$ and $\gamma_i(2) = p_j$ and γ_j is continuous. We construct a path between p_i and p_j. Define $\gamma_{ij} : I_{ij} = [0,2] \to X$ such that

$$\gamma_{ij}(c) = \begin{cases} \gamma_i(c), \text{ if } c \in I_i \\ \gamma_j(c), \text{ if } c \in I_j \end{cases}$$

Then note that $I_{ij} = I_i \cup I_j$, which is union of closed sets in $\mathbb{R}$ and γ_{ij} when restricted to I_i is γ_i and when restricted to I_j is γ_j which are both continuous by assumption. Also note that $I_i \cap I_j = \{1\}$ and $\gamma_i(1) = q = \gamma_j(1)$. Then γ_{ij} is continuous by ***Gluing Lemma***(Prop. 24). Then γ_{ij} is a path between p_i to p_j through q.

(d) Suppose $p = (p_1 \ldots, p_k), q = (q_1 \ldots, q_k) \in X_1 \times \ldots \times X_k$. Then there exists a path $\gamma_i : [0,1] \to X_i$ such that $\gamma_i(0) = p_i$ and $\gamma_i(1) = q_i$. Then construct $\gamma : [0,1] \to X_1 \times \ldots \times X_k$ such that $\pi_i \circ \gamma = \gamma_i$. Then by Characteristic property of product topology(Prop. 28(a)), γ is a continuous map such that $\gamma(0) = p$ and $\gamma(1) = q$. Thus $X_1 \times \ldots \times X_k$ is path-connected.

(e) It follows from (a) and the fact that quotient maps are surjective.

(f) We need to show that the path-components are disjoint and their union is X. To see that distinct path-components are disjoint, suppose U, V are path-components that are not disjoint. Then there is a point in common, and (c) implies that $U \cup V$ is connected. By maximality, $U \cup V = U = V$, so U and V are not distinct.

To see that union of the components is X, let $x \in X$ be arbitrary. There is a trivial path containing x, namely $\gamma : [0,1] \to X$ such that $\gamma(c) = x$ for all $c \in [0,1]$. Note that this path is continuous and surjective since its a constant function. Then $\{x\}$ is path-connected. If U is union of all path-connected sets containing x, then U is path-connected by (c), and it certainly is maximal, so it is a path-component containing x.

(g) Follows directly from (f).

(h) Follows from Prop. 47(l) and Prop. 50. □

Remark 12. *Prop. 50 can rephrased as: "Each component is a disjoint union of path-components", since path-components are disjoint sets by Prop. 51(f).*

Example 9. *Some examples of path connected spaces in $\mathbb{R}^n$.*

1. *The* **unit closed ball** $\bar{\mathbb{B}}_0(1) \subseteq \mathbb{R}^n$ *is path-connected. For any two points* $x, y \in \bar{\mathbb{B}}_0(1)$*, the straight-line path* $\gamma : [0,1] \to \mathbb{R}^n$ *defined by* $\gamma(t) = (1-t)x + ty$ *lies in* $\bar{\mathbb{B}}_0(1)$ *since* $\|\gamma(t)\| \leq (1-t)\|x\| + t\|y\| \leq 1$*. A similar argument shows that every open ball* $\mathbb{B}_x(r)$ *and every closed ball* $\bar{\mathbb{B}}_x(r)$ *in* $\mathbb{R}^n$ *is path-connected.*

2. *Define* **punctured euclidean space** *to be the space* $\mathbb{R}^n - \{0\}$*. If* $n > 1$*, this space is path-connected: Given* x, y *different from* 0*, we can join* x *and* y *by straight-line path between them if that path does not go through the origin. Otherwise, we can choose a point* x *not on the line joining* x *and* y*, and take the broken-line path from* x *to* z*, and then from* z *to* y*.*

3. *The* **unit sphere** $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n$ *is path-connected for* $n > 1$*. The map* $g : \mathbb{R}^n - \{0\} \to \mathbb{S}^{n-1}$ *defined by* $g(x) = x/\|x\|$ *is continuous and surjective, then by Prop. 51(a),* $\mathbb{S}^{n-1}$ *is path-connected.*

Example 10 (***Topologist's Sine curve***)**.** *Let* $S = \{x \times \sin(1/x) : 0 < x \leq 1\}$*. Because* S *is the image of* $(0,1]$ *under a continuous map,* S *is connected. Therefore its closure* $\bar{S} \in \mathbb{R}^2$ *is also connected by Prop. 47(f). The set* $\bar{S}$ *is called the* ***topologist's sine curve****. Note that* $\bar{S} = S \cup \{0 \times [-1,1]\}$*. We show that* $\bar{S}$ *is not path-connected.*

Suppose there is a path $\gamma : [c,d] \to \bar{S}$ *beginning at the origin and ending at a point of* S*. The set of those* t *for which* $\gamma(t) \in 0 \times [-1,1]$ *is closed, so is has a largest element* b*. Then* $\gamma : [b,c] \to \bar{S}$ *is a path that maps* b *into the vertical interval* $0 \times [-1,1]$ *and the set* $(b,c]$ *to points of* S*. Replace* $[b,c]$ *by* $[0,1]$ *for convenience; let* $\gamma(t) = (x(t), y(t))$*. Then* $x(0) = 0$*, while* $x(t) > 0$ *and* $y(t) = sin(1/x(t))$ *for* $t > 0$*. We show that there is a sequence of points* $t_n \to 0$ *such that* $y(t_n) = (-1)^n$*. Then the sequence* $y(t_n)$ *does not converge, contradicting continuity of* γ*. To find* t_n*, we proceed as follows: Given* n*, choose* u *with* $0 < u < x(1/n)$ *such that* $\sin(1/u) = (-1)^n$*. Then use the intermediate value theorem to find* t_n *with* $0 < t_n < 1/n$ *such that* $x(t_n) = u$*.*

Definition 33. *A topological space is said to be* ***locally (path-)connected*** *if it admits a basis of (path-)connected open subsets. To be more precise, this means that for any* $p \in X$ *and any neighborhood* U *of* p*, there is a (path-)connected neighborhood of* p *contained in* U*.*

Proposition 52 (***Properties of Locally Connected Spaces***)**.** *Let* X *be a locally connected topological space.*

(a) The components of X *are open in* X*.*

(b) Every open subset of X *is locally connected.*

Proof. ***(a)*** Let $A \subseteq X$ be a component of X, i.e., A is a maximal connected subset of X. Then for any $p \in A$, there exists a connected neighborhood U of p, and $U \subseteq A$ by Prop. 47(c). Thus every point of A has a neighborhood in A, so A is open.

(b) Since X has a basis of connected open subsets, any open subset U will also admit a basis of connected open subsets of U by subspace topology, then by Def. 33, U is locally connected. □

Proposition 53 (***Properties of Locally Path-Connected Spaces***)**.** *Let* X *be a locally path-connected topological space.*

(a) X *is locally connected.*

(b) Every open subset of X *is locally path-connected.*

(c) The path-components of X *are open in* X*.*

(d) The path components of X are equal to its components.

(e) X is connected iff it is path-connected.

Proof. ***(a)*** Every locally path-connected space admits a basis of path-connected open subsets, which are also connected open subsets by Prop. 49.

(b) Since X has a basis of path-connected open subsets, any open subset U will also admit a basis of path-connected open subsets of U by subspace topology, then by Def. 33, U is locally path-connected.

(c) Let $A \subseteq X$ be a path-component of X, i.e., A is a maximal path-connected subset of X. Then for any $p \in A$, there exists a path-connected neighborhood U of p, and $U \subseteq A$ by Prop. 49 and 47(c). Thus every point of A has a neighborhood in A, so A is open.

(d) Let $p \in X$, and A and B be component and path-components containing p, respectively. By Remark. 12, $B \subseteq A$ and A can be written as disjoint union of path-components, each of which is open in X, by (c), thus in A. If $A - B \neq \emptyset$, then the sets B and $A - B$ form a separation of A which contradicts the fact that A is a component. Thus $A = B$.

(e) X is connected iff it has exactly one component, which by (d) is the same as having exactly one path component, which in turn is equivalent to being path-connected. □

11 Compactness

Definition 34. *A topological space X is said to be* ***compact*** *if every open cover of X has a finite subcover. A* ***compact subset*** *of a topological space is one that is a compact space in the subspace topology.*

Proposition 54 (*Properties of Compact Spaces*). *Let X and Y be topological spaces.*

(a) If $F : X \to Y$ is continuous and X is compact, then $F(X)$ is compact.

*(b) (**Topological Invariance of Compactness**). Every space homeomorphic to a compact space is compact.*

(c) Let U be a subspace of X. Then U is compact iff every covering of U by sets open in X contains a finite subcollection covering U.

(d) Every closed subspace of a compact subspace is compact.

(e) If Z is compact subspace of a Hausdorff space X and x_0 is not in Z, then there exist disjoint open sets U, V of X containing x_0 and Z respectively.

(f) Every compact subspace of a Hausdorff space is closed.

(g) Any union of finitely many compact subspaces of X is compact.

(h) Let $F : X \to Y$ be bijective continuous map. If X is compact and Y is Hausdorff, then F is a homeomorphism.

*(i) **(The tube lemma).** Consider the product space $X \times Y$, where Y is compact. If N is open set of $X \times Y$ containing the slice $x_0 \times Y$ of $X \times Y$, then N contains some tube $W \times Y$ about $x_0 \times Y$, where W is a neighborhood of x_0 in X.*

(j) The product of finitely many compact spaces is compact.

(k) Every quotient of a compact space is compact.

(l) If X is Hausdorff and K and L are disjoint compact subsets of X, then there exist disjoint open subsets $U, V \subseteq X$ such that $K \subseteq U$ and $L \subseteq V$.

(m) Every compact subset of a metric space is bounded.

Proof. ***(a)*** Let $\mathcal{A}$ be an open cover of $F(X)$. Then each of $F^{-1}(A)$, $A \in \mathcal{A}$ is open in X by continuity of F and the collection of such sets form an open covering of X. But X is compact, so $F^{-1}(A_1), \ldots, F^{-1}(A_m)$, for some $A_1 \ldots, A_m \in \mathcal{A}$ cover X, so that $A_1, \ldots, A_m$ cover $F(X)$.

(b) If X and Y are homeomorphic, the there exists a homeomorphism between. Since it is bijective and continuous both sides X and Y are both compact if one of them is.

(c) ($\implies$)Let $\mathcal{A} = \{A_\alpha\}_{\alpha \in J}$ is a covering of U by open set in X. Then the collection $\{A_\alpha \cap U : \alpha \in J\}$ is a covering of U by sets open in U; hence a finite subcollection $\{A_{\alpha_1} \cap U, \ldots, A_{\alpha_n} \cap U\}$ covers U. Then $\{A_{\alpha_1}, \ldots, A_{\alpha_n}\}$ is a finite subcollection of $\mathcal{A}$(open set in X) that covers U.
($\impliedby$)Let $\mathcal{A}' = \{A'_\alpha\}$ be a covering of U by open sets in U. For each α, choose a set A_α open in X such that $A'_\alpha = A_\alpha \cap U$. The collection $\mathcal{A} = \{A_\alpha\}$ is a covering of U by open sets in X. By assumption, some finite subcollection $\{A_{\alpha_1}, \ldots, A_{\alpha_n}\}$ covers U. Then $\{A'_{\alpha_1}, \ldots, A'_{\alpha_n}\}$ is a finite subcollection of $\mathcal{A}'$ that covers U.

(d) Let U be a closed subspace of the compact space X. Given a covering $\mathcal{A}$ of U by set open in X, the collection $\mathcal{B} = \mathcal{A} \cup \{X - U\}$ is an open covering of X. Some finite subcollection of $\mathcal{B}$ covers X. If this subcollection contains the set $X - U$, discard $X - U$; otherwise, leave the subcollection alone. The resulting collection is a finite subcollection of $\mathcal{A}$ that covers U.

(e) Given that $x_0 \notin Z$, then $x_0 \in X - Z$. For each point $z \in Z$ consider the disjoint neighborhoods U_z, V_z of x_0 and y respectively(using the Hausdorff condition). Then the collection $\{V_z : z \in Z\}$ is an open covering of Z with open sets in X; therefore, finitely many $V_{z_1}, \ldots, V_{z_n}$ cover Z. The open set $V = V_{z_1} \cup \ldots \cup V_{z_n}$ contains Z, and it is disjoint from the open set $U = U_{z_1} \cap \ldots \cap U_{z_n}$ which is neighborhood of x_0.

(f) By (e) every point not in the compact subspace Z of a Hausdorff space X has a disjoint neighborhood U, so that $X - Z$ is open.

(g) Let $U_1, \ldots, U_n \subseteq X$ be compact subspaces of X. Let $\mathcal{A}$ be covering of $U = \bigcup_{i=1}^n U_i$ by sets open in X. But $U_i \subseteq U$, so $\mathcal{A}$ also covers each U_i. Then a finite subcollection $\mathcal{A}_i$ of $\mathcal{A}$ covers U_i. It follows that the finite subcollection $\mathcal{A}' = \bigcup_{i=1}^n \mathcal{A}_i$ covers U.

(h) We will prove that images of closed sets of X under F are closed in Y; this will prove the continuity of the map F^{-1}. If A is closed in X, then A is compact by (d). Therefore, by (a), $F(A)$ is compact. Since Y is Hausdorff, $F(A)$ is closed in Y, by (f).

(i) Consider the collection $\mathcal{A}$ of basis elements $U \times V \subseteq N$ such that $x_0 \in U$. Then $\mathcal{A}$ is an open cover of the slice $S = x_0 \times Y$ which is compact, since it is homeomorphic to Y. Then there are finitely many such basis elements $\mathcal{A}' = \{U_1 \times V_1, \ldots, U_n \times V_n\}$ that cover the slice S. Define $W = \cap_{i=1}^n U_i$. The set W is open, and it contains x_0. We assert that $\mathcal{A}'$ which were chosen to cover the slice S actually cover the tube $W \times Y$. Let $x \times y \in W \times Y$. Consider a point $x_0 \times y$ of the slice S having the same $y-$coordinate as this point. Now $x_0 \times y$ belongs to $U_i \times V_i$ for some i, so that $y \in V_i$. But $x \in U_j$ for every j(because $x \in W$). Therefore, we have $x \times y \in U_i \times V_i$. Since all the sets $U_i \times V_i$ lie in N and since they cover $W \times Y$, the tube $W \times Y$ lies in N also.

(j) We shall prove that the product of two compact spaces is compact; the result follows by induction for any finite product. Let X and Y be compact spaces. Let $\mathcal{A}$ is an open covering of $X \times Y$. Given $x_0 \in X$, the slice $x_0 \times Y$ is compact and may therefore be covered by finitely many elements $A_1, \ldots, A_m$ of $\mathcal{A}$. Their union $N = \cup_{i=1}^m A_i$ is an open set containing $x_0 \times Y$, then by the tube lemma(i), the open set N contains a tube $W \times Y$ about $x_0 \times Y$, where W is open in X. Then $W \times Y$ is covered by finitely many elements $A_1, \ldots, A_m$ of $\mathcal{A}$. Thus, for each $x \in X$, we can choose a neighborhood W_x of x such that $W_x \times Y$ can be covered by finitely many elements of $\mathcal{A}$. The collection of all the neighborhoods W_x is an open covering of X; therefore by compactness of X there exists a finite subcollection $\{W_1, \ldots, W_n\}$ covering X. The union of the tubes $W_1 \times Y, \ldots, W_n \times Y$ is all of $X \times Y$; since each may be covered by finitely many elements of $\mathcal{A}$, so may $X \times Y$ be covered.

(k) It follows from (a) and the fact that quotient maps are surjective.

(l) Let $x \in L$, then by (e) there exist disjoint open sets $U_x, V_x \subseteq X$ containing x and K respectively. Then the collection $\mathcal{U} = \{U_x : x \in L\}$ is an open cover of L, so there exists a finite subcollection of $U_1, \ldots, U_n$ that covers L. Then the set $V = \cap_{i=1}^n V_i$ is an open set containing K which is disjoint from each U_i. Thus the result follows by taking $U = \cup_{i=1}^n U_i$ and $V = \cap_{i=1}^n V_i$.

(m) Suppose X is a metric space and $A \subseteq X$ is compact. Let x be any point of X, and consider the collection of open ball $\{\mathbb{B}_x(n) : n \in \mathbb{N}\}$ as an open cover of A. By compactness, A is covered by finitely many of these balls. This means that the largest ball $\mathbb{B}_x(n_{\max})$ contains all of A, so A is bounded. □

Lemma 5. *Every closed, bounded interval in $\mathbb{R}$ is compact.*

Proposition 55 (***Heine-Borel***)**.** *The compact subsets of $\mathbb{R}^n$ are exactly the closed and bounded ones.*

Proof. If $K \subseteq \mathbb{R}^n$ is compact, it follows from (f) and (m) of Prop. 54 that it is closed and bounded. Conversely, suppose $K \subseteq \mathbb{R}^n$ is closed and bounded. Then there is some $R > 0$ such that K is contained in the cube $[-R, R]^n$. Now, $[-R, R]$ is compact by Lemma 5, and thus $[-R, R]^n$ is compact by Prop. 54(j). Because K is a closed subset of a compact set, it is compact by Prop. 54(d). □

Proposition 56 (***Extreme value theorem***)**.** *If X is a compact space and $F : X \to \mathbb{R}$ is continuous, then $F(X)$ is bounded and attains its maximum and minimum values on X.*

Proof. By Prop. 54(a), $F(X)$ is compact subset of $\mathbb{R}$, so by (f) and (m) of Prop. 54 it is closed and bounded. In particular, it is contains its supremum and infimum. □

Definition 35. *Suppose (M_1, d_1) and (M_2, d_2) are metric spaces, and $F : M_1 \to M_2$ is a map. Then F is said to be* ***uniformly continuous*** *if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in M_1, d_1(x, y) < \delta$ implies $d_2(F(x), F(y)) < \epsilon$. It is said to be* ***Lipschitz continuous*** *if there is a constant $C \geq 0$ such that $d_2(F(x), F(y)) \leq C d_1(x, y)$ for all $x, y \in M_1$. Any such C is called a* ***Lipschitz constant for F****. We say that F is* ***locally Lipschitz continuous*** *if every point $x \in M_1$ has a neighborhood on which F is Lipschitz continuous.*

Proposition 57. *For maps between metric spaces, we have the following:*

(a) Lipschitz continuous $\implies$ uniformly continuous $\implies$ continuous.

(b) Lipschitz continuous $\implies$ locally Lipschitz continuous $\implies$ continuous.

Proof. Suppose (M_1, d_1) and (M_2, d_2) are metric spaces, and $F : M_1 \to M_2$ is a map.

(a) (Lipschitz continuous $\implies$ uniformly continuous). For a given $\epsilon > 0$, choose $\delta = \epsilon / C$. Then for any $x, y \in M_1$ and $d_1(x, y) < \delta$, we have $d_2(F(x), F(y)) \leq C d_1(x, y) < C\delta = \epsilon$.

(Uniformly continuous $\implies$ continuous). Follows from Def. 35 and Prop. 17.

(b) (Lipschitz continuous $\implies$ locally Lipschitz continuous). For any point $p \in M_1$, M_1 is the neighborhood of p on which F is Lipschitz continuous.

(Locally Lipschitz continuous $\implies$ continuous). For any point $p \in M_1$, there is a neighborhood U of p on which F is Lipschitz continuous. Then it follows from (a) that F is continuous on U. □

Proposition 58. *Suppose (M_1, d_1) and (M_2, d_2) are metric spaces and $F : M_1 \to M_2$ is a map. Let K be any compact subset of M_1.*

(a) If F is continuous, then $F|_K$ is uniformly continuous.

(b) If F is locally Lipschitz continuous, then $F|_K$ is Lipschitz continuous.

Proof. **(a)** Assume F is continuous and let $\epsilon > 0$ be given. For each $x \in K$, by continuity there is a positive number $\delta(x)$ such that $d_1(x,y) < 2\delta \implies d_2(F(x),F(y)) < \epsilon/2$. Because the open balls $\{\mathbb{B}_x(\delta(x)) : x \in K\}$ cover K, by compactness there are finitely many points $x_1, \ldots, x_n \in K$ such that $K \subseteq \mathbb{B}_{x_1}(\delta(x_1)) \cup \ldots \cup \mathbb{B}_{x_n}(\delta(x_n))$. Let $\delta = \min\{\delta(x_1), \ldots, \delta(x_n)\}$. Suppose $x, y \in K$ satisfy $d_1(x,y) < \delta$. There is some i such that $x \in \mathbb{B}_{x_i}(\delta(x_i))$, and then $d_1(y,x_i) \leq d_1(y,x) + d_1(x,x_i) < 2\delta(x_i)$. Thus both $x, y \in \mathbb{B}_{x_i}(2\delta(x_i))$. Then by assumption $d_2(F(x),F(y)) \leq d_2(F(x),F(x_i)) + d_2(F(x_i),F(y)) < \epsilon$.

(b) Assume F is locally Lipschitz continuous. Because F is continuous, Prop. 54(a) and (m) show that $F(K)$ is compact and therefore bounded. Let $D = \operatorname{diam} F(K)$. For each $x \in K$, there is a positive number $\delta(x)$ such that F is Lipschitz continuous on $\mathbb{B}_x(2\delta(x))$, with Lipschitz constant $C(x)$. By compactness, there are points $x_1, \ldots, x_n \in K$ such that $K \subseteq \mathbb{B}_{x_1}(\delta(x_1)) \cup \ldots \cup \mathbb{B}_{x_n}(\delta(x_n))$. Let $C = \max\{C(x_1), \ldots, C(x_n)\}$ and $\delta = \min\{\delta(x_1), \ldots, \delta(x_n)\}$, and let $x, y \in K$ be arbitrary. On the one hand, if $d_1(x,y) < \delta$, then by the same argument as in (a), x and y lie in of the balls on which F is Lipschitz continuous, so $d_2(F(x),F(y)) \leq C d_1(x,y)$. On the other hand, if $d_1(x,y) \geq \delta$, then $d_2(F(x),F(y)) \leq D \leq (D/\delta) d_1(x,y)$. Therefore, $\max\{C, D/\delta\}$ is a Lipschitz constant for F on K. □

Definition 36. *A sequence of point $(x_i)_{i=1}^{\infty}$ in metric space M is a* ***Cauchy sequence*** *if for every $\epsilon > 0$, there exists an integer N such that $m, n \geq N$ implies $d(x_m, x_n) < \epsilon$. A metric space M is said to be* ***complete*** *if every Cauchy sequence in M converges to a point of M.*

Remark 13. *Every convergent sequence is a Cauchy sequence. Suppose $x_n \to x$, then for a given $\epsilon > 0$, there exists an integer $N > 0$, such that $n > N$ implies $d(x_n, x) < \epsilon/2$. Then for $m, n > N$, we have $d(x_n, d_m) \leq d(x_n, x) + d(x, x_m) < \epsilon$.*

Definition 37. *A space X is said to be* ***limit point compact*** *if every infinite subset of X has a limit point in X, and* ***sequentially compact*** *if every sequence of points in X has a subsequence that converges to a point in X.*

Proposition 59. *Compactness implies limit point compactness.*

Proof. Suppose X is compact and let $S \subseteq X$ be an infinite subset. If S has no limit point in X, then every point $x \in X$ has a neighborhood U such that $U \cap S$ is either empty or $\{x\}$. Finitely many of these neighborhoods cover X. But since each such neighborhood contains at most one point of S, this implies that S is finite, which is a contradiction. □

Proposition 60. *For first countable Hausdorff spaces, limit point compactness implies sequential compactness.*

Proof. Suppose X is first countable, Hausdorff, and limit point compact, and let $(p_n)_{n \in \mathbb{N}}$ be any sequence of points in X. If the sequence takes on only finitely many values, the it has a constant subsequence, which is certainly convergent. So we may suppose it takes on infinitely many values. By hypothesis the set of values has a limit point $p \in X$. If p is equal to p_n for infinitely many values of n, again there is a constant subsequence and we are done; so by discarding finitely many terms at the beginning of the sequence if necessary we may assume that $p_n \neq p$ for all n. Because X is first countable, Remark 2 shows that there is a descending neighborhood basis at p, say $(B_n)_{n \in \mathbb{N}}$. For such a neighborhood basis, it is easy to see than any subsequence (p_{n_i}) such that $p_{n_i} \in B_i$ converges to p. Since p is a limit point, we can choose n_1 such that $p_1 \in B_1$. Suppose by induction that we have chosen $n_1 < n_2 < \ldots < n_k$ with $p_{n_i} \in B_i$. By Prop. 16, the sequence takes infinitely many values in B_{k+1}, so we can choose some $n_{k+1} > n_k$ such that $p_{n_{k+1}} \in B_{k+1}$. This completes the induction, and prove that there is a subsequence (p_{n_i}) converging to p. □

Proposition 61. *For metric spaces and second-countable topological spaces, sequential compactness implies compactness.*

Proof. Suppose first that X is second countable and sequentially compact, and let $\mathcal{U}$ be an open cover of X. By Prop. 12, $\mathcal{U}$ has a countable subcover $\mathcal{U}' = \{U_i\}_{i\in\mathbb{N}}$. Assume no finite subcollection of $\mathcal{U}'$ covers X. Then for each i there exists $q_i \in X$ such that $q_i \notin U_1 \cup \ldots \cup U_i$. By hypothesis, the sequence (q_i) has a convergent subsequence $(q_{i_k}) \to q \in X$. Now, $q \in U_m \in \mathcal{U}'$, since $\mathcal{U}'$ covers X, and then convergence of the subsequence means that $q_{i_k} \notin U_m$ as soon as $i_k \geq m$, which is a contradiction. This proves that second countable sequentially compact spaces are compact.

Let M be a sequentially compact metric space. We will show that M is second countable, which by the above argument implies that M is compact. From Prop. 13, it suffices to show that M is separable. The key idea is to show first that sequential compactness implies the following weak form of compactness for metric spaces: *for each $\epsilon > 0$, the open cover of M consisting of all ϵ-balls has a finite subcover.*

Suppose this is not true for some ϵ. Construct a sequence as follows. Let $q_1 \in M$ be arbitrary. Since $\mathbb{B}_{q_1}(\epsilon) \neq M$, there is a point $q_2 \notin \mathbb{B}_{q_1}(\epsilon)$. Similarly, since $\mathbb{B}_{q_1}(\epsilon) \cup \mathbb{B}_{q_1}(\epsilon) \neq M$, there is a point q_3 in neither of two preceding ϵ-balls. Proceeding by induction, we construct a sequence (q_n) such that for each n,

$$q_{n+1} \notin \cup_{i=1}^{n} \mathbb{B}_{q_i}(\epsilon). \qquad (\dagger)$$

Replacing this sequence by a convergent sequence (which still satisfies $(\dagger)$), we can assume $q_n \to q \in M$. Since convergent sequences are Cauchy, as soon as n is large enough we have $d(q_{n+1}, q_n) < \epsilon$, which contradicts $(\dagger)$.

Now, for each n, let F_n be a finite set of points in M such that the balls of radius $1/n$ around these points cover M. The set $\cup_n F_n$ is countable, and dense in M. This shows that M is separable and completes the proof. □

Corollary (***Equivalent Formulations of Compactness***)**.** *For metric spaces and second-countable Hausdorff spaces, limit point compactness, sequential compactness, and compactness are all equivalent properties.*

Proposition 62 (***Bolzano-Weierstrass Theorem***)**.** *Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*

Proof. Every bounded sequence is contained in some compact cube in $\mathbb{R}^n$. Then by the preceding corollary it has a convergent subsequence. □

Proposition 63. *Let (M, d) be a metric space and (x_n) is a Cauchy sequence in M. Then (x_n) is convergent iff it has a convergent subsequence.*

Proof. ($\Longrightarrow$)Immediately follows from the fact that for a convergent sequence any subsequence is convergent.
($\Longleftarrow$)Suppose (x_{n_k}) is a subsequence of Cauchy (x_n) such that $x_{n_k} \to x \in M$. Let $\epsilon > 0$ be given. Then there exists an integer N such that $d(x_{n_k}, x) < \epsilon/2$ and $d(x_n, x_m) < \epsilon/2$ for $n_k, n, m > N$. Then for $n_k > n > N$, we have $d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \epsilon$, i.e., $x_n \to x$. □

Proposition 64. *Endowed with the Euclidean metric, a subset of $\mathbb{R}^n$ is a complete metric space iff it is closed in $\mathbb{R}^n$. In particular, $\mathbb{R}^n$ is complete.*

Proposition 65. *Every compact metric space is complete.*

Proposition 66 (***Closed Map Lemma***)**.** *Suppose X is a compact space, Y is a Hausdorff space, and $F : X \to Y$ is a continuous map.*

(a) F is a closed map.

(b) If F is surjective, it is a quotient map.

(c) If F is injective, it is a topological embedding.

(d) If F is bijective, it is a homeomorphism.

Proof. ***(a)*** Let $C \subseteq X$ be closed. Then by Prop. 54(d) C is compact in X. By continuity of F, we note that $F(C)$ is compact in Y. Then by Prop. 54(f) it is closed in Y.

(b), ***(c)***, ***(d)*** follow immediately from Prop. 44 and noting that F is a closed map. □

Definition 38. *If X and Y are topological spaces, a map $F : X \to Y$ (continuous or not) is said to be* ***proper*** *if for every compact set $K \subseteq Y$, the preimage $F^{-1}(K)$ is compact.*

Proposition 67 (***Sufficient Conditions for Properness***)**.** *Suppose X and Y are topological spaces, and $F : X \to Y$ is a continuous map.*

(a) If X is compact and Y is Hausdorff, then F is proper.

(b) If F is a closed map with compact fibers, then F is proper.

(c) If F is a topological embedding with closed image, then F is proper.

(d) If Y is Hausdorff and F has a continuous left inverse (i.e., a continuous map $G : Y \to X$ such that $G \circ F = Id_X$), then F is proper.

(e) If F is proper and $A \subseteq X$ is a subset that is saturated w.r.t. F, then $F|_A : A \to F(A)$ is proper.

Proof. ***(a)*** Let $K \subseteq Y$ be compact. Since X is Hausdorff then by Prop. 54(f) it is closed in Y. Then by continuity of F, $F^{-1}(K)$ is closed in X. Since $F^{-1}(K)$ is a closed subset of a compact space X, by Prop. 54(d), it is compact.

(b) Let $K \subseteq Y$ be compact, and let $\mathcal{U}$ be a cover of $F^{-1}(K)$ by open subsets of X. If $y \in K$ is arbitrary, the fiber $F^{-1}(y)$ is covered by finitely many of the sets in $\mathcal{U}$, say $U_1, \ldots, U_k$. The set $A_y = X - (\bigcup_{i=1}^n U_i)$ is closed in X disjoint from $F^{-1}(y)$, so $V_y = Y - F(A_y)$ is open in Y and contains y. Then $F^{-1}(V_y) = X - F^{-1}(F(A_y)) \subseteq X - A_y = \bigcup_{i=1}^n U_i$ by elementary set theory. Because K is compact, it is covered by finitely many of the sets V_y. Thus $F^{-1}(K)$ is covered by finitely many sets of the form $F^{-1}(V_y)$, each of which is covered by finitely many of the sets in $\mathcal{U}$, so it follows that $F^{-1}(K)$ is compact.

(c) Follows from (b) because a topological embedding with closed image is a closed map, and its fibers are singletons, which are certainly compact.

(d) Suppose $K \subseteq Y$ is compact. Any point $x \in F^{-1}(K)$ satisfies $x = G(F(x)) \in G(K)$, i.e., $F^{-1}(K) \subseteq G(K)$. Since K is closed in Y (because Y is Hausdorff), it follows that $F^{-1}(K)$ is a closed subset of the compact set $G(K)$(by continuity of G), so it is compact.

(e) Let $K \subseteq F(A)$ be compact. The fact that A is saturated means that $A = F^{-1}(F(A))$. But $F(A) \supseteq K$ implies $A = F^{-1}(F(A)) \supseteq F^{-1}(K)$, i.e., $F^{-1}(K) \subseteq A$. Then we have $(F|_A)^{-1}(K) = F^{-1}(K)$, which is compact because F is proper. □

Definition 39. *A topological space X is said to be* ***locally compact*** *if every point has a neighborhood contained in a compact subset of X. A subset of X is said to be* ***precompact in X*** *if its closure in X is compact.*

Proposition 68. *For a Hausdorff space X, the following are equivalent:*

(a) X is locally compact.

(b) Each point of X has a precompact neighborhood.

(c) X has a basis of precompact open subsets.

Proof. Note that ***(c)*** $\implies$ ***(b)*** $\implies$ ***(a)***, so all we have to prove is ***(a)*** $\implies$ ***(c)***. It suffices to show that if X is locally compact Hausdorff space, then each point $x \in X$ has a ***neighborhood basis*** (Def. 14) of precompact open subsets.

Let $K \subseteq X$ be compact set containing a neighborhood U of x. The collection $\mathcal{V}$ of all neighborhoods of x contained in U is clearly a neighborhood basis at x. Because X is Hausdorff, K is closed in X. If $V \in \mathcal{V}$, then $\bar{V} \subseteq K$, since $V \subseteq K = \bar{K}$. Therefore $\bar{V}$ is compact because a closed subset of a compact set is compact. Thus $\mathcal{V}$ is the required neighborhood basis. □

Proposition 69. *Every open or closed subspace of a locally compact Hausdorff space is itself a locally compact Hausdorff space.*

Proof. Let X be a locally compact Hausdorff space. Note that every subspace of X is Hausdorff, so only local compactness needs to be proved.

Suppose $U \subseteq X$ is open in X. Let $x \in U$, then U is a neighborhood of X. If W is any precompact neighborhood of x, then $\bar{W} - U = \bar{W} \cap (X - U)$ is closed in $\bar{W}$ and therefore compact. Because compact subsets can be separated by open subsets in a Hausdorff space by Prop. 54(e), there are disjoint open subsets Y containing x and Y' containing $\bar{W} - U$. Let $V = Y \cap W$. We show that V is a precompact neighborhood of X in U, i.e., $\bar{V}$ is compact and $\bar{V} \subseteq U$. Because $\bar{V} \subseteq \bar{W}$, $\bar{V}$ is compact. Since $V \subseteq Y \subseteq X - Y'$, we have $\bar{V} \subseteq \bar{Y} \subseteq \overline{X - Y'} = X - Y'$, and thus $\bar{V} = \bar{V} \cap \bar{W} \subseteq \bar{W} \cap (X - Y') = \bar{W} - Y'$. Now the fact that $\bar{W} - U \subseteq Y'$ means that $\bar{W} - Y' \subseteq U$, so $\bar{V} \subseteq U$.

Suppose $Z \subseteq X$ is closed. Any $x \in Z$ has a precompact neighborhood U in X. Since $\overline{U \cap Z}$ is closed subset of the compact set $\bar{U}$, it is compact. Since $\overline{U \cap Z} \subseteq \bar{Z} = Z$ it follows that $U \cap Z$ is a precompact neighborhood of $x \in Z$. Thus Z is locally compact. □

Proposition 70 (***Proper Continuous Maps are Closed***)**.** *Suppose X is a topological space and Y is a locally compact Hausdorff space. Then every proper continuous map $F : X \to Y$ is closed.*

Proof. Let $K \subseteq X$ be closed. To show that $F(K)$ is closed in Y, we show that it contains all of its limit points. Let y be a limit point of $F(K)$, and let U be a precompact neighborhood of y. Then y is also a limit point of $F(K) \cap \bar{U}$. Because F is proper, $F^{-1}(\bar{U})$ is compact, which implies that $F^{-1}(\bar{U}) \cap K$ is compact, since it is a closed subset of the compact set $F^{-1}(\bar{U})$. Because F is continuous, then $F(K \cap F^{-1}(K)) = F(K) \cap \bar{U}$ is compact and therefore closed in Y. In particular, $y \in F(K) \cap \bar{U} \subseteq F(K)$, so $F(K)$ is closed. □

Proposition 71 (***Baire Category Theorem***)**.** *In a locally compact Hausdorff space or a complete metric space, every countable union of nowhere dense sets has empty interior.*

Proposition 72. *In a locally compact Hausdorff space or a complete metric space, every nonempty countable closed subset contains at least one isolated point.*

Definition 40. *A sequence $(K_i)_{i=1}^{\infty}$ of compact subsets of a topological space X is called an* ***exhaustion of X by compact sets*** *if $X = \bigcup_i K_i$ and $K_i \subseteq Int\, K_{i+1}$ for each i.*

Proposition 73. *A second-countable, locally compact Hausdorff space admits an exhaustion by compact sets.*

References

[1] John M. Lee, Introduction to Smooth Manifolds.