

Differential Calculus

Jayadev Naram

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1 Total Derivatives in Finite-Dimensional Vector spaces

Definition 1. Let V, W be finite-dimensional vector spaces, which we may assume to be endowed with norms. If $U \subseteq V$ is an open subset $a \in U$, a map $F : U \rightarrow W$ is said to be **differentiable at a** if there exists a linear map $L : V \rightarrow W$ such that

$$\lim_{v \rightarrow 0} \frac{\|F(a+v) - F(a) - Lv\|}{\|v\|} = 0. \quad (1)$$

Remark 1. It can be easily seen that

$$\lim_{v \rightarrow 0} \frac{\|F(a+v) - F(a) - Lv\|}{\|v\|} = 0 \Leftrightarrow \lim_{v \rightarrow 0} \frac{F(a+v) - F(a) - Lv}{\|v\|} = 0. \quad (2)$$

Proposition 1. Suppose $F : U \rightarrow W$ is differentiable at $a \in U$. Then the linear map L satisfying (1) is unique.

Proof. Let $L_1 : V \rightarrow W$ and $L_2 : V \rightarrow W$ satisfy (1). Define $A = L_1 - L_2$, then we have

$$\begin{aligned} \|Av\| &= \|(F(a+v) - F(a) - L_2v) - (F(a+v) - F(a) - L_1v)\| \\ &\leq \|F(a+v) - F(a) - L_2v\| + \|F(a+v) - F(a) - L_1v\| \end{aligned}$$

Dividing by $\|v\|$ and taking limit we get

$$\Rightarrow \lim_{v \rightarrow 0} \frac{\|Av\|}{\|v\|} \leq \lim_{v \rightarrow 0} \left(\frac{\|F(a+v) - F(a) - L_1v\|}{\|v\|} + \frac{\|F(a+v) - F(a) - L_2v\|}{\|v\|} \right) = 0$$

Therefore we have $Av = 0 \forall v \in V$ which implies that $A = 0$, i.e., $L_1 = L_2$. \square

Definition 2. If F is differentiable at a , the linear map L satisfying (1) is denoted by $DF(a)$ and is called the **total derivative of F at a** .

Remark 2. Condition (1) can also be written as

$$F(a+v) - F(a) = DF(a)v + R(v), \quad (3)$$

where the remainder term $R(v)$ satisfies $\|R(v)\|/\|v\| \rightarrow 0$ as $v \rightarrow 0$. Thus the total derivative represents the “best linear approximation” to $F(a+v) - F(a)$ near a . Note that $\|R(v)\|/\|v\| \rightarrow 0$ implies that eventually $\|R(v)\|/\|v\| \leq 1$, i.e., $\|R(v)\| \leq \|v\|$.

Proposition 2. Suppose V, W are finite-dimensional vector spaces, $U \subseteq V$ is an open subset, $a \in U$, and $F : U \rightarrow W$ is a map. If F is differentiable at a , then it is continuous at a .

Proof. In (3) take norm and apply limit $v \rightarrow 0$ on both sides

$$\begin{aligned} 0 \leq \lim_{v \rightarrow 0} \|F(a+v) - F(a)\| &= \lim_{v \rightarrow 0} \|DF(a)v + R(v)\| \leq \lim_{v \rightarrow 0} (\|DF(a)v\| + \|R(v)\|) \\ &\leq \lim_{v \rightarrow 0} (\|DF(a)\| \|v\| + \|v\|) = 0, \end{aligned}$$

where $\|DF(a)\|$ is the operator norm. Thus F is continuous at a . \square

Proposition 3. Suppose V, W, X are finite-dimensional vector spaces. Then

- (a) If $T : V \rightarrow W$ is a linear map, then T is differentiable at every point $v \in V$, with total derivative equal to T itself: $DT(v) = T$.
- (b) If $B : V \times W \rightarrow X$ is a bilinear map, then B is differentiable at every point $(v, w) \in V \times W$, and $DB(v, w)(x, y) = B(v, y) + B(x, w)$.

Proof. (a) Setting $L = T$ in (1) and using the linearity of T , we see that T is differentiable everywhere with the total derivative equal to T itself.

(b) We use (3) to show that bilinear map is differentiable. Note that

$$B(v+x, w+y) = B(v, w) + B(v, y) + B(x, w) + B(x, y).$$

But $B(x, y) \leq \|B\| \|x\| \|y\|$ where $\|B\|$ is operator norm which is finite by continuity of B . Then comparing with (3)

$$B(v+x, w+y) - B(v, w) = DB(v, w)(x, y) + R(x, y),$$

where $DB(v, w)(x, y) = B(v, y) + B(x, w)$ and $R(x, y) \leq \|B\| \|x\| \|y\| \rightarrow 0$ as $(x, y) \rightarrow 0$. \square

Proposition 4 (The Chain Rule for Total Derivatives). Suppose V, W, X are finite-dimensional vector spaces, $U \subseteq V$ and $\tilde{U} \subseteq W$ are open subsets, and $F : U \rightarrow \tilde{U}$ and $G : \tilde{U} \rightarrow X$ are maps. If F is differentiable at $a \in U$ and G is differentiable at $F(a) \in \tilde{U}$, then $G \circ F$ is differentiable at a and $D(G \circ F)(a) = DG(F(a)) \circ DF(a)$.

Proof. Let $A = DF(a)$ and $B = DG(F(a))$. We need to show that

$$\lim_{v \rightarrow 0} \frac{\|G(F(a+v)) - G(F(a)) - BAv\|}{\|v\|} = 0. \quad (4)$$

Let us write $b = F(a)$ and $w = F(a+v) - F(a)$. With these substitutions, we can rewrite the quotient in (4) as

$$\begin{aligned} \frac{\|G(b+w) - G(b) - BAv\|}{\|v\|} &= \frac{\|G(b+w) - G(b) - Bw + Bw - BAv\|}{\|v\|} \\ &\leq \frac{\|G(b+w) - G(b) - Bw\|}{\|v\|} + \frac{\|B(w - Av)\|}{\|v\|} \quad (\dagger) \end{aligned}$$

The differentiability of F at a means that for any $\epsilon > 0$, we can ensure that

$$\|w - Av\| = \|F(a+v) - F(a) - Av\| \leq \epsilon \|v\|$$

as long as v lies in a small enough neighborhood of 0. Moreover, as $v \rightarrow 0$, $\|v\| = \|F(a+v) - F(a)\| \rightarrow 0$ by continuity of F . Therefore by differentiability of G at b means that by making $\|v\|$ even smaller if necessary, we can also achieve

$$\|G(b+w) - G(b) - Bw\| \leq \epsilon \|w\|.$$

Also note that $\|B(w - Av)\| \leq \|B\|\|w - Av\|$. Putting all of these estimates together, we see that for $\|v\|$ sufficiently small, (\dagger) is bounded by

$$\begin{aligned} \epsilon \frac{\|w\|}{\|v\|} + \|B\| \frac{\|w - Av\|}{\|v\|} &= \epsilon \frac{\|w - Av + Av\|}{\|v\|} + \|B\| \frac{\|w - Av\|}{\|v\|} \\ &\leq \epsilon \frac{\|w - Av\|}{\|v\|} + \frac{\|Av\|}{\|v\|} + \|B\| \frac{\|w - Av\|}{\|v\|} \\ &\leq \epsilon^2 + \epsilon\|A\| + \epsilon\|B\|, \end{aligned}$$

which can be made as small as desired. \square

Lemma 1. *The addition operation $+: \mathbb{R}^2 \rightarrow \mathbb{R}$ defined as $(+)(x, y) = x + y$ is differentiable and $D(+)(a, b) = +$. The multiplication operation $\times: \mathbb{R}^2 \rightarrow \mathbb{R}$ defined as $(\times)(x, y) = xy$ is differentiable and $D(\times)(a, b)(x, y) = bx + ay$. The reciprocal operation $h: \mathbb{R} \rightarrow \mathbb{R}$ defined as $h(x) = 1/x$ is differentiable and $Dh(x) = -\frac{1}{x^2}$.*

Proposition 5. *Suppose V, W are finite-dimensional vector spaces, $U \subseteq V$ is an open subset, a is a point in U , and $F, G: U \rightarrow W$ and $f, g: U \rightarrow \mathbb{R}$ are maps. Then*

(a) *If F is a constant map, then F is differentiable at a and $DF(a) = 0$.*

(b) *If F and G are differentiable at a , then $F + G$ is also, and*

$$D(F + G)(a) = DF(a) + DG(a).$$

(c) *If f and g are differentiable at a , then fg is also, and*

$$D(fg)(a) = f(a)Dg(a) + g(a)Df(a).$$

(d) *If f and g are differentiable at a and $g(a) \neq 0$, then f/g is differentiable at a , and*

$$D(f/g)(a) = \frac{g(a)Df(a) - f(a)Dg(a)}{g(a)^2}.$$

Proof. (a) Let $F(v) = c \in W$ for all $v \in V$. Then setting $L = 0$ in (1) satisfies the equation showing that F is differentiable at a and $DF(a) = 0$.

(b) Note that $F + G = (+) \circ (F, G)$, then using chain rule

$$\begin{aligned} D(F + G)(a) &= D((+) \circ (F, G))(a) = D(+)(F(a), G(a)) \circ D(F, G)(a) \\ &= D((+)(F(a), G(a))) \circ (DF(a), DG(a)) \\ &= (+)(DF(a), DG(a)) = DF(a) + DG(a). \end{aligned}$$

(c) Note that $fg = (\times) \circ (f, g)$, then using chain rule

$$\begin{aligned} D(fg)(a) &= D((\times) \circ (f, g))(a) = D((\times)(f(a), g(a))) \circ D(f, g)(a) \\ &= D((\times)(f(a), g(a))) \circ (Df(a), Dg(a)) \\ &= g(a)Df(a) + f(a)Dg(a) \end{aligned}$$

(d) Note that $f/g = (\times) \circ (f, 1/g)$ and $1/g = h \circ g$, so by chain rule

$$D(1/g)(a) = -\frac{1}{g(a)^2}Dg(a).$$

Then

$$\begin{aligned}
D(f/g)(a) &= D((\times) \circ (f, 1/g))(a) = D(\times)(f(a), 1/g(a)) \circ D(f, 1/g)(a) \\
&= D(\times)(f(a), 1/g(a))(Df(a), D(1/g)(a)) \\
&= \frac{1}{g(a)} Df(a) - f(a) \frac{1}{g(a)^2} Dg(a) \\
&= \frac{g(a) Df(a) - f(a) Dg(a)}{g(a)^2}.
\end{aligned}$$

□

2 Total and Partial Derivatives in \mathbb{R}^n

Definition 3. Suppose $U \subseteq \mathbb{R}^n$ is open and $f : U \rightarrow \mathbb{R}$ is a real-valued function. For any $a = (a^1, \dots, a^n) \in U$ and any $j \in \{1, \dots, n\}$, the **j-th partial derivative of f at a** is defined to be the ordinary derivative of f w.r.t. x^j while holding the other variables fixed:

$$\frac{\partial f}{\partial x^j}(a) = \lim_{h \rightarrow 0} \frac{f(a + he_j) - f(a)}{h}$$

if the limit exists.

Definition 4. For a vector-valued function $F : U \rightarrow \mathbb{R}^m$, we can write the coordinates of $F(x)$ as $F(x) = (F^1(x), \dots, F^m(x))$. This defines m functions $F^1, \dots, F^m : U \rightarrow \mathbb{R}$ called the **component functions of F**. The partial derivatives of F are defined simply to be the partial derivatives $\partial F^i / \partial x^j$ of its component functions. The matrix $(\partial F^i / \partial x^j)$ of partial derivatives is called the **Jacobian matrix of F**, and its determinant is called the **Jacobian determinant of F**.

Definition 5. If $F : U \rightarrow \mathbb{R}^m$ is a function for which each partial derivative exists at each point in U and the functions $\partial F^i / \partial x^j : U \rightarrow \mathbb{R}$ so defined are all continuous, then F is said to be of class C^1 or **continuously differentiable**. If this is the case, we can differentiate the functions $\partial F^i / \partial x^j$ to obtain **second-order partial derivatives**

$$\frac{\partial^2 F^i}{\partial x^k \partial x^j} = \frac{\partial}{\partial x^k} \left(\frac{\partial F^i}{\partial x^j} \right),$$

if they exist. Continuing this way leads to higher-order partial derivatives: the **partial derivatives of F of order k** are the (first) partial derivatives of those of order $k - 1$, when they exist.

Definition 6. If $U \subseteq \mathbb{R}^n$ is an open subset and $k \geq 0$, a function $F : U \rightarrow \mathbb{R}^m$ is said to be of class C^k or **k times continuously differentiable** if all the partial derivatives of F of order less than or equal to k exist and are continuous functions on U .

Remark 3. Thus a function of class C^0 is just a continuous function. Because existence and continuity of derivatives are local properties, clearly F is C^k iff it has the property in a neighborhood of each point in U .

Definition 7. A function that is of class C^k for every $k \geq 0$ is said to be of **class C^∞** , or **smooth**, or **infinitely differentiable**. If U and V are open subsets of Euclidean spaces, a function $F : U \rightarrow V$ is called a **diffeomorphism** if it is smooth and bijective and its inverse function is also smooth.

Proposition 6. Suppose $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are open subsets and $F : U \rightarrow V$ is a diffeomorphism. Then $m = n$, and for each $a \in U$, the total derivative is invertible, with $DF(a)^{-1} = D(F^{-1})(F(a))$.

Proof. Because $F^{-1} \circ F = Id_U$, the chain rule implies that for each $a \in U$,

$$Id_{\mathbb{R}^n} = D(Id_U)(a) = D(F^{-1} \circ F)(a) = D(F^{-1})(F(a)) \circ DF(a).$$

Similarly, $F \circ F^{-1} = Id_V$ implies that for each $F(a) \in V$, we have

$$DF(F^{-1}(F(a))) \circ D(F^{-1})(F(a)) = DF(a) \circ D(F^{-1})(F(a)) = Id_{\mathbb{R}^m}.$$

This implies that $DF(a)$ is invertible with inverse $D(F^{-1})(F(a))$, and therefore $m = n$. \square

Definition 8 (Smoothness on Arbitrary Domains). If $A \subseteq \mathbb{R}^n$ is an arbitrary subset, a function $F : A \rightarrow \mathbb{R}^m$ is said to be **smooth on A** if it admits a smooth extension to an open neighborhood of each point, or more precisely, if for every $x \in A$, there exists an open neighborhood $U_x \subseteq \mathbb{R}^n$ and a smooth function $\tilde{F} : U_x \rightarrow \mathbb{R}^m$ that agrees with F on $U_x \cap A$. The notion of diffeomorphism extends to arbitrary subsets in the obvious way: given arbitrary subsets $A, B \subseteq \mathbb{R}^n$, a **diffeomorphism from A to B** is a smooth bijective map $f : A \rightarrow B$ with smooth inverse.

Definition 9. If $U \subseteq \mathbb{R}^n$ is open, the set of all real-valued functions of class C^k on U is denoted by $C^k(U)$, and the set of all smooth real-valued functions by $C^\infty(U)$. Sums, constant multiples, and products of functions are defined pointwise: for $f, g : U \rightarrow \mathbb{R}$ and $c \in \mathbb{R}$,

$$\begin{aligned}(f + g)(x) &= f(x) + g(x), \\ (cf)(x) &= c(f(x)), \\ (fg)(x) &= f(x)g(x).\end{aligned}$$

Proposition 7 (Equality of Mixed Partial Derivatives). If U is an open subset of \mathbb{R}^n and $F : U \rightarrow \mathbb{R}^m$ is a function of class C^2 , then the mixed second-order partial derivatives of F do not depend on the order of differentiation:

$$\frac{\partial^2 F^i}{\partial x^j \partial x^k} = \frac{\partial^2 F^i}{\partial x^k \partial x^j}.$$

Corollary. If $F : U \rightarrow \mathbb{R}^m$ is smooth, then the mixed partial derivatives of F of any order are independent of the order of differentiation.

Proposition 8. Let $U \subseteq \mathbb{R}^n$ be open, and suppose $F : U \rightarrow \mathbb{R}^m$ is differentiable at $a \in U$. Then all of the partial derivatives of F at a exist, and $DF(a)$ is the linear map whose matrix is the Jacobian of F at a :

$$DF(a) = \left(\frac{\partial F^j}{\partial x^i}(a) \right).$$

Proof. Let $B = DF(a)$, and for $v \in \mathbb{R}^n$ small enough that $a + v \in U$, let $R(v) = F(a + v) - F(a) - Bv$. The fact that F is differentiable at a implies that each component of the vector-valued function $R(v)/\|v\|$ goes to zero as $v \rightarrow 0$. The i -th partial derivative of F^j at a , if it exists, is

$$\frac{\partial F^j}{\partial x^i}(a) = \lim_{t \rightarrow 0} \frac{F^j(a + te_i) - F^j(a)}{t} = \lim_{t \rightarrow 0} \frac{B_i^j t + R^j(te_i)}{t} = B_i^j + \lim_{t \rightarrow 0} \frac{R^j(te_i)}{t}.$$

The norm of the quotient on the right above is $\|R^j(te_i)\|/\|te_i\|$, which approaches zero as $t \rightarrow 0$. It follows that $\partial F^j/\partial x^i(a)$ exists and is equal to B_i^j as claimed. \square

Proposition 9. Suppose $U \subseteq \mathbb{R}^n$ is open. Then $F : U \rightarrow \mathbb{R}^m$ is differentiable at $a \in U$ iff each of its component functions F^1, \dots, F^m is differentiable at a and

$$DF(a) = \begin{pmatrix} DF^1(a) \\ \vdots \\ DF^m(a) \end{pmatrix}$$

Proof. Using the fact that $y = (y_1, \dots, y_m) \rightarrow 0 \Leftrightarrow y_i \rightarrow 0$ for each i , from Remark 1 we see that

$$\begin{aligned} \lim_{v \rightarrow 0} \frac{\|F(a+v) - F(a) - DF(a)v\|}{\|v\|} = 0 &\Leftrightarrow \lim_{v \rightarrow 0} \frac{F(a+v) - F(a) - DF(a)v}{\|v\|} = 0 \\ &\Leftrightarrow \lim_{v \rightarrow 0} \frac{F^i(a+v) - F^i(a) - DF^i(a)v}{\|v\|} = 0, \forall i \\ &\Leftrightarrow \lim_{v \rightarrow 0} \frac{\|F^i(a+v) - F^i(a) - DF^i(a)v\|}{\|v\|} = 0, \forall i. \end{aligned}$$

□

Proposition 10. Let $U \subseteq \mathbb{R}^n$ be open. If $F : U \rightarrow \mathbb{R}^n$ is of class C^1 , then it is differentiable at each point of U .

Proposition 11. Let $U \subseteq \mathbb{R}^n$ be an open subset, and suppose $f, g \in C^\infty(U)$ and $c \in \mathbb{R}$.

(a) Then $f + g$, cf , and fg are smooth.

(b) If g never vanishes on U , then f/g is smooth.

Proof. The result follows immediately by noting that each of the partial derivatives of $f + g$, cf , fg and f/g of any order are continuous as they can be written as sum, product, quotient of partial derivatives of f and g which are assumed to be continuous. □

Proposition 12 (The Chain Rule for Partial Derivatives). Let $U \subseteq \mathbb{R}^n$ and $\tilde{U} \subseteq \mathbb{R}^m$ be open subsets, and let $x = (x^1, \dots, x^n)$ denote the standard coordinates on U and $y = (y^1, \dots, y^m)$ those on \tilde{U} .

(a) A composition of C^1 functions $F : U \rightarrow \tilde{U}$ and $G : \tilde{U} \rightarrow \mathbb{R}^p$ is again of class C^1 , with partial derivatives given by

$$\frac{\partial(G^i \circ F)}{\partial x^j}(x) = \sum_{k=1}^m \frac{\partial G^i}{\partial y^k}(F(x)) \frac{\partial F^k}{\partial x^j}(x).$$

(b) If F and G are smooth, then $G \circ F$ is smooth.

Proof. (a) From the chain rule of total derivative (Prop. 4) and the Jacobian matrix formulation of total derivative (Prop. 8), the matrix of $D(G \circ F)$ will be the product of the Jacobian matrices of G and F . Since $H = G \circ F : U \rightarrow \mathbb{R}^p$, the components of $H = (H^1, \dots, H^p)$ can be written as $H^i = G^i \circ F : U \rightarrow \mathbb{R}$. Then we have

$$\begin{aligned} (\partial H^i / \partial x^j) &= (\partial G^i / \partial y^k)(\partial F^k / \partial x^j), \\ \Rightarrow \frac{\partial H^i}{\partial x^j}(x) &= \frac{\partial(G^i \circ F)}{\partial x^j}(x) = \sum_{k=1}^m \frac{\partial G^i}{\partial y^k}(F(x)) \frac{\partial F^k}{\partial x^j}(x). \end{aligned}$$

Then each component $\partial H^i / \partial x^j$ is continuous because it is sum of product of continuous functions. Thus $G \circ F$ is also C^1 .

(b) Repeated application of chain rule shows that $G \circ F$ is smooth. □

Proposition 13. Suppose $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ are arbitrary subsets, and $F : A \rightarrow \mathbb{R}^m$ and $G : B \rightarrow \mathbb{R}^p$ are smooth maps (according to Def. 8) such that $F(A) \subseteq B$. Then $G \circ F : A \rightarrow \mathbb{R}^p$ is smooth.

Proof. Let $x \in A$, then by smoothness of F , there exists a neighborhood U of x and a smooth map $\tilde{F} : U \rightarrow \mathbb{R}^m$ such that $\tilde{F}|_{U \cap A} = F$. But $F(x) \in B$, so by smoothness of G , we find a neighborhood V of $F(x)$ and a smooth map $\tilde{G} : V \rightarrow \mathbb{R}^p$ such that $\tilde{G}|_{V \cap B} = G$. Define $\tilde{U} = U \cap A \cap F^{-1}(V \cap B)$. Then \tilde{U} is a neighborhood of x , and $\tilde{G} \circ \tilde{F} : \tilde{U} \rightarrow \mathbb{R}^p$ is a smooth map (by Prop. 12) such that $\tilde{G} \circ \tilde{F}|_{\tilde{U}} = G \circ F$. \square

Definition 10. Suppose $f : U \rightarrow \mathbb{R}$ is a smooth real-valued function on an open subset $U \subseteq \mathbb{R}^n$ and $a \in U$. For each vector $v \in \mathbb{R}^n$, we define the **directional derivative of f in the direction v at a** to be the number

$$D_v f(a) = \left. \frac{d}{dt} \right|_{t=0} f(a + tv). \quad (5)$$

Remark 4. This definition makes sense for any vector v ; we do not require v to be a unit vector as one sometimes does in elementary calculus.

Remark 5. Since $D_v f(a)$ is the ordinary derivative of the composite function $t \mapsto a + tv \mapsto f(a + tv)$, by chain rule it can be written more concretely as

$$D_v f(a) = \sum_{i=1}^n v^i \frac{\partial f}{\partial x^i}(a) = Df(a)v.$$

Proposition 14 (Differentiation Under an Integral Sign). Let $U \subseteq \mathbb{R}^n$ be an open subset, let $a, b \in \mathbb{R}$, and let $f : U \times [a, b] \rightarrow \mathbb{R}$ be continuous function such that partial derivatives $\partial f / \partial x^i : U \times [a, b] \rightarrow \mathbb{R}$ exist and are continuous on $U \times [a, b]$ for $i = 1, \dots, n$. Define $F : U \rightarrow \mathbb{R}$ by

$$F(x) = \int_a^b f(x, t) dt.$$

Then F is of class C^1 , and its partial derivatives can be computed by differentiating under the integral sign:

$$\frac{\partial F}{\partial x^i}(x) = \int_a^b \frac{\partial f}{\partial x^i}(x, t) dt.$$

For any m -tuple $I = (i_1, \dots, i_m)$ of indices with $1 \leq i_j \leq n$, we let $|I| = m$ denote the number of indices in I , and

$$\partial_I = \frac{\partial^m}{\partial x^{i_1} \dots \partial x^{i_m}},$$

$$(x - a)^I = (x^{i_1} - a^{i_1}) \dots (x^{i_m} - a^{i_m}).$$

Proposition 15 (Taylor's Theorem). Let $U \subseteq \mathbb{R}^n$ be an open subset, and let $a \in U$ be fixed. Suppose $f \in C^{k+1}(U)$ for some $k \geq 0$. If W is any convex subset of U containing a , then for all $x \in W$,

$$f(x) = P_k(x) + R_k(x), \quad (6)$$

where P_k is the **k -th order Taylor polynomial of f at a** , defined by

$$P_k(x) = f(a) + \sum_{m=1}^k \frac{1}{m!} \sum_{I: |I|=m} \partial_I f(a) (x - a)^I, \quad (7)$$

and R_k is the **k -th remainder term**, given by

$$R_k(x) = \frac{1}{k!} \sum_{I: |I|=k+1} (x-a)^I \int_0^1 (1-t)^k \partial_I f(a+t(x-a)) dt. \quad (8)$$

Proof. For $k = 0$ (where we interpret P_0 to mean $f(a)$), this is just the fundamental theorem of calculus (Prop. 14) applied to the function $u(t) = f(a+t(x-a))$, together with the chain rule. Assume the result holds for some k , integration by parts applied to the integral in the remainder term yield

$$\begin{aligned} & \int_0^1 (1-t)^k \partial_I f(a+t(x-a)) dt \\ &= \left[-\frac{(1-t)^{k+1}}{k+1} \partial_I f(a+t(x-a)) \right]_{t=0}^{t=1} + \int_0^1 \frac{(1-t)^{k+1}}{k+1} \frac{\partial}{\partial t} (\partial_I f(a+t(x-a))) dt \\ &= \frac{1}{k+1} \partial_I f(a) + \frac{1}{k+1} \sum_{j=1}^n (x^j - a^j) \int_0^1 (1-t)^{k+1} \frac{\partial}{\partial x^j} \partial_I f(a+t(x-a)) dt. \end{aligned}$$

When we insert this into (6), we obtain the analogous formula with k replaced by $k+1$. \square

Corollary. Suppose $U \subseteq \mathbb{R}^n$ is an open subset, $a \in U$, and $f \in C^{k+1}(U)$ for some $k \geq 0$. If W is a convex subset of U containing a on which all of the $(k+1)$ -st partial derivatives of f are bounded in absolute value by a constant M , then for all $x \in W$,

$$|f(x) - P_k(x)| \leq \frac{n^{k+1} M}{(k+1)!} |x-a|^{k+1},$$

where P_k is the k -th Taylor polynomial of f at a , defined by (7).

Proof. There are n^{k+1} terms on the right-hand side of (8), each term is bounded in absolute value by $(1/(k+1)!)|x-a|^{k+1}M$. \square

Proposition 16 (Lipschitz Estimate for C^1 Functions). Let $U \subseteq \mathbb{R}^n$ be an open subset, and suppose $F : U \rightarrow \mathbb{R}^m$ is of class C^1 . Then F is Lipschitz continuous on every compact convex subset $K \subseteq U$. The Lipschitz constant can be taken to be $\sup_{x \in K} \|DF(x)\|$.

Proof. Since $\|DF(x)\|$ is a continuous function of x , it is bounded on the compact set K . Let $M = \sup_{x \in K} \|DF(x)\|$. For arbitrary $a, b \in K$, we have $a+t(b-a) \in K$ for all $t \in [0, 1]$ because K is convex. By the fundamental theorem of calculus applied to each component of F , together with the chain rule,

$$\begin{aligned} F(b) - F(a) &= \int_0^1 \frac{d}{dt} F(a+t(b-a)) dt \\ &= \int_0^1 DF(a+t(b-a))(b-a) dt. \\ \implies \|F(b) - F(a)\| &\leq \int_0^1 \|DF(a+t(b-a))\| \|b-a\| dt \\ &\leq \int_0^1 M \|b-a\| dt = M \|b-a\|. \end{aligned}$$

\square

Corollary. If $U \subseteq \mathbb{R}^n$ is an open subset and $F : U \rightarrow \mathbb{R}^m$ is of class C^1 , then f is locally Lipschitz continuous.

Proof. Each point of U is contained in a ball whose closure is contained in U , and Prop. 16 shows that the restriction of F to such a ball is Lipschitz continuous. \square

3 The Inverse Function Theorem and Related Results

Definition 11. Let (X, d) be a metric space. A map $G : X \rightarrow X$ is said to be a **contraction** if there is a constant $\lambda \in (0, 1)$ such that $d(G(x), G(y)) \leq \lambda d(x, y)$ for all $x, y \in X$. A **fixed point** of a map $G : X \rightarrow X$ is a point $x \in X$ such that $G(x) = x$.

Remark 6. Clearly, every contraction is continuous.

Proposition 17 (Contraction Lemma). Let X be a nonempty complete metric space. Every contraction $G : X \rightarrow X$ has a unique fixed point.

Proof. Uniqueness is immediate, for if x, x' are both fixed points of G , the contraction property implies that $d(x, x') = d(G(x), G(x')) \leq \lambda d(x, x')$, which is possible only if $x = x'$.

To prove the existence of a fixed point, let x_0 be an arbitrary point in X , and define a sequence $(x_n)_{n=0}^\infty$ inductively by $x_{n+1} = G(x_n)$. For any $i \geq 1$ we have $d(x_i, x_{i+1}) = d(G(x_{i-1}), G(x_i)) \leq \lambda d(x_{i-1}, x_i)$, and therefore by induction $d(x_i, x_{i+1}) \leq \lambda^i d(x_0, x_1)$. If N is a positive integer and $j \geq i \geq N$,

$$\begin{aligned} d(x_i, x_j) &\leq d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) + \dots + d(x_{j-1}, x_j) \\ &\leq (\lambda^i + \dots + \lambda^{j-1})d(x_0, x_1) \leq \lambda^i \left(\sum_{n=0}^{\infty} \lambda^n \right) d(x_0, x_1) \\ &\leq \lambda^N \left(\frac{1}{1-\lambda} \right) d(x_0, x_1), \end{aligned}$$

where we have used that $\lambda^N \geq \lambda^i$ for $i \geq N$. Since the last expression can be made as small as desired by choosing N large, the sequence (x_n) is Cauchy and therefore converges to a limit $x \in X$. Because G is continuous,

$$x_n \rightarrow x \implies G(x_n) \rightarrow G(x), \text{ but } G(x) = \lim_{n \rightarrow \infty} G(x_n) = \lim_{n \rightarrow \infty} x_{n+1} = x,$$

so x is the desired fixed point. □

Proposition 18 (Inverse Function Theorem). Suppose U, V are open subsets of \mathbb{R}^n , and $F : U \rightarrow V$ is a smooth function. If $DF(a)$ is invertible, i.e., Jacobian determinant is nonzero, at some point $a \in U$, then there exists connected neighborhoods $U_0 \subseteq U$ of a and $V_0 \subseteq V$ of $F(a)$ such that $F|_{U_0} : U_0 \rightarrow V_0$ is a diffeomorphism.

Proof. We begin by making some simple modifications to the function F to streamline the proof. First, the function F_1 defined by $F_1(x) = F(x + a) - F(a)$ is smooth on a neighborhood of 0 and satisfies $F_1(0) = 0$ and $DF_1(0) = DF(a)$; clearly, F is a diffeomorphism on a connected neighborhood of a iff F_1 is a diffeomorphism on a connected neighborhood of 0. Second, the function $F_2 = DF_1(0)^{-1} \circ F_1$ is smooth on the same neighborhood of 0 and satisfies $F_2(0) = 0$ and $DF_2(0) = I_n$; and F_2 is a diffeomorphism on a connected neighborhood of 0 iff F_1 is a diffeomorphism and therefore also F . Henceforth, replacing F by F_2 , we assume that F is defined in a neighborhood U of 0, $F(0) = 0$ and $DF(0) = I_n$. Because the determinant of $DF(x)$ is a continuous function of x , by shrinking U if necessary, we may assume that $DF(x)$ is invertible for each $x \in U$.

Let $H(x) = x - F(x)$ for each $x \in U$. Then $DH(0) = I_n - I_n = 0$. Because the matrix entries of $DH(x)$ are continuous functions of x , there is a number $\delta > 0$ such that $\mathbb{B}_0(\delta) \subseteq U$ and for all $x \in \mathbb{B}_0(\delta)$, we have $\|DH(x)\| \leq \frac{1}{2}$. If $x, x' \in \mathbb{B}_0(\delta)$, the Lipschitz estimate for smooth functions (Prop. 16) implies that

$$\|H(x) - H(x')\| \leq \frac{1}{2} \|x - x'\|. \quad (9)$$

In particular, taking $x' = 0$, this implies

$$\|H(x)\| \leq \frac{1}{2}\|x\|. \quad (10)$$

Since $x' - x = F(x') - F(x) + H(x') - H(x)$, it follows that

$$\|x' - x\| \leq \|F(x') - F(x)\| + \|H(x') - H(x)\| \leq \|F(x') - F(x)\| + \frac{1}{2}\|x' - x\|,$$

and rearranging gives

$$\|x' - x\| \leq 2\|F(x') - F(x)\| \quad (11)$$

for all $x, x' \in \bar{\mathbb{B}}_0(\delta)$. In particular, this shows that F is injective on $\bar{\mathbb{B}}_0(\delta)$.

Now let $y \in \mathbb{B}_0(\delta/2)$ be arbitrary. We will show that there exists a unique point $x \in \mathbb{B}_0(\delta)$ such that $F(x) = y$. Let $G(x) = y + H(x) = y + x - F(x)$, so that $G(x) = x$ iff $F(x) = y$. If $\|x\| \leq \delta$, (10) implies

$$\|G(x)\| \leq \|y\| + \|H(x)\| < \frac{\delta}{2} + \frac{1}{2}\|x\| \leq \delta, \quad (12)$$

so G maps $\bar{\mathbb{B}}_0(\delta)$ to itself. It follows from (9) that $\|G(x') - G(x)\| = \|H(x) - H(x')\| \leq \frac{1}{2}\|x - x'\|$, so G is a contraction. Since $\bar{\mathbb{B}}_0(\delta)$ is a complete metric space, the contraction lemma implies that G has a unique fixed point $x \in \bar{\mathbb{B}}_0(\delta)$. From (12), $\|x\| = \|G(x)\| < \delta$, so in fact $x \in \mathbb{B}_0(\delta)$, thus proving the claim.

Let $V_0 = \mathbb{B}_0(\delta/2)$ and $U_0 = \mathbb{B}_0(\delta) \cap F^{-1}(V_0)$. Then U_0 is open in \mathbb{R}^n , and the argument above shows that $F : U_0 \rightarrow V_0$ is bijective, so $F^{-1} : V_0 \rightarrow U_0$ exists. Substituting $x = F^{-1}(y)$ and $x' = F^{-1}(y')$ into (11) shows that F^{-1} is continuous. Thus $F : U_0 \rightarrow V_0$ is a homeomorphism, and it follows that U_0 is connected because V_0 is.

The only thing that remains to be proved is that F^{-1} is smooth. If we knew it were smooth, Prop. 6 would imply that $D(F^{-1})(y) = DF(x)^{-1}$, where $x = F^{-1}(y)$. We begin by showing that F^{-1} is differentiable to each point of V_0 , with total derivative given by this formula.

Let $y \in V_0$ be arbitrary, and set $x = F^{-1}(y)$ and $L = DF(x)$. We need to show that

$$\lim_{y' \rightarrow y} \frac{F^{-1}(y') - F^{-1}(y) - L^{-1}(y' - y)}{\|y' - y\|} = 0.$$

Given $y' \in V_0 - \{y\}$, write $x' = F^{-1}(y') \in U_0 - \{x\}$. Then

$$\begin{aligned} \frac{F^{-1}(y') - F^{-1}(y) - L^{-1}(y' - y)}{\|y' - y\|} &= L^{-1} \left(\frac{L(x' - x) - (y' - y)}{\|y' - y\|} \right) \\ &= \frac{\|x' - x\|}{\|y' - y\|} L^{-1} \left(- \frac{F(x') - F(x) - L(x' - x)}{\|x' - x\|} \right). \end{aligned}$$

The factor $\|x' - x\|/\|y' - y\|$ above is bounded due to (11), and because L^{-1} is linear and therefore bounded, $\|L^{-1}\|$ is bounded. As $y' \rightarrow y$, it follows that $x' \rightarrow x$ by continuity of F^{-1} , and then the term in the bracket of last equation goes to zero because $L = DF(x)$ and F is differentiable. This completes the proof that F^{-1} is differentiable.

By Prop. 8, the partial derivatives of F^{-1} are defined at each point $y \in V_0$. Observe that the formula $D(F^{-1})(y) = DF(F^{-1}(y))^{-1}$ implies that the matrix-valued function $y \mapsto D(F^{-1})(y)$ can be written as the composition

$$y \xrightarrow{F^{-1}} F^{-1}(y) \xrightarrow{DF} DF(F^{-1}(y)) \xrightarrow{i} DF(F^{-1}(y))^{-1}, \quad (13)$$

where i is the matrix inversion. In the composition, F^{-1} is continuous; DF is smooth because its component functions are the partial derivatives of F ; and i is smooth because Cramer's rule expresses the entries of an inverse matrix as rational functions of entries of the matrix. Because

$D(F^{-1})$ is composition of continuous functions, it is continuous. Thus, the partial derivatives of F^{-1} are continuous, so F^{-1} is of class C^1 .

Now assume by induction that we have shown that F^{-1} is of class C^k . This means that each of the functions in (13) is of class C^k . Because $D(F^{-1})$ is a composition of C^k functions, it is itself C^k ; this implies that partial derivatives of F^{-1} are of class C^k , so F^{-1} itself is of class C^{k+1} . Continuing by induction, we conclude that F^{-1} is smooth. \square

Corollary. Suppose $U \subseteq \mathbb{R}^n$ is an open subset, and $F : U \rightarrow \mathbb{R}^m$ is a smooth function whose Jacobian determinant is nonzero at every point in U . Then

(a) F is an open map.

(b) If F is injective, then $F : U \rightarrow F(U)$ is a diffeomorphism.

Proof. (a) For each $a \in U$, the fact that the Jacobian determinant of F is nonzero implies that $DF(a)$ is invertible, so the inverse function theorem implies that there exists open subsets $U_a \subseteq U$ containing a and $V_a \subseteq F(U)$ containing $F(a)$ such that F restricts to a diffeomorphism $F|_{U_a} : U_a \rightarrow V_a$. In particular, this means that each point of $F(U)$ has a neighborhood contained in $F(U)$, so $F(U)$ is open. If $U_0 \subseteq U$ is an arbitrary open subset, the same argument with U replaced by U_0 shows that $F(U_0)$ is also open.

(b) If F is injective, then the inverse map $F^{-1} : F(U) \rightarrow U$ exists; on a neighborhood of each point $F(a) \in F(U)$ F^{-1} defined above is equal to the inverse of $F|_{U_a}$, so it is smooth. \square

Proposition 19 (Implicit Function Theorem). Let $U \subseteq \mathbb{R}^n \times \mathbb{R}^k$ be an open subset, and let $(x, y) = (x^1, \dots, x^n, y^1, \dots, y^k)$ denote the standard coordinates on U . Suppose $\Phi : U \rightarrow \mathbb{R}^k$ is a smooth function, $(a, b) \in U$, and $c = \Phi(a, b)$. If the $k \times k$ matrix $(\partial\Phi^i(a, b)/\partial y^j)$ is nonsingular, then there exists neighborhoods $V_0 \subseteq \mathbb{R}^n$ of a and $W_0 \subseteq \mathbb{R}^k$ of b and a smooth function $F : V_0 \rightarrow W_0$ such that $\Phi^{-1}(c) \cap (V_0 \times W_0)$ is the graph of F , i.e., $\Phi(x, y) = c$ for $(x, y) \in V_0 \times W_0$ iff $y = F(x)$.

Proof. Consider the smooth function $\Psi : U \rightarrow \mathbb{R}^n \times \mathbb{R}^k$ defined by $\Psi(x, y) = (x, \Phi(x, y))$. Its total derivative at (a, b) is

$$D\Psi(a, b) = \begin{pmatrix} I_n & 0 \\ \frac{\partial\Phi^i}{\partial x^j}(a, b) & \frac{\partial\Phi^i}{\partial y^j}(a, b) \end{pmatrix},$$

which is nonsingular because it is block lower triangular and the two blocks on the main diagonal are nonsingular. Thus by inverse function theorem there exists connected neighborhood U_0 of (a, b) and Y_0 of (a, c) such that $\Psi : U_0 \rightarrow Y_0$ is a diffeomorphism. Since $\Psi : U_0 \rightarrow Y_0$ is defined by $\Psi(x, y) = (x, \Phi(x, y))$, the inverse map $\Psi^{-1} : Y_0 \rightarrow U_0$ will be of the form $\Psi^{-1}(x, y) = (x, B(x, y))$ for smooth function $B : Y_0 \rightarrow \mathbb{R}^k$. Shrinking U_0 and Y_0 if necessary, we may assume that $U_0 = V \times W$ is a product neighborhood.

The two compositions $\Psi \circ \Psi^{-1}$ and $\Psi^{-1} \circ \Psi$ give

$$\begin{aligned} (x, y) &= (\Psi \circ \Psi^{-1})(x, y) = \Psi(x, B(x, y)) = (x, \Phi(x, B(x, y))), \forall (x, y) \in Y_0 \\ (x, y) &= (\Psi^{-1} \circ \Psi)(x, y) = \Psi^{-1}(x, \Phi(x, y)) = (x, B(x, \Phi(x, y))) \forall (x, y) \in U_0. \end{aligned} \quad (14)$$

If $\Phi(x, y) = c$, then the second equation of (14) gives $y = B(x, c)$. This suggests that we define $F(x) = B(x, c)$ for all $x \in \mathbb{R}^n$ for which $(x, c) \in Y_0$. Now let $V_0 = \{x \in V : (x, c) \in Y_0\}$ and $W_0 = W$, then $F : V_0 \rightarrow W_0$ defined by $F(x) = B(x, c)$.

Let $x \in V_0$. If $\Phi(x, y) = c$ then $y = B(x, c) = F(x)$, so the graph of F is contained in $\Phi^{-1}(c)$. Conversely, suppose $y = F(x)$ and in the first equation of (14) we set $(x, y) = (x, c)$, then $c = \Phi(x, B(x, c)) = \Phi(x, F(x)) = \Phi(x, y)$. This completes the proof. \square

Proposition 20. *The implicit function theorem is equivalent to the inverse function theorem.*

Proof. (\implies) Already shown above.

(\impliedby) Let $F : U \rightarrow V$ be a smooth map defined such that $U, V \subseteq \mathbb{R}^n$ are open subsets such that at some point $p \in U$ the Jacobian determinant is nonzero. Finding a local inverse for $y = F(x)$ near p amounts to solving the equation $G(x, y) = F(x) - y = 0$ for x in terms of y near $(p, F(p))$. Note that $\partial G^i / \partial x^j = \partial F^i / \partial x^j$. Hence,

$$\det \left[\frac{\partial G^i}{\partial x^j}(p, F(p)) \right] = \det \left[\frac{\partial F^i}{\partial x^j}(p, F(p)) \right] \neq 0.$$

By the implicit function theorem, x can be expressed in terms of y locally near $(p, F(p))$, i.e., there is a smooth function $x = H(y)$ defined in a neighborhood of $F(p)$ in \mathbb{R}^n such that $G(x, y) = F(x) - y = F(H(y)) - y = 0$. Thus, $y = F(H(y))$. Since $y = F(x)$, $x = H(y) = H(F(x))$. Therefore, F and H are inverse functions defined near p and $F(p)$ respectively and H is smooth by implicit function theorem. \square

References

- [1] John M. Lee, Introduction to Smooth Manifolds.