

# Real Analysis

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### 1 The Real Numbers

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$\mathbb{R}$	Set of all real numbers
$\mathbb{C}$	Set of all complex numbers
$\mathbb{N}$	Set of all natural numbers
$\mathbb{Q}$	Set of all rational numbers
$\mathbb{R} - \mathbb{Q}$	Set of all irrational numbers

Table 1: Standard notation for various sets.

**Definition 1.1.** A subset  $A$  of  $\mathbb{R}$  is said to be **bounded above** if there is some  $x \in \mathbb{R}$  such that  $a \leq x$  for all  $a \in A$ . Any such number  $x$  is called an **upper bound** for  $A$ .

**Axiom 1.2** (The Least Upper Bound Axiom or The Completeness Axiom). *Any nonempty set of real numbers with an upper bound has a least upper bound.*

**Definition 1.3.** Let  $A \subseteq \mathbb{R}$  be a nonempty set that is bounded above. Then the **supremum** of the set, denoted by  $\sup A$ , is the least upper bound, i.e.,  $\sup A = s \in \mathbb{R}$  such that

- (i)  $s$  is an upper bound for  $A$ ;
- (ii) if  $x$  is any upper bound for  $A$ , then  $s \leq x$ .

**Remark 1.4.** If  $A \subseteq \mathbb{R}$  is unbounded above, then  $\sup A = +\infty$  and if  $A = \emptyset$ , then  $\sup A = -\infty$  as every real number is an upper bound for  $A$ .

**Proposition 1.5** (Characterization of the Supremum). *Let  $A \subseteq \mathbb{R}$  be a nonempty set that is bounded above. Then the following statements about  $s = \sup A$  are all equivalent:*

- (a) if  $x$  is any upper bound for  $A$ , then  $s \leq x$ ;
- (b) if  $y < s$ , then we must have  $y < a \leq s$  for some  $a \in A$ ;
- (c) for every  $\varepsilon > 0$ , there is an  $a \in A$  such that  $a > s - \varepsilon$ .

*Proof.* (a)  $\implies$  (b) Suppose for some  $y < s$ , there is no  $a \in A$  such that  $y < a \leq s$ . Then for all  $a \in A$ , we have  $a \leq y < s$ . But then  $y$  is an upper bound with  $y < s$  which contradicts (a).

(b)  $\implies$  (c) Consider an arbitrary  $y \in \mathbb{R}$  such that  $y < s$ . Setting  $\varepsilon = s - y$  and using (b) we have that there is some  $a \in A$  such that  $s - \varepsilon = y < a$ .

(c)  $\implies$  (a) Let  $x$  be an upper bound for  $A$ , i.e.,  $a \leq x$  for all  $a \in A$  and suppose that  $x < s$ . By taking  $\varepsilon = s - x$  and applying (c), there must be some  $a \in A$ , such that  $s - \varepsilon = x < a$  which is a contradiction that  $x$  is an upper bound for  $A$ . Then  $s \leq x$ .  $\square$

**Theorem 1.6.** If  $A$  is a nonempty subset of  $\mathbb{R}$  that is bounded below, then  $A$  has a greatest lower bound called the infimum of  $A$  which is denoted by  $\inf A$ , i.e., there is a number  $m \in \mathbb{R}$  satisfying:

- (i)  $m$  is a lower bound for  $A$ ;
- (ii) if  $x$  is a lower bound for  $A$ , then  $x \leq m$ .

*Proof.* Consider the set  $-A = \{-a : a \in A\}$  which is bounded above as  $A$  is bounded below. By the completeness axiom,  $-A$  must have a least upper bound  $m = \sup(-A)$ . Note that  $-m = -\sup(-A)$  then the greatest lower bound for  $A$ , so that  $\inf A = -\sup(-A)$ .  $\square$

**Remark 1.7.** As we have established that  $\inf A = -\sup(-A)$ , we have  $\inf A = -\infty$  if  $A$  is unbounded below, and  $\inf \emptyset = +\infty$ . In case a set  $A$  is both bounded above and bounded below, we simply say that  $A$  is bounded.

**Proposition 1.8** (Characterization of the Infimum). Let  $A \subseteq \mathbb{R}$  be a nonempty set that is bounded below. Then the following statements about  $m = \inf A$  are all equivalent:

- (a) if  $x$  is any lower bound for  $A$ , then  $x \leq m$ ;
- (b) if  $y > m$ , then we must have  $m \leq a < y$  for some  $a \in A$ ;
- (c) for every  $\varepsilon > 0$ , there is an  $a \in A$  such that  $a < m + \varepsilon$ .

Proof similar to that of Prop. 1.5.

**Proposition 1.9.** Let  $A$  be a bounded subset of  $\mathbb{R}$  containing at least two points. Then

- (a)  $-\infty < \inf A < \sup A < +\infty$ .
- (b) If  $B$  is a nonempty subset of  $A$ , then  $\inf A \leq \inf B \leq \sup B \leq \sup A$ .
- (c) If  $B$  is the set of all upper bounds for  $A$ , then  $B$  is nonempty, bounded below and  $\inf B = \sup A$ .

*Proof.* (a) The boundedness of  $A$  implies that  $-\infty < \inf A \leq \sup A < +\infty$ . Since there are at least two points in  $A$ ,  $\inf A \neq \sup A$ .

(b) and (c) trivially hold from the definitions of infimum and supremum.  $\square$

**Definition 1.10.** A sequence  $(x_n)$  of real numbers is said to converge to  $x \in \mathbb{R}$  if, for every  $\varepsilon > 0$ , there is a positive integer  $N$  such that

$$n \geq N \implies |x_n - x| \leq \varepsilon.$$

In this case, we call  $x$  the limit of the sequence  $(x_n)$  and write  $x = \lim_{n \rightarrow \infty} x_n$ .

**Proposition 1.11.** Let  $A$  be a nonempty subset of  $\mathbb{R}$  that is bounded above. Then there is a sequence  $(x_n)$  of elements of  $A$  that converges to  $\sup A$ .

*Proof.* Using Prop. 1.5(c), we set  $\varepsilon_n = 1/n$  and  $y_n = \sup A - \varepsilon_n$ . Then for all  $n \in \mathbb{N}$ , there exists an element  $x_n \in A$  such that  $x_n > \sup A - \varepsilon = y_n$ . But then  $|x_n - \sup A| < |y_n - \sup A| = \varepsilon_n = 1/n$ . This shows that  $\lim_{n \rightarrow \infty} |x_n - \sup A| = 0$ , i.e.,  $\lim_{n \rightarrow \infty} x_n = \sup A$ .  $\square$

**Lemma 1.12** (Archimedean property in  $\mathbb{R}$ ). If  $x$  and  $y$  are positive real numbers, then there is some positive integer  $n$  such that  $nx > y$ .

*Proof.* Suppose that no such  $n$  existed, i.e., suppose that  $nx \leq y$  for all  $n \in \mathbb{N}$ . Then  $A = \{nx : n \in \mathbb{N}\}$  is bounded above by  $y$ , and so  $s = \sup A$  is finite. Now, since  $s - x < s$ , we must have some element of  $A$  in between, i.e.,  $s - x < nx \leq s$  for some  $n \in \mathbb{N}$ . But then  $s < (n+1)x \in A$  which is a contradiction, hence there is some  $n \in \mathbb{N}$  such that  $nx > y$ .  $\square$

**Theorem 1.13.** *If  $a$  and  $b$  are real numbers with  $a < b$ , then there is a rational number  $r \in \mathbb{Q}$  with  $a < r < b$ .*

*Proof.* We set  $x = b - a > 0$ ,  $y = 1$ , and apply Lemma 1.12 to get a positive integer  $q$  such that  $q(b - a) > 1$ , i.e.,  $qb > qa + 1$ . But if  $qa$  and  $qb$  differ by more than 1, there must be some integer in between, i.e., there is some  $p \in \mathbb{Z}$  with  $qa < p < qb$ . Thus,  $a < \frac{p}{q} < b$ .  $\square$

**Corollary 1.13.1.** *If  $a$  and  $b$  are real numbers with  $a < b$ , then there is also an irrational number  $x \in \mathbb{R} - \mathbb{Q}$  with  $a < x < b$ .*

*Proof.* We set  $x = \frac{1}{\sqrt{2}}(b - a) > 0$ ,  $y = 1$ , and apply Lemma 1.12 to get a positive integer  $q$  such that  $\frac{1}{\sqrt{2}}qb > \frac{1}{\sqrt{2}}qa + 1$ . But then there is some  $p \in \mathbb{Z}$  with  $\frac{1}{\sqrt{2}}qa < p < \frac{1}{\sqrt{2}}qb$ . Thus,  $a < \frac{\sqrt{2}p}{q} < b$ .  $\square$

**Corollary 1.13.2.** *Given  $a < b$ , there are, in fact, infinitely many distinct rationals between  $a$  and  $b$ . The same goes for irrationals, too.*

*Proof.*  $\square$

**Remark 1.14.** The least upper bound axiom holds in  $\mathbb{Z}$  since for any nonempty set that is bounded above, there exists a least upper bound which is the maximum value of the set itself. But this axiom does not hold in  $\mathbb{Q}$ . Consider the set  $A = \{p/q \in \mathbb{Q} : p^2 < 2q^2\}$ . This is bounded above in  $\mathbb{Q}$  as 2 is an upper bound. But  $\sup A = \sqrt{2} \notin \mathbb{Q}$  so  $A$  has not least upper bound in  $\mathbb{Q}$ .

**Proposition 1.15.** *The following statements are all equivalent:*

- (a) *(The least upper bound property). Any nonempty set of real numbers with an upper bound has a least upper bound.*
- (b) *A monotone, bounded sequence of real numbers converges.*
- (c) *(The nested interval property). If  $(I_n)$  is a sequence of closed, bounded, nonempty intervals in  $\mathbb{R}$  with  $I_1 \supset I_2 \supset \dots$ , then  $\cap_{n=1}^{\infty} I_n \neq \emptyset$ . If, in addition,  $\text{length}(I_n) \rightarrow 0$ , then  $\cap_{n=1}^{\infty} I_n$  contains precisely one point.*

*Proof.* (a)  $\implies$  (b) Let  $(x_n) \subset \mathbb{R}$  be monotone and bounded. We first suppose that  $(x_n)$  is increasing. Now, since  $(x_n)$  is bounded, we may set  $x = \sup_n x_n \in \mathbb{R}$ . Suppose  $\varepsilon > 0$ , then from (a) we must have  $x_N > x - \varepsilon$  for some  $N$ . But then, for any  $n \geq N$ , we have  $x - \varepsilon < x_N \leq x_n \leq x$ , i.e.,  $|x - x_n| < \varepsilon$  for all  $n \geq N$ . Consequently,  $(x_n)$  converges and  $x = \sup_n x_n = \lim_{n \rightarrow \infty} x_n$ . Finally, if  $(x_n)$  is decreasing, consider the increasing sequence  $(-x_n)$ . From the previous arguments,  $(-x_n)$  converges to  $\sup_n (-x_n) = -\inf_n x_n$ . It then follows that  $(x_n)$  converges to  $\inf_n x_n$ .

(b)  $\implies$  (c) Let  $I_n = [a_n, b_n]$ . Then  $I_n \supset I_{n+1}$  means that  $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$  for all  $n$ . Then from (b), we have  $a = \lim_{n \rightarrow \infty} a_n = \sup_n a_n$  and  $b = \lim_{n \rightarrow \infty} b_n = \inf_n b_n$  both exist and satisfy  $a \leq b$ . Thus we must have  $\cap_{n=1}^{\infty} I_n = [a, b]$ . Finally, if  $b_n - a_n = \text{length}(I_n) \rightarrow 0$ , then  $a = b$  and so  $\cap_{n=1}^{\infty} I_n = \{a\}$ .

(c)  $\implies$  (a) Let  $A$  be a nonempty subset of  $\mathbb{R}$  that is bounded above. Specifically, let  $a_1 \in A$  and let  $b_1$  be an upper bound for  $A$ . For later reference, set  $I_1 = [a_1, b_1]$ . Now consider the point  $x_1 = (a_1 + b_1)/2$ . If  $x_1$  is an upper bound for  $A$ , we set  $I_2 = [a_1, x_1]$ ; otherwise, there is an element  $a_2 \in A$  with  $a_2 > x_1$ . In this case, set  $I_2 = [a_2, b_1]$ . In either event,  $I_2$  is a closed subinterval of  $I_1$  of the form  $[a_2, b_2]$ , where  $a_2 \in A$  and  $b_2$  is an upper bound for  $A$ . Moreover,  $\text{length}(I_2) \leq \text{length}(I_1)/2$ . We now start the process all over again, using  $I_2$  in the place of  $I_1$ , and obtain  $I_3 = [a_3, b_3] \supset I_2$ , where  $a_3 \in A$  and  $b_3$  is an upper bound for  $A$ , with  $\text{length}(I_3) \leq \text{length}(I_2)/2 \leq \text{length}(I_1)/4$ . By induction, we get a sequence of nested closed intervals  $I_n = [a_n, b_n]$ , where  $a_n \in A$  and  $b_n$  is an upper bound for  $A$ , with  $\text{length}(I_n) \leq \text{length}(I_1)/2^{n-1} \rightarrow 0$  as  $n \rightarrow \infty$ . The single point  $b \in \bigcap_{n=1}^{\infty} I_n$  is the least upper bound for  $A$  since  $b = \sup_n a_n = \inf_n b_n$ .  $\square$

## References

- [1] Carothers, N.-L. Real analysis. Cambridge University Press, 2000.