Fourier Series And The Fourier Transform

Et Applications

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Summary

Fourier Series

Signals

- Define a signal to be a real valued (Riemann) integrable function on some time interval T.
- $f : T \to \mathbb{R}$, $T = [a, b]$, $a, b \in \mathbb{R}$
- $\forall [t_0, t_1] \subseteq T$, $\left| \int_{t_0}^{t_1} f(t)dt \right| < \infty$

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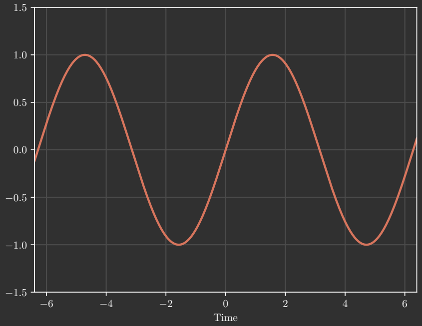
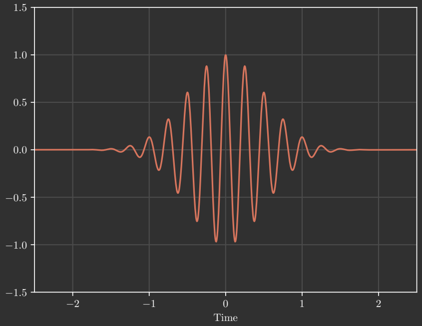
$$f(t) = \sin(t)$$



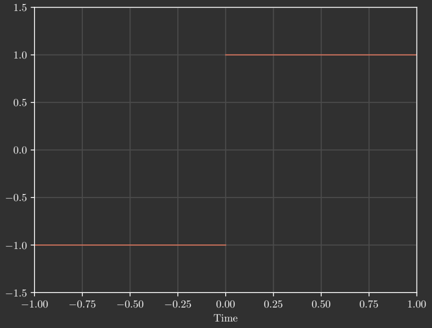
Signals

$$f(t) = \cos(25x)e^{-2x^2}$$



Signals

$$f(t) = \begin{cases} 1 & t \geq 0 \\ -1 & t < 0 \end{cases}$$



Fourier Series

- In the interest of brevity we will limit our study of signals to those defined on the interval $T = [0, 1]$
- All such signals can be expressed as a (possibly infinite) sum of sinusoidal signals with integer valued frequencies;

$$f(t) \sim \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi int}$$

$$\hat{f}(n) := \int_0^1 f(t) e^{-2\pi int} dt$$

$$\lim_{n \to \pm\infty} \hat{f}(n) = 0$$

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- Since we specified signals to be real valued functions, their Fourier series admit the following form.

$$f(t) \sim -\frac{a_0}{2} + \sum_{n\in\mathbb{N}} a_n \cos(2\pi n t) + \sum_{n\in\mathbb{N}} a_n \sin(2\pi n t)$$

$$a_n := 2\int_0^1 f(t)\cos(2\pi n t)dt$$

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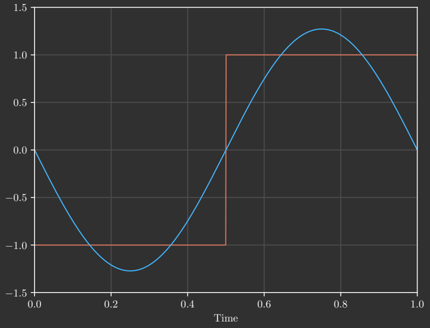

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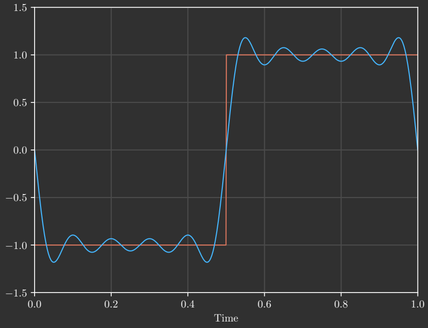

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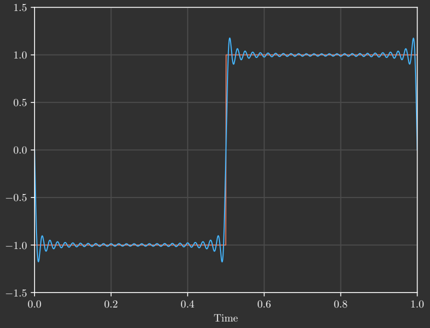

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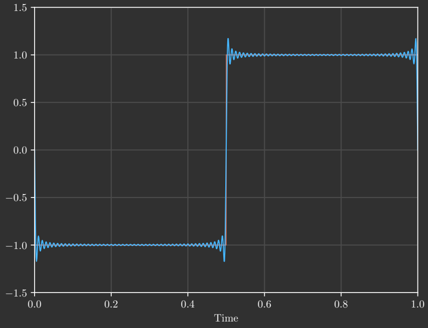

Fourier series on the n-cube

- Fourier analysis of this nature generalises near seamlessly to higher dimensions.
- Extend the definition of signals to real valued functions on the n-cube; $T^n = [0,1]^n$.

$$f(\bar{t}) \sim \sum_{\bar{x}\in\mathbb{N}^n} \hat{f}(\bar{x})e^{2\pi i\bar{t}}, \; \bar{x} = (x_1, x_2, \ldots, x_n)$$

$$\hat{f}(\bar{x}) = \int_{T^n} f(\bar{t})e^{-2\pi i\bar{t}\cdot\bar{x}}d\bar{t}$$

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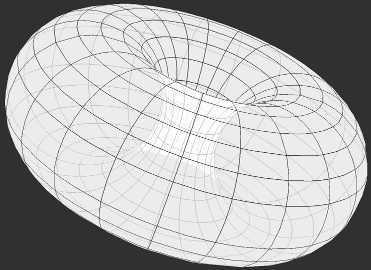
Periodicity

- Observe that the Fourier Series as defined on signals with domain T^n extend them to the entirety of $\mathbb{R}^n$.
- In particular, if we identify the endpoints of our intervals, our space is homeomorphic to the n-torus.



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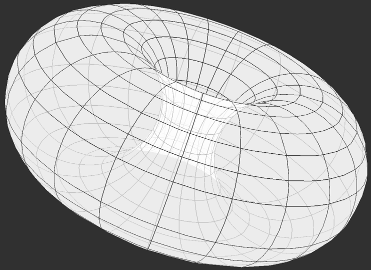

Results

- The rate of convergence of Fourier Series is governed by the smoothness of the signal.

$$f \in C^k \implies \hat{f}(n) \in O\left(\frac{1}{n^{k+1}}\right)$$

- For piece-wise continuous signals f with left and right first order derivatives everywhere on T;
- The Fourier series of f converges pointwise to f.
- The Cesaro means of the Fourier series converge pointwise to f.
- If f is continuous on T then the Fourier series of f converges to f uniformly with respect to $t \in T$.
- There exists a sequence of polynomials which converge to f uniformly with respect to $t \in T$

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Methodology

- Integration techniques E.g. integration by parts, exploiting periodicity.
- Convolution, $f * g = \int_0^1 f(t)g(1-t)dt$.
- In particular, convolution with the Dirichlet and Fejer Kernels.

$$D_N(x) = \sum_{n=-N}^{N} e^{2\pi int} = \frac{\sin\left(2\pi(N+\frac{1}{2})x\right)}{\sin(\pi x)}$$

$$F_N(x) = \frac{1}{N}\sum_{n=0}^{N-1} D_n(x) = \frac{1}{N}\left(\frac{1-\cos(2\pi Nx)}{1-\cos(2\pi x)}\right)$$

- Epsilon-Delta proofs.

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The Fourier Transform

Motivating the Fourier Transform

- Information is often presented and interpreted in a naturally ordered manner;
- Sound pressure is temporal.
- Images have spatial composition.
- It's difficult to deduce the qualitative effect of the various components of some data set on human perception with analysis over the natural domain.
- Trends in the frequency domain are much more reliably and accurately correlated with perception.

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- We may consider real valued integrable functions on $\mathbb{R}$ signals with arbitrarily large period.
- In turn, the 'distance' between frequencies becomes arbitrarily small.
- In particular, the function $\hat{f}(n)$ on the integers is extended to a continuous function over the reals.

$$\mathcal{F}f(s) = \int_{-\infty}^{\infty} f(t)e^{-2\pi i s t}dt$$

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The Fourier Transform on $\mathbb{R}^n$

- Data is often multidimensional;
- time
- space
- colour
- We can extend the fourier transform to higher dimensions analogously to the Fourier Series.

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Discrete information

- Signals realised in the world at large are discrete, not continuous.
- Implementations of the Fourier transform need to be designed appropriately.
- In particular, a discrete analogue is needed.
- "It Works to the extent that a large fraction of the world's economy depends on it working, and that's not a bad measure of success. Consider this a proof by economics."
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- Choose N to be the number of samples (and the number of frequencies).
- f is now defined on a discrete set of times with cardinality N

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DFT Matrix

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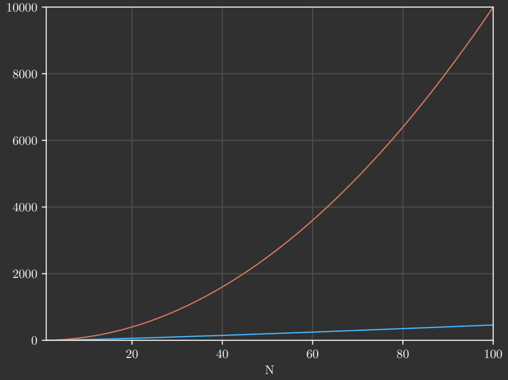
FFT

- Fast Fourier Transforms (FFTs) are a class of algorithms which implement the Discrete Fourier Transform.
- Typically, these are divide and conquer algorithms which achieve $O(N \log N)$ complexity through the use of recursion.
- "The most important numerical algorithm of our lifetime"
  - Gilbert Strang

- N^2 in orange.
- $N \log N$ in blue.



Applications

Huffman Encoding

- Huffman Encoding is a scheme to save memory by exploiting the frequency of symbols.
- A tree is constructed based on the frequency with which symbols occur and they are then assigned a binary encoding based on this tree.
- Symbols that ocur more frequently are assigned shorter encodings.
- If one is willing to replace a significant portion of a string with a single symbol, the space required to store information can be reduced considerably.
- This (amongst other entropy encoding techniques) is the basis of all lossy Fourier Transformation compression Algorithms.
- The key is to determine what information can be discarded with little effect.

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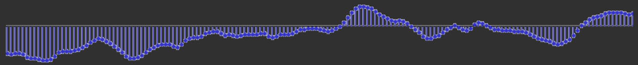
Audio

- Audio is a natural candidate for the Fourier transform in light of it's well known association with frequency.
- The Fourier transform of an audio signal conveys the amplitude of each frequency present.



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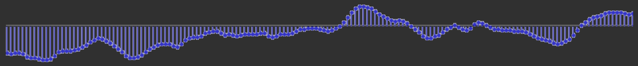

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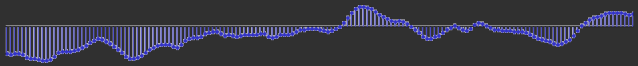

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Psychoacoustics

- Humans only perceive a narrow band of frequencies, $\sim$ 20Hz - 20kHz
- Moreover, within this band of frequencies our capacity to process the volume of data available to us is severly limited.
- Current (experimentally determined) models posit the existence of 24 'critical bands', frequency ranges within which one can only discern a single dominant frequency in some temporal neighbourhood.
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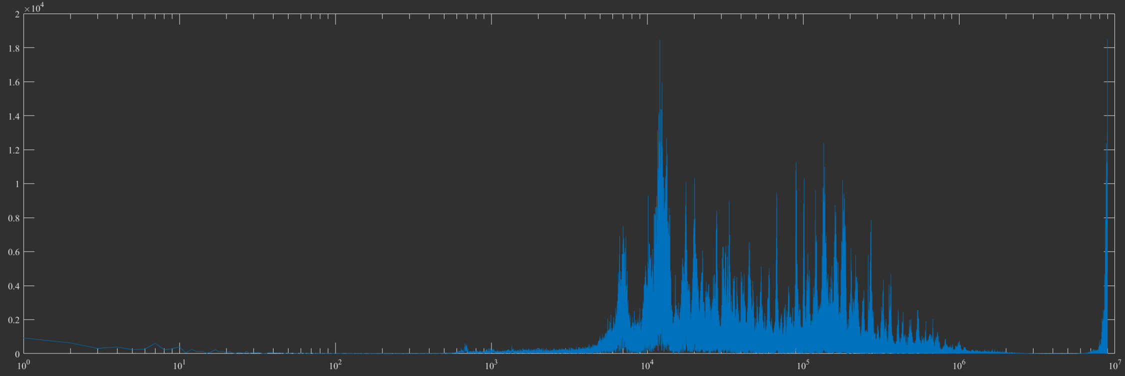
The Ecstasy of Gold FFT



JPEG Compression

- Compression of images operates in much the same way.
- The image is interpreted as a 2D array of tuples, typically representing information about hue and brightness.
- The array is divided into smaller arrays with dimensions $\sim 2^6 \times 2^6$.
- A 2 dimensional FFT is performed on each of the sub-arrays, and certain frequencies are discarded according to some experimentally determined and refined scheme.
- In particular the result is multiplied by a proprietary quantisation matrix.
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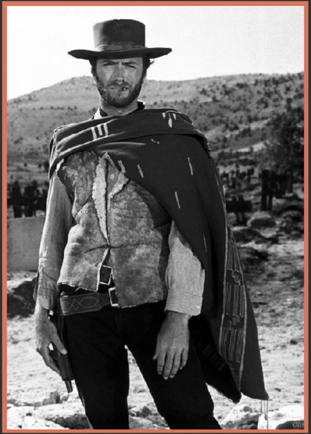
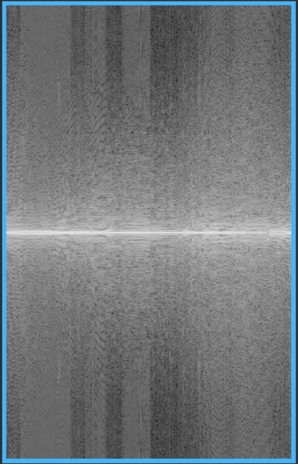


Blondie FFT



Lossy Compression

- Such compression algorithms are termed 'lossy'.
- This is because information originally encoded is lost in the process, and can not be recovered via Fourier inversion.
- Typically this is an acceptable trade off, observable discrepancies between the original and are minor and scarce.
- MP3 compression ratios vary depending on user selected parameters which govern the quality of the output, namely bit and sample rates.
- At 128 kilobits per second and 44.1kHz (the modal choice of parameters) compression ratios are on average 11 to 1.
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Acknowledgments

Special thanks to Prof. Daniel Daners for his guidance and support throughout the semester.

References

- Katznelson, Y. *An Introduction to Harmonic Analysis.* 1st ed. (John Wiley Sons, Inc, 1968).
- Osgood, B. G. *Lectures on the Fourier Transform and Its Applications.* (AMS, 2019).
- Raissi, R. *The Theory Behind Mp3.* http://www.mp3-tech.org/programmer/docs/mp3_theory.pdf.

Thank You For Your Time