SCDL3991 Report

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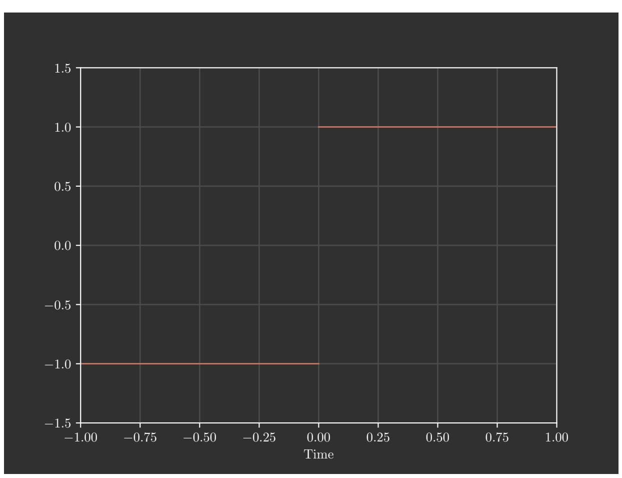
Abstract

Fourier Analysis is used widely in modern industry and in various areas of mathematical study. In this paper we investigate and prove results regarding the convergence of Fourier Series. We then outline the functionality of the Fourier Transform and its discrete cousins made possible by the existence of spectral decompositions of complex signals. This is then examined in the context of audio processing.

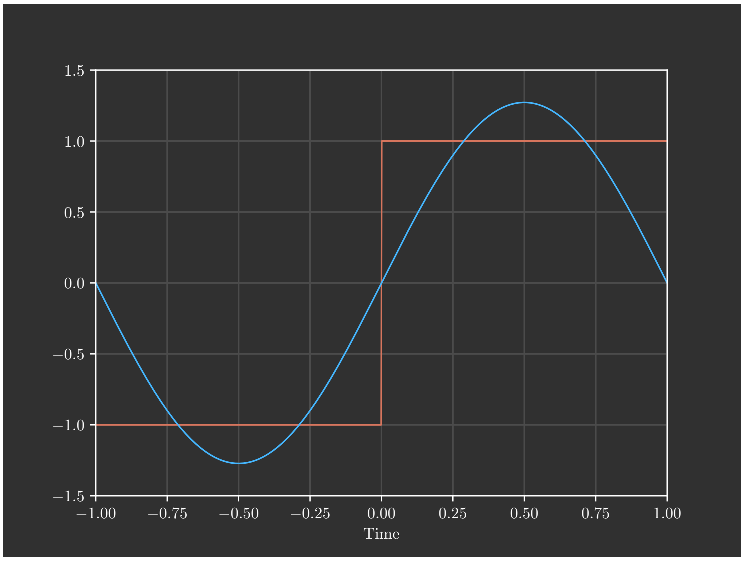
1 Introduction

Fourier Analysis is possibly the mathematical technology of greatest practical importance to the modern world. In particular, the Fourier Transform and its derivatives are ubiquitous in all manner of industries. We seek to make rigorous the foundations of these techniques and determine under which conditions they are most effective.

Below is the sign function on the interval $[-1, 1]$. We define it as follows; $f(t) = \begin{cases} -1 & t < 0 \\ 1 & t \geq 0 \end{cases}$

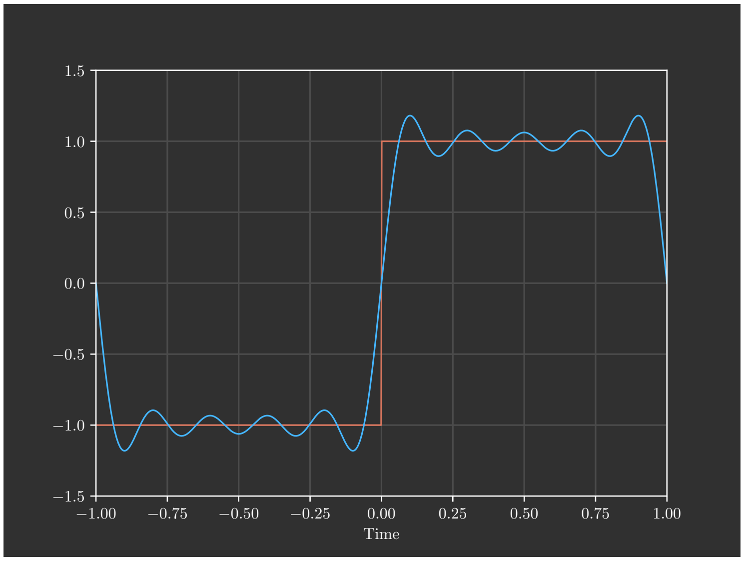


The first order Fourier approximation to the signal is a simple sinusoid.

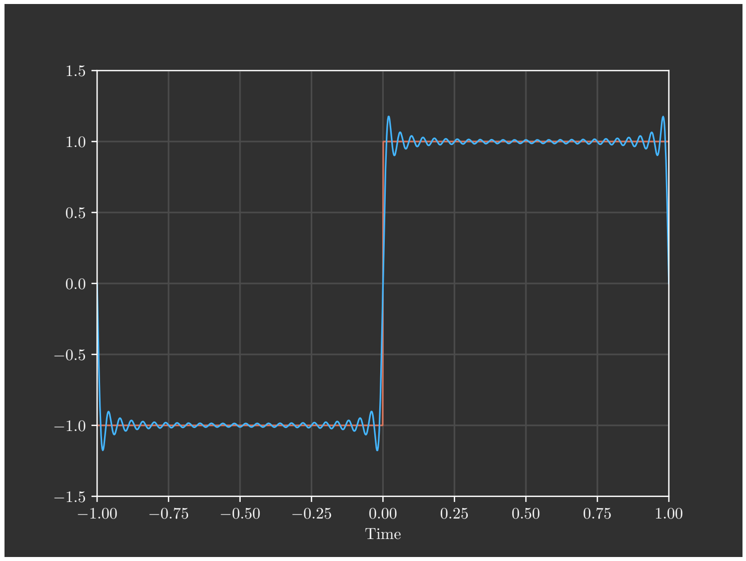


Higher order approximations are more complex and quickly come to resemble the signal closely.

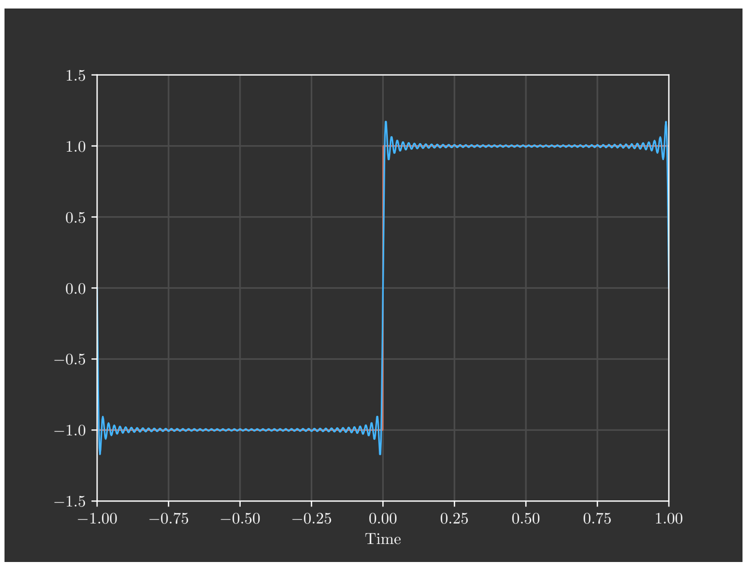
The 10th order approximation;



The 50th order approimation;



And lastly the 100th order approximation.



Over the smooth and continuous intervals of the function, the approximations converge quickly. At the discontinuities however, an error is encountered which persists in arbitrarily high order approximations (and converges to approximately 9% [4] of the difference between the left and right limits at the discontinuity). This behaviour is called the Gibbs Phenomenon and is emblematic of the convergence issues which plague Fourier Analysis of non smooth and discontinuous functions.

1.1 Signals

The primary subject of our study is the signal. We define a signal to be any Riemann integrable function from a subset of $\mathbb{R}^n$ to $\mathbb{C}$. Virtually all pertinent quantitative information in the world at large admits such an encoding or model. Such functions can (to varying degrees) be expressed as (possibly infinite) series of sinusoids. For the purpose of investigating this claim analytically we will limit our study to periodic functions on the unit hyper-cube. Note that such approximations are sufficient since any signal on any (hyper)rectangular region may be made periodic on the unit hyper-cube by a process of translation, scaling and periodic extension.

1.2 Fourier Series

The discrete frequencies of the approximating series vary according to the conventions of authors, in this paper we will assume integral multiples of 2π. The coefficients of the Fourier Series of a one dimensional signal f are defined as follows;

$$\hat{f}(n) = \int_0^1 f(t)e^{-2i\pi nt}dt \tag{1}$$

for natural numbers n. The Nth order Fourier series of f is defined as follows (an analogous definition of the Fourier series of an n dimensional signal can be found in the appendices);

$$S_N(t) = \sum_{n=-N}^{N} \hat{f}(n)e^{2i\pi nt} \tag{2}$$

There is, however, no absolute sense in which the limit of the sequence of partial sums is equal to the approximated signal. The nature of its convergence upon the signal is dependent on the properties of the signal, in particular its differentiability and continuity.

2 Fourier Series Convergence

We list and prove the convergence properties of the Fourier Series of signals with varying degrees of differentiability and continuity.

2.1 The Riemann Lebesgue Lemma

The Fourier Coefficients of Riemann integrable functions converge to zero. Smooth functions converge more quickly with clear asymptotic bounds.

Theorem 2.1. *If f is a signal on the interval $[a, b]$, then*

$$\lim_{n\to\infty} \hat{f}(n) = \int_a^b f(t)e^{-2i\pi nt}dt = 0 \tag{3}$$

moreover, if f is periodic,

$$f \in \mathbb{C}^k \implies \hat{f} \in O\left(\frac{1}{n^k}\right) \tag{4}$$

Proof. We first demonstrate the lemma for simple functions, then reduce all signals to simple functions by proxy of their lower Darboux sums.

Define the indicator function of an interval I, χ_I to be the function which is unit valued on I and zero valued elsewhere. A simple function is defined as a sum of scalar multiples of indicator functions over disjoint intervals. Of relevance is that the Riemann sums of a function are described by the integrals of simple functions. Let g be the simple function over disjoint sub intervals I of $[a, b]$ in the set T, and $\lambda_I \chi_I$, $I \in T$, $\lambda_I \in \mathbb{R}$ be the scaled characteristic functions of those sub intervals;

$$g(t) := \sum_{I \in T} \lambda_I \chi_I(t)$$

then the Fourier coefficients of g are as follows

$$
\begin{aligned}
|\hat{g}(n)| &= \left|\int_a^b \sum_{I\in T} \lambda_I \chi_I(t) e^{-2i\pi nt} dt\right| \\
|\hat{g}(n)| &= \left|\sum_{I\in T} \int_I \lambda_I e^{-2i\pi nt} dt\right| \\
|\hat{g}(n)| &\leq \max_{I\in T}(|\lambda_I|) \sum_{I\in T} \left|\int_I e^{-2i\pi nt} dt\right| \\
|\hat{g}(n)| &\leq \frac{\max_{I\in T}(|\lambda_I|)|T|}{\pi n}
\end{aligned}
$$

Therefore,

$$\lim_{n\to\infty} \hat{g}(n) = 0 \tag{5}$$

For more general signals f, let g_N be the sequence of simple functions representing the values attained by progressively finer regular meshes in the lower Darboux sums of f. Since f is Riemann integrable, $\forall\varepsilon > 0\ \exists N_\varepsilon \in \mathbb{N}$ such that $\forall N \geq N_\varepsilon\ \int_a^b (f(t) - g_N(t))dt \leq \frac{\varepsilon}{2}$. By the result of (5), $\forall\varepsilon > 0$ $\exists n_\varepsilon \in \mathbb{N}$ such that $\forall n \geq n_\varepsilon\ \hat{g}_N(n) \leq \frac{\varepsilon}{2}$. We now have the means to construct an epsilon delta proof of the first result. First fix ε then choose N and n as above (in that order).

$$
\begin{aligned}
\hat{f}(n) &= \int_a^b f(t) e^{-2i\pi nt} dt \\
\hat{f}(n) &= \int_a^b (f(t) - g_N(t)) e^{-2i\pi nt} dt + \int_a^b g_N(t) e^{-2i\pi nt} dt \\
|\hat{f}(n)| &\leq \left|\int_a^b (f(t) - g_N(t)) e^{-2i\pi nt} dt\right| + \left|\int_a^b g_N(t) e^{-2i\pi nt} dt\right| \\
|\hat{f}(n)| &\leq \int_a^b (f(t) - g_N(t)) dt + |\hat{g_N}(n)| \\
|\hat{f}(n)| &\leq \varepsilon
\end{aligned}
$$

As simple corollaries, $\forall \lambda \in \mathbb{R}$

$$\lim_{n\to\infty}\int_a^b f(t)e^{i\lambda nt+\phi}dt = \lim_{n\to\infty}\int_a^b f(t)\sin(\lambda nt+\phi)dt = \lim_{n\to\infty}\int_a^b f(t)\cos(\lambda nt+\phi)dt = 0 \tag{6}$$

The generalised statement of these results for functions of n variables can be found in the appendices. Suppose now that $f \in C^k$

$$\begin{aligned}
\hat{f}(n) &= \int_a^b f(t)e^{-2i\pi nt}dt \\
\hat{f}(n) &= -\frac{1}{2i\pi n}\Big| f(t) - e^{-2i\pi nt}\Big|_a^b + \frac{1}{2i\pi n}\int_0^1 f'(t)e^{-2i\pi nt}dt \\
\hat{f}(n) &= \frac{1}{2i\pi n}\int_0^1 f'(t)e^{-2i\pi nt}dt \\
\hat{f}(n) &= \vdots \text{ (Theorem 6.8 justifies iteration of this process)} \\
\hat{f}(n) &= \left(\frac{1}{2i\pi n}\right)^k \int_0^1 f^k(t)e^{-2i\pi nt}dt
\end{aligned}$$

By the Riemann Lebesgue lemma, $\exists n_1 \in \mathbb{N}$ such that $\forall n > n_1$, $|\int_0^1 f^k(t)e^{-2i\pi nt}dt| \leq 1$. Therefore, $\forall n > n_1$, $|\hat{f}(n)| \leq \frac{1}{n^k}$ and hence, $\hat{f}(n) \in O\left(\frac{1}{n^k}\right)$ □

2.2 Piecewise Differentiable

Theorem 2.2. *If a signal f is piecewise continous with finitely many discontinuities and has left and right derivatives at every point, then its Fourier Series converges pointwise to the average of the left and right limits of the function for every value in its preimage.*

$$\lim_{N\to\infty}\sum_{n=-N}^{N}\hat{f}(n)e^{2i\pi nt} = \frac{f(t^-)+f(t^+)}{2} \tag{7}$$

Proof. We first introduce convolution, which will allow us to approach the sequence of partial sums analytically. The convolution of 2 signals f and g periodic on the interval $[0,1]$ is defined as follows.

$$f * g(t) = \int_0^1 f(x)g(t-x)dx \tag{8}$$

Note that this operation is commutative;

$$
\begin{aligned}
(f * g)(t) &= \int_0^1 f(x)g(t-x)dx \\
&= -\int_t^{t-1} f(t-u)g(u)du \\
&= \int_0^1 f(t-x)g(x)dx \\
&= (g * f)(t)
\end{aligned}
\tag{9}
$$

We now introduce the Dirichlet Kernels, a family of series which when convolved with signals yields their Fourier Series.

$$
D_N(t) = \sum_{n=-N}^{N} e^{2i\pi nt} \tag{10}
$$

Note that the periodicity of the complex exponential implies periodicity of the kernel. Consider the convolution of such an f with the Nth Dirichlet Kernel.

$$
\begin{aligned}
(f * D_N)(t) &= \int_0^1 f(x) \sum_{n=-N}^{N} e^{2i\pi n(t-x)}dx \\
&= \int_0^1 f(x) \sum_{n=-N}^{N} e^{2i\pi n} e^{-2i\pi nt}dx \\
&= \sum_{n=-N}^{N} \int_0^1 f(x)e^{2i\pi nt}e^{-2i\pi nx}dx \\
&= \sum_{n=-N}^{N} e^{2i\pi nt} \int_0^1 f(x)e^{-2i\pi nx}dx \\
&= \sum_{n=-N}^{N} e^{2i\pi nt}\hat{f}(n) \\
&= S_N(t)
\end{aligned}
$$

It can be shown by induction (Theorem 6.2) that the Dirichlet Kernel as defined admits the following closed form

$$
D_N(t) = \frac{\sin\left((2\pi(N+\frac{1}{2})t\right)}{\sin(\pi t)} \tag{11}
$$

From this it is deduced;

$$
\begin{aligned}
(f * D_N)(t) &= (D_N * f)(t) \\
S_N(t) &= \int_0^1 \frac{\sin\left((2\pi(N+\frac{1}{2})x\right)}{\sin(\pi x)} f(t-x)dx \\
S_N(t) &= \int_{-\frac{1}{2}}^0 \frac{\sin\left((2\pi(N+\frac{1}{2})x\right)}{\sin(\pi x)} f(t-x)dx + \int_0^{\frac{1}{2}} \frac{\sin\left((2\pi(N+\frac{1}{2})x\right)}{\sin(\pi x)} f(t-x)dx \\
S_N(t) &= \int_{-\frac{1}{2}}^0 \frac{\sin\left((2\pi(N+\frac{1}{2})x\right)}{\sin(\pi x)} (f(t-x) - f(t^-))dx + f(t^-)\int_{-\frac{1}{2}}^0 \frac{\sin\left((2\pi(N+\frac{1}{2})x\right)}{\sin(\pi x)} dx \\
&\quad + \int_0^{\frac{1}{2}} \frac{\sin\left((2\pi(N+\frac{1}{2})x\right)}{\sin(\pi x)} (f(t-x) - f(t^+)dx + f(t^+)\int_0^{\frac{1}{2}} \frac{\sin\left((2\pi(N+\frac{1}{2})x\right)}{\sin(\pi x)} dx
\end{aligned}
$$

By Theorem 6.4,

$$
\begin{aligned}
S_N(t) - \frac{f(t^-)+f(t^+)}{2} &= \int_{-\frac{1}{2}}^0 \sin\left((2\pi(N+\frac{1}{2})x\right)\left(\frac{x}{\sin(\pi x)}\right)\left(\frac{f(t-x)-f(t^-)}{x}\right)dx \\
&\quad + \int_0^{\frac{1}{2}} \sin\left((2\pi(N+\frac{1}{2})x\right)\left(\frac{x}{\sin(\pi x)}\right)\left(\frac{f(t-x)-f(t^+)}{x}\right)dx
\end{aligned}
$$

The remaining integrals will be proven to converge to zero for large N.

Since f admits left and right derivatives everywhere, $\forall \varepsilon > 0$ we may choose $\delta_\varepsilon \in \mathbb{R}^-$ such that $\forall \delta$, $0 > \delta > \delta_\varepsilon$ $|\frac{f(t-\delta)-f(t^-)}{\delta} + f'(t^-)| \leq \varepsilon$. Additionally, $\frac{x}{\sin(\pi x)}$ is well defined in this neighbourhood ($\lim_{x\to 0} \frac{x}{\sin(\pi x)} = \pi^{-1}$). Therefore, δ may be chosen such that;

$$
\left|\int_\delta^0 \sin\left((2\pi(N+\frac{1}{2})x\right)\left(\frac{x}{\sin(\pi x)}\right)\left(\frac{f(t-x)-f(t^-)}{x}\right)\right| < \varepsilon
$$

By the Riemann Lebesgue Lemma;

$$
\lim_{N\to\infty}\left|\int_{-\frac{1}{2}}^\delta \sin\left((2\pi(N+\frac{1}{2})x\right)\left(\frac{x}{\sin(\pi x)}\right)\left(\frac{f(t-x)-f(t^-)}{x}\right)dx\right| < \varepsilon
$$

Therefore;

$$
\lim_{N\to\infty}\int_{-\frac{1}{2}}^0 \sin\left((2\pi(N+\frac{1}{2})x\right)\left(\frac{x}{\sin(\pi x)}\right)\left(\frac{f(t-x)-f(t^-)}{x}\right)dx = 0
$$

The remaining integral may be treated in much the same way. Choose an appropriate δ such that;

$$\left|\int_0^\delta \sin\left((2\pi(N+\frac{1}{2})x\right)\left(\frac{x}{\sin(\pi x)}\right)\left(\frac{f(t-x)-f(t^+)}{x}\right)dx\right| < \varepsilon$$

By the Riemann Lebesgue Lemma;

$$\lim_{N\to\infty}\left|\int_\delta^{\frac{1}{2}} \sin\left((2\pi(N+\frac{1}{2})x\right)\left(\frac{x}{\sin(\pi x)}\right)\left(\frac{f(t-x)-f(t^+)}{x}\right)dx\right| < \varepsilon$$

Therefore;

$$\lim_{N\to\infty}\int_0^{\frac{1}{2}} \sin\left((2\pi(N+\frac{1}{2})x\right)\left(\frac{x}{\sin(\pi x)}\right)\left(\frac{f(t-x)-f(t^+)}{x}\right)dx = 0$$

Thus we conclude;

$$\lim_{N\to\infty} S_N(t) = \frac{f(t^-)+f(t^+)}{2}$$

□

3 Continuous

The Cesaro means of a series are the averages of its partial sums. For a sequence of partial sums S_N, $\forall N \in \mathbb{N}$,

$$\sigma_N = \frac{1}{N}\sum_{n=0}^{N-1} S_n$$

Theorem 3.1. *If f is a continuous signal, then the Cesaro means of f's Fourier Series converge uniformly to f.*

$$\lim_{N\to\infty} \sigma_N(t) \rightrightarrows f(t) \tag{12}$$

Proof. As in the previous proof, we will introduce a family of kernels to aid our analysis. The Fejer family of Kernels is defined as follows; $\forall N \in \mathbb{N}$, $N \geq 1$

$$F_N(t) = \frac{1}{N}\sum_{n=0}^{N-1} D_n(t)$$

Consider the convolution of the Nth Fejer Kernel with f.

$$(F_N * f)(t) = \int_0^1 F_N(x)f(t-x)dx$$
$$(F_N * f)(t) = \frac{1}{N}\int_0^1 \sum_{n=0}^{N-1} D_n(x)f(t-x)dx$$
$$(F_N * f)(t) = \frac{1}{N}\sum_{n=0}^{N-1} \int_0^1 D_n(x)f(t-x)dx$$
$$(F_N * f)(t) = \frac{1}{N}\sum_{n=0}^{N-1} (D_n * f)(t)$$
$$(F_N * f)(t) = \frac{1}{N}\sum_{n=0}^{N-1} S_n(t)$$
$$(F_N * f)(t) = \sigma_N(t)$$

$$\sigma_N(t) = (F_N * f)(t)$$
$$\sigma_N(t) = \int_0^1 F_N(t)f(t-x)dx + f(t)\int_0^1 F_N(t)$$

By Theorem 6.6

$$\sigma_N(t) - f(t) = \int_0^1 F_N(t)(f(t-x) - f(t))dx$$
$$|\sigma_N(t) - f(t)| \leq \int_0^1 |F_N(t)(f(t-x) - f(t))|dx$$

Since f is periodic and continuous, it exhibits uniform continuity over the interval $[0,1]$ (since $[0,1]$ is closed we may simply take the smallest δ neighbourhood required to ensure every point is within ε of its neighbours). Hence we may choose δ such that for all x with $|x| \leq \delta$, $|f(t-x) - f(t)| \leq \varepsilon$. Additionally, the closure of the interval and f's continuity guarantee that f admits a maximum and a minimum, thus we may define $M := |\max_{x\in[0,1]}(f(x)) - \min_{x\in[0,1]}(f(x))|$. M estimates from above the difference between any two points in the image of f.

$$\begin{aligned}|\sigma_N(t) - f(t)| \leq &\int_{-\delta}^{\delta} |F_N(x)(f(t-x) - f(t))|dx \\ &+ \int_{-\frac{1}{2}}^{-\delta} |F_N(x)(f(t-x) - f(t))|dx + \int_{\delta}^{\frac{1}{2}} |F_N(x)(f(t-x) - f(t))|dx\end{aligned}$$

By corollary of Theorem 6.5 $F_N(t) \geq 0$

$$|\sigma_N(t) - f(t)| \leq \varepsilon \int_{-\delta}^{\delta} F_N(x)dx + M \int_{-\frac{1}{2}}^{-\delta} F_N(x)dx + M \int_{\delta}^{\frac{1}{2}} F_N(x)dx$$

By Theorem 6.7, the fist integral tends to one and hence the associated term vanishes. Additionally, the corollaries state that the remaining integrals tend to zero. The independece of our estimate from t proves that the convergence is uniform. $\square$

4 The Fourier Transform and its Derivatives

Having justified the discretised frequency decomposition of signals, we now define a continuous analogue for non periodic signals on the entirety of $\mathbb{R}$. We may no longer integrate over a scaled interval and instead integrate over the reals. The Fourier Transform of a one dimensional signal is as follows (the definition of the n dimensional Fourier transform is given in the appendices.)

$$\mathcal{F}f(s) = \int_{-\infty}^{\infty} f(t)e^{-2i\pi nst}dt \tag{13}$$

In the literature, the transform is described as taking a function from the time domain to the frequency domain. The transform mimics the Fourier coefficients defined for Fourier series, and indeed the transform as defined above reflects the magnitude (and in some sense, direction) of the contribution of a particular frequency to the image of f. The natural domain need not be time, though the connection between frequency and time is likely more apparent than that between frequency and any other semi ordered pre-image.

Note that a signal can be recovered from its Fourier Transform by the Inverse Fourier Transform, a result known as the Fourier Inversion Theorem. It will not be proved here but Walter Rudin's Real and Complex Analysis deals with the matter closely and is provided in reference [6].

4.1 Audio

Audio is typically encoded as sound pressure over a time domain. This means it is typically loosely continuous (insofar as it can be considered indiscrete, this will be addressed) and integrable, without singularities of any kind. It is, however, typically confined to a finite interval. We may extend such a signal to a function over the reals by declaring it to be zero valued outside of the original bounds. The Fourier transform can then be applied to obtain a spectral decomposition of the audio. However, unless we prescribe such a signal with a well behaved function, we are unlikely to be able to describe arbitrary sounds with continuous functions. Instead, audio is encoded discretely, as a set of high frequency samples. Any practical implementation of the Fourier transform then needs to be discretised to fit the domain of real world data.

4.2 The Discrete Fourier Transform

Having motivated a discretised Fourier Transform we proceed to define one. The domain clearly needs to be discrete and finite, so the image should be considered accordingly. If only finitely many frequencies are to be admitted in the image of the transform, then smaller frequencies should be prioritised, since the Riemann Lebesgue Lemma insinuates that larger frequencies have smaller magnitude (and therefore contribute less in some sense). These limitations amount to choices of two of the following three parameters, the length of time to be sampled, the frequency of the samples and the total number of samples. Judicious choice of which variables to control and what level of precision to demand are paramount to maximising the efficacy of the Discrete Fourier Transform.

Consder a list of N uniformly spaced samples over a time frame of length L. Let the sampling rate be $B = \frac{N}{L}$ The sample points are then $t_n = \frac{n}{B}$, and the N frequencies, $s_n = \frac{n}{L}$, $n \in \{0, 1, 2, ..., N-1\}$.

$$\underline{t} = \begin{bmatrix} t_0 \\ t_1 \\ \vdots \\ t_{N-1} \end{bmatrix} \qquad \underline{f}(\underline{t}) = \begin{bmatrix} f(t_0) \\ f(t_1) \\ \vdots \\ f(t_{N-1}) \end{bmatrix}$$

Note that insofar as we may choose the time sample to be non negative (as is conventional), we may (and do) choose the distinct frequencies to be non negative.

The most natural discretisation of an integral is a Riemann sum. The given sampling of times provides a mesh with which to do so. A possible discretisation of the Fourier transform is then;

$$\underline{\mathcal{F}} f(s_m) = \int_0^{N-1} \underline{f}(\underline{t}) e^{-2\pi s_m t_n} dt$$

$$\underline{\mathcal{F}} f(s_m) = \frac{1}{B} \sum_{n=0}^{N-1} \underline{f}(\underline{t_n}) e^{-2\pi s_m t_n}$$

$$\underline{\mathcal{F}} f(s_m) = \frac{1}{B} \sum_{n=0}^{N-1} \underline{f}(\underline{t_n}) e^{-\frac{2\pi mn}{N}}$$

New notation and convention are introduced in the interest of clarity. Since the information communicated by the transform is to some degree relative, i.e. the observations of relevance to be made are those regarding the comparative contributions to an image produced by the various frequencies, the factor of $\frac{1}{B}$ is discarded. Additionally, the exponentials of the discretised transform now take the form of roots of unity (and powers thereof). Let

$$\omega_N = e^{\frac{2\pi}{N}} \tag{14}$$

With these adjustments, the discretised Fourier transform is defined as such;

$$\underline{\mathcal{F}f}(s_m) := \sum_{n=0}^{N-1} \underline{f}(\underline{t_n})\omega_N^{-mn} \tag{15}$$

As defined, the discrete Fourier Transform lends itself rather naturally to matrix form.

$$\begin{bmatrix} \underline{\mathcal{F}f}[s_0] \\ \underline{\mathcal{F}f}[s_1] \\ \vdots \\ \underline{\mathcal{F}f}[s_{N-2}] \\ \underline{\mathcal{F}f}[s_{N-1}] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_N^{-1\cdot 1} & \omega_N^{-1\cdot 2} & \cdots & \omega_N^{-1\cdot(N-1)} \\ 1 & \omega_N^{-2\cdot 1} & \omega_N^{-2\cdot 2} & \cdots & \omega_N^{-2\cdot(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_N^{-(N-1)\cdot 1} & \omega_N^{-(N-1)\cdot 2} & \ldots & \omega_N^{(N-1)^2} \end{bmatrix} \begin{bmatrix} \underline{f}[t_0] \\ \underline{f}[t_1] \\ \vdots \\ \underline{f}[t_{N-2}] \\ \underline{f}[t_{N-1}] \end{bmatrix} \tag{16}$$

The properties of the matrix, in particular its eigenvectors, are not presently especially well understood, and there is a small existing body of literature detailing what little is known. For virtue of admitting an $N \times N$ matrix form however, it is clear that naive implementations of the Fourier Transform have $O(N^2)$ complexity.

Additionally:

Linearity of the Fourier Transform as an operator is immediately apparent

Invertibility follows from the injectivity of the matrix.

4.3 Fast Fourier Transforms

The Discrete Fourier Transform is itself to a degree an approximation, thus introducing error by approximating it is justifiable if we are able to significantly improve the asymptotic complexity of implementation and the error is manageable and well understood. In general these algorithms are able to obtain $O(N\log(N))$ complexity through the use of divide and conquer techniques (typically the algorithm is called recursively on half of its input).

4.4 Applications in Audio Analysis

We now motivate the application of the Fast Fourier Transform and other Disctre Fourier Transform derivatives to audio data. Previously it was established that Fourier Analysis of audio naturally decomposed sounds into their constant sounds (cords) into their consitutent frequencies. We can exploit this by finely partitioning a signal and performing Fast Fourier Transforms on each interval.

4.4.1 Filtering

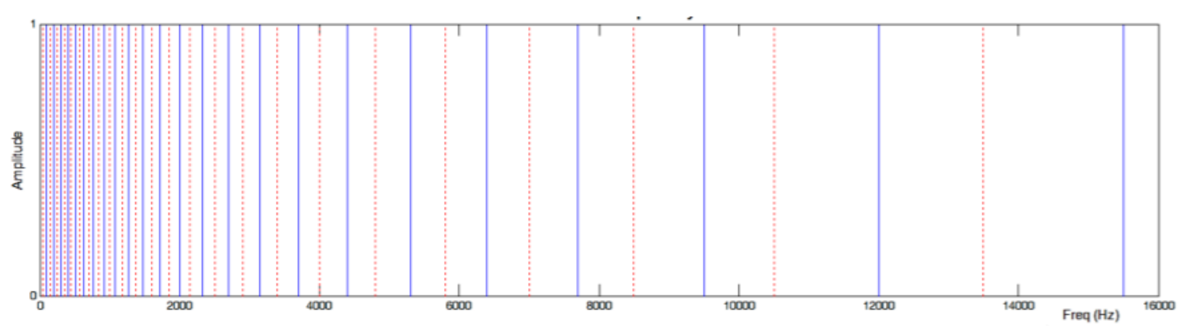
Suppose some audio is blemished by the presence of an undesired frequency. Such perturbations can arise from data corruption in transfer and processing and are difficult to correct in the time domain. The presence of multiple frequencies at any given time prevent simple excision of the whine. In the frequency domain however, a constant errant sound manifests itself as a peak. The peak is isolated and therefore especially easy to remove if it is particularly discordant. In essence, the greater the magnitude of the problem, the easier it is to correct. Having removed any errors, or indeed any distinct element of the frequency domain deemed unsavoury, the invertibility and linearity of the Discrete Fourier Transformation yield a new signal free of the unwanted elements.

4.4.2 Compression

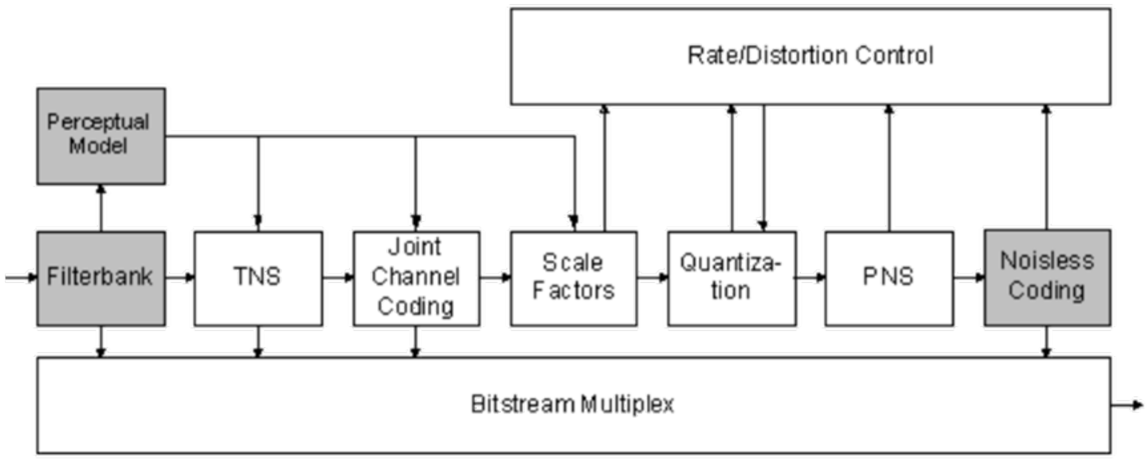
The MP3 (MPEG-1 Audio Layer 3) is an algorithmic protocol for the compression of audio data. It employs the techniques above in conjunction with established results from the field of psychoacoustics and entropy encoding to reduce the size of data. The principal strategy is to systematically identify and remove information which is functionally inperceivable.

As in the filtering process, the Fast Fourier Transform is performed on many disjoint fine intervals which cover the domain. Then all frequencies which lie outside the range of human hearing (20Hz - 20kHz, [5]), are filtered.

However, more sophisticated results from the field of psychoacoustics can be used to justify the deletion of more information still.



In particular, the MP3 algorithm relies on the purported existence of 24 critical bands of frequencies [1], within which, only a single dominant frequency can be heard in any given window of approximately half a second. Using this technique approximately 90% of data can be deleted with minimal impact on perceived quality. A diagrammatic outline of the procedure is provided by MPEG [3].



The perceptual model and quantisation stages are those discussed above, the remainder are technical optimisations pertaining to the MP3 data structure/s.

5 Discussion

The theoretical results proved in this paper justify the extensive use of Fourier Analysis in the world today. That integrable periodic functions admit such accurate trigonometric approximations is in and of itself remarkable, but is of inordinate practical importance to both the modelling and analysis of physical phenomena. Joseph Fourier first developed such methods for the purpose of resolving differential equations describing heat diffusion, and Fourier Analysis (particularly the use of Fourier Series) remains an effective means to solve differential equations and associated eigenfunction problems. These methods, however, have long since been used in other fields, such as signal processing for communications and both the compression and correction of data (as outlined in the context of audio in this report).

The results proved in this report however are limited, insofar as they only accurately describe approximations and transformations of single variable functions. Phenomena in the world at large are often dependent on many variables. Higher dimensional Fourier Series and Transformations are promising areas of further study of great practical importance. Moreover, the scale of existing implementations of the Discrete Fourier Transform means that even slight improvements to its performance (or even special cases thereof in wide use) would translate to significant improvements in efficiency and accuracy. As mentioned in 4.2 the algebraic properties of the Discrete Fourier Transform are not particularly well understood and little literature exists on the matter, new insights could possibly lead to improvements of existing Fast Fourier Transform algorithms.

6 Appendices

6.1 The n dimensional Fourier Series

[4] Suppose f is an n dimensional signal on the unit n-cube, $C^n = [0,1]^n$. Then its Fourier coefficients are, $\forall \bar{n} \in \mathbb{N}^n$

$$\hat{f}(\bar{n}) = \int_{C^n} f(\bar{t}) e^{-2\pi i \bar{n}\cdot\bar{t}} d\bar{t}. \tag{17}$$

The Fourier Series of f is;

$$f(\bar{t}) \sim \sum_{\bar{n}\in\mathbb{N}^n} \hat{f}(\bar{n}) e^{2i\pi\bar{n}\cdot\bar{t}} \tag{18}$$

6.2 The Riemann Lebesgue Lemma in n dimensions

[2]

Theorem 6.1. $||\bar{n}|| \in \mathbb{N}^n$

$$\lim_{\bar{n}\to\infty} \hat{f}(\bar{n}) = 0 \tag{19}$$

6.3 The Dirichlet Kernel

Theorem 6.2.

$$D_N(t) = \frac{\sin\left((2\pi(N+\frac{1}{2})t\right)}{\sin(\pi t)} \tag{20}$$

Proof. Trivially this is true for $N = 0$. Suppose it's true for some $N \in \mathbb{N}$

$$\begin{aligned}
D_N(t) &= \frac{\sin\left((2\pi(N+\frac{1}{2})t\right)}{\sin(\pi t)} \\
D_{N+1}(t) &= \frac{\sin\left((2\pi(N+\frac{1}{2})t\right)}{\sin(\pi t)} + e^{2i\pi(N+1)t} + e^{-2i\pi(N+1)t} \\
D_{N+1}(t) &= \frac{\sin\left((2\pi(N+\frac{1}{2})t\right)}{\sin(\pi t)} + 2\cos(2\pi(N+1)t) \\
D_{N+1}(t) &= \frac{\sin\left((2\pi(N+1-\frac{1}{2})t\right)}{\sin(\pi t)} + 2\frac{\cos(2\pi(N+1)t)\sin(\pi t)}{\sin(\pi t)} \\
D_{N+1}(t) &= \frac{\sin\left((2\pi(N+1)t\right)\cos(\pi t)}{\sin(\pi t)} - \frac{\cos\left((2\pi(N+1)t\right)\sin(\pi t)}{\sin(\pi t)} + 2\frac{\cos(2\pi(N+1)t)\sin(\pi t)}{\sin(\pi t)} \\
D_{N+1}(t) &= \frac{\sin\left((2\pi(N+1)t\right)\cos(\pi t)}{\sin(\pi t)} + \frac{\cos(2\pi(N+1)t)\sin(\pi t)}{\sin(\pi t)} \\
D_{N+1}(t) &= \frac{\sin\left((2\pi(N+1+\frac{1}{2})t\right)}{\sin(\pi t)}
\end{aligned}$$

The result is proved true by induction. $\square$

Theorem 6.3.

$$\int_0^1 D_N(t)dt = 1 \tag{21}$$

Proof. We will use the series form of the kernel to prove the result

$$\int_0^1 D_N(t)dt = \int_0^1 \sum_{n=-N}^{N} e^{2i\pi nt}dt$$
$$\int_0^1 D_N(t)dt = \int_0^1 dt + \int_0^1 \sum_{n=1}^{N}(e^{2i\pi nt} + e^{-2i\pi nt})dt$$
$$\int_0^1 D_N(t)dt = 1 + \sum_{n=1}^{N} \frac{1}{2i\pi n}\Big|e^{2i\pi nt} - e^{-2i\pi nt}\Big|_0^1$$
$$\int_0^1 D_N(t)dt = 1$$

□

Theorem 6.4.

$$\int_0^{\frac{1}{2}} D_N(t)dt = \frac{1}{2} \tag{22}$$

Proof. Again we will exploit the series representation to prove the result.

$$\int_0^{\frac{1}{2}} D_N(t)dt = \int_0^{\frac{1}{2}} \sum_{n=-N}^{N} e^{2i\pi nt}dt$$
$$\int_0^{\frac{1}{2}} D_N(t)dt = \int_0^{\frac{1}{2}} dt + \int_0^{\frac{1}{2}} \sum_{n=1}^{N}(e^{2i\pi nt} + e^{-2i\pi nt})dt$$
$$\int_0^{\frac{1}{2}} D_N(t)dt = \frac{1}{2} + \sum_{n=1}^{N} \frac{1}{2i\pi n}\Big|e^{2i\pi nt} - e^{-2i\pi nt}\Big|_0^{\frac{1}{2}}$$
$$\int_0^{\frac{1}{2}} D_N(t)dt = \frac{1}{2} + \sum_{n=1}^{N} \frac{1}{2i\pi n}(e^{i\pi n} - e^{-i\pi n})$$
$$\int_0^{\frac{1}{2}} D_N(t)dt = \frac{1}{2}$$

With the final equality owing to n taking integer values. As a simple corollary;

$$\int_{-\frac{1}{2}}^{0} D_N(t)dt = \int_{-\frac{1}{2}}^{0} \sum_{n=-N}^{N} e^{2i\pi nt}dt$$
$$\int_{-\frac{1}{2}}^{0} D_N(t)dt = -\int_{\frac{1}{2}}^{0} \sum_{n=-N}^{N} e^{-2i\pi nt}dt$$
$$\int_{-\frac{1}{2}}^{0} D_N(t)dt = \int_{0}^{\frac{1}{2}} \sum_{n=-N}^{N} e^{2i\pi nt}dt$$
$$\int_{-\frac{1}{2}}^{0} D_N(t)dt = \frac{1}{2}$$

□

6.4 The Fejer Kernel

Theorem 6.5.

$$F_N(t) = \frac{1}{N}\left(\frac{1-\cos(2\pi Nt)}{1-\cos(2\pi t)}\right) \tag{23}$$

Proof. We will prove the result by induction. $F_1(t) = D_0(t) = 1$. Having proved it true for the base case we now suppose it to be true for some N

$$F_N(t) = \frac{1}{N}\left(\frac{1-\cos(2\pi Nt)}{1-\cos(2\pi t)}\right)$$
$$F_{N+1}(t) = \frac{1}{N+1}\left(\frac{1-\cos(2\pi Nt)}{1-\cos(2\pi t)}\right) + \frac{D_N}{N+1}$$
$$F_{N+1}(t) = \frac{1}{N+1}\left(\frac{1-\cos(2\pi Nt)}{1-\cos(2\pi t)} + \frac{\sin(2\pi(N+\frac{1}{2})t)}{\sin(\pi t)}\right)$$
$$F_{N+1}(t) = \frac{1}{N+1}\left(\frac{1-\cos(2\pi Nt)}{1-\cos(2\pi t)} + \frac{\sin(2\pi(N+\frac{1}{2})t)\sin(\pi t)}{\sin^2(\pi t)}\right)$$
$$F_{N+1}(t) = \frac{1}{N+1}\left(\frac{1-\cos(2\pi Nt)}{1-\cos(2\pi t)} + \frac{2\sin(2\pi(N+\frac{1}{2})t)\sin(\pi t)}{1-\cos(2\pi t)}\right)$$
$$F_{N+1}(t) = \frac{1}{N+1}\left(\frac{1-\cos(2\pi Nt)}{1-\cos(2\pi t)} + \frac{\cos(2\pi Nt)-\cos(2\pi(N+1)t)}{1-\cos(2\pi t)}\right)$$
$$F_{N+1}(t) = \frac{1}{N+1}\left(\frac{1-\cos(2\pi(N+1)t)}{1-\cos(2\pi t)}\right)$$

The result is proved true by induction.

As a simple corollary;

$$F_N(t) = \frac{1}{N}\left(\frac{\sin^2(\pi Nt)}{\sin^2(\pi t)}\right) \tag{24}$$

This implies the Fejer Kernel is non negative and even. □

Theorem 6.6.

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} F_N(t)dt = 1 \tag{25}$$

Proof.

$$\begin{aligned}
\int_{-\frac{1}{2}}^{\frac{1}{2}} F_N(t)dt &= \frac{1}{N}\int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{n=0}^{N-1} D_n(t)dt \\
\int_{-\frac{1}{2}}^{\frac{1}{2}} F_N(t)dt &= \frac{1}{N}\sum_{n=0}^{N-1} \int_{-\frac{1}{2}}^{\frac{1}{2}} D_n(t)dt \\
\int_{-\frac{1}{2}}^{\frac{1}{2}} F_N(t)dt &= \frac{1}{N}\sum_{n=0}^{N-1} 1 \\
\int_{-\frac{1}{2}}^{\frac{1}{2}} F_N(t)dt &= 1
\end{aligned}$$

□

Theorem 6.7. $\forall \delta \in (0, \frac{1}{2}]$,

$$\lim_{N\to\infty} \int_{\delta}^{\delta} F_N(t)dt = 1$$

Proof.

$$\begin{aligned}
\int_{-\frac{1}{2}}^{\frac{1}{2}} F_N(t)dt &= 1 \\
\int_{-\frac{1}{2}}^{-\delta} F_N(t)dt + \int_{\delta}^{\frac{1}{2}} F_N(t)dt &= 1 - \int_{-\delta}^{\delta} F_N(t)dt \\
2\left|\int_{\delta}^{\frac{1}{2}} F_N(t)dt\right| &= \left|1 - \int_{-\delta}^{\delta} F_N(t)dt\right|
\end{aligned}$$

We now invoke the closed form of the Fejer Kernel

$$\begin{aligned}
\left|1 - \int_{-\delta}^{\delta} F_N(t)dt\right| &= \frac{2}{N}\left|\int_{\delta}^{\frac{1}{2}} \frac{\sin^2(\pi N t)}{\sin^2(\pi t)}dt\right| \\
\left|1 - \int_{-\delta}^{\delta} F_N(t)dt\right| &\leq \frac{2}{N}\int_{\delta}^{\frac{1}{2}} \left|\frac{\sin^2(\pi N t)}{\sin^2(2\pi t)}\right| dt \\
\left|1 - \int_{-\delta}^{\delta} F_N(t)dt\right| &\leq \frac{2}{N\sin^2(\pi t)}\int_{\delta}^{\frac{1}{2}} \sin^2(\pi N \delta)dt \\
\left|1 - \int_{-\delta}^{\delta} F_N(t)dt\right| &\leq \frac{1}{N\sin^2(\pi \delta)}
\end{aligned}$$

□

Additionally, we have proved;

$$\lim_{N\to\infty}\int_{-\frac{1}{2}}^{-\delta} F_N(t)dt = \lim_{N\to\infty}\int_{\delta}^{\frac{1}{2}} F_N(t)dt = 0 \tag{26}$$

since

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} F_N(t)dt = 1$$

6.5 Periodic Functions

Theorem 6.8. *The derivatives of differentiable periodic functions are also periodic*

Proof. Suppose f is such a function with period t.

$$f'(x+t) = \lim_{h\to 0}\frac{f(x+t)-f(x+t-h)}{h}$$
$$f'(x+t) = \lim_{h\to 0}\frac{f(x)-f(x-h)}{h}$$
$$f'(x+t) = f'(x)$$

□

6.6 The n dimensional Fourier Transform

[4] Suppose f is an n dimensional signal on the unit n-cube, $C^n = [0,1]^n$. Then its Fourier Transform is;

$$Ff(\bar{s}) = \int_{\mathbb{R}^n} f(t)e^{-2i\pi\bar{s}\cdot\bar{t}}d\bar{t}$$

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