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BIOMECHANICS OF OPTIMAL FLIGHT IN SKI-JUMPING

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Abstract—The flight in a vertical plane of a ski-jumper after take-off was studied with the purpose of maximising flight distance. To solve the problem of optimal flight (how a jumper must change his angle of attack to obtain the longest jump) the basic theorem of the optimal control theory—Pontriagin's maximum principle—was applied. The calculations were based on data from wind tunnel experiments. It was shown that the maximum flight distance is achieved when the angle of attack is gradually increased according to a convex function the form of which depends on the individual aerodynamic parameters.

NOMENCLATURE

F_D, F_L	drag and lift forces
C_D, C_L	undimensioned drag and lift coefficients
A	frontal surface area of jumper at $\alpha = 90$
ρ	density of air
g	gravitational constant ($= 9.81\ ms^{-2}$)
m	mass of the jumper-ski system
x (or ξ), y (or η)	horizontal and vertical coordinates
l, L	current and final lengths of the jump
$R(\xi, \eta)$	equation defining the shape of the landing slope
t', T'	current and final times
t, T	dimensionless current and final times
V'	velocity of C.G.
V	undimensioned velocity of C.G.
V_n	velocity component perpendicular to the ramp (at the take-off instant)
α, θ, β	angles of attack, of path, of landing slope
α^0	optimal angle of attack
α^*	angle of attack supplying the nought to the Hamiltonian derivative
p_v, p_θ, p_x, p_y	impulses of the conjugated system of equations
$a, b, d, a', b', d', D, B, S$	constant coefficients of equations
H	Hamiltonian

Subscript '0' used as an initial index (take-off from the edge of the ramp). Partial derivatives are denoted by ∂.

INTRODUCTION

The flight length in ski-jumping depends not only on the tangential and normal velocity components (due, respectively, to runway speed and push-off thrust) but is substantially influenced by the skier's posture in the air and by the change of this orientation to the flow of air (angle of attack) during the flight (Komi *et al.*, 1974; Tani and Iuchi, 1971). By varying the departure position at the instant of take-off and by regulating the relative locations of the legs, arms and trunk the athlete can control his flight path in the air by changing his angle of attack. The problem is formulated in the following way: how must the skier control his body in flight during 2.5–4.5 s in order to land as far down the hill as possible?

BASIC EQUATIONS

The results of many film analyses (Komi *et al.*, 1974) have ascertained a relatively static flying position of each jumper. This facilitates describing the displacement and velocity patterns of the jumper-ski system and allows the use of individual experimental wind-tunnel characteristics. Because of this, the posture of the skier was assumed to be fixed during flight.

If there are no side wind and other disturbing effects, the center of gravity (C.G.) of the skier describes a curve within a vertical plane. Let us define x- and y-axis in this plane with origin at the center of gravity of the jumper-ski system at the instant of take-off, the x-axis being in the horizontal direction forward and the y-axis in the vertical direction upward as shown in Fig. 1. If we denote the velocity of the C. G. by V' and the direction of the flight path by θ, then, as is known (Tani and Iuchi, 1971), the equations of displacements in the directions parallel and normal to the flight path are

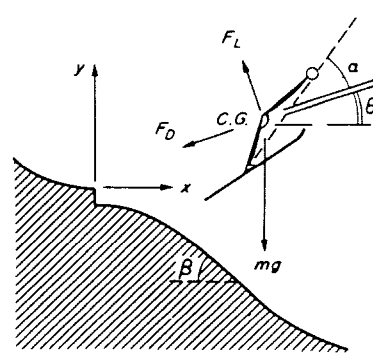

Fig. 1. Profile contour of a jumping hill and the corresponding coordinate system. Forces affecting the skier in the air, flight angles.

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expressed by

$$m\frac{\mathrm{d}V'}{\mathrm{d}t'} = -F_D - mg\sin\theta$$

$$mV'\frac{\mathrm{d}\theta}{\mathrm{d}t'} = F_L - mg\cos\theta \qquad (1)$$

$$\frac{\mathrm{d}x}{\mathrm{d}t'} = V'\cos\theta, \quad \frac{\mathrm{d}y}{\mathrm{d}t'} = V'\sin\theta$$

respectively, in which t' is the time after the instant of take-off; x and y are the coordinates of the C.G.; m is the mass of the system; g is the gravitational acceleration; F_D and F_L are the drag and lift, that is, the components of the aerodynamic force acting in the direction opposite to and in the upward direction normal to the flight path, respectively.

The forces F_D and F_L can be described in the forms

$$F_D = \tfrac{1}{2}\rho V^2 A C_D(\alpha), \quad F_L = \tfrac{1}{2}\rho V^2 A C_L(\alpha) \qquad (2)$$

where ρ is the density of air; A the conventional frontal surface area (perpendicular to the velocity); C_D and C_L the drag and lift coefficients, respectively, which depends on α, the angle of attack. The coefficients C_D and C_L were determined by wind tunnel experiments.

The angle of attack was defined as the angle between the longitudinal axis of the body (passing through the head and feet) and the direction of the flight path (Fig. 1). Then from the known fixed posture of the jumper the angle of attack of the skis could be easily derived.

STATEMENT AND SOLUTION OF THE PROBLEM

The problem of optimal control consists in determining the control $\alpha(t')$ which could lead to the achievement of a maximal jump length measured according to the competition rules from the edge of the take-off ramp (point O). The longest jump is equivalent to the achievement of the maximum value of abscissa $x(T')$ at the instant of landing T'. The initial conditions of the flight are

$$t' = 0, \quad x = y = 0, \quad V' = V_0', \quad \theta = \theta_0.$$

A problem formulated like that is the variational problem with a free right bound and time unfixed (Pontriagin *et al.*, 1961) which requires maximising the function

$$J = x(T'). \qquad (3)$$

It is worth transferring to undimensioned variables by introducing V and t for velocity and time, respectively, where $V = V'/V_0'$ and $t = t'/(V_0'/g)$. The substitution of these values into /1/ generates the following equations

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -C_D\sigma V^2 - \sin\theta$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = C_L\sigma V - \frac{\cos\theta}{V}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = V\frac{V_0^2}{g}\cos\theta \qquad (4)$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = V\frac{V_0^2}{g}\sin\theta$$

where $A\rho V_0^2/2mg$ is denoted by σ.

Now the initial conditions are

$$t = 0, \quad x = y = 0, \quad V = 1, \quad \theta = \theta_0. \qquad (5)$$

According to Pontriagin's maximum principle (Pontriagin *et al.*, 1961) let us consider the Hamiltonian of this problem

$$H = \sum p_k f_k.$$

Here $f_k(k = 1, 4)$ are the right parts of the flight equations and p_k are so-called impulses which must satisfy the system of equations canonically conjugated to the system (4)

$$\frac{\mathrm{d}p_x}{\mathrm{d}t} = -\frac{\partial H}{\partial x} \qquad \frac{\mathrm{d}p_y}{\mathrm{d}t} = -\frac{\partial H}{\partial y}$$

$$\frac{\mathrm{d}p_v}{\mathrm{d}t} = -\frac{\partial H}{\partial V} \qquad \frac{\mathrm{d}p_\theta}{\mathrm{d}t} = -\frac{\partial H}{\partial \theta}.$$

We have

$$H = p_x V\frac{V_0^2}{g}\cos\theta + p_y V\frac{V_0^2}{g}\sin\theta - p_v(C_D\sigma V^2 + \sin\theta) + p_\theta\left(C_L\sigma V - \frac{\cos\theta}{V}\right). \qquad (6)$$

Since the Hamilton function obviously does not depend on x and y, it can be shown according to the computing algorithm (Krylov and Tichernousko, 1972) that the impulses p_x and p_y do not depend on the time and are equal to its terminal values

$$p_x = \cos\beta, \quad p_y = -\cos\beta\,\mathrm{ctg}\,\theta(T) \qquad (7)$$

(for landing in the middle of a slope tilted at an angle β with respect to the horizontal—see Fig. 1).

Thus, the conjugate system for impulses includes only two equations

$$\begin{aligned}\frac{\mathrm{d}p_v}{\mathrm{d}t} &= -p_x\frac{V_0^2}{g}\cos\theta - p_y\frac{V_0^2}{g}\sin\theta + 2p_vC_D\sigma V - p_\theta\left(C_L\sigma + \frac{\cos\theta}{V^2}\right)\\ \frac{\mathrm{d}p_\theta}{\mathrm{d}t} &= p_x V\frac{V_0^2}{g}\sin\theta - p_y V\frac{V_0^2}{g}\cos\theta + p_v\cos\theta - p_\theta\frac{\sin\theta}{V}\end{aligned} \qquad (8)$$

with the boundary conditions

$$t = T, \quad p_v = 0, \quad p_\theta = 0. \qquad (9)$$

Now we have a closed system of equations with corresponding boundary conditions. This Cauchy problem has only one solution (Pontriagin, 1965).

According to the maximum principle the desired optimal control $\alpha^0(t)$ supplies the maximum to

Hamiltonian H for each t within the range $(0, T)$ provided V, θ, p_v, p_θ are solutions of systems (4), (7) and (8) with the given initial conditions (5), (9).

INITIAL PARAMETERS FOR THE COMPUTATION

Knowing the values of C_D and C_L as functions of the posture used, equations (1) and (2) can be integrated numerically to obtain the flight path with a constant angle of attack (steady flight), i.e. to get the dependences of V and θ on t. Then the coordinates can be calculated as follows

$$x = \int_0^t V\cos\theta\,dt, \qquad y = \int_0^t V\sin\theta\,dt.$$

But a flight with a fixed angle of attack (α = constant) is not realistic (Tani and Iuchi, 1971). The numerical characteristic curves $C_D(\alpha)$ and $C_L(\alpha)$ were calculated from the data described by Grozin (1971) following wind tunnel experiments for a variety of postures (Fig. 2 shows an example of Grozin's data). C_L is plotted against C_D, with α as a parameter. In this example, the angle between body and skis was 20–23°. For calculations, the relationships $C_D(\alpha)$ and $C_L(\alpha)$ were presented in convenient quadratic forms as

$$\begin{aligned} C_D &= a\alpha^2 + b\alpha + d \\ C_L &= a'\alpha^2 + b'\alpha + d' \end{aligned} \tag{10}$$

with constant coefficients a, b, d, a', b', d'.

The following values were used for calculations.

Air density was assumed to be $\rho = 1.22\ \mathrm{kg\,m^{-3}}$. The typical characteristic skier ratio of surface area (for the initial position at $\alpha = 90°$) to mass $A/m = 0.01\ \mathrm{m^2\,kg^{-1}}$ was chosen.

The profile contour of the landing slope was numerically described, following a diagram showing the design features of the largest jumping hill in Planica (Yugoslavia), by the equation $R(\xi, \eta) = 0$, where ξ and η are, respectively, the x and y coordinates of the landing curve of the slope.

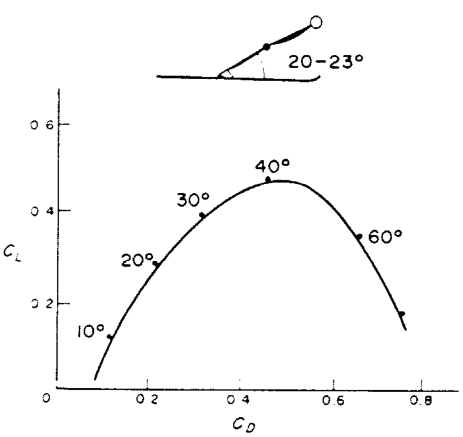

Fig. 2. C_L/C_D graph with α as a parameter, for 20–23° angle between body and skis (Grozin, 1971).

RESULTS AND DISCUSSION

In the problem of optimal control of the flight of a ski-jumper described above the designed systems of equations were numerically solved with the aid of computers using the method of consecutive approach (Krylov *et al.*, 1972). On each computing step of the procedure the unknown control parameter, i.e. the angle of attack α_0 was determined by maximising the Hamiltonian. This requires comparing its value at the point α^* suppling the nought of the Hamiltonian partial derivative

$$\alpha^* = \frac{Dp_v V - Bp_\theta}{Sp_\theta},$$

(where D, B, S are constants depending on the parameters a, b, a', b' of (10)) with the values at the boundary points of the available range for the angle of attack α_{min} and α_{max}.

The completion of the computing procedure was determined by the instant of landing of the skier T, at the intersection of the flight path $(x(t), y(t))$ with the profile contour of the landing hill $R(\xi, \eta) = 0$ within the given accuracy range $R(x(T), y(T)) \simeq 0$.

Figure 3 shows the pattern of optimal change in the angle of attack affording the maximal flight length of 174 m with initial speed $31.9\ \mathrm{m\,s^{-1}}$. The initial data for the calculations were taken from the record jump by Volf in 1969, who actually flew 165 m. For the sake of clarity the vertical coordinates of the jumping hill and flight path were stretched by a factor of two in the diagram.

The computations in the same conditions were also

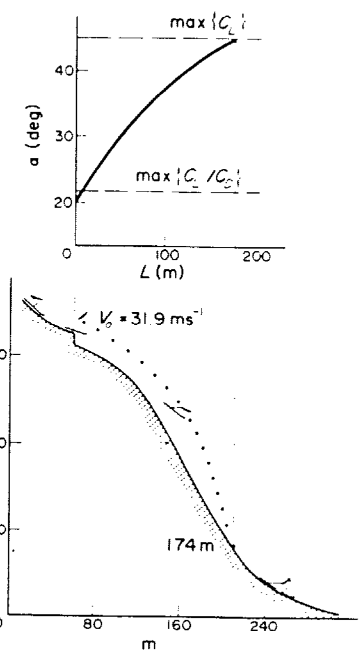

Fig. 3. Optimal flight path on the Planica jumping hill and corresponding optimal angle of attack.

made for three different values of the speed of departure $V_0 = 20, 30, 40\ \mathrm{m\,s^{-1}}$ but identical take-off angle (initial angle of the flight) $\theta_0 = -0.0902$ rad (-5.17°). Four corresponding optimal control curves $\alpha^\circ(x)$ which produce flight lengths of 39 m (for $V_0 = 20\ \mathrm{ms^{-1}}$), 113 m ($30\ \mathrm{ms^{-1}}$), 174 m ($31.9\ \mathrm{ms^{-1}}$) and 245 m ($40\ \mathrm{ms^{-1}}$) are shown in Fig. 4. In order to facilitate comparisons the abscissae of all four plots were expressed as a dimensionless ratio l/L (the ratio of the current flight distance l to the final one L).

Unlike the discrete on-off character of control in the problem of optimal downhill skiing (Remizov, 1980) the optimal control curve of a ski jump is a continuous, convex, increasing function.

It is to be noted that up to now it has been assumed by practitioners that a greater length is achieved when a constant angle of attack corresponding approximately to the maximal aerodynamic quality (maximal ratio C_L/C_D) is maintained (Grozin, 1971; Komi *et al.*, 1974; Tani and Iuchi 1971). According to our calculations this would result, for an initial speed of $30\ \mathrm{ms^{-1}}$, in a loss of 13 m in comparison with the optimal trajectory. The flight distance of the same jumper taking off with similar initial conditions and keeping a constant angle of attack for the maximal lift coefficient would result in a loss of 10 m in comparison with optimal flight path.

Obviously the jumper can alter his angle of attack, which is determined by the angular momentum of the jumper at take-off, changes in that angular momentum by the torque applied about the C.G. by the aerodynamic forces during flight (center of pressure of air $\neq$ C.G.), changes in the moment of inertia of the jumper, and action and reaction effects associated with rotation of different parts of the body, caused by muscular activity. The purpose of this paper, however, is not to determine the causal factors that alter the angle of attack, but to seek the optimal pattern of the latter.

The results discussed here make it possible to formulate some conclusions which would be useful for ski-jumping practice.

(1) During the early flight the skier should take a position with a small angle of attack (more leaning forward). This will reduce drag. In the second half of the optimal flight the angle of attack is to be gradually increased up to the angle of maximal value of the lift coefficient in order to achieve the longest jump. That is, in the later stages of the flight the gliding (lifting) property is more important than the drag effect.

(2) For take-off velocities of $20\text{–}25\ \mathrm{ms^{-1}}$, typical of competitions in the smaller jumping hills, the drag factor affects the flying distance much less than in the case of the larger hills (initial velocities of $27\text{–}33\ \mathrm{ms^{-1}}$). That is why the optimal flights on the medium and, all the more so on smaller jumping hills, are realised with fairly large angles of attack ($\alpha = 30^\circ$) in the early flight, while the best results on large jumping hills are attained by using a more leaning forward posture (a smaller angle of attack, $15\text{–}23^\circ$) in the beginning of the flight.

It is possible to evaluate how various initial factors affect the optimal flight length by the computation of numerous flight paths. These results are presented in Fig. 5. We consider the influence of the initial speed V_0, the parametrical ratio of skier A/m, and the take-off velocity component perpendicular to the take-off ramp V_h. One finds:

(1) The dependence of the optimal flight distance upon the initial speed is shown by the solid line (here A/m and V_h are constant and equal, respectively, to $0.01\ \mathrm{m^2\,kg^{-1}}$ and $2.8\ \mathrm{ms^{-1}}$).

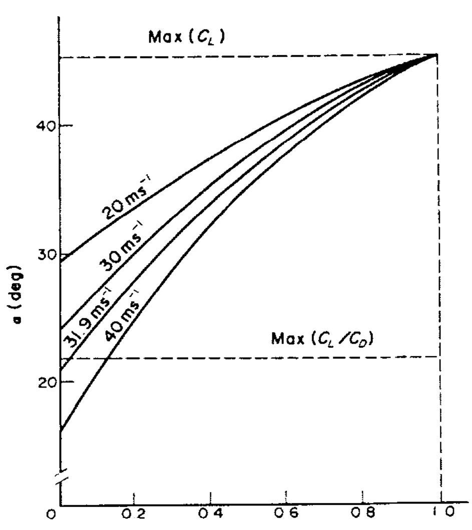

Fig. 4. Optimal angle of attack during the flight for different initial speeds (plotted against the relative current length of flight).

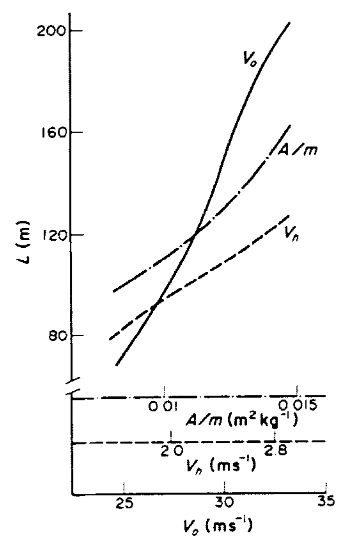

Fig. 5. Dependence of the length of optimal flight on the speed of departure V_0 (solid line); on the perpendicular component of take-off velocity V_h (dashed line) and on the parametrical ratio A/m (dotted line) for the Planica jumping hill.

(2) The same dependence on the perpendicular take-off component V_k is shown by the dashed line ($V_0 = 31.9\ \mathrm{ms}^{-1}$, $A/m = 0.01\ \mathrm{m}^2\,\mathrm{kg}^{-1}$ were held constant).

(3) The same dependence on the parametrical ratio of the conventional frontal area of the skier to his mass A/m is shown by the dotted line ($V_0 = 30\ \mathrm{ms}^{-1}$, $V_k = 2.8\ \mathrm{ms}^{-1}$ were held constant). Note that each curve has its own abscissa axis.

The graph demonstrates a substantional influence of the jumper's dimension on the result. From the sports viewpoint, the ideal ski jumper should have a flattened body with a large frontal area (sailing property) and small weight. Such body proportions provide less drag in the early flight and more lift near the end. For practical purposes any modification of the posture and clothes of the jumper to increase the sailing surface area would bring a marked improvement in results.

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