Mathematical and Logical Foundations of Computer Science – Rules

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1 Propositional Logic: Proof Rules

1.1 Constructive Natural Deduction

$$\frac{\bot}{A}\ [\bot E] \qquad \frac{}{\top}\ [\top I] \qquad \frac{\begin{array}{c}\overline{A}^{\,1}\\ \vdots\\ B\end{array}}{A \to B}\ 1\ [\to I] \qquad \frac{A \to B \quad A}{B}\ [\to E]$$

$$\frac{\begin{array}{c}\overline{A}^{\,1}\\ \vdots\\ \bot\end{array}}{\neg A}\ 1\ [\neg I] \qquad \frac{\neg A \quad A}{\bot}\ [\neg E]$$

$$\frac{A}{A \vee B}\ [\vee I_L] \qquad \frac{A}{B \vee A}\ [\vee I_R] \qquad \frac{A \vee B \quad A \to C \quad B \to C}{C}\ [\vee E]$$

$$\frac{A \quad B}{A \wedge B}\ [\wedge I] \qquad \frac{A \wedge B}{B}\ [\wedge E_R] \qquad \frac{A \wedge B}{A}\ [\wedge E_L]$$

1.2 Constructive Sequent Calculus (Derived Rules)

$$\frac{\Gamma_1, \Gamma_2 \vdash A \quad \Gamma_1, B, \Gamma_2 \vdash C}{\Gamma_1, A \to B, \Gamma_2 \vdash C}\ [\to L] \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ [\to R]$$

$$\frac{\Gamma_1, \Gamma_2 \vdash A}{\Gamma_1, \neg A, \Gamma_2 \vdash B}\ [\neg L] \qquad \frac{\Gamma, A \vdash \bot}{\Gamma \vdash \neg A}\ [\neg R]$$

$$\frac{\Gamma_1, A, \Gamma_2 \vdash C \quad \Gamma_1, B, \Gamma_2 \vdash C}{\Gamma_1, A \vee B, \Gamma_2 \vdash C}\ [\vee L] \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ [\vee R_1] \qquad \frac{\Gamma \vdash A}{\Gamma \vdash B \vee A}\ [\vee R_2]$$

$$\frac{\Gamma_1, A, B, \Gamma_2 \vdash C}{\Gamma_1, A \wedge B, \Gamma_2 \vdash C}\ [\wedge L] \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ [\wedge R]$$

$$\frac{}{\Gamma_1, A, \Gamma_2 \vdash A}\ [Id] \qquad \frac{\Gamma \vdash B \quad B, \Gamma \vdash A}{\Gamma \vdash A}\ [Cut]$$

$$\frac{\Gamma_1, B, A, \Gamma_2 \vdash C}{\Gamma_1, A, B, \Gamma_2 \vdash C}\ [X] \qquad \frac{\Gamma_1, \Gamma_2 \vdash B}{\Gamma_1, A, \Gamma_2 \vdash B}\ [W] \qquad \frac{\Gamma_1, A, A, \Gamma_2 \vdash B}{\Gamma_1, A, \Gamma_2 \vdash B}\ [C]$$

1.3 Classical Natural Deduction

It includes all the Constructive Natural Deduction rules, plus:

$$\frac{}{A \vee \neg A}\ [LEM] \qquad \frac{\neg\neg A}{A}\ [DNE]$$

1.4 Classical Sequent Calculus – 1st version

It includes all the Constructive Sequent Calculus rules, plus:

$$\frac{}{\Gamma \vdash A \vee \neg A}\ [LEM] \qquad \frac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A}\ [DNE]$$

1.5 Classical Sequent Calculus – 2nd version (Derived Rules)

$$\frac{\Gamma_1, \Gamma_2 \vdash A, \Delta_1 \quad \Gamma_1, B, \Gamma_2 \vdash \Delta_2}{\Gamma_1, A \to B, \Gamma_2 \vdash \Delta_1, \Delta_2}\ [\to L] \qquad \frac{\Gamma, A \vdash \Delta_1, B, \Delta_2}{\Gamma \vdash \Delta_1, A \to B, \Delta_2}\ [\to R] \qquad \frac{\Gamma_1, \Gamma_2 \vdash A, \Delta}{\Gamma_1, \neg A, \Gamma_2 \vdash \Delta}\ [\neg L]$$

$$\frac{\Gamma_1, A, \Gamma_3 \vdash \Delta_1 \quad \Gamma_2, B, \Gamma_4 \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \vee B, \Gamma_3, \Gamma_4 \vdash \Delta_1, \Delta_2}\ [\vee L] \qquad \frac{\Gamma \vdash \Delta_1, A, B, \Delta_2}{\Gamma \vdash \Delta_1, A \vee B, \Delta_2}\ [\vee R] \qquad \frac{\Gamma, A \vdash \Delta_1, \Delta_2}{\Gamma \vdash \Delta_1, \neg A, \Delta_2}\ [\neg R]$$

$$\frac{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}{\Gamma_1, A \wedge B, \Gamma_2 \vdash \Delta}\ [\wedge L] \qquad \frac{\Gamma_1 \vdash \Delta_1, A, \Delta_3 \quad \Gamma_2 \vdash \Delta_2, B, \Delta_4}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, A \wedge B, \Delta_3, \Delta_4}\ [\wedge R] \qquad \frac{}{\Gamma_1, A, \Gamma_2 \vdash \Delta_1, A, \Delta_2}\ [Id]$$

$$\frac{\Gamma_1 \vdash B, \Delta_1 \quad \Gamma_2, B \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ [Cut] \qquad \frac{\Gamma_1, B, A, \Gamma_2 \vdash \Delta}{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}\ [X_L] \qquad \frac{\Gamma \vdash \Delta_1, B, A, \Delta_2}{\Gamma \vdash \Delta_1, A, B, \Delta_2}\ [X_R]$$

$$\frac{\Gamma_1, \Gamma_2 \vdash \Delta}{\Gamma_1, A, \Gamma_2 \vdash \Delta}\ [W_L] \qquad \frac{\Gamma_1, A, A, \Gamma_2 \vdash \Delta}{\Gamma_1, A, \Gamma_2 \vdash \Delta}\ [C_L] \qquad \frac{\Gamma \vdash \Delta_1, \Delta_2}{\Gamma \vdash \Delta_1, A, \Delta_2}\ [W_R] \qquad \frac{\Gamma \vdash \Delta_1, A, A, \Delta_2}{\Gamma \vdash \Delta_1, A, \Delta_2}\ [C_R]$$

2 Predicate Logic: Proof Rules

2.1 Natural Deduction

The Natural Deduction rules for Predicate Logic include all Proposition Logic rules plus the following rules:

$$\frac{P[x\backslash y]}{\forall x.P}\ [\forall I] \qquad \frac{\forall x.P}{P[x\backslash t]}\ [\forall E] \qquad \frac{P[x\backslash t]}{\exists x.P}\ [\exists I] \qquad \frac{\exists x.P \quad \begin{matrix}\overline{P[x\backslash y]}^{\,1} \\ \vdots \\ Q\end{matrix}}{Q}\ 1\ [\exists E]$$

Side conditions:

- for $[\forall I]$: y must not be free in any not-yet-discharged hypothesis or in $\forall x.P$
- for $[\forall E]$: $\mathtt{fv}(t)$ must not clash with $\mathtt{bv}(P)$
- for $[\exists I]$: $\mathtt{fv}(t)$ must not clash with $\mathtt{bv}(P)$
- for $[\exists E]$: y must not be free in Q or in not-yet-discharged hypotheses or in $\exists x.P$

2.2 Sequent Calculus (Derived Rules)

The Sequent Calculus rules for Predicate Logic include all Propositional Logic rules plus the following rules:

$$\frac{\Gamma \vdash P[x\backslash y]}{\Gamma \vdash \forall x.P}\ [\forall R] \qquad \frac{\Gamma_1, P[x\backslash t], \Gamma_2 \vdash Q}{\Gamma_1, \forall x.P, \Gamma_2 \vdash Q}\ [\forall L] \qquad \frac{\Gamma \vdash P[x\backslash t]}{\Gamma \vdash \exists x.P}\ [\exists R] \qquad \frac{\Gamma_1, P[x\backslash y], \Gamma_2 \vdash Q}{\Gamma_1, \exists x.P, \Gamma_2 \vdash Q}\ [\exists L]$$

Side conditions:

- for $[\forall R]$: y must not be free in Γ or $\forall x.P$
- for $[\forall L]$: $\mathtt{fv}(t)$ must not clash with $\mathtt{bv}(P)$
- for $[\exists R]$: $\mathtt{fv}(t)$ must not clash with $\mathtt{bv}(P)$
- for $[\exists L]$: y must not be free in Γ_1, Γ_2, Q, or $\exists x.P$

3 Predicate Logic: Semantics

Given a signature: $\langle\langle f_1^{k_1}, \ldots, f_n^{k_n}\rangle, \langle p_1^{j_1}, \ldots, p_m^{j_m}\rangle\rangle$

- of function symbols f_i of arity k_i, for $1 \leq i \leq n$
- of predicate symbols p_i of arity j_i, for $1 \leq i \leq m$

a model M is a structure $\langle D, \langle \mathcal{F}_{f_1}, \ldots, \mathcal{F}_{f_n}\rangle, \langle \mathcal{R}_{p_1}, \ldots, \mathcal{R}_{p_m}\rangle\rangle$

1. of a non-empty domain D
2. interpretations $\mathcal{F}_{f_i} \in D^n \to D$ for function symbols f_i
3. interpretations $\mathcal{R}_{p_i} \subseteq D^n$ for function symbols p_i

A variable valuation v is a partial function from variables to D

Given a model M and a variable valuation v, we assign meaning to terms and formulas as follows:

- Meaning of terms:
  - $[\![x]\!]_v^M = v(x)$
  - $[\![f(t_1, \ldots, t_n)]\!]_v^M = \mathcal{F}_f(\langle [\![t_1]\!]_v^M, \ldots, [\![t_n]\!]_v^M\rangle)$
- Meaning of formulas:
  - $\vDash_{M,v} p(t_1, \ldots, t_n)$ iff $\langle [\![t_1]\!]_v^M, \ldots, [\![t_n]\!]_v^M\rangle \in \mathcal{R}_p$
  - $\vDash_{M,v} \neg P$ iff $\neg\vDash_{M,v} P$
  - $\vDash_{M,v} P \land Q$ iff $\vDash_{M,v} P$ and $\vDash_{M,v} Q$
  - $\vDash_{M,v} P \lor Q$ iff $\vDash_{M,v} P$ or $\vDash_{M,v} Q$
  - $\vDash_{M,v} P \to Q$ iff $\vDash_{M,v} Q$ whenever $\vDash_{M,v} P$
  - $\vDash_{M,v} \forall x.P$ iff for every $d \in D$ we have $\vDash_{M,(v,x\mapsto d)} P$
  - $\vDash_{M,v} \exists x.P$ iff there exists a $d \in D$ such that $\vDash_{M,(v,x\mapsto d)} P$