

Problems for Week 4

January 26, 2022

Problem 1. Alex and Bob each flips a fair coin twice. Denote “1” as head, and “0” as tail. Let X be the maximum of the two numbers Alex gets, and let Y be the minimum of the two numbers Bob gets.

1. Find and sketch the joint PMF $P_{X,Y}(x,y)$
2. Find the marginal PMF $P_X(x)$ and $P_Y(y)$
3. Find the conditional PMF $P_{X|Y}(x|y)$. Does $P_{X|Y}(x|y) = P_X(x)$? Why

Problem 2. Find the marginal CDFs $F_X(x)$ and $F_Y(y)$ and determine whether or not X and Y are independent, if

$$F_{X,Y}(x,y) = \begin{cases} x - 1 - \frac{e^{-y} - e^{-xy}}{y}, & \text{if } 1 \leq x \leq 2, y \geq 0 \\ 1 - \frac{e^{-y} - e^{-2y}}{y}, & \text{if } x > 2, y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Problem 3. Let X and Y have a joint PDF

$$f_{X,Y}(x,y) = \begin{cases} c(x+y), & \text{if } x \in [0,1], y \in [0,1] \\ 0, & \text{otherwise.} \end{cases}$$

1. Find c , $f_Y(y)$ and $\mathbb{E}[Y]$
2. Find $f_{Y|X}(y,|x)$
3. Find $\mathbb{P}(Y > X | X > 1/2)$.

Problem 4. Suppose that X and Y are independent and both have the same density

$$f(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Let us find $\mathbb{P}(X + Y \leq 1)$.

Problem 5 (Mean Squared Error). Recall we define $MSE(\hat{\Theta}) = \mathbb{E}[(\hat{\Theta} - \theta)^2]$. Show that

$$MSE(\hat{\Theta}) = \text{Var}(\hat{\Theta}) + B(\hat{\Theta})^2 \tag{1}$$