Solution to Problems for Week 1

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Problem 1

Consider an experiment of rolling a die twice. The outcome of this experiment is an ordered pair whose first element is the first value rolled and whose second element is the second value rolled.

1. Find the sample space.
2. Find the event A that the value on the first roll is greater than or equal to the value on the second roll.
3. Find the event B that the first roll is a six.
4. Let C be the event that the first valued rolled and the second value rolled differ by two. Find $A \cap C$.

Solution

1. The sample space is

$$\Omega = \{(1,1),(1,2),\ldots,(1,6),\ldots,(6,1),(6,2),\ldots,(6,6)\}.$$

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4. According to the definition we have

$$C = \{(1,3),(3,1),(2,4),(4,2),(3,5),(5,3),(4,6),(6,4)\}$$

and

$$A \cap C = \{(3,1),(4,2),(5,3),(6,4)\}.$$

Problem 2

1. Show that $\mathbb{P}(\cup_{k=1}^{n} A_k) \leq \sum_{k=1}^{n} \mathbb{P}(A_k)$
2. Show that $\mathbb{P}(\cap_{k=1}^{n} A_k) \geq 1 - \sum_{k=1}^{n} \mathbb{P}(A_k^c)$

Solution

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According to our lecture, we know that

$$\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B) \tag{1}$$

We use mathematic induction to prove this result. Suppose the inequality holds for $n = m$, i.e.,

$$\mathbb{P}(\cup_{k=1}^{m} A_k) \leq \sum_{k=1}^{m} \mathbb{P}(A_k). \tag{2}$$

We now show that it holds with $n = m + 1$. Applying Eq. (1) with $A = \cup_{k=1}^{m} A_k$ and $B = A_{m+1}$ we know

$$\mathbb{P}(\cup_{k=1}^{m+1} A_k) \leq \mathbb{P}(\cup_{k=1}^{m} A_k) + \mathbb{P}(A_{m+1}) \leq \sum_{k=1}^{m} \mathbb{P}(A_k) + \mathbb{P}(A_{m+1}).$$

The shows the stated inequality.

Solution

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$$\mathbb{P}(\cap_{k=1}^n A_k) \geq 1 - \sum_{k=1}^n \mathbb{P}(A_k^c).$$

The complement of $\cap_{k=1}^n A_k$ is $\cup_{k=1}^n A_k^c$ (a point not in the intersection should belong to A_k^c for some k). Therefore

$$\mathbb{P}(\cap_{k=1}^n A_k) = 1 - \mathbb{P}(\cup_{k=1}^n A_k^c) \geq 1 - \sum_{k=1}^n \mathbb{P}(A_k^c), \tag{3}$$

where we have used

$$\mathbb{P}(\cup_{k=1}^n A_k^c) \leq \sum_{k=1}^n \mathbb{P}(A_k^c). \tag{4}$$

Problem 3

Prove that

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

Solution

We have

$$A \cup B = B \cup \left(A \cap B^c\right).$$

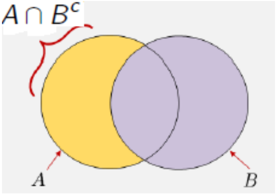
Note B and $A \cap B^c$ are disjoint. Then

$$\mathbb{P}(A \cup B) = \mathbb{P}(B) + \mathbb{P}(A \cap B^c).$$

Furthermore,

$$\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c).$$

$$\mathbb{P}(A \cup B) = \mathbb{P}(B) + \left(\mathbb{P}(A) - \mathbb{P}(A \cap B)\right)$$



Problem 4

Consider tossing a coin. The event space is

$$\mathcal{F} = \{\emptyset, \{H\}, \{T\}, \Omega\}.$$

We define two functions as follows

$$\begin{aligned}
&\mathbb{P}_1[\emptyset] = 0, \quad \mathbb{P}_1[\{H\}] = 1/2, \quad \mathbb{P}_1[\{T\}] = 1/2, \quad \mathbb{P}_1[\Omega] = 1 \\
&\mathbb{P}_2[\emptyset] = 0, \quad \mathbb{P}_2[\{H\}] = 1/3, \quad \mathbb{P}_2[\{T\}] = 1/3, \quad \mathbb{P}_2[\Omega] = 1
\end{aligned}$$

1. Is $\mathbb{P}_1$ a probability law?
2. Is $\mathbb{P}_2$ a probability law?

Solution

1. $\mathbb{P}_1$ is a probability law as it satisfies all the requirements.

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$$1 = \mathbb{P}_2[\Omega] \neq \mathbb{P}_2[\{H\}] + \mathbb{P}_2[\{T\}] = 2/3.$$

Problem 5

You toss a fair coin 5 times. What is the probability that you see at least two heads.

Solution

The sample space has $2^5 = 32$ outcomes

$$\Omega = \{(x_1, x_2, \ldots, x_5) : x_i \in \{H, T\}\}$$

.

$$\begin{aligned}
B_2 &= \{\text{have 2 heads in 5 tosses}\} \\
B_3 &= \{\text{have 3 heads in 5 tosses}\} \\
B_4 &= \{\text{have 4 heads in 5 tosses}\} \\
B_5 &= \{\text{have 5 heads in 5 tosses}\}
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Then

$$|B_2| = \binom{5}{2}, \quad |B_3| = \binom{5}{3}, \quad |B_4| = \binom{5}{4}, \quad |B_5| = \binom{5}{5}.$$

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Then

$$|B_2| = \binom{5}{2}, \quad |B_3| = \binom{5}{3}, \quad |B_4| = \binom{5}{4}, \quad |B_5| = \binom{5}{5}.$$

Since this is an experiment with equally likely outcomes, we know

$$\mathbb{P}(\{\text{at least 2 heads}\}) = \frac{\binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5}}{32} = \frac{26}{32}.$$

Problem 6

Let the events A and B have

$$\mathbb{P}(A) = x, \quad \mathbb{P}(B) = y, \quad \mathbb{P}(A \cup B) = z.$$

Find the probabilities $\mathbb{P}(A \cap B), \mathbb{P}(A^c \cap B^c)$ and $\mathbb{P}(A \cap B^c)$

Solution

1. By Eq. (1) we know

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) = x + y - z.$$

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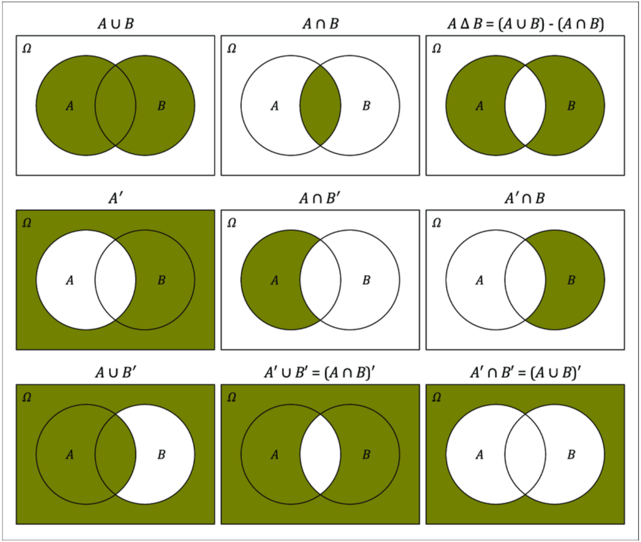
3. Furthermore,

$$\mathbb{P}(A \cap B^c) + \mathbb{P}(A \cap B) = \mathbb{P}(A),$$

from which we know

$$\mathbb{P}(A \cap B^c) = \mathbb{P}(A) - \mathbb{P}(A \cap B) = x - (x + y - z) = z - y.$$


A ∪ B
A ∩ B
A Δ B = (A ∪ B) - (A ∩ B)
A′
A ∩ B′
A′ ∩ B
A ∪ B′
A′ ∪ B′ = (A ∩ B)′
A′ ∩ B′ = (A ∪ B)′
Ω
A
B