Solution to Problems for Week 4

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Problem 1

Alex and Bob each flips a fair coin twice. Denote "1" as head, and "0" as tail. Let X be the maximum of the two numbers Alex gets, and let Y be the minimum of the two numbers Bob gets.

1. Find and sketch the joint PMF $P_{X,Y}(x, y)$
2. Find the marginal PMF $P_X(x)$ and $P_Y(y)$
3. Find the conditional PMF $P_{X|Y}(x|y)$. Does $P_{X|Y}(x|y) = P_X(x)$? Why

Solution

1. We know $X(\Omega) = \{0, 1\}$, $Y(\Omega) = \{0, 1\}$. Note $X = 0$ if Alex gets two 0, which happens with probability $1/4$. $Y = 0$ if Bob gets either $(0, 0), (0, 1)$ or $(1, 0)$, which happens with probability $3/4$

$$P_{X,Y}(0,0) = \mathbb{P}(\{X = 0, Y = 0\}) = \mathbb{P}(\{X = 0\})\mathbb{P}(\{Y = 0\}) = \frac{1}{4}\frac{3}{4} = \frac{3}{16}$$
$$P_{X,Y}(1,0) = \mathbb{P}(\{X = 1, Y = 0\}) = \mathbb{P}(\{X = 1\})\mathbb{P}(\{Y = 0\}) = \frac{3}{4}\frac{3}{4} = \frac{9}{16}$$
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2. The marginal PMF is

$$P_X(0) = P_{X,Y}(0,0) + P_{X,Y}(0,1) = \frac{3}{16} + \frac{1}{16} = \frac{4}{16}$$
$$P_X(1) = P_{X,Y}(1,0) + P_{X,Y}(1,1) = \frac{9}{16} + \frac{3}{16} = \frac{12}{16}$$
$$P_Y(0) = P_{X,Y}(0,0) + P_{X,Y}(1,0) = \frac{3}{16} + \frac{9}{16} = \frac{12}{16}$$

Solution

1. By $P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}$, we know

$$P_{X|Y}(0|0) = \frac{P_{X,Y}(0,0)}{P_Y(0)} = \frac{3/16}{12/16} = \frac{1}{4}$$

$$P_{X|Y}(1|0) = \frac{P_{X,Y}(1,0)}{P_Y(0)} = \frac{9/16}{12/16} = \frac{3}{4}$$

$$P_{X|Y}(0|1) = \frac{P_{X,Y}(0,1)}{P_Y(1)} = \frac{1/16}{4/16} = \frac{1}{4}$$

$$P_{X|Y}(1|1) = \frac{P_{X,Y}(1,1)}{P_Y(1)} = \frac{3/16}{4/16} = \frac{3}{4}$$

Solution

1. By $P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}$, we know

$$P_{X|Y}(0|0) = \frac{P_{X,Y}(0,0)}{P_Y(0)} = \frac{3/16}{12/16} = \frac{1}{4}$$
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$$P_{X|Y}(1|1) = \frac{P_{X,Y}(1,1)}{P_Y(1)} = \frac{3/16}{4/16} = \frac{3}{4}$$

Then it is clear that

$$P(X|Y)(x|y) = P_X(x) \quad \forall x, y \in \{0, 1\}.$$

Therefore, X and Y are independent.

Problem 2

Find the marginal CDFs $F_X(x)$ and $F_Y(y)$ and determine whether or not X and Y are independent, if

$$F_{X,Y}(x,y) = \begin{cases} x - 1 - \frac{e^{-y} - e^{-xy}}{y}, & \text{if } 1 \leq x \leq 2, y \geq 0 \\ 1 - \frac{e^{-y} - e^{-2y}}{y}, & \text{if } x > 2, y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Solution

Note $e^{-\infty} = 0$. Therefore

$$F_X(x) = F_{X,Y}(x,\infty) = \begin{cases} x-1, & \text{if } 1 \le x \le 2 \\ 1, & \text{if } x > 2 \\ 0, & \text{otherwise.} \end{cases}$$

Solution

Note $e^{-\infty} = 0$. Therefore

$$F_X(x) = F_{X,Y}(x, \infty) = \begin{cases} x - 1, & \text{if } 1 \leq x \leq 2 \\ 1, & \text{if } x > 2 \\ 0, & \text{otherwise.} \end{cases}$$

$$F_Y(y) = F_{X,Y}(\infty, y) = \begin{cases} 1 - \frac{e^{-y} - e^{-2y}}{y}, & \text{if } y \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Solution

Note $e^{-\infty} = 0$. Therefore

$$F_X(x) = F_{X,Y}(x, \infty) = \begin{cases} x - 1, & \text{if } 1 \leq x \leq 2 \\ 1, & \text{if } x > 2 \\ 0, & \text{otherwise.} \end{cases}$$

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No they are not independent because

$$F_{X,Y}(x, y) \neq F_X(x) F_Y(y) \text{ if } 1 \leq x \leq 2, y \geq 0.$$

Problem 3

Let X and Y have a joint PDF

$$f_{X,Y}(x,y) = \begin{cases} c(x+y), & \text{if } x \in [0,1], y \in [0,1] \\ 0, & \text{otherwise.} \end{cases}$$

1. Find $c, f_Y(y)$ and $\mathbb{E}[Y]$
2. Find $f_{Y|X}(y,|x)$
3. Find $\mathbb{P}(Y > X | X > 1/2)$.

Solution

1. We know $\int_{[0,1]\times[0,1]} f_{X,Y}(x,y)dxdy = 1$. Then $(\int_0^1 xdx = \int_0^1 dx^2/2 = \frac{1}{2}(x^2|_0^1) = 1/2)$

$$\int_0^1 \int_0^1 c(x+y)dxdy = c\int_0^1 xdx + c\int_0^1 ydy = c(1/2+1/2) = 1 \implies c = 1$$

$$f_X(x) = \int_{\mathbb{R}} f_{X,Y}(x,y)dy = \int_0^1 (x+y)dy = x + 1/2$$

$$f_Y(y) = \int_{\mathbb{R}} f_{X,Y}(x,y)dx = \int_0^1 (x+y)dx = y + 1/2$$

$$\mathbb{E}[Y] = \int_{\mathbb{R}} yf_Y(y)dy = \int_0^1 y(y+1/2)dy = 1/3 + 1/4 = 7/12$$

Solution

1. We know $\int_{[0,1]\times[0,1]} f_{X,Y}(x,y)dxdy = 1$. Then $(\int_0^1 xdx = \int_0^1 dx^2/2 = \frac{1}{2}(x^2|_0^1) = 1/2)$

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$$\mathbb{E}[Y] = \int_{\mathbb{R}} y f_Y(y)dy = \int_0^1 y(y+1/2)dy = 1/3 + 1/4 = 7/12$$

2. By the definition of conditional probability we know

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{x+y}{x+1/2}$$

Solution

1.

$$\begin{aligned}\mathbb{P}(\{Y > X | X > 1/2\}) &= \frac{\mathbb{P}(\{Y > X\} \cap \{X > 1/2\})}{\mathbb{P}(\{X > 1/2\})} \\ &= \frac{\mathbb{P}(\{1/2 < X < Y\})}{1 - F_X(1/2)} \\ &= \frac{\int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{y} (x+y)dxdy}{1 - \int_0^{\frac{1}{2}} (x + 1/2)dx} = \frac{3}{10}\end{aligned}$$

Solution

1.

$$\mathbb{P}(\{Y > X | X > 1/2\}) = \frac{\mathbb{P}(\{Y > X\} \cap \{X > 1/2\})}{\mathbb{P}(\{X > 1/2\})}$$
$$= \frac{\mathbb{P}(\{1/2 < X < Y\})}{1 - F_X(1/2)}$$
$$= \frac{\int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{y} (x+y)dxdy}{1 - \int_0^{\frac{1}{2}} (x + 1/2)dx} = \frac{3}{10}$$

Here are more detailed calculations (optional)

$$\int_{1/2}^{y} (x+y)dx = \int_{1/2}^{y} d(x^2/2 + xy) = (x^2/2 + xy)_{1/2}^{y} = \frac{3y^2}{2} - \frac{1}{8} - \frac{y}{2}.$$

$$\int_{1/2}^{1} \left(\frac{3y^2}{2} - \frac{1}{8} - \frac{y}{2}\right) dy = \int_{1/2}^{1} \frac{1}{2} dy^3 - \frac{1}{8}\int_{1/2}^{1} dy - \frac{1}{4}\int_{1/2}^{1} dy^2 = \frac{7}{16} - \frac{1}{16} - \frac{3}{16} = \frac{3}{16}$$

$$\int_0^{1/2} (x + 1/2)dx = \int_0^{1/2} d(x^2/2 + x/2) = (x^2/2 + x/2)_0^{1/2} = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}.$$

Problem 4

Suppose that X and Y are independent and both have the same density

$$f(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Let us find $\mathbb{P}(X + Y \leq 1)$.

Solution

1. Using independence, the joint density is

$$f(x,y) = f_X(x) f_Y(y) = \begin{cases} 4xy, & \text{if } x \in [0,1], y \in [0,1] \\ 0, & \text{otherwise.} \end{cases}$$

Solution

1. Using independence, the joint density is

$$f(x,y) = f_X(x)f_Y(y) = \begin{cases} 4xy, & \text{if } x \in [0,1], y \in [0,1] \\ 0, & \text{otherwise.} \end{cases}$$

2. Now (note $\int_0^{1-x} ydy = \frac{1}{2}\int_0^{1-x} dy^2 = \frac{1}{2}y^2|_0^{1-x} = \frac{(1-x)^2}{2}$)

$$\begin{aligned} \mathbb{P}(X+Y \le 1) &= \iint_{x+y\le 1} f(x,y)dydx \\ &= 4\int_0^1 x \int_0^{1-x} ydydx = 4\int_0^1 \frac{x(1-x)^2}{2}dx \\ &= 4\int_0^1 \frac{x+x^3-2x^2}{2}dx = 2\Big(\frac{x^2}{2}\Big|_0^1 + \frac{x^4}{4}\Big|_0^1 - \frac{2x^3}{3}\Big|_0^1\Big) \\ &= \frac{1}{6}. \end{aligned}$$

Problem 5

Show that

$$\mathsf{MSE}(\hat{\Theta}) = \mathbb{E}[(\hat{\Theta} - \theta)^2] = \mathsf{Var}(\hat{\Theta}) + B(\hat{\Theta})^2 \tag{1}$$

Solution

$$
\begin{aligned}
\mathbb{E}[(\hat{\Theta} - \theta)^2] &= \mathbb{E}\left[\left(\hat{\Theta} - \mathbb{E}[\hat{\Theta}] + \mathbb{E}[\hat{\Theta}] - \theta\right)^2\right] \\
&= \underbrace{\mathbb{E}\left[\left(\hat{\Theta} - \mathbb{E}[\hat{\Theta}]\right)^2\right]}_{:=\mathrm{Var}(\hat{\Theta})} + \mathbb{E}\left[\left(\underbrace{\mathbb{E}[\hat{\Theta}] - \theta}_{:=B(\hat{\Theta})}\right)^2\right] + 2\mathbb{E}\left[\left(\hat{\Theta} - \mathbb{E}[\hat{\Theta}]\right)\left(\mathbb{E}[\hat{\Theta}] - \theta\right)\right]
\end{aligned}
$$

Solution

$$\begin{aligned}\mathbb{E}[(\hat{\Theta}-\theta)^2] &= \mathbb{E}\left[\left(\hat{\Theta}-\mathbb{E}[\hat{\Theta}]+\mathbb{E}[\hat{\Theta}]-\theta\right)^2\right] \\ &= \underbrace{\mathbb{E}\left[\left(\hat{\Theta}-\mathbb{E}[\hat{\Theta}]\right)^2\right]}_{:=\mathsf{Var}(\hat{\Theta})} + \mathbb{E}\left[\big(\underbrace{\mathbb{E}[\hat{\Theta}]-\theta}_{:=B(\hat{\Theta})}\big)^2\right] + 2\mathbb{E}\left[\left(\hat{\Theta}-\mathbb{E}[\hat{\Theta}]\right)\left(\mathbb{E}[\hat{\Theta}]-\theta\right)\right]\end{aligned}$$

It is clear

$$\mathbb{E}\left[\left(\hat{\Theta}-\mathbb{E}[\hat{\Theta}]\right)\left(\mathbb{E}[\hat{\Theta}]-\theta\right)\right] = \left(\mathbb{E}[\hat{\Theta}]-\theta\right)\underbrace{\mathbb{E}\left[\left(\hat{\Theta}-\mathbb{E}[\hat{\Theta}]\right)\right]}_{=0}.$$

We can combine the above two inequalities to derive the stated bound.