

UNIVERSITY OF BIRMINGHAM

School of Computer Science

06 30181

Mathematical Foundations of Computer Science

Main Summer Examinations 2019

Time allowed: 2:00

[Answer all questions]

Note

Answer ALL questions. Each question will be marked out of 20. The paper will be marked out of 80, which will be rescaled to a mark out of 100.

Part A**Question 1 [Linear Algebra]**

(a) Consider the following parametric representations of a plane E and a line L in 3D:

$$E : X = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} + s \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad L : Y = \begin{pmatrix} 1 \\ 3 \\ -8 \end{pmatrix} + u \cdot \begin{pmatrix} 5 \\ 2 \\ -3 \end{pmatrix}$$

(i) Find the point P where E and L intersect. **[4 marks]**

(ii) Show that L is orthogonal to E . **[2 marks]**

(iii) Use the previous item to describe E in normal form. **[2 marks]**

(b) Find the inverse of the following matrix over $\text{GF}(2)$:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

[6 marks]

(c) Let V be an algebra of vectors and let $B = \{\vec{v}_1, \dots, \vec{v}_n\} \subseteq V$ be a collection of elements of V .

(i) When do we say that B generates V ? **[3 marks]**

(ii) Assuming B generates V , why are we interested in the elements of B being “linearly independent” of each other? **[3 marks]**

Question 2 [Set Theory]

- (a) (i) List the elements of the set $\{(x, y) \in \mathbb{N} \times \mathbb{N} \mid x + 2y = 10\}$. **[1 mark]**
- (ii) Describe the cardinality of each of the following sets, giving brief justifications:
- The set of numbers that can be stored as a `float` in Java. **[2 marks]**
 - The set of valid Java programs. **[2 marks]**
 - The set \mathbb{R} of real numbers. **[3 marks]**

- (b) Consider the following binary relation \approx on $\mathbb{Z} \times \mathbb{Z}$:

$$(a, b) \approx (c, d) \stackrel{\text{def}}{\iff} a^2 + b^2 = c^2 + d^2$$

- (i) Check that \approx is an equivalence relation. **[3 marks]**
- (ii) By interpreting the elements of $\mathbb{Z} \times \mathbb{Z}$ as the coordinates of points in the plane, sketch a diagram which illustrates the equivalence classes of \approx . **[2 marks]**
- (iii) What is the minimum number of elements in each equivalence class? **[1 mark]**
- (c) (i) Consider the function $r: \mathbb{R} \rightarrow \mathbb{Z}$ which rounds a real number to the nearest integer. Is it injective, surjective, bijective? Give brief justifications for your answers. **[3 marks]**
- (ii) Compare r with the built-in Java method

`int round(float f)` **[3 marks]**

Part B

Question 3 [Languages and Automata]

- (a) (i) Describe two kinds of problem for which you would specify a language (e.g. by means of a regular expression) when using a tool to solve them. **[3 marks]**
- (ii) For the alphabet $\Sigma = \{a, b, c\}$, give a regular expression for those words in which a and b appear at least once. **[4 marks]**
- (b) For the alphabet $\Sigma = \{a, b\}$, obtain a deterministic finite automaton for $(ba)^* \mid bb(ab)^*$, by first writing down automata for $(ba)^*$ and $bb(ab)^*$, then combining, then determinizing. **[4 marks]**
- (c) (i) What does it mean for a grammar to be ambiguous? **[2 marks]**
- (ii) Show that this grammar, for the alphabet $\Sigma = \{a, b\}$, is ambiguous.

$$M ::= abMb \mid \varepsilon \mid abb$$

[3 marks]

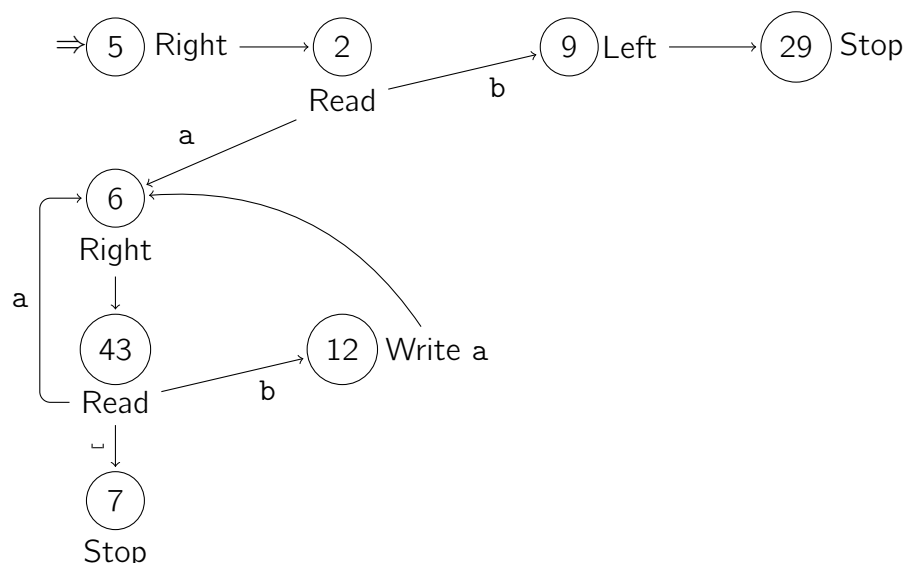
- (iii) Show that there is no regular expression that generates this language. **[4 marks]**

Question 4 [Computability and Complexity]

- (a) (i) Andy is implementing a programming language for his final year project. Certain programs cause a `VeryBadException`. He tells his supervisor, Brenda, that he intends to detect all such programs. She replies that this problem is undecidable. What does this mean? **[3 marks]**

- (ii) What are the implications of Church's thesis for Brenda's comment? **[2 marks]**

- (b) Here is a Turing machine with tape alphabet $\{a, b, \sqcup\}$.



At the start of execution, the head is to the left of a block of $n \geq 1$ letters that are all a or b, on an otherwise blank tape.

- (i) When the input block is ab, trace the execution. (The number of steps, including Stop, is less than 10.)

[3 marks]

- (ii) For a general input block, what is the contents of the tape and the head position when execution stops?

[3 marks]

- (iii) Give the worst case number of steps (including Stop) and show that it is $O(n)$.

[3 marks]

- (c) (i) What does it mean for a search problem to be an \mathcal{NP} search problem? **[3 marks]**

- (ii) Consider the following problem. Given a finite list of numbers with no repetition, select at least half of these numbers that appear in increasing order, if possible. For example, given $[3, 8, 4, 5, 2, 7, 9]$, acceptable solutions would be $[3, 4, 5, 7]$ or $[4, 5, 7, 9]$ or $[3, 4, 5, 7, 9]$. But given $[3, 8, 9, 4, 5, 2, 1]$ you would say "Impossible". Explain why this is an \mathcal{NP} search problem. **[3 marks]**

Do not complete the attendance slip, fill in the front of the answer book or turn over the question paper until you are told to do so

Important Reminders

- Coats/outwear should be placed in the designated area.
- Unauthorised materials (e.g. notes or Tippex) must be placed in the designated area.
- Check that you do not have any unauthorised materials with you (e.g. in your pockets, pencil case).
- Mobile phones and smart watches must be switched off and placed in the designated area or under your desk. They must not be left on your person or in your pockets.
- You are not permitted to use a mobile phone as a clock. If you have difficulty seeing a clock, please alert an Invigilator.
- You are not permitted to have writing on your hand, arm or other body part.
- Check that you do not have writing on your hand, arm or other body part – if you do, you must inform an Invigilator immediately
- Alert an Invigilator immediately if you find any unauthorised item upon you during the examination.

Any students found with non-permitted items upon their person during the examination, or who fail to comply with Examination rules may be subject to Student Conduct procedures.