Solution to Problems for Week 2

Yunwen Lei

School of Computer Science
University of Birmingham

Problem 1

The table below shows some information about 200 students and their most frequent method of travel to school.

	Walk	Not walk	Total
Male	56	40	96
Female	72	32	104
Total	128	72	200

1. Find the probability that a randomly chosen student is female
2. Find the probability that a randomly chosen student is female and walks to school
3. Given the student is female what is the probability that they walk to school?

Solution

	Walk	Not walk	Total
Male	56	40	96
Female	72	32	104
Total	128	72	200

Solution

	Walk	Not walk	Total
Male	56	40	96
Female	72	32	104
Total	128	72	200

1. $\mathbb{P}(\text{Female}) = \frac{104}{200} = \frac{13}{25}$
2. $\mathbb{P}(\text{Female} \cap \text{Walk}) = \frac{72}{200} = \frac{9}{25}$
3. $\mathbb{P}(\text{Walk}|\text{Female}) = \frac{\mathbb{P}(\text{Female} \cap \text{Walk})}{\mathbb{P}(\text{Female})} = \frac{72}{104} = \frac{9}{13}$

Problem 2

1. There are 8 counters in a bag.
2. Five of the counters are red and 3 of the counters are blue.
3. Two counters are taken at random from the bag.
4. Compute the probability that one counter of each colour are taken.

Solution

Let $A = \{\text{the first counter is red}\}$ and $B = \{\text{the second counter is red}\}$. Then

$$\mathbb{P}(A \cap B^c) = \mathbb{P}(A)\mathbb{P}(B^c|A) = \frac{5}{8}\frac{3}{7} = \frac{15}{56}$$

Solution

Let $A = \{\text{the first counter is red}\}$ and $B = \{\text{the second counter is red}\}$. Then

$$\mathbb{P}(A \cap B^c) = \mathbb{P}(A)\mathbb{P}(B^c|A) = \frac{5}{8}\frac{3}{7} = \frac{15}{56}$$

$$\mathbb{P}(A^c \cap B) = \mathbb{P}(A^c)\mathbb{P}(B|A^c) = \frac{3}{8}\frac{5}{7} = \frac{15}{56}.$$

Solution

Let $A = \{\text{the first counter is red}\}$ and $B = \{\text{the second counter is red}\}$. Then

$$\mathbb{P}(A \cap B^c) = \mathbb{P}(A)\mathbb{P}(B^c|A) = \frac{5}{8}\frac{3}{7} = \frac{15}{56}$$

$$\mathbb{P}(A^c \cap B) = \mathbb{P}(A^c)\mathbb{P}(B|A^c) = \frac{3}{8}\frac{5}{7} = \frac{15}{56}.$$

Therefore

$$\mathbb{P}(A \cap B^c) + \mathbb{P}(A^c \cap B) = \frac{15}{28}.$$

Problem 3

For three events A, B and C, we know that

- A and C are independent
- B and C are independent
- A and B are disjoint
- $\mathbb{P}(A \cup C) = \frac{2}{3}, \mathbb{P}(B \cup C) = \frac{3}{4}, \mathbb{P}(A \cup B \cup C) = \frac{11}{12}$.

Compute $\mathbb{P}(A), \mathbb{P}(B)$ and $\mathbb{P}(C)$.

Solution

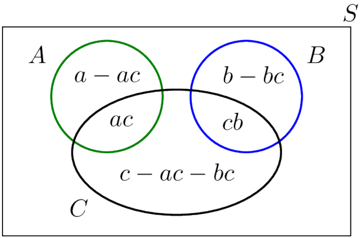
We assume $\mathbb{P}(A) = a, \mathbb{P}(B) = b, \mathbb{P}(C) = c$. Now

$$\mathbb{P}(A \cup C) = a + c - ac = \frac{2}{3}$$

$$\mathbb{P}(B \cup C) = b + c - bc = \frac{3}{4}$$

$$\mathbb{P}((A \cup B) \cap C) = \mathbb{P}(A \cap C) + \mathbb{P}(B \cap C)$$

$$\mathbb{P}(A \cap C) = \mathbb{P}(A)\mathbb{P}(C), \ \mathbb{P}(B \cap C) = \mathbb{P}(B)\mathbb{P}(C)$$



$P(A) = a, P(B) = b, P(C) = c$

Solution

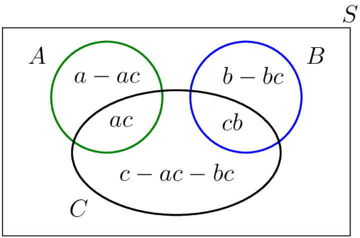
We assume $\mathbb{P}(A) = a, \mathbb{P}(B) = b, \mathbb{P}(C) = c$. Now

$$\mathbb{P}(A \cup C) = a + c - ac = \frac{2}{3}$$

$$\mathbb{P}(B \cup C) = b + c - bc = \frac{3}{4}$$

$$\mathbb{P}((A \cup B) \cap C) = \mathbb{P}(A \cap C) + \mathbb{P}(B \cap C)$$

$$\mathbb{P}(A \cap C) = \mathbb{P}(A)\mathbb{P}(C), \ \mathbb{P}(B \cap C) = \mathbb{P}(B)\mathbb{P}(C)$$



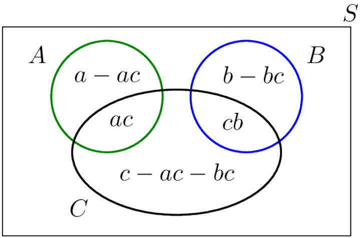
$P(A) = a, P(B) = b, P(C) = c$

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A \cup B) + \mathbb{P}(C) - \mathbb{P}((A \cup B) \cap C) = a + b + c - ac - bc = \frac{11}{12}$$

Solution

We assume $\mathbb{P}(A) = a, \mathbb{P}(B) = b, \mathbb{P}(C) = c$. Now

$$\mathbb{P}(A \cup C) = a + c - ac = \frac{2}{3}$$
$$\mathbb{P}(B \cup C) = b + c - bc = \frac{3}{4}$$
$$\mathbb{P}((A \cup B) \cap C) = \mathbb{P}(A \cap C) + \mathbb{P}(B \cap C)$$
$$\mathbb{P}(A \cap C) = \mathbb{P}(A)\mathbb{P}(C), \ \mathbb{P}(B \cap C) = \mathbb{P}(B)\mathbb{P}(C)$$



$P(A) = a, P(B) = b, P(C) = c$

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A \cup B) + \mathbb{P}(C) - \mathbb{P}((A \cup B) \cap C) = a + b + c - ac - bc = \frac{11}{12}$$

Subtract the third equation from the sum of the first and second equations:

$$\frac{11}{12} - \frac{8+9}{12} = \mathbb{P}(A \cup B \cup C) - \mathbb{P}(B \cup C) - \mathbb{P}(A \cup C)$$
$$= a + b + c - ac - bc - (a + c - ac) - (b + c - bc) = -c.$$

Then $c = 1/2$, which then gives $a = 1/3$ and $b = 1/2$.

Solution

$$\mathbb{P}(A \cup C) = a + c - ac = \frac{2}{3}$$
$$\mathbb{P}(B \cup C) = b + c - bc = \frac{3}{4}$$
$$c = 1/2$$

Then

Solution

$$\mathbb{P}(A \cup C) = a + c - ac = \frac{2}{3}$$
$$\mathbb{P}(B \cup C) = b + c - bc = \frac{3}{4}$$
$$c = 1/2$$

Then

$$\frac{2}{3} = a + 1/2 - a/2 \Longrightarrow a = 1/3$$
$$\frac{3}{4} = b + 1/2 - b/2 \Longrightarrow b = 1/2$$

Problem 4

Consider two independent fair coin tosses, and the following events:

$$H_1 = \{\text{1st toss is a head}\}$$
$$H_2 = \{\text{2nd toss is a head}\}$$
$$D = \{\text{the two tosses have different results}\}$$

Show that these events are pairwise independent but not independent.

Solution

The events H_1 and H_2 are independent, by definition. To see that H_1 and D are independent, we note that

$$\mathbb{P}(D|H_1) = \frac{\mathbb{P}(H_1 \cap D)}{\mathbb{P}(H_1)} = \frac{1/4}{1/2} = \mathbb{P}(D)$$

Solution

The events H_1 and H_2 are independent, by definition. To see that H_1 and D are independent, we note that

$$\mathbb{P}(D|H_1) = \frac{\mathbb{P}(H_1 \cap D)}{\mathbb{P}(H_1)} = \frac{1/4}{1/2} = \mathbb{P}(D)$$

Similarly, H_2 and D are independent. On the other hand, we have

$$\mathbb{P}(H_1 \cap H_2 \cap D) = 0 \neq \mathbb{P}(H_1)\mathbb{P}(H_2)\mathbb{P}(D).$$

Problem 5

1. A computer manufacturer uses chips from three sources.
2. Chips from source A, B and C are defective with probabilities $0.005, 0.001$ and 0.01, respectively.
3. The proportions of chips from A, B, C are $0.5, 0.1, 0.4$, respectively.
4. A randomly selected chip is found to be defective.
5. Compute the probability that the chips are from A.

Solution

Let D be the event for chip defective. Then

$$\begin{aligned}\mathbb{P}(D) &= \mathbb{P}(D|A)\mathbb{P}(A) + \mathbb{P}(D|B)\mathbb{P}(B) + \mathbb{P}(D|C)\mathbb{P}(C) \\ &= 0.005 \times 0.5 + 0.001 \times 0.1 + 0.01 \times 0.4 = 0.0066.\end{aligned}$$

Solution

Let D be the event for chip defective. Then

$$\begin{aligned}\mathbb{P}(D) &= \mathbb{P}(D|A)\mathbb{P}(A) + \mathbb{P}(D|B)\mathbb{P}(B) + \mathbb{P}(D|C)\mathbb{P}(C) \\ &= 0.005 \times 0.5 + 0.001 \times 0.1 + 0.01 \times 0.4 = 0.0066.\end{aligned}$$

By the Bayes' lemma we know

$$\mathbb{P}(A|D) = \frac{\mathbb{P}(A)\mathbb{P}(D|A)}{\mathbb{P}(D)} = \frac{0.005 \times 0.5}{0.0066} = \frac{25}{66}.$$

Problem 6

1. Let A_1 and A_2 be a partition of the sample space Ω.
2. Assume $\mathbb{P}(B) > 0$ and $\mathbb{P}(A_1|B) < \mathbb{P}(A_1)$.
3. Prove that $\mathbb{P}(A_2|B) > \mathbb{P}(A_2)$.

Solution

1. We prove it by contraction. Let us assume $\mathbb{P}(A_2|B) \leq \mathbb{P}(A_2)$

Solution

1. We prove it by contraction. Let us assume $\mathbb{P}(A_2|B) \leq \mathbb{P}(A_2)$
2. Since $\mathbb{P}(A_1|B) < \mathbb{P}(A_1)$, we know

$$\mathbb{P}(A_1 \cap B) = \mathbb{P}(A_1|B)\mathbb{P}(B) < \mathbb{P}(A_1)\mathbb{P}(B).$$

Solution

1. We prove it by contraction. Let us assume $\mathbb{P}(A_2|B) \leq \mathbb{P}(A_2)$
2. Since $\mathbb{P}(A_1|B) < \mathbb{P}(A_1)$, we know

$$\mathbb{P}(A_1 \cap B) = \mathbb{P}(A_1|B)\mathbb{P}(B) < \mathbb{P}(A_1)\mathbb{P}(B).$$

3. The condition $\mathbb{P}(A_2|B) \leq \mathbb{P}(A_2)$ implies

$$\mathbb{P}(A_2 \cap B) = \mathbb{P}(A_2|B)\mathbb{P}(B) \leq \mathbb{P}(A_2)\mathbb{P}(B)$$

Solution

1. We prove it by contraction. Let us assume $\mathbb{P}(A_2|B) \leq \mathbb{P}(A_2)$
2. Since $\mathbb{P}(A_1|B) < \mathbb{P}(A_1)$, we know

$$\mathbb{P}(A_1 \cap B) = \mathbb{P}(A_1|B)\mathbb{P}(B) < \mathbb{P}(A_1)\mathbb{P}(B).$$

3. The condition $\mathbb{P}(A_2|B) \leq \mathbb{P}(A_2)$ implies

$$\mathbb{P}(A_2 \cap B) = \mathbb{P}(A_2|B)\mathbb{P}(B) \leq \mathbb{P}(A_2)\mathbb{P}(B)$$

4. Note $A_1 \cap B$ and $A_2 \cap B$ is a partition of B. We then get

$$\mathbb{P}(B) = \mathbb{P}(A_1 \cap B) + \mathbb{P}(A_2 \cap B) < \mathbb{P}(A_1)\mathbb{P}(B) + \mathbb{P}(A_2)\mathbb{P}(B) = \mathbb{P}(B).$$

5. This leads to a contradiction!