UNIVERSITY OF BIRMINGHAM

School of Computer Science

Mathematical and Logical Foundations of Computer Science

Second Class Test 2021/22

This test is designed to be solved in about one hour and is worth 8% of your total grade.

Mathematical and Logical Foundations of Computer Science

Question 1 [Relations, functions, induction & linear equations]

(a) Let $V = \{1, 2, 3, 4, 5, 6, 7\}$ and let $E = \{(1, 7), (2, 1), (4, 1), (6, 5), (6, 6)\}$ be a binary relation on V. Find the equivalence closure of this relation and state the equivalence classes. (It may help to draw a diagram.) **[4 marks]**

(b) The coefficients of the following system are taken from GF(2). Solve it using Gaussian elimination.

$$\begin{aligned} x_1 + x_2 + x_4 &= 0 \\ x_1 + x_3 + x_4 &= 1 \\ x_2 + x_5 &= 0 \\ x_1 + x_2 + x_3 + x_5 &= 0 \\ x_1 + x_3 &= 0 \end{aligned}$$

[5 marks]

(c) Let S be the smallest subset of $\{\mathtt{a}, \mathtt{b}\}^*$ such that all of the following conditions are satisfied:

- $\varepsilon \in S$.
- If $w \in S$, then $\mathtt{aab}w \in S$.
- If $w \in S$, then any anagram of w is also in S. (An anagram of w is a string that arises from w by a permutation of the letters.)

(i) Show that aababaabaa is in S. **[2 marks]**

(ii) Argue that ababab is not in S by giving a property and proving that all elements of S satisfy this property. **[5 marks]**

(d) Consider the following Java methods. Do they represent functions? If yes, are they injective, surjective, or bijective? Justify your answers.

```
int doubleInt(int number) { return number * 2; }
float addOneToFloat(float number) { return number + 1.0; }
```

[4 marks]

Question 2 [SAT & Predicate Logic]

(a) Let p, q, r, s be atoms capturing the states of four cells, which can either be filled or empty: p is true if the cell is filled, and false if the cell is empty, and similarly for the other atoms. Consider the following formula:

$$
\begin{array}{l}
(p \vee \neg q) \wedge (p \vee r) \wedge (p \vee s) \wedge (q \vee \neg p) \wedge (q \vee r) \wedge (q \vee s) \\
\wedge(\neg r \vee \neg p) \wedge (\neg r \vee \neg q) \wedge (\neg r \vee s) \wedge (\neg s \vee \neg p) \wedge (\neg s \vee \neg q) \wedge (\neg s \vee r)
\end{array}
$$

- (i) Using DPLL, find a valuation that shows that the above formula is satisfiable. Justify your answer as we did in the SAT lecture. **[6 marks]**
- (ii) Is the formula valid? Justify your answer. **[2 marks]**
- (iii) Explain in one sentence what property about the states of the four cells p, q, r, and s, this formula captures. **[2 marks]**

(b) Consider the following signature:

- Function symbols: `zero` (arity 0); `succ` (arity 1)
- Predicate symbols: $<$ (arity 2)

We will use infix notation for the binary symbol $<$. Consider the following formulas that capture properties of the above symbols:

- let S_1 be $\forall y.(\exists x.x < y) \to 0 < y$
- let S_2 be $\forall x.x < \mathtt{succ}(x)$

For simplicity we write 0 for `zero`, 1 for `succ(zero)`, 2 for `succ(succ(zero))`, etc.

- (i) Provide a constructive Natural Deduction proof of:

$$S_1 \to S_2 \to 0 < 2$$

(Hint: you can prove this formula without $[\forall I]$ and $[\exists E]$.) **[6 marks]**

- (ii) Explain why the following tree is not a Natural Deduction proof. Justify your answer.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overline{S_2}^{\,1}}{x < \mathtt{succ}(x)}\,[\forall E]}{\forall y.y < \mathtt{succ}(x)}\,[\forall I]}{\exists x.\forall y.y < x}\,[\exists I]}{S_2 \to \exists x.\forall y.y < x}\,1\,[\to I]
$$

(Hint: keep in mind that the $\forall$ and $\exists$ rules have side conditions.) **[4 marks]**