Solution to Problems for Week 3

Yunwen Lei

School of Computer Science
University of Birmingham

Problem 1

Two dice are tossed. Let X be the absolute difference in the number of dots facing up.

1. Find the PMF of X.
2. Find the CDF of X.

Solution

We have the table

	1	2	3	4	5	6
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2	1	0	1	2	3
4	3	2	1	0	1	2
5	4	3	2	1	0	1
6	5	4	3	2	1	0

Solution

We have the table

	1	2	3	4	5	6
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2	1	0	1	2	3
4	3	2	1	0	1	2
5	4	3	2	1	0	1
6	5	4	3	2	1	0

It is clear

$$P_X(0) = \mathbb{P}(\{X = 0\}) = 6/36, \quad P_X(1) = \mathbb{P}(\{X = 1\}) = 10/36.$$

Solution

We have the table

	1	2	3	4	5	6
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2	1	0	1	2	3
4	3	2	1	0	1	2
5	4	3	2	1	0	1
6	5	4	3	2	1	0

It is clear

$$P_X(0) = \mathbb{P}(\{X = 0\}) = 6/36, \quad P_X(1) = \mathbb{P}(\{X = 1\}) = 10/36.$$

In a similar way we know

$$P_X(2) = \frac{8}{36}, \quad P_X(3) = \frac{6}{36}, \quad P_X(4) = \frac{4}{36}, \quad P_X(5) = \frac{2}{36}.$$

Solution

Then

$$F_X(0) = \mathbb{P}(\{X \leq 0\}) = P_X(0) = \frac{6}{36}$$
$$F_X(1) = \mathbb{P}(\{X \leq 1\}) = P_X(0) + P_X(1) = \frac{6+10}{36} = \frac{16}{36}$$
$$F_X(2) = \mathbb{P}(\{X \leq 2\}) = P_X(0) + P_X(1) + P_X(2) = \frac{6+10+8}{36} = \frac{24}{36}$$

Solution

Then

$$F_X(0) = \mathbb{P}(\{X \leq 0\}) = P_X(0) = \frac{6}{36}$$
$$F_X(1) = \mathbb{P}(\{X \leq 1\}) = P_X(0) + P_X(1) = \frac{6+10}{36} = \frac{16}{36}$$
$$F_X(2) = \mathbb{P}(\{X \leq 2\}) = P_X(0) + P_X(1) + P_X(2) = \frac{6+10+8}{36} = \frac{24}{36}$$

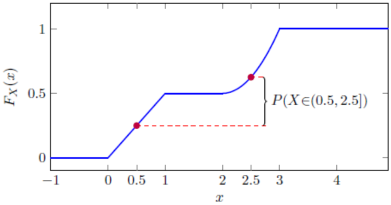
Then we get the following CDF

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{6}{36} & \text{if } x \in [0,1) \\ \frac{16}{36} & \text{if } x \in [1,2) \\ \frac{24}{36} & \text{if } x \in [2,3) \\ \frac{30}{36} & \text{if } x \in [3,4) \\ \frac{34}{36} & \text{if } x \in [4,5) \\ \frac{36}{36} & \text{if } x \geq 5. \end{cases}$$

Problem 2

Consider a continuous random variable X with a CDF given by

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ 0.5x, & \text{if } 0 \le x \le 1 \\ 0.5, & \text{if } 1 \le x \le 2 \\ 0.5(1 + (x-2)^2), & \text{if } 2 \le x \le 3 \\ 1, & \text{otherwise.} \end{cases}$$



1. Compute $\mathbb{P}(X \in (0.5, 2.5))$
2. Compute the PDF

Solution

1. $$\begin{aligned}\mathbb{P}(0.5 < X \leq 2.5) &= F_X(2.5) - F_X(0.5)\\ &= 0.5 * (1 + 0.5^2) - 0.5 * 0.5 = 0.625 - 0.25 = 0.375\end{aligned}$$

Solution

1. $$\begin{aligned}\mathbb{P}(0.5 < X \le 2.5) &= F_X(2.5) - F_X(0.5) \\ &= 0.5 * (1 + 0.5^2) - 0.5 * 0.5 = 0.625 - 0.25 = 0.375\end{aligned}$$

2. We can differentiate the CDF to obtain (recall that we can take integration of PDF to get CDF, and we can take derivative of CDF to give PDF)

$$f_X(x) = \begin{cases} 0' = 0, & \text{if } x < 0 \\ 0.5x' = 0.5, & \text{if } 0 \le x \le 1 \\ 0.5' = 0, & \text{if } 1 \le x \le 2 \\ (0.5(1 + (x-2)^2)' = x - 2, & \text{if } 2 \le x \le 3 \\ 1' = 0, & \text{otherwise.} \end{cases}$$

Problem 3

Let X have a CDF F_X. Denote $Y_1 = \max\{X, 0\}$ and $Y_2 = \min\{X, 0\}$. Compute the CDF of Y_1 and Y_2.

Solution

If $y < 0$, then (note that $\max\{X, 0\} \geq 0$ which is impossible to be less than y)

$$\mathbb{P}(Y_1 \leq y) = \mathbb{P}(\max\{X, 0\} \leq y) = 0.$$

If $y \geq 0$, then (note that $\max\{X, 0\} \leq y$ means the intersection of $A_1 = \{X \leq y\}$ and $A_2 = \{0 \leq y\}$. A_2 is the whole sample space and therefore the intersection is A_1)

$$\mathbb{P}(Y_1 \leq y) = \mathbb{P}(\max\{X, 0\} \leq y) =$$

Solution

If $y < 0$, then (note that $\max\{X, 0\} \geq 0$ which is impossible to be less than y)

$$\mathbb{P}(Y_1 \leq y) = \mathbb{P}(\max\{X, 0\} \leq y) = 0.$$

If $y \geq 0$, then (note that $\max\{X, 0\} \leq y$ means the intersection of $A_1 = \{X \leq y\}$ and $A_2 = \{0 \leq y\}$. A_2 is the whole sample space and therefore the intersection is A_1)

$$\mathbb{P}(Y_1 \leq y) = \mathbb{P}(\max\{X, 0\} \leq y) = \mathbb{P}(X \leq y) = F_X(y).$$

If $y \geq 0$, then (note that $\min\{X, 0\} \leq 0$ which is always to be less than y)

$$\mathbb{P}(Y_2 \leq y) = \mathbb{P}(\min\{X, 0\} \leq y) = 1.$$

If $y < 0$, then (note that $\min\{X, 0\} \leq y$ means the union of $A_1 = \{X \leq y\}$ and $A_2 = \{0 \leq y\}$. A_2 is empty and therefore the union is just A_1)

$$\mathbb{P}(Y_2 \leq y) = \mathbb{P}(\min\{X, 0\} \leq y) =$$

Solution

If $y < 0$, then (note that $\max\{X, 0\} \geq 0$ which is impossible to be less than y)

$$\mathbb{P}(Y_1 \leq y) = \mathbb{P}(\max\{X, 0\} \leq y) = 0.$$

If $y \geq 0$, then (note that $\max\{X, 0\} \leq y$ means the intersection of $A_1 = \{X \leq y\}$ and $A_2 = \{0 \leq y\}$. A_2 is the whole sample space and therefore the intersection is A_1)

$$\mathbb{P}(Y_1 \leq y) = \mathbb{P}(\max\{X, 0\} \leq y) = \mathbb{P}(X \leq y) = F_X(y).$$

If $y \geq 0$, then (note that $\min\{X, 0\} \leq 0$ which is always to be less than y)

$$\mathbb{P}(Y_2 \leq y) = \mathbb{P}(\min\{X, 0\} \leq y) = 1.$$

If $y < 0$, then (note that $\min\{X, 0\} \leq y$ means the union of $A_1 = \{X \leq y\}$ and $A_2 = \{0 \leq y\}$. A_2 is empty and therefore the union is just A_1)

$$\mathbb{P}(Y_2 \leq y) = \mathbb{P}(\min\{X, 0\} \leq y) = \mathbb{P}(X \leq y) = F_X(y).$$

Therefore

$$F_{Y_1}(y) = \begin{cases} 0, & \text{if } y < 0 \\ F_X(y), & \text{otherwise.} \end{cases} \qquad F_{Y_2}(y) = \begin{cases} F_X(y), & \text{if } y < 0 \\ 1, & \text{otherwise.} \end{cases}$$

Problem 4

The lifetime, X years, of a certain type of battery has probability density function given by

$$f_X(x) = \begin{cases} \frac{k}{x^2}, & \text{if } 1 \leq x \leq a \\ 0, & \text{otherwise,} \end{cases}$$

where k and a are positive constants.

1. Compute the value of k.
2. Compute the CDF.
3. Compute the probability of $X \in (a/4, a/2)$.

Solution

The CDF can be computed by ($x \in (1, a)$)

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt = \int_1^x \frac{k}{t^2}dt = -k\int_1^x dt^{-1}$$
$$= k(1 - x^{-1}).$$

Solution

The CDF can be computed by ($x \in (1, a)$)

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt = \int_1^x \frac{k}{t^2}dt = -k\int_1^x dt^{-1}$$
$$= k(1 - x^{-1}).$$

To determine k, we use the fact $F_X(a) = 1$

$$1 = k(1 - 1/a) = \frac{k(a-1)}{a} \Longrightarrow k = \frac{a}{a-1}.$$

Solution

The CDF can be computed by $(x \in (1, a))$

$$
\begin{aligned}
F_X(x) &= \int_{-\infty}^{x} f_X(t)dt = \int_{1}^{x} \frac{k}{t^2} dt = -k \int_{1}^{x} dt^{-1} \\
&= k(1 - x^{-1}).
\end{aligned}
$$

To determine k, we use the fact $F_X(a) = 1$

$$
1 = k(1 - 1/a) = \frac{k(a-1)}{a} \implies k = \frac{a}{a-1}.
$$

Since $f_X(x) = 0$ for $x \leq 1$ we always have $F_X(x) = 0$ if $x \leq 1$. Since $F_X(x) > F_X(a)$ if $x \geq a$ we have $F_X(x) = 1$ if $x \geq a$. Therefore the CDF is

$$
F_X(x) = \begin{cases} 0, & \text{if } x \leq 1 \\ \frac{a(x-1)}{(a-1)x}, & \text{if } x \in (1, a) \\ 1, & \text{otherwise.} \end{cases}
$$

Solution

If $a \geq 4$, we know $(a/4 \geq 1)$

$$\mathbb{P}(X \in (a/4, a/2)) = F_X(a/2) - F_X(a/4) = \frac{a(a/2-1)}{(a-1)a/2} - \frac{a(a/4-1)}{(a-1)a/4}$$
$$= \frac{a-2}{a-1} - \frac{4(a/4-1)}{a-1} = \frac{2}{a-1}.$$

Solution

If $a \geq 4$, we know $(a/4 \geq 1)$

$$\mathbb{P}(X \in (a/4, a/2)) = F_X(a/2) - F_X(a/4) = \frac{a(a/2-1)}{(a-1)a/2} - \frac{a(a/4-1)}{(a-1)a/4}$$
$$= \frac{a-2}{a-1} - \frac{4(a/4-1)}{a-1} = \frac{2}{a-1}.$$

If $a \in (2,4)$, we know $(a/4 < 1 < a/2)$

$$\mathbb{P}(X \in (a/4, a/2)) = F_X(a/2) - F_X(a/4) = \frac{a-2}{a-1} - 0 = \frac{a-2}{a-1}.$$

Solution

If $a \geq 4$, we know $(a/4 \geq 1)$

$$\mathbb{P}(X \in (a/4, a/2)) = F_X(a/2) - F_X(a/4) = \frac{a(a/2-1)}{(a-1)a/2} - \frac{a(a/4-1)}{(a-1)a/4}$$
$$= \frac{a-2}{a-1} - \frac{4(a/4-1)}{a-1} = \frac{2}{a-1}.$$

If $a \in (2,4)$, we know $(a/4 < 1 < a/2)$

$$\mathbb{P}(X \in (a/4, a/2)) = F_X(a/2) - F_X(a/4) = \frac{a-2}{a-1} - 0 = \frac{a-2}{a-1}.$$

If $a \in [1,2]$, we know $(a/2 \leq 1)$

$$\mathbb{P}(X \in (a/4, a/2)) = F_X(a/2) - F_X(a/4) = 0 - 0 = 0.$$

Solution

If $a \geq 4$, we know $(a/4 \geq 1)$

$$\mathbb{P}(X \in (a/4, a/2)) = F_X(a/2) - F_X(a/4) = \frac{a(a/2-1)}{(a-1)a/2} - \frac{a(a/4-1)}{(a-1)a/4}$$
$$= \frac{a-2}{a-1} - \frac{4(a/4-1)}{a-1} = \frac{2}{a-1}.$$

If $a \in (2,4)$, we know $(a/4 < 1 < a/2)$

$$\mathbb{P}(X \in (a/4, a/2)) = F_X(a/2) - F_X(a/4) = \frac{a-2}{a-1} - 0 = \frac{a-2}{a-1}.$$

If $a \in [1,2]$, we know $(a/2 \leq 1)$

$$\mathbb{P}(X \in (a/4, a/2)) = F_X(a/2) - F_X(a/4) = 0 - 0 = 0.$$

Therefore,

$$\mathbb{P}(X \in (a/4, a/2)) = \begin{cases} \frac{2}{a-1}, & \text{if } a \geq 4 \\ \frac{a-2}{a-1}, & \text{if } a \in (2,4) \\ 0, & \text{otherwise.} \end{cases}$$

Problem 5

Suppose that X has PDF

$$f_1(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{1}{(1+x)^2}, & \text{otherwise.} \end{cases}$$

Suppose Y has PDF

$$f_2(y) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{1}{1+y}, & \text{otherwise.} \end{cases}$$

1. Is f_1 a well-defined PDF?
2. Is f_2 a well-defined PDF?

Solution

The integral of f_1 is

$$\int_{-\infty}^{\infty} f_1(x)dx = \int_0^{\infty} \frac{1}{(1+x)^2}dx = -\int_0^{\infty} d(1+x)^{-1} = -(1+x)^{-1}|_0^{\infty} = 1$$

Solution

The integral of f_1 is

$$\int_{-\infty}^{\infty} f_1(x)dx = \int_0^{\infty} \frac{1}{(1+x)^2}dx = -\int_0^{\infty} d(1+x)^{-1} = -(1+x)^{-1}|_0^{\infty} = 1$$

The integral of f_2 is

$$\int_{-\infty}^{\infty} f_2(y)dy = \int_0^{\infty} \frac{1}{1+y}dx = \int_0^{\infty} d\log(y+1) = \log(y+1)|_0^{\infty} = \infty.$$

We know a well defined density function should have the integration 1 when taking the integration over the range. Therefore, f_1 is a well-defined PDF while f_2 is not.