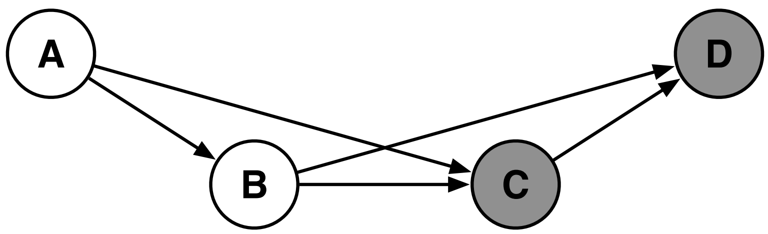
CSE 250a. Assignment 6

Out: *Tue Nov 01*
Due: *Tue Nov 08* (by 11:59 PM, Pacific Time, via gradescope)
Grace period: 48 hours

6.1 EM algorithm



(a) **Posterior probability**

Consider the belief network shown above, with observed nodes C and D and hidden nodes A and B. Compute the posterior probability $P(a, b|c, d)$ in terms of the CPTs of the belief network—that is, in terms of $P(a)$, $P(b|a)$, $P(c|a, b)$ and $P(d|b, c)$.

(b) **Posterior probability**

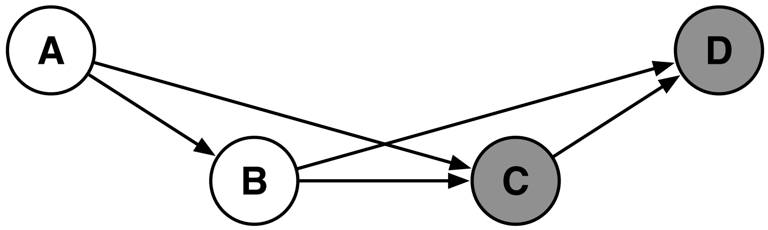
Compute the posterior probabilities $P(a|c, d)$ and $P(b|c, d)$ in terms of your answer from part (a); that is, for this problem, you may assume that $P(a, b|c, d)$ is given.

(c) **Log-likelihood**

Consider a partially complete data set of *i.i.d.* examples $\{c_t, d_t\}_{t=1}^{T}$ drawn from the joint distribution of the above belief network. The log-likelihood of the data set is given by:

$$\mathcal{L} = \sum_t \log P(C\!=\!c_t, D\!=\!d_t).$$

Compute this log-likelihood in terms of the CPTs of the belief network. You may re-use work from earlier parts of the problem.



(d) **EM algorithm**

Give the EM updates to estimate CPTs that maximize the log-likelihood in part (c); in particular, complete the numerator and denominator in the below expressions for the update rules. Simplify your answers as much as possible, expressing them in terms of the posterior probabilities $P(a, b|c_t, d_t)$, $P(a|c_t, d_t)$, and $P(b|c_t, d_t)$, as well as the functions $I(c, c_t)$, and $I(d, d_t)$.

$$P(A\!=\!a) \;\leftarrow\; \overline{\qquad\qquad\qquad\qquad}$$

$$P(B\!=\!b|A\!=\!a) \;\leftarrow\; \overline{\qquad\qquad\qquad\qquad}$$

$$P(C\!=\!c|A\!=\!a, B\!=\!b) \;\leftarrow\; \overline{\qquad\qquad\qquad\qquad}$$

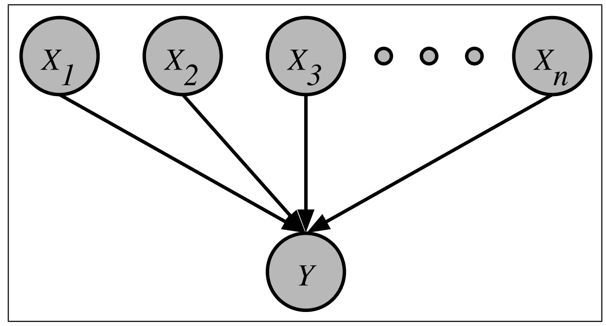
$$P(D\!=\!d|B\!=\!b, C\!=\!c) \;\leftarrow\; \overline{\qquad\qquad\qquad\qquad}$$

6.2 EM algorithm for noisy-OR

Consider the belief network on the right, with binary random variables $X \in \{0,1\}^n$ and $Y \in \{0,1\}$ and a noisy-OR conditional probability table (CPT). The noisy-OR CPT is given by:

$$P(Y=1|X) \;=\; 1 - \prod_{i=1}^{n}(1-p_i)^{X_i},$$

which is expressed in terms of the noisy-OR parameters $p_i \in [0,1]$.



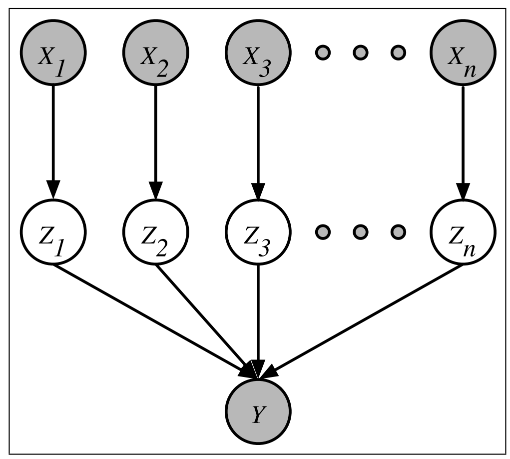
In this problem, you will derive and implement an EM algorithm for estimating the noisy-OR parameters p_i. It may seem that the EM algorithm is not suited to this problem, in which all the nodes are observed, and the CPT has a parameterized form. In fact, the EM algorithm can be applied, but first we must express the model in a different but equivalent form.

Consider the belief network shown to the right. In this network, a binary random variable $Z_i \in \{0,1\}$ intercedes between each pair of nodes X_i and Y. Suppose that:

$$\begin{aligned} P(Z_i\!=\!1|X_i\!=\!0) &= 0, \\ P(Z_i\!=\!1|X_i\!=\!1) &= p_i. \end{aligned}$$

Also, let the node Y be *determined* by the logical-OR of Z_i. In other words:

$$P(Y\!=\!1|Z) \;=\; \begin{cases} 1 \text{ if } Z_i\!=\!1 \text{ for any } i, \\ 0 \text{ if } Z_i\!=\!0 \text{ for all } i. \end{cases}$$



(a) Show that this "extended" belief network defines the same conditional distribution $P(Y|X)$ as the original one. In particular, starting from

$$P(Y\!=\!1|X) \;=\; \sum_{Z\in\{0,1\}^n} P(Y\!=\!1, Z|X),$$

show that the right hand side of this equation reduces to the noisy-OR CPT with parameters p_i. To perform this marginalization, you will need to exploit various conditional independence relations.

(b) Consider estimating the noisy-OR parameters p_i to maximize the (conditional) likelihood of the observed data. The (normalized) log-likelihood in this case is given by:

$$\mathcal{L} \;=\; \frac{1}{T}\sum_{t=1}^{T} \log P(Y\!=\!y^{(t)}|X\!=\!\vec{x}^{(t)}),$$

where $(\vec{x}^{(t)}, y^{(t)})$ is the tth joint observation of X and Y, and where for convenience we have divided the overall log-likelihood by the number of examples T. From your result in part (a), it follows that we can estimate the parameters p_i in either the original network or the extended one (since in both networks they would be maximizing the same equation for the log-likelihood).

Notice that in the extended network, we can view X and Y as observed nodes and Z as hidden nodes. Thus in this network, we can use the EM algorithm to estimate each parameter p_i, which simply defines one row of the "look-up" CPT for the node Z_i.

Compute the posterior probability that appears in the E-step of this EM algorithm. In particular, for joint observations $x \in \{0,1\}^n$ and $y \in \{0,1\}$, use Bayes rule to show that:

$$P(Z_i\!=\!1, X_i\!=\!1|X\!=\!x, Y\!=\!y) \;=\; \frac{y x_i p_i}{1-\prod_j (1-p_j)^{x_j}}$$

(c) For the data set $\{\vec{x}^{(t)}, y^{(t)}\}_{t=1}^{T}$, show that the EM update for the parameters p_i is given by:

$$p_i \;\leftarrow\; \frac{1}{T_i}\sum_t P\left(Z_i\!=\!1, X_i\!=\!1|X\!=\!x^{(t)}, Y\!=\!y^{(t)}\right),$$

where T_i is the number of examples in which $X_i\!=\!1$. (You should derive this update as a special case of the general form presented in lecture.)

(d) Download the data files on the course web site, and use the EM algorithm to estimate the parameters p_i. The data set[1] has $T = 267$ examples over $n = 23$ inputs. To check your solution, initialize all $p_i = 0.05$ and perform 256 iterations of the EM algorithm. At each iteration, compute the log-likelihood shown in part (b). (If you have implemented the EM algorithm correctly, this log-likelihood will always increase from one iteration to the next.) Also compute the number of mistakes $M \leq T$ made by the model at each iteration; a mistake occurs either when $y_t = 0$ and $P(y_t = 1|\vec{x}_t) \geq 0.5$ (indicating a false positive) or when $y_t\!=\!1$ and $P(y_t\!=\!1|\vec{x}_t) \leq 0.5$ (indicating a false negative). The number of mistakes should generally decrease as the model is trained, though it is not guaranteed to do so at each iteration. Complete the following table:

iteration	number of mistakes M	log-likelihood $\mathcal{L}$
0	175	-0.95809
1	56	
2		-0.40822
4		
8		
16		
32		
64	37	
128		
256		-0.31016

You may use the already completed entries of this table to check your work.

(e) Turn in your source code. As always, you may program in the language of your choice.

[1]For those interested, more information about this data set is available at *http://archive.ics.uci.edu/ml/datasets/SPECT+Heart*. However, be sure to use the data files provided on Canvas, as they have been specially assembled for this assignment.

6.3 Auxiliary function

In class we derived an auxiliary function for *maximizing* the log-likelihood in belief networks with hidden variables. In this problem you will derive an auxiliary function for *minimizing* a simpler function that is nearly quadratic near its minimum, but nearly linear far away from its minimum.

(a) Consider the function $f(x) = \log\cosh(x)$. Show that the minimum occurs at $x = 0$.

(b) Show that $f''(x) \leq 1$ for all x.

(c) Consider the function $Q(x, y) = f(y) + f'(y)(x - y) + \frac{1}{2}(x - y)^2$. Plot $f(x)$, $Q(x, -2)$, and $Q(x, 3)$ as a function of x.

(d) Prove that $Q(x, y)$ is an auxiliary function for $f(x)$. In particular, show that it satisfies:
(i) $Q(x, x) = f(x)$
(ii) $Q(x, y) \geq f(x)$
Hint: use part (b), and note that $f(x) = f(y) + \int_y^x du f'(u) = f(y) + \int_y^x du \left[f'(y) + \int_y^u dv f''(v)\right]$.

(e) Derive the form of the update rule $x_{n+1} = \text{argmin}_x Q(x, x_n)$.

(f) Write a simple program to show that your update rule in (e) converges numerically for the initial guesses $x_0 = -2$ and $x_0 = 3$. **Turn in your source code as well as plots (or tables) of x_n versus n.**