MATH 211: HOMEWORK 6

Book problems

Sec 5.1: 2ab
Sec 5.2: 2ab, 4ab
Sec 5.3: 9abc

Problem 1

Consider the spring problem $x''(t) = -kx(t)$ where $x : [1, 6] \to \mathbb{R}$ and $x(1) = 0$, $x'(1) = 1$. Let $k = 5$.

(a) Rewrite this in the form

$$y'(t) = Ay(t), y(1) = y_0$$

for $y(t) = \begin{pmatrix} x(t) \\ p(t) \end{pmatrix}$, $p(t) = x'(t)$, and A a matrix.

(b) Show the energy $E = \frac{1}{2}\Bigg([p(t)]^2 + k[x(t)]^2\Bigg)$ is constant in time. This implies the trajectory should lie on an ellipse in the coordinates (x, p).

(c) Use Euler's method with step size $h = .01$ to numerically approximate $y(t)$. Plot the error $|\tilde{\omega}_i - \omega_i|$ against t_i's where $t_i = 1 + ih$, ω_i are the Euler's method iterative values, and $\tilde{\omega}_i$ are the Euler's method values evaluated at the time steps t_i, but with h 10 times smaller. In other words, there are 9 Euler steps in between each $\tilde{\omega}_i$. We use these $\tilde{\omega}_i$ as the *true* solution, though obviously as it is Euler's method it will be limited in accuracy as well.

(d) If you think of $y(t)$ as a particle in $\mathbb{R}^2$ (also called phase space), make a movie of the Euler's method just computed showing the particle move in phase space, plotting two separate scatter points corresponding to ω_i and $\tilde{\omega}_i$. Draw the ellipse from part (b) corresponding to the trajectory on the same plot so the viewer can see how well Euler's method stays on the trajectory. Make the screen size $[-3, 3] \times [3, 3]$.

Problem 2

Consider the problem

$$\begin{pmatrix} x'(t) \\ p'(t) \end{pmatrix} = \begin{pmatrix} p(t) \\ g(x(t)) \end{pmatrix}.$$

Date: today.

Find the third order Taylor's method, with initial conditions $x(a) = x_0$, $p(a) = p_0$. In other words, find a formula for all $\omega_i = (x_i, p_i)$ iterations with time step size h. *Your answer for ω_{i+1} should only be in terms of ω_i (i.e. x_i and p_i) and g and its derivatives.*

Problem 3

Write down the Taylor's method of order 3 algorithm, i.e. the relation of the ω_i steps for the following IVP:

$$y'(t) = e^{-ty}, \qquad y(0) = 1.$$

Let h be the time step with starting time being 0. Your answer should be of the form

$$\omega_0 = 1$$
$$\omega_{i+1} = \omega_i + \cdots .$$