MATH 211: HOMEWORK 7

Book problems

Sec 5.4: 27, 30, 31, 32

Problem 1

Suppose we have $n = 10$ particles living in $\mathbb{R}$ with positions $x_1(t), \cdots x_n(t) \in \mathbb{R}$. Let $x = (x_1, \cdots x_n)$. Suppose the initial conditions are $x_j(0) = j$ and $x_j'(0) = 0$ for all j, except we let $x_1(0) = .5$. Let $u_j = x_j - j$ be the displacement. Then the force equations are

$$\begin{aligned}
u_1''(t) &= -10(u_1(t) - u_2(t)) \\
u_n''(t) &= -10(u_n(t) - u_{n-1}(t)) \\
u_j''(t) &= -10(2u_j(t) - u_{j-1}(t) - u_{j+1}(t)) \quad \textit{otherwise.}
\end{aligned}$$

Note this corresponds to springs attached together in 1D.

(a) Express your solution as an Initial Value Problem, i.e., $y'(t) = f(t, y(t))$, and $y(0) = y_0$ for some y_0 and f.

(b) For time range $[0, 8]$, time-step $h = .05$, approximate the solutions using 4^{th} order Runge-Kutta, and make a movie of the positions $x_j(t)$. Fix the viewing window at $[0, 11] \times [-1, 1]$.

(c) Using the same Runge-Kutta scheme, approximate $y(8)$ for the following time-steps: $h \in 2^{-s}$, $s = 4, 5, \cdots 13$. Treat the last as your *true* solution $y(t)$. Show the loglog plot of your error at time $t = 8$, i.e. if your simulation approximation of $y(8)$ is ω_h, give the loglog plot of h vs $\|y(8) - \omega_h)\|$, where $\| \cdot \|$ is defined here by

$$\|y\| = \sum_{j=1}^{n} |y_j|.$$

In addition, plot the line corresponding to the global error you expect on the same loglog plot. To do this, plot h vs h^p in the loglog plot, where p is the order you expect. The error curve and the line should have roughly the same slope.

Problem 2

Suppose we have a ball hanging on a spring with a time-dependent force acting on it. The ball's position is denoted $x(t)$ as a function of t. The equation of motion is given by

$$x''(t) = -3x(t) - \sin(t), \qquad\qquad x(0) = 0, x'(0) = 1.$$

Date: today.

(a) Rewrite the problem into a form $y'(t) = f(t, y(t))$, $y(0) = y_0$. y will be vector-valued.
(b) Write down the second order Taylor algorithm for this problem, i.e. the equations $\omega_0 = y_0$ and $\omega_{i+1} = \omega_i + h\phi(t_i, \omega_i)$. Write out fully what ϕ is. Here $t_{i+1} - t_i = h$ for all i, and $t_0 = 0$.
(c) Find α and β such that $\omega_{i+1} = \omega_i + hf(t_i + \alpha, \omega_i + \beta f(t_i, \omega_i))$ has the same order of convergence as the Taylor algorithm in part (b).